



The parabolic Anderson model on the hypercube

Luca Avena^{a,*}, Onur Gün^b, Marion Hesse^b

^a *Universiteit Leiden, Niels Bohrweg 1, 2333 CA Leiden, Netherlands*

^b *Weierstrass Institute, Mohrenstrasse 39, 10117 Berlin, Germany*

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Abstract

We consider the parabolic Anderson model $\frac{\partial}{\partial t} v_n = \kappa \Delta_n v_n + \xi_n v_n$ on the n -dimensional hypercube $\{-1, +1\}^n$ with random i.i.d. potential ξ_n . We study v_n at the location of the k th largest potential, $x_{k,2^n}$. Our main result is that, for a certain class of potential distributions, the solution exhibits a phase transition: for short time scales $v_n(t_n, x_{k,2^n})$ behaves like a system without diffusion and grows as $\exp\{(\xi_n(x_{k,2^n}) - \kappa)t_n\}$, whereas, for long time scales the growth is dictated by the principal eigenvalue and the corresponding eigenfunction of the operator $\kappa \Delta_n + \xi_n$, for which we give precise asymptotics. Moreover, the transition time depends only on the difference $\xi_n(x_{1,2^n}) - \xi_n(x_{k,2^n})$. One of our main motivations is to investigate the mutation–selection model of population genetics on a random fitness landscape, which is given by the ratio of v_n to its total mass, with ξ_n corresponding to the fitness landscape. We show that the above mentioned phase transition translates to the mutation–selection model as follows: a population initially concentrated at $x_{k,2^n}$ moves completely to $x_{1,2^n}$ on time scales where the transition of growth rates occurs. The class of potentials we consider includes the Random Energy Model (REM) which is studied in the statistical physics literature as one of the main examples of a random fitness landscape.

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* Corresponding author.

E-mail addresses: l.avena@math.leidenuniv.nl (L. Avena), Onur.Guen@wias-berlin.de (O. Gün), Hesse@wias-berlin.de (M. Hesse).

1. Introduction and main results

1.1. The model

Consider the n -dimensional hypercube $\Sigma_n = \{-1, 1\}^n$, $n \in \mathbb{N}$. For $x \in \Sigma_n$, we use the notation $x = (x^{(1)}, \dots, x^{(n)})$, where $x^{(i)}$ denotes the spin of x at site i , $i = 1, \dots, n$. The Hamming distance on Σ_n is defined by

$$d(x, y) := \#\{i : x^{(i)} \neq y^{(i)}\}. \quad (1.1)$$

We declare that x and y are neighbours, denoted by $x \sim y$, if $d(x, y) = 1$.

Our model is described through the following system of differential equations with random potential:

$$\frac{\partial}{\partial t} v_n(t, x, y) = \kappa \Delta_n v_n(t, x, y) + \xi_n(x) v_n(t, x, y), \quad t \geq 0, x \in \Sigma_n, \quad (1.2)$$

with *localized initial condition* $v_n(0, \cdot, y) = \delta_y(\cdot)$.¹ Here, $\xi_n := \{\xi_n(x) : x \in \Sigma_n\}$ is a given *random potential*, $\kappa > 0$ is the *diffusion constant* and Δ_n , acting on the second coordinate (i.e. the first spatial coordinate), denotes the *Laplace operator* on Σ_n :

$$\Delta_n f(x) := \frac{1}{n} \sum_{z \sim x} (f(z) - f(x)), \quad x \in \Sigma_n,$$

where f is a function on Σ_n .

The solution of (1.2) admits the Feynman–Kac representation

$$v_n(t, x, y) = \mathbb{E}_x[\exp(\int_0^t \xi_n(X_s) ds) \mathbb{1}\{X_t = y\}], \quad (1.3)$$

where $(X_s : s \geq 0)$ is distributed as a simple random walk on Σ_n with generator $\kappa \Delta_n$ and $\mathbb{E}_x$ stands for its expectation when the walk starts at x , i.e., $X_0 = x$. Since the simple random walk on Σ_n is time reversible, we can conclude from (1.3) that

$$v_n(t, x, y) = v_n(t, y, x).$$

We will also deal with the *de-localized model*. To this end let $v_n(t, \cdot)$ be the solution of (1.2) with *de-localized initial condition* $v_n(0, \cdot) \equiv 1$. It is trivial to see that $v_n(t, y) = \sum_{x \in \Sigma_n} v_n(t, x, y)$ and that $v_n(t, y)$ admits the Feynman–Kac representation

$$v_n(t, y) = \mathbb{E}_y[\exp(\int_0^t \xi_n(X_s) ds)].$$

Literature overview. Eq. (1.2) and its variants are often referred to as the Parabolic Anderson Model (PAM). This model originates as the parabolic version of the Anderson localization problem and has found a wide range of applications in many areas such as chemical kinetics, magnetism, turbulence and population dynamics, the last being one of the motivations of this article. PAM is also attractive for mathematicians since it yields precise solutions based on the Feynman–Kac representation and the spectral analysis of the Hamiltonian operator $\kappa \Delta + \xi$. We refer the readers to the recent book [15] and the references therein for the applications of PAM and a survey of mathematical results. The main feature of (1.2) is the competition between the diffusion term that flattens the solution and the potential part that creates peaks. A characteristic of this competition is the *intermittency effect*, namely, the total mass of the solution v_n is

¹ We emphasize that the third coordinate of v_n is left invariant under the action of (1.2).

carried by a few separated regions with small diameters. Indeed, the rigorous mathematical research on PAM started with the seminal paper [12] in which intermittency was proved on the d -dimensional lattice under minimal conditions on the potential. In a follow-up paper [13] the same authors gave a description of the shape of relevant islands in terms of a variational problem in the growth rate. The potentials considered in [13] consisted of distributions with upper tails that are double exponential or slightly heavier/lighter. The asymptotic size of the islands is finite for double exponential tails and shrinks to a single site for heavier tails. This geometric picture was made precise in [11]. The growth of the solution for much heavier tails has random first order terms, and results in this direction were achieved in [19] for potentials with Pareto and exponential distributions. Later, [16] proved single site localization for the same kind of potentials and the evolution of the localization point was investigated in the context of ageing in [18].

In this work we consider PAM on the n -dimensional hypercube for a class of potentials that lead to single site localization. We will describe the growth of the solution and provide localization results. Our point of view focuses on solutions starting from the site of an extremal potential and on how the growth and localization change with time, observing a phase transition in time. Moreover, we will explore the fact the normalized solution of PAM corresponds to a mutation–selection model and explain our localization results in terms of the latter.

It is worth emphasizing that, contrary to our setup, the state space for all the results described above is the d -dimensional lattice. There are only a few works about PAM on different graphs, one being [7] where the authors study PAM on complete graph with exponentially distributed random potential. This work has been an inspiration for us as it also proves a phase transition on the growth depending on the time scales. Nonetheless working on the complete graph is much easier than on the hypercube since the exact asymptotics of the whole spectrum of eigenvalues and eigenvectors are readily available.

Overview of our results. Our first main result is an exact description of the behaviour of the solution at the location of the k th largest potential. Let $x_{1,2^n}, x_{2,2^n}, \dots, x_{2^n,2^n}$ denote the locations of the largest potential, second largest potential and so on. We denote by λ_1 the principal eigenvalue of the operator $\kappa \Delta_n + \xi_n$. For the potential we essentially assume that, for any fixed k , almost surely $\xi_n(x_{k,2^n}) \sim \theta n$ for some $\theta > 0$ and the gap between the extremal points $\xi_n(x_{1,2^n}) - \xi_n(x_{k,2^n})$ stays order of (random) constant (see Section 1.2). The behaviour of $v_n(t_n, x_{k,2^n})$ goes through a transition on time scales of order $n \log n$. To this end let $c_n = \frac{1}{2} n \log n$ and let $t_n/c_n \rightarrow \alpha \in [0, \infty]$, as $n \rightarrow \infty$. We prove that (see Section 1.3) for $\alpha < (\xi(x_{1,2^n}) - \xi(x_{k,2^n}))^{-1}$

$$v_n(t_n, x_{k,2^n}) \sim \exp\{(\xi(x_{k,2^n}) - \kappa)t_n\},$$

and for $\alpha > (\xi(x_{1,2^n}) - \xi(x_{k,2^n}))^{-1}$

$$v_n(t_n, x_{k,2^n}) \sim \exp\{\lambda_1 t_n\} \exp\{-c_n(1 + o(1))\}. \quad (1.4)$$

Hence, in the short time regime the solution at all the high peaks grows like a system without diffusion, more precisely, when the potential is shifted down by κ and the diffusion is removed. However, on the long time regimes, with the observation that (see Lemma 2.1)

$$\lambda_1 = \xi_{1,2^n} - \kappa + O(1/n^2),$$

we see that the growth is much faster. We also mention that the second term in (1.4) is the decay of the principal eigenfunction at $x_{k,2^n}$. The class of potentials we consider includes the Random Energy Model (REM) of spin glasses introduced in [4], where the potential field is formed by i.i.d. Gaussian random variables with mean 0 and variance n (see Section 1.4).

Relation to the mutation–selection model. The mutation–selection model on the hypercube with random fitness landscape is given by the solution $u_n(t, \cdot, y)$ of the following PDE

$$u_n(t, x, y) = \kappa \Delta_n u_n(t, x, y) + [\xi_n(x) - \bar{\xi}(t)] u_n(t, x, y), \quad t \geq 0, x \in \Sigma_n \quad (1.5)$$

with the localized initial condition $u_n(t, \cdot, y) = \delta_y(\cdot)$, where $\bar{\xi}(t)$ represents the mean fitness

$$\bar{\xi}(t) := \sum_{x \in \Sigma_n} u_n(t, x, y) \xi_n(x). \quad (1.6)$$

Let us briefly explain the biological meaning of the mathematical objects appearing in (1.5). Haploid genotypes are identified with linear arrangement of n sites $x = (x^{(1)}, \dots, x^{(n)})$ with each site taking values -1 or $+1$. In the multilocus context sites correspond to *loci* and the variables $x^{(i)}$ to *alleles*. From the viewpoint of molecular evolution, x corresponds to a DNA (or RNA) sequence where the nucleotides are lumped into purines (say, $+1$) and pyrimidines (say, -1). In biology literature the hypercube Σ_n is usually called the sequence space. Then the mutation–selection model given in (1.5) describes the evolution of an infinite population of haploids under the effect of mutation and selection. The population evolves in continuous time (non-overlapping generations) with mutation and selection occurring independently (parallel). $\xi_n(x)$ is the Malthusian fitness of *type* x and forms a fitness landscape, which in our case is random. Site mutations happen with rate κ/n (hence, a total rate of κ). From (1.6) it follows that $\sum_x u_n(t, x, y) = 1$, and $u_n(t, x, y)$ corresponds to the frequency of type x at time t under this evolution. Finally, note that the localized initial condition means that initially the population consists of only type y . Here, the competition between diffusion and potential discussed in PAM, corresponds to a competition between mutation and selection, two driving forces of Darwinian evolution. The mutation–selection model dates back to Wright [20]. We refer readers to the classical book [2] for an introduction to population genetics and to [10] for an excellent survey that involves the statistical physics methods used to solve mutation–selection models for a wide range of landscapes.

The motivation to consider a random fitness landscape is the following. Realistic landscapes are expected to be complex with structures such as valleys and hills [3]. Random fitness landscapes naturally form a class of complex landscapes. The simplest choice, that is an i.i.d. landscape, is also known as the *House of Cards model* and was introduced in [14] by Kingman.

It is well-known that (see e.g. [17]) the linear system v_n can be transformed to u_n via normalization by its total mass, that is,

$$u_n(t, x, y) = \frac{v_n(t, x, y)}{v_n(t, y)}.$$

In a way $v_n(t, x, y)$ can be thought as *absolute* frequencies. Hence, behaviour of the mutation–selection model is related to the localization properties of the PAM. Indeed, we will prove that (see Section 1.3) the phase transition exhibited in growth rates of v_n is translated to the behaviour of u_n as follows: on short time scales $u_n(t_n, x_{k,2^n}, x_{k,2^n}) \rightarrow 1$, whereas, on long time scales $u_n(t_n, x_{1,2^n}, x_{k,2^n}) \rightarrow 1$. In other words, a population initially consisted of $x_{k,2^n}$ type individuals stays that way for a certain threshold in time, after which it is invaded by the best fit type $x_{1,2^n}$.

The coupled model where the reproduction events are followed by mutation is known as quasispecies model, introduced by Eigen in [5]. Main feature of this model is the existence of an *error threshold*, that is, for a *single peak* landscape (a *master* sequence has a fitness $\sigma > 1$

and the rest has the same fitness 1) in the limit as the genome length $n \rightarrow \infty$ and time $t \rightarrow \infty$ the population is essentially randomly distributed over the space if the mutation rate is above a certain value, whereas for the mutation rates below this critical value the population consists of individuals close to the master sequence, what Eigen calls a *quasispecies*. Similar results were proven in [9] and [8] for the REM landscape. We have to emphasize that our model is actually not in the direction of these results. In the quasispecies models we have mentioned the mutation rate and the fitness at the highest peak are on the same scale. In our case the fitness of the highest peak is on the scale of n while the mutation rate is kept constant. Hence, we do not have the quasispecies picture. Instead, we focus on studying the evolution on intermediate time scales, that is, before the equilibrium. The phase transition we observe is on the time scale of observation rather than on the mutation rate.

The class of random fitness distributions we study gives rise to landscapes that are heavy-tailed for the dynamics of the population: population is concentrated on edge of the landscape. Moreover, the movement of population frequency from k th largest potential to the largest resembles the situation called “punctuated equilibrium” in the evolutionary theory, where the evolution stalls at an intermediate stage for a very long time. It is also interesting to note that the time the transition occurs is inversely proportional to the difference between the k th largest and the largest potential. Hence, the closer the potential to the global potential the longer the population will be stuck at this “local minima”.

Structure of the paper. In the rest of this section, we first describe precisely the potentials we study (Section 1.2), then we state our main results on the growth rates and localization (Section 1.3). Finally, we show that the REM landscape satisfies our assumptions on the potential field (Section 1.4). Section 2 is devoted to the proofs of the main results. Along these proofs we will make use of two lemmas on spectral properties of the operator $\kappa \Delta_n + \xi_n$. These lemmas are proved in the concluding Section 3.

Notation. Throughout the paper we use the notations o , O , Θ , $\ll$, $\gg$, $\sim$ for any two sequences f_n, g_n as follows. We write $f_n = o(g_n)$, $f_n \ll g_n$ or $g_n \gg f_n$ if $f_n/g_n \rightarrow 0$ as $n \rightarrow \infty$; $f_n = O(g_n)$ if $\limsup_{n \rightarrow \infty} f_n/g_n < \infty$; $f_n = \Theta(g_n)$ if there are positive constants C_1, C_2 such that $C_1 g_n \leq f_n \leq C_2 g_n$ for all n large enough; and $f_n \sim g_n$ if $f_n/g_n \rightarrow 1$ as $n \rightarrow \infty$. Moreover, for constants in our estimates we use the letter C freely as long as it does not appear at the end result.

1.2. The potential

For each n , $\xi_n = \{\xi_n(x) : x \in \Sigma_n\}$ is a collection of i.i.d. real-valued random variables whose common cumulative distribution function is denoted by G_n . We assume that G_n is continuous, i.e. $\xi_n(x)$ has no atoms. We will further require mild assumptions which allow to control the typical growth of the extremes of the potential and of its neighbourhoods (Assumptions (R1) and (L)), as well as the gaps between these extremes (see (R2)). To state these assumptions we first introduce the required ingredients together with a classical coupling construction which will be useful along the proofs. Define

$$\varphi_n(r) := \log \frac{1}{1 - G_n(r)}, \quad r \in \mathbb{R},$$

and its left-continuous inverse

$$\psi_n(s) := \min\{r : \varphi_n(r) \geq s\}, \quad s > 0.$$

Let $\eta_n = \{\eta_n(x) : x \in \Sigma_n\}$ be an i.i.d. field of mean 1 exponential random variables. Then $\psi_n(\eta_n) \stackrel{d}{=} \xi_n$, and from now on we assume without loss of generality that $\xi_n = \psi_n(\eta_n)$. Note that since ψ_n is strictly increasing the sites ordered according to their potentials coincide for the two fields. More precisely, we can label the vertices of Σ_n by $x_{1,2^n}, \dots, x_{2^n,2^n}$ so that

$$\xi_n(x_{1,2^n}) =: \xi_{1,2^n} > \xi_n(x_{2,2^n}) =: \xi_{2,2^n} > \dots > \xi_n(x_{2^n,2^n}) =: \xi_{2^n,2^n}$$

and

$$\eta_n(x_{1,2^n}) =: \eta_{1,2^n} > \eta_n(x_{2,2^n}) =: \eta_{2,2^n} > \dots > \eta_n(x_{2^n,2^n}) =: \eta_{2^n,2^n}.$$

Note that the above inequalities are strict because G_n is continuous. Let $\sigma_i, i \in \mathbb{N}$, be an independent sequence of random variables where σ_i is exponentially distributed with intensity i . It is well-known that (see e.g. Section I.6 of [6])

$$(\eta_{1,2^n}, \eta_{2,2^n}, \dots, \eta_{2^n,2^n}) \stackrel{d}{=} (\sigma_1 + \dots + \sigma_{2^n}, \sigma_2 + \dots + \sigma_{2^n}, \dots, \sigma_{2^n}).$$

From now on we describe the field η_n (and in turn the field ξ_n) through the sequence $(\sigma_i, i \in \mathbb{N})$. Namely, η_n is given by its order statistics $(\eta_{1,2^n}, \eta_{2,2^n}, \dots, \eta_{2^n,2^n})$ coupled to $(\sigma_i, i \in \mathbb{N})$ via $\eta_{i,2^n} = \sigma_i + \dots + \sigma_{2^n}$. We denote by P and E the distribution and expectation in this common probability space, respectively.

Since

$$P(\eta_{1,2^n} \geq nc \log 2) \leq 2^{-n(c-1)}, \quad \forall c > 1,$$

and

$$P(\eta_{1,2^n} \leq nc \log 2) \leq \exp(-2^{n(1-c)}), \quad \forall c < 1,$$

by an application of Borel–Cantelli lemma P -a.s.

$$\lim_{n \rightarrow \infty} \frac{\eta_{1,2^n}}{n} = \log 2. \tag{1.7}$$

We have $\eta_{1,2^n} - \eta_{k,2^n} = \sigma_1 + \dots + \sigma_{k-1}$. Hence, P -a.s. for any $k \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \frac{\eta_{k,2^n}}{n} = \log 2.$$

Therefore, the extremes of the field η_n all grow like $n \log 2$ and the gap between extremal points are (random) constants, i.e.,

$$\eta_{k,2^n} - \eta_{l,2^n} = \sigma_k + \dots + \sigma_{l-1}, \quad \text{for any } k < l. \tag{1.8}$$

We now list our assumptions for the field ξ_n . The first set of assumptions is about the extremes of the field and concerns only the right-tail of the distribution of ξ_n in terms of ψ_n . The following assumption identifies the growth rate of the extremes.

Assumption (R1) (Growth of Extremes). For any $a > 0$

$$\psi_n(an) \sim f(a)n$$

where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a strictly increasing continuous function. We define $\theta := f(\log 2)$.

Hence, by (1.7), $\xi_{1,2^n}$ grows like θn . The choice of this growth rate is arbitrary but it makes the representation cleaner and this is the actual case for REM.

Our second assumption on the right-tail is more crucial, it guarantees that, like in the exponential field, the gaps between extremes are order of (random) constants.

Assumption (R2) (*Gaps Between Extremes*). For any sequence $s_n \sim \theta n$, for any $c \in \mathbb{R}$,

$$\psi_n(s_n + c) - \psi_n(s_n) \rightarrow g(c),$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is such that $g(c) \neq 0$ for any $c \neq 0$.

Therefore, by (1.7) and (1.8), P -a.s.

$$\xi_{k,2^n} - \xi_{l,2^n} = g(\sigma_k + \cdots + \sigma_{l-1}) + o(1), \quad \text{for any } k < l.$$

Recall that $g(c) > 0$ for $c > 0$, that is, the gap above does not vanish. For convenience we define

$$\Delta \xi_{k,l} := g(\sigma_k + \cdots + \sigma_{l-1}), \quad \text{for } k < l,$$

and set $\Delta \xi_{k,l} = 0$ for $k \geq l$. Note that $\Delta \xi_{k,l}$ does not depend on n .

For further reference, we sum up the implications of Assumptions (R1) and (R2) in a lemma

Lemma 1.1. *Let Assumptions (R1) and (R2) be satisfied. Then P -a.s. for any $k, l \in \mathbb{N}$*

(i)

$$\lim_{n \rightarrow \infty} \frac{\xi_{k,2^n}}{n} = \theta;$$

(ii)

$$\xi_{k,2^n} - \xi_{l,2^n} = \Delta \xi_{k,l} + o(1), \quad \text{for any } k < l.$$

Our last assumption concerns the left tail of the distribution of ξ_n .

Assumption (L) (*Small Sizes Control*). There exists a sequence $l_n \ll n$ for which

$$\sum_{n \in \mathbb{N}} n G_n(-l_n) < \infty.$$

Essentially, the above assumption yields that there are enough path between extremal points that avoid sites with potential smaller than $-l_n$. Moreover, it guarantees that the neighbours of extremal points also have potential not smaller than $-l_n$.

1.3. Results

Let

$$c_n := \frac{1}{2} n \log n,$$

and consider time scales t_n such that

$$\frac{t_n}{c_n} \xrightarrow{n \rightarrow \infty} \alpha \in [0, \infty].$$

We denote by λ_1 the principal eigenvalue² of the operator $\kappa \Delta_n + \xi_n$.

² Note that with a slight abuse of notation we drop the dependence on n in λ_1 .

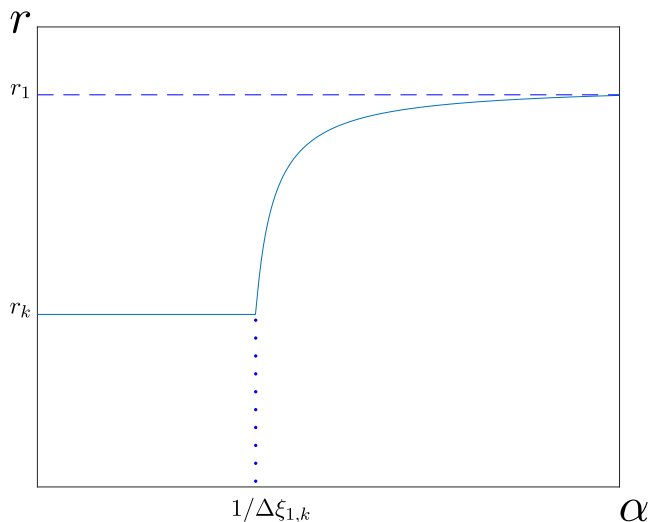


Fig. 1. Plot of growth rate with respect to time. α -axis is time normalized by c_n , that is, $t_n = \alpha c_n$. r -axis is the growth rate shifted by θn . Here $r_1 = \xi_{1,2^n} - \kappa$ and $r_k = \xi_{k,2^n} - \kappa$.

Theorem 1.2 (Growth Rates). Let Assumptions (R1), (R2) and (L) be satisfied. Then P -a.s. for any $k \in \mathbb{N} \setminus \{1\}$ as $n \rightarrow \infty$

$$v_n(t_n, x_{k,2^n}) \sim \begin{cases} \exp\{(\xi_{k,2^n} - \kappa)t_n\} & \text{if } \alpha < 1/\Delta\xi_{1,k}, \\ \exp\{\lambda_1 t_n - c_n(1 + o(1))\} & \text{if } \alpha > 1/\Delta\xi_{1,k}. \end{cases}$$

Moreover, for any $\alpha \in [0, \infty]$

$$v_n(t_n, x_{1,2^n}) \sim \exp\{\lambda_1 t_n\}. \quad (1.9)$$

In words, on the short time scales the solution grows by the single peak, which can be seen as the model with no diffusion and potential $\xi_{k,2^n} - \kappa$, and on the other hand, for longer time scales the growth is larger which is determined by the principal eigenvalue and a correction term given by the decay of the principal eigenfunction at $x_{k,2^n}$ (see Lemma 2.2). We also mention that (see Lemma 2.1)

$$\lambda_1 = \xi_{1,2^n} - \kappa + \Theta(1/n^2).$$

Let us consider the time scale of phase transition and for simplicity take $t_n = \alpha c_n$. Then the growth rate, to be precise the ratio of the term in the exponentials to the time scale t_n , is $\xi_{k,2^n} - \kappa$ for $\alpha < 1/\Delta\xi_{1,k}$ and $\xi_{1,2^n} - \kappa - 1/\alpha + o(1)$ for $\alpha > 1/\Delta\xi_{1,k}$, see also Fig. 1.

Theorem 1.3 (Localization). Let Assumptions (R1), (R2) and (L) be satisfied. Then P -a.s. for any $k \in \mathbb{N} \setminus \{1\}$ as $n \rightarrow \infty$

(i) if $\alpha < 1/\Delta\xi_{1,k}$

$$u_n(t_n, x_{k,2^n}, x_{k,2^n}) \longrightarrow 1;$$

(ii) if $\alpha > 1/\Delta\xi_{1,k}$

$$u_n(t_n, x_{1,2^n}, x_{k,2^n}) \longrightarrow 1.$$

Moreover, for any $\alpha \in [0, \infty]$ P -a.s. as $n \rightarrow \infty$

$$u_n(t_n, x_{1,2^n}, x_{1,2^n}) \longrightarrow 1. \quad (1.10)$$

1.4. REM landscape

Our main application is the REM landscape, that is, ξ_n is a collection of i.i.d. centred Gaussian random variables with variance n .

Proposition 1.4. *The REM landscape satisfies Assumptions (R1), (R2) and (L).*

Proof. Let Z denote a standard normal random variable. Then

$$1 - G_n(r) = P(Z \geq r/\sqrt{n}) = \frac{1}{2\sqrt{\pi}} \int_{r/\sqrt{n}}^{\infty} e^{-x^2/2} dx.$$

We use the following trivial bounds

$$e^{-r^2/2} \left(\frac{1}{r} - \frac{1}{r^3} \right) \leq \int_r^{\infty} e^{-x^2/2} dx \leq e^{-r^2/2} \frac{1}{r}, \quad \forall r > 0.$$

Using the above and the definitions of φ_n and ψ_n , we get that for any sequence $s_n \sim an$, $a > 0$ and $c \in \mathbb{R}$

$$\psi_n(s_n + c) = \sqrt{2ns_n} + \frac{c}{\sqrt{2s_n/n}} - \frac{\log \sqrt{2\pi}}{\sqrt{2s_n/n}} - \frac{\log \sqrt{2s_n}}{\sqrt{2s_n/n}} + o(1). \quad (1.11)$$

Setting $c = 0$, Assumption (R1) follows with $f(a) = \sqrt{2a}$. Then by definition $\theta = \sqrt{2 \log 2}$. Moreover,

$$\psi_n(s_n + c) - \psi_n(s_n) = c/\sqrt{2a} + o(1),$$

which yields, for $a = \theta$, Assumption (R2) with $g(c) = c/\sqrt{2\theta}$. Using (1.11) we have

$$G_n(-l_n) \leq e^{-l_n^2/(2n)}.$$

Setting $l_n = n^c$, for some $c \in (1/2, 1)$, Assumption (L) is satisfied. $\square$

2. Proof of Theorems 1.3 and 1.2

We describe the growth of $v_n(t_n, \cdot, x_{k,2^n})$ by using spectral properties of the operator $\kappa \Delta_n + \xi_n$ with zero boundary conditions on certain vertices of extremal potential. To this end we have two main ingredients: firstly, precise descriptions of principal eigenvalue, spectral gap and localization of the principal eigenvector for the aforementioned operators; secondly, a general mechanism allowing us to turn these spectral properties to estimates for v_n . For the latter, we follow the general framework established in [11].

We introduce the spectral objects we use for our estimates. For $l \in \mathbb{N}$ set $\Gamma_l = \{x_{1,2^n}, \dots, x_{l,2^n}\}$. For $x_{i,2^n} \in \Gamma_l$, $i \in \{1, \dots, l\}$, consider the principal eigenvalue and (positive) eigenfunction of the operator $\kappa \Delta_n + \xi_n$ with zero boundary conditions on $\Gamma_l \setminus \{x_{i,2^n}\}$, denoted by $\lambda_{i,l}$ and $v_{i,l}$, respectively, where $v_{i,l}$ is normalized so that $v_{i,l}(x_{i,2^n}) = 1$. Let $g_{i,l}$ denote the corresponding spectral gap, that is, the difference between the principal eigenvalue and the

second largest eigenvalue. We write λ_i and v_i for $\lambda_{i,i}$ and $v_{i,i}$, respectively. Note that as before we do not use n in the notation for eigenvalues and eigenvectors. Finally, for $A \subseteq \Sigma_n$ we define the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A\},$$

and write simply τ_x for $\tau_{\{x\}}$.

The following two lemmas contain the main spectral results. We postpone their proofs to the next section.

Lemma 2.1. *Let Assumptions (R1), (R2) and (L) be satisfied. Then P -a.s. for any $k \in \mathbb{N}$ for all $1 \leq i \leq l \leq k$*

$$\lambda_{i,l} = \xi_{i,2^n} - \kappa + \Theta(1/n^2)$$

and

$$g_{i,l} \geq \xi_{i,2^n} - \xi_{l+1,2^n}.$$

Lemma 2.2. *Let Assumptions (R1), (R2) and (L) be satisfied. Then P -a.s. for any $k \in \mathbb{N}$ the followings are true:*

- (i) $\sum_{x \neq x_i} v_{i,l}(x) \rightarrow 0$ and $\|v_{i,l}\|_2^2 \rightarrow 1$, for all $1 \leq i \leq l \leq k$;
- (ii) $v_{i,l}(x_{k,2^n}) = \exp(-c_n(1 + o(1)))$, for all $1 \leq i \leq l < k$.

Now we describe a general mechanism that allows us to use these spectral properties to get certain estimates. For the following argument, randomness is not relevant and one can take a general connected graph Σ with Δ denoting the generator of the nearest neighbour simple random walk. Consider a potential $V : \Sigma \rightarrow \mathbb{R}$ and subsets $\mathcal{T}, \Lambda \subset \Sigma$ such that $\mathcal{T} \cap \Lambda = \emptyset$. Let λ_z be the principal eigenvalue of the operator $\kappa\Delta + V$ on $(\Sigma \setminus (\mathcal{T} \cup \Lambda)) \cup \{z\}$ with zero boundary conditions (which is equivalent to set V to $-\infty$ on $(\mathcal{T} \cup \Lambda) \setminus \{z\}$). For $z \in \mathcal{T}$, let v_z be the corresponding (positive) eigenfunction normalized so that $v_z(z) = 1$. Then v_z admits the probabilistic representation (see e.g. [15], Chapter 2):

$$v_z(x) = \mathbb{E}_x[\exp(\int_0^{\tau_z} [V(X_s) - \lambda_z] ds) \mathbb{1}\{\tau_z = \tau_{\mathcal{T}} < \tau_{\Lambda}\}]. \quad (2.1)$$

Define

$$\omega(t, x, y) = \mathbb{E}_x\left[\exp\left(\int_0^t V(X_s) ds\right) \mathbb{1}\{X_t = y\} \mathbb{1}\{\tau_{\mathcal{T}} \leq t\} \mathbb{1}\{\tau_{\Lambda} > t\}\right]. \quad (2.2)$$

Lemma 2.3. *For any $t > 0$*

$$\omega(t, x, y) \leq \sum_{z \in \mathcal{T}} \omega(t, z, y) v_z(x) \|v_z\|_2^2. \quad (2.3)$$

This lemma is a version of Theorem 4.1 in [11] but since the results in [11] are written for $\mathbb{Z}^d$ for the sake of completeness we give a proof.

Proof of Lemma 2.3. We start by claiming that for any $u \in [0, t]$ and $z \in \mathcal{T}$

$$\mathbb{E}_z\left[\exp\left(\int_0^{t-u} V(X_s) ds\right) \mathbb{1}\{X_{t-u} = y\} \mathbb{1}\{\tau_{\Lambda} > t-u\}\right] \leq e^{-\lambda_z u} \|v_z\|_2^2 \omega(t, z, y). \quad (2.4)$$

Indeed, note that $\omega(t, z, y)$ admits the following lower bound

$$\begin{aligned}\omega(t, z, y) &= \mathbb{E}_z \left[\exp \left(\int_0^t V(X_s) ds \right) \mathbb{1}\{X_t = y\} \mathbb{1}\{\tau_A > t\} \right] \\ &\geq \mathbb{E}_z \left[\exp \left(\int_0^u V(X_s) ds \right) \mathbb{1}\{X_u = z\} \mathbb{1}\{\tau_A > u\} \mathbb{1}\{\tau_{\mathcal{T} \setminus \{z\}} > u\} \right] \\ &\quad \times \mathbb{E}_z \left[\exp \left(\int_0^{t-u} V(X_s) ds \right) \mathbb{1}\{X_{t-u} = y\} \mathbb{1}\{\tau_A > t - u\} \right],\end{aligned}\quad (2.5)$$

where the equality uses that $z \in \mathcal{T}$ and the inequality follows by Markov property. Then, by the spectral decomposition

$$\mathbb{E}_z \left[\exp \left(\int_0^u V(X_s) ds \right) \mathbb{1}\{X_u = z\} \mathbb{1}\{\tau_A > u\} \mathbb{1}\{\tau_{\mathcal{T} \setminus \{z\}} > u\} \right] \geq e^{u\lambda_z} \|v_z\|_2^{-2}, \quad (2.6)$$

which implies (2.4) through (2.5). Since we use the type of estimate in (2.6) throughout the rest of this section, let us explain in detail how to derive it. Let

$$h(t, x) := \mathbb{E}_x \left[\exp \left(\int_0^t V(X_s) ds \right) \mathbb{1}\{X_t = z\} \mathbb{1}\{\tau_A > t\} \mathbb{1}\{\tau_{\mathcal{T} \setminus \{z\}} > t\} \right].$$

Then h solves the parabolic equation

$$\frac{\partial}{\partial t} h(t, x) = \kappa \Delta_n h(t, x) + \xi_n(x) h(t, x), \quad t \geq 0, \quad x \in (\Sigma \setminus (\mathcal{T} \cup A)) \cup \{z\}$$

with initial condition $h(0, \cdot) = \delta_z(\cdot)$ and boundary conditions

$$h(t, x) = 0, \quad \forall t \geq 0, \quad x \in \mathcal{T} \cup A \setminus \{z\}.$$

Therefore, h can be given using the spectrum of $\kappa \Delta + \xi$ with zero boundary conditions on $\mathcal{T} \cup A \setminus \{z\}$. We already defined λ_z and v_z as the principal eigenvalue and eigenvector, respectively. Let $\lambda^{(i)}$ and $v^{(i)}$, $i = 2, \dots, |V| - |A| - |\mathcal{T}| + 1 := L$, denote the rest of eigenvalues and the corresponding eigenvectors, respectively, in the spectrum. Here, eigenvectors have the usual l_2 normalization: $\|v^{(i)}\|_2^2 = 1$. Then, with the initial condition $h(0, \cdot) = \delta_z(\cdot)$, we get

$$h(t, x) = e^{\lambda_z t} \frac{v_z(x)v_z(z)}{\|v_z\|_2^2} + \sum_{i=2}^L e^{\lambda^{(i)} t} v^{(i)}(x)v^{(i)}(z).$$

For $x = z$ all the coefficients in the second sum become $(v^{(i)}(z))^2$, thus, non-negative. Since we chose $v_z(z) = 1$, we arrive at that $h(t, x) \geq e^{t\lambda_z} \|v_z\|_2^{-2}$, that is, (2.6).

Having established (2.4), we now continue with the proof of (2.3). By the definition (2.2) we have

$$\begin{aligned}\omega(t, x, y) &= \sum_{z \in \mathcal{T}} \mathbb{E}_x \left[\exp \left(\int_0^{\tau_{\mathcal{T}}} V(X_s) ds \right) \mathbb{1}\{X_{\tau_{\mathcal{T}}} = z\} \mathbb{1}\{\tau_{\mathcal{T}} \leq t\} \mathbb{1}\{\tau_A > t\} \right. \\ &\quad \left. \times \mathbb{E}_z \left[\exp \left(\int_0^{t-\tau_{\mathcal{T}}} V(X_s) ds \right) \mathbb{1}\{X_{t-\tau_{\mathcal{T}}} = y\} \mathbb{1}\{\tau_A > t - \tau_{\mathcal{T}}\} \right] \right]\end{aligned}$$

Since on $\mathbb{1}\{X_{\tau_{\mathcal{T}}} = z\}$ we have $\tau_{\mathcal{T}} = \tau_z$, we can replace $\mathbb{1}\{X_{\tau_{\mathcal{T}}} = z\}$ by $\mathbb{1}\{\tau_z = \tau_{\mathcal{T}}\}$, and $\tau_{\mathcal{T}} \leq t < \tau_A$ implies that $\tau_{\mathcal{T}} < \tau_A$ and we get an upper bound if we replace

$\mathbb{1}\{\tau_T \leq t\} \mathbb{1}\{\tau_A > t\}$ by $\mathbb{1}\{\tau_T < \tau_A\}$. Hence,

$$\begin{aligned} \omega(t, x, y) &\leq \sum_{z \in \mathcal{T}} \mathbb{E}_x \left[\exp \left(\int_0^{\tau_z} V(X_s) ds \right) \mathbb{1}\{\tau_z = \tau_T < \tau_A\} \right. \\ &\quad \left. \times \mathbb{E}_z \left[\exp \left(\int_0^{t-\tau_z} V(X_s) ds \right) \mathbb{1}\{X_{t-\tau_z} = y\} \mathbb{1}\{\tau_A > t - \tau_z\} \right] \right] \\ &\leq \sum_{z \in \mathcal{T}} \mathbb{E}_x \left[\exp \left(\int_0^{\tau_z} V(X_s) ds \right) \mathbb{1}\{\tau_z = \tau_T < \tau_A\} e^{-\lambda_z \tau_z} \|v_z\|_2^2 \omega(t, z, y) \right] \\ &= \sum_{z \in \mathcal{T}} \omega(t, z, y) \|v_z\|_2^2 v_z(x). \end{aligned}$$

For the inequality on the second line we used (2.4) and for the equality on the third line we used the representation of v_z given in (2.1). $\square$

We divide the expectation in the Feynman–Kac representation (1.3) of $v_n(t, x, x_{k,2^n})$ into two parts: expectation along the paths that visit Γ_{k-1} and those that do not. Namely,

$$v(t, x, x_{k,2^n}) = \omega_k(t, x, x_{k,2^n}) + \tilde{\omega}_k(t, x, x_{k,2^n})$$

where

$$\omega_k(t, x, y) := \mathbb{E}_x \left[e^{\int_0^t \xi_n(X_s) ds} \mathbb{1}\{X_t = y\} \mathbb{1}\{\tau_{\Gamma_{k-1}} \leq t\} \right], \quad (2.7)$$

and

$$\tilde{\omega}_k(t, x, y) := \mathbb{E}_x \left[e^{\int_0^t \xi_n(X_s) ds} \mathbb{1}\{X_t = y\} \mathbb{1}\{\tau_{\Gamma_{k-1}} > t\} \right].$$

We first prove the following.

Lemma 2.4. *P-a.s. for any $k \in \mathbb{N}$ as $n \rightarrow \infty$*

$$\tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) \sim e^{\lambda_k t_n}$$

Proof of Lemma 2.4. Using the spectral decomposition of the operator $\kappa \Delta_n + \xi_n$ with zero boundary conditions on Γ_{k-1} , as discussed before, and part (i) of Lemma 2.2 we get

$$\tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) \geq e^{\lambda_k t_n} \|v_k\|_2^{-2} \sim e^{\lambda_k t_n}$$

We use the spectral gap to get the upper bound

$$\tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) \leq e^{\lambda_k t_n} \|v_k\|_2^{-2} + e^{\lambda_k t_n} e^{-g_{k,k} t_n} \|\delta_{x_k}\|_2^2.$$

Since $\|\delta_{x_{k,2^n}}\|_2 = 1$, using Lemma 1.1 part (iii), Lemmas 2.1 and 2.2 part (ii) we are finished with the proof. $\square$

Note that for $k = 1$, $\tilde{\omega}_1(t, x, y) = v_n(t, x, y)$. Hence, the above lemma gives

$$v_n(t_n, x_{1,2^n}, x_{1,2^n}) \sim e^{\lambda_1 t_n}. \quad (2.8)$$

We need the following result. Recall that we have defined $c_n = \frac{1}{2} n \log n$.

Lemma 2.5. *P-a.s. for any $k \in \mathbb{N}$ and for any $i = 1, \dots, k-1$ as $n \rightarrow \infty$*

$$v_n(t_n, x_{i,2^n}, x_{k,2^n}) \leq e^{\lambda_1 t_n} \exp(-2c_n(1 + o(1))) + e^{\lambda_i t_n} \exp(-c_n(1 + o(1))). \quad (2.9)$$

Proof of Lemma 2.5. We prove (2.9) using strong induction. For $k = 1$ there is nothing to check. Now assume that (2.9) is true for $1, \dots, k-1$. Let $i \in \{2, \dots, k-1\}$. We use Lemma 2.3 with $\mathcal{T} = \Gamma_{i-1}$ and $\Lambda = \emptyset$. In this case the corresponding ω defined in (2.2) coincides with ω_i defined as in (2.7). Note that we have $\omega_i(t, x, y) = v_n(t, x, y)$ if $x \in \Gamma_{i-1}$. Hence, using Lemma 2.3

$$w_i(t_n, x_{i,2^n}, x_{k,2^n}) = w_i(t_n, x_{k,2^n}, x_{i,2^n}) \leq \sum_{j=1}^{i-1} v_n(t_n, x_{j,2^n}, x_{i,2^n}) v_{j,i-1}(x_{k,2^n}) \|v_{j,i-1}\|_2^2$$

By part (ii) of Lemma 2.2, since $i < k$, we have $v_{j,i-1}(x_{k,2^n}) = \exp(-c_n(1 + o(1)))$ for $j = 1, \dots, i-1$ and by part (i) of the same Lemma we have $\|v_{j,i-1}\|_2^2 \sim 1$. By the strong induction step we have $v(t_n, x_{j,2^n}, x_{i,2^n}) \leq e^{\lambda_1 t_n} \exp(-c_n(1 + o(1)))$. Hence,

$$w_i(t_n, x_{i,2^n}, x_{k,2^n}) \leq e^{\lambda_1 t_n} \exp(-2c_n(1 + o(1))).$$

Now we use Lemma 2.3 with $\mathcal{T} = \{x_{i,2^n}\}$ and $\Lambda = \Gamma_{i-1}$. Since $\mathbb{1}\{X_t = x_{i,2^n}\} \mathbb{1}\{\tau_{x_{i,2^n}} \leq t\} = \mathbb{1}\{X_t = x_{i,2^n}\}$, $\omega(t, \cdot, x_{i,2^n})$ defined in (2.2) coincides with $\tilde{\omega}_i(t, \cdot, x_{i,2^n})$, and using Lemma 2.3 we get

$$\tilde{\omega}_i(t_n, x, x_{i,2^n}) \leq \tilde{\omega}_i(t_n, x_{i,2^n}, x_{i,2^n}) v_i(x) \|v_i\|_2^2. \quad (2.10)$$

Hence, by parts (i) and (ii) of Lemmas 2.2 and 2.4 we have

$$\tilde{\omega}_i(t_n, x_{k,2^n}, x_{i,2^n}) \leq e^{\lambda_i t_n} \exp(-c_n(1 + o(1))).$$

Since $v_n = \omega_i + \tilde{\omega}_i$, we have proved that (2.9) holds true for $i = 2, \dots, k-1$. For $i = 1$ recall that $v_n = \tilde{\omega}_1$. Similar to how we arrived at (2.10) we get

$$v_n(t_n, x, x_{1,2^n}) \leq v_n(t_n, x_{1,2^n}, x_{1,2^n}) v_1(x) \|v_1\|_2^2. \quad (2.11)$$

Hence, using once again part (ii) of Lemmas 2.2 and 2.4 we have

$$v_n(t_n, x_{1,2^n}, x_{k,2^n}) \leq e^{\lambda_1 t_n} \exp(-c_n(1 + o(1)))$$

This completes the proof. $\square$

Lemma 2.6. *P-a.s. for any $k \in \mathbb{N}$ as $n \rightarrow \infty$*

$$v_n(t_n, x_{k,2^n}, x_{k,2^n}) \leq e^{\lambda_1 t_n} \exp(-2c_n(1 + o(1))) + e^{\lambda_k t_n}. \quad (2.12)$$

Proof of Lemma 2.6. Using Lemma 2.3 with $\Gamma = \Gamma_{k-1}$ and $\Lambda = \emptyset$ we get

$$\omega_k(t_n, x_{k,2^n}, x_{k,2^n}) \leq \sum_{i=1}^{k-1} v_n(t_n, x_{i,2^n}, x_{k,2^n}) v_{i,k-1}(x_{k,2^n}).$$

Hence, using part (ii) of Lemmas 2.2 and 2.5 we have

$$\omega_k(t_n, x_{k,2^n}, x_{k,2^n}) \leq e^{\lambda_1 t_n} \exp(-2c_n(1 + o(1))).$$

Finally, Lemma 2.4 finishes the proof. $\square$

Lemma 2.7. *P-a.s. for any $k \in \mathbb{N}$ as $n \rightarrow \infty$*

$$v_n(t_n, x_{1,2^n}, x_{k,2^n}) \geq e^{\lambda_1 t_n} \exp(-c_n(1 + o(1))).$$

Proof of Lemma 2.7. Note that the description in (2.1) gives

$$v_1(x) = \mathbb{E}_x \left[\exp \left(\int_0^{\tau_{x_{1,2^n}}} [\xi_n(X_s) - \lambda_1] ds \right) \right].$$

Hence, using the Feynman–Kac formula, the spectral decomposition of $\kappa \Delta_n + \xi_n$ we get

$$\begin{aligned} v_n(t_n, x, x_{1,2^n}) &= \mathbb{E}_x \left[e^{\int_0^{t_n} \xi_n(X_s) ds} \mathbb{1}_{\{X_{t_n} = x_{1,2^n}\}} \right] \\ &= \mathbb{E}_x \left[e^{\int_0^{\tau_{x_{1,2^n}}} \xi_n(X_s) ds} E_{x_{1,2^n}} [e^{\int_0^{t_n - \tau_{x_{1,2^n}}} \xi_n(X_s) ds} \mathbb{1}_{\{X_{t_n - \tau_{x_{1,2^n}}} = x_{1,2^n}\}}] \right] \\ &\geq \mathbb{E}_x \left[e^{\int_0^{\tau_{x_{1,2^n}}} \xi_n(X_s) ds} e^{\lambda_1(t_n - \tau_{x_{1,2^n}})} \|v_1\|_2^{-2} \right] \\ &= e^{t_n \lambda_1} \|v_1\|_2^{-2} \mathbb{E}_x \left[e^{\int_0^{\tau_{x_{1,2^n}}} (\xi_n(X_s) - \lambda_1) ds} \right] = e^{t_n \lambda_1} \|v_1\|_2^{-2} v_1(x). \end{aligned}$$

Then, parts (i) and (ii) of Lemma 2.2 finish the proof. $\square$

Note that, Lemmas 2.5 and 2.7 yield that P -a.s. for any $k \in \mathbb{N} \setminus \{1\}$ as $n \rightarrow \infty$

$$v_n(t_n, x_{1,2^n}, x_{k,2^n}) = e^{\lambda_1 t_n} \exp(-c_n(1 + o(1))) \quad (2.13)$$

Now we are ready to prove the main results.

Proof of Theorem 1.3 and 1.2. We first prove the statements for $k = 1$, namely, (1.9) and (1.10). Recalling (2.11) and that $v_n(t, x_{1,2^n}) = \sum_{x \in \Sigma_n} v(t, x, x_{1,2^n})$ we have

$$\sum_{x \neq x_{1,2^n}} v_n(t, x, x_{1,2^n}) \leq v_n(t, x_{1,2^n}, x_{1,2^n}) \sum_{x \neq x_{1,2^n}} v_1(x)$$

Hence, using part (i) of Lemma 2.2 as $n \rightarrow \infty$ we have $v_n(t_n, x_{1,2^n}) \sim v_n(t_n, x_{1,2^n}, x_{1,2^n})$, that is, (1.10). Finally, (2.8) finishes the proof of (1.9). Now we assume $k \in \mathbb{N} \setminus \{1\}$. Using Lemma 2.3 with $\mathcal{T} = \Gamma_{k-1}$ and $\Lambda = \emptyset$ and Lemma 2.5 we have as $n \rightarrow \infty$

$$\begin{aligned} \omega_k(t_n, x, x_{k,2^n}) &\leq v_n(t_n, x_{1,2^n}, x_{k,2^n}) v_{1,k-1}(x) + \sum_{i=2}^{k-1} v_n(t_n, x_{i,2^n}, x_{k,2^n}) v_{i,k-1}(x) \\ &\leq v_n(t_n, x_{1,2^n}, x_{k,2^n}) v_{1,k-1}(x) + \\ &\quad \left[e^{\lambda_1 t_n} \exp(-2c_n(1 + o(1))) + e^{\lambda_2 t_n} \exp(-c_n(1 + o(1))) \right] \sum_{i=2}^{k-1} v_{i,k-1}(x). \end{aligned} \quad (2.14)$$

The same reasoning we used to get (2.10) yields

$$\tilde{\omega}_k(t_n, x, x_{k,2^n}) \leq \tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) v_k(x) \|v_k\|_2^2.$$

We are now in shape to conclude the proofs, for which we treat short and long time regimes separately.

Short time regime: The key point is that in this time regime, by Lemma 2.1,

$$e^{\lambda_1 t_n} \exp(-c_n(1 + o(1))) \ll e^{\lambda_k t_n}. \quad (2.15)$$

Recall that $v_n = \omega_k + \tilde{\omega}_k$. Due to (2.15), Lemmas 2.4 and 2.6 yield

$$v_n(t_n, x_{k,2^n}, x_{k,2^n}) \sim e^{\lambda_k t_n}.$$

For the second item on the right hand side of the last inequality in (2.14), (2.15) gives

$$\left[e^{\lambda_1 t_n} \exp(-2c_n(1+o(1))) + e^{\lambda_2 t_n} \exp(-c_n(1+o(1))) \right] \ll e^{\lambda_k t_n}.$$

By part (i) of Lemma 2.2 we have $\sum_x \sum_{i=2}^{k-1} v_{i,k-1}(x) \leq k(1+o(1))$. Via Lemma 2.5 and (2.15) we get $v_n(t_n, x_{1,2^n}, x_{k,2^n}) \ll e^{\lambda_k t_n}$. Hence, applying again part (i) and (ii) of Lemma 2.2 we have

$$\sum_x \omega_k(t_n, x, x_k) \ll v_n(t_n, x_k, x_k)$$

Hence, by part (i) of Lemma 2.2

$$\sum_{x \neq x_{k,2^n}} \tilde{\omega}_k(t_n, x, x_{k,2^n}) \leq \tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) \sum_{x \neq x_{k,2^n}} v_k(x) \|v_k\|_2^2 \ll \tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n})$$

and we get $v_n(t_n, x_{k,2^n}) \sim \tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n})$. Since $\tilde{\omega}_k(t, x_k, x_k) \leq v_n(t, x_k, x_k)$ we reach at

$$u(t_n, x_{k,2^n}, x_{k,2^n}) = \frac{v_n(t_n, x_{k,2^n}, x_{k,2^n})}{v_n(t_n, x_{k,2^n})} \longrightarrow 1.$$

This finishes the proof of the statement in Theorem 1.3 concerning short time scales. By Lemma 2.4 we have $\tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) \sim e^{\lambda_k t_n}$ and by Lemma 2.1 $\lambda_k = \xi_k - \kappa + \Theta(1/n^2)$. Since on short time scales $t_n \ll n^2$ we have $e^{\lambda_k, 2^n 2^n} \sim e^{(\xi_k, 2^n - \kappa)t_n}$. Hence, we are finished with the proof of Theorem 1.2 for short time scales.

Long time regime: In this time regime, by Lemma 2.1,

$$e^{\lambda_1 t_n} \exp(-c_n(1+o(1))) \gg e^{\lambda_k t_n}.$$

By Lemma 2.1 and part (ii) of Lemma 1.1 there exist random positive constants C and C' such that $\lambda_1 - \lambda_2 > C$ and $\lambda_1 - \lambda_k > C'$. By the latter and Lemma 2.1, in this regime we have $t_n > C''n \log n$ for some random positive constant C'' . Therefore, for some $\varepsilon_n \rightarrow 0$,

$$\frac{e^{\lambda_2 t_n} \exp(-c_n(1+o(1)))}{e^{\lambda_1 t_n} \exp(-c_n(1+o(1)))} \leq e^{-C t_n} e^{\varepsilon_n n \log n} \leq e^{-(C''n - \varepsilon_n)n \log n}.$$

By part (i) of Lemma 2.2 and Hölder's inequality

$$\sum_{x \in \Sigma_n} \sum_{i=2}^{k-1} v_{i,k-1}(x) \leq 2k2^{n/2}.$$

Then, since $n \ll n \log n$, using (2.13) we get

$$\begin{aligned} & \sum_x \left(e^{\lambda_1 t_n} \exp(-2c_n(1+o(1))) + e^{\lambda_2 t_n} \exp(-c_n(1+o(1))) \right) \sum_{i=2}^{k-1} v_{i,k-1}(x) \\ & \ll v_n(t_n, x_{1,2^n}, x_{k,2^n}). \end{aligned}$$

Using part (i) of Lemma 2.2 we have

$$\sum_{x \neq x_{1,2^n}} v_n(t_n, x_{1,2^n}, x_{k,2^n}) v_{1,k-1}(x) \ll v_n(t_n, x_{1,2^n}, x_{k,2^n}),$$

and conclude through (2.14) that $\sum_x \omega_k(t_n, x, x_{k,2^n}) \sim v_n(t_n, x_{1,2^n}, x_{k,2^n})$. Once more using part (i) of Lemmas 2.2, 2.4 and 2.7 we get

$$\sum_x \tilde{\omega}_k(t_n, x, x_{k,2^n}) \leq \tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) \sum_x v_k(x) \sim \tilde{\omega}_k(t_n, x_{k,2^n}, x_{k,2^n}) \sim e^{\lambda_k t_n} \ll v_n(t_n, x_{1,2^n}, x_{k,2^n}).$$

Hence, $v_n(t_n, x_{k,2^n}) \sim v_n(t_n, x_{1,2^n}, x_{k,2^n})$, and (2.13) finishes the proof for the long time scales. $\square$

3. Proof of spectral results

In this section we prove Lemmas 2.1 and 2.2. For proving the results about eigenvalues we first give a general result for a given potential on Σ_n which is similar in spirit to the cluster expansion result given in [13] (Lemma 2.18, on page 45). Let $V : \Sigma_n \rightarrow [-\infty, \infty)$ be a potential and $A \subset \Sigma_n$ be such that

$$d_{\min}(A) = \min\{d(x, y) : x, y \in A, x \neq y\} > 2.$$

We set

$$N := \max_A V, \quad M := \max_{\Sigma_n \setminus A} V.$$

Lemma 3.1. *If*

$$M \leq N - \kappa,$$

then

$$N - \kappa \leq \lambda_1 < \gamma \tag{3.1}$$

for any $\gamma > N - \kappa$ with

$$\frac{\kappa}{\gamma - (N - \kappa)} < \frac{n(\gamma - M)}{\kappa}. \tag{3.2}$$

Proof of Lemma 3.1. The lower bound in (3.1) follows easily by replacing V with $-\infty$ everywhere except at the site in A where the maximum value N is reached and using the fact that λ_1 is non-decreasing in V .

For the upper bound we will show that any $\gamma > N - \kappa$ that satisfies (3.2) is in the resolvent set of $\kappa \Delta_n + V$. This is enough because if $\gamma > 0$ satisfies (3.2) then so does any $\gamma' > \gamma$. We denote by $\mathcal{R}_\gamma$ the resolvent at γ . Using the probabilistic representation of the resolvent, since we are on a finite space, it is enough to check that

$$\mathcal{R}_\gamma \mathbf{1}(x) = \mathbb{E}_x \left[\int_0^\infty dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] < \infty \tag{3.3}$$

for any $x \in \Sigma_n$ where $\mathbf{1}$ denotes the constant function of 1. We define hitting times $0 \leq \sigma_0 < \tau_0 < \sigma_1 < \tau_1 < \dots$ by

$$\sigma_0 = \inf\{t \geq 0 : X(t) \in A\},$$

and for $i \in \mathbb{N} \cup \{0\}$

$$\tau_i = \inf\{t \geq \sigma_i : X(t) \notin A\},$$

$$\sigma_{i+1} = \inf\{t \geq \tau_i : X(t) \in A\}.$$

Note that, since $d_{\min}(A) > 2$, $\tau_i - \sigma_i$, $i = 0, 1, \dots$, is an independent sequence of exponential random variables with rate κ . Using these stopping times we can write

$$\mathcal{R}_\gamma \mathbf{1}(x) = \mathbb{E}_x \left[\int_0^{\sigma_0} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] + \sum_{i=0}^{\infty} \mathbb{E}_x \left[\int_{\sigma_i}^{\sigma_{i+1}} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right].$$

Note that $V(x) \leq M \leq N - \kappa < \gamma$ for $x \notin A$. Since $X(t) \notin A$ for $t \in [0, \sigma_0]$

$$\mathbb{E}_x \left[\int_0^{\sigma_0} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] \leq \int_0^{\infty} dt e^{\int_0^t (M - \gamma) ds} = \int_0^{\infty} dt e^{-t(\gamma - M)} = \frac{1}{\gamma - M} < \infty.$$

For $i = 0, 1, \dots$ we have

$$\begin{aligned} \mathbb{E}_x \left[\int_{\sigma_i}^{\sigma_{i+1}} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] &= \mathbb{E}_x \left[e^{\int_0^{\sigma_0} (V(X_s) - \gamma) ds} \right. \\ &\quad \times \mathbb{E}_{X(\sigma_0)} [e^{\int_0^{\sigma_i} (V(X_s) - \gamma) ds}] \\ &\quad \times \mathbb{E}_{X(\sigma_i)} \left[\int_0^{\sigma_1} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] \left. \right]. \end{aligned}$$

Since $V(X_s) < \gamma$ for $s \in [0, \sigma_0]$ we have $e^{\int_0^{\sigma_0} (V(X_s) - \gamma) ds} < 1$. By the strong Markov property

$$\mathbb{E}_{X(\sigma_0)} \left[e^{\int_0^{\sigma_i} (V(X_s) - \gamma) ds} \right] \leq \left(\max_{x \in A} \mathbb{E}_x \left[e^{\int_0^{\sigma_1} (V(X_s) - \gamma) ds} \right] \right)^i.$$

Hence,

$$\begin{aligned} \mathbb{E}_x \left[\int_{\sigma_i}^{\sigma_{i+1}} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] &\leq \left(\max_{z \in A} \mathbb{E}_z \left[e^{\int_0^{\sigma_1} (V(X_s) - \gamma) ds} \right] \right)^i \\ &\quad \times \max_{z \in A} \mathbb{E}_z \left[\int_0^{\sigma_1} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right]. \end{aligned}$$

Therefore, to finish the proof of (3.3) it is enough to check that

$$\max_{x \in A} \mathbb{E}_x \left[e^{\int_0^{\sigma_1} (V(X_s) - \gamma) ds} \right] < 1 \tag{3.4}$$

and

$$\max_{x \in A} \mathbb{E}_x \left[\int_0^{\sigma_1} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] < \infty. \tag{3.5}$$

For the former we write

$$\mathbb{E}_x \left[e^{\int_0^{\sigma_1} (V(X_s) - \gamma) ds} \right] = \mathbb{E}_x \left[e^{\int_0^{\tau_0} (V(X_s) - \gamma) ds} \mathbb{E}_{X(\tau_0)} [e^{\int_0^{\sigma_0} (V(X_s) - \gamma) ds}] \right].$$

Since $d_{\min}(A) > 2$ any $z \notin A$ has at most one neighbour that is in A . Hence, σ_0 , for the walk starting from any $z \notin A$, is stochastically bounded from below by an exponential random variable with rate κ/n . Hence, using once again that $V \leq M < \gamma$ on A^c , we can conclude that for any $z \notin A$

$$\mathbb{E}_z \left[e^{\int_0^{\sigma_0} (V(X_s) - \gamma) ds} \right] < \mathbb{E}_z \left[e^{-(\gamma - M)\sigma_0} \right] \leq \frac{\kappa/n}{\kappa/n + \gamma - M} < \frac{\kappa}{n(\gamma - M)}.$$

Also, recall that starting from $x \in A$, τ_0 is distributed as an exponential random variable with rate κ . Hence,

$$\begin{aligned} \max_{x \in A} \mathbb{E}_x \left[e^{\int_0^{\sigma_1} (V(X_s) - \gamma) ds} \right] &= \max_{x \in A} \mathbb{E}_x \left[e^{\int_0^{\tau_0} (V(X_s) - \gamma) ds} \mathbb{E}_{X(\tau_0)} [e^{\int_0^{\sigma_0} (V(X_s) - \gamma) ds}] \right] \\ &\leq \max_{x \in A} \mathbb{E}_x \left[e^{\int_0^{\tau_0} (V(X_s) - \gamma) ds} \right] \frac{\kappa}{n(\gamma - M)} \\ &= \max_{x \in A} \frac{\kappa}{\kappa + \gamma - V(x)} \frac{\kappa}{n(\gamma - M)} \\ &= \frac{\kappa}{\kappa + \gamma - N} \frac{\kappa}{n(\gamma - M)}. \end{aligned} \quad (3.6)$$

By (3.2) the last quantity above is less than 1 and thus, (3.4) is satisfied. Now it remains to check (3.5). To this end we write

$$\mathbb{E}_x \left[\int_0^{\sigma_1} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] = \mathbb{E}_x \left[\left(\int_0^{\tau_0} + \int_{\tau_0}^{\sigma_0} \right) dt e^{\int_0^t (V(X_s) - \gamma) ds} \right]. \quad (3.7)$$

For any $x \in A$ for a $c > 0$ appropriately chosen $V(x) - \gamma \leq N - \gamma < c < \kappa$. Hence,

$$\mathbb{E}_x \left[\int_0^{\tau_0} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] \leq \mathbb{E}_x \left[\frac{e^{\tau_0 c} - 1}{c} \right].$$

Now since τ_0 is distributed as an exponential random variable with rate κ and $c < \kappa$ the above quantity is finite. The second integral on the right hand side of (3.7) is equal to

$$\mathbb{E}_x \left[e^{\int_0^{\tau_0} (V(X_s) - \gamma) ds} \mathbb{E}_{X(\tau_0)} \left[\int_0^{\sigma_0} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] \right]. \quad (3.8)$$

Since $V \leq M < \gamma$ on A^c for any $z \notin A$

$$\mathbb{E}_z \left[\int_0^{\sigma_0} dt e^{\int_0^t (V(X_s) - \gamma) ds} \right] \leq \int_0^{\infty} dt e^{-t(\gamma - M)} < \infty.$$

We have already seen in (3.6) that $\max_{x \in A} \mathbb{E}_x \left[e^{\int_0^{\tau_0} (V(X_s) - \gamma) ds} \right] = \frac{\kappa}{\gamma - (N - \kappa)}$. Thus, (3.8) is finite and (3.5) is satisfied. This completes the proof of the lemma. $\square$

The key ingredient of the proofs of Lemmas 2.1 and 2.2 is the next result. For $\delta \in (0, 1)$ define

$$A_n^\delta := \{x : \eta(x) \geq n\delta \log 2\}.$$

Let

$$I(x) := x \log x + (1 - x) \log(1 - x) + \log 2, \quad x \in [0, 1]$$

be the Cramer's rate function.

Lemma 3.2.

- (i) Let $\delta > 1/2$ and ω^δ be the smallest solution of $I(\omega^\delta) = 2(1 - \delta) \log 2$. Then P -a.s. for any $c < \omega^\delta$

$$d_{\min}(A_n^\delta) := \min \left\{ d(x, y) : x, y \in A_n^\delta, x \neq y \right\} \geq cn,$$

for all n large enough.

(ii) P -a.s. for any $i, k \in \mathbb{N}$ with $i \neq k$

$$d(x_i, x_k) \sim n/2,$$

for all n large enough.

Proof of 3.2. Since $P(\eta(x) \geq n\delta \log 2) = 2^{-\delta n}$ the statement of part (i) is same as part (ii) of Lemma 3.7 in [1], with δ is replaced by γ in the notation used in the aforementioned article. For part (ii) note that for any i, k the distribution of $d(x_i, x_k)$ is that of a Binomial random variable with parameters n and $1/2$, conditioned to be non-zero. Hence, the result follows from the strong law of large numbers. $\square$

Proof of Lemma 2.1. For $\delta \in (1/2, 1)$ let

$$A_n^\delta = \{x : \eta(x) \geq n\delta \log 2\}$$

Then by Lemma 3.2 for some $c \in (0, \omega^\delta)$, P -a.s.

$$d_{\min}(A_n^\delta) := \min\{d(x, y) : x, y \in A_n^\delta, x \neq y\} \geq cn$$

for all n large enough. We use Lemma 3.1 with V given by $V \equiv \xi$ on $\Sigma_n \setminus \Gamma_l \cup \{x_{i,2^n}\}$ and $V \equiv -\infty$ on $\Gamma_l \setminus \{x_{i,2^n}\}$. Part (i) of Lemma 1.1 and the fact that f in Assumption (R1) is strictly increasing imply that P -a.s. $x_{i,2^n} \in A_n^\delta$ for n large enough. This yields

$$N = \max_{A_n^\delta} V = \xi_{i,2^n}.$$

Hence, with $\gamma = \xi_{i,2^n} - \kappa + \varepsilon_n$,

$$\lambda_{i,l} \leq \xi_{i,2^n} - \kappa + \varepsilon_n$$

if

$$\frac{\kappa}{\varepsilon_n} < \frac{n(\xi_{i,2^n} - \kappa + \varepsilon_n - M)}{\kappa}. \quad (3.9)$$

By the definition of A_n^δ and Assumption (R1) we have $M \leq \psi_n(n\delta \log 2) \leq \delta_1 \theta n$ for some $\delta_1 \in (0, 1)$. By Lemma 1.1 $\xi_i \geq \delta_2 \theta n$ for some $\delta_2 \in (\delta_1, 1)$. Thus, (3.9) is satisfied if

$$\frac{\kappa}{\varepsilon_n} < \frac{(\delta_2 - \delta_1)\theta n^2 - \kappa n}{\kappa}.$$

Hence, we can choose the sequence ε_n so that $\varepsilon_n = C/n^2$. Therefore,

$$\lambda_{i,l} \leq \xi_{i,2^n} - \kappa + O(1/n^2).$$

Now we prove the lower bound for $\lambda_{i,l}$. Let l_n be as in Assumption (L). We first claim that P -a.s.

$$\xi_n(y) \geq -l_n, \quad \forall y \sim x_{k,2^n}, \quad \forall k \in \mathbb{N}. \quad (3.10)$$

Note that the random variables $\xi_n(y)$, $y \sim x_{k,2^n}$, are independent and have the distribution of ξ_n conditioned on not being the k th largest. We have the following obvious bound for the latter

$$P(\xi_n(y) \leq -l_n \mid y \neq x_{k,2^n}) = \frac{P(\xi_n(y) \leq -l_n, y \neq x_{k,2^n})}{P(y \neq x_{k,2^n})} \leq \frac{G_n(-l_n)}{P(y \neq x_{k,2^n})} = \frac{G_n(-l_n)}{1 - 1/2^n}.$$

Consequently,

$$P(\exists y \sim x_{k,2^n} \text{ s.t. } \xi_n(y) \leq -l_n) \leq CnG_n(-l_n).$$

By [Assumption \(L\)](#) the last quantity above is summable in n , and an application of Borel–Cantelli lemma proves (3.10). Using (3.10) we have $\lambda_{i,l}$ is bounded below by the principal eigenvalue, $\tilde{\lambda}$, of $\kappa \Delta_n + V$ on $x_{i,2^n} \cup \{y : y \sim x_{i,2^n}\}$ with zero boundary conditions, where $V(x_{i,2^n}) = \xi_{i,2^n}$ and $V(y) = -l_n$ for $y \sim x_{i,2^n}$. Since $\xi_{i,2^n} - \kappa \gg -l_n$, the principal eigenvalue of the operator one gets by collapsing the neighbours of $x_{i,2^n}$ to a single state with potential $-l_n$ is same as $\tilde{\lambda}$. The matrix representation of the states operator is

$$\begin{bmatrix} \xi_{i,2^n} - \kappa & \kappa \\ \kappa/n & -\kappa(1 - 1/n) - l_n \end{bmatrix}.$$

Using the fact that $l_n \ll n$ (by [Assumption \(L\)](#)), a simple calculation shows that the principal eigenvalue of the above matrix is

$$\xi_i - \kappa + \frac{C\kappa^2}{n\xi_{i,2^n}} + o(n^{-2}).$$

Finally, since $\xi_{i,2^n} \sim \theta n$ we have the right upper and conclude that

$$\lambda_{i,l} = \xi_{i,2^n} - \kappa + \Theta(n^{-2}).$$

For the spectral gap, using the min–max formula, we have that the second largest eigenvalue is bounded above by the principal eigenvalue of $\kappa \Delta_n + \xi_n$ with zero boundary conditions on Γ_l . With the same exact proof above we get that this principal eigenvalue is $\xi_{l+1,2^n} - \kappa + o(1)$ (since N in this case is $\xi_{l+1,2^n}$). Hence, we are finished with the proof of the spectral gap. $\square$

Proof of Lemma 2.2 part (i). Since $v_{i,l}$ is the principal eigenfunction of a symmetric operator, by Perron–Frobenius theorem its values are non-negative. Therefore, recalling that $v_{i,l}(x_i) = 1$, $\sum_{x \neq x_1} v_{i,l}(x) \rightarrow 0$ implies $\|v_{i,l}\|_2 \rightarrow 1$.

Now we show that $\sum_{x \neq x_1} v_{i,l}(x) \rightarrow 0$. For $\delta \in (1/2, 1)$ let A_n^δ and ω^δ be as in [Lemma 3.2](#). We again set $\xi = \infty$ on $\Gamma_l \setminus \{x_i\}$, and define $B_n := B(x_i, cn - 3)$ for some $c \in (0, \omega^\delta)$. We will first prove that

$$\max_{x \notin B_n} v_{i,l}(x) \leq \exp(-c_n(1 + o(1))), \quad (3.11)$$

where as before $c_n = \frac{1}{2}n \log n$. We write

$$\begin{aligned} v_{i,l}(x) &= \mathbb{E}_x[\exp(\int_0^{\tau_{x_i,2^n}} (\xi(X_s) - \lambda_{i,l}) ds) \mathbb{1}\{\tau_{x_i,2^n} = \tau_{\Gamma_l}\} \mathbb{1}\{\tau_{A_n^\delta \setminus \{x_i,2^n\}} > \tau_{x_i,2^n}\}] \\ &\quad + \mathbb{E}_x[\exp(\int_0^{\tau_{x_i,2^n}} (\xi(X_s) - \lambda_{i,l}) ds) \mathbb{1}\{\tau_{x_i,2^n} = \tau_{\Gamma_l}\} \mathbb{1}\{\tau_{A_n^\delta \setminus \{x_i,2^n\}} \leq \tau_{x_i,2^n}\}] \end{aligned} \quad (3.12)$$

Since $X(s) \notin A_n^\delta$ for any $s \in [0, \tau_{x_i,2^n})$ on the event $\tau_{A_n^\delta \setminus \{x_i,2^n\}} > \tau_{x_i,2^n}$, using the definition of A_n^δ and [Assumption \(R1\)](#) we have $\xi(X_s) \leq \theta'n$ on the same event, for some $\theta' < \theta$. Hence, the first expectation on the right hand side of (3.12) is bounded above by

$$\mathbb{E}_x[\exp(\int_0^{\tau_{x_i,2^n}} (\theta'n - \lambda_{i,l}) ds)]$$

By [Lemma 2.1](#) we have $\lambda_{i,l} = \xi_{i,2^n} - \kappa + o(1)$. As a result, via [Lemma 1.1](#), $\lambda_{i,l} > \theta'n$ for large enough n . Finally, since $x \notin B_n$, $\tau_{x_i,2^n}$ is stochastically bounded below by the sum of $cn - 3$ i.i.d. exponentials with rate κ , and this yields

$$\mathbb{E}_x[\exp(\int_0^{\tau_{x_i,2^n}} (\theta'n - \lambda_{i,l}) ds)] \leq \left[\frac{\kappa}{\kappa + \lambda_{i,l} - \theta'n} \right]^{cn-3}.$$

Using the fact that $\lambda_{i,l} = \xi_{i,2^n} - \kappa + o(1)$ and [Lemma 1.1](#) we get

$$\left[\frac{\kappa}{\kappa + \lambda_{i,l} - \theta'n} \right]^{cn-3} = \exp(-cn \log n(1 + o(1))). \quad (3.13)$$

On the event $\mathbb{1}\{\tau_{A_n^\delta \setminus \{x_{i,2^n}\}} \leq \tau_{x_{i,2^n}}\}$ we have $\xi(X_s) \leq \theta'n < \lambda_{i,l}$ for any $s \in [0, \tau_{A_n^\delta \setminus \{x_{i,2^n}\}})$. Hence, using the strong Markov property the second term on the right hand side of [\(3.12\)](#) is bounded above by

$$\max_{x \in A_n^\delta \setminus \{x_i\}} v_{i,l}(x).$$

Since $d_{\min}(A_n^\delta) \geq cn$ and $x_i \in A_n^\delta$ for n large enough, if $x \in A_n^\delta \setminus \{x_i\}$ then for any $y \sim x$ we have $y \notin A_n^\delta$ and $d(y, B_n) > 1$. Hence,

$$\begin{aligned} \max_{x \in A_n^\delta \setminus \{x_{i,2^n}\}} v_{i,l}(x) &\leq \frac{\kappa}{\kappa + \lambda_{i,l} - \xi_{l+1,2^n}} \frac{\kappa}{\kappa + \lambda_{i,l} - \theta'n} \max_{x \notin B_n} v_{i,l}(x) \\ &\leq \frac{C}{n} \max_{x \notin B_n} v_{i,l}(x) \end{aligned}$$

for some positive constant C . For the first inequality in the above display we used the fact that $\xi(x) \leq \xi_{l+1,2^n}$ for $x \in A_n^\delta \setminus \{x_{i,2^n}\}$ and $\xi(x) \leq \theta'n$ for $x \notin A_n^\delta$, and for the second inequality we used both parts of [Lemmas 1.1](#) and [2.1](#). Hence, together with [\(3.12\)](#) and [\(3.13\)](#) we get

$$\max_{x \notin B_n} v_{i,l}(x) \leq \exp(-cn \log n(1 + o(1))) + \frac{C}{n} \max_{x \notin B_n} v_{i,l}(x).$$

Then

$$\begin{aligned} \max_{x \notin B_n} v_{i,l}(x) &\leq (1 - C/n)^{-1} \exp(-cn \log n(1 + o(1))) \\ &= \exp(-cn \log n(1 + o(1))). \end{aligned}$$

Since $\omega^\delta \rightarrow 1/2$ as $\delta \rightarrow 1$, by part (i) of [Lemma 3.2](#), as $\delta \rightarrow 1$, we can choose c as close to $1/2$ as we wish. Hence, we are finished with the proof of [\(3.11\)](#).

By [\(3.11\)](#) we have

$$\sum_{x \in B_n^c} v_{i,l}(x) \leq 2^n \exp\left(-\frac{1}{2}n \log n(1 + o(1))\right) \rightarrow 0.$$

Hence, it remains to prove that

$$\sum_{x \in B_n \setminus \{x_i\}} v_{i,l}(x) \rightarrow 0.$$

As before $\xi(x) \leq \theta'n < \lambda_{i,l}$ for large n , $x \notin A_n^\delta$, and $B_n \cap A_n^\delta = \{x_{i,2^n}\}$. Therefore, by [\(3.11\)](#) and the strong Markov property, for any $x \in B_n \setminus \{x_{i,2^n}\}$

$$\mathbb{E}_x \left[\exp\left(\int_0^{\tau_{x_{i,2^n}}} (\xi(X_s) - \lambda_{i,l}) ds\right) \mathbb{1}\{\tau_{x_{i,2^n}} = \tau_{\Gamma_l}\} \mathbb{1}\{\tau_{B_n^c} < \tau_{x_{i,2^n}}\} \right] \leq \exp(-c_n(1 + o(1))).$$

As a result, it is enough to check that

$$\sum_{x \in B_n \setminus \{x_{i,2^n}\}} \mathbb{E}_x \left[\exp\left(\int_0^{\tau_{x_{i,2^n}}} (\xi(X_s) - \lambda_{i,l}) ds\right) \mathbb{1}\{\tau_{x_{i,2^n}} = \tau_{\Gamma_l}\} \mathbb{1}\{\tau_{B_n^c} > \tau_{x_{i,2^n}}\} \right] \rightarrow 0.$$

We now construct a stochastic lower bound for $\tau_{x_{i,2^n}}$ through the Coupon collector's problem. Let x be such that $d(x_{i,2^n}, x) = r$. Without loss of generality we can assume that $x_{i,2^n} =$

$(+1, +1, \dots, +1)$. Then the number of -1 's in the configuration of x is exactly r . Now we reject all the jumps that switches a $+1$ to a -1 , in other words, we consider only the spin sites with -1 and wait for all of them to become $+1$. Then this waiting time, denoted by τ' , is a lower bound. Observe that the first time a -1 becomes a $+1$ is distributed as an exponential random variable with rate $\kappa r/n$ (recall that per spin the jump rate is κ/n); after this event there are now $r-1$ spins with a -1 sign and the first time one of them becomes a $+1$ is distributed as an exponential random variable with rate $\kappa(r-1)/n$; proceeding like this we wait finally an exponential time with rate κ/n for the last -1 sign to become a $+1$. Hence, τ' is given by

$$\tau' = \frac{\alpha_r}{\kappa r/n} + \frac{\alpha_{r-1}}{\kappa(r-1)/n} + \dots + \frac{\alpha_1}{\kappa/n}$$

where $\alpha_1, \dots$ is an i.i.d. sequence of exponential random variables with rate 1. Hence, for any x s.t. $d(x_{i,2^n}, x) = r$

$$\begin{aligned} & \mathbb{E}_x \left[\exp \left(\int_0^{\tau_{x_{i,2^n}}} (\xi(X_s) - \lambda_{i,l}) ds \right) \mathbb{1}\{\tau_{x_{i,2^n}} = \tau_{\Gamma_l}\} \mathbb{1}\{\tau_{B_n^c} > \tau_{x_{i,2^n}}\} \right] \\ & \leq \mathbb{E}[\exp(-\tau'(\lambda_{i,l} - \theta'n))] \\ & = \prod_{j=1}^r \frac{\kappa j/n}{\kappa j/n + \lambda_{i,l} - \theta'n} \leq \prod_{j=1}^r \frac{j}{Cn^2} \end{aligned}$$

for some positive constant C . Note that for the last step once again we used that $\lambda_{i,l} \sim \theta n$. For the last term above we use the following upper bound

$$\begin{aligned} \prod_{j=1}^r \frac{j}{Cn^2} &= C' \exp(-2r \log n + \sum_{j=1}^r \log j) \\ &\leq C'' \exp(-2r \log n + r \log r) = C'' n^{-2r} r^r. \end{aligned}$$

for some positive constants C', C'' . Since

$$|\{x : d(x_{i,2^n}, x) = r\}| = \binom{n}{r},$$

in order to finish the proof it is enough to check that

$$\sum_{r=1}^{cn} n^{-2r} r^r \binom{n}{r} \rightarrow 0.$$

Since $c < 1$, using Sterling's approximation we get

$$\begin{aligned} \sum_{r=1}^{cn} n^{-2r} r^r \binom{n}{r} &\leq C \sum_{r=1}^{cn} n^{-2r} r^r \frac{n^n}{(n-r)^{n-r} r!} e^{-r} \\ &= C \sum_{r=1}^{cn} n^{-r} \frac{r^r}{r!} \left[1 + \frac{r}{n-r}\right]^{n-r} e^{-r} \\ &\leq C \sum_{r=1}^{cn} n^{-r} \frac{r^r}{r!} \end{aligned}$$

Since $r^r \leq r! e^r$ for any $r \in \mathbb{N}$ we get

$$\sum_{r=1}^{cn} n^{-r} \frac{r^r}{r!} \leq \sum_{r=1}^{cn} n^{-r} e^r \leq \sum_{r=1}^{\infty} (e/n)^r = \frac{1}{1 - e/n} - 1 \xrightarrow{n \rightarrow \infty} 0.$$

Hence, we are finished with the proof of part (i). $\square$

Proof of Lemma 2.2 part (ii). Since $i \neq k$ and $x_{k,2^n} \in A_n^\delta$ for n large enough, $x_{k,2^n} \notin B_n$, where $B_n = B(x_{i,2^n}, cn - 3)$ as described in the proof of Lemma 2.2 part (i). Hence, via (3.11) we have the right upper bound for $v_{i,l}(x_{k,2^n})$.

For l_n given in Assumption (L) for some $\delta \in (0, 1)$ we define

$$L_n := \{x \in \Sigma_n : -l_n \leq \xi_n(x) \leq \psi_n(n\delta \log 2)\}$$

and

$$p_n := P(\xi_n(x) \notin L_n). \quad (3.14)$$

For $x \neq x_{i,2^n}$ we denote by $H(x)$ the number of nearest neighbour paths $x = y_0 \rightarrow y_1 \rightarrow \dots \rightarrow y_d = x_{i,2^n}$, where $d = d(x, x_{i,2^n})$, such that, $\xi_n(y_j) \in L_n$ for all $j = 1, \dots, d-1$. It is understood that $H(x) = 1$ for x such that $d(x, x_{i,2^n}) = 1$. Note that $H(x)$ and $H(z)$ have identical distributions for any x, z with $d(x, x_{i,2^n}) = d(z, x_{i,2^n})$. We label the expectation of any such distribution by $\mathcal{H}(d)$, that is, $\mathcal{H}(d) = E[H(x)]$ for some x with $d(x, x_{i,2^n}) = d$. Finally, we define

$$S(x) := \{y \sim x : d(y, x_i) = d(x, x_{i,2^n}) - 1 \text{ and } y \in L_n\}.$$

Any nearest neighbour path from x to $x_{i,2^n}$ of length $d(x, x_{i,2^n})$ that stays in L_n in between should move to a vertex in $S(x)$ in its first step. Hence,

$$H(x) = \sum_{y \in S(x)} H(y).$$

Since $\mathbb{1}\{y \in S(x)\}$, $y \sim x$, and $H(y')$ are independent events for any $y' \sim x$ we get

$$E[H(x)] = E[|S(x)|]E[H(y)]$$

For x , with $d = d(x, x_i)$, $|S(x)|$ is a Binomial random variable with parameters d and $(1 - p_n)$. This yields

$$\mathcal{H}(d) = d(1 - p_n)\mathcal{H}(d - 1),$$

From $\mathcal{H}(1) = 1$ it readily follows that

$$\mathcal{H}(d) = d!(1 - p_n)^{d-1}.$$

Since $H(x) \leq d!$, for any $\theta \in (0, 1)$ we get

$$\begin{aligned} \mathcal{H}(d) &= E[H(x)] = E[H(x)\mathbb{1}\{H(x) \leq \theta\mathcal{H}(d)\}] + E[H(x)\mathbb{1}\{H(x) \geq \theta\mathcal{H}(d)\}] \\ &\leq \theta\mathcal{H}(d) + d!P(H(x) \geq \theta\mathcal{H}(d)). \end{aligned}$$

Hence,

$$P(H(x) \geq \theta\mathcal{H}(d)) \geq (1 - \theta) \frac{\mathcal{H}(d)}{d!} = (1 - \theta)(1 - p_n)^{d-1}.$$

Let $\theta = \theta_n = o(1)$. Since $d(x, x_i) \leq n$ for any $x \in \Sigma_n$

$$P(H(x) \leq \theta_n\mathcal{H}(d)) \leq 1 - (1 - \theta_n)(1 - p_n)^{n-1} \leq C(\theta_n + np_n + \theta_n np_n).$$

Recalling (3.14)

$$p_n = G(-l_n) + P(\xi_n(x) \geq \psi_n(n\delta \log 2)).$$

The second term above decay as $2^{-\delta'n}$ for some $\delta' > 0$. Hence, using [Assumption \(L\)](#) we have $\sum_n np_n < \infty$. Now we choose $\theta_n = n^{-1-a}$ for some $a > 0$ and get

$$\sum_n (\theta_n + np_n + \theta_n np_n) < \infty.$$

Hence, by an application of Borel–Cantelli lemma we reach at that P -a.s. $H(x) \geq \theta_n H(r)$. [Assumption \(R1\)](#) and part (i) of [Lemma 1.1](#) imply $\xi_n(x_{l,2^n}) \gg \psi_n(\delta \log 2n)$. Thus, P -a.s. $L_n \cap \Gamma_l = \emptyset$ for n large enough.

Let $d = d(x_{k,2^n}, x_{i,2^n})$. Since $H(x_{k,2^n}) \geq n^{-1-a} \mathcal{H}(d) = n^{-1-a} d!(1 - p_n)^{d-1}$ and the probability of any nearest neighbour path of length d is n^{-d} , using the probabilistic representation of $v_{i,l}$ we get that P -a.s. for n large enough

$$\begin{aligned} v_{i,l}(x_{k,2^n}) &= \mathbb{E}_{x_{k,2^n}} \left[\exp \left(\int_0^{\tau_{x_{i,2^n}}} (\xi_n(X_s) - \lambda_{i,l}) ds \mathbb{1}_{\{\tau_{x_{i,2^n}} = \tau_{\Gamma_l}\}} \right) \right] \\ &\geq \frac{n^{-1-a} d!(1 - p_n)^{d-1}}{n^d} \prod_{j=0}^{d-1} \frac{\kappa}{\kappa + \lambda_{i,l} - \xi_n(y_j)} \\ &\geq \frac{n^{-1-a} d!(1 - p_n)^{d-1}}{n^d} \frac{\kappa}{\kappa + \lambda_{i,l} - \xi_{k,2^n}} \left[\frac{\kappa}{\kappa + \lambda_{i,l} + l_n} \right]^{d-1}. \end{aligned} \quad (3.15)$$

$d = \text{dist}(x_i, x_k) \sim n/2$ by part (ii) of [Lemma 3.2](#). This and the fact that $p_n \rightarrow 0$ yield for the first term above

$$\frac{n^{-1-a} d!(1 - p_n)^{d-1}}{n^d} = \exp(-O(n)).$$

$\lambda_{i,l} + \kappa = \xi_{i,2^n} + o(1)$ by [Lemma 2.1](#); $\xi_{i,2^n} \sim \theta n$ by part (i) of [Lemma 1.1](#); by part (ii) of [Lemma 1.1](#) $\xi_{i,2^n} - \xi_{k,2^n} = C + o(1)$ for some random positive constant C . Hence, using the fact that $d = \text{dist}(x_i, x_k) \sim n/2$ and $l_n \ll n$ we can conclude that

$$\begin{aligned} v_{i,l}(x_{k,2^n}) &\geq C' \exp \left(-d \log(\xi_{i,2^n} + o(1) + l_n) + O(n) \right) \\ &= \exp \left(-\frac{n}{2} \log n(1 + o(1)) \right). \end{aligned}$$

This gives the right lower bound and we are finished with the proof of part (ii) of [Lemma 2.2](#). $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

- [1] G. Ben Arous, J. Černý, The arcsine law as a universal aging scheme for trap models, *Comm. Pure Appl. Math.* 61 (3) (2008) 289–329.
- [2] J.F. Crow, M. Kimura, *An Introduction to Population Genetics Theory*, Harper & Row, New York, 1970.

- [3] J. de Visser, J. Krug, Empirical fitness landscapes and the predictability of evolution, *Nat. Rev. Genet.* (15) (2014) 480–490.
- [4] B. Derrida, Random-energy model: an exactly solvable model of disordered systems, *Phys. Rev. B* (3) 24 (5) (1981) 2613–2626.
- [5] M. Eigen, Self-organization of matter and the evolution of macromolecules, *Naturwissenschaften* 58 (10) (1971) 465–523.
- [6] W. Feller, *An Introduction to Probability Theory and its Applications*. Vol. II, John Wiley & Sons, Inc., New York-London-Sydney, 1966.
- [7] K. Fleischmann, S.A. Molchanov, Exact asymptotics in a mean field model with random potential, *Probab. Theory Related Fields* 86 (2) (1990) 239–251.
- [8] S. Franz, L. Peliti, Error threshold in simple landscapes, *J. Phys. A* 30 (13) (1997) 4481–4487.
- [9] S. Franz, L. Peliti, M. Sellitto, An evolutionary version of the random energy model, *J. Phys. A* 26 (13) (1993) L1195.
- [10] W. Gabriel, E. Baake, Biological evolution through mutation, selection, and drift: An introductory review, 1999.
- [11] J. Gärtner, W. König, S. Molchanov, Geometric characterization of intermittency in the parabolic anderson model, *Ann. Probab.* 35 (2) (2007) 439–499.
- [12] J. Gärtner, S.A. Molchanov, Parabolic problems for the Anderson model. I. Intermittency and related topics, *Comm. Math. Phys.* 132 (3) (1990) 613–655.
- [13] J. Gärtner, S.A. Molchanov, Parabolic problems for the Anderson model. II. Second-order asymptotics and structure of high peaks, *Probab. Theory Related Fields* 111 (1) (1998) 17–55.
- [14] J.F.C. Kingman, A simple model for the balance between selection and mutation, *J. Appl. Probab.* 15 (1) (1978) 1–12.
- [15] W. König, *The Parabolic Anderson Model, Random Walk in Random Potential*, Birkhäuser, Basel, 2016.
- [16] W. König, H. Lacoin, P. Mörters, N. Sidorova, A two cities theorem for the parabolic Anderson model, *Ann. Probab.* 37 (1) (2009) 347–392.
- [17] P.A.P. Moran, Global stability of genetic systems governed by mutation and selection, *Math. Proc. Camb. Phil. Soc.* 80 (2) (1976) 331–336.
- [18] P. Mörters, M. Ortgieste, N. Sidorova, Ageing in the parabolic anderson model, *Ann. Inst. Henri Poincaré Probab. Stat.* 47 (4) (2011) 969–1000.
- [19] R. van der Hofstad, P. Mörters, M. Ortgieste, N. Sidorova, Weak and almost sure limits for the parabolic anderson model with heavy-tailed potentials, *Ann. Appl. Probab.* 18 (6) (2008) 2450–2494.
- [20] S. Wright, The roles of mutation, inbreeding, crossbreeding and selection in evolution, in: *Proceeding of the Sixth International Congress of Genetics*, Vol. 1, 1932, pp. 356–366.