Symmetrizations

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Chapter 5
Symmetrizations

Gabriele Bianchi and Paolo Gronchi

Abstract In this chapter we present the operation of symmetrization of sets and briefly touch on the strictly connected operation of rearrangement of functions. We define the main known symmetrizations and, for each of them, we describe its main properties. We also define an abstract setting for dealing with symmetrizations and we present some characterizations of Minkowski and Steiner symmetrizations and of polarization in terms of their properties. We also present shadow systems, and the topic of convergence of successive symmetrals.

5.1 Introduction

The idea of replacing an object by one that retains some of its features but is in some sense more symmetrical has been extremely fruitful over the years. The object may be a set or a function, for example, and the process is then often called symmetrization or rearrangement, respectively. Steiner symmetrization, introduced by Jakob Steiner around 1836 in his attempt to prove the isoperimetric inequality, is still today a potent tool for establishing crucial inequalities in geometry. The influence of such inequalities, which often have analytical versions, extends far beyond geometry to other areas such as analysis and PDEs, and even outside mathematics, to economics and finance.

The topic received a huge boost in 1951 from the classic text of Pólya and Szegő [74]. By this time, many other types of symmetrization had been introduced, with similar applications. The general idea is to find a symmetrization that preserves

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one physical quantity, while not increasing (or sometimes not reducing) another. As well as volume, surface area, and mean width, the book by Pólya and Szegő considers electrostatic capacity, principal frequency (the first eigenvalue of the Laplacian), and torsional rigidity, thereby extending the scope to mathematical physics.

A beautiful and quite recent survey on rearrangements by Talenti [86] contains a comprehensive bibliography, conveniently divided between the main periods of development.

In Section 5.3 we introduce the abstract setting used to deal with symmetrizations and present in detail all the known ones, describing for each of them the main properties. Section 5.4 is devoted to the notion of shadow systems, a very fruitful way of seeing Steiner symmetrization very effective in proving inequalities. In Section 5.5 we present new expressions for Steiner and Minkowski symmetrizations that bring to light the dual relationship between them. We also present some containment relations between different symmetrals. Section 5.6 presents characterizations of Steiner and Minkowski symmetrizations and of polarization. Section 5.7 deals with the convergence of sequences of iterated symmetrizations of a set to a ball. We summarize what is known up to date, which questions remain unanswered and how certain properties can be generalized to different symmetrization processes and to the class of compact sets. The last section, Section 5.8, first briefly present the connections between rearrangements and symmetrization and then it describes a recent unifying proof of the Pólya–Szegő inequality valid for many rearrangements.

5.2 Preliminaries

Throughout this chapter we use the notation and many of the notions which have been introduced in Chapter 1, but we need to introduce some more notation.

Let D^n be the open unit ball in $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$ we write $[x, y]$ for the line segment with endpoints x and y. If X is a set, we denote by $\operatorname{conv} X$, $\operatorname{clo} X$, and $\dim X$ the *convex hull*, *closure*, and *dimension* (that is, the dimension of the affine hull) of X, respectively.

Throughout the paper, the term *subspace* means a linear subspace. The Grassmannian of k-dimensional subspaces in $\mathbb{R}^n$ is denoted by $\mathcal{G}(n, k)$. If H is a subspace of $\mathbb{R}^n$, then $X|H$ is the (orthogonal) projection of X on H and $x|H$ is the projection of a vector $x \in \mathbb{R}^n$ on H. Moreover, $X^\dagger$ denotes the *reflection* of X in H, i.e., the image of X under the map that takes $x \in \mathbb{R}^n$ to $2(x|H) - x$. If $x \in \mathbb{R}^n \setminus \{o\}$, then $x^\perp$ is the $(n-1)$-dimensional subspace orthogonal to x.

If $X^\dagger = X$, we say X is *H-symmetric*. If $H = \{o\}$, we instead write $-X = (-1)X$ for the reflection of X in the origin and *o-symmetric* for $\{o\}$-symmetric. A set X is called *rotationally symmetric* with respect to the i-dimensional subspace H if for all $x \in H$, $X \cap (H^\perp + x)$ is a union of $(n-i-1)$-dimensional spheres, each with center at x. If $\dim H = n-1$, then a compact convex set is rotationally symmetric with respect to H if and only if it is H-symmetric. The term *H-symmetric spherical cylinder* will always mean a set of the form $D_r(x) + s(B^n \cap H^\perp) = D_r(x) \times s(B^n \cap H^\perp)$, where

$s > 0$ and $D_r(x) \subset H$ is the ball with $\dim D = \dim H$, center x, and radius $r > 0$. Of course, H-symmetric spherical cylinders are rotationally symmetric with respect to both H and $H^\perp$. The phrase *translate orthogonal to* H means translate by a vector in $H^\perp$.

We write $\mathcal{H}^k$ for k-dimensional Hausdorff measure in $\mathbb{R}^n$, where $k \in \{1, \dots, n\}$. We denote by $\mathcal{C}^n$, $\mathcal{M}^n$, and $\mathcal{L}^n$ the class of non-empty compact sets, $\mathcal{H}^n$-measurable sets, and $\mathcal{H}^n$-measurable sets of finite $\mathcal{H}^n$-measure, respectively, in $\mathbb{R}^n$. Let $\mathcal{K}^n$ be the class of *convex bodies*, i.e. non-empty compact convex subsets of $\mathbb{R}^n$, and let $\mathcal{K}^n_n$ be the class of members of $\mathcal{K}^n$ with interior points. For $K \in \mathcal{K}^n$, $S(K)$ denotes its *surface area*, defined in Section 1.9. If $K \in \mathcal{K}^n_n$ then $S(K) = \mathcal{H}^{n-1}(\partial K)$. By κ_n we denote the volume $\mathcal{H}^n(B^n)$ of the unit ball in $\mathbb{R}^n$.

The *Blaschke addition* $K \sharp L$ of $K, L \in \mathcal{K}^n_n$ is a convex body whose surface area measure is

$$S_{n-1}(K \sharp L, \cdot) = S_{n-1}(K, \cdot) + S_{n-1}(L, \cdot).$$

The existence of this body is a consequence of Minkowski's existence theorem, Theorem 1.46. The body $K \sharp L$ is determined up to a translation.

Let $H \in \mathcal{G}(n, i)$, $i \in \{0, \dots, n\}$. If $p \in \mathbb{R}^n$, write $p = (x, y)$, where $x \in H$ and $y \in H^\perp$ satisfy $p = x + y$. Suppose that $s, t \in \mathbb{R}$ and $K, L \in \mathcal{K}^n$. The *fiber combination* $(s \circ K) \uplus_H (t \circ L)$ of K and L relative to H, defined by

$$(s \circ K) \uplus_H (t \circ L) = \{(x, sy + tz) : (x, y) \in K, (x, z) \in L\}, \tag{5.1}$$

was introduced by McMullen [65], who noted that $(s \circ K) \uplus_H (t \circ L) \in \mathcal{K}^n$, $(s \circ K) \uplus_H (t \circ L) = sK + tL$ if $i = 0$, and $K \uplus_H L = K \cap L$ if $i = n$. (We have adapted the definition in [65] to suit our purposes.)

We recall the *Brunn–Minkowski inequality for intrinsic volumes*.

Theorem 5.1 *If* $K, L \in \mathcal{K}^n$ *and* $i \in \{1, \dots, n\}$ *then*

$$V_i(K + L)^{\frac{1}{i}} \geq V_i(K)^{\frac{1}{i}} + V_i(L)^{\frac{1}{i}} \tag{5.2}$$

When $i = n$ this is the Brunn–Minkowski inequality, Theorem 3.23. When $i \neq n$ this is a consequence of the fact that $V_i(K)$ equals, up to a positive multiplicative constant, the mixed volume $V(K[i], B^n[n-i])$ and that the $1/i$-power of this mixed volume is concave with respect to K, as stated in the Generalized Brunn–Minkowski inequality, Theorem 3.24.

When dealing with relationships between sets in $\mathbb{R}^n$ or functions on $\mathbb{R}^n$, the term *essentially* means up to a set of $\mathcal{H}^n$-measure zero. If $A \in \mathcal{M}^n$,

$$\Theta(A, x) = \lim_{r \to 0^+} \frac{\mathcal{H}^n\big(A \cap (x + rD^n)\big)}{\mathcal{H}^n(x + rD^n)}, \tag{5.3}$$

is the *density* of A at x, provided the limit exists. Moreover, we define

$$A^* = \{x \in \mathbb{R}^n : \Theta(A, x) = 1\}.$$

Elements of A^* are called *Lebesgue density points*, or simply density points, of A. Note that $A^* = A$, essentially, by the Lebesgue density theorem. Given $A \in \mathcal{M}(\mathbb{R}^n)$ and $C \in \mathcal{K}^n_n$ containing o in its interior, let $\overline{\mathcal{M}}^*_C(A)$ and $\overline{\mathcal{M}}_{*C}(A)$ denote, respectively, its *upper* and *lower anisotropic outer Minkowski content* with respect to C, i.e.,

$$\begin{aligned} \overline{\mathcal{M}}^*_C(A) &= \limsup_{\varepsilon \to 0+} \frac{\mathcal{H}^n(A + \varepsilon C) - \mathcal{H}^n(A)}{\varepsilon}, \\ \overline{\mathcal{M}}_{*C}(A) &= \liminf_{\varepsilon \to 0+} \frac{\mathcal{H}^n(A + \varepsilon C) - \mathcal{H}^n(A)}{\varepsilon}. \end{aligned} \tag{5.4}$$

We observe that the limits are unchanged if C is replaced by $\operatorname{int} C$. When $C = B^n$ they are called upper and lower outer Minkowski content. When the two limits coincide we denote them by $\overline{\mathcal{M}}_C(A)$. When $A \in \mathcal{K}^n$ these limits coincide and $\overline{\mathcal{M}}_C(A)$ coincides with the surface area of A and with its perimeter. In Chapter 3.3.1 essentially the same concept is considered, where the lim sup and the lim inf are denoted as the upper and lower (outer) relative surface area of A with respect to C.

Let $\mathcal{M}(\mathbb{R}^n)$ (or $\mathcal{M}_+(\mathbb{R}^n)$) denote the set of real-valued (or non-negative, respectively) measurable functions on $\mathbb{R}^n$ and let $\mathcal{S}(\mathbb{R}^n)$ denote the set of functions f in $\mathcal{M}(\mathbb{R}^n)$ such that $\mathcal{H}^n(\{x : f(x) > t\}) < \infty$ for $t > \operatorname{ess\,inf} f$. By $\mathcal{V}(\mathbb{R}^n)$, we denote the set of functions f in $\mathcal{M}_+(\mathbb{R}^n)$ such that $\mathcal{H}^n(\{x : f(x) > t\}) < \infty$ for $t > 0$. Members of $\mathcal{S}(\mathbb{R}^n)$ have been called *symmetrizable* and those of $\mathcal{V}(\mathbb{R}^n)$ are often said to *vanish at infinity*. If $f \in \mathcal{M}(\mathbb{R}^n)$, we denote its *graph* by G_f and define its *subgraph* $K_f \subset \mathbb{R}^{n+1}$ by

$$K_f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \geq t\}. \tag{5.5}$$

5.3 *i*-Symmetrization: Properties and Examples

Let $i \in \{0, \dots, n-1\}$ and let $H \in \mathcal{G}(n,i)$ be fixed. Let $\mathcal{B} \subset \mathcal{C}^n$ be a class of non-empty compact sets in $\mathbb{R}^n$ and let $\mathcal{B}_H$ denote the subclass of members of $\mathcal{B}$ that are H-symmetric. We call a map $\diamondsuit : \mathcal{B} \to \mathcal{B}_H$ an *i-symmetrization* on $\mathcal{B}$ (with respect to H). If $K \in \mathcal{B}$, the corresponding set $\diamondsuit K$ is called a *symmetral*. We consider the following properties, where it is assumed that the class $\mathcal{B}$ is appropriate for the properties concerned and that they hold for all $K, L \in \mathcal{B}$. Recall that $K^\dagger$ is the reflection of K in H.

1. (*Monotonicity* or *strict monotonicity*) $K \subset L \Rightarrow \diamondsuit K \subset \diamondsuit L$ (or $\diamondsuit K \subset \diamondsuit L$ and $K \neq L \Rightarrow \diamondsuit K \neq \diamondsuit L$, respectively).

2. (*F-preserving*) $F(\diamondsuit K) = F(K)$, where $F : \mathcal{B} \to [0, \infty)$ is a set function. In particular, we can take $F = V_j$, $j = 1, \dots, n$, the jth intrinsic volume, though we generally prefer to write *mean width preserving*, *surface area preserving*, and *volume* (or *measure*) *preserving* when $j = 1$, $n-1$, and n, respectively.

3. (*Idempotent*) $\diamondsuit^2 K = \diamondsuit(\diamondsuit K) = \diamondsuit K$.

4. (*Invariance on H-symmetric sets*) $K^\dagger = K \Rightarrow \diamondsuit K = K$.

5. (*Invariance on H-symmetric spherical cylinders*) If $K = D_r(x)+s(B^n \cap H^\perp)$, where $s > 0$ and $D_r(x) \subset H$ is the i-dimensional ball with center x and radius $r > 0$, then $\Diamond K = K$.

6. (*Projection invariance*) $(\Diamond K)|H = K|H$.

7. (*Invariance under translations orthogonal to H of H-symmetric sets*) If K is H-symmetric and $y \in H^\perp$, then $\Diamond(K + y) = \Diamond K$.

In this chapter sometimes we will also consider sets in $\mathcal{L}^n$, since some of the symmetrizations that we deal with can be defined also in this class. When $\mathcal{B} \subset \mathcal{L}^n$ the property of being monotonic has to be intended up to sets of measure zero: $\Diamond$ is monotonic if $K \subset L$, essentially, implies $\Diamond K \subset \Diamond L$, essentially.

With the application to symmetrizations of sets in $\mathcal{L}^n$ in mind we introduce another property of i-symmetrizations.

8. (*Smoothing*) For each $d > 0$ and bounded $K \in \mathcal{L}^n$,

$$(\Diamond^* K) + dB^n \subset \Diamond^*(K + dB^n) \subset \Diamond(K + dB^n), \tag{5.6}$$

essentially, where, for $A \in \mathcal{L}^n$, $\Diamond^* A$ is defined by

$$\Diamond^* A = (\Diamond A)^*.$$

We remark that $(\Diamond^* K) + dB^n = (\Diamond^* K) + dD^n$ is open (see [11, Lemma 2.1] for a proof). Lemma 4.4 in [11] proves that the definition of smoothing can be rephrased in other ways.

Lemma 5.2 *The following statements are equivalent.*

i) $\Diamond$ is smoothing (in the sense of (5.6)*).*

ii) For each $d > 0$ and bounded $K \in \mathcal{L}^n$, we have

$$(\Diamond^* K) + dD^n \subset \Diamond^*(K + dD^n). \tag{5.7}$$

iii) For each $d > 0$ and bounded $K \in \mathcal{L}^n$, (5.7) *holds essentially.*

Sometimes when dealing with symmetrizations defined on $\mathcal{B} \subset \mathcal{C}^n$ or $\mathcal{K}^n$, we refer to smoothing as satisfying, for $d > 0$ and $K \in \mathcal{B}$,

$$(\Diamond K) + dB^n \subset \Diamond(K + dB^n). \tag{5.8}$$

Remark If $\Diamond$ is measure preserving and smoothing then $\mathcal{H}^n((\Diamond^* K) + \varepsilon B^n) \le \mathcal{H}^n(\Diamond^*(K + \varepsilon B^n)) = \mathcal{H}^n(K + \varepsilon B^n)$ and $\mathcal{H}^n(\Diamond^* K) = \mathcal{H}^n(K)$, for $K \in \mathcal{L}^n$ and $\varepsilon > 0$. This implies

$$\frac{\mathcal{H}^n((\Diamond^* K) + \varepsilon B^n) - \mathcal{H}^n(\Diamond^* K)}{\varepsilon} \le \frac{\mathcal{H}^n(K + \varepsilon B^n) - \mathcal{H}^n(K)}{\varepsilon},$$

and that $\Diamond$ reduces the upper and lower outer Minkowski content (and the perimeter, when K and $\Diamond^* K$ are set whose outer and lower Minkowski content coincide with the perimeter).

Invariance under translations orthogonal to H of H-symmetric sets, as well as being idempotent, are satisfied by most natural symmetrizations, so in discussing examples we shall only mention these properties when they do not hold.

Two special cases are of particular importance: $i = 0$ and $i = n - 1$. If $i = 0$, then $H = \{o\}$ and 0-symmetrization is the same as the o-symmetrization. One example of 0-symmetrization is *central symmetrization*, given for $K \in \mathcal{K}^n$ by

$$\Diamond K = \Delta K = \frac{1}{2}K + \frac{1}{2}(-K). \tag{5.9}$$

The central symmetral ΔK differs from the ubiquitous *difference body* $DK = K + (-K)$ only by a dilatation factor of $1/2$. It is a particular instance of *Minkowski symmetrizations* that we will define in a moment. Other examples of 0-symmetrizations are the *pth central symmetrization*, given for $K \in \mathcal{K}^n$ and $p \geq 1$ by

$$\Diamond K = \Delta_p K = \left(2^{-1/p} K\right) +_p \left(2^{-1/p}(-K)\right)$$

(here $+_p$ denotes the general L_p addition introduced in Section 3.4.1) and the M-symmetrization. For its definition and for more on 0-symmetrizations we refer the reader to Gardner, Hug, and Weil [38] and to Bianchi, Gardner, and Gronchi [7, Section 4].

The other case of particular importance is $i = n - 1$, to which we will devote more attention.

5.3.1 Steiner Symmetrization

The prime example of an $(n-1)$-symmetrization is *Steiner symmetrization*. If $K \in \mathcal{C}^n$, the *Steiner symmetral* of K with respect to $H \in \mathcal{G}(n, n-1)$ is the set $S_H K$ such that for each line G orthogonal to H and meeting K, the set $G \cap S_H K$ is a (possibly degenerate) closed line segment with midpoint in H and $\mathcal{H}^1$-measure equal to that of $G \cap K$. An extension of this definition to Lebesgue measurable subsets of $\mathbb{R}^n$ is possible and is presented below in the more general setting of Schwarz symmetrization. We list some of its properties.

a) If $K \in \mathcal{C}^n$, then $S_H K \in \mathcal{C}^n$, and if $K \in \mathcal{K}^n$, then $S_H K \in \mathcal{K}^n$. The first claim is elementary and we prove the second one. Let $u \in \mathbb{S}^{n-1}$ be orthogonal to H. There are two functions $f, g : K|H \to \mathbb{R}$ such that we can describe $K \in \mathcal{K}^n$ as

$$K = \{x + tu \in \mathbb{R}^n : g(x) \leq t \leq f(x)\}, \quad x \in K|H.$$

Since K is convex, the function g is convex and f is concave. We have

$$S_H K = \{x + tu \in \mathbb{R}^n : -(f(x) - g(x))/2 \leq t \leq (f(x) - g(x))/2\},$$

and this shows that $S_H K$ is convex.

b) On $\mathcal{K}^n$, Steiner symmetrization is strictly monotonic, invariant on H-symmetric sets, volume preserving (by Fubini's theorem), and projection invariant.
c) On $\mathcal{K}^n$ and for $j \in \{1, \dots, n-1\}$, the jth intrinsic volume V_j is generally reduced (meaning not increased and not always preserved) by S_H. In particular, Steiner symmetrization generally reduces the surface area. See Section 5.4 and Theorem 5.10 for a proof.
d) On $\mathcal{C}^n$, Steiner symmetrization is monotonic but not strictly monotonic (if $H = \{x_1 = 0\}$, then $B^n \subsetneq B^n \cup \{(2, 0, \dots, 0), (-2, 0, \dots, 0)\}$ but $S_H(B^n) = B^n = S_H(B^n \cup \{(2, 0, \dots, 0), (-2, 0, \dots, 0)\})$) and it is not invariant on H-symmetric sets (same example).
e) On $\mathcal{K}^n$, S_H is smoothing, in the sense of (5.8). This is a consequence of the inclusion proved in the next theorem. Choosing $L = dB^n$ and using $S_H dB^n = dB^n$, (5.10) yields (5.8).

Theorem 5.3 *If $K, L \in \mathcal{C}^n$, then*

$$S_H K + S_H L \subset S_H(K + L). \tag{5.10}$$

Proof Let G be a line orthogonal to H. It is enough to prove

$$(S_H K + S_H L) \cap G \subset S_H(K + L) \cap G\,.$$

For $y \in H$ let $G_y = G + y$. We can write

$$(S_H K + S_H L) \cap G = \bigcup_{y \in H} (S_H K \cap G_y) + (S_H L \cap G_{-y})\,.$$

Since all the segments $(S_H K \cap G_y) + (S_H L \cap G_{-y})$ are contained in G and centered in H, their union equals the largest of its elements, $(S_H K \cap G_{\bar{y}}) + (S_H L \cap G_{-\bar{y}})$. Its length equals the sum of the lengths of the two segments

$$\mathcal{H}^1(S_H K \cap G_{\bar{y}}) + \mathcal{H}^1(S_H L \cap G_{-\bar{y}}).$$

Now, $S_H(K + L) \cap G$ is a segment whose length is not smaller than

$$\begin{aligned}\mathcal{H}^1\left((K \cap G_{\bar{y}}) + (L \cap G_{-\bar{y}})\right) &\geq \mathcal{H}^1(K \cap G_{\bar{y}}) + \mathcal{H}^1(L \cap G_{-\bar{y}}) \\ &= \mathcal{H}^1(S_H K \cap G_{\bar{y}}) + \mathcal{H}^1(S_H L \cap G_{-\bar{y}})\,.\end{aligned}$$

This concludes the proof. □

Section 5.4 presents many other properties of the Steiner symmetrization. It also contains a proof of (5.10) valid for convex bodies and which uses shadow systems.

5.3.2 Schwarz Symmetrization

Let $i \in \{0, \dots, n-1\}$ and $K \in \mathcal{C}^n$. The *Schwarz symmetral* of K with respect to $H \in \mathcal{G}(n,i)$ is the set $S_H K$ such that for each $(n-i)$-dimensional plane G orthogonal to H and meeting K, the set $G \cap S_H K$ is a (possibly degenerate) $(n-i)$-dimensional closed ball with center in H and $\mathcal{H}^{n-i}$-measure equal to that of $G \cap K$. See [37, p. 62] and also [43, p. 178] (where the process is referred to as Schwarz rounding). When $i = n-1$ it coincides with the Steiner symmetrization. It is convenient to use the same notation for Steiner and Schwarz symmetrizations.

An extension of this definition to measurable subsets A of $\mathbb{R}^n$ is possible. Let G be a $(n-i)$-dimensional plane G orthogonal to H and meeting A. If $G \cap A$ is not $\mathcal{H}^{n-i}$-measurable then $G \cap S_H A = \emptyset$. If $\mathcal{H}^{n-i}(G \cap A) = \infty$ then $G \cap S_H A = G$. If $\mathcal{H}^{n-i}(G \cap A) < \infty$ then $G \cap S_H A$ is defined as in the case of compact sets, i.e., it is a (possibly degenerate) $(n-i)$-dimensional closed ball with center in H and $\mathcal{H}^{n-i}$-measure equal to that of $G \cap A$. See [37, p. 62], [85, p. 106] and [2, p. 182].

In the literature on rearrangements of functions this symmetrization is often indicated as the $(n-i, n)$-Steiner symmetrization. It is at the heart of the definition of symmetric decreasing rearrangement of a function. Indeed, the symmetric decreasing rearrangement of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the function whose subgraph is the Schwarz symmetrization, with respect to the x_{n+1}-axis, of the subgraph of u.

We list some of its properties.

a) $S_H K$ is rotationally symmetric with respect to H.
b) If $K \in \mathcal{C}^n$, then $S_H K \in \mathcal{C}^n$.
c) On $\mathcal{K}^n$ and on $\mathcal{C}^n$, Schwarz symmetrization is monotonic, volume preserving (again by Fubini's theorem), and projection invariant.
d) On $\mathcal{K}^n$, Schwarz symmetrization is invariant on H-symmetric spherical cylinders, but it is not invariant on H-symmetric sets.
e) On $\mathcal{K}^n_n$, Schwarz symmetrization is strictly monotonic, but on $\mathcal{C}^n$ and on $\mathcal{K}^n$ it is not (for instance, when H is the x_n-axis then $S_H(B^n \cap \{x_1 = 0\}) = S_H(B^n \cap \{x_1 = 0, x_2 = 0\})$).
f) The Schwarz symmetrization, for $i \in \{0, \dots, n-2\}$, can be viewed as a limit, in the Hausdorff distance, of a sequence of Steiner symmetrizations. This issue will be treated properly in Section 5.7 but we anticipate that there exist sequences of hyperplanes (H_m) containing H such that, for each $K \in \mathcal{C}^n$, the sequence $(S_{H_m} \dots S_{H_2} S_{H_1} K)$ of iterated Steiner symmetrizations of K converges to $S_H K$, i.e.
$$\lim_{m \to \infty} \delta(S_{H_m} \dots S_{H_2} S_{H_1} K, S_H K) = 0.$$
g) The existence of the approximating sequence presented in item f) allows to prove that properties valid for Steiner symmetrization and which are maintained in the passage to the limit, are also valid for Schwarz symmetrization. For instance, it can be used to prove that if $K \in \mathcal{K}^n$ then $S_H K \in \mathcal{K}^n$. Indeed, if (H_m) is as in item f), then, for $m \in \mathbb{N}$, $S_{H_m} \dots S_{H_2} S_{H_1} K \in \mathcal{K}^n$, since Steiner symmetrization maintains convexity. Since the limit of a sequence of convex bodies is a convex body, so is $S_H K$.

h) The same idea can be used to prove that Schwarz symmetrization generally reduces the jth intrinsic volume V_j for $j \in \{1, \dots, n-1\}$. Moreover, together with the fact that Steiner symmetrization satisfies (5.10) and is monotonic, the same method gives that Theorem 5.3 is valid also for Schwarz symmetrization and that S_H is smoothing on $\mathcal{C}^n$, in the sense of (5.8).

i) Schwarz symmetrization is smoothing as a map on $\mathcal{L}^n$. The result is valid for each i, and in particular, for Steiner symmetrization.

Theorem 5.4 *Let* $i \in \{0, \dots, n-1\}$. *The map* $S_H : \mathcal{L}^n \to \mathcal{L}^n_H$ *is smoothing.*

Proof Let $A \in \mathcal{L}^n$ be bounded and $d > 0$.
Step 1. Let $D^{n-i} = D^n \cap H^\perp$. We prove that, for $x \in H$ and $r > 0$,

$$S_H\left(A \cap (x + H^\perp)\right) + rD^{n-i} \subset S_H\left((A \cap (x + H^\perp)) + rD^{n-i}\right). \tag{5.11}$$

The set $(A \cap (x + H^\perp)) + rD^{n-i}$ is $\mathcal{H}^{n-i}$-measurable, because it is open in the relative topology on $x + H^\perp$. If $A \cap (x + H^\perp)$ is not $\mathcal{H}^{n-i}$-measurable, the set on the left-hand side in (5.11) is the empty set and the inclusion holds true. If it is measurable, $\mathcal{H}^{n-i}\,(A \cap (x + H^\perp)) < \infty$, since A is bounded. Both sets in (5.11) are $(n-i)$-dimensional balls in $x + H^\perp$ with center in x. The one on the left has radius

$$r_1 = \mathcal{H}^{n-i}\left(A \cap (x + H^\perp)\right)^{\frac{1}{n-i}} / \kappa_{n-i}^{\frac{1}{n-i}} + d,$$

while the one on the right has radius

$$r_2 = \mathcal{H}^{n-i}\left(A \cap (x + H^\perp) + rD^{n-i}\right)^{\frac{1}{n-i}} / \kappa_{n-i}^{\frac{1}{n-i}}.$$

The Brunn–Minkowski inequality in $\mathbb{R}^{n-i}$ implies $r_1 \le r_2$ and (5.11).

Step 2. For $x \in H$, denote by Π_x the orthogonal projection onto $x + H^\perp$. If L is any set in $\mathbb{R}^n$, then

$$(L + dD^n) \cap (x + H^\perp) = \bigcup_{y \in H, |y-x|<d} \Pi_x\left((L \cap (y + H^\perp)) + r_y D^{n-i}\right), \tag{5.12}$$

where $r_y = \sqrt{d^2 - |y-x|^2}$. Indeed, $p \in (L + dD^n) \cap (x + H^\perp)$ if and only if $p|H = x$ and there is a $z \in L$ such that $p \in z + dD^n$. If $z|H = y$, then this holds if and only if $|y - x| < d$ and

$$|p - \Pi_x z| < r_y,$$

that is, $p \in \Pi_x(z + r_y D^{n-i})$.

Step 3. We prove that, for $x \in H$,

$$(S_H A + dD^n) \cap \left(x + H^\perp\right) \subset S_H(A + dD^n) \cap \left(x + H^\perp\right).$$

We use (5.12) with $L = S_H A$, (5.11), the fact that the action of S_H is the same for each y, the pointwise monotonicity of S_H, and (5.12) with $L = A$, to obtain

$$\begin{aligned}
(S_H A + dD^n) \cap \left(x + H^\perp\right) &= \bigcup_{y\in H,\,|y-x|<d} \Pi_x \left(\left[S_H A \cap (y + H^\perp)\right] + r_y D^{n-i}\right)\\
&= \bigcup_{y\in H,\,|y-x|<d} \Pi_x \left(S_H \left[A \cap (y + H^\perp)\right] + r_y D^{n-i}\right)\\
&\subset \bigcup_{y\in H,\,|y-x|<d} \Pi_x \left(S_H \left[(A \cap (y + H^\perp)) + r_y D^{n-i}\right]\right)\\
&= \bigcup_{y\in H,\,|y-x|<d} S_H \left(\Pi_x \left[(A \cap (y + H^\perp)) + r_y D^{n-i}\right]\right)\\
&\subset S_H \left(\bigcup_{y\in H,\,|y-x|<d} \Pi_x \left[(A \cap (y + H^\perp)) + r_y D^{n-i}\right]\right)\\
&= S_H \left((A + dD^n) \cap (x + H^\perp)\right)\\
&= S_H (A + dD^n) \cap (x + H^\perp).
\end{aligned}$$

Step 4. Step 3 proves

$$S_H A + dD^n \subset S_H (A + dD^n).$$

Since $S^*_H A + dD^n \subset \operatorname{clo}(S_H A) + dD^n = S_H A + dD^n$, it also proves that $S^*_H A + dD^n \subset S^*_H (A + dD^n)$ essentially. □

Remark 5.5 (Blaschke's proof of the Brunn–Minkowski inequality in $\mathcal{K}^n$) Convexity of the Schwarz 1-symmetrization of a compact convex set is equivalent to the Brunn–Minkowski inequality in $\mathcal{K}^n$. Thus, the proof of item g) above provides another proof of the Brunn–Minkowski inequality besides those given in Chapter 3. Bonnesen and Fenchel [20] refers to this proof as Blaschke's proof of the Brunn–Minkowski inequality.

Let us explain this equivalence. Let $K, L \in \mathcal{K}^n$, let $M \subset \mathbb{R}^n \times \mathbb{R}$ be defined as

$$M = \operatorname{conv}\Big(K \times \{0\}, L \times \{1\}\Big).$$

and let H be the x_{n+1}-axis. The proof lies in the following three observations:

i) for $t \in [0, 1]$,
$$M \cap \{x_{n+1} = t\} = \big((1 - t)K + tL\big) \times \{t\};$$

ii) if one describes $S_H M$ as
$$S_H M = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x| \leq f(t)\},$$
then the convexity of $S_H M$ is equivalent to the concavity of the "profile" $[0, 1] \ni t \to f(t)$;

iii) Since $\mathcal{H}^n(S_H M \cap \{x_{n+1} = t\}) = \mathcal{H}^n(M \cap \{x_{n+1} = t\})$, $\kappa_n f(t)^n = \mathcal{H}^n(M \cap \{x_{n+1} = t\}) = \mathcal{H}^n\big((1 - t)K + tL\big)$.

We can conclude that the convexity of $S_H M$ is equivalent to the concavity of the function $\mathcal{H}^n\big((1-t)K+tL\big)^{1/n}$ on $[0,1]$.

5.3.3 Minkowski Symmetrization

We shall consider *Minkowski symmetrization* in the following general form. Let $i \in \{0,\dots,n-1\}$ and let $H \in \mathcal{G}(n,i)$. The *Minkowski symmetral* of $K \in \mathcal{C}^n$ is defined by

$$M_H K = \frac{1}{2}K + \frac{1}{2}K^{\dagger}, \tag{5.13}$$

where $K^{\dagger}$ is the reflection of K in H. The case $i=0$ corresponds to $K^{\dagger} = -K$ and $M_H K = \Delta K$, the central symmetral. We list some of its properties.

a) If $K \in \mathcal{C}^n$, then $M_H K \in \mathcal{C}^n$ and if $K \in \mathcal{K}^n$, then $M_H K \in \mathcal{K}^n$.
b) Minkowski symmetrization is Minkowski additive, i.e., for $K, L \in \mathcal{C}^n$, $M_H(K+L) = M_H K + M_H L$.
c) On $\mathcal{K}^n$, Minkowski symmetrization is strictly monotonic, but on $\mathcal{C}^n$ it is only monotonic. Indeed, for instance, when $H = \{o\}$ or $H = e_1^{\perp}$ and $C = ([-1,-\frac{1}{3}] \cup [\frac{1}{3},1]) \times [-1,1]^{n-1}$, then $C \subsetneq [-1,1]^n$ and

$$M_H C = [-1,1]^n = M_H([-1,1]^n).$$

d) On $\mathcal{K}^n$, Minkowski symmetrization is idempotent, but on $\mathcal{C}^n$ it is not. For instance $M_H^2\{-1,1\} = M_H\{-1,0,1\}) = \{-1,-1/2,0,1/2,1\}$.
e) Since Minkowski addition commutes with projections and $K^{\dagger}|H = K|H$, it is projection invariant. On $\mathcal{K}^n$, M_H is clearly also invariant on H-symmetric sets, but the same is false on $\mathcal{C}^n$ as the example of the H-symmetric set C above shows.
f) Minkowski symmetrization generally increases the volume, by the Brunn–Minkowski inequality. On $\mathcal{K}^n$, since the first intrinsic volume V_1 is linear with respect to Minkowski addition, M_H is mean width preserving, but for $j \in \{2,\dots,n-1\}$, it generally increases the jth intrinsic volume V_j, by Theorem 5.1, the Brunn–Minkowski inequality for intrinsic volumes.

5.3.4 Minkowski–Blaschke Symmetrization

There is an extension of Minkowski symmetrization analogous to Schwarz symmetrization that we shall call *Minkowski–Blaschke symmetrization*, though it has been referred to by other names. For example, Bonnesen and Fenchel [20, pp. 79–80] call it stiffening and attribute it to Blaschke [19, p. 137]. If $i \in \{1,\dots,n-2\}$ and $H \in \mathcal{G}(n,i)$, the support function $h_K(u)$ of $K \in \mathcal{K}^n$ at a point $u \in \mathbb{S}^{n-1}$ is replaced by the average of h_K over the subsphere of $\mathbb{S}^{n-1}$ orthogonal to H and containing u. More precisely, the Minkowski–Blaschke symmetral $\overline{M}_H K$ of K is defined for

$u \in \mathbb{S}^{n-1}$ via

$$h_{\overline{M}_H K}(u) = \frac{1}{\mathcal{H}^{n-i-1}(\mathbb{S}^{n-1} \cap (H^\perp + u))} \int_{\mathbb{S}^{n-1} \cap (H^\perp + u)} h_K(v)\, \mathrm{d}v$$

if $\mathcal{H}^{n-i-1}(\mathbb{S}^{n-1} \cap (H^\perp + u)) \neq 0$, and by $h_{\overline{M}_H K}(u) = h_K(u)$ otherwise. We can extend the definition to $i = n-1$ if we interpret it to mean that $\overline{M}_H K = M_H K$ in this case. We list some of its properties.

a) The Minkowski–Blaschke symmetral is rotationally symmetric with respect to H.
b) The Minkowski–Blaschke symmetral belongs to $\mathcal{K}^n$. Indeed, $\overline{M}_H K$ can be seen as the limit of Minkowski averages of rotated copies of K.
c) Minkowski–Blaschke symmetrization is strictly monotonic, mean width preserving (as can be shown by integration in spherical coordinates), invariant on H-symmetric spherical cylinders, and projection invariant, but it is not invariant on H-symmetric sets.

5.3.5 Fiber Symmetrization

If $H \in \mathcal{G}(n,i)$, $i \in \{0, \ldots, n-1\}$, we define the *fiber symmetral* $F_H K$ of $K \in \mathcal{C}^n$ with respect to H by

$$\begin{aligned} F_H K &= \bigcup_{x \in H} \left(\frac{1}{2}(K \cap (H^\perp + x)) + \frac{1}{2}(K \cap (H^\perp + x))^\dagger \right) \\ &= \bigcup_{x \in H} \left(\frac{1}{2}(K \cap (H^\perp + x)) + \frac{1}{2}(K^\dagger \cap (H^\perp + x)) \right). \end{aligned} \tag{5.14}$$

Thus each non-degenerate section of $F_H K$ by an $(n-i)$-dimensional subspace orthogonal to H is the Minkowski symmetral of the corresponding section of K. Then $F_H K = M_H K = \Delta K$ when $i = 0$ and, if $K \in \mathcal{K}^n$, $F_H K = S_H K$ when $i = n-1$. When $K \in \mathcal{K}_n$, then

$$F_H K = \left(\frac{1}{2} \circ K \right) \text{⋒}_H \left(\frac{1}{2} \circ K^\dagger \right).$$

We list some of its properties.

a) If $K \in \mathcal{C}^n$, $F_H K \in \mathcal{C}^n$ and, if $K \in \mathcal{K}^n$, $F_H K \in \mathcal{K}^n$.
b) Fiber symmetrization is monotonic on $\mathcal{C}^n$ and strictly monotonic on $\mathcal{K}^n$, it is invariant on H-symmetric sets, and projection invariant.
c) Fiber symmetrization generally increases the volume. Indeed, the Brunn–Minkowski inequality gives $\mathcal{H}^{n-i}(F_H K \cap (H^\perp + x)) \geq \mathcal{H}^{n-i}(K \cap (H^\perp + x))$, for $x \in H$, and integration of this inequality with respect to x gives the inequality between the volumes.

5.3.6 Blaschke Symmetrization

Finally, for $H \in \mathcal{G}(n,i)$, $i \in \{0,\dots,n-1\}$, we can define the *Blaschke symmetral* of $K \in \mathcal{K}_n^n$ by

$$B_H K = \left(2^{-1/(n-1)} K\right) \sharp \left(2^{-1/(n-1)} K^\dagger\right).$$

Since surface area measure is positively homogeneous of degree $n-1$, we may equivalently define $B_H K$ by

$$S_{n-1}(B_H K, \cdot) = \frac{1}{2} S_{n-1}(K, \cdot) + \frac{1}{2} S_{n-1}(K^\dagger, \cdot). \tag{5.15}$$

These formulas define $B_H K$ up to translation. We define the Blaschke sum so that the centroids of $B_H K$ and $K|H$ coincide. When $i = 0$, we have $K^\dagger = -K$ and then the body $B_H K$ is often called the *Blaschke body* of K and denoted by ∇K. We list some of the properties.

a) Blaschke symmetrization is invariant on H-symmetric sets.
b) When $n = 2$, then up to translation and on $\mathcal{K}_n^n$, B_H coincides with Δ ($i = 0$) or M_H ($i = 1$), whose properties have already been discussed.
c) When $n \geq 3$, B_H is neither monotonic nor (except when $i = 0$) projection invariant, regardless of the position chosen for the Blaschke sum, as proved in Bianchi, Gardner, and Gronchi [7, Theorem 3.1]. We present the part of the argument that shows the claim regarding monotonicity.

Theorem 5.6 ([7]) *Let $H \in \mathcal{G}(n,i)$, $i \in \{0,\dots,n-1\}$. Blaschke symmetrization B_H in $\mathbb{R}^n$, $n \geq 3$, is not monotonic.*

Proof Let T^n be an n-dimensional cone in $\mathbb{R}^n$ with centroid at the origin, x_n-axis as its axis, and radius and height (i.e., width in the direction e_n) both equal to 1. Let and $H = \{o\}$ when $i = 0$, and let H be the subspace of $\mathbb{R}^n$ spanned by $e_1,\dots,e_i$, when $i \geq 1$. For each i, we have $(T^n)^\dagger = -T^n$ and, therefore, $B_H T^n = \nabla T^n$, the Blaschke body of T^n. We claim that when $n \geq 3$, the height of ∇T^n is less than 1. Suppose the claim is true. Let $0 < s < 1$ and let $L_s \subset T^n$ be the spherical cylinder with base of radius s contained in the base of T^n, the x_n-axis as its axis, and with maximal height $w = w(s)$. The set L_s is centrally symmetric, so $B_H L_s = \nabla L_s$ is a translate of L_s and the height of ∇L_s is w; since $w \to 1$ as $s \to 0$, when s is sufficiently small it is not possible that $\nabla L_s \subset \nabla T^n$. To prove the claim, let $n \geq 3$ and recall that the surface area of the curved part of the boundary of an n-dimensional cone of radius r and height h is $r^{n-2}\sqrt{h^2 + r^2}\kappa_{n-1}$. Therefore the surface area of the curved part of the boundary of T^n is $\sqrt{2}\kappa_{n-1}$, while the area of the base of T^n is κ_{n-1}. The surface area measure $S_{n-1}(T^n, \cdot)$ consists of a point mass at $-e_n$ and a multiple of $(n-2)$-dimensional Lebesgue measure on the $(n-2)$-dimensional sphere of latitude in $\mathbb{S}^{n-1}$ whose points have vertical angle $\pi/4$ with the positive x_n-axis. From this and (5.15) it is easy to see that ∇T^n is an o-symmetric truncated double cone of radius a, say, with the x_n-axis as axis, such that the top of ∇T^n is an $(n-1)$-dimensional ball B

of radius h contained in the plane $\{x_n = a - h\}$, for some $0 < h < a$. By (5.15), $V_{n-1}(B) = \kappa_{n-1}/2$, whence $h = 2^{-1/(n-1)}$, and the surface area of the curved part of the boundary of ∇T^n contained in $\{x_n \geq 0\}$ is $\sqrt{2}\kappa_{n-1}/2$. From the latter we see that

$$\sqrt{2}a^{n-1}\kappa_{n-1} - \sqrt{2}h^{n-1}\kappa_{n-1} = \sqrt{2}\kappa_{n-1}/2$$

and hence $a = 1$. Thus the height of ∇T^n is $2(a - h) = 2(1 - 2^{-1/(n-1)})$, which is less than 1 when $n \geq 3$. This proves the claim. □

d) Blaschke symmetrization preserves surface area, by definition. It is a consequence of the Kneser–Süss inequality [81, p. 460] that Blaschke symmetrization generally increases volume. This can be proved using the formula for the volume given in Theorem 1.47, the formula that expresses certain mixed volumes in terms of an integral, and Minkowski's Inequality (contained in Theorem 3.19, Definition 3.20 and Theorem 3.28, respectively). They imply the following inequalities

$$\begin{aligned}
V_n(B_H K) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_{B_H K}(u)\,\mathrm{d}S_{n-1}(B_H K, u)\\
&= \frac{1}{2n}\int_{\mathbb{S}^{n-1}} h_{B_H K}(u)\,\mathrm{d}S_{n-1}(K, u) + \frac{1}{2n}\int_{\mathbb{S}^{n-1}} h_{B_H K}(u)\,\mathrm{d}S_{n-1}(K^\dagger, u)\\
&= \frac{1}{2}V(K[n-1], B_H K) + \frac{1}{2}V(K^\dagger[n-1], B_H K)\\
&\geq \frac{1}{2}V_n(K)^{\frac{n-1}{n}} V_n(B_H K)^{\frac{1}{n}} + \frac{1}{2}V_n(K^\dagger)^{\frac{n-1}{n}} V_n(B_H K)^{\frac{1}{n}}\\
&= V_n(K)^{\frac{n-1}{n}} V_n(B_H K)^{\frac{1}{n}},
\end{aligned}$$

from which we derive $V_n(B_H K) \geq V_n(K)$.

5.3.7 Polarization

Let $i = n-1$, H be an oriented affine hyperplane and H^+, H^- be the closed halfspaces bounded by H. The *polarization*, or *two-point symmetrization*, of a subset A of $\mathbb{R}^n$ is the set

$$P_H A = \big((A \cup A^\dagger) \cap H^+\big) \cup \big((A \cap A^\dagger) \cap H^-\big). \tag{5.16}$$

This is not an $(n-1)$-symmetrization as defined in Section 5.3, because in general $P_H K$ is not H-symmetric. Note that if we invert the orientation of H, we switch H^+ and H^- and the result of the symmetrization is the same, up to a reflection with respect to H.

The rearrangement P_H associated to this set operation is again called polarization. The superlevel set $\{x : P_H f \geq t\}$ is the (set) polarization of $\{x : f(x) \geq t\}$, for each $t > \operatorname{ess\,inf} f$. The rearrangement can also be defined as

$$P_H f(x) = \begin{cases} \max\{f(x), f(x^\dagger)\} & \text{if } x \in H^+, \\ \min\{f(x), f(x^\dagger)\} & \text{if } x \in H^-. \end{cases}$$

We list some of the properties of polarization.

a) If $C \in \mathcal{C}^n$, $P_H C \in \mathcal{C}^n$. If $K \in \mathcal{K}^n$, $P_H K$ is not necessarily convex. If $A \in \mathcal{L}^n$, $P_H A \in \mathcal{L}^n$.
b) Polarization is monotonic, invariant on H-symmetric sets, and projection invariant, but it is not invariant with respect to translations orthogonal to H of H-symmetric sets.
c) Polarization preserves volume. Indeed if $A \in \mathcal{L}^n$ one can write

$$A = (A \cap A^\dagger) \cup \Big((A \triangle A^\dagger) \cap H^+\Big) \cup \Big((A \triangle A^\dagger) \cap H^-\Big)$$

and

$$P_H A = (A \cap A^\dagger) \cup \Big((A \triangle A^\dagger) \cap H^+\Big) \cup \Big((A \triangle A^\dagger) \cap H^-\Big)^\dagger,$$

where the unions on the right-hand sides of these formulas are disjoint.

d) On $\mathcal{K}^n$, polarization is perimeter preserving. This is essentially due to the following simple structure of the polarization of a convex set. Let $K \in \mathcal{K}^n$, $x \in K|H$ and let p_x be the midpoint of the segment $K \cap (x + H^\perp)$. Then

$$P_H K \cap (x + H^\perp) = \begin{cases} K \cap (x + H^\perp) & \text{if } p_x \in H^+, \\ (K \cap (x + H^\perp))^\dagger & \text{if } p_x \notin H^+. \end{cases}$$

Thus the boundary of K can be decomposed in two disjoint parts so that the boundary of $P_H K$ is the disjoint union of one of these parts and of the reflection with respect to H of the other part. More precisely, if $E_1 = \{x \in K|H : p_x \in H^+\}$ and $E_2 = (K|H) \setminus E_1$ then

$$\partial K = (\partial K \cap (E_1 + H^\perp)) \cup (\partial K \cap (E_2 + H^\perp))$$

and

$$\partial P_H K = (\partial K \cap (E_1 + H^\perp)) \cup (\partial K \cap (E_2 + H^\perp))^\dagger.$$

e) P_H is smoothing.

Theorem 5.7 *For $A \subset \mathbb{R}^n$ and $d > 0$, we have $P_H A + dB^n \subset P_H(A + dB^n)$, and the analogous formula with D^n substituting B^n holds true. Moreover, if $A \in \mathcal{L}^n$ then $P_H^* A + dD^n \subset P_H^*(A + dD^n)$, essentially.*

Proof First observe that, if $x \in \mathbb{R}^n$,

$$P_H(x + dB^n) = \begin{cases} x + dB^n & \text{if } x \in H^+, \\ (x + dB^n)^\dagger & \text{if } x \in H^-. \end{cases} \tag{5.17}$$

Let $z \in P_H A$.
Assume first $z \in H^+$. Then $z \in A$ or $z^\dagger \in A$. If $z \in A$ then $z + dB^n \subset A + dB^n$ and, by (5.17) and by the monotonicity of P_H,

$$z + dB^n = P_H(z + dB^n) \subset P_H(A + dB^n).$$

If $z^\dagger \in A$ then $z^\dagger + dB^n \subset A + dB^n$ and, by (5.17) and by the monotonicity of P_H,

$$z + dB^n = (z^\dagger + dB^n)^\dagger = P_H(z^\dagger + dB^n) \subset P_H(A + dB^n).$$

Assume now $z \in H^-$. Then both z and $z^\dagger$ are in A. The set $\{z, z^\dagger\} + dB^n \subset A + dB^n$ and, being symmetric, it coincides with its polarization. Thus

$$z + dB^n \subset \{z, z^\dagger\} + dB^n = P_H\left(\{z, z^\dagger\} + dB^n\right) \subset P_H(A + dB^n).$$

This concludes the proof of $P_H A + dB^n \subset P_H(A + dB^n)$. The inclusion $P_H A + dD^n \subset P_H(A + dD^n)$ is proved in the same way.
Let $A \in \mathcal{L}^n$. Since $P_H^* A + dD^n \subset \mathrm{clo}(P_H A) + dD^n = P_H A + dD^n$, the previous inclusion proves $P_H^* A + dD^n \subset P_H^*(A + dD^n)$ essentially.

f) Both Steiner and Schwarz symmetrization, for any i, can be approximated in C^n by sequences of polarizations in the Hausdorff metric. See Van Schaftingen [90] for the explicit construction of one sequence with this property. See also Burchard and Fortier [25] for the importance and the history of these results. This property has many consequences. In the literature it has been used to prove that certain properties and inequalities which are valid for the rearrangement associated to polarization are also valid for the rearrangements associated to Steiner and Schwarz symmetrization.

5.3.8 Some Inclusions between the Symmetrizations

In Section 5.5 we describe a duality between fiber (Steiner, if $i = n-1$) and Minkowski symmetrization. For $i = 0, \dots, n-1$ the fiber symmetral of $K \in \mathcal{K}^n$ is the *union* of all H-symmetric compact convex sets such that some translate orthogonal to H *is contained in* K, while the Minkowski symmetral of K is the *intersection* of all H-symmetric compact convex sets such that some translate orthogonal to H *contains* K. Bianchi, Gardner, and Gronchi [7, Section 5] introduces two new symmetrizations, the inner and outer rotational symmetrizations, which display exactly the same duality in a rotationally symmetric setting.

Let $H \in \mathcal{G}(n, i)$, $i \in \{1, \dots, n-1\}$ and $K \in \mathcal{K}^n$. The *inner rotational symmetral* $I_H K$ of K is such that for each $(n-i)$-dimensional plane G orthogonal to H and meeting K, the set $G \cap I_H K$ is a (possibly degenerate) $(n-i)$-dimensional ball with center in H and radius equal to that of the (possibly degenerate) largest $(n-i)$-dimensional ball contained in $G \cap K$. The *outer rotational symmetral* $O_H K$ of K is the intersection of all rotationally symmetric with respect to H convex bodies for which some translate orthogonal to H contains K.

We list some of their properties and we refer to [7] for their proofs.

a) The symmetrals $I_H K$ and $O_H K$ are rotationally symmetric with respect to H.
b) The symmetral $I_H K \in \mathcal{K}^n$ and $I_H K \subset S_H K$. If $i = n-1$, then $I_H K = S_H K$.
c) The inclusion $M_H K \subset O_H K$ holds true. If $i = n-1$, $O_H K = M_H K$.
d) The symmetrization I_H is monotonic, idempotent, invariant on H-symmetric spherical cylinders, and projection invariant, but not strictly monotonic or invariant on H-symmetric sets.
e) I_H generally reduces V_j for $j \in \{1, \dots, n-1\}$ and also for $j = n$ when $i \in \{1, \dots, n-2\}$.
f) The symmetrization O_H is strictly monotonic, idempotent, and invariant on H-symmetric spherical cylinders, but not invariant on H-symmetric sets unless $i = n-1$. It is also projection invariant.
g) O_H generally increases V_j for $j \in \{2, \dots, n\}$ and also for $j = 1$ when $i \in \{1, \dots, n-2\}$.

Some of the applications of the symmetrizations stem from containment relations. The following theorem summarizes all the known inclusions between the various known symmetrals. Some are well known and some are observed and proved in [7] or [9]. It should be added that $F_H K = M_H K$ when $i = 0$.

Theorem 5.8 *If* $H \in \mathcal{G}(n,i)$, $i \in \{1, \dots, n-1\}$, *and* $K \in \mathcal{K}^n$, *then*

$$I_H K \subset F_H K \subset M_H K \subset O_H K \tag{5.18}$$

and

$$I_H K \subset S_H K \subset \overline{M}_H K \subset O_H K. \tag{5.19}$$

When $i = n-1$, *we have* $I_H K = F_H K = S_H K$ *and* $M_H K = \overline{M}_H K = O_H K$.

5.4 Shadow Systems

In order to prove many of the properties of Steiner symmetrization, it is useful to look at it as a continuous transformation of a convex body into its symmetral. We introduced $S_H K$ as a union of segments orthogonal to H. Each of those segments, suitably translated (remaining orthogonal to H) can be restored in K. We can imagine letting these chords move with constant speed, with each of them staying on the line containing it, so that at time 0 they form K and at time 1 they form $S_H K$.

This idea seems to add nothing new, but it doesn't. This continuous movement which transforms K into $S_H K$ is a particular case of some movements introduced and studied by Rogers and Shephard in 1958 [77]. Given a subset A of $\mathbb{R}^n$, a direction $v \in \mathbb{R}^n$ and a real valued function α defined on A, we set

$$K_t = \operatorname{conv}[\{x + t\alpha(x)v \, : \, x \in A\}] \, , t \in \mathbb{R}.$$

In 1964 Shephard [84] observed that such a family of convex sets can be seen as a *shadow system*, a family of projections of the same $(n+1)$-dimensional convex body onto a fixed hyperplane along a varying direction. If we define

$$\tilde{K} = \operatorname{conv}[\{(x, \alpha(x)e_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \, : \, x \in A\}]$$

we may recognize that K_t is the projection (not an orthogonal one!) of $\tilde{K}$ onto $\mathbb{R}^n = e_{n+1}^{\perp}$ along the direction of $e_{n+1} - tv$.

This idea of visualizing shadow systems as projections of a larger body immediately clarifies that two or more shadow systems moving in the same direction v may generate other shadow systems by taking their projections onto a fixed subspace or their Minkowski addition or their convex hulls. In formulas, if $\{K_t\}_{t\in\mathbb{R}}$ and $\{L_t\}_{t\in\mathbb{R}}$ are shadow systems in the same direction v and H is a subspace, then

$$\{K_t|H\}_{t\in\mathbb{R}}, \quad \{K_t + L_t\}_{t\in\mathbb{R}}, \quad \text{and} \quad \{\operatorname{conv}(K_t \cup L_t)\}_{t\in\mathbb{R}}$$

are also shadow systems. While it is now clear that many examples of shadow systems can be produced, we have still to explain the convenience and advantages of this new concept. And so we come to the main feature of shadow systems, stated in the following theorem by Rogers and Shephard.

Theorem 5.9 *If $\{K_t\}_{t\in\mathbb{R}}$ is a shadow system, then $V_n(K_t)$ is a convex function of t.*

Proof A definitely concise and elegant proof of the statement is based on the formula

$$V_n(K_t) = (n+1)V(\tilde{K}[n], [0, e_{n+1} - tv]) \, ,$$

the inclusion

$$[0, e_{n+1} - \frac{t_1 + t_2}{2}v] \subset \frac{1}{2}[0, e_{n+1} - t_1 v] + \frac{1}{2}[0, e_{n+1} - t_2 v]$$

and the monotonicity and continuity of mixed volumes (see Proposition 1.40).

For completeness, we present a second proof where we infer the convexity of $V_n(K_t)$ from that of $\mathcal{H}^1(K_t \cap G)$, for any line G parallel to the direction of movement v. Since $K_t = \operatorname{conv}[\{x + t\alpha(x)v \, : \, x \in A\}]$, any element in $K_t \cap G$ is an appropriate convex combination of points in $\{x + t\alpha(x)v \, : \, x \in A\}$. Every such convex combination is a linear function of t. Thus, the left and right endpoints of $K_t \cap G$ can be seen as the minimum and the maximum, respectively, of linear functions of t. This implies the desired convexity. □

If we go back to the shadow system associated to Steiner symmetrization, the continuous movement connecting K to $S_H K$ can be explicitly written down. If $H = v^{\perp}$ and

$$K = \{x + sv \in \mathbb{R}^n : x \in K|H,\ g(x) \le s \le f(x)\}\,,$$

then, for $t \in [0,2]$,

$$K_t = \{x + sv \in \mathbb{R}^n : x \in K|H,\ g(x) \le s + t\,\frac{f(x)+g(x)}{2} \le f(x)\} \tag{5.20}$$

and this shows that $K = K_0$, $S_H K = K_1$ and $K_2 = K^\dagger$, the reflection of K in H. For $t \notin [0,2]$ the previous expression is not valid since it does not represent, in general, a convex body. Theorem 5.9 ensures the convexity of $V_n(K_t)$ for all t, but in the interval $[0,2]$ we already know that the volume is constant. In order to see Theorem 5.9 play a leading role, we have to change the functional at hand.

For every hyperplane $u^\perp$, we have seen that $\{K_t|u^\perp\}_{0\le t\le 2}$ is a shadow system and Theorem 5.9 states the convexity of $\mathcal{H}^{n-1}(K_t|u^\perp)$ with respect to t. In words: The brightness of K_t in any direction is a convex function of t. *Cauchy's surface area formula* (see [81, (5.73)])

$$S(K) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} V_{n-1}(K|u^\perp)\,\mathrm{d}u \tag{5.21}$$

shows that the average brightness of a convex body L is, up to a constant depending on n, its surface area. Therefore, the function $S(K_t)$ is convex in t. Since it assumes the same value at 0 and 2, we deduce that its value at 1 is not larger than that at the extreme points, which means that the surface area of $S_H K$ is not larger than the surface area of K. Formally, this statement reads as follows.

Theorem 5.10 *For every hyperplane H and convex body K,*

$$S(S_H K) \le S(K)\,.$$

This result can be generalized in various ways. From the above argument, stressed for the first time in [77], we can deduce the following principle.

Whenever we have a functional F which is

- *convex with respect to the parameter t of every shadow system, and*
- *invariant under reflections,*

we can conclude that $F(S_H K) \le F(K)$.

This can be used, together with results on the convergence to a ball of sequences of successive Steiner symmetrizations (see Section 5.7), to prove the classical isoperimetric inequality for convex sets. Indeed, Rogers and Shephard's complete argument gave the following principle.

Theorem 5.11 *If a functional F on $\mathcal{K}^n$ is*

- *convex with respect to the parameter t of every shadow system,*
- *invariant under reflections, and*
- *continuous with respect to the Hausdorff distance,*

then, in the class of convex bodies with prescribed volume, F attains its minimum at a ball.

Other applications can be listed. Cauchy's surface area formula is a special case of *Kubota's integral recursion* (see [37, (A.46) and (A.47)]), which says that the i-th intrinsic volume of K is, up to a constant depending only on n and i, the average of the volumes of the projections of K onto i-dimensional subspaces. Such averages are intended as integrals over the Grassmannian $\mathcal{G}(n,i)$ with respect to the Haar measure. In formula

$$V_i(K) = \frac{\kappa_n\binom{n}{i}}{\kappa_i\kappa_{n-i}} \int_{\mathcal{G}(n,i)} V_i(K|E)\,\mathrm{d}E\,. \tag{5.22}$$

Therefore, all intrinsic volumes of a shadow system $\{K_t\}_{t\in I}$ are convex functions of t. Since they are also continuous and reflection invariant, we can conclude that Steiner symmetrization does not increase any intrinsic volume and hence arrive to the corresponding isoperimetric inequalities.

Theorem 5.12 (Isoperimetric inequalities) *For $K \in \mathcal{K}^n_n$ and $i = 1, 2, \ldots, n-1$*

$$\frac{V_i(K)}{V_n(K)^{\frac{i}{n}}} \ge \frac{V_i(B^n)}{V_n(B^n)^{\frac{i}{n}}}\,.$$

The discussion of the equality case is usually more involved (and sometimes it is still an open problem). It can be proved that in Theorem 5.10 and Theorem 5.12 equality holds if and only if K is a ball. In both cases the characterization of balls can be proved showing that the intrinsic volume $V_i(K_t)$ is, for $i < n$, a strictly convex function of t. A possible proof is postponed until after the next theorem.

The convexity of the intrinsic volumes of a shadow system with respect to its parameter t is a special case of a more general result showed by Shephard in [84].

Theorem 5.13 *If $\{K^1_t\}_{t\in I}$, $\{K^2_t\}_{t\in I}$, ..., $\{K^n_t\}_{t\in I}$ are shadow systems along the same direction v, then the mixed volume $V(K^1_t, K^2_t, \ldots, K^n_t)$ is a convex function of t.*

Proof A proof of this result could follow the path already made for the volume of a shadow system, using the formula

$$(n+1)V(\tilde{K}^1, \tilde{K}^2, \ldots, \tilde{K}^n, [0, e_{n+1} + tv]) = V(K^1_t, K^2_t, \ldots, K^n_t)\,,$$

where $\tilde{K}^j$ denotes the $(n+1)$-dimensional body whose shadows on $e^{\perp}_{n+1}$ are in $\{K^j_t\}_{t\in I}$. □

Strict convexity of the $V_i(K_t)$, $i < \dim(K)$. In order to prove the strict convexity, following the previous proof of the convexity of mixed volumes, it is enough to show that

$$V(\tilde{K}[i], B^n[n-i], [0, w_1 + w_2]) < V(\tilde{K}[i], B^n[n-i], [0, w_1] + [0, w_2]),$$

where we abbreviate $w_1 = e_{n+1} + t_1 v$ and $w_2 = e_{n+1} + t_2 v$. By Theorem 3.19 and Definition 3.20, this inequality can be written in terms of an integral with respect to the ith area measure of $\tilde{K}$:

$$\frac{1}{n+1} \int_{\mathbb{S}^n} \left(h_{[0,w_1]}(u) + h_{[0,w_2]}(u) - h_{[0,w_1+w_2]}(u) \right) \mathrm{d}S(\tilde{K}[i], B^n[n-i], u) > 0.$$

Since the integrand function is non-negative and vanishes only on a neighborhood of a great circle, some fine analysis on the support of the ith area measure of $\tilde{K}$ (see, for example, [81, Theorem 4.5.3]) implies the strict inequality.

In (5.10) we have encountered the following property of Steiner symmetrization.

Theorem 5.14 *If* $K, L \in C^n$ *then*

$$S_H K + S_H L \subset S_H(K + L).$$

Here we provide another proof of this inclusion, valid if K and L are convex bodies, which uses shadow systems.

Proof Let us consider the shadow systems $\{K_t\}_{t \in [0,2]}$ and $\{L_t\}_{t \in [0,2]}$ which connect K and L to their reflections on H passing through their Steiner symmetral, respectively. As we observed, $\{K_t + L_t\}_{t \in [0,2]}$ is a shadow system and a proof of Theorem 5.9 used the fact that $\mathcal{H}^1((K_t + L_t) \cap G)$ is a convex function of t for every line G orthogonal to H. Since for $t = 0$ and $t = 2$ it attains the same value, we deduce that

$$\mathcal{H}^1((S_H K + S_H L) \cap G) \le \mathcal{H}^1((K + L) \cap G),$$

for every line G orthogonal to H. By the definition of Steiner symmetrization, this means

$$S_H(S_H K + S_H L) \subset S_H(K + L),$$

which is the thesis, since $S_H(S_H K + S_H L) = S_H K + S_H L$. □

5.4.1 Sas–Macbeath Inequality

Any convex body $K \in \mathcal{K}^n_n$ can be approximated by polytopes contained in it with an increasing number of vertices, but how close can the approximation be if the number of vertices is bounded?

To be a little more precise, let us denote by $M_m(K)$ the maximal volume of a polytope contained in K with at most m vertices. The ratio $M_m(K)/V_n(K)$ is affine invariant and less than or equal to 1 (it is 1 if and only if K is a polytope with at most m vertices). In 1939 Sas [80] for $n = 2$ and in 1951 Macbeath [61] for all n proved the following inequality.

Theorem 5.15 (Sas–Macbeath Inequality) *For every* $K \in \mathcal{K}^n_n$ *and* $m > n$

$$\frac{M_m(K)}{V_n(K)} \geq \frac{M_m(B^n)}{V_n(B^n)} \, .$$

Proof Consider the shadow system $\{K_t\}_{t\in[0,2]}$ connecting K to its reflection on H and a polytope $P \subset K_0 = K$ with at most m vertices. If we let the vertices of P move solidly to the chord on which they are located, then each vertex has a constant speed and $\{P_t\}_{t\in[0,2]}$ (the convex hulls of these vertices at time t) form a shadow system. Hence, the $V_n(P_t)$ is a convex function of t and so is the maximum over all polytopes with at most m vertices, that is $M_m(K_t)$. Since $V_n(K_t)$ is constant and $M_m(K)$ is continuous with respect to the Hausdorff distance, the argument by Rogers and Shephard concludes the proof. □

In Theorem 5.15 equality holds for ellipsoids but a complete characterization is still missing. In 1939 Sas [80] proved that in the plane, for every m, equality holds if and only if K is an ellipse. In 1917 Blaschke [16] characterized 3-dimensional ellipsoids as the only minimizers for $m = 4$. In 1986 Bianchi [5] proved the corresponding result for $m = 5$. For $n > 3$ or $n = 3$ and $m > 5$ the problem is still open.

If we denote by $M^i_m(K)$ the maximal i-th intrinsic volume of a polytope contained in K with at most m vertices, then we define a similar functional:

$$\frac{M^i_m(K)}{V_n(K)^{\frac{i}{n}}} \, ,$$

where the power on the volume of K is chosen so that the functional is scaling invariant and convex along shadow systems. However, such a functional is not affine invariant and has no upper bound.

Remark If a functional F on $\mathcal{K}^n$ is convex along shadow systems and upper bounded then it is also $\mathrm{SL}(n)$ invariant. Indeed, let H be a hyperplane, let $v \neq o$ a vector in $H^\perp$ and let α be a linear function defined on H. We define, for $t \in \mathbb{R}$, the shadow system

$$K_t = \mathrm{conv}(\{x + t\alpha(x|H)v : x \in K\}).$$

For each t, K_t is an affine image of K. If a functional F is convex along such a shadow system and bounded from above, then $F(K_t)$ has to be constant and this implies that F is invariant with respect to shears. Since shears generate the whole group $\mathrm{SL}(n)$, the statement is proved. If F is also scaling and reflection invariant, then it is affine invariant.

Theorem 5.11 implies the following result.

Theorem 5.16 *For every* $K \in \mathcal{K}^n_n$ *and* $m > n$

$$\frac{M^i_m(K)}{V_n(K)^{\frac{i}{n}}} \geq \frac{M^i_m(B^n)}{V_n(B^n)^{\frac{i}{n}}} \, .$$

5.4.2 Sylvester's Functional

Instead of taking the maximal volume of a polytope contained in K with at most m vertices, we can focus on its average. Clearly we have to declare how to evaluate the average. The most natural way is to define

$$S_n(K;m) = \frac{1}{V_n(K)^{m+1}} \int_K \dots \int_K V_n(\mathrm{conv}[x_1, x_2, \dots, x_m])\, \mathrm{d}x_1 \dots \mathrm{d}x_m\,,$$

which is the expected volume of a random polytope from K divided by $V_n(K)$ (to ensure scaling invariance), where the vertices are selected uniformly and independently in K.

If we consider the shadow system $\{K_t\}_{t\in[0,2]}$ defined in (5.20), then the volume of K_t is constant and the functional can be rewritten as

$$S_n(K_t;m) = \frac{1}{V_n(K)^{m+1}} \int_K \dots \int_K V_n(\mathrm{conv}[\bigcup_{j=1}^{m} x_j + t\alpha(x_j|H)v])\, \mathrm{d}x_1 \dots \mathrm{d}x_m\,.$$

Therefore, the functional is convex with respect to t. Furthermore, it is continuous with respect to the Hausdorff distance, reflection invariant and affine invariant. As a consequence of Theorem 5.11 we deduce Blaschke-Groemer inequality (see [17] and [42])

Theorem 5.17 (Blaschke–Groemer inequality) *For every $K \in \mathcal{K}_n^n$ and $m > n$*

$$S_n(K;m) \geq S_n(B^n;m)\,,$$

where equality holds if and only if K is an ellipsoid.

Proof We already sketched a proof of the inequality, but here the discussion of the equality cases is a bit more involved than the one for intrinsic volumes in Theorem 5.12. Indeed, $S_n(K;m)$ is affine invariant, and so the strict convexity along every shadow system cannot be proved. Blaschke and Groemer based their arguments on the characterization of ellipsoids as the only convex bodies which are affine images of their Steiner symmetrals along every direction. Such a characterization is due to Brunn [23] in an equivalent form:

If all midpoints of every family of parallel chords of K are contained in a hyperplane, then K is an ellipsoid.

Hence, if K is not an ellipsoid, then we can choose a direction v and m of the midpoints of chords of K parallel to v so that their convex hull is a polytope of positive volume. Following such a polytope along the shadow system (5.20) give rise to a shadow system $\{P_t\}_{t\in[0,2]}$ with $V_n(P_t)$ strictly convex. A continuity argument leads to the conclusion. □

For $n = 2$, $S_n(K;m)$ is closely related to the so-called Sylvester's four-point problem, which appeared in the Educational Times of 1864, question 1491:

> *Show that the chance of four points forming the apices of a reentrant quadrilateral is* $1/4$ *if they be taken at random in an indefinite plane, but* $1/4 + e^2 + x^2$*, where e is a finite constant and x a variable quantity, if they be limited by an area of any magnitude and of any form.*

Sylvester's four-point problem captured the attention of many mathematicians who proposed as many different solutions. For a historical overview of the problem and its developments, we refer to the interesting article by Pfiefer [73].

Here, we just mention that Groemer introduced a power $p \geq 1$ of the volume in the functional and observed that this does not affect minimizers. Schöpf [83] extended to all $p > 0$.

Nowadays, the search of maximizers of

$$S_n(K; m, p) = \frac{1}{V_n(K)^{m+p}} \int_K \dots \int_K V_n(\mathrm{conv}[x_1, x_2, \dots, x_m])^p \, \mathrm{d}x_1 \dots \mathrm{d}x_m \,,$$

is called *Sylvester's problem* and it is solved only in the plane. For $p = 1$, Dalla and Larman [34] proved that triangles are maximizers and Giannopoulos [39] proved they are the only ones. Campi, Colesanti and Gronchi [27] showed that parallelograms are maximizers among centrally symmetric figures and Saroglou [79] proved the uniqueness of such maximizers.

In higher dimensions the main result is due to Bárány and Buchta [3], who proved that for every $K \in \mathcal{K}^n$ there exists $\bar{m}$, depending on K, such that $S_n(K; m) \leq S_n(T; m)$, for all $m \geq \bar{m}$, where T is a simplex.

Hartzoulaki and Paouris in [45] replaced the volume in Sylvester's functional with intrinsic volumes and the p-power with an arbitrary increasing function. They proved that

$$\int_K \dots \int_K f\left(V_i(\mathrm{conv}[x_1, x_2, \dots, x_m])\right) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \,,$$

$1 \leq i \leq n - 1$, is minimal when K is a ball, among bodies of given volume. They further proved that the ball is the only minimizer if f is convex and strictly increasing.

5.4.3 Busemann's Functional

A slight modification of Sylvester's functional appears in the Busemann formula, which expresses the volume of a convex body in terms of the areas of its central sections. Assume $K \in \mathcal{K}^n_n$ contains the origin in its interior: The *Busemann intersection formula* (see [26]) states that

$$V_n(K)^{n-1} = \frac{n!}{2} \int_{\mathbb{S}^{n-1}} V_{n-1}(K \cap u^\perp)^{n+1} B_{n-1}(K \cap u^\perp; n-1, 1) \, \mathrm{d}u \,,$$

where, for $m \geq n$,

$$B_n(K;m,p) = \frac{1}{V_n(K)^{m+p}} \int_K \dots \int_K V_n(\operatorname{conv}[o,x_1,\dots,x_m])^p \, \mathrm{d}x_1 \dots \mathrm{d}x_m \,.$$

The functional $B_n(K;m,p)$ is called Busemann's functional and differs from Sylvester's in having fixed a point in the origin.

It is no more translation invariant, but it clearly remains $\mathrm{GL}(n)$ invariant, continuous with respect to Hausdorff distance, and convex along shadow systems.

In [26] Busemann proved that $B_n(K;n,1)$ attains its minimum if and only if K is an origin symmetric ellipsoid. Such a result is known as *Busemann random simplex inequality*.

Theorem 5.18 (Busemann random simplex inequality) *For every $m \geq n$, $p \geq 1$ and $K \in \mathcal{K}_n^n$*

$$B_n(K;m,p) \geq B_n(B^n;m,p)\,,$$

where equality holds if and only if K is an origin symmetric ellipsoid.

The proof of the equality case is similar to the one for Theorem 5.17

When $m = n$ Busemann's functional may be represented in a different way, highlighting a Minkowski sum of segments. Indeed,

$$V_n(\operatorname{conv}[o,x_1,\dots,x_n]) = \frac{1}{n!} V_n\left(\sum_{i=1}^n [o,x_i]\right) .$$

This fact suggests a different extension that we shall meet in next paragraph.

5.4.4 Zonotopes associated to *K*

Bourgain, Meyer, Milman, and Pajor [21], in connection with some comparisons between norms in the local theory of Banach spaces, considered the functional

$$I(K;m,p) = \frac{1}{V_n(K)^{m+p}} \int_K \dots \int_K V_n\left(\sum_{i=1}^m [0,x_i]\right)^p \mathrm{d}x_1 \dots \mathrm{d}x_m \,,$$

for $m \geq n$, and the more general version

$$I(K_1,K_2,\dots,K_m;p) \\ = \frac{1}{(V_n(K_1)\dots V_n(K_m))^{\frac{m+p}{m}}} \int_{K_1} \dots \int_{K_m} V_n\left(\sum_{i=1}^m [0,x_i]\right)^p \mathrm{d}x_1 \dots \mathrm{d}x_m \,,$$

where each point is chosen from a different convex body in $\mathcal{K}_n^n$.

As we have already observed, if we consider the shadow system $\{K_t\}_{t\in[0,2]}$ defined in (5.20) and leave every point in K move solidly to the chord on which it lays, then the volume of K_t is constant and each segment $[o,x_i]$ is a shadow system.

Since the Minkowski sum of shadow systems is a shadow system, $I(K_t; m, p)$ is a convex function of t for all $p \geq 1$. Standard arguments provide the continuity with respect to Hausdorff distance and $\mathrm{GL}(n)$ invariance. Hence, Theorem 5.11 recovers a result proved in [21].

Theorem 5.19 (BMMP zonotope inequality) *For every* $m \geq n$, $p \geq 1$ *and* $K \in \mathcal{K}_n^n$

$$I(K; m, p) \geq I(B^n; m, p)\,,$$

where equality holds if and only if K is an origin symmetric ellipsoid.

The equality condition was proved by Campi and Gronchi [30].

In fact, Bourgain, Meyer, Milman and Pajor [21] proved the inequality for all $p \geq 0$ and also for the functional involving more bodies:

$$I(K_1, \ldots, K_m; p) \geq I(B_1, \ldots, B_m; p)\,,$$

where B_i is the ball with the same volume as K_i centered at the origin.

The Minkowski sum of a finite number of segments is called a *zonotope*. The simplest zonotope is a parallelotope, the sum of n affinely independent segments, that is an affine image of the n-dimensional cube. By increasing the number of segments, zonotopes can approximate the unit ball. A set which is the limit, in the Hausdorff metric, of a sequence of zonotopes is called a *zonoid*. Zonoids play a basic role in the Brunn–Minkowski theory of convex bodies and appear in different contexts of the mathematical literature. We refer to [82] for an exhaustive review on this topic.

5.4.5 The L_p Busemann–Petty Inequality

An easy way to associate a zonoid to a body K with positive volume is to sum all segments joining the origin to a point in K. Due to the link between Minkowski addition and sums of support functions (see Proposition 1.28), we present the zonoid ΓK by means of its support function:

$$h_{\Gamma K}(x) = \frac{2\pi\kappa_n}{\kappa_{n+1} V_n(K)} \int_K |\langle x, z\rangle| \,\mathrm{d}z\,,$$

where the constant in front of the integral is such that $\Gamma(\lambda K) = \lambda \Gamma K$, for all $\lambda > 0$, and $\Gamma B^n = B^n$.

This body (usually with a different normalization) is known in the literature as the *centroid body* of K. Centroid bodies were first defined and investigated by Petty [71], but the concept had previously appeared in work of Dupin, in connection with problems for floating bodies (see Gardner [37, Chap. 9] and Schneider [81, Sect. 7.4] for references). When K is an origin symmetric body, the boundary of ΓK is, up to a dilatation, the locus of the centroids of all the halves of K obtained by cutting K with hyperplanes through the origin.

One of the basic results obtained by Petty [71] is an integral representation of the volume of ΓK by means of Busemann's functional $B_n(K; n, 1)$. Using the Busemann random simplex inequality, Petty proved the well known *Busemann–Petty centroid inequality*: For $K \in \mathcal{K}_n^n$,

$$V_n(\Gamma K) \geq V_n(K)\,, \tag{5.23}$$

where equality holds if and only if K is an origin symmetric ellipsoid.

Petty [72] proved that the Busemann–Petty centroid inequality implies the *Petty projection inequality*:

$$V_n(K)^{n-1} V_n(\Pi^\circ K) \leq \kappa_n^n\,, \tag{5.24}$$

where equality holds if and only if K is an ellipsoid. Here, $\Pi^\circ K$ is the *polar projection body* of K, i.e., the polar body of the *projection body* ΠK of K, that can be defined by

$$h_{\Pi K}(x) = V_{n-1}(K|x^\perp)\,.$$

A short way to show that (5.23) implies (5.24) was obtained by Lutwak in [54].

Zhang [93] proved a reverse form of (5.24), known as *Zhang projection inequality*:

$$V_n(K)^{n-1} V_n(\Pi^\circ K) \geq \frac{1}{n^n}\binom{2n}{n}\,,$$

with equality if and only if K is a simplex.

The L_p extension of the classical Brunn–Minkowski theory for convex bodies was initiated by Lutwak [55], in which the idea of Firey [35] of the p-Minkowski addition for sets is widely developed.

As already seen in Section 3.4, if $p \geq 1$ and K, L are convex bodies containing the origin in their interior, the p-sum of K and L is the convex body $K +_p L$ defined by

$$h_{K+_pL}(x)^p = h_K(x)^p + h_L(x)^p\,.$$

Bianchini and Colesanti [14] observed that the p-sum of shadow systems is again a shadow system, since the projection of a p-sum is the p-sum of the projections. Notice that the p-sum is the Minkowski sum for $p = 1$ and tends to the convex hull as p tends to infinity. Taking into account the p-sum of segments we can define

$$h_{\Gamma_p K}(x)^p = \frac{\kappa_2 \kappa_{p-1} \kappa_n}{\kappa_{n+p} V_n(K)} \int_K |\langle x, z\rangle|^p \, \mathrm{d}z\,, \tag{5.25}$$

where the constant is so that $\Gamma_p(\lambda K) = \lambda \Gamma_p K$, for all $\lambda > 0$, and $\Gamma_p B^n = B^n$. The body $\Gamma_p K$ is known as the *L_p centroid body* of K.

For $p = 1$, $\Gamma_1 K$ is the centroid body of K, while for $p = 2$, the body defined by (5.25) is also well known. Indeed, up to a constant, $\Gamma_2 K$ is the ellipsoid of inertia (or Legendre ellipsoid) of K, i.e., the ellipsoid having the same moments of inertia as K about every axis. Many results concerning this body, which is fundamental in classical mechanics, can be found in the literature (see, e.g., Milman and Pajor [70] and Lindenstrauss and Milman [50] for references). In 1918 Blaschke [18] proved that, for $n = 3$,

$$V_n(\Gamma_2 K) \geq V_n(K)\,, \tag{5.26}$$

where equality holds if and only if K is an origin symmetric ellipsoid. In 1937 this result was extended by John [46] in all dimensions; other proofs were given by Petty [71] and by Lutwak, Yang, and Zhang [56].

Inequalities (5.23) and (5.26) are special instances of the more recent *L_p Busemann–Petty centroid inequality*.

Theorem 5.20 (L_p Busemann–Petty centroid inequality) *For $p \geq 1$ and $K \in \mathcal{K}_n^n$*

$$V_n(\Gamma_p K) \geq V_n(K)\,,$$

where equality holds if and only if K is an origin symmetric ellipsoid.

Theorem 5.20 was first proved by Lutwak, Yang, and Zhang [57], while Campi and Gronchi [28] presented a proof based on shadow systems.

If we consider the shadow system $\{K_t\}_{t\in[0,2]}$ defined in (5.20), each segment joining o to a moving point $x \in K$ is a shadow system, and their p-sum (better their p-integral) $\Gamma_p K_t$ is a shadow system too. Therefore, not only the volume of $\Gamma_p K_t$ is convex but also its intrinsic volumes, its diameter, its Sylvester's functional, etc., are convex.

Focusing on Steiner symmetrization, this implies $F(S_H K) \leq F(K)$ for many functionals F.

In addition to this, the use of shadow systems allows to recover arguments scattered over the years and in the literature to the search of maximizers (when the functional is bounded from above). Such arguments are confined, at least until now, to the plane (in the symmetric or non-symmetric case) or to the case of polyhedra with few vertices or to zonotopes. See, for example, Campi and Gronchi [29].

5.4.6 The Blaschke–Santaló Inequality

As in Chapter 1, the polar body K° of $K \in \mathcal{K}_n^n$ is the convex body defined by

$$K^\circ = \{x \in \mathbb{R}^n | \langle x, y\rangle \leq 1, \forall y \in K\}\,.$$

Notice that the polar body of K strongly depends on the location of the origin. If K is an origin-symmetric convex body, then the product

$$V_n(K)V_n(K^\circ)$$

is called the *volume product* of K, and it is invariant under linear transformations.

For a general convex body K, the volume product is defined as the minimum, for $x \in K$, of $V_n(K)V_n((K-x)^\circ)$. Aleksandrov [1] proved that $V_n((K-x)^\circ)^{-\frac{1}{n}}$ is a strictly concave function of x. The unique point $s(K)$ where this minimum is attained is called the *Santaló point* of K and is characterized by the fact that $(K - z)^\circ$ has its centroid at the origin if and only if $z = s(K)$.

A sharp upper bound for the volume product of a convex body $K \in \mathcal{K}^n$ with centroid at the origin is given by the *Blaschke–Santaló inequality*.

Theorem 5.21 (Blaschke–Santaló inequality) *For $K \in \mathcal{K}^n_n$,*

$$V_n(K)V_n(K^\circ) \leq \kappa_n^2 ,$$

where equality holds if and only if K is an ellipsoid centered at the origin.

It was proved by Blaschke [15] for $n \leq 3$ and by Santaló for all n. A sharpening of this inequality was proved by Meyer and Pajor [67].

A different proof of the Blaschke–Santaló inequality relies on the following result.

Theorem 5.22 *If K_t, $t \in [0, 1]$, is a shadow system of convex bodies in $\mathcal{K}^n_n$, then $V_n(K^\circ_t)^{-1}$ is a convex function of t.*

This was proved by Campi and Gronchi [31] for centrally symmetric bodies and by Meyer and Reisner [68] in full generality. We sketch here a proof in the symmetric case.

One of the main ingredients in the proof of Theorem 5.22 is a consequence of the Borell–Brascamp–Lieb inequality, which deals with p-means of functions and their integrals. It can be interpreted as an inverse Hölder inequality, and its links with other well-known inequalities are widely described in the survey article by Gardner [36].

Theorem 5.23 (Borell–Brascamp–Lieb inequality) *If $0 < \lambda < 1$, $-1/n \leq p \leq \infty$, and f, g, h are non-negative integrable functions on $\mathbb{R}^n$ satisfying*

$$h((1-\lambda)x + \lambda y) \geq [(1-\lambda)f(x)^p + \lambda g(y)^p]^{1/p} ,$$

for all x, $y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \left[(1-\lambda)\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{\frac{p}{np+1}} + \lambda\left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{\frac{p}{np+1}}\right]^{\frac{np+1}{p}} .$$

We shall use a corollary that expresses the concavity of an integral in terms of that of the integrand and the dimension of the space of integration. Let us recall the definition of p-concave function.

Definition 5.24 Let $p \neq 0$. A non-negative function f on $\mathbb{R}^n$ is called p-concave on a convex set L if

$$f((1-\lambda)x + \lambda y) \geq [(1-\lambda)f(x)^p + \lambda f(y)^p]^{1/p}$$

for all x, $y \in L$ and $0 < \lambda < 1$.

Note that if $p < 0$, then f is p-concave if and only if f^p is convex. The above definition can be extended to the case $p = 0$ by continuity.

Corollary 5.25 *Let $F(x, y)$ be a non-negative p-concave function on $\mathbb{R}^n \times \mathbb{R}^m$, $p \geq -1/n$. If, for every y in $\mathbb{R}^m$, the integral*

$$\int_{\mathbb{R}^n} F(x, y)\,\mathrm{d}x$$

exists, then it is a $\frac{p}{np+1}$-concave function of y.

Proof Take $y_0, y_1 \in \mathbb{R}^m$ and fix $\lambda \in (0, 1)$. Let $y_\lambda = (1-\lambda)y_0 + \lambda y_1$, and

$$f(x) = F(x, y_0)\,, \quad g(x) = F(x, y_1)\,, \quad h(x) = F(x, y_\lambda)\,.$$

For every $x_0, x_1 \in \mathbb{R}^n$, we have that

$$h((1-\lambda)x_0 + \lambda x_1) = F((1-\lambda)x_0 + \lambda x_1, y_\lambda) \geq \big((1-\lambda)f^p(x_0) + \lambda g^p(x_1)\big)^{1/p}\,,$$

where we used the p-concavity of F.

The Borell–Brascamp–Lieb inequality now gives the desired conclusion. □

We are now ready to prove Theorem 5.22.

Proof (of Theorem 5.22) Let $\{K_t\}_{t\in[0,1]}$ be a shadow system along the direction v, with speed function α on K_0, and originated by the $(n+1)$-dimensional convex body $\tilde{K}$, such that

$$K_t = \mathrm{conv}\{x + \alpha(x)t\,v : x \in K_0\}$$

can be thought of as the projection along the direction $e_{n+1} - tv$ of $\tilde{K}$ onto $e_{n+1}^\perp$. The support functions h_{K_t}, $t \in [0, 1]$, and that of $\tilde{K}$ are clearly related. Precisely, for $u \in e_{n+1}^\perp$, we have

$$h_{K_t}(u) = h_{\tilde{K}}(u + t\langle u, v\rangle e_{n+1})\,. \tag{5.27}$$

We know that

$$V_n(K_t^\circ) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_{K_t}^{-n}(z)\,\mathrm{d}z\,.$$

Let $D^{n-1} = \{x \in v^\perp : |x| \leq 1\}$; thus ${\mathbb{S}^{n-1}}_+ = \{z \in \mathbb{S}^{n-1} : \langle z, v\rangle \geq 0\}$ is the graph of the function $\sqrt{1-|x|^2}$, $x \in D^{n-1}$. Consequently,

$$\int_{\mathbb{S}^{n-1}} h_{K_t}^{-n}(z)\,\mathrm{d}z = 2\int_{D^{n-1}} \frac{h_{K_t}^{-n}(x + \sqrt{1-|x|^2}v)}{\sqrt{1-|x|^2}}\,\mathrm{d}x\,,$$

where we took into account that K_t is origin-symmetric. By (5.27),

$$h_{K_t}(x + \sqrt{1-|x|^2}v) = h_{\tilde{K}}(x + \sqrt{1-|x|^2}v + t\sqrt{1-|x|^2}e_{n+1})\,.$$

Therefore, we obtain that

$$V_n(K_t^\circ) = \frac{2}{n}\int_{D^{n-1}} \frac{h_{\tilde{K}}^{-n}\left(x/\sqrt{1-|x|^2}+v+te_{n+1}\right)}{(1-|x|^2)^{\frac{n+1}{2}}}\,\mathrm{d}x\,,$$

where we used also the homogeneity of the support function.

By the change of variable $y = x/\sqrt{1-|x|^2}$ in the latter integral (with Jacobian $(1-|x|^2)^{-(n+1)/2}$), we conclude that

$$V_n(K_t^\circ) = \frac{2}{n}\int_{\mathbb{R}^{n-1}} h_{\tilde{K}}^{-n}(y+v+te_{n+1})\,\mathrm{d}y\,.$$

Since the function $h_{\tilde{K}}$ is convex in $\mathbb{R}^{n+1}$, by Corollary 5.25, we infer that $V_n(K_t^\circ)$ is p-concave, with respect to t, with $p = (-1/n)/(1-(n-1)/n) = -1$. □

In [60] Lutwak and Zhang dealt with the functional

$$G_p(K) = V_n(\Gamma_p^\circ K)V_n(K)\,, \tag{5.28}$$

where $\Gamma_p^\circ K$ is the polar of the L_p centroid body of K. Using Steiner symmetrization they proved the so-called L_p *Blaschke–Santaló inequality*.

Theorem 5.26 (**L_p Blaschke–Santaló inequality**) *For all* $p \geq 1$ *and* $K \in \mathcal{K}_n^n$

$$V_n(\Gamma_p^\circ K)V_n(K) \leq V_n(\Gamma_p^\circ B^n)V_n(B^n),$$

and equality holds if and only if K *is an ellipsoid centered at the origin.*

The name of this inequality comes from the fact that it implies the Blaschke–Santaló inequality for centrally symmetric bodies as p tends to infinity.

One of the main questions still open in convex geometry is the problem of finding a sharp lower bound for the volume product of a convex body (see the survey article [54]).

It was conjectured by Mahler [62] that the minimum of the volume product is attained when K is a simplex, that is

$$V_n(K)V_n(K^\circ) \geq \frac{(n+1)^{n+1}}{(n!)^2}\,. \tag{5.29}$$

In 1939 Mahler [63] proved the conjecture in the plane and in 1991 Meyer [66] showed that equality holds only for triangles.

For centrally symmetric convex bodies the inequality

$$V_n(K)V_n(K^\circ) \geq \frac{4^d}{d!} \tag{5.30}$$

is a conjecture as well, where the value on the right-hand side is the volume product of a parallelotope. It was proved in the plane by Mahler [63] and Reisner [76] characterized parallelograms as the only minimizers. Saint Raymond [78] showed that in higher dimension there are convex bodies, other than parallelotopes and their polars, giving equality in (5.30). He also proved that the conjecture holds true in all dimensions for the affine images of convex sets symmetric with respect to the coordinate hyperplanes (called unconditional bodies). Barthe and Fradelizi [4] generalized to all bodies whose hyperplanes of symmetries have a one-point intersection. Inequality (5.30) was proved by Reisner [75], [76] for all zonoids. Different proofs were presented by Gordon, Meyer and Reisner [40] and by Campi and Gronchi [32] using shadow systems.

Bourgain and Milman [22] proved that there exists a constant c, not depending on the dimension, such that

$$V_n(K)V_n(K^\circ) \geq c^n \kappa_n^2 \,.$$

In [31] Campi and Gronchi dealt with the lower bound of (5.28). It is easy to check that G_p is continuous, (-1)-concave along shadow systems and $\mathrm{GL}(n)$ invariant. Besides, $G_p(K)$ tends to zero as K moves away from the origin. If c_K denotes the centroid of K, Campi and Gronchi proved that in the two-dimensional case the functionals

$$\min_{x\in K} G_p(K-x)\,, \quad \max_{x\in\mathbb{R}^n} G_p(K-x)\,, \quad G_p(K-c_K)$$

are minimized by triangles (or parallelograms, in the symmetric case).

5.4.7 The Affine Quermassintegrals Inequality

The intrinsic volumes (or their close relatives, the quermassintegrals) of a convex body $K \in \mathcal{K}_n^n$ are not invariant under volume preserving affine transformations. An affine invariant version was defined by Lutwak in [52] by replacing the L^1 norm in Kubota's formula (5.22) by the L^{-n} norm:

$$\Phi_i(K) = \frac{\kappa_n}{\kappa_i}\left(\int_{\mathcal{G}(n,i)} V_i^{-n}(K|E)\,\mathrm{d}E\right)^{-\frac{1}{n}} \,.$$

The affine invariance of $\Phi_i(K)$ was shown by Grinberg [41].

Conjectured by Lutwak [53] in 1988, the *affine quermassintegrals inequality* has been proved only recently by Milman and Yehudayoff [69] and it compares the affine quermassintegral of a convex body K with that of B_K, a ball with the same volume as K.

Theorem 5.27 (Affine Quermassintegrals inequality) *For every $K \in \mathcal{K}_n^n$ and $i = 1, 2, \dots, n-1$,*

$$\Phi_i(B_K) \leq \Phi_i(K)\,, \tag{5.31}$$

with equality for a given i if and only if K is an ellipsoid.

For origin-symmetric convex bodies, $\Phi_1^{-n}(K)$ is proportional to the volume of the polar body K°, and so the case $i = 1$ of (5.31) amounts to the Blaschke—Santaló inequality. For general convex bodies the Blaschke–Santaló inequality is stronger.

On the other extreme, $\Phi_{n-1}^{-n}(K)$ is proportional to the volume of the polar projection body $\Pi^\circ K$ and the case $i = n-1$ of (5.31) is equivalent to Petty projection inequality.

The elegant proof of Milman and Yehudayoff goes through shadow systems to arrive to the familiar looking inequality

$$\Phi_i(S_H K) \leq \Phi_i(K)\,,$$

but they need the definition of a new body, the Projection Rolodex of K, (a subset of a vector bundle over a lower-dimensional Grassmannian) and appropriate measures on Grassmannians. Last but not least, they solved the equality cases with a ten pages proof. In short: The arguments they used are surely too involved to fit into these notes.

5.5 Duality between Steiner and Minkowski Symmetrals

The following theorem shows a duality between Steiner (or, more generally, fiber) and Minkowski symmetrization. We recall that $F_H K = \triangle K = M_H K$ if $i = 0$ and that, if $i = n-1$, Steiner and fiber coincide. It is proved in [7] and the proof is taken from there.

Theorem 5.28 *Let $H \in \mathcal{G}(n,i)$, $i \in \{0,\dots,n-1\}$, and for $K \in \mathcal{K}^n$ and $y \in H^\perp$, let*

$$K_y = K + y \qquad \text{and} \qquad K_y^\dagger = (K_y)^\dagger = K^\dagger - y. \tag{5.32}$$

Then for $K \in \mathcal{K}^n$, we have

$$F_H K = \bigcup_{y \in H^\perp} (K_y \cap K_y^\dagger) \tag{5.33}$$

and

$$M_H K = \bigcap_{y \in H^\perp} \operatorname{conv}(K_y \cup K_y^\dagger). \tag{5.34}$$

Proof Let $z \in F_H K$. Then, using (5.14), we have $z = ((x+a)/2) + (x-b)/2$, where $x \in H, a, b \in H^\perp$, and $x+a, x+b \in K$. Let $y = -(a+b)/2$. Then $z = (x+a)+y \in K+y$ and $z = (x-b)-y \in K^\dagger - y$. Therefore $z \in K_y \cap K_y^\dagger$, so $F_H K$ is contained in the right-hand side of (5.33). For the reverse inclusion, note first that $K_y \cap K_y^\dagger$ is H-symmetric. From the invariance of F_H on H-symmetric sets and the fact that F_H is monotonic and invariant on translations orthogonal to H of H-symmetric sets, we obtain

$$K_y \cap K_y^\dagger = F_H(K_y \cap K_y^\dagger) \subset F_H K_y = F_H K$$

for all $y \in H^\perp$. This proves (5.33).

To prove (5.34), let $y \in H^\perp$ and let $Q_y = \mathrm{conv}(K_y \cup K_y^\dagger)$. Then since K_y and $K_y^\dagger$ are contained in Q_y and the latter is convex,

$$M_H K = M_H K_y = \frac{1}{2}K_y + \frac{1}{2}K_y^\dagger \subset Q_y,$$

so $M_H K \subset \cap_{y \in H^\perp} Q_y$. To prove the reverse containment in (5.34), observe that if $v \in \mathbb{S}^{n-1}$, then by (5.32) and $h_{K\pm y}(v) = h_K(v) \pm \langle y, v\rangle$, we obtain

$$\begin{aligned} h_{\cap_{y\in H^\perp} Q_y}(v) &\le \min_{y\in H^\perp} h_{Q_y}(v) = \min_{y\in H^\perp} \max\{h_{K_y}(v), h_{K_y^\dagger}(v)\} \\ &= \min_{y\in H^\perp} \max\{h_K(v) + \langle y, v\rangle, h_{K^\dagger}(v) - \langle y, v\rangle\} \\ &= \frac{1}{2}h_K(v) + \frac{1}{2}h_{K^\dagger}(v) = h_{M_H K}(v), \end{aligned} \tag{5.35}$$

as required, where the first equality in (5.35) results from observing that the minimum occurs when the two expressions are equal, i.e., when $\langle y, v\rangle = (h_{K^\dagger}(v) - h_K(v))/2$. □

Corollary 5.29 ([7]) *If $H \in \mathcal{G}(n,i)$, $i \in \{0,\dots,n-1\}$, and $K \in \mathcal{K}^n$, then the fiber symmetral $F_H K$ (and therefore the Steiner symmetral $S_H K$, if $i = n-1$) is the union of all H-symmetric compact convex sets such that some translate orthogonal to H is contained in K, and the Minkowski symmetral $M_H K$ is the intersection of all H-symmetric compact convex sets such that some translate orthogonal to H contains K.*

Proof Let us prove the claim regarding $F_H K$. For each $y \in H^\perp$, the set $K_y \cap K_y^\dagger$ is H-symmetric and its translation by $-y$ is contained in K. On the other hand, if $M \in \mathcal{K}^n$ is H-symmetric and $M + w \subset K$ for some $w \in H^\perp$, then $M \subset K_{-w} \cap K_{-w}^\dagger$. The proof of the other claim is analogous. □

5.5.1 Inclusions between General Symmetrizations and Known Ones.

The containment results presented here have a crucial role in proving many of the results for general i-symmetrizations described in the next sections. All of them are proved in [7], which also contains a critical discussion of the necessity of each hypothesis. The first one is a corollary of Theorem 5.28.

Corollary 5.30 *Let $H \in \mathcal{G}(n,i)$, $i \in \{0,\dots,n-1\}$, and let $\mathcal{B} = \mathcal{K}^n$ or $\mathcal{B} = \mathcal{K}^n_n$. Suppose that $\diamond : \mathcal{B} \to \mathcal{B}_H$ is monotonic, invariant on H-symmetric sets, and invariant under translations orthogonal to H of H-symmetric sets. Then*

$$F_H K \subset \diamond K \subset M_H K \tag{5.36}$$

for all $K \in \mathcal{B}$.

Proof Let $K \in \mathcal{B}$ and let $y \in H^{\perp}$. The set $K_y \cap K_y^{\dagger}$ is H-symmetric. Hence, using the monotonicity and invariance property of $\Diamond$, we have

$$K_y \cap K_y^{\dagger} = \Diamond(K_y \cap K_y^{\dagger}) \subset \Diamond K_y = \Diamond K. \tag{5.37}$$

This formula and (5.33) prove the inclusion on the left. The set $\operatorname{conv}(K_y \cup K_y^{\dagger})$ is H-symmetric and $K \subset \operatorname{conv}(K_y \cup K_y^{\dagger}) - y$. Again, using the monotonicity and invariance property of $\Diamond$, we have

$$\Diamond K \subset \Diamond \left(\operatorname{conv}(K_y \cup K_y^{\dagger}) - y\right) = \Diamond \operatorname{conv}(K_y \cup K_y^{\dagger}) = \operatorname{conv}(K_y \cup K_y^{\dagger}). \tag{5.38}$$

This formula and (5.34) prove the inclusion on the right. □

We present, without proof, two more results. The first one proves for $i = n-1$ the same inclusions of Corollary 5.30 under assumptions of a different nature.

Theorem 5.31 *Let $H \in \mathcal{G}(n, n-1)$, let $\mathcal{B} = \mathcal{K}^n$ or $\mathcal{B} = \mathcal{K}^n_n$, and let $F : \mathcal{B} \to [0, \infty)$ be a strictly increasing set function invariant under translations orthogonal to H of H-symmetric sets. If $\Diamond : \mathcal{B} \to \mathcal{B}_H$ is monotonic, F-preserving, and either invariant on H-symmetric spherical cylinders or projection invariant, then $\Diamond$ is invariant under translations orthogonal to H of H-symmetric sets and*

$$S_H K \subset \Diamond K \subset M_H K \tag{5.39}$$

for all $K \in \mathcal{B}$.

The second result corresponds to Corollary 5.30 for symmetrizations whose symmetral is rotationally symmetric with respect to H, like S_H and $\overline{M}_H$.

Theorem 5.32 *Let $H \in \mathcal{G}(n, i)$, $i \in \{1, \dots, n-1\}$, let $\mathcal{B} = \mathcal{K}^n$ or $\mathcal{B} = \mathcal{K}^n_n$, and let $\Diamond : \mathcal{B} \to \mathcal{B}_H$ be invariant on H-symmetric spherical cylinders and invariant under translations orthogonal to H of H-symmetric sets. Consider the expression*

$$I_H K \subset \Diamond K \subset O_H K. \tag{5.40}$$

The left-hand inclusion holds for all $K \in \mathcal{B}$ if, in addition to the assumptions stated before (5.40)*, $\mathcal{B} = \mathcal{K}^n_n$ and $\Diamond$ is monotonic. The right-hand inclusion holds for all $K \in \mathcal{B}$ if, in addition to the assumptions stated before* (5.40)*, $\Diamond$ is strictly monotonic and idempotent.*

5.6 Characterization of Minkowski and Steiner Symmetrizations and of Polarization

In this section we present some characterizations proved in [7] and in [10]. We refer to these papers for a critical discussion of the necessity of each hypothesis.

5.6.1 Characterizations of Minkowski Symmetrization

Theorem 5.33 *Let $H \in \mathcal{G}(n,i)$, $i \in \{0,\dots,n-1\}$, and let $\mathcal{B} = \mathcal{K}^n$ or $\mathcal{B} = \mathcal{K}^n_n$. Suppose that $\Diamond : \mathcal{B} \longrightarrow \mathcal{B}_H$ is monotonic. Assume in addition either that*

(i) *$i = n-1$ and $\Diamond$ is mean width preserving and either invariant on H-symmetric spherical cylinders or projection invariant, or that*

(ii) *$i \in \{1,\dots,n-1\}$ and $\Diamond$ is mean width preserving, invariant on H-symmetric sets, and invariant under translations orthogonal to H of H-symmetric sets, or that*

(iii) *$i = 0$ and $\Diamond$ is invariant on o-symmetric sets and invariant under translations of o-symmetric sets.*

Then $\Diamond$ is Minkowski symmetrization with respect to H.

Proof Let $K \in \mathcal{B}$. Let us prove part (i). Theorem 5.31 with $F = V_1$ proves

$$\Diamond K \subset M_H K.$$

Since both $\Diamond$ and M_H preserve mean width we have $V_1(\Diamond K) = V_1(M_H K) = V_1(K)$. Since V_1 is strictly monotonic the previous inclusion is an equality.

Let us prove part (ii). Corollary 5.30 proves again $\Diamond K \subset M_H K$. The conclusion follows as before.

Part (iii) is an immediate consequence of Corollary 5.30, since when $i = 0$ we have $F_H = M_H$. □

The next result exploits the linearity of M_H with respect to Minkowski addition.

Theorem 5.34 *Let $H \in \mathcal{G}(n,i)$, $i \in \{0,\dots,n-1\}$. If $\Diamond : \mathcal{K}^n \longrightarrow \mathcal{K}^n_H$ is monotonic, invariant on H-symmetric sets, and linear (i.e., $\Diamond(K+L) = \Diamond K + \Diamond L$ for all $K, L \in \mathcal{K}^n$), then $\Diamond$ is Minkowski symmetrization with respect to H.*

For the proof we refer to [7].

5.6.2 Characterizations of Steiner Symmetrization

Here we present one characterization valid for $(n-1)$-symmetrizations defined in C^n and one valid for $(n-1)$-symmetrizations defined in $\mathcal{K}^n_n$.

Theorem 5.35 (i) *Let $H \in \mathcal{G}(n,n-1)$. Suppose that $\Diamond : C^n \longrightarrow C^n_H$ is an $(n-1)$-symmetrization that is monotonic, volume preserving, and invariant on H-symmetric spherical cylinders. Then*

$$\mathcal{H}^{n-1}\left((\Diamond K) \cap (H^\perp + x)\right) = \mathcal{H}^{n-1}\left(K \cap (H^\perp + x)\right) \tag{5.41}$$

for all $K \in C^n$ and $\mathcal{H}^{n-1}$-almost all $x \in H$.

(ii) *Suppose that in addition to the assumptions in* (i), *$(\Diamond K) \cap (x + H^\perp)$ is a line segment for $\mathcal{H}^{n-1}$-almost all $x \in H$. Then $\Diamond$ is essentially Steiner symmetrization on C^n, in the sense that for all $K \in C^n$, $(\Diamond K) \cap G = (S_H K) \cap G$ for $\mathcal{H}^{n-1}$-almost all lines G orthogonal to H.*

Theorem 5.36 *Let* $H \in \mathcal{G}(n, n-1)$ *and let* $\diamond : \mathcal{K}^n_n \to (\mathcal{K}^n_n)_H$ *be an* $(n-1)$*-symmetrization. If* $\diamond$ *is monotonic, volume preserving, and either invariant on* H*-symmetric spherical cylinders or projection invariant, then* $\diamond$ *is Steiner symmetrization with respect to* H*.*

We refer to [7] for the detailed proofs. Here we only explain the main ideas.

To prove Theorem 5.35 (i) we argue as follows. For $x \in H$ let $D_r(x)$ be the $(n-1)$-dimensional ball in H with center x and radius r. Let s be such that $K|H^\perp \subset [-s, s]$ (we identify $H^\perp$ with $\mathbb{R}$ so that $[-s, s]$ is a shorthand for $s(B^n \cap H^\perp)$). The set $D_r(x) + [-s, s]$ is an H-symmetric spherical cylinder which contains $K \cap (D_r(x) + H^\perp)$. For $L \in \mathcal{C}^n$ let

$$m_{r,L}(x) = \mathcal{H}^n\big(L \cap (D_r(x) + [-s, s])\big).$$

The three assumption on $\diamond$ in part (i) imply

$$m_{r,\diamond K} \geq m_{r,K}. \tag{5.42}$$

Indeed, monotonicity and invariance on H-symmetric spherical cylinders imply

$$\begin{aligned}\diamond\big(K \cap (D_r(x) + [-s, s])\big) &\subset (\diamond K) \cap \diamond(D_r(x) + [-s, s]) \\ &= (\diamond K) \cap (D_r(x) + [-s, s]).\end{aligned}$$

From this inclusion, using the fact that $\diamond$ is volume preserving, we obtain

$$\begin{aligned}m_{r,\diamond K}(x) = \mathcal{H}^n\big((\diamond K) \cap (D_r(x) + [-s, s])\big) &\geq \mathcal{H}^n\big(\diamond(K \cap (D_r(x) + [-s, s]))\big) \\ &= \mathcal{H}^n\big(K \cap (D_r(x) + [-s, s])\big) = m_{r,K}(x).\end{aligned}$$

This proves (5.42). Dividing both sides in (5.42) by $\mathcal{H}^{n-1}(D_r(x))$ and passing to the limit as $r \to 0$ we obtain (5.41) with the inequality (left-hand side ≥ right-hand side), for $\mathcal{H}^{n-1}$-almost all $x \in H$. Integrating this inequality over H, using Fubini's theorem and the volume invariance of $\diamond$, we conclude that (5.41) holds with equality, for $\mathcal{H}^{n-1}$-almost all $x \in H$.

The proof of part Theorem 5.35 (ii) follows directly from (5.41) and the definition of $S_H K$.

Theorem 5.36, under the assumption that $\diamond$ is invariant on H-symmetric spherical cylinders, is a corollary of Theorem 5.35, while under the assumption that $\diamond$ is projection invariant it requires other ideas, which we do not describe here.

We conclude this section with a characterization of Schwarz symmetrization from [9].

Theorem 5.37 *Let* $i \in \{1, \ldots, n-2\}$*, let* $H \in \mathcal{G}(n, i)$*, and let* $\diamond_H$ *be an* i*-symmetrization on* $\mathcal{K}^n_n$*. Suppose that* $\diamond_H$ *is monotonic, volume preserving, rotationally symmetric, and invariant on* H*-symmetric cylinders. Then* $\diamond_H$ *is Schwarz symmetrization with respect to* H*.*

5.6.3 Characterizations of Polarization

We consider the four maps Id, †, P_H, or $P_H^\dagger = \dagger \circ P_H$, where Id and † denote the identity map and reflection in H, respectively. In this section we present two results from [10] which characterize these four maps among maps $\lozenge : \mathcal{E} \subset \mathcal{L}^n \to \mathcal{L}^n$.

Theorem 5.38 *Let $H = u^\perp$, $u \in \mathbb{S}^{n-1}$, be oriented with $u \in H^+$, let $\mathcal{E} = \mathcal{C}^n$ or $\mathcal{L}^n$, and suppose that $\lozenge : \mathcal{E} \to \mathcal{L}^n$ is monotonic, measure preserving, perimeter preserving on convex bodies, and invariant on H-symmetric unions of two disjoint balls. Then $\lozenge$ essentially equals* Id, †, $\lozenge_{P_H}$, *or* $\lozenge^\dagger_{P_H}$.

We say that $\lozenge$ *is perimeter preserving on convex bodies* if, for each $K \in \mathcal{K}^n_n$, $\lozenge K$ is a set of finite perimeter such that $S(\lozenge K) = S(K) = 2V_{n-1}(K)$, where S denotes perimeter in the sense of De Giorgi. The condition of invariance on H-symmetric union of two disjoint balls may seem peculiar, but it is much weaker than the natural assumption that $\lozenge$ is invariant on all H-symmetric sets.

For maps $\lozenge : \mathcal{K}^n \to \mathcal{L}^n$, invariance on H-symmetric unions of two disjoint balls is not available. The next result resorts to a different and rather strong condition; we say that $\lozenge$ is *convexity preserving away from H* if $\lozenge K$ is essentially convex (that is, $\lozenge K$ coincides with a convex set up to a set of $\mathcal{H}^n$-measure zero) for all $K \in \mathcal{K}^n$ with $K \cap H = \emptyset$.

Theorem 5.39 *Let $H = u^\perp$, $u \in \mathbb{S}^{n-1}$, be oriented with $u \in H^+$ and let $\lozenge : \mathcal{K}^n_n \to \mathcal{L}^n$ be monotonic, measure preserving, invariant on H-symmetric sets, perimeter preserving on convex bodies, and convexity preserving away from H. Then $\lozenge$ essentially equals* Id, †, $\lozenge_{P_H}$, *or* $\lozenge^\dagger_{P_H}$.

5.7 Convergence of Successive Symmetrals

Around 1836, Jacob Steiner introduced the symmetrization process that today bears his name in an attempt to prove the isoperimetric inequality. As we have explained, many other inequalities have since been proved by the same method. Two are the main features that have allowed to prove that a certain functional is minimized (or maximized) by a ball:

1) The functional at hand is always decreased (or increased) by a symmetrization;
2) There are sequences of hyperplanes such that the corresponding sequence of successive symmetrals of a convex body converges to a ball of the same volume.

In this section we focus on item 2. Given an i-symmetrization $\lozenge$ we focus on the convergence of successive applications of $\lozenge$ through a sequence of i-dimensional subspaces. (In this section convergence always means convergence in the Hausdorff metric.) We refer to this as a *symmetrization process*. We try to summarize what is known up to date, to explain how the answer depends on the specific symmetrization, whether it depends on the class, $\mathcal{K}^n$ or $\mathcal{C}^n$, of the initial seed, to describe which questions still remain unanswered.

It is well known that in the plane a sequence of successive Steiner symmetrals may converge to a ball or to a regular polygon, and convergence may sometimes seem simple or even obvious. In fact, in the literature there are few examples of Steiner symmetrization processes which do not converge (see [13, Section 1], [25, Lemma 6.3], [6, Examples 2.1], [88, Section 4]). These examples exhibit the same behavior and here we describe one in dimension two. We remark that in this example the directions orthogonal to the sequence of the lines of symmetrization form a dense subset of $\mathbb{S}^1$.

Choose a sequence (α_m) in $(0, \pi/2)$ with

$$\sum_{m=1}^{\infty} \alpha_m = \infty \, , \qquad \prod_{m=1}^{\infty} \cos \alpha_m = r > 0 \, . \tag{5.43}$$

Let $\beta_k = \sum_{m=1}^{k} \alpha_m$ and $v_m = (\cos \beta_m, \sin \beta_m)$. Besides, let $K \in \mathcal{K}_2^2$ have area smaller than πr^2 and contain a horizontal segment L, with L of length 1 and o as midpoint. The sequence (K_m) defined as

$$K_m = S_{v_m^\perp} \dots S_{v_1^\perp} K \, , \; m \in \mathbb{N}, \tag{5.44}$$

does not converge. Indeed, let $L_m = S_{v_m^\perp} \dots S_{v_1^\perp} L$. Since β_m diverges, the segments L_m (which are contained in K_m) spin in circles forever while their length decreases monotonically to r. Moreover, the closure of $\cup_{m \in \mathbb{N}} L_m$ contains rD^n. If (K_m) were converging, the limit would contain rD^n, but the area of K_m, which is equal to the area of K, is strictly less than the area of rD^n. Thus, a contradiction.

Bianchi, Burchard, Gronchi, and Volčič [6] coined the term convergence in shape for the behaviour of this example; a sequence (H_m) *converges in shape* if there exists a sequence of isometries (I_m) such that $I_m S_{H_m} \dots S_{H_1} C$ is a convergent sequence, for each compact set C. The same paper proves that the sequence above has indeed this property, and, furthermore, poses the following question:

Do Steiner processes always converge in shape?

Some partial answer is given in [6], but the question in full generality is still open. This question suggests that convergence of Steiner processes is not rare. To be more precise, let us introduce some definitions, following Coupier and Davydov [33] and widening the field of interest to all symmetrizations encountered so far. Let $\Diamond$ be an i-symmetrization on $\mathcal{B}$, where $\mathcal{B} = \mathcal{K}^n$ or C^n. A sequence (H_m) of i-dimensional subspaces is called *weakly $\Diamond$-universal* for $\mathcal{B}$ if, for any $k \in \mathbb{N}$, and $C \in \mathcal{B}$, the sequence

$$\Diamond_{H_m} \Diamond_{H_{m-1}} \dots \Diamond_{H_k} C \tag{5.45}$$

converges, as m tends to infinity, to an origin-symmetric ball, which may depend on C and k. The sequence is called *$\Diamond$-universal* if it is weakly $\Diamond$-universal and the limit ball is independent of k. We remark that every weakly Steiner-universal sequence is in fact universal, since Steiner processes maintain the volume.

In [88] Ulivelli introduces a more general concept. The sequence (H_m) is called $\Diamond$-*stable* for $\mathcal{B}$ if, for any $k \in \mathbb{N}$ and $C \in \mathcal{B}$, the sequence in (5.45) converges. (We do not prescribe the limit set in this case).

The first example of a Steiner-universal sequence in $\mathcal{K}^n$, $n = 2, 3$, goes back to Blaschke [19]. The sequence he chose is a periodic one. For $n = 3$ he chose three planes U_1, U_2 and U_3, so that their normal vectors span the whole space and at least two of the angles between these vectors are irrational multiples of π. He defined the process as a cyclical repetition of the three symmetrizations S_{U_1}, S_{U_2} and S_{U_3}, i.e. as S_{U_1}, S_{U_2}, S_{U_3}, S_{U_1}, S_{U_2}, S_{U_3}, ... The universality of the sequence was the right ingredient that Blaschke needed to present Schwarz symmetrization as a limit of sequences of Steiner symmetrizations, as explained in Section 5.3.2, and to prove the Brunn–Minkowski inequality.

The argument used by Blaschke was extended by Klain [47, Theorem 5.1] to prove that every sequence of hyperplanes chosen from a finite set $\mathcal{F}$ is Steiner-stable in $\mathcal{K}^n$ and the limit is symmetric under reflection in each hyperplane occurring infinitely often in the sequence. Klain [47, Corollary 5.4] uses this result to construct Steiner-universal sequences in $\mathcal{K}^n$, as described in the next theorem. We say that $v_1, \ldots, v_n \in \mathbb{R}^n$ form an irrational basis of $\mathbb{R}^n$ if they span $\mathbb{R}^n$ and the angle between any two of them is an irrational multiple of π.

Theorem 5.40 *Let (H_m) be a sequence chosen from a finite set $\mathcal{F} = \{v_1^\perp, \ldots, v_n^\perp\} \subset \mathcal{G}(n, n-1)$. Assume that $v_1, \ldots, v_n$ form an irrational basis of $\mathbb{R}^n$ and that, for each i, $v_i^\perp$ appears infinitely many times in (H_m). Then (H_m) is Steiner-universal in $\mathcal{K}^n$.*

The main steps in the proofs by Blaschke and Klain are similar:

1) existence of convergent subsequences;
2) existence of a continuous functional F such that $F(S_{H_m}K) \le F(K)$, with equality if and only if $S_{H_m}K = K$ or $V_n(K) = 0$;
3) the limit of a convergent subsequence is symmetric under reflection in each hyperplane occurring infinitely many times;
4) the hypothesis on the vectors implies that this limit is a ball.

We try to explain these items one by one in a sketch of the proof of Theorem 5.40.

Proof 1) The existence of convergent subsequences follows by (now) standard compactness arguments. The monotonicity of Steiner symmetrization and Blaschke selection theorem, Theorem 1.17, offer a simple conclusion.

2) Blaschke chose as F the surface area, while Klain uses the layering function

$$F(K) = \int_0^\infty V_n(K \cap rB^n) e^{-r^2} \, dr .$$

Another possible choice is $F(K) = \int_K |x|^2 \, dx$.

3) We follow Klain's argument as interpreted by A. Burchard and presented in [6, Theorem 6.1] and [9, Theorem 5.6]. For simplicity, we present only the argument which assumes $K \in \mathcal{K}_n^n$. Let $K_m = S_{H_m} \ldots S_{H_1} K$. Since Steiner symmetrization preserves volume, we have

$$V_n(K_m) = V_n(K). \tag{5.46}$$

The main idea is to construct a subsequence along which the subspaces $v_j^\perp \in \mathcal{F}$ appear in a particular order. With each index m, we associate a permutation π_m of $\{1,\dots,n\}$ that indicates the order in which the subspaces $v_1^\perp,\dots,v_n^\perp$ appear for the first time among those H_j with $j \geq m$. Since there are only finitely many permutations, we can pick a subsequence (H_{m_p}) such that the permutation π_{m_p} is the same for each p. By relabeling the subspaces, we may assume that this permutation is the identity. Passing to a further subsequence, we may assume that every subspace in $\mathcal{F}$ appears in each segment $H_{m_p}, H_{m_p+1},\dots,H_{m_{p+1}-1}$.
By Blaschke selection theorem, there is a subsequence (again denoted by (K_{m_p})) that converges in the Hausdorff metric to some $L \in \mathcal{K}_n^n$.
We show by induction that L is $v_j^\perp$-symmetric for $j = 1,\dots,n$. For $j = 1$, observe that $H_{m_p} = v_1^\perp$ for each p. Therefore K_{m_p} is $v_1^\perp$-symmetric for each p and the same is true for L. Suppose that L is $v_r^\perp$-symmetric for $r = 1,\dots,j-1$. Let m'_p be the index where $v_j^\perp$ appears for the first time after H_{m_p}. Then for $m_p+1 \leq m \leq m'_p-1$, $H_m = v_r^\perp$ for some $r = 1,\dots,j-1$. Steiner symmetrization does not increase the symmetric difference distance of two sets (because it does not change their volume and it does not decrease the volume of their intersection). Using this, the inductive hypothesis and the convergence of (K_{m_p}) to L, one proves that $(K_{m'_p-1})$ too converges in the Hausdorff metric to L as $p \longrightarrow \infty$. Thus

$$S_{v_j^\perp} L = \lim_{p\to\infty} S_{v_j^\perp} K_{m'_p-1} = \lim_{p\to\infty} K_{m'_p}.$$

We also have

$$F(S_{v_j^\perp} L) = \lim_{p\to\infty} F(K_{m'_p}) \geq \lim_{p\to\infty} F(K_{m_{p+1}}) = F(L)\,.$$

Therefore, $S_{v_j^\perp} L = L$, i.e. L is $v_j^\perp$-symmetric and this concludes the inductive step.
Once that the symmetry of L with respect to each $v_j^\perp$ is proved, the fact that Steiner symmetrization does not increase the symmetric difference distance can be used again to prove that the entire sequence (K_m) converges to L.

4) Blaschke and Klain used roughly the same hypothesis on the hyperplanes mutual position. Burchard, Chambers, and Dranovski [24] tackled the problem of characterizing the sets of reflections with respect to hyperplanes in $\mathbb{R}^n$ that generate a dense subgroup of $\mathrm{O}(n)$. They proved that the set of reflections in $v_i^\perp$, $i = 1,2,\dots,n$, generate a dense subgroup of $\mathrm{O}(n)$ if the v_i's

   i) span $\mathbb{R}^n$,
   ii) cannot be partitioned into two mutually orthogonal non-empty subsets, and
   iii) at least two of them form an angle that is an irrational multiple of π.

   Conditions i) and ii) are also necessary, but, when $n > 2$, iii) is not, and [24] explains that iii) implies the right necessary and sufficient condition. i.e. that the group generated by the reflections is not a finite Coxeter group in $\mathrm{O}(n)$. □

The extension of Klain's result to compact sets was first obtained in [6]. Now it can also be seen as a consequence of Bianchi, Gardner and Gronchi [9, Theorem 7.3] which proves that a sequence of hyperplanes is Steiner-universal in C^n if and only if it is Steiner-universal in $\mathcal{K}_n^n$.

To complete the picture regarding Steiner symmetrization, we recall some results on the rate of convergence to a ball and on random sequences.

In 1986 Mani-Levitska [64] was the first to deal with a sequence of hyperplanes uniformly, independently and randomly chosen. He proved that the sequence of related Steiner symmetrizations almost surely rounds every convex body with positive volume and conjectured that the same holds for compact sets. The conjecture was settled by van Schaftingen [89] and extended to measurable sets by Volčič [91].

The first result we know of on the rate of convergence (that is, on the determination of the least number of successive symmetrals required to transform a set K of volume κ_n within a certain distance from B^n) goes back to Hadwiger [44], even though the estimate was very rough. Such results often require very delicate analysis, as evidenced by the deep work of Bourgain, Klartag, Lindenstrauss, Milman, and others. (See [48], [49], and the references given there.) Klartag [49] proves that there exist

$$m = [cn^4 \log^2(1/\varepsilon) + 1]$$

Steiner symmetrizations that transform $K \in \mathcal{K}_n^n$ in a convex body with a Hausdorff distance less than ε from B^n. (Here c is some numerical constant.) Bianchi and Gronchi [12] established a lower bound on the rate of convergence by constructing bodies "hard to be rounded". In every dimension n and for each positive integer m, they constructed origin symmetric convex bodies whose Hausdorff distance from the ball centered in the origin of the same volume cannot be decreased by any sequence of m successive Steiner symmetrizations. Coupier and Davydov [33] complements and strengthens the results of [49], [64] and [91]. In particular, [33] gives an estimate on the speed of convergence to a sphere of random Steiner symmetrizations.

Coupier and Davydov [33] also proves the following result, which maybe was one of the reasons for their choice of the term universal. Its proof is nice and elegant and we report it here.

Theorem 5.41 *A sequence (H_k) of hyperplanes is Steiner-universal in $\mathcal{K}_n^n$ if and only if it is Minkowski-universal in $\mathcal{K}_n^n$.*

Proof Let $K \in \mathcal{K}_n^n$, and assume that (H_k) is a Steiner-universal sequence of hyperplanes. Since Minkowski symmetrization increases the volume and is monotonic, the sequence $V_n(M_{H_k} \dots M_{H_1} K)$ is nondecreasing and bounded and, as $k \longrightarrow \infty$,

$$V_n(M_{H_k} \dots M_{H_1} K) \longrightarrow V, \tag{5.47}$$

for a suitable $V > 0$. By the Blaschke selection Theorem, there exists a subsequence (λ_k) such that, as $k \longrightarrow \infty$,

$$M_{H_{\lambda_k}} \dots M_{H_1} K \longrightarrow E,$$

for a suitable $E \in \mathcal{K}_n^n$ with $V_n(E) = V$. For any positive integer k, m, $m > k$, the inclusion between Steiner and Minkowski symmetrization (5.18) implies

$$\begin{aligned} S_{H_{\lambda_m}} \dots S_{H_{\lambda_{k+1}}} M_{H_{\lambda_k}} \dots M_{H_1} K \subset M_{H_{\lambda_m}} \dots M_{H_{\lambda_{k+1}}} M_{H_{\lambda_k}} \dots M_{H_1} K \\ = M_{H_{\lambda_m}} \dots M_{H_1} K. \end{aligned}$$

Passing to the limit with respect to m, and using the Steiner universality of the sequence (H_k), we obtain that a ball with volume $V_n(M_{H_{\lambda_k}} \dots M_{H_1} K)$ is contained in E. The arbitrariness of k and (5.47) imply that E contains a ball of volume V. Since $V = V_n(E)$, a comparison of the volume forces E to be a ball of volume V. Therefore, any convergent subsequence has the same limit, that is $M_{H_k} \dots M_{H_1} K$ converges to a ball. The same argument can be repeated for the sequence

$$M_{H_k} \dots M_{H_l} K,$$

for any $l \in \mathbb{N}$. Observe that the limiting ball has the same mean width as K, since Minkowski symmetrization is mean width preserving, and therefore it does not depend on l.

The reverse is completely analogous. Assume that (H_k) is a Minkowski-universal sequence of hyperplanes. Since Steiner symmetrization decreases the mean width, the sequence $V_1(S_{H_k} \dots S_{H_1} K)$ is nonincreasing and positive and, as $k \to \infty$,

$$V_1(S_{H_k} \dots S_{H_1} K) \to W$$

for a suitable W. There exists a subsequence of convex bodies converging, as $k \to \infty$, to a suitable $E \in \mathcal{K}^n$, with $V_1(E) = W$. For any positive integer k, m, $m > k$, the inclusion between Steiner and Minkowski symmetrization implies

$$M_{H_{\lambda_m}} \dots M_{H_{\lambda_{k+1}}} S_{H_{\lambda_k}} \dots S_{H_1} K \supset S_{H_{\lambda_m}} \dots S_{H_1} K\,.$$

The Minkowski universality of the sequence (H_k) yields, letting $m \to \infty$, that a ball with the same mean width as that of $S_{H_{\lambda_k}} \dots S_{H_1} K$ contains E. A comparison of mean widths forces E to be a ball with mean width W. Therefore, any convergent subsequence has the same limit, that is, $S_{H_k} \dots S_{H_1} K$ converges to a ball with the same volume of K, since Steiner symmetrization is volume preserving. □

Bianchi, Gardner, and Gronchi [9] studies some of the questions touched on in this section for other known symmetrizations and for general i-symmetrizations. It proves that Klain's Theorem is valid in $\mathcal{K}^n$ for fiber, Schwarz, and Minkowski–Blaschke symmetrizations, as well as for any i-symmetrization satisfying certain hypotheses, and it is valid in C^n for Schwarz symmetrization (see [9, Section 5]). It also proves results in the spirit of Coupier and Davydov; for $i = 1, \dots, n-1$, a sequence of subspaces in $\mathcal{G}(n, i)$ is Minkowski-universal in $\mathcal{K}_n^n$ if and only if it is so in C^n, and the same holds true for Steiner-universal and for Schwarz-universal sequences (see [9, Section 7]). These results, together with a study of the problem of which reflections with respect to i-dimensional subspaces generate a dense subgroup of $\mathrm{O}(n)$, carried out in [8], enable the authors to create universal sequences in $\mathcal{K}^n$ and in $\mathcal{C}^n$ for many of the symmetrizations mentioned above (see [9, Section 6]).

Ulivelli [87] further extends Klain's Theorem by proving its validity in $\mathcal{C}^n$ for Minkowski symmetrization (as a consequence of the proof that a Minkowski symmetrizations process with seed $C \in \mathcal{C}^n$ converges if and only if the one with seed $\operatorname{conv}(C)$ converges). The same author proves in [88] that the family $\mathcal{F}$ of all i-symmetrizations $\Diamond$ which are monotonic, invariant on H-symmetric sets and invariant under translations orthogonal to H of H-symmetric sets share the same behavior with respect to symmetrizations processes. Since $\mathcal{F}$ contains Steiner, Minkowski and fiber symmetrizations, this means that if a sequence (H_k) is Minkowski-universal or Minkowski-stable, then it is also $\Diamond$-universal or $\Diamond$-stable, for every $\Diamond \in \mathcal{F}$. A similar result is proved also for convergence in shape.

5.8 Rearrangements and a Proof of the Pólya–Szegő Inequality for Smoothing Rearrangements

A familiar version of the Pólya–Szegő inequality states that if $\Phi : [0,\infty) \to [0,\infty)$ is convex and $\Phi(0) = 0$ (i.e. a Young function) and $f \in \mathcal{V}(\mathbb{R}^n)$ is Lipschitz, then $f^\#$ is Lipschitz and

$$\int_{\mathbb{R}^n} \Phi(|\nabla f^\#(x)|)\,\mathrm{d}x \le \int_{\mathbb{R}^n} \Phi(|\nabla f(x)|)\,\mathrm{d}x; \tag{5.48}$$

see, e.g., [2]. Here $f^\#$ denotes the *symmetric decreasing rearrangement* of f, the function whose superlevel sets have the same $\mathcal{H}^n$-measure as those of f and such that, for $t > 0$, $\{x : f^\#(x) > t\}$ is a ball centered at the origin o of $\mathbb{R}^n$. The subgraph of $f^\#$ is the Schwarz symmetrization of the subgraph of f with respect to the x_{n+1}-axis.

The map that takes f to $f^\#$ is the primary example of a rearrangement. Other examples are the rearrangements associated to Steiner and Schwarz symmetrizations (often called (k,n)*-Steiner* rearrangements), and *polarization*. The Pólya–Szegő inequality (5.48) holds for each of the just-mentioned rearrangements.

Diverse variants and applications of the Pólya–Szegő inequality have generated a very substantial literature, surveyed by Talenti who in [86, p. 126] provides over fifty references.

At the heart of the definition of the rearrangement Tf of a function f there is the formula

$$\{x : Tf(x) > t\} = \Diamond_T\{x : f(x) > t\} \tag{5.49}$$

where $\Diamond_T$ denotes a map from $\mathcal{L}^n$ to itself which is monotonic and measure preserving. The superlevel set $\{x : Tf(x) > t\}$ depends only on $\{x : f(x) > t\}$ and this relation, the map $\Diamond_T$, is the same for each t. A rearrangement is a map from function spaces and one may wonder which properties of this map make (5.49) valid. An answer is given in the following theorem, proved in [10], but before stating it let us define two properties.

Let $X \subset \mathcal{M}(\mathbb{R}^n)$ and $T : X \to X$. We say that:

1) T is *equimeasurable* if $\mathcal{H}^n(\{x : Tf(x) > t\}) = \mathcal{H}^n(\{x : f(x) > t\})$ for $t \in \mathbb{R}$;
2) T is *monotonic* if $f, g \in X$, $f \leq g$, essentially, implies $Tf \leq Tg$.

Theorem 5.42 *Let $X = \mathcal{S}(\mathbb{R}^n)$ or $\mathcal{V}(\mathbb{R}^n)$, let $T : X \to X$ be equimeasurable and monotonic. Then there exists a map $\lozenge_T : \mathcal{L}^n \to \mathcal{L}^n$ for which* (5.49) *is valid. This map is defined for $A \in \mathcal{L}^n$ by*

$$\lozenge_T A = \{x : T1_A(x) = 1\},$$

and it is measure preserving and monotonic. Moreover T is defined, essentially, by $\lozenge_T$.

We recall that the term essentially means up to a set of $\mathcal{H}^n$-measure zero. We can thus define the notion of rearrangement as follows.

Let $X \subset \mathcal{M}(\mathbb{R}^n)$. A map $T : X \to X$ is called a *rearrangement* if it is equimeasurable and monotonic.

Bianchi, Gardner, Gronchi, and Kiderlen [10] and [11] have studied rearrangements in an abstract setting, based on the properties that they satisfy, independent from the specific rearrangement. For the convenience of the reader, we now state five results proved in [10] as Lemmas 4.1, 4.5, 4.7, Theorem 4.8 and the remarks that follow it, and Theorem 4.9, respectively.

Proposition 5.43 (i) *If $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is equimeasurable, then* $\operatorname{ess\,inf} Tf = \operatorname{ess\,inf} f$ *for $f \in \mathcal{S}(\mathbb{R}^n)$.*
(ii) *If $T : \mathcal{M}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^n)$ is a rearrangement, then* $\operatorname{ess\,inf} Tf \geq \operatorname{ess\,inf} f$ *for $f \in \mathcal{M}(\mathbb{R}^n)$. Hence, $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*
(iii) *In either case, $T : \mathcal{V}(\mathbb{R}^n) \to \mathcal{V}(\mathbb{R}^n)$ and T is essentially the identity on constant functions.*

Proposition 5.44 *Let $X = \mathcal{M}(\mathbb{R}^n)$, $\mathcal{M}_+(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$, or $\mathcal{V}(\mathbb{R}^n)$, and let $T : X \to X$ be equimeasurable.*
(i) *The induced map $\lozenge_T : \mathcal{L}^n \to \mathcal{L}^n$ given by*

$$\lozenge_T A = \{x : T1_A(x) = 1\}$$

for $A \in \mathcal{L}^n$ is well defined and measure preserving.
(ii) *If $X = \mathcal{M}_+(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$, or $\mathcal{V}(\mathbb{R}^n)$, then T essentially maps characteristic functions of sets in $\mathcal{L}^n$ to characteristic functions of sets in $\mathcal{L}^n$, in the sense that for each $A \in \mathcal{L}^n$,*

$$T1_A = 1_{\lozenge_T A},$$

essentially.

Proposition 5.45 *Let $X = \mathcal{S}(\mathbb{R}^n)$ or $\mathcal{V}(\mathbb{R}^n)$ and let $T : X \to X$ be a rearrangement. For $X = \mathcal{S}(\mathbb{R}^n)$, $A \in \mathcal{L}^n$, and $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 0$, we have*

$$T(\alpha 1_A + \beta) = \alpha T1_A + \beta,$$

essentially. When $X = \mathcal{V}(\mathbb{R}^n)$, (5.45) *holds, essentially, if* $\beta = 0$.

Proposition 5.46 *Let* $X = \mathcal{M}(\mathbb{R}^n)$, $\mathcal{M}_+(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$, *or* $\mathcal{V}(\mathbb{R}^n)$ *and let* $T : X \to X$ *be a rearrangement.*
(i) *The map* $\Diamond_T : \mathcal{L}^n \to \mathcal{L}^n$ *defined by* (5.44) *is monotonic.*
(ii) *If* $X = \mathcal{S}(\mathbb{R}^n)$ *or* $\mathcal{V}(\mathbb{R}^n)$ *and* $f \in X$, *then*

$$\{x : Tf(x) \geq t\} = \Diamond_T\{x : f(x) \geq t\} \quad \textit{and} \quad \{x : Tf(x) > t\} = \Diamond_T\{x : f(x) > t\},$$

essentially, for $t > \operatorname{ess\,inf} f$. *Moreover,* T *is essentially determined by* $\Diamond_T$, *since*

$$Tf(x) = \max\left\{\sup\{t \in \mathbb{Q},\ t > \operatorname{ess\,inf} f : x \in \Diamond_T\{z : f(z) \geq t\}\}, \operatorname{ess\,inf} f\right\},$$

essentially.

Proposition 5.47 *Let* $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *be a rearrangement and let* $f \in \mathcal{S}(\mathbb{R}^n)$. *If* $\varphi : \mathbb{R} \to \mathbb{R}$ *is right-continuous and increasing (i.e., non-decreasing), then* $\varphi \circ f \in \mathcal{S}(\mathbb{R}^n)$ *and*

$$\varphi(Tf) = T(\varphi \circ f),$$

essentially.

We recall that the symbol A^*, for a subset A of $\mathbb{R}^n$, denotes the set of points of $\mathbb{R}^n$ of density 1 for A.

We say that a rearrangement T is *smoothing* if the associated map $\Diamond_T$ is smoothing, i.e. if

$$(\Diamond_T^* A) + dB^n \subset \Diamond_T^*(A + dB^n), \tag{5.50}$$

essentially, for each $d > 0$ and bounded measurable set A, where $\Diamond_T^* A$ is defined by

$$\Diamond_T^* A = (\Diamond_T A)^*.$$

We recall (see Lemma 5.2) that in this definition one can equivalently require the pointwise inclusion

$$(\Diamond_T^* A) + dD^n \subset \Diamond_T^*(A + dD^n). \tag{5.51}$$

Bianchi, Gardner, Kiderlen, and Gronchi [11] prove that the notion of smoothing is equivalent to the rearrangement reducing the modulus of continuity.

Theorem 5.48 *Let* $X = \mathcal{S}(\mathbb{R}^n)$ *or* $\mathcal{V}(\mathbb{R}^n)$. *The rearrangement* $T : X \to X$ *is smoothing if and only if* T *reduces the modulus of continuity, that is,* T *is such that* $\omega_d(Tf) \leq \omega_d(f)$ *for* $d > 0$ *and* $f \in X$, *where*

$$\omega_d(f) = \operatorname*{ess\,sup}_{|x-y| \leq d} |f(x) - f(y)|.$$

All special rearrangements mentioned in the lines following (5.48) are smoothing, as we have proved in Section 5.3. Bianchi, Gardner, Gronchi, and Kiderlen [11] proves the Pólya–Szegő inequality for all smoothing rearrangements.

Theorem 5.49 *Let $X = \mathcal{S}(\mathbb{R}^n)$ or $\mathcal{V}(\mathbb{R}^n)$, let $T : X \to X$ be a rearrangement, and let $\Phi : [0, \infty) \to [0, \infty)$ be convex with $\Phi(0) = 0$. If T is smoothing and $f \in X$ is Lipschitz then Tf coincides with a Lipschitz function $\mathcal{H}^n$-almost everywhere on $\mathbb{R}^n$, and*

$$\int_{\{x:\, Tf(x)\geq a\}} \Phi\left(|\nabla Tf(x)|\right)\, \mathrm{d}x \leq \int_{\{x:\, f(x)\geq a\}} \Phi\left(|\nabla f(x)|\right)\, \mathrm{d}x \tag{5.52}$$

for each $a > \operatorname{ess\,inf} f$. Hence

$$\int_{\mathbb{R}^n} \Phi\left(|\nabla Tf(x)|\right)\, \mathrm{d}x \leq \int_{\mathbb{R}^n} \Phi\left(|\nabla f(x)|\right)\, \mathrm{d}x, \tag{5.53}$$

where the integrals may be infinite.

The analogous result is valid also for functions in the Orlicz space $W^{1,\Phi}(\mathbb{R}^n)$, the same function space where (5.48) is valid, and for Young functions Φ with values in $[0, +\infty]$.

Example We have observed that smoothing rearrangements reduce outer Minkowski content (and perimeter, when the two concepts coincide) and one may wonder if the Pólya–Szegő inequality is valid for every rearrangement T such that $\Diamond_T$ has this property, but this is not the case. Let B be a ball containing o in its interior and with a center different from o. For $A \in \mathcal{L}^n$ define $\Diamond_T A = \lambda B$ where λ is chosen so that $\mathcal{H}^n(A) = \mathcal{H}^n(\lambda B)$. This map clearly does not increase the outer Minkowski content, by the isoperimetric inequality. On the other hand if, say, $f(x) = \max\{1 - |x|, 0\}$ and $p > 1$ then

$$\int_{\mathbb{R}^n} |\nabla Tf(x)|^p\, \mathrm{d}x > \int_{\mathbb{R}^n} |\nabla f(x)|^p\, \mathrm{d}x. \tag{5.54}$$

This can be computed directly but it also comes from the study of the equality cases in (5.48). Indeed, if (5.54) were false then (5.54) would hold with the equality, because f coincides with its symmetric decreasing rearrangement $f^{\#}$. This contradicts the fact that equality in (5.54), for that particular f, holds only for functions which, up to a translation, coincide with their symmetric decreasing rearrangement, and Tf does not satisfy this property.

5.8.1 Proof of the Pólya–Szegő Inequality

The method of proof of Theorem 5.49 is new and we present it here. We divide the proof in four steps and, for each step, we first describe the relevant ideas and then write and prove the relative lemmas (with one exception).

Step 1. *The function Tf coincides with a Lipschitz function $\mathcal{H}^n$-almost everywhere on $\mathbb{R}^n$*. Assume that L is the Lipschitz constant for f. Then Tf is Lipschitz with a Lipschitz constant not larger than L, as T reduces the modulus of continuity, by Theorem 5.48.

In the proof we use the following abbreviations: let $K_{f,a} = K_f \cap \{x_{n+1} \geq a\}$, $K_{Tf,a} = K_{Tf} \cap \{x_{n+1} \geq a\}$ and $K^*_{Tf,a} = (K_{Tf,a})^*$.

Step 2. *Let $C \in \mathcal{K}^{n+1}_{n+1}$ be an o-symmetric convex body of revolution about the x_{n+1}-axis, supported by the hyperplanes $\{x_{n+1} = \pm 1\}$. This body in a later step is chosen to represent Φ. We prove, for $d > 0$,*

$$\mathcal{H}^{n+1}\big(K^*_{Tf,a} + d \operatorname{int} C\big) \leq \mathcal{H}^{n+1}\big(K_{f,a} + d \operatorname{int} C\big). \tag{5.55}$$

We prove this slice by slice, where by this we mean that we prove formula (5.58) *below, for $t > \operatorname{ess\,inf} f$. Taking the $\mathcal{H}^n$-measures of both sides of* (5.58)*, integrating with respect to t, and using Fubini's theorem and the fact that $\lozenge_T$ is measure preserving, we obtain* (5.55)*. In order to prove* (5.58) *we need $\{x : f(x) \geq a\}$ bounded, and Lemma 5.51 proves that this is the case for any Lipschitz $f \in \mathcal{S}(\mathbb{R}^n)$ and $a > \operatorname{ess\,inf} f$.*

In formula (5.58) we are slightly abusing notation by extending the action of $\lozenge_T$ to horizontal hyperplanes in R^{n+1}. To make this rigorous, if E is a subset of the hyperplane $\{x_{n+1} = t\} = \mathbb{R}^n + te_{n+1}$ in $\mathbb{R}^{n+1}$such that $E|\mathbb{R}^n \in \mathcal{L}^n$, we shall define

$$\lozenge_T E = (\lozenge_T(E \,|\, \mathbb{R}^n)) + te_{n+1}. \tag{5.56}$$

The action of $\lozenge^*_T$ can be extended in a similar fashion. Note that we have, for $t > \operatorname{ess\,inf} f$,

$$K_{Tf} \cap \{x_{n+1} = t\} = \lozenge_T(K_f \cap \{x_{n+1} = t\}),$$

essentially, and

$$\big(K_{Tf} \cap \{x_{n+1} = t\}\big)^* = \lozenge^*_T(K_f \cap \{x_{n+1} = t\}), \tag{5.57}$$

where here and below, sets of Lebesgue density points are taken with respect to the appropriate horizontal hyperplane identified with $\mathbb{R}^n$.

Lemma 5.50 *Let $X = \mathcal{S}(\mathbb{R}^n)$ or $\mathcal{V}(\mathbb{R}^n)$, let $T : X \to X$ be a rearrangement, and let $d > 0$. Let $C \in \mathcal{K}^{n+1}_{n+1}$ be an o-symmetric convex body of revolution about the x_{n+1}-axis, supported by the hyperplanes $\{x_{n+1} = \pm 1\}$. If T is smoothing, $a > d + \operatorname{ess\,inf} f$, and $f \in \mathcal{S}(\mathbb{R}^n)$ is such that $\{x : f(x) \geq a\}$ is bounded, then*

$$\big(K^*_{Tf,a} + d \operatorname{int} C\big) \cap \{x_{n+1} = t\} \subset \lozenge^*_T\big((K_{f,a} + d \operatorname{int} C) \cap \{x_{n+1} = t\}\big) \tag{5.58}$$

for $t > \operatorname{ess\,inf} f$.

Proof Let $d > 0$ and let $C = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : |x_{n+1}| \leq 1, |x| \leq g(x_{n+1})\}$, for a suitable concave function g defined on $[-1, 1]$. For $t \in \mathbb{R}$, denote by Π_t the orthogonal projection onto $\{x_{n+1} = t\}$. If L is any set in $\mathbb{R}^{n+1}$, then

$$(L + d \operatorname{int} C) \cap \{x_{n+1} = t\} = \bigcup_{t-d<s<t+d} \Pi_t\left((L \cap \{x_{n+1} = s\}) + r_s D^n\right), \tag{5.59}$$

where $r_s = d\,g((t-s)/d)$ and $D^n = \operatorname{int} B^n$. Indeed, $p \in (L + d \operatorname{int} C) \cap \{x_{n+1} = t\}$ if and only if $p \,|\, \langle e_{n+1} \rangle = te_{n+1}$ and there is a $z \in L$ such that $p \in z + d \operatorname{int} C$. If $z \,|\, \langle e_{n+1} \rangle = se_{n+1}$, then this holds if and only if $t - d < s < t + d$ and

$$|p - \Pi_t z| < d\,g\left(\frac{t-s}{d}\right),$$

that is, $p \in \Pi_t(z + r_s D^n)$.

Applying (5.59) with L replaced by $L \cap \{x_{n+1} \ge a\}$, we obtain

$$\begin{aligned} &((L \cap \{x_{n+1} \ge a\}) + d \operatorname{int} C) \cap \{x_{n+1} = t\} \\ &= \bigcup_{t-d<s<t+d} \Pi_t\left((L \cap \{x_{n+1} \ge a\} \cap \{x_{n+1} = s\}) + r_s D^n\right) \\ &= \bigcup_{t-d<s<t+d,\ s \ge a} \Pi_t\left((L \cap \{x_{n+1} = s\}) + r_s D^n\right). \end{aligned} \tag{5.60}$$

Let $f \in X$ satisfy the hypotheses of the lemma. It is not difficult to prove that

$$K_f^* \cap \{x_{n+1} = s\} \subset \left(K_f \cap \{x_{n+1} = s\}\right)^*, \tag{5.61}$$

where the set of Lebesgue density points on the right is formed with respect to the hyperplane $\{x_{n+1} = s\} = \mathbb{R}^n + se_{n+1}$, identified with $\mathbb{R}^n$. (See [11, Lemma 5.1] for a detailed proof.) We have

$$K_{Tf,a}^* \subset K_{Tf}^* \cap \{x_{n+1} \ge a\}^* \subset K_{Tf}^* \cap \{x_{n+1} \ge a\}.$$

From this and (5.61) with f replaced by Tf, we obtain

$$K_{Tf,a}^* \cap \{x_{n+1} = s\} \subset K_{Tf}^* \cap \{x_{n+1} = s\} \subset (K_{Tf} \cap \{x_{n+1} = s\})^*, \tag{5.62}$$

whenever $s \ge a$, while the set on the left is clearly empty if $s < a$. We use (5.59) with $L = (K_{Tf} \cap \{x_{n+1} \ge a\})^*$, (5.62), (5.57), the fact that T is smoothing as expressed by (5.51) with $A = K_f \cap \{x_{n+1} = s\}$, the fact that the action of $\diamondsuit_T^*$ as extended by (5.56) is the same for each t, the pointwise monotonicity of $\diamondsuit_T^*$, and (5.60) with $L = K_f$, to obtain

$$\begin{aligned}
&\left(K^*_{Tf,a} + d\ \operatorname{int} C\right) \cap \{x_{n+1} = t\}\\
&= \bigcup_{t-d<s<t+d} \Pi_t \left([K^*_{Tf,a} \cap \{x_{n+1} = s\}] + r_s D^n \right)\\
&\subset \bigcup_{t-d<s<t+d,\ s\geq a} \Pi_t \left([K_{Tf} \cap \{x_{n+1} = s\}]^* + r_s D^n \right)\\
&= \bigcup_{t-d<s<t+d,\ s\geq a} \Pi_t \left(\left[\Diamond^*_T \left(K_f \cap \{x_{n+1} = s\} \right) \right] + r_s D^n \right)\\
&\subset \bigcup_{t-d<s<t+d,\ s\geq a} \Pi_t \left(\Diamond^*_T \left[(K_f \cap \{x_{n+1} = s\}) + r_s D^n \right] \right)\\
&= \bigcup_{t-d<s<t+d,\ s\geq a} \Diamond^*_T \left(\Pi_t \left[(K_f \cap \{x_{n+1} = s\}) + r_s D^n \right] \right)\\
&\subset \Diamond^*_T \left(\bigcup_{t-d<s<t+d,\ s\geq a} \Pi_t \left[(K_f \cap \{x_{n+1} = s\}) + r_s D^n \right] \right)\\
&= \Diamond^*_T \left((K_{f,a} + d\ \operatorname{int} C) \cap \{x_{n+1} = t\} \right).
\end{aligned}$$

Lemma 5.51 *Let $f \in \mathcal{S}(\mathbb{R}^n)$ be Lipschitz. If $a > \operatorname{ess\,inf} f$, then $\{x : f(x) \geq a\}$ is bounded.*

Proof Let $\varepsilon > 0$ be such that $a - \varepsilon > \operatorname{ess\,inf} f$ and let L be the Lipschitz constant of f.

Suppose that $\{x : f(x) \geq a\}$ is unbounded. Then there are points x_k in this set with $|x_{k+1}| > |x_k| + 2\varepsilon/(1 + L)$ for $k \in \mathbb{N}$. The Lipschitz property implies that $f(x) \geq a - \varepsilon$ whenever $x \in B\left(x_k, \varepsilon/(1 + L)\right)$, $k \in \mathbb{N}$. As these balls are disjoint, $\mathcal{H}^n(\{x : f(x) \geq a - \varepsilon\}) = \infty$, contradicting $f \in \mathcal{S}(\mathbb{R}^n)$. □

Recall that $\overline{\mathcal{M}}_C(A)$ is the anisotropic outer Minkowski content of $A \in \mathcal{M}(\mathbb{R}^n)$ with respect to C defined in (5.4). We will apply this notion in $\mathbb{R}^{n+1}$. Also recall that h_C is the support function of C and G_f denotes the graph of $f \in \mathcal{M}(\mathbb{R}^n)$.

Step 3. *Since*

$$\mathcal{H}^{n+1}(K^*_{Tf,a}) = \mathcal{H}^{n+1}(K_{Tf,a}) = \mathcal{H}^{n+1}(K_{f,a}),$$

by the equimeasurability of T, (5.55) *gives*

$$\frac{\mathcal{H}^{n+1}(K^*_{Tf,a} + d\ \operatorname{int} C) - \mathcal{H}^{n+1}(K^*_{Tf,a})}{d} \leq \frac{\mathcal{H}^{n+1}\left(K_{f,a} + d\ \operatorname{int} C\right) - \mathcal{H}^{n+1}(K_{f,a})}{d}.$$

Passing to the limit as $d \to 0$, if they exist, we obtain an inequality between the respective anisotropic outer Minkowski contents, i.e.

$$\overline{\mathcal{M}}_C(K^*_{Tf,a}) \leq \overline{\mathcal{M}}_C(K_{f,a}). \tag{5.63}$$

Lemma 5.52 below, applied to Tf and to f, proves that the limits exist and that this inequality can be expressed in terms of integrals over the graphs of Tf and f. The inequality (5.63) *becomes*

$$\int_{G_{Tf}\cap\{x_{n+1}>a\}} h_C(\nu(x))\,\mathrm{d}\mathcal{H}^n(x) \leq \int_{G_f\cap\{x_{n+1}>a\}} h_C(\nu(x))\,\mathrm{d}\mathcal{H}^n(x), \tag{5.64}$$

where $\nu(x)$ denotes the outer unit normal to the graph. The integral on the right can be written as

$$\int_{\{y:f(y)>a\}} h_C\left(\frac{(-\nabla f(y),1)}{\sqrt{1+|\nabla f(y)|^2}}\right)\sqrt{1+|\nabla f(y)|^2}\,\mathrm{d}y$$
$$= \int_{\{y:f(y)>a\}} h_C(-\nabla f(y),1)\,\mathrm{d}y,$$

where we used the 1*-homogeneity of h_C. Similarly, the integral on the left can be rewritten in the same form, with f replaced by Tf. Consequently,* (5.64) *yields*

$$\int_{\{y:Tf(y)>a\}} h_C(-\nabla Tf(y),1)\,\mathrm{d}y \leq \int_{\{y:f(y)>a\}} h_C(-\nabla f(y),1)\,\mathrm{d}y. \tag{5.65}$$

In order to apply Lemma 5.52 we have to assume $\mathcal{H}^n(\{x : f(x) = a\}) = \mathcal{H}^n(\{x : Tf(x) = a\}) = 0$. In the next step we show that, for the purpose of proving Theorem 5.49, this is not restrictive.

Lemma 5.52 *Let $f \in \mathcal{S}(\mathbb{R}^n)$ be Lipschitz and let C be as in Lemma 5.50. Let $a > \operatorname{ess\,inf} f$ be such that $\mathcal{H}^n(\{x : f(x) = a\}) = 0$. Then*

$$\overline{\mathcal{M}}_C\big(K^*_{f,a}\big) = \overline{\mathcal{M}}_C\big(K_{f,a}\big) = \int_{G_f\cap\{x_{n+1}>a\}} h_C(\nu(x))\,\mathrm{d}\mathcal{H}^n(x)$$
$$+\,\mathcal{H}^n(\{x : f(x) \geq a\}), \tag{5.66}$$

where $\nu(x)$ denotes the outer unit normal to K_f at x.

The proof of this lemma is quite technical and we do not include it here. It consists essentially in proving that we can apply to our situation a result proved by L. Lussardi and E. Villa. This result, [51, Remark 4.2, Theorem 4.4, and Remark 4.5], states the following. (Recall that if $E \subset \mathbb{R}^{n+1}$ is $\mathcal{H}^{n+1}$-measurable, its density $\Theta(E,x)$ at x is defined by (5.3) with n replaced by $n+1$, and, for $t \in [0,1]$, define $E^t = \{x \in \mathbb{R}^{n+1} : \Theta(E,x) = t\}$.) If $E \subset \mathbb{R}^{n+1}$ is a Borel set whose boundary is countably $\mathcal{H}^n$-rectifiable and bounded, $\mathcal{H}^n(\partial E \cap E^0) = 0$, and E has the property that there exist $\gamma > 0$ and a probability measure μ in $\mathbb{R}^{n+1}$ absolutely continuous with respect to $\mathcal{H}^n$, such that for each $x \in \partial E$ and $r \in (0,1)$,

$$\mu(x + rD^n) \geq \gamma r^n, \tag{5.67}$$

then E has finite perimeter, the anisotropic outer Minkowski content of E with respect to C is defined, and

$$\overline{\mathcal{M}}_C(E) = \int_{\partial^e E} h_C(\nu(x))\, \mathrm{d}\mathcal{H}^n(x), \tag{5.68}$$

where $\partial^e E = \mathbb{R}^{n+1} \setminus (E^0 \cup E^1)$ denotes the essential boundary of E.

Step 4: conclusion. *Lemma 5.53 below proves that, given any $M > 0$, it is possible to choose C so that*

$$h_C(y, 1) = 1 + b\, \Phi(|y|), \quad \forall y \in \mathbb{R}^n : |y| < M, \tag{5.69}$$

for some $b > 0$. If we choose M larger than the Lipschitz constant of f and of Tf, we have

$$\max \{|\nabla f(x)|, |\nabla T f(x)|\} \leq M$$

for $\mathcal{H}^n$-almost all $x \in \mathbb{R}^n$. Assume $\mathcal{H}^n(\{x : f(x) = a\}) = 0$. Note that, by the equimeasurability of T, this is equivalent to $\mathcal{H}^n(\{x : Tf(x) = a\}) = 0$. Under this assumption inequality (5.65) *is valid and, using* (5.69)*, we can rewrite it in terms of Φ as*

$$\int_{\{y: Tf(y) > a\}} 1 + b\Phi(|\nabla T f(x)|)\, \mathrm{d}x \leq \int_{\{y: f(y) > a\}} 1 + b\Phi(|\nabla f(x)|)\, \mathrm{d}x. \tag{5.70}$$

Since $\mathcal{H}^n(\{y : Tf(y) > a\}) = \mathcal{H}^n(\{y : f(y) > a\})$, the terms 1 *in the integrands give the same contribution, they cancel each other, and* (5.70) *implies the Pólya–Szegő inequality* (5.52).

If $\mathcal{H}^n(\{x : f(x) = a\}) > 0$ we argue by approximation. The set of values t such that $\mathcal{H}^n(\{x : f(x) = t\}) = 0$ is dense in $(\operatorname{ess\,inf} f, \infty)$, so there is an increasing sequence $\{a_m\}$ contained in $(\operatorname{ess\,inf} f, a)$ and converging to a such that $\mathcal{H}^n(\{x : f(x) = a_m\}) = \mathcal{H}^n(\{x : Tf(x) = a_m\}) = 0$ for each m. The validity of (5.52) *with $a = a_m$, for each m, implies, in the limit, its validity for a.*

Finally, by Proposition 5.43, we have $\operatorname{ess\,inf} Tf = \operatorname{ess\,inf} f$. Letting $a \to \operatorname{ess\,inf} f$ in (5.52)*, we arrive at* (5.53).

Lemma 5.53 *Let $\Phi : [0, \infty) \to [0, \infty)$ be convex with $\Phi(0) = 0$ and let $M > 0$. Then there exist $b > 0$ and an o-symmetric convex body $C \subset \mathbb{R}^{n+1}$ of revolution about the x_{n+1}-axis, such that*

$$h_C(y, 1) = 1 + b\, \Phi(|y|), \tag{5.71}$$

for $y \in \mathbb{R}^n$ with $|y| \leq M$. In particular, C is supported by the hyperplanes $\{x_{n+1} = \pm 1\}$ and hence satisfies the conditions in Lemma 5.50.

Proof Define

$$\Psi(t) = \begin{cases} \Phi(t), & \text{if } 0 \le t \le M, \\ mt + q, & \text{if } t \ge M, \end{cases}$$

where $m > 0$ and $q \le 0$ are such that $\Psi : [0, \infty) \to [0, \infty)$ is convex. Then, for $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, define

$$g(y, t) = \begin{cases} |t|\,(1 + b\,\Psi(|y|/|t|)\,, & \text{if } t \neq 0, \\ b\,m|y|, & \text{if } t = 0, \end{cases} \tag{5.72}$$

$$= \begin{cases} |t|\,(1 + b\,\Phi(|y|/|t|)\,, & \text{if } |t| \ge |y|/M, \\ b\,m|y| + (1 + b\,q)|t|, & \text{if } |t| \le |y|/M, \end{cases} \tag{5.73}$$

where $b > 0$. It is enough to show that b can be chosen so that $g = h_C$ is the support function of a convex body C, since the origin symmetry and symmetry about the x_{n+1}-axis of C, and (5.71), then follow directly from the definition of g. To this end, note that from (5.72), the positive homogeneity of g follows immediately, and the subadditivity of g for $t > 0$ or for $t < 0$ is a routine exercise using the triangle inequality and the convexity of Ψ. It is then enough to observe that if b is small enough to ensure that $1 + b\,q > 0$, then the function $b\,m|y| + (1 + b\,q)|t|$ in (5.73) coincides with the support function of the o-symmetric spherical cylinder with the x_{n+1}-axis as its axis, with height $2(1 + b\,q)$ and radius $b\,m$. □

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