

# Learning Controllers from Data via Approximate Nonlinearity Cancellation

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**Abstract**—We introduce a method to deal with the data-driven control design of nonlinear systems. We derive conditions to design controllers via (approximate) nonlinearity cancellation. These conditions take the compact form of data-dependent semi-definite programs. The method returns controllers that can be certified to stabilize the system even when data are perturbed and disturbances affect the dynamics of the system during the execution of the control task, in which case an estimate of the robustly positively invariant set is provided.

## I. INTRODUCTION

Automating the control design process is important to cope with complex dynamical plants whose dynamics is poorly known. Data-driven control is a notable example of such an automated synthesis. Namely, data-driven control refers to the procedure of designing controllers for an unknown system starting solely from measurements collected from the plant and some priors about the plant itself (linear vs. nonlinear parametrization, nature of the noise, etc.). In this paper we study the problem of designing controllers for nonlinear systems from data.

*Related literature.* System identification followed by control design for the identified system is a classical way to indirectly perform data-driven control [1]. By *direct* data-driven control instead it is meant a procedure in which no intermediate step of identifying the system model is taken, earlier examples being the iterative feedback tuning (IFT) [2], and the virtual reference feedback tuning (VRFT) [3]. Recent times have seen a renewed interest in direct data-driven control, viewed as compact data-dependent conditions which, once verified, automatically return controllers without *explicitly* identifying the plant. One of the focus points in these data-driven control results is how to deal with perturbations and noise affecting the data and the resulting noise-induced uncertainty. Assuming a process noise with bounded  $\ell_\infty$  norm, [4] defines a set of system's matrices pairs consistent with the data and, using an extended Farkas' lemma, derives conditions under which stability of all systems in the set hold. These conditions can be checked using polynomial optimization techniques.

The papers [5], [6] highlight the relevance of a result in [7], about representing the behavior of a linear time-invariant system via a single input-output trajectory, and use this result to develop *data-enabled*, rather than model-based, predictive control, providing probabilistic guarantees on performance for systems subject to stochastic disturbances.

The result of [7] has also been used in [8] to obtain a data-dependent representation for linear systems based on which linear matrix inequalities only depending on data are introduced and used to provide solutions to problems such as state- and output-feedback stabilization as well as the linear quadratic regulator synthesis. The presence of deterministic noise with bounded energy affecting the data is dealt with a matrix elimination result to get rid of the resulting noise-induced uncertainty in the representation.

If the samples of process noise are i.i.d. and Gaussian, then [9] provides a quantification in probability of the confidence region, which [10] exploits to give data-dependent conditions for minimizing the worst case cost of the LQ problem over all the system's matrices in the confidence region. The technical tool for this study is an extension of the S-lemma provided in [11]. A new matrix S-lemma is introduced in [12] to provide non-conservative conditions for designing controllers from data affected by disturbances satisfying quadratic bounds. Other results to deal with disturbances use a full-block S-procedure and linear fractional representations [13], the classical S-procedure [14] and Petersen's lemma [15].

The majority of the available results consider linear systems. Unsurprisingly, deriving solutions for *nonlinear* systems is harder. Earlier representative results of data-driven control of nonlinear systems include the nonlinear extension of VRFT [16], the design of controllers in the form of kernel functions tuned using data via set-membership identification techniques [17], and the so-called model-free control [18], [19].

A way to deal with nonlinear systems is to exploit some structure, when it is a priori known the class to which the system belongs. Data-driven control of second-order Volterra systems is studied in [20] and data-dependent LMI-based stabilization of bilinear systems in [21], the latter being motivated by Carleman bilinearization of general nonlinear systems. A point-to-point optimal control problem for bilinear systems is formulated in the recent work [22]. The data-driven control design for polynomial systems is the subject of [23], [24]. While [23] uses Rantzer's dual Lyapunov's theory and moments based techniques, [24] uses Lyapunov second method and a particular parametrization of the Lyapunov function to obtain SOS programs whose feasibility directly provide stabilizing controllers. See [15] for additional results on the data-driven control design of polynomial systems based on Petersen's lemma. When the system is not polynomial, the approach in [24] returns a state-dependent matrix condition rather than an SOS condition. If such a state-dependent matrix condition can be solved at each time step along a trajectory of the system, then a control sequence that steers that trajectory to the origin is obtained. This idea is pursued in [25].

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*Contribution.* We introduce a method to deal with the data-driven control design of nonlinear systems building up on and strengthening the results of [8] in several directions.

We first consider nonlinear vector fields that are expressed as combinations of known basis functions (not necessarily polynomials). We derive conditions to design controllers that stabilize the closed-loop system via nonlinearity cancellations (Section III). This approach returns formulas for controller design which retain the same simplicity and compactness of the formulas established in [8], namely semidefinite programs (SDP) only depending on data. Conceptually, however, the approach taken here is different from the one in [8]. In fact, [8] considers a first-order Taylor's expansion and treats the nonlinearity as a remainder, thus searching for linear control laws. Here, the idea is to learn from data what basis functions (in the considered library) form the dynamics, and to cancel them out through the control input. Consequently, the control law here is inherently nonlinear. Next, we make the crucial observation that, were exact cancellation unfeasible, we can formulate an SDP that minimizes the norm of the matrix by which the nonlinearities enter the dynamics (Section III-B). This idea is suggested by a regularization procedure in which the hard constraint of the first approach, corresponding to an exact nonlinearity cancellation, is lifted to the cost function, which leads to an *approximate* nonlinearity cancellation. (In different contexts, this idea has been pursued in [26]–[28]). In general, the design based on an approximate nonlinearity cancellation does not return globally stabilizing controllers, whence the need to explicitly characterize the region of attraction of the closed-loop system (Section IV).

The idea of cancelling out the nonlinearities (approximately or exactly) has points in common with the popular *feedback linearization*, see [29] for basic concepts and results, and [30] for specific results dealing with the “approximate” case. In the latter case, the idea is to find a control law for which the closed-loop dynamics is nearly linear in *some* coordinates. In contrast with [30], our approach works in the *original* coordinates. This allows us to explicitly determine region of attractions and invariant sets (and to try to *maximize* such sets by minimising the remainder, which is what we attempt to do through nonlinearity cancellation). Connections with feedback linearization are further discussed in Section VII.

To present the main ideas, we choose to give the results first for data that are not perturbed. The results are then extended to the case in which data are perturbed by process disturbances (Section VI). We show how our approach can accommodate the presence of process disturbances not only during the data collection phase, but also during the execution of the control task and provide estimates of *robustly positively invariant sets* [31] for the closed-loop system. The results are also extended to systems with neglected nonlinearities (Section VI-B), thus significantly enlarging the class of systems the approach can cope with.

*Outline.* The framework is set in Section II. The main results are discussed in Sections III and IV, with some extensions in Section V. Control design in the presence of disturbances and neglected nonlinearities is studied in Section VI. Some additional discussion is finally provided in Section VII.

*Notation.* Throughout the paper,  $\succ$  ( $\succeq$ ) and  $\prec$  ( $\preceq$ ) denote positive and negative (semi)-definiteness, respectively;  $\mathbb{S}^{n \times n}$  denotes the set of  $n \times n$  real-valued symmetric matrices;  $M^\top$  is the transpose of  $M$ . We let  $\|x\|$  denote the 2-norm of a vector  $x$ , and let  $\|A\|$  be the induced 2-norm of a matrix  $A$ .

## II. FRAMEWORK

We consider a discrete-time system in the form

$$x^+ = A_* Z_*(x) + Bu \quad (1)$$

( $x^+$  denotes forward shifting, i.e.,  $x^+(k) = x(k+1)$ ,  $k \in \mathbb{N}$ ) where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  is the control input. Both  $x$  and  $u$  are assumed to be measured.<sup>1</sup>  $Z_* : \mathbb{R}^n \rightarrow \mathbb{R}^R$  is a vector-valued continuous function,  $A_* \in \mathbb{R}^{n \times R}$ ,  $B \in \mathbb{R}^{n \times m}$  are constant matrices. Any nonlinear system  $x^+ = f(x) + Bu$  with  $f$  continuous (but otherwise arbitrary) can be written as in (1). In this paper,  $A_*$ ,  $B$  are regarded unknown while the following standing assumption is made for  $Z_*$ .

*Assumption 1:* We know a continuous function  $Z : \mathbb{R}^n \rightarrow \mathbb{R}^S$  such that  $Z_*(x) = TZ(x)$  for some matrix  $T \in \mathbb{R}^{R \times S}$ .  $\square$

Under Assumption 1, system (1) reads equivalently as

$$x^+ = AZ(x) + Bu \quad (2)$$

with  $A \in \mathbb{R}^{n \times S}$ , and  $A, B$  unknown.

Assumption 1 means that we choose a library of functions capable of describing the dynamics of the system (the case of neglected nonlinearities will be discussed in Section VI). This assumption is satisfied in many practical cases such as with mechanical and electrical systems where information about the dynamics can be derived from first principles, but the exact systems parameters may be unknown. We allow  $Z$  to contain terms not present in  $Z_*$ , which may arise from an imprecise knowledge of the system dynamics. In this paper, we will directly consider the case where  $Z$  contains *both* linear and nonlinear functions, i.e.,

$$Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix}, \quad (3)$$

with  $Q : \mathbb{R}^n \rightarrow \mathbb{R}^{S-n}$  containing only nonlinear functions. The special case where  $Z(x) = x$  reduces the analysis to that of linear systems, which have been the subject of numerous investigations, as reviewed in the Introduction. In contrast,  $Z(x) = Q(x)$  accounts for purely nonlinear systems, and just leads to simplified algorithms and results. We will exemplify this point in connection with Theorem 1. Let

$$\mathbb{D} := \{x(k), u(k)\}_{k=0}^T \quad (4)$$

be a dataset collected from the system with an experiment, meaning that we have a set of state and input samples that satisfy  $x(k+1) = AZ(x(k)) + Bu(k)$  for  $k = 0, \dots, T-1$ ,  $T > 0$ . The problem of interest is to determine, using  $\mathbb{D}$ , a control law  $u = KZ(x)$  that stabilizes the system around

<sup>1</sup>We will not address the case of input/output data, i.e., the case in which we only measure a function  $y = h(x)$  of the state. This case can be approached as in [8, Section VI] by considering a state-space representation with extended state vector  $\chi(k) := [y(k-1)^\top \dots y(k-n)^\top u(k-1)^\top \dots u(k-n)^\top]^\top$ . The analysis is similar to the linear case but there are several technical aspects that must be addressed, we will report this analysis elsewhere.

the origin (globally or locally, both cases will be considered). Note that we might consider a control law  $u = KH(x)$  with  $H$  different from  $Z$ . As it will become clear soon, we focus on  $u = KZ(x)$  as our approach is based on nonlinearity cancellation/minimization.

The framework can be modified and/or extended in several directions: (i) Continuous-time systems can be handled with similar arguments (Section V-A); (ii) The analysis extends to a more general class of nonlinear systems (Section V-B); (iii) Noisy data and neglected nonlinearities are considered in Section VI.

### III. EXACT NONLINEARITY CANCELLATION

We start by considering the scenario in which there exists a controller  $K$  that linearizes the closed-loop dynamics, namely the scenario in which there exists a controller  $K$  such that

$$u = KZ(x) \implies x^+ = Mx \quad (5)$$

for some matrix  $M$  (which we will also require to be Schur.<sup>2</sup>)

#### A. Data-based closed-loop representation and control design for exact nonlinearity cancellation

Consider the dataset  $\mathbb{D}$  in (4), and define

$$U_0 := [u(0) \ u(1) \ \cdots \ u(T-1)] \in \mathbb{R}^{m \times T}, \quad (6a)$$

$$X_0 := [x(0) \ x(1) \ \cdots \ x(T-1)] \in \mathbb{R}^{n \times T}, \quad (6b)$$

$$X_1 := [x(1) \ x(2) \ \cdots \ x(T)] \in \mathbb{R}^{n \times T}, \quad (6c)$$

$$Z_0 := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \\ Q(x(0)) & Q(x(1)) & \cdots & Q(x(T-1)) \end{bmatrix} \in \mathbb{R}^{S \times T}. \quad (6d)$$

All the results of this paper rest on the following lemma. An analogous result was established in [32, Lemma 1] for the case of polynomial systems.

*Lemma 1:* Consider any matrices  $K \in \mathbb{R}^{m \times S}$ ,  $G \in \mathbb{R}^{T \times S}$  such that

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} G. \quad (7)$$

Let  $G$  be partitioned as  $G = [G_1 \ G_2]$ , where  $G_1 \in \mathbb{R}^{T \times n}$  and  $G_2 \in \mathbb{R}^{T \times (S-n)}$ . Then, system (1) under the control law  $u = KZ(x)$  results in the closed-loop dynamics

$$x^+ = Mx + NQ(x) \quad (8)$$

where  $M := X_1 G_1$  and  $N := X_1 G_2$ .  $\square$

*Proof.* The closed-loop dynamics resulting from the control law  $u = KZ(x)$  is given by

$$x^+ = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_S \end{bmatrix} Z(x) \quad (9a)$$

$$= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} GZ(x) = X_1 GZ(x). \quad (9b)$$

The second identity follows from (7) while the last one follows because the elements of  $X_1, Z_0$  and  $U_0$  satisfy the relation

<sup>2</sup>A matrix  $M$  is said to be Schur if all its eigenvalues have modulus less than one. For continuous-time systems, a matrix  $M$  is said to be Hurwitz if all its eigenvalues have negative real part.

$x(k+1) = AZ(x(k)) + Bu(k)$ ,  $k = 0, \dots, T-1$ , which, in compact form, gives  $X_1 = AZ_0 + BU_0$ .  $\blacksquare$

Arrived at this stage, it is simple to derive a convex program (specifically a semi-definite program (SDP)) that searches for a controller  $K$  that cancels out the nonlinearities and renders the closed-loop system globally asymptotically stable.

*Theorem 1:* Consider a nonlinear system as in (1), along with the following SDP in the decision variables  $P_1 \in \mathbb{S}^{n \times n}$ ,  $Y_1 \in \mathbb{R}^{T \times n}$ , and  $G_2 \in \mathbb{R}^{T \times (S-n)}$ :

$$Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{(S-n) \times n} \end{bmatrix}, \quad (10a)$$

$$\begin{bmatrix} P_1 & (X_1 Y_1)^\top \\ X_1 Y_1 & P_1 \end{bmatrix} \succ 0, \quad (10b)$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n \times (S-n)} \\ I_{S-n} \end{bmatrix}, \quad (10c)$$

$$X_1 G_2 = 0_{n \times (S-n)}. \quad (10d)$$

If the SDP is feasible then the control law  $u = KZ(x)$  with

$$K = U_0 [Y_1 P_1^{-1} \ G_2] \quad (11)$$

linearizes the closed-loop dynamics, and renders the origin a globally asymptotically stable equilibrium.  $\square$

*Proof.* Suppose that (10) is feasible. Let  $G_1 = Y_1 P_1^{-1}$  and note that the two constraints (10a) and (10c) together yield

$$Z_0 [G_1 \ G_2] = I_S. \quad (12)$$

This relation, combined with (11), gives

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} [G_1 \ G_2], \quad (13)$$

which is (7). By Lemma 1, we conclude that the closed-loop dynamics satisfies  $x^+ = Mx + NQ(x)$  with  $M = X_1 G_1$  and  $N = X_1 G_2$ . By (10d),  $N = 0$ . Hence,  $K$  linearizes the closed-loop dynamics. Finally, note that (10b) is equivalent to  $P_1 \succ 0$  and  $(X_1 Y_1)^\top P_1^{-1} (X_1 Y_1) - P_1 \prec 0$ . The latter, in turn, is equivalent to  $(X_1 Y_1 P_1^{-1})^\top P_1^{-1} (X_1 Y_1 P_1^{-1}) - P_1^{-1} \prec 0$ . By recalling that  $Y_1 P_1^{-1} = G_1$  and  $X_1 G_1 = M$ , we conclude that  $M$  is Schur. (This also shows that  $V(x) = x^\top P_1^{-1} x$  is a Lyapunov function for the closed-loop system.)  $\blacksquare$

In Theorem 1, the decision variable  $G_2$  represents the same quantity that appears in Lemma 1. The decision variables  $Y_1, P_1$  are instead related to  $G_1$  in Lemma 1 via  $Y_1 = G_1 P_1$  with  $P_1$  a positive definite matrix, that is  $Y_1$  defines a change of variable relative to  $G_1$ . As it emerges from the proof of Theorem 1, this change of variable is instrumental to arrive at a convex formulation of the design program.

Some remarks are in order.

Theorem 1 gives an extension to nonlinear systems of the results in [8]. In fact, in the limit case where  $Z(x) = x$  we have  $S = n$  and (10) reduces to the first two constraints (10a)-(10b), which appeared in [8, Theorem 3]. The conditions (10c)-(10d) implement the linearization constraint, and (10a)-(10b) ensure a stable behavior for the linear dynamics. Note in particular that (10c), together with (10a), forms a consistency relation which makes it possible to parametrize the closed-loop dynamics through data alone. The other extreme case occurs when  $Z$  contains only nonlinear functions, i.e., when

$Z(x) = Q(x)$ . In this case, (10) reduces to the two constraints (10c)-(10d). This corresponds to a situation where the system has stable open-loop linear dynamics and the controller is only responsible for canceling out all the nonlinearities.

As a second remark, we observe that a necessary condition for the SDP (10) to be feasible is that  $Z_0$  has full row rank (this is indeed necessary to have both (10a) and (10c) fulfilled). This requirement can be viewed as a condition on the *richness* of the data, and is the natural generalization of the condition on the rank of  $X_0$  that appears in the linear case [8, Theorem 3], [33, Theorem 16]. This condition is weaker than having  $[U_0; Z_0]$  full row rank, which is instead necessary to identify  $A, B$  from data, and this shows that learning a control law is in general easier than identifying the dynamics of the system. Note that Lemma 1 indeed gives a data-based *closed-loop* representation of the system dynamics, without any explicit estimate of the system matrices. Having  $[U_0; Z_0]$  full row rank brings certain advantages, though. In fact, in this case, *any* controller that linearizes the closed-loop dynamics can be parametrized through the data. In particular, in this situation we obtain an “*if and only if*” result, meaning that (10) is feasible and returns a stabilizing and linearizing controller whenever such a controller exists. We state the result here but discuss it in Appendix A to maintain continuity.

**Theorem 2:** Suppose there exists a stabilizing and linearizing feedback controller, *i.e.*, a controller  $K = [\bar{K} \quad \hat{K}]$  such that

$$A + BK = [\bar{A} + B\bar{K} \quad 0_{n \times (S-n)}] \quad (14a)$$

$$\bar{A} + B\bar{K} \text{ is Schur} \quad (14b)$$

having partitioned  $A = [\bar{A} \quad \hat{A}]$  with  $\bar{A} \in \mathbb{R}^{n \times n}$ . Let  $[U_0; Z_0]$  have full row rank. Then (10) is feasible and  $K$  can be written as in (11) for some  $Y_1, P_1, G_2$  satisfying (10).  $\square$

**Example 1.** Consider the Euler discretization of an inverted pendulum

$$x_1^+ = x_1 + T_s x_2 \quad (15a)$$

$$x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m \ell^2}\right) x_2 + \frac{T_s}{m \ell^2} u, \quad (15b)$$

where  $T_s$  is the sampling time,  $m$  is the mass to be balanced,  $\ell$  is the distance from the base to the center of mass of the balanced body,  $\mu$  is the coefficient of rotational friction, and  $g$  is the acceleration due to gravity. The states  $x_1, x_2$  are the angular position and velocity, respectively,  $u$  is the applied torque. The system has an unstable equilibrium in  $(x, u) = (0, 0)$ , corresponding to the pendulum upright position, which we want to stabilize. Suppose that the parameters are  $T_s = 0.1$ ,  $m = 1$ ,  $\ell = 1$ ,  $g = 9.8$  and  $\mu = 0.01$ .

We choose  $Z(x) = [x_1 \quad x_2 \quad \sin(x_1)]^\top$ , and regard all the parameters  $T_s, m, \ell, g, \mu$  as unknown (here, a correct choice for  $Z(x)$  simply derives from physical considerations, namely Lagrange's equations of motion). We collect data by running an experiment with input uniformly distributed in  $[-0.5, 0.5]$ , and with an initial state within the same interval. We collect  $T = 10$  samples (corresponding to the motion of the pendulum that oscillates around the upright position). The SDP (10) is feasible and we obtain  $K = [-23.5641 \quad -10.3901 \quad -9.8]$ .

The resulting control law indeed cancels out the nonlinearity ensuring global asymptotic stability.  $\blacksquare$

**Example 2.** Consider the polynomial system

$$x_1^+ = x_2 + x_1^3 + u \quad (16a)$$

$$x_2^+ = 0.5x_1. \quad (16b)$$

Suppose that we choose

$$Z(x) = [x^\top \quad x_1^2 \quad x_2^2 \quad x_1 x_2 \quad x_1^3 \quad x_2^3 \quad x_1 x_2^2 \quad x_1^2 x_2]^\top, \quad (17)$$

*i.e.*, we capture the nonlinearity by including all the possible monomials up to degree 3. The equilibrium of the unforced system ( $u = 0$ ) is only locally asymptotically stable (*e.g.*, any initial condition such that  $x_1(0) > 1$  and  $x_2(0) \geq 0$  leads to a divergent solution). We collect data by running an experiment with input uniformly distributed in  $[-0.5, 0.5]$ , and with an initial state within the same interval. We collect  $T = 10$  samples. The SDP is feasible and returns the controller

$$K = \begin{bmatrix} \underbrace{0}_{x_1} & \underbrace{-1.0007}_{x_2} & \underbrace{0}_{x_1^2} & \underbrace{0}_{x_2^2} & \underbrace{0}_{x_1 x_2} & \underbrace{-1}_{x_1^3} & \underbrace{0}_{x_2^3} & \underbrace{0}_{x_1 x_2^2} & \underbrace{0}_{x_1^2 x_2} \end{bmatrix} \quad (18)$$

The SDP correctly assigns the value  $-1$  to the sixth entry of  $K$ , and automatically discovers that no other nonlinearities are present. The resulting control law is  $u = -1.0007x_2 - x_1^3$  and ensures global asymptotic stability.  $\blacksquare$

The examples show that even a few samples may suffice to learn a stabilizing control policy. In fact, in terms of number of data points, the only necessary condition in (10) comes from having  $Z_0$  full row rank, and this condition can be met even with  $T = S$  samples. The situation may be different with noisy data as we discuss in Section VI. As a second remark, note that this approach differs from the approach in [8], which considers *linear* control laws. This new approach considers *nonlinear* control laws; this is indeed essential to achieve nonlinearity cancellation (or nonlinearity minimization, if cancellation is impossible, as we discuss in Section IV).

## B. Nonlinearity cancellation as a minimization problem

A variant of (10) consists in approaching the design problem as a *minimization* problem, namely as the problem of finding a controller that minimizes the nonlinearity in closed loop with respect to a chosen norm. We state the result using the induced 2-norm  $\|\cdot\|$  but we can consider other norms.

**Theorem 3:** Consider a nonlinear system as in (1) along with the following SDP in the decision variables  $P_1 \in \mathbb{S}^{n \times n}$ ,  $Y_1 \in \mathbb{R}^{T \times n}$ , and  $G_2 \in \mathbb{R}^{T \times (S-n)}$ :

$$\text{minimize}_{P_1, Y_1, G_2} \|X_1 G_2\| \quad (19a)$$

$$\text{subject to} \quad Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{(S-n) \times n} \end{bmatrix}, \quad (19b)$$

$$\begin{bmatrix} P_1 & (X_1 Y_1)^\top \\ X_1 Y_1 & P_1 \end{bmatrix} \succ 0, \quad (19c)$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n \times (S-n)} \\ I_{S-n} \end{bmatrix}. \quad (19d)$$

If this SDP is feasible and the solution achieves zero cost (*i.e.*,  $\|X_1 G_2\| = 0$ ) then the control law  $u = KZ(x)$  with  $K$  given

by (11) linearizes the closed-loop dynamics, and renders the origin a globally asymptotically stable equilibrium.  $\square$

*Proof.* The proof is analogous to that of Theorem 1.  $\blacksquare$

**Example 3.** We consider again system (16) under the same setting as before. The SDP (19) is feasible and we obtain

$$K = \begin{bmatrix} 0.0001 & -1.0007 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \quad (20)$$

As before, the program correctly assigns the value  $-1$  to the sixth entry of  $K$ . Note that when nonlinearity cancellation is possible, (10) and (19) are *equivalent* in the sense that their feasible sets coincide. This controller differs from the one in (18) since there are infinitely many stabilizing and linearizing controllers and neither (10) nor (19) involve constraints other than stability and linearization.  $\blacksquare$

#### IV. APPROXIMATE NONLINEARITY CANCELLATION

##### A. Control design for approximate nonlinearity cancellation

There is a simple yet important difference between (10) and its lifted version (19). The difference is that the latter is always feasible when the former is feasible and this implies that we can always use (19) in place of (10) when exact nonlinearity cancellation is possible. However, (19) can be used even when exact cancellation is impossible. The next result addresses this scenario. It shows in particular that, in this case, we can still have stability guarantees using (19).

**Theorem 4:** Consider a nonlinear system as in (1), along with the SDP (19). Assume that

$$\lim_{|x| \rightarrow 0} \frac{|Q(x)|}{|x|} = 0. \quad (21)$$

If the SDP is feasible then  $u = KZ(x)$ , with  $K$  as in (11), renders the origin an asymptotically stable equilibrium.  $\square$

*Proof.* Suppose that (19) is feasible. Let  $G_1 = Y_1 P_1^{-1}$ , and note that the two constraints (19b) and (19d) together yield  $Z_0 [G_1 \ G_2] = I_S$ . This identity, along with (11), gives (7). By Lemma 1, we have that the closed-loop dynamics satisfies  $x^+ = Mx + NQ(x)$ , where  $M = X_1 G_1$  and  $N = X_1 G_2$ . Although  $N$  can be different from zero, (19c) ensures that  $M$  is Schur. Asymptotic stability thus follows from (21).  $\blacksquare$

Condition (21) ensures that the linear dynamics dominates the nonlinear dynamics around the origin. In turn, as shown in the next subsection, this ensures that we can obtain an estimate of the *region of attraction*. This condition is satisfied for many systems of practical relevance, for instance is satisfied by any polynomial system. Condition (21) can be replaced by asking that  $Q$  is differentiable at  $x = 0$  and satisfies  $Q(0) = 0$ . In fact, in this case  $Q$  admits a Taylor's expansion at  $x = 0$ , namely we have

$$Q(x) = \underbrace{\left[ \frac{\partial Q}{\partial x} \right]_{x=0}}_{=:F} x + r(x) \quad (22a)$$

with  $r : \mathbb{R}^n \rightarrow \mathbb{R}^{S-n}$  a differentiable function of the state such that  $\lim_{|x| \rightarrow 0} \frac{|r(x)|}{|x|} = 0$ . Thus, system (1) can be equivalently represented as

$$x^+ = \bar{A}x + \hat{A}Q(x) + Bu \quad (23a)$$

$$= (\bar{A} + \hat{A}F)x + \hat{A}r(x) + Bu \quad (23b)$$

where we have partitioned  $A$  as  $A = [\bar{A} \ \hat{A}]$  with  $\bar{A} \in \mathbb{R}^{n \times n}$ . Hence, Theorem 4 becomes applicable with  $Q$  replaced by  $r$ , where  $r$  can be determined from  $Q$ . As an example, for the inverted pendulum where  $Q(x) = \sin(x_1)$  this reasoning leads to  $r(x) = \sin(x_1) - x_1$ , which gives  $\lim_{|x| \rightarrow 0} \frac{|r(x)|}{|x|} = 0$  (for the inverted pendulum Theorem 4 reduces in any case to Theorem 3 since exact cancellation is possible).

We point out that there exists a counterpart of Theorem 2, which provides conditions under which we can parametrize *all* feedback controllers that ensure local stability through a stable linear dynamics. We state the result here but prove it in Appendix B to maintain continuity.

**Theorem 5:** Suppose that there exists a feedback controller,  $K = [\bar{K} \ \hat{K}]$  such that  $\bar{A} + B\bar{K}$  is Schur, having partitioned  $A = [\bar{A} \ \hat{A}]$  with  $\bar{A} \in \mathbb{R}^{n \times n}$ . Let  $\begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}$  have full row rank. Then (19) is feasible and  $K$  can be written as in (11) for some  $P_1, Y_1, G_2$  satisfying (19).  $\square$

##### B. Estimating the region of attraction

**Definition 1:** A set  $\mathcal{S}$  is called *positively invariant* (PI) for the system  $x^+ = f(x)$  if for every  $x(0) \in \mathcal{S}$  the solution is such that  $x(t) \in \mathcal{S}$  for  $t > 0$ . Let  $\bar{x}$  be an asymptotically stable equilibrium point for the system  $x^+ = f(x)$ . A set  $\mathcal{R}$  defines a *region of attraction* (ROA) for the system relative to  $\bar{x}$  if for every  $x(0) \in \mathcal{R}$  we have  $\lim_{t \rightarrow \infty} x(t) = \bar{x}$ .  $\square$

Building on Theorem 4, we can give estimates of the ROA for the closed-loop system relative to the equilibrium  $\bar{x} = 0$ . Consider the same conditions as in Theorem 4 and note that  $V(x) := x^\top P_1^{-1}x$  is a Lyapunov function for the linear part of the dynamics. In particular,

$$V(x^+) - V(x) = \underbrace{(Mx + NQ(x))^\top P_1^{-1}(Mx + NQ(x)) - x^\top P_1^{-1}x}_{=:h(x)} \quad (24)$$

where the matrices  $M, N$  and  $P_1$  are all computable from data. We immediately obtain the following result.

**Proposition 1:** Consider the same setting as in Theorem 4. Let  $\mathcal{V} := \{x : h(x) < 0\}$  with  $h(x)$  as in (24), and consider the Lyapunov function  $V(x) = x^\top P_1^{-1}x$ . Then, any sub-level set  $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$  of  $V$  contained in  $\mathcal{V} \cup \{0\}$  is a PI set for the closed-loop system and defines an estimate of the ROA relative to  $\bar{x} = 0$ .  $\square$

We exemplify Theorem 4 and Proposition 1.

**Example 4.** Consider the nonlinear system

$$x_1^+ = x_2 + x_1^3 + u \quad (25a)$$

$$x_2^+ = 0.5x_1 + 0.2x_2^2 \quad (25b)$$

under the same experimental setting as before, in particular let  $Z(x)$  be as in (17). Exact nonlinearity cancellation is now impossible. Nonetheless, the SDP (19) is feasible and returns the controller  $K$  in (28) at the top of the next page. For this controller, we numerically determine  $\mathcal{V} = \{x : h(x) < 0\}$  over which the Lyapunov function  $V(x) = x^\top P_1^{-1}x$  decreases and the largest sub-level set  $\mathcal{R}_\gamma$  of  $V$  contained in  $\mathcal{V} \cup \{0\}$  which gives an estimate of the ROA. These two sets are displayed in Figure 1 (Left). We note that the SDP (19) almost assigns

$$K = \begin{bmatrix} \underbrace{-0.0113}_{x_1} & \underbrace{-1.0862}_{x_2} & \underbrace{0.0005}_{x_1^2} & \underbrace{0}_{x_2^2} & \underbrace{0.0039}_{x_1 x_2} & \underbrace{-1.0010}_{x_1^3} & \underbrace{-0.0130}_{x_2^3} & \underbrace{0.0119}_{x_1 x_2^2} & \underbrace{-0.0010}_{x_1^2 x_2} \end{bmatrix} \quad (28)$$

$$M = \begin{bmatrix} -0.0113 & -0.0862 \\ 0.5000 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0.0005 & 0 & 0.0039 & -0.0010 & -0.0130 & 0.0119 & -0.0010 \\ 0 & 0.2000 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (29)$$

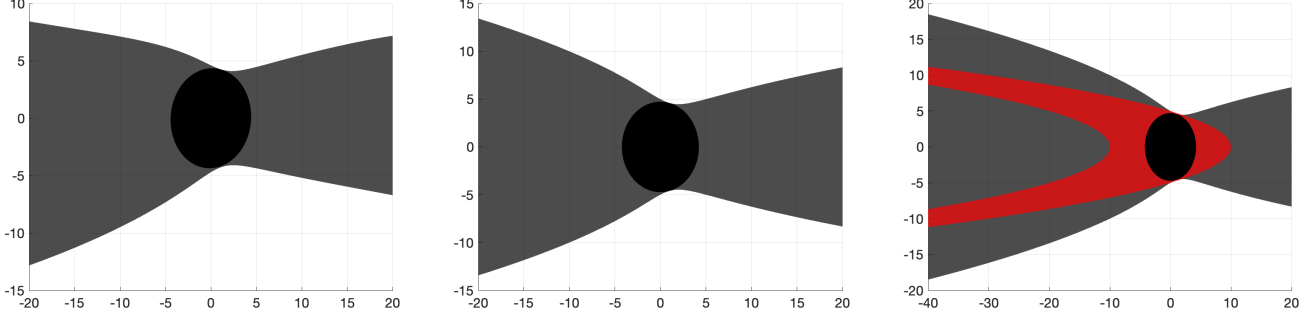


Fig. 1. Results for Example 4. Left: Sets  $\mathcal{V}$  and  $\mathcal{R}_\gamma$  in grey and black color, respectively, for the controller  $K$  in (28) (we recall that  $\mathcal{R}_\gamma$  is a valid estimate for the ROA); Middle: Sets  $\mathcal{V}$  and  $\mathcal{R}_\gamma$  in grey and black color, respectively, for the controller  $K$  in (30); Right: Sets  $\mathcal{V}$ ,  $\mathcal{R}_\gamma$  and  $\mathcal{R}$  (exact ROA) for the controller  $K$  in (30). The set  $\mathcal{R}$  is displayed in red color.

the value  $-1$  to the sixth entry of  $K$ , thus reducing the effect of the nonlinearity on the first state component. Specifically, we obtain  $M$  and  $N$  reported in (29), and  $N$  has minimum norm  $\|N\| = 0.2$  (this value cannot be further reduced since the term  $0.2x_2^2$  cannot be canceled out).

The approach that we just described for estimating the ROA is fully automatic and is generically applicable. Note, however, that once we compute a controller  $K$  then we can pursue *any* approach (data- or model-based) to estimate the ROA. In fact, the SPD (19) returns the *exact* description of the closed-loop dynamics:  $x^+ = [M \ N] Z(x)$  (we stress that this expression does not correspond to identifying open-loop dynamics of the system). From this description, we can then indeed apply any technique to find Lyapunov functions and estimate the ROA, see for instance [34, Section 8.2]. To illustrate this point, suppose that (19) returns

$$K = \begin{bmatrix} \underbrace{0}_{x_1} & \underbrace{-1}_{x_2} & \underbrace{0}_{x_1^2} & \underbrace{0}_{x_2^2} & \underbrace{0}_{x_1 x_2} & \underbrace{-1}_{x_1^3} & \underbrace{0}_{x_2^3} & \underbrace{0}_{x_1 x_2^2} & \underbrace{0}_{x_1^2 x_2} \end{bmatrix} \quad (30)$$

(this is indeed what we obtain with a variant of (19), see next (32)), from which we have  $M = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}$  and  $N = \begin{bmatrix} 0 & 0 & 0_{1 \times 5} \\ 0 & 0.2 & 0_{1 \times 5} \end{bmatrix}$ , or, equivalently,

$$x_1^+ = 0 \quad (31a)$$

$$x_2^+ = 0.5x_1 + 0.2x_2^2. \quad (31b)$$

From the closed-loop dynamics we conclude that the exact ROA is given by the set  $\mathcal{R} := \{x : |0.5x_1 + 0.2x_2^2| < 5\}$ . In fact, the solution to system (31) is given by  $x_1(t) = 0$  for  $t \geq 1$  and  $x_2(t) = b^{-1}(b(ax_1(0) + bx_2(0)^2))^{2^{t-1}}$  for  $t \geq 2$ , with  $a = 0.5$  and  $b = 0.2$ . Hence, the solution converges asymptotically if and only if  $|b(ax_1(0) + bx_2(0)^2)| < 1$ , from which one infers the ROA  $\mathcal{R}$  specified above. This is a situation where it is simple to exactly compute by inspection the ROA, which gives a better result with respect to the automatic procedure, cf.

Figure 1 (Middle, Right). The automatic procedure, however, is applicable even when an exact description of the closed-loop dynamics is not available, as it is the case when noisy data are being measured, a case examined in Section VI. ■

We conclude this section with a few remarks.

As a first comment, note that the SDP (19) can also be used to infer the stability properties of any controller  $K$  for which a solution to (7) exists. This can be done by regarding (11) as an additional constraint to (19), *i.e.*, by adding the constraint

$$U_0 [Y_1 \ G_2] = K \begin{bmatrix} P_1 & 0_{n \times (S-n)} \\ 0_{(S-n) \times n} & I_{S-n} \end{bmatrix}$$

which is convex. This can be useful whenever a controller is inferred based on physical intuition and we want to determine closed-loop stability properties *before* inserting the controller into the loop. For the same reason, by adding the constraint  $U_0 [Y_1 \ G_2] = 0$  we infer the ROA for the open-loop system.

As a final observation, we mention a particularly effective variant of (19):

$$\text{minimize}_{P_1, Y_1, G_2, X, V} \quad \text{trace}(X) + \text{trace}(V) \quad (32a)$$

$$\text{subject to} \quad (19b) - (19d) \quad (32b)$$

$$\begin{bmatrix} X & X_1 G_2 \\ (X_1 G_2)^\top & V \end{bmatrix} \succeq 0. \quad (32c)$$

This SDP uses the trace as a convex envelope of the rank [35], hence it searches for solutions yielding a *sparse* nonlinear term  $N = X_1 G_2$ , which can be useful to analyse properties of the closed-loop system, including the ROA.

Applied to Example 4 the SDP (32) indeed systematically returns a controller with third-to-ninth entries as in (30). If we further regularize (32) by enforcing a sparsity term for  $X_1 Y_1$ , the SDP exactly returns (30) (systematically for different datasets). In a sense, the cost function in (32) is analogous to regularization terms used in regression algorithms to penalize complex models [36]. The difference is that (32) promotes

low-complexity (sparse) *closed-loop* systems (the term  $X_1 G_2$ ), and this favours low-complexity (sparse) control laws.

## V. EXTENSIONS

### A. Continuous-time systems

Continuous-time systems can be treated in a similar way to the discrete-time case, we will report the main differences. Suppose that we have a continuous-time system

$$\dot{x} = AZ(x) + Bu \quad (33)$$

and that we make an experiment on it. Sampling the observed trajectory with sampling time  $T_s > 0$  we collect data matrices  $U_0, X_0, Z_0, X_1$  with  $U_0, X_0$  and  $Z_0$  as in (6a), (6b) and (6d), respectively, and with  $X_1 := [\dot{x}(0) \ \dot{x}(T_s) \ \dots \ \dot{x}((T-1)T_s)]$ . It is readily seen that these data matrices satisfy the relation  $X_1 = AZ_0 + BU_0$ . As a consequence, the same analysis carried out in Section III and IV carries over to the present case. The only modification occurs in the Lyapunov stability condition which reads  $X_1 Y_1 + (X_1 Y_1)^\top \prec 0$  instead of (19c) (or (10b)). In fact, recalling that the matrix  $M$  that dictates the linear dynamics in closed loop is given by  $M = X_1 Y_1 P_1^{-1}$ , the above Lyapunov inequality gives  $P_1^{-1} M + M^\top P_1^{-1} \prec 0$ , and this implies that  $M$  is Hurwitz (with Lyapunov function  $V(x) = x^\top P_1^{-1} x$ ). Hence, (19) ((10) is analogous) becomes

$$\text{minimize}_{P_1, Y_1, G_2} \|X_1 G_2\| \quad (34a)$$

$$\text{subject to} \quad (19b), (19d) \quad (34b)$$

$$X_1 Y_1 + (X_1 Y_1)^\top \prec 0, \quad (34c)$$

and the (continuous-time) control law is given by  $u = KZ(x)$  with  $K$  as in (11). For estimating the ROA we can proceed as in Section IV-B, we omit the details.

Continuous-time controllers can be directly implemented via analog devices. Alternatively, they can be discretized and digitally implemented [37]. As for the latter, in the context of data-driven control, continuous-time controller redesign for digital implementation (usually known as *emulation* approach) has been recently studied in [38]–[40], albeit the analysis there is restricted to linear dynamics. Extending the ideas of [40] to nonlinear systems is currently under study.

### B. A more general class of nonlinear systems

We now turn our attention to the case of systems

$$x^+ = A_\star Z_\star(\xi) \quad (35)$$

where  $\xi := \begin{bmatrix} x \\ u \end{bmatrix}$ ,  $A_\star \in \mathbb{R}^{n \times R}$  is an unknown constant matrix and where  $Z_\star : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^R$  is a continuous function of the state and the input. System (35) is more general than (1) as it allows *both* the state  $x$  and the input  $u$  to enter the dynamics nonlinearly. We rephrase Assumption 1 as follows:

*Assumption 2:* We know a continuous function  $Z : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^S$  such that  $Z_\star(\xi) = \mathcal{T}Z(\xi)$  for some matrix  $\mathcal{T} \in \mathbb{R}^{R \times S}$ .  $\square$

Under this assumption, (35) can be equivalently written as  $x^+ = AZ(\xi)$  with  $A \in \mathbb{R}^{n \times S}$  an unknown matrix. As before, we allow  $Z(\xi)$  to contain both  $\xi$  and the nonlinear function  $\mathcal{Q} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{S-n-m}$ , namely we consider

$$Z(\xi) = \begin{bmatrix} \xi \\ \mathcal{Q}(\xi) \end{bmatrix}. \quad (36)$$

The presence of  $\mathcal{Q}(\xi)$  makes it difficult to adopt a similar design as in the previous sections, unless one regards the control input  $u$  as a state variable and extends the dynamics to include the controller dynamics. This “adding one integrator” tool, which has been widely used in control theory, reduces the design of the controller for (35) to the case with constant input vector fields previously studied, as we detail below.

Let us add the controller dynamics in the form  $u^+ = v$ , with  $v \in \mathbb{R}^m$  a new control input. This extension leads to

$$\xi^+ = \mathcal{A}Z(\xi) + \mathcal{B}v, \quad (37)$$

where

$$\mathcal{A} := \begin{bmatrix} \bar{A} & \hat{A} \\ 0_{m \times (n+m)} & 0_{m \times (S-n-m)} \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} 0_{n \times m} \\ I_m \end{bmatrix} \quad (38)$$

having partitioned  $A$  as  $A = [\bar{A} \ \hat{A}]$  with  $\bar{A} \in \mathbb{R}^{n \times (n+m)}$ . We therefore arrived at a representation which allows us to proceed as in the previous sections. We collect the dataset  $\{x(k), u(k), v(k)\}_{k=0}^T$  from the system and define the data matrices

$$V_0 := [v(0) \ v(1) \ \dots \ v(T-1)] \in \mathbb{R}^{m \times T},$$

$$\Xi_0 := [\xi(0) \ \xi(1) \ \dots \ \xi(T-1)] \in \mathbb{R}^{(n+m) \times T},$$

$$\Xi_1 := [\xi(1) \ \xi(2) \ \dots \ \xi(T)] \in \mathbb{R}^{(n+m) \times T},$$

$$Z_0 := \begin{bmatrix} \xi(0) & \xi(1) & \dots & \xi(T-1) \\ Q(\xi(0)) & Q(\xi(1)) & \dots & Q(\xi(T-1)) \end{bmatrix} \in \mathbb{R}^{S \times T},$$

which satisfy the identity  $\Xi_1 = \mathcal{A}Z_0 + \mathcal{B}V_0$ .

The next result parallels Theorem 4, we omit the proof.

*Corollary 1:* Consider a nonlinear system as in (35), and assume that

$$\lim_{|\xi| \rightarrow 0} \frac{|\mathcal{Q}(\xi)|}{|\xi|} = 0. \quad (39)$$

Consider the following SDP in the decision variables  $Y_1 \in \mathbb{R}^{T \times (n+m)}$ ,  $G_2 \in \mathbb{R}^{T \times (S-n-m)}$ ,  $P_1 \in \mathbb{S}^{(n+m) \times (n+m)}$ :

$$\text{minimize}_{P_1, Y_1, G_2} \|\Xi_1 G_2\| \quad (40a)$$

$$\text{subject to} \quad Z_0 Y_1 = \begin{bmatrix} P_1 \\ 0_{(S-n-m) \times (n+m)} \end{bmatrix} \quad (40b)$$

$$\begin{bmatrix} P_1 & (\Xi_1 Y_1)^\top \\ \Xi_1 Y_1 & P_1 \end{bmatrix} \succ 0 \quad (40c)$$

$$Z_0 G_2 = \begin{bmatrix} 0_{(n+m) \times (S-n-m)} \\ I_{S-n-m} \end{bmatrix}. \quad (40d)$$

If this SDP is feasible then the dynamical controller

$$u^+ = \begin{bmatrix} \bar{\kappa} & \hat{\kappa} \end{bmatrix} \begin{bmatrix} \xi \\ \mathcal{Q}(\xi) \end{bmatrix} \quad \text{with} \quad (41)$$

$$\begin{bmatrix} \bar{\kappa} & \hat{\kappa} \end{bmatrix} = V_0 \begin{bmatrix} Y_1 P_1^{-1} & G_2 \end{bmatrix}$$

renders the origin of the closed-loop system an asymptotically stable equilibrium.  $\blacksquare$

As before, we can replace (39) with  $\mathcal{Q}(\xi)$  differentiable at  $\xi = 0$  and  $\mathcal{Q}(0) = 0$ , so that  $\mathcal{Q}(\xi) = [\partial \mathcal{Q} / \partial \xi]_{\xi=0} \xi + r(\xi)$ , with  $r(\xi)$  differentiable and such that  $\lim_{|\xi| \rightarrow 0} |r(\xi)| / |\xi| = 0$ . In this way, we can choose  $Z(\xi) = \begin{bmatrix} \xi \\ r(\xi) \end{bmatrix}$  instead of (36). Further, we can use the Lyapunov function  $V(\xi) = \xi^\top P_1^{-1} \xi$

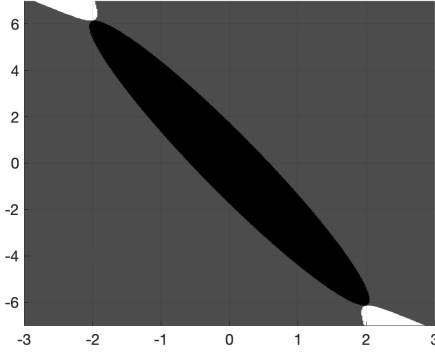


Fig. 2. Results for Example 5. The grey set represents the set  $\mathcal{V}$  where  $V(\xi^+) - V(\xi)$  is negative. Here,  $\mathcal{Z}(\xi) = [\xi^\top \sin \xi_1 - \xi_1 (\cos \xi_1 - 1) \xi_3]^\top$  and  $V(\xi) = \xi^\top P_1^{-1} \xi$ , with  $P_1^{-1} = \begin{bmatrix} 0.2159 & 0.0689 & 0.0123 \\ 0.0689 & 0.0240 & 0.0039 \\ 0.0123 & 0.0039 & 0.0009 \end{bmatrix}$ . The black set is a Lyapunov sublevel set  $\mathcal{R}_\gamma$  contained in  $\mathcal{V}$ , hence it provides an estimate of the ROA for the system, and  $\gamma = 0.076$ . Both sets  $\mathcal{V}$  and  $\mathcal{R}_\gamma$  are projected onto the plane  $\{\xi : \xi_3 = 0\}$ .

to estimate the ROA of the closed-loop system (35), (41), similarly to what has been done in Proposition 1.

**Example 5.** Consider the Euler discretization of an inverted pendulum

$$x_1^+ = x_1 + T_s x_2 \quad (42a)$$

$$x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m \ell^2}\right) x_2 + \frac{T_s}{m \ell} \cos x_1 u, \quad (42b)$$

where now the force is applied at the base, and this results in a state-dependent input vector field  $[0 \quad \frac{T_s}{m \ell} \cos x_1]^\top$ . The parameters  $T_s, m, \ell, \mu, g$  and the states  $x_1, x_2$  are the same as in Example 1. The problem is again that of stabilizing the unstable equilibrium in  $(x, u) = (0, 0)$ .

The vector  $\mathcal{Q}(\xi)$  suggested by physical considerations is  $[\sin \xi_1 \cos \xi_1 \xi_3]^\top$ , which is zero at  $\xi = 0$  and differentiable. Hence, the function  $r(\xi) = [\sin \xi_1 - \xi_1 (\cos \xi_1 - 1) \xi_3]^\top$  satisfies  $\lim_{|\xi| \rightarrow 0} \frac{|r(\xi)|}{|\xi|} = 0$ . Here,  $r(\xi)$  is a preferred choice over  $\mathcal{Q}(\xi)$  because it yields a controllable linear part, which is necessary for the feasibility of the SDP. We collect data by running an experiment with input uniformly distributed in  $[-0.5, 0.5]$ , and with an initial state within the same interval. We collect  $T = 10$  samples corresponding to the motion of the pendulum that oscillates around the upright position. The SDP (40) is feasible and we obtain  $\mathcal{K} = [-17.6197 \quad -5.6815 \quad -0.3012 \quad 0 \quad 0]$ . The controller locally asymptotically stabilizes the closed-loop system around the origin. For this controller, we numerically determine the set  $\mathcal{V} = \{\xi : V(\xi^+) - V(\xi) = H(\xi) < 0\}$ , with  $H(\xi) := (\Xi_1 G_1 \xi + \Xi_1 G_2 \mathcal{Q}(\xi))^\top P_1^{-1} (\Xi_1 G_1 \xi + \Xi_1 G_2 \mathcal{Q}(\xi)) - \xi^\top P_1^{-1} \xi$ , over which the Lyapunov function  $V(\xi) = \xi^\top P_1^{-1} \xi$  decreases. Any sub-level set  $\mathcal{R}_\gamma$  of  $V$  contained in  $\mathcal{V} \cup \{0\}$  gives an estimate of the ROA for the closed-loop system. The set  $\mathcal{V}$  and a sublevel set of  $V$  are displayed in Figure 2. The 0 values taken on by the last two entries of  $\mathcal{K}$  (which correspond to the subvector  $\tilde{\mathcal{K}}$  in (41)) is a byproduct of the minimization of  $\|\Xi_1 G_2\|$ , which in turn imposes a small value of  $\|V_0 G_2\|$ , in view of the addition of the integrator. ■

Corollary 1 is a direct extension of Theorem 4 and allows the designer to deal with a more general class of nonlinear

systems, including systems with state-dependent input vector fields. Nevertheless, if it is known that the input vector field is state-independent, it is preferable to use the design proposed by Theorem 4, which might guarantee a global stabilization result by a static feedback in case the solution attains a zero cost, as formalized in Theorem 3.

## VI. ROBUSTNESS TO DISTURBANCES AND NEGLECTED NONLINEARITIES

In this section, we discuss robustness to disturbances and/or neglected nonlinearities. Consider a system in the form

$$x^+ = AZ(x) + Bu + Ed \quad (43)$$

where  $d \in \mathbb{R}^s$  is an unknown signal that accounts for process disturbances and/or neglected nonlinearities (when  $Z$  does *not* include all the nonlinearities present in the system), whereas  $E \in \mathbb{R}^{n \times s}$  is a known matrix that specifies which channel the signal  $d$  enters. If such information is not available then we simply let  $E = I_n$ . Because of  $d$ , the previous tools must be modified to maintain stability guarantees. While the tools we use to study process disturbances and neglected nonlinearities are similar, we will tackle the two cases separately.

### A. Process disturbances: noisy data and robust invariance

We start with the case where  $d$  is a process disturbance. The presence of  $d$  affects the analysis in two different directions. First, it affects controller design since it corrupts the data.<sup>3</sup> Second, it leads to notions other than Lyapunov stability and ROA. We will address both the questions.

Similarly to the disturbance-free case, suppose we perform an experiment on the system, and we collect state and input samples satisfying  $x(k+1) = AZ(x(k)) + Bu(k) + Ed(k)$ ,  $k = 0, \dots, T-1$ . These samples are then grouped into the data matrices  $U_0, X_0, X_1, Z_0$  as in (14). Furthermore, let

$$D_0 := [d(0) \quad d(1) \quad \dots \quad d(T-1)] \quad (44)$$

be the (unknown) data matrix that collects the samples of  $d$ . Our first step is to establish an analogue of Lemma 1.

**Lemma 2:** Consider any matrices  $K \in \mathbb{R}^{m \times S}$ ,  $G \in \mathbb{R}^{T \times S}$  satisfying (7). Let  $G$  be partitioned as  $G = [G_1 \quad G_2]$ , where  $G_1 \in \mathbb{R}^{T \times n}$ . System (43) under the control law  $u = KZ(x)$  results in the closed-loop dynamics

$$x^+ = \Psi x + \Xi Q(x) + Ed \quad (45)$$

where  $\Psi := (X_1 - ED_0)G_1$  and  $\Xi := (X_1 - ED_0)G_2$ . □

*Proof.* Similarly to (9), we have

$$x^+ = [B \quad A] \begin{bmatrix} K \\ I_S \end{bmatrix} Z(x) + Ed \quad (46a)$$

$$= [B \quad A] \begin{bmatrix} U_0 \\ Z_0 \end{bmatrix} GZ(x) + Ed \quad (46b)$$

$$= (X_1 - ED_0)GZ(x) + Ed. \quad (46c)$$

<sup>3</sup>By following [8, Section V-A], the analysis can be further extended to the case of measurement noise, namely when we measure  $y = x + w$  instead of  $x$ ,  $w$  being a noise signal. We omit the details due to space limitations.



The last identity follows as  $X_1, U_0, Z_0, D_0$  satisfy the relation  $X_1 = AZ_0 + BU_0 + ED_0$ . ■

The closed-loop dynamics now depends on  $D_0$ , and (19) no longer gives stability guarantees. In fact, the constraint (19c) ensures that  $M = X_1G_1$  is Schur. However, by Lemma 2 the matrix of interest is now  $\Psi = (X_1 - ED_0)G_1$ , and stability of  $M$  does not ensure that also  $\Psi$  is stable. To have stability, we need to modify (19c) accounting for the uncertainty induced by  $D_0$ . A simple and effective way to achieve this is to ensure that  $(X_1 - ED)G_1$  is stable for all the matrices  $D$  in a given set  $\mathcal{D}$  to which  $D_0$  is deemed to belong. We will consider the set

$$\mathcal{D} := \{D \in \mathbb{R}^{s \times T} : DD^\top \preceq \Delta\Delta^\top\} \quad (47)$$

with  $\Delta$  a design parameter, and enforce, in place of (19c),

$$Y_1^\top (X_1 - ED)^\top P_1^{-1} (X_1 - ED) Y_1 - P_1 + \Omega \prec 0 \quad \forall D \in \mathcal{D} \quad (48)$$

where  $Y_1$  and  $P_1 \succ 0$  are decision variables which satisfy the identity  $Y_1 P_1^{-1} = G_1$ , while  $\Omega \succ 0$  is a free design parameter we will comment on shortly. By enforcing (48) we guarantee that  $(X_1 - ED)G_1$  is stable for all  $D \in \mathcal{D}$ , hence we ensure stability of  $(X_1 - ED_0)G_1$  if  $D_0 \in \mathcal{D}$ . The choice of the set  $\mathcal{D}$  clearly reflects our prior information or guess about  $d$ . For instance, if we know that  $|d| \leq \delta$  for some  $\delta > 0$  then we let  $\Delta := \delta\sqrt{T}I_s$ . Stochastic disturbances can also be accounted for (possibly, with other choices of  $\Delta$ ), see Section VI-C. In general, large sets  $\mathcal{D}$  make condition  $D_0 \in \mathcal{D}$  easier to hold but make (48) more difficult to satisfy. We proceed by making the assumption  $D_0 \in \mathcal{D}$  explicit.

*Assumption 3:*  $D_0 \in \mathcal{D}$ . ■

A final comment regards  $\Omega$ . The condition  $\Omega \succ 0$  ensures that  $Y_1^\top (X_1 - ED)^\top P_1^{-1} (X_1 - ED) Y_1 - P_1$  is bounded away from singularity, as we vary  $D$ , by a *known* quantity, and this is key to have an explicit expression for the ROA. There is no loss of generality in considering (48) instead of

$$Y_1^\top (X_1 - ED)^\top P_1^{-1} (X_1 - ED) Y_1 - P_1 \prec 0 \quad \forall D \in \mathcal{D}. \quad (49)$$

Indeed, for any  $\Omega \succ 0$  there exist  $Y_1, P_1 \succ 0$  that satisfy (48) if and only if there exist  $Y_1, P_1 \succ 0$  that satisfy (49).

Condition (48) cannot be implemented directly as it involves *infinitely* many constraints. The next result provide a tractable (and convex) condition for (48). Following [41, Lemma A.4]<sup>4</sup>, we could actually establish the *equivalence* between the next (50) and (48). Here, we will only show that (50) implies (48), which is enough for our purposes.

*Lemma 3:* Suppose that there exist  $Y_1 \in \mathbb{R}^{T \times n}$ ,  $P_1 \in \mathbb{S}^{n \times n}$ , and a scalar  $\epsilon > 0$  such that

$$\begin{bmatrix} P_1 - \Omega & (X_1 Y_1)^\top & Y_1^\top \\ X_1 Y_1 & P_1 - \epsilon E \Delta \Delta^\top E^\top & 0_{n \times T} \\ Y_1 & 0_{T \times n} & \epsilon I_T \end{bmatrix} \succ 0 \quad (50)$$

with  $\Omega \succ 0$  and  $\Delta$  given. Then, (48) holds. ■

<sup>4</sup>Lemma A.4 in [41], also known as the *Petersen's lemma*, permits to study matrix inequalities which involve uncertainty, like (48), and gives conditions under which such inequalities can be equivalently assessed considering only the 'boundary' of the uncertainty, like (50) does. We refer the reader to [15] for a recent discussion on the use of Petersen's lemma in data-driven control of linear and polynomial systems.

*Proof.* See Appendix C. ■

*Theorem 6:* Consider a nonlinear system as in (43) with  $Z$  satisfying the condition (21) and with  $d$  a process disturbance. For a given  $\Omega \succ 0$  and  $\Delta$ , suppose that the following SDP (this is just (19) with (19c) replaced by (50) to account for robust stability)

$$\text{minimize}_{P_1, Y_1, G_2} \quad \|X_1 G_2\| \quad (51a)$$

$$\text{subject to} \quad (19b), (50), (19d) \quad (51b)$$

is feasible. If Assumption 3 holds then the control law  $u = KZ(x)$  with  $K$  in (11) renders the origin an asymptotically stable equilibrium for the closed-loop system. ■

*Proof.* Suppose that (51) is feasible. Let  $G_1 = Y_1 P_1^{-1}$  and note that the two constraints (19b) and (19d) together yield  $Z_0 [G_1 \ G_2] = I_s$ . This relation, combined with (11), gives (7). In view of Lemma 2, the closed-loop dynamics satisfies  $x^+ = \Psi x + \Xi Q(x) + Ed$ , with  $\Psi = (X_1 - ED_0)G_1$ . Next, we prove that  $\Psi$  is Schur. By Lemma 3 and since  $D_0 \in \mathcal{D}$  by hypothesis, (48) holds for  $D = D_0$ . We have in particular  $P_1^{-1} Y_1^\top (X_1 - ED_0)^\top P_1^{-1} (X_1 - ED_0) Y_1 P_1^{-1} - P_1^{-1} \prec 0$ . By recalling that  $Y_1 P_1^{-1} = G_1$ , we conclude that  $\Psi$  is Schur. The result follows from (21). ■

Building on Theorem 6 it is possible to characterize regions of attractions as well as robust invariant sets [31]. We start with the ROA as a preliminary step for robust invariance. Consider the closed-loop dynamics  $x^+ = \Psi x + \Xi Q(x)$  where we set  $d \equiv 0$  since we consider the ROA, and let  $V(x) := x^\top P_1^{-1} x$ . We have

$$V(x^+) - V(x) = \underbrace{(\Psi x + \Xi Q(x))^\top P_1^{-1} (\Psi x + \Xi Q(x)) - x^\top P_1^{-1} x}_{=: s(x)} \quad (52)$$

with  $\Psi = (X_1 - ED_0)G_1$ ,  $\Xi = (X_1 - ED_0)G_2$ . Although  $\Psi$  and  $\Xi$  are unknown, we can upper bound  $s(x)$  with a quantity that is computable from data alone. First consider  $x^\top \Phi x$  where  $\Phi := P_1^{-1} - \Psi^\top P_1^{-1} \Psi$ . By Theorem 6, (48) holds for  $D = D_0$ , i.e.,  $P_1 \Phi P_1 - \Omega \succ 0$ . Premultiplying this inequality left and right by  $P_1^{-1}$  gives  $\Phi - P_1^{-1} \Omega P_1^{-1} \succ 0$ , thus  $x^\top \Phi x \geq x^\top \underline{\Phi} x$  for all  $x$ , where  $\underline{\Phi} := P_1^{-1} \Omega P_1^{-1}$ . Accordingly,

$$V(x^+) - V(x) \leq -x^\top \underline{\Phi} x + (2\Psi x + \Xi Q(x))^\top P_1^{-1} \Xi Q(x).$$

Bearing in mind the expressions of  $\Psi$  and  $\Xi$ , and the fact that  $\|D_0\| \leq \|\Delta\|$ , we can write

$$V(x^+) - V(x) \leq \underbrace{-x^\top \underline{\Phi} x + \ell_1(x) + \ell_2(x) + \ell_3(x) + \ell_4(x)}_{=: \ell(x)} \quad (53)$$

having set

$$\begin{aligned} \ell_1(x) &:= (2X_1 G_1 x + X_1 G_2 Q(x))^\top P_1^{-1} X_1 G_2 Q(x), \\ \ell_2(x) &:= \|\Delta\| \|(2X_1 G_1 x + X_1 G_2 Q(x))^\top P_1^{-1} E\| |G_2 Q(x)|, \\ \ell_3(x) &:= \|\Delta\| \|2G_1 x + G_2 Q(x)\| |E^\top P_1^{-1} X_1 G_2 Q(x)|, \\ \ell_4(x) &:= \|\Delta\|^2 \|E^\top P_1^{-1} E\| \|2G_1 x + G_2 Q(x)\| |G_2 Q(x)|, \end{aligned}$$

which are all computable from data alone.

*Proposition 2:* Consider the same setting as in Theorem 6. Let  $\mathcal{L} := \{x : \ell(x) < 0\}$ , with  $\ell(x)$  as in (53), and consider

the Lyapunov function  $V(x) = x^\top P_1^{-1}x$ . Then, any sub-level set  $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$  of  $V$  contained in  $\mathcal{L} \cup \{0\}$  is a PI set for the closed-loop system with  $d \equiv 0$  and defines an estimate of the ROA relative to  $\bar{x} = 0$ .  $\square$

We now consider robust invariance [31, Definition 2.2].

**Definition 2:** A set  $\mathcal{S}$  is called *robustly positively invariant* (RPI) for the system  $x^+ = f(x, d)$  if for every  $x(0) \in \mathcal{S}$  and all  $d(t) \in \mathcal{I}$ , with  $\mathcal{I}$  a compact set, the solution is such that  $x(t) \in \mathcal{S}$  for  $t > 0$ .  $\square$

Unlike local stability and invariance, which pose conditions on the disturbance only relatively to the data collection phase robust invariance constrains  $d$  for *all* times  $t \geq 0$ . This calls for strengthening Assumption 3 in the sense of Definition 2.<sup>5</sup>

**Assumption 4:**  $|d| \leq \delta$  for some known  $\delta > 0$ .  $\square$

Assumption 4 is indeed stronger than Assumption 3 in the sense that it implies Assumption 3 once we set  $\Delta := \delta\sqrt{T}I_s$ . We can now proceed with the analysis of robust invariance. Consider the closed-loop system  $x^+ = \Psi x + \Xi Q(x) + Ed$  with  $d$  satisfying Assumption 4, and let  $V(x) := x^\top P_1^{-1}x$ . It is simple to verify that we now have

$$V(x^+) - V(x) \leq \ell(x) + g(x, \delta), \quad (55)$$

where  $\ell(x)$  is as in (53), and where

$$g(x, \delta) := r_1(x)\delta + r_2(x)\delta + r_3\delta^2, \quad (56a)$$

$$r_1(x) := 2|(X_1G_1x + X_1G_2Q(x))^\top P_1^{-1}E|, \quad (56b)$$

$$r_2(x) := 2\|\Delta\| \|E^\top P_1^{-1}E\| |G_1x + G_2Q(x)|, \quad (56c)$$

$$r_3 := \|E^\top P_1^{-1}E\|. \quad (56d)$$

Let

$$\mathcal{X} := \{x : \ell(x) + g(x, \delta) \leq 0\} \quad (57)$$

and let  $\mathcal{X}^c$  be its complement.

**Theorem 7:** Consider a nonlinear system as in (43) with  $Z$  satisfying (21) and with  $d$  a process disturbance for which Assumption 4 holds. For a given  $\Omega \succ 0$ , suppose that (51) is feasible with  $\Delta := \delta\sqrt{T}I_s$ , and consider the control law  $u = KZ(x)$  where  $K$  is as in (11). Let  $V(x) := x^\top P_1^{-1}x$ , and define  $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$ , with  $\gamma > 0$  arbitrary. Let  $\mathcal{Z} := \mathcal{R}_\gamma \cap \mathcal{X}^c$  ( $\mathcal{Z}$  is the subset of  $\mathcal{R}_\gamma$  for which the Lyapunov difference  $V(x^+) - V(x)$  can be positive; it is nonempty for any choice of  $\gamma > 0$ ). If

$$V(x) + \ell(x) + g(x, \delta) \leq \gamma \quad \forall x \in \mathcal{Z} \quad (58)$$

then  $\mathcal{R}_\gamma$  is an RPI set for the closed-loop system.  $\square$

**Proof.** As shown in Theorem 6, feasibility of (51), along with  $D_0 \in \mathcal{D}$ , ensures that  $V(x) = x^\top P_1^{-1}x$  is a Lyapunov function for the linear part of the dynamics, and (21) ensures that  $\mathcal{L} = \{x : \ell(x) < 0\}$ , with  $\ell(x)$  as in (53), is nonempty (if  $\mathcal{L}$  is empty then (58) never holds). Then, assume that (58) holds and let  $x \in \mathcal{R}_\gamma$ . We divide the analysis in two cases. First assume that  $x \notin \mathcal{Z}$ . Since  $x \in \mathcal{R}_\gamma$  then  $x \notin \mathcal{X}^c$ . Then  $x \in \mathcal{X}$ , so that  $V(x^+) - V(x) \leq \ell(x) + g(x, \delta) \leq 0$ , and this implies  $x^+ \in \mathcal{R}_\gamma$ . Next, assume that  $x \in \mathcal{Z}$ . In view of (58) we have  $V(x^+) \leq \gamma$ , thus  $x^+ \in \mathcal{R}_\gamma$ .  $\blacksquare$

<sup>5</sup>As an example, a Gaussian disturbance may satisfy the condition  $D_0 \in \mathcal{D}$  but is not bounded in the sense of Definition 2. Set invariance for unbounded disturbances is studied in [42]. We will not pursue this problem here.

Equations (53) and (55) suggest that from a practical point of view it might be convenient to *regularize* the objective function in (51) so as to mitigate the effect of the disturbance. As shown in the subsequent numerical examples, a convenient choice is the following one:

$$\text{minimize}_{P_1, Y_1, G_2} \|X_1G_2\| + \lambda_1\|P_1\| + \lambda_2\|G_2\| \quad (59a)$$

$$\text{subject to} \quad (19b), (50), (19d) \quad (59b)$$

where  $\lambda_1, \lambda_2 \geq 0$  are weighting parameters. Penalizing  $\|P_1\|$  increases the smallest eigenvalue of  $\Phi$ , while penalizing  $\|G_2\|$  decreases the various terms  $\ell_i$  and  $r_i$  in (53) and (55). Notice that penalizing  $\|P_1\|$  might increase the terms  $\ell_i$  and  $r_i$ , but while these quantities depend on  $P_1^{-1}$ ,  $\Phi$  depends on  $P_1^{-2}$ , so penalizing  $\|P_1\|$  can still be advantageous.

Since (59) has the same feasible set as (51) it is understood that all the results of this section as well as those to follow remain true if (51) is replaced with (59).

**Example 6.** We consider again the inverted pendulum of Example 1, this time assuming that a disturbance  $d$  acts on the control channel, namely we have  $E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and the second equation is modified as

$$x_2^+ = \frac{T_s g}{\ell} \sin x_1 + \left(1 - \frac{T_s \mu}{m\ell^2}\right) x_2 + \frac{T_s}{m\ell^2} u + d.$$

We collect data by running an experiment with input uniformly distributed in  $[-0.5, 0.5]$ , and with an initial state within the same interval. We consider a disturbance uniformly distributed in  $[-\delta, \delta]$ . We collect  $T = 30$  samples and solve (59) with  $\lambda_1 = \lambda_2 = 0.1$ ,  $\Omega = I_2$  and  $\Delta = \delta\sqrt{T}$ . Figure 3 reports results for  $\delta = 0.01$ . We observe the following: (i) (59) remains feasible up to  $\delta \approx 0.1$  but for such large values we get empty estimates of ROA/RPI. (ii) the regularization is in fact needed to get nonempty estimates of ROA/RPI, and even small values for  $\lambda_1, \lambda_2$  suffice. This permits to preserve the baseline strategy of nonlinearity minimization. In fact, the controller we obtain is  $K = [-23.9436 \quad -11.4581 \quad -9.8564]$ , which generates the term  $-9.8564 \sin(x_1)$  that approximately cancels out the nonlinearity. (iii) Compared with the disturbance-free case, here we need a larger number of samples to get nonempty estimates of ROA/RPI, although (59) remains feasible even for  $T = 10$ . Intuitively, collecting more samples can indeed help to get more information on the system's dynamics; we will elaborate on this point in Section VI-C.  $\blacksquare$

## B. Neglected nonlinearities

A similar analysis can be carried out in case of neglected nonlinearities. The difference is that now  $d$  will be a function of the state  $x$ , say  $d = d(x)$ . The *combination* of neglected nonlinearities and genuine disturbances is also possible, but we omit the details for brevity. Thus, the analysis which follows only considers invariance instead of *robust* invariance.

In order to handle the case of neglected nonlinearities, we assume some knowledge on the *strength* of such nonlinearities (Assumption 5 is essentially the counterpart of Assumption 4).

**Assumption 5:** We know a set  $\mathcal{Q} \subseteq \mathbb{R}^n$  and a scalar  $\delta > 0$  such that  $|d(x)| \leq \delta$  for all  $x \in \mathcal{Q}$ .  $\square$

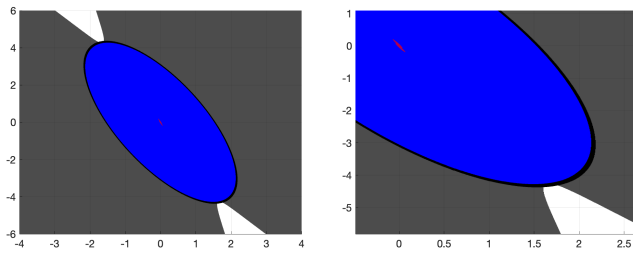


Fig. 3. Results for Example 6. We take  $Z(x) = [x_1 \ x_2 \ \sin(x_1) - x_1]^\top$  and solve (59) with  $\lambda_1 = \lambda_2 = 0.1$ ,  $\Omega = I_2$  and  $\Delta = \delta\sqrt{T}$ , with  $T = 30$  and  $\delta = 0.01$ . Left: the grey set represents the set  $\mathcal{X}$  in (57), while the blue set is the RPI set  $\mathcal{R}_\gamma$ ; here,  $P_1^{-1} = \begin{bmatrix} 0.1901 & 0.0664 \\ 0.0664 & 0.0475 \end{bmatrix}$  and  $\gamma = 0.4440$ . The black set wrapping  $\mathcal{R}_\gamma$  is the ROA, which is larger than the RPI set. The red set around the origin is  $\mathcal{Z}$ ; here,  $\max_{x \in \mathcal{Z}} V(x) + \ell(x) + g(x, \delta) = 0.001$ . States originating in  $\mathcal{Z}$  do not exit  $\mathcal{R}_\gamma$ . In particular, any sub-level set  $\mathcal{R}_\gamma = \{x : V(x) \leq \gamma\}$  with  $\gamma \in [0.0010, 0.4440]$  is an RPI set for the closed-loop system. Right: zoom showing  $\mathcal{R}_\gamma$  close to the border of  $\mathcal{X}$ .

**Theorem 8:** Consider a nonlinear system as in (43) with  $Z$  satisfying (21) and with  $d = d(x)$  a nonlinear function of the state for which Assumption 5 holds. Consider an experiment on the system such that  $x(k) \in \mathcal{Q}$  for  $k = 0, \dots, T-1$ . For a given  $\Omega \succ 0$ , suppose that (51) is feasible with  $\Delta = \delta\sqrt{T}I_s$ . Let  $V(x) := x^\top P_1^{-1}x$  and  $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$  with  $\gamma > 0$  arbitrary. Finally, let  $\mathcal{X}$  be as in (57) and  $\mathcal{Z} := \mathcal{R}_\gamma \cap \mathcal{X}^c$ . If  $\mathcal{R}_\gamma \subseteq \mathcal{Q}$  and

$$V(x) + \ell(x) + g(x, \delta) \leq \gamma \quad \forall x \in \mathcal{Z} \quad (60)$$

then  $\mathcal{R}_\gamma$  is a PI set for the closed-loop system.  $\square$

*Proof.* Under the stated conditions we have  $D_0 \in \mathcal{D}$ . Thus, the feasibility of (51) guarantees that  $V(x) = x^\top P_1^{-1}x$  is a Lyapunov function for the linear part of the dynamics, and (21) ensures that  $\mathcal{L} = \{x : \ell(x) < 0\}$ , with  $\ell(x)$  as in (53), is nonempty (otherwise (60) would never hold). Then, assume that (60) holds and let  $x \in \mathcal{R}_\gamma$ . Since  $x \in \mathcal{R}_\gamma$  then  $x \in \mathcal{Q}$ , and therefore  $|d(x)| \leq \delta$ . Hence, exactly as in (55), we have  $V(x^+) - V(x) \leq \ell(x) + g(x, \delta)$  where  $g(x, \delta)$  is as in (56). The rest of the proof is analogous to that of Theorem 7. Assume that  $x \notin \mathcal{Z}$ . Since  $x \in \mathcal{R}_\gamma$  then  $x \notin \mathcal{X}^c$ . Thus  $x \in \mathcal{X}$ , and hence  $V(x^+) - V(x) \leq \ell(x) + g(x, \delta) \leq 0$ , which implies  $x^+ \in \mathcal{R}_\gamma$ . Next, assume that  $x \in \mathcal{Z}$ . In view of (60), we have  $V(x^+) \leq \gamma$ , thus  $x^+ \in \mathcal{R}_\gamma$ .  $\blacksquare$

We can also have asymptotic stability under a strengthened Assumption 5. Here we report a prototypical result.

**Theorem 9:** Consider the same setting as in Theorem 8, and suppose that  $|d(x)| \leq \delta(x)$  for all  $x$ , where  $\delta(x) : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a known continuous function such that  $\lim_{|x| \rightarrow 0} \frac{\delta(x)}{|x|} = 0$ . Let  $\ell(x)$  be as in (53) and  $g(x, \delta(x))$  as in (56) with  $\delta$  replaced by  $\delta(x)$ . Finally, define  $\mathcal{W} := \{x : \ell(x) + g(x, \delta(x)) < 0\}$ . Then, the origin is an asymptotically stable equilibrium for the closed-loop system, and any set  $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$  contained in  $\mathcal{W} \cup \{0\}$  is a PI set and defines an estimate of the ROA relative to  $\bar{x} = 0$ .  $\square$

*Proof.* Analogously to (55), the Lyapunov function satisfies  $V(x^+) - V(x) \leq \ell(x) + g(x, \delta(x))$  for all  $x$ . Then the result follows immediately.  $\blacksquare$

**Example 7.** Consider the previous example, but this time assume that we purposely neglect the nonlinearity and design

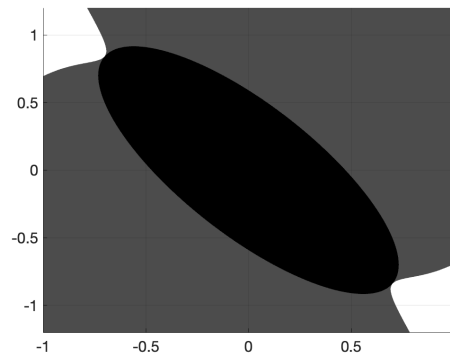


Fig. 4. Results for Example 7 in which we consider a linear control law. The grey set represents the set  $\mathcal{W}$ , while the black set represents the set  $\mathcal{R}_\gamma$  which defines the ROA; here,  $P_1^{-1} = \begin{bmatrix} 0.2116 & 0.1291 \\ 0.1291 & 0.1351 \end{bmatrix}$  and  $\gamma = 0.0473$ .

a linear control law. Specifically, the dynamics of the inverted pendulum can be written as

$$\begin{aligned} x_1^+ &= x_1 + T_s x_2, \\ x_2^+ &= \frac{T_s g}{\ell} x_1 + \left(1 - \frac{T_s \mu}{m \ell^2}\right) x_2 + \frac{T_s}{m \ell^2} u + d, \\ d &= \frac{T_s g}{\ell} (\sin x_1 - x_1). \end{aligned}$$

In this case, the type of dynamics is known, hence we focus on Theorem 9. We consider  $\delta(x) = 2|\sin x_1 - x_1|$ , thus  $|d(x)| \leq \delta(x)$  for all  $x$  (we over-approximate  $d$  by more than 100%). We run an experiment with input and initial state uniformly distributed in  $[-0.1, 0.1]$ . This ensures that up to  $T = 10$  the state  $x_1$  remains close to the equilibrium, so that  $d$  remains small. In particular, with this choice,  $x_1$  never exceeds  $\pm 0.06$  ( $\approx \pm 3.5^\circ$ ), and  $\delta(x) \leq 3 \cdot 10^{-5} =: c$ . Thus we take  $T = 10$ , set  $\Omega = I_2$ ,  $\Delta = c\sqrt{T}$  and solve (59).

Note that (59) now involves only the variables  $P_1, Y_1$ , thus only the two constraints (19b) and (50) are present. We get  $K = [-19.0204 \ -10.7947]$  and the ROA in Figure 4. As expected, the outcome is worse than the one obtained when we exploit the knowledge of the nonlinearities and we consider a nonlinear control law. Another shortcoming is that we now need to run the experiment close to the equilibrium point in order to keep  $d$  small, which is not needed when we take the nonlinearity into account.  $\blacksquare$

### C. Results in probability

All previous results rest on the assumption that  $D_0 \in \mathcal{D}$ . Clearly, once the experiment is performed and the data are collected, whether  $D_0 \in \mathcal{D}$  or not is a *deterministic* property (yes or no). Yet, certifying that  $D_0$  actually belongs to  $\mathcal{D}$  can be a difficult task. It turns out that we can establish results that relate closed-loop stability with the *probability* that  $D_0 \in \mathcal{D}$ . We focus on the case of process disturbances, in particular we give a probabilistic version of Theorem 6.

**Theorem 10:** Consider a nonlinear system as in (43) with  $Z$  satisfying (21) and with  $d$  a process disturbance. For a given  $\Omega \succ 0$  and  $\Delta$ , suppose that (51) is feasible. If  $D_0 \in \mathcal{D}$  with probability at least  $p$  then the control law  $u = KZ(x)$ ,

with  $K$  as in (11), renders the origin an asymptotically stable equilibrium with probability at least  $p$ .  $\square$

*Proof.* The result is a direct consequence of the *law of total probability* [43, Theorem 3, pp. 28]. Given two events  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , let  $P(\mathcal{E}_1)$  and  $P(\mathcal{E}_1|\mathcal{E}_2)$  denote the probability of  $\mathcal{E}_1$  and the conditional probability of  $\mathcal{E}_1$  given  $\mathcal{E}_2$ . Let  $\mathcal{E}_1$  denote the event that  $K$  is stabilizing and  $\mathcal{E}_2$  denote the event  $D_0 \in \mathcal{D}$ . We have  $P(\mathcal{E}_1) = P(\mathcal{E}_1|\mathcal{E}_2)P(\mathcal{E}_2) + P(\mathcal{E}_1|\mathcal{E}_2^c)P(\mathcal{E}_2^c)$ , with  $\mathcal{E}^c$  the complement of  $\mathcal{E}$ . Then,  $P(\mathcal{E}_1) \geq P(\mathcal{E}_1|\mathcal{E}_2)P(\mathcal{E}_2)$  and the result follows because  $P(\mathcal{E}_1|\mathcal{E}_2) = 1$  by Theorem 6.  $\blacksquare$

Theorem 10 allows us to extend our range of application to cases where bounds on  $d$  are known only with a limited accuracy, as exemplified in the next Proposition 3. Theorem 10 has another interesting use. For disturbances obeying the *law of large numbers* [43, Section 5] we can repeat the same experiment multiple times and average the data so as to filter out noise. Specifically, suppose we make  $N$  experiments on system (43), each of length  $T$ , and let  $(U_0^{(r)}, D_0^{(r)}, Z_0^{(r)}, X_1^{(r)})$ , with  $r = 1, \dots, N$ , be the dataset resulting from the  $r$ -th experiment. Given  $N$  matrices  $S^{(r)}$ , with  $r = 1, \dots, N$ , let  $\underline{S} := \frac{1}{N} \sum_{r=1}^N S^{(r)}$  denote their average. Since each dataset satisfies the relation  $X_1^{(r)} = AZ_0^{(r)} + BU_0^{(r)} + ED_0^{(r)}$ , if we average  $N$  datasets we obtain the relation

$$\underline{X}_1 = A\underline{Z}_0 + B\underline{U}_0 + E\underline{D}_0. \quad (62)$$

Because the dynamics are nonlinear, (62) does *not* represent a valid trajectory of the system in the sense that it cannot result from a single experiment on (43). Yet, and this is the crucial point, the dataset  $(\underline{U}_0, \underline{D}_0, \underline{Z}_0, \underline{X}_1)$  gives a data-based parametrization of the closed loop in the sense of Lemma 2. Specifically, for any  $K, G$  satisfying

$$\begin{bmatrix} K \\ I_s \end{bmatrix} = \begin{bmatrix} \underline{U}_0 \\ \underline{Z}_0 \end{bmatrix} G \quad (63)$$

we have (cf. (9))

$$A + BK = (\underline{X}_1 - E\underline{D}_0)G. \quad (64)$$

Hence, Lemma 2, and consequently Theorems 6 and 10, apply to  $(\underline{U}_0, \underline{D}_0, \underline{Z}_0, \underline{X}_1)$  with no modifications, with the advantage that  $\underline{D}_0$  will have a reduced norm in expectation thanks to the law of large numbers. While the law of large numbers gives an asymptotic result, recent results in non-asymptotic statistics permit us, for relevant classes of disturbance, to get high-confidence bounds on  $\|\underline{D}_0\|$  even with a *finite* number of experiments. As an example, we give the following result.<sup>6</sup>

**Proposition 3:** Consider  $N$  experiments, each of length  $T$ , on system (43), and assume that the disturbances  $d(k) \in \mathbb{R}^s$  are i.i.d. zero-mean random vectors with covariance matrix  $\Sigma$  such that  $|d(k)| \leq \delta$  *almost surely* (i.e., with probability 1). Then, for all  $\mu > 0$ ,

$$\|\underline{D}_0\| \leq \sqrt{T \left( \frac{\|\Sigma\|}{N} + \mu \right)} \quad (65)$$

<sup>6</sup>The notation used in the sequel is standard, e.g., see [43]. Independent and identically distributed random vectors are abbreviated as i.i.d.. We will denote by  $\mathcal{N}(\mu, \Sigma)$  the multivariate normal (Gaussian) distribution with mean  $\mu$  and covariance matrix  $\Sigma$ .

with probability at least  $1 - 2s \exp\left(-\frac{TN\mu^2}{2\delta^2(\|\Sigma\| + N\mu)}\right)$ .

Let instead the disturbances  $d(k)$  be i.i.d. random vectors drawn from  $\mathcal{N}(0, \Sigma)$ . Then, for all  $\mu > 0$ ,

$$\|\underline{D}_0\| \leq \sqrt{\frac{T}{N}} \left( \lambda_{\max}(\Sigma^{1/2})(1 + \mu) + \sqrt{\frac{\text{trace}(\Sigma)}{T}} \right) \quad (66)$$

with probability at least  $1 - \exp(-T\mu^2/2)$ , where  $\lambda_{\max}$  denotes the maximum eigenvalue.  $\square$

*Proof.* Since the disturbances  $d(k)$  are independent then the vectors which form the columns of  $\underline{D}_0$  are also independent. This can be easily verified, for instance, through the so-called *characteristic function*, e.g., see [43, Theorem 28, pp. 131]. It is also easy to verify that these vectors have zero mean and covariance matrix  $\Sigma/N$ . The bounds (65) and (66) follow from Corollary 6.20 and Theorem 6.1 in [44], respectively.  $\blacksquare$

Under the assumption on the disturbances stated in Proposition 3, we can choose  $\Delta = \eta I_s$  with  $\eta$  equal to the right-hand side of (65) or (66), and control  $\eta$  via  $T, \mu$  and  $N$ . This may lead us to satisfy, with a certain probability, the condition  $\|\underline{D}_0\| \leq \eta$  (thus  $\underline{D}_0 \in \mathcal{D}$ ) with  $\eta$  small. As a result, we may render (51) easier to satisfy and have stability guarantees (in probability). Specifically, by applying Theorem 10, if (51), with  $X_1, Z_0$  replaced by  $\underline{X}_1, \underline{Z}_0$ , is feasible then the control law  $u = KZ(x)$ , where  $K$  is given by (11) with  $U_0$  replaced by  $\underline{U}_0$ , will asymptotically stabilize the origin with the same probability as condition  $\|\underline{D}_0\| \leq \eta$  is satisfied.

A second advantage of having  $\|\underline{D}_0\| \leq \eta$  with  $\eta$  small is that, by virtue of (53) and (55), we may have (in probability) less conservative estimates for the ROA and RPI sets compared to the ones obtained with deterministic (worst-case) bounds for the disturbance.

**Example 8.** We consider again Example 6 under the same experimental setup for the disturbance, but now we repeat the experiment  $N = 100$  times, each time using the same input pattern. For the uniform distribution it holds that  $\Sigma = \delta^2/3$ . With  $\mu = 4 \cdot 10^{-5}$ , Proposition 3 implies  $\|\underline{D}_0\| \leq 0.0348$  with probability at least 99.48%. The bound is much tighter compared to the worst-case bound  $\|\underline{D}_0\| \leq \delta\sqrt{T} = 0.0548$  obtained by only exploiting the property  $|d| \leq \delta$ .

We solve (59) (recall that (59) has the same feasible set as (51)) using the same parameters as in Example 6 but now with the average matrices  $\underline{U}_0, \underline{Z}_0, \underline{X}_1$ , and  $\Delta = 0.0348$ . We obtain  $K = [-20.9897 \quad -11.1369 \quad -9.8222]$ . Theorem 10 implies that  $K$  is stabilizing with probability at least 99.48% ( $K$  is indeed stabilizing as  $\|\underline{D}_0\| = 0.0050 < \Delta$ ). The RPI set obtained with  $\Delta = 0.0348$  is much larger than the one obtained in Example 6 with the worst-case value  $\Delta = \delta\sqrt{T}$ ; compare the new Figure 5 with Figure 3.  $\blacksquare$

**Example 9.** We conclude the section with some simulation results for the polynomial system of Example 4. The system has “more unstable” dynamics than the pendulum system, and we obtain non-negligible RPI sets only for  $|d| \leq 0.001$ . For the same setting as in Example 4 and a disturbance uniformly distributed the SDP (59) returns the RPI set in Figure 6 (Left). With averaging, we already improve the estimate for  $N = 10$ , see Figure 6 (Right). With averaging, we also systematically obtain non-negligible RPI sets up to  $|d| \leq 0.01$ .  $\blacksquare$

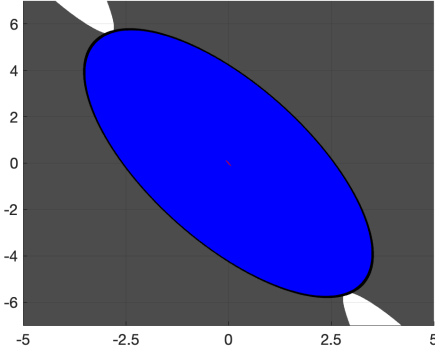


Fig. 5. Simulation results for the pendulum in case of repeated experiments. See the caption of Figure 3 for a description of the various sets.

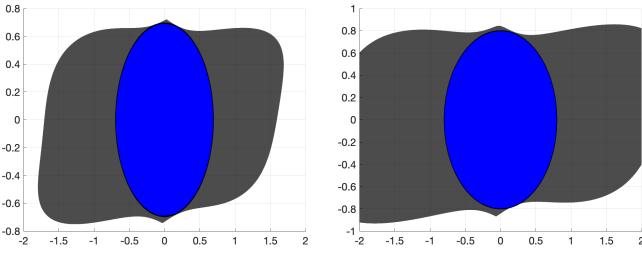


Fig. 6. Simulation results for the polynomial system of Example 4 with a disturbance uniformly distributed in  $[-0.001, 0.001]^2$  which gives the bound  $\delta = 0.0014$ . We consider trajectories of length  $T = 50$  and solve (59) with  $\lambda_1 = \lambda_2 = 0.1$ . Left: results without averaging. The grey set represents the set  $\mathcal{X}$  in (57), while the blue set is the RPI set. Right: results with averaging ( $N = 10$ ). We took  $\mu = 5 \cdot 10^{-7}$  which gives  $\Delta = 0.0052I_2$  and certifies stability with 98.86% probability.

## VII. DISCUSSION

### A. Approximate nonlinearity cancellation and ROA size

Exact nonlinearity cancellation leads to global asymptotic stability in the case no noise is affecting the data used in the design (Theorem 1). When an exact cancellation of the nonlinearities is not possible, an approximate one should be considered, as studied in Theorem 4. In general this result returns a local asymptotic stabilizer. Here, we would like to stress that this does not imply that it does not exist a global stabilizer attaining the same cost as the feasible solutions of the SDP (19) appearing in Theorem 4. We illustrate this point by revisiting system (25) in Example 4, which was used to demonstrate Theorem 4 and its follow-up, Proposition 1.

We observe that, were the model of the system known, one could design a global asymptotic stabilizer given by  $u = -x_2 - 0.1x_1^2 - x_1^3 - 0.08x_1x_2^2 - 0.016x_2^4$ . This controller returns a closed-loop system whose linear part  $M$  is Schur and whose nonlinear part  $N$  has norm equal to 0.2, the optimal value attained by the SDP (19). Hence, if one would include quartic monomials in  $Z(x)$ , it could be numerically verified whether or not the global asymptotic stabilizer is a feasible solution to the SDP (19). However, there is no analytic guarantee that the SDP will return exactly the global stabilizer, and in general it will not.

### B. Nonlinearity cancellation and coordinate transformations

In model-based design, the possibility of cancelling the nonlinearity is eased by the existence of a normal form revealed by a suitable coordinate transformation. In this section we comment on how the techniques investigated so far lend themselves to be used along with such coordinate transformations obtainable for systems having a uniform relative degree equal to the dimension of the state space.

Consider the discrete time nonlinear system with output

$$x^+ = f(x, u) \quad (67a)$$

$$y = h(x) \quad (67b)$$

where  $u, y \in \mathbb{R}$  for the sake of simplicity. We assume that both the state  $x$  and the output  $y$  are available for measurements. A prior information about the system is that it satisfies

$$\begin{aligned} \frac{\partial h \circ f_0^i \circ f(x, u)}{\partial u} &= 0, \quad \forall (x, u) \in \mathbb{R}^{n+1}, \quad 0 \leq i \leq n-2 \\ \frac{\partial h \circ f_0^{n-1} \circ f(x, u)}{\partial u} &\neq 0, \quad \forall (x, u) \in \mathbb{R}^{n+1} \end{aligned} \quad (68)$$

where  $f_0(x) = f(x, 0)$ ,  $f_0^d = \underbrace{f_0 \circ f_0 \circ \dots \circ f_0}_{d \text{ times}}$ ,

$$\begin{bmatrix} h(x) \\ h \circ f_0(x) \\ \vdots \\ h \circ f_0^{n-1}(x) \end{bmatrix} =: \Phi_0(x) \quad (69)$$

is a global coordinate transformation [45], [29]. The transformation  $\Phi_0$  depends on the system's dynamics, which is not available; nevertheless it can be implemented bearing in mind the interpretation of its entries as the value of the output at a given time and at future time instants, namely, at any time  $k$ , we have that

$$w(k) := \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+n-1) \end{bmatrix} = \Phi_0(x(k)),$$

so that in the coordinates  $w$  the system's dynamics can be written as

$$w(k+1) = \begin{bmatrix} w_2(k) \\ w_3(k) \\ \vdots \\ w_n(k) \\ h \circ f_0^{n-1} \circ f(x(k), u(k)) \end{bmatrix}, \quad y(k) = w_1(k). \quad (70)$$

Note that the last entry of the vector field on the right-hand side has been deliberately left to depend on the original state  $x$  rather on the new one  $z$ , which turns out to be useful to obtain a *causal* control policy. The point of this transformation is that, were the system's dynamics known, one could design a static feedback controller that stabilizes the system via exact nonlinearity cancellation. When the dynamics are unknown, one can still achieve exact nonlinearity cancellation by modifying the techniques proposed in Section III-A, provided that the following assumption holds:

**Assumption 6:** A vector-valued function  $Q : \mathbb{R}^n \rightarrow \mathbb{R}^{S-n}$  is known for which  $h \circ f_0^{n-1} \circ f(x, u) = a^\top Q(x) + bu$  for some (unknown) quantities  $a \in \mathbb{R}^{S-n}$ ,  $b \in \mathbb{R} \setminus \{0\}$ .

Asking for  $h \circ f_0^{n-1} \circ f(x, u)$  to take this specific form is clearly demanding, but one can in principle collect the discrepancy between  $h \circ f_0^{n-1} \circ f(x, u)$  and  $a^\top Q(x) + bu$  into a mismatch function and treat it as a disturbance, analogously to what has been discussed in Section VI-B.

Under the assumption above, a controller can be designed following the construction in the previous subsection with suitable modifications. We start defining the matrix of input samples  $U_0$  as in (6a), and

$$\begin{aligned} W_0 &:= [w(0) \ w(1) \ \cdots \ w(T-1)] \in \mathbb{R}^{n \times T}, \\ W_1 &:= [w(1) \ w(2) \ \cdots \ w(T)] \in \mathbb{R}^{n \times T}, \\ Q_0 &:= [Q(x(0)) \ Q(x(1)) \ \cdots \ Q(x(T-1))] \in \mathbb{R}^{(S-n) \times T}, \\ Z_0 &:= [W_0^\top \ Q_0^\top]^\top \in \mathbb{R}^{S \times T}, \end{aligned} \quad (71a)$$

which satisfy the identity  $W_1 = A_c W_0 + B_c(a^\top Q_0 + bU_0)$ , where the pair  $(A_c, B_c)$  is in the Brunovsky canonical form [46]. Note that since both the state  $x$  and the output  $y$  are assumed to be available for measurements, the matrices of data  $W_0, W_1, Q_0$  are known. In particular, the matrix  $W_0$  (similarly for  $W_1$ ) comprises output samples:

$$W_0 = \begin{bmatrix} y(0) & y(1) & \cdots & y(T-1) \\ y(1) & y(2) & \cdots & y(T) \\ \vdots & \vdots & \ddots & \vdots \\ y(n-1) & y(n) & \cdots & y(n+T-2) \end{bmatrix}.$$

We have the following result.

**Corollary 2:** Consider the nonlinear system with output (67). Assume that the conditions (68) hold and that  $\Phi_0$  in (69) is a global coordinate transformation. Let Assumption 6 hold. If there exist decision variables  $G_1 \in \mathbb{R}^{T \times n}$ ,  $k_1 \in \mathbb{R}$ , and  $G_2 \in \mathbb{R}^{T \times (S-n)}$  such that

$$Z_0 G_1 = \begin{bmatrix} I_n \\ 0_{(S-n) \times n} \end{bmatrix}, \quad (72a)$$

$$W_1 G_1 = A_c + B_c [k_1 \ \underbrace{0 \ \cdots \ 0}_{n-1 \text{ times}}], \quad (72b)$$

$$k_1 \in (-1, 1), \quad (72c)$$

$$Z_0 G_2 = \begin{bmatrix} 0_{n \times (S-n)} \\ I_{S-n} \end{bmatrix}, \quad (72d)$$

$$W_1 G_2 = 0_{n \times (S-n)}, \quad (72e)$$

then  $u = K [Q(x)]$ , with  $K = U_0 G$ , linearizes the closed-loop system and renders the origin a globally asymptotically stable equilibrium.  $\square$

**Proof.** Conditions (72a), (72d) along with the definition of the controller gain  $K$ , show that the identity (7) holds. Thus, the closed-loop system is of the form

$$w^+ = A_c w + B_c(a^\top Q(x) + bu) \quad (73a)$$

$$= A_c w + B_c(a^\top Q(x) + bU_0 G [Q(x)]) \quad (73b)$$

$$= W_1 G [Q(x)] = W_1 G_1 w \quad (73c)$$

where the third equality follows from the identities  $B_c b U_0 G = W_1 G - A_c W_0 G - B_c a^\top Q_0 G$ , (72a) and (72d), and the last one

from (72e). Hence, the controller  $u = K [Q(x)]$  linearizes the closed-loop system. Finally, by (72b), the closed-loop system coincides with  $w^+ = (A_c + B_c [k_1 \ 0 \ \cdots \ 0])w$ , where the matrix  $A_c + B_c [k_1 \ 0 \ \cdots \ 0]$  is Schur since all its eigenvalues are given by the solutions of the equation  $\lambda^n = k_1$  and  $|k_1| < 1$ .  $\blacksquare$

The control law only uses the variables  $y, x$  and as such it is implementable. In fact, bearing in mind (72a) and (72d), the identity  $W_1 G = A_c W_0 G + B_c(a^\top Q_0 G + bU_0 G)$  is equivalent to

$$\begin{bmatrix} A_c + B_c [k_1 \ 0 \ \cdots \ 0] & 0_{n \times (S-n)} \\ B_c [0_{n \times n} \ a^\top] + B_c b U_0 G \end{bmatrix} = \begin{bmatrix} A_c & 0_{n \times (S-n)} \\ B_c [0_{n \times n} \ a^\top] + B_c b U_0 G \end{bmatrix}$$

from which we deduce that  $U_0 G = b^{-1} [k_1 \ 0 \ \cdots \ 0] - a^\top$ , that is  $U_0 G_1 w$  only depends on the first component of  $w$ , which is the output  $y$ .

The method discussed in this section is essentially a feed-back linearization method, cf. [29]. The advantage with respect to the main approach of Sections III and IV is the possibility to linearize exactly the dynamics in coordinates different from the original ones, in which case global asymptotic stability follows at once. A main disadvantage is Assumption 6. Moreover, the main approach is directly applicable when exact linearization is impossible. In contrast, to estimate ROA and RPI sets with the method discussed in this section we should also know the map  $\Phi_0$  in (69), which is needed to characterize invariant sets in the original  $x$ -coordinates.

**Example 10.** Consider the polynomial system

$$x_1^+ = x_2^2 + x_1^3 + u \quad (74a)$$

$$x_2^+ = 0.5x_1 + 0.2x_2^2 \quad (74b)$$

$$y = x_2 \quad (74c)$$

Exact cancellation based on Theorem 1 is not possible for this system. On the other hand, the conditions of Corollary 2 hold.

In particular, notice that

$$h \circ f_0^{n-1} \circ f(x, u) = \frac{1}{20}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_1^3 + \frac{1}{25}x_1x_2^2 + \frac{1}{125}x_2^4 + \frac{1}{2}u.$$

Hence, if we choose

$$Q(x) = [x_1^2 \ x_2^2 \ x_1x_2 \ x_1^3 \ x_2^3 \ x_1x_2^2 \ x_1^2x_2 \ x_1^4 \ x_2^4 \ x_1x_2^3 \ x_1^2x_2^2 \ x_1^3x_2]$$

then Assumption 6 is satisfied. The choice of such a  $Q(x)$  can be guided by some prior knowledge, namely that the nonlinearity in the last equation of the system in the new coordinates is a polynomial of degree no larger than 4. On the other hand, using  $Q(x)$  instead of  $[Q(x)]$  is motivated by the fact that, if this were not the case, then the matrix  $Z_0$  would be rank deficient (this is a test that can be carried out from the collected data). This is because each column  $i$  of  $W_0$  is equal to  $[y(i-1) \ y(i)]^\top = [x_2(i-1) \ 0.5x_1(i-1) + 0.2x_2(i-1)^2]^\top$  and it would be expressible as a linear combination of the entries of column  $i$  of  $Q_0$  if the latter would include  $x$ .

Applying Corollary 2, we find that the SDP (72) is feasible and returns the solution  $k_1 = 0.372$  and

$$K = \begin{bmatrix} 0.7423 & 0 & -0.1 & -1 & 0 & -1 & 0 & -0.08 & 0 & 0 \\ & & & & & & & -0.016 & 0 & 0 \end{bmatrix}$$

which linearizes the closed-loop system in the coordinates  $w$ , and renders the origin a globally asymptotically stable equilibrium. ■

### VIII. CONCLUSIONS

We have introduced a method to design Lyapunov-based stabilizing controllers for nonlinear systems from data, which reduces the design to the solution of data-dependent SDP. The method is certified to provide a solution in the presence of perturbed data as well as estimates of the region of attraction of the closed-loop system. Both deterministic and stochastic perturbations on the data are studied. We also extended the results to deal with the presence of neglected nonlinearities. Possible future research should focus on output feedback control design, the inclusion of criteria to maximize the region of attraction and the design of more general (non quadratic) Lyapunov functions.

### APPENDIX

#### A. A parametrization of all stabilizing and linearizing feedback controllers

Suppose that  $\begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}$  has full row rank. In this case, we can prove that any stabilising and linearising feedback controller can be parametrised as in (11) for some  $Y_1, P_1, G_2$  satisfying (10). Note in particular that this implies that the SDP is feasible. This result is as a generalization of [8, Theorem 3] where an analogous result for linear system is provided under the condition that  $\begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$  has full row rank. In the linear case, the latter condition reduces to a design condition for controllable dynamics, see [7, Theorem 1], [47, Theorem 1]. We are not aware of analogous results for nonlinear systems.

*Proof of Theorem 2.* Consider any stabilizing and linearizing feedback controller  $K$ . We have

$$A + BK = X_1 G \quad (75)$$

for some  $G \in \mathbb{R}^{T \times S}$  satisfying (7). Note that  $G$  exists as  $\begin{bmatrix} U_0 \\ Z_0 \end{bmatrix}$  has full row rank by hypothesis. By partitioning  $K = \begin{bmatrix} \bar{K} & \hat{K} \end{bmatrix}$  with  $\bar{K} \in \mathbb{R}^{m \times n}$  and  $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$  with  $G_1 \in \mathbb{R}^{T \times n}$ , we have  $X_1 G_1 = \bar{A} + B\bar{K}$  and  $X_1 G_2 = \hat{A} + B\hat{K} = 0$ , where the matrix  $X_1 G_1$  is Schur and  $X_1 G_2 = 0$  by the assumption that  $K$  is stabilizing and linearizing. Hence, there exists a matrix  $P_1 \succ 0$  such that  $(X_1 G_1)^\top P_1^{-1} X_1 G_1 - P_1^{-1} \prec 0$ . This implies  $(X_1 Y_1)^\top P_1^{-1} X_1 Y_1 - P_1 \prec 0$  with  $Y_1 = G_1 P_1$ , which is the constraint (10b). Since  $Z_0 G = I_S$  and  $Y_1 = G_1 P_1$  we have

$$Z_0 \begin{bmatrix} Y_1 & G_2 \end{bmatrix} = \begin{bmatrix} P_1 & 0_{n \times (S-n)} \\ 0_{(S-n) \times n} & I_{S-n} \end{bmatrix}, \quad (76)$$

which matches the constraints (10a) and (10c). Thus, all the constraints in (10) are satisfied, hence the program is feasible.

As for the form of the controller, by (7) we have  $K = U_0 G$  which in terms of  $Y_1, G_2$  reads as (11). ■

#### B. A parametrisation of all (locally) stabilising feedback controllers

*Proof of Theorem 5.* The identity (75) is still valid because independent of the properties of  $K$ . Furthermore, we can still

write  $X_1 G_1 = \bar{A} + B\bar{K}$  and  $X_1 G_2 = \hat{A} + B\hat{K}$ . (The only difference with respect to Theorem 2 is that now  $X_1 G_2$  might be different from zero.) Observe now that, by assumption,  $X_1 G_1$  is Schur. Thus, there exists a matrix  $P_1 \succ 0$  such that  $(X_1 G_1)^\top P_1^{-1} X_1 G_1 - P_1^{-1} \prec 0$ . By defining  $Y_1 = G_1 P_1$ , this is equivalent to (19c). Recalling that  $Z_0 G = I_S$  we have again the identity (76). Thus, all the constraints in (19) are satisfied and the program is feasible.

As for the form of the controller, by (7) we have  $K = U_0 G$  which in terms of  $Y_1, G_2$  reads as (11). ■

#### C. Proof of Lemma 3

Lemma 3 is a direct consequence of the following result.

*Lemma 4:* Let  $B \in \mathbb{R}^{n \times p}$ ,  $C \in \mathbb{R}^{q \times n}$  be given matrices, and let  $\mathcal{D} := \{D \in \mathbb{R}^{q \times p} : DD^\top \preceq \Delta\Delta^\top\}$ . Then, for arbitrary  $\epsilon > 0$  it holds that

$$BD^\top C + C^\top DB^\top \preceq \epsilon^{-1}BB^\top + \epsilon C^\top \Delta\Delta^\top C \quad \forall D \in \mathcal{D}.$$

*Proof.* A completion of squares

$$\left(\sqrt{\epsilon^{-1}}B - \sqrt{\epsilon}C^\top D\right)\left(\sqrt{\epsilon^{-1}}B - \sqrt{\epsilon}C^\top D\right)^\top \succeq 0$$

gives the result. ■

*Proof of Lemma 3.* Let (50) hold. By a Schur complement, this is equivalent to

$$\begin{bmatrix} P_1 - \Omega & (X_1 Y_1)^\top \\ X_1 Y_1 & P_1 \end{bmatrix} - \epsilon^{-1} \underbrace{\begin{bmatrix} Y_1^\top \\ 0_{n \times T} \end{bmatrix}}_{:=B} \begin{bmatrix} Y_1 & 0_{T \times n} \end{bmatrix} - \epsilon \underbrace{\begin{bmatrix} 0_{n \times s} \\ E \end{bmatrix}}_{:=C^\top} \Delta\Delta^\top \begin{bmatrix} 0_{s \times n} & E^\top \end{bmatrix} \succ 0.$$

An application of Lemma 4 gives

$$\begin{bmatrix} P_1 - \Omega & Y_1^\top (X_1 - ED)^\top \\ (X_1 - ED)Y_1 & P_1 \end{bmatrix} \succ 0 \quad \forall D \in \mathcal{D}. \quad (77)$$

This is equivalent to (48) after another Schur complement, and this gives the result. ■

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