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Exploiting the potential of dynamic asymmetry in dragging to foster students' understanding of functions and their Cartesian graphs

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Abstract
The notion of function is considered central in mathematical thinking and in mathematics curricula, especially at the high school level. In particular, being able to interpret a function's behavior by looking at its Cartesian graph and to construct a graph starting from the function's properties are processes that are widely considered essential in mathematical thinking. These processes are based on two key aspects: 1) understanding basic notions in functional thinking, such as variable and covariation; 2) re-constructing the relationship between variables from a Cartesian graph of a function, and, vice versa, constructing a Cartesian graph of a function based on the relationship between the variables. However, many studies have shown that mastering these concepts is complex. We designed and implemented a sequence of activities in a digital interactive environment, capitalizing on its *dynamic asymmetry potential* (DAP), the potential to let students experience both the variations and the asymmetry in the behavior of varying objects that can be dragged on the screen. Through examples from an implementation of these activities, we show how students can come to appreciate specific aspects of the notion of function, tightly related to the DAP and to recognize them in the traditional static Cartesian graph representation.

Keywords
Cartesian graph; covariation; dragging; dynamic asymmetry potential; real function.

1. Some aspects of "function" that experts can "see" in static representations

"Function" is considered one of the "big ideas" around which mathematics curriculum should be built, because of its centrality in mathematical thinking. Such a belief was held by leading mathematicians and reformers like Felix Klein, who wrote:

> "We, who used to be called the reformers, would put the function concept at the very center of teaching, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into teaching as early as possible with constant use of the graphical

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> method, the representation of functional relations in the x y system, which is used today as a matter of course in every practical application of mathematics." (Klein, 1908 in McCallum, 2019, p. 79)

However, as highlighted by Thompson and Carlson (2017), the meaning of "concept of function" depends on whom we envision holding it, and it is well known that there are "many meanings and ways of thinking that different individuals and groups hold that could fit under the heading concept of function" (p. 421). In this chapter we focus on interpreting a function's behavior, recognizing its properties through different representations of the covariation between its variables, and eventually by looking at its Cartesian graph. Such processes, typically developed in secondary school, are widely considered to be essential in mathematical thinking, and they are based on two key aspects: 1) understanding basic notions in functional thinking, such as variable and covariation of variables, i.e., the relation between variations of two variables (e.g., for small increase of the independent variable how does the dependent variable change?); 2) re-constructing the relationship between variables from a Cartesian graph of a function, and, vice versa, constructing a Cartesian graph of a function based on the relationship between the variables.

A Cartesian graph of a function from real numbers into real numbers consists of the set of points (x, f(x)) with real coordinates on the Cartesian plane, where the independent variable x belongs to the domain of the function and the dependent variable f(x) is its image. This description suggests that a subset of the Cartesian plane representing the graph of a function is made up of points whose coordinates incorporate the functional relation between the two variables. Specifically, these coordinates are two numbers belonging to the same set (the set of real numbers): indeed, f "takes in" and "puts out" only ordinary numbers which are represented as points belonging to x-axis and y-axis.

Expert mathematicians and mathematics educators have interiorized such ways of thinking that allow them, for example, to "see" in a Cartesian graph of a function the variables and dynamic aspects of the covariational link relating them. On the other hand, students who are introduced to functions in lower secondary school through "standard definitions" and graphs typically do not perceive these aspects in a Cartesian graph. With "standard definition" we mean, for example "a relationship between inputs and outputs is a function if there is no more than one output for each input" (Dietiker et al., 2013), "a mathematical correspondence that assigns exactly one element of one set to each element of the same or another set" (Merriam-Webster, 2017), or similar ones; as opposed to definitions "with dynamic elements", such as "a variable quantity regarded in relation to one or more other variables in terms of which it may be expressed or on which its values depend" (Top ranked result from a Google search performed on 11 December 2017), or Euler's definition: "Those quantities that depend on others in this way, namely those that undergo a change when others change, are called functions of these quantities. This definition applies rather widely and includes all ways in which one quantity can be determined by others" (Euler, 1755; Euler & Blanton, 2000) (for a list of definitions, representations and their discussion see McCallum, 2019, p. 81).

1.1 Looking at the graph of a function through the eyes of an expert

To give the reader an idea of covariational aspects that expert mathematicians and mathematics educators "see" in Cartesian graphs of functions we include a few excerpts we collected from interviews conducted with six experts (four university professors or senior researchers in mathematics education, and two mathematics graduate students), each speaking a different language (English, Italian, Hebrew, Greek, German, Chinese-Cantonese). We asked each expert to answer in his/her own language, and then translate it into English, maintaining the translation as literal as possible, so we could also identify words and linguistic constructs referring to covariational aspects they saw in the graphs. The interviews were conducted remotely on Google Meet, and they were video-recorded. At the beginning of the interview the researcher would send the interviewee files containing the graphs in Figure 1; she would then ask the interviewees to share their screen as they answered her questions: "How would you describe the function represented by the graph shown below?", and: "What does a graph of a function convey to you about it?". She did not tell the interviewees that she was interested in their dynamic perception of covariation in the Cartesian graphs, so the number of the excerpts reported from each language is not an indication of that language offering more "dynamic" terminology, but simply of the expert being more explicit about his/her perception of covariation and of "dynamism" in the static graph.
In answering the first question, the experts mentioned:

"The graph of a function is to let you see the relationship between the variables. A picture/image." (translation from Chinese-Cantonese)

"The relation between two variables. The way two magnitudes associate each other." (translation from Greek)

"The graph of a function shows how this function evolves." (translation from German)

"[...] how a function converges, [...], sometimes you can assume that there is an exponential or logarithmic behavior, [...] when the things oscillate, if they oscillate "strictly" [...] overall you can see lots of things: the slope, the variations, sometimes the curvature [...]. The graph tells me its behavior, the values it takes on, information about how it changes, so its derivative." (translation from Italian)

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“When I look at a graph what I think about is a global behavior of a behavior in the region I am looking at. The behavior of a relationship between – in this case we are talking about x and y – as x changes how y changes with it. I see something going up and down [moves finger up and down along an imaginary y-axis] as this thing sweeps [moving hand in open extended position from left to right along an imaginary x-axis]. It tells you what you can see of a function, not what the function is, completely.” (English)

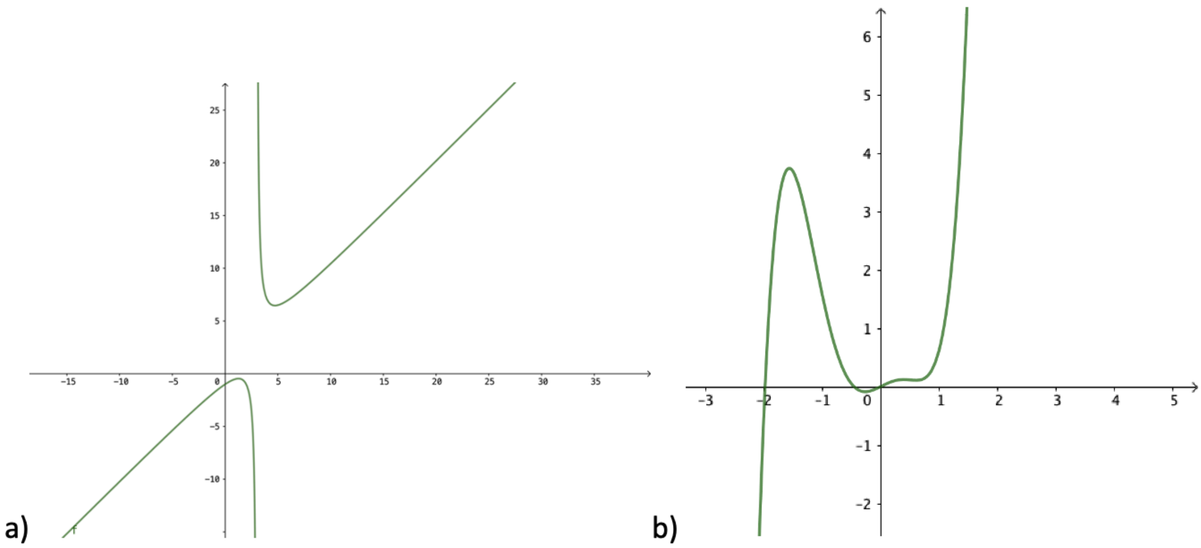

Figure 1 a) and b): The two graphs assigned to the experts who were interviewed

The experts were then asked to open graph a) and b), respectively, and share their screen.
In graph a) the following excerpts suggest the experts’ perception of covariation and of “dynamism” in the Cartesian graph.

“The graph here is getting closer to this straight line. Right up corner and left down corner [pointing to the lower left and upper right corners of the graph]. Here there is a vertical line, closer to this vertical line, going up and going down [pointing to the invisible asymptote].” (translation from Chinese-Cantonese)

“The two variables seem to behave proportionally [...] but x approaching 4 there is an abnormality. When x is close to 4 the y variable becomes extremely small from the left and extremely large from the right.” (translation from Greek)

In the description of the German interviewee we noticed constructs to which we struggled to find a literal translation, but that seem to refer to dynamic types of behavior of the function. For example, “polar zone” (German: Polstelle) referring to the function’s behavior around its vertical asymptote, and something like “slope ratio” (German: änderungen des

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Steigungsverhältnis) that refers to “changes of change”, an idea that is closely related to the notion of derivative. The interviewee also spoke dynamically about the function’s behavior near the “pole” (vertical asymptote), both addressing changes in the dependent variable and in its change (slope of the function) in relation to x “moving from left to right”: “If x is less than 5 and the pole, there the behavior of the graph doesn’t increase but it decreases, but the slope ratio from right directly towards the pole becomes infinitely large [...] from left to right the slope decreases and over here [pointing to the right part of the graph] the slope increases, and both when x goes to negative and positive infinity we move towards a constant slope.”

Similar dynamic elements seem to be present in the English-speaking expert’s view: “As I sweep from all the way on the left to all the way on the right [moving the cursor across the x-axis in the static graph], x is increasing, I see y [moves the cursor along the curve in the static graph] increasing, it looks pretty linearly until it gets to zero, then it tapers off, and suddenly drops and comes back in [moving the cursor off the bottom of the screen then back down from the top along the graph] [...] I think of that as if it wrapped around my head [gestures with his hand around his head] as if it's continuous, which the definition doesn't let it be, but that's what I picture. Then it drops down a certain amount and starts increasing again. That's a global behavior.”

In graph b) other aspects we noticed of the experts’ perception of covariation and of “dynamism” in the static graph are the following.

In Chinese the part of the graph on the right is described to “go up without bound” (and analogously on the left to “go down without bound”). Moreover, what in English could be referred to as “fluctuation” of the dependent variable in correspondence of values between -1 and 1 of the independent variable is translated literally as “up up down down”, as a sort of “wavy motion”. The expert, looking at this graph also said: “x and y seem to vary very quickly [pointing to a local maximum] and here x and y vary slowly little little.”

Reference to “fluctuation” also appears in other languages, described also as “going up and down” (Greek), “changing behavior” (Italian), “wiggling” (English), “moving from local minima to local maxima.” (Hebrew)

Other dynamic references that appeared in different languages were: the function’s “cutting the x-axis” (Hebrew), “going under” or “changing sign” (English and Italian), being “increasing” or “decreasing” (Italian), and “inflection points, where the slope changes sign.” (German and Italian)

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1.2 Focus of this chapter

Being able to speak (and therefore think) about covariation, perceiving dynamism in Cartesian graphs, is an important skill that is intrinsically related to functional thinking, and necessary for making sense of concepts in Calculus such as derivative or integral of a function. However, mastering these concepts is known to be complex, as shown by several studies in the literature. As we will discuss in the next section, secondary school students frequently fail to develop this kind of language and way of "seeing" (and thinking).

The problem we addressed in our research and that we present in this chapter is the following: since the idea that covariation is "incorporated" into a Cartesian graph of a function usually gets lost for most students, how can we help students "see" what experts do, guiding them through a sequence of activities with appropriately designed dynamic interactive digital artifacts?

To address this question, we designed and implemented a sequence of activities in a digital interactive environment capitalizing on its potential to let students experience both the variations and the asymmetry in the behavior of varying objects that can be dragged on the screen. Thanks to digital resources, we will explore new ways to introduce functions, before giving a formal definition, focusing on specific meanings and ways of thinking that we see as key in addressing and overcoming some well-known difficulties students encounter. Specifically, through appropriately designed digital resources assigned to students in a certain sequence, we try to foster high school students' skill of "seeing" covariation in a Cartesian graph, like experts do.

The second section of this chapter gives an overview of students' difficulties with covariational aspects of functions and Cartesian graphs of functions; it is followed by a section presenting a review of the main studies in the literature that provide the groundwork for our sequence of activities with dynamic interactive digital artifacts. In the fourth section we introduce our theoretical framework and define the *dynamic asymmetry potential* (DAP) and the proposed sequence of activities that make use of it. In the fifth section we show how students can come to appreciate specific aspects of the notion of function, tightly related to the DAP and recognize them in the traditional static Cartesian graph representation.

In the discussion we share thoughts about curricular changes - somewhat in line with those proposed over a century ago by Felix Klein (see Weigand et al., 2019) - that we believe should occur, at least at the secondary school level, to offer students a learning pathway that takes advantage of digital resources such as those discussed. In such a pathway we see advantages both in terms of addressing and overcoming common student difficulties with certain meanings of function, and in terms of mathematical learning in general, since research has shown that dynamic interactive applets and textbooks can provide an environment for students to communicate in new, meaningful, and inclusive ways.

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2. Students' difficulties with function as covariation and Cartesian graphs

There is a wide literature describing a variety of possible difficulties related to the learning of functions. We focus mainly on two such difficulties that are deeply related to each other, namely the difficulty that students often have in seeing function as covariation and the difficulty in reading and interpreting Cartesian graphs.

When students begin to explore relationships between quantities that co-vary or they are asked to interpret what a function represents in reality, e.g., how the speed of a body varies with respect to time, they are expected to evaluate how one variable changes, while imagining subsequent changes in the other (Carlson et al., 2002). In this chapter we refer to covariation in functions as an asymmetric relationship that links two variables, such that one's variations depend on the variations of the other one.

We acknowledge that there are also studies in mathematics education dealing with the study of covariational reasoning intended as the ability to identify the relationships of invariance between two generic quantities (Arzarello, 2019; Bagossi, 2022).

Thompson and Carlson have contributed significantly to developing the current view of covariational reasoning as a theoretical construct, which is presented in the literature (Thompson & Carlson, 2017). They revised prior covariational frameworks and elaborated a detailed description of the covariational aspects characterizing functions. Functional dependency is seen as a correspondence, and functions as entities that accept an input and produce an output; but it is also seen as a process involving *two quantities varying together,* linked by an asymmetric relation. Some representations of functions emphasize one of these two aspects over the other.

The covariational view of functions has been found to be essential also for understanding other concepts of calculus that are related to functions, such as limits and derivatives (e.g., Kaput, 1992; Cottrill et al., 1996). For example, students who have acquired a dynamic view of function as covariation might consider the algebraic formula $A(l) = l^2$ as a means of determining the area of a square for any set of possible side lengths. They would be able to see that as the value of the side length increases, the value of the area also increases, and that as the value l increases, the rate of growth of $A(l)$ increases more and more (Thompson, 1994). In line with this idea, Nagle and colleagues (2017) highlighted the importance of fostering students' dynamic imagery to include simultaneous movement in both the x and y coordinates to avoid the common conception of the limit of a function as the value reached near a certain x-value.

The covariational view is often difficult to develop and many students tend to make mistakes in contexts where they are asked to explore a function that models a dynamic situation, such

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as the one about the area model described above, or to construct functions or graphs of functions that model real situations (Thompson & Carlson, 2017).
Lack of covariational reasoning also has serious implications in the manipulation of a Cartesian graph; such a representation is extremely rich in meaning and powerful but, at the same time, the interpretation of a Cartesian graph requires a deep understanding of the relations between its elements. As stated by Sierpinska (1988): "the most fundamental conception of a function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another" (p. 572).
As discussed in the first section of this chapter, a challenging hurdle for university and secondary school students is to see the Cartesian graph as a collection of points on the plane that describes a relationship between two variables. The coordination of the information that each point carries, connecting the function's domain and its image, gets lost in the learning process. Indeed, very often students identify and refer to "a point on the curve in the Cartesian plane", "the graph", "the independent variable", "f(x)" and "the function" as if they were all the same thing (Antonini et al., 2020). This is a delicate and key issue, because distinguishing between these elements and understanding their relationships is at the core of the concept of functional dependency.

Although digital technology has recently brought many affordances, what still happens is that very often students at different school levels associate the function to a static image, which is often hard to interpret for them. In particular, they tend to see Cartesian graphs as pictures representing a physical situation, e.g., a mountain, or a path of an object, e.g., the road traveled by a car, or the passing of time, even when neither variable is or depends on time at all (e.g., Leinhardt et al., 1990), and the variations and the relation between variations related to functional dependency remain hidden (Carlson & Oehrtman, 2005; Johnson et al., 2020).

These studies suggest that students are "blind" to what experts can "see" in a graph; so, students need the opportunity to engage in activities on functions that emphasize covariation for them to be able to "see" what experts see in graphs (but also in other representations of functions, such as input/output tables, algebraic expressions...).

3. Background studies inspiring the activities' design

At the end of the last century, the widespread availability of computers gave rise to the design of new approaches to functions, based on representations realized by implementing technological artifacts. Initially, the use of computers in calculus lessons involved programming and implementing numerical algorithms, then software for graphical tools appeared. For example, Cuoco (1995) described innovative approaches to functions that have been designed to support students' development of computational models for representing functions. For example, Function Machines focused on building input-output machines, while Logo was a language in which a sequence of instructions could be performed on a specific input to produce an output. Cuoco found that following an approach to functions in Logo "students who are able to model a situation with a Logo procedure are already viewing the function at hand as a process" (Cuoco, 1995, p. 12).

Schwarz and Dreyfus (1995) created an entire curriculum on functions, using a computer microworld called Triple Representation Model, to foster the use of different representations of functions (tables, graphs and algebra). Following such an approach students seemed to be able to cope with partial data about functions and to coordinate different representations.

Tall (1996) studied potentialities of magnifying the graph of a function represented on a computer screen. Such a magnification makes the graph look always less curved, until it becomes almost straight. When it looks visibly straight, the slope of the curve representing the function is the same as the slope of the line on the screen.

Another example is the SimCalc Project (Kaput Center, 2016), focused on the development of a mathematical curriculum based on the exploration of motion in animated worlds modeled through functions. Indeed, motion, in the sense of the variation of space in time, can provide good grounds for introducing functional dependency, since motion is a foundational metaphor for the notion of variation: "In our everyday conceptual system, change is understood metaphorically in terms of motion" (Lakoff & Núñez, 2000, p. 406).

Laborde (1999) describes how such a metaphor of motion underlies the representation of functions in the Cartesian plane: the Cartesian graph can be interpreted as the trajectory of a point, representing the dependent variable, that moves in relation to the variation of another moving point, representing the independent variable. This view of the Cartesian graph in terms of covariation can seem very natural because it involves the temporal dimension, that characterizes out daily lives. While an expert mathematician sees covariation in the graph, it is very hard for students to acquire such a perspective, as described in the literature (e.g., Dubinsky & Harel, 1992; Leinhardt et al., 1990).

A main reason may be attributed to the approaches traditionally adopted to introduce functions and graphs. According to Falcade et al. (2007), even if students associate the concept of function to the idea of correspondence, they frequently do not perceive the covariation of the two variables. To foster this understanding through a dynamic interpretation of the notion of function, Laborde and Mariotti (2001) designed a sequence of activities in Cabri, in which students first experience variables' movements, to provide "a qualitative experience of covariation, and, in particular, an experience of functional dependency not primarily based on a numerical setting" (Falcade et al., 2007, p. 318).

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Other research has made use of dynamic interactive artifacts for studying graphs of functions. For example, Hazzan and Goldenberg (1997) proposed an approach to functions built on dynamic geometry, trying to shadow the purely numerical aspects characterizing this mathematical concept. Their approach consisted in analyzing geometrical constructions by looking at the functional dependency that linked the geometric objects involved, such as points and lines.

Hoffkamp (2011) analyzed the role of dynamic aspects of functional dependency shown simultaneously in different representations in students' development of variational and covariational reasoning. Moreover, Ng (2016) found an interplay between students' gestures and their dragging actions used to convey dynamic and temporal relationships in functions.

The original studies by Goldenberg and colleagues (1992) have inspired the research of other authors such as Sinclair and colleagues (2009), and Antonini and colleagues (2020). More specifically, they have found that certain digital interactive artifacts that for experts in mathematics are representations of functions can be effective in helping students experience and talk about covariation. Indeed, Antonini et al. (2020) used discourse analysis to analyze students' written narratives about similar dynamic digital interactive artifacts; the study shows how pointwise, local and global properties of function can emerge from the interaction with such artifacts. In particular, the authors found that students' written discourse may focus on the behavior of isolated points, neighborhoods of points, points as limits; or it may capture variations and invariants in specific intervals; or it may express properties of the function that are general. Antonini and Lisarelli (2021) also advanced theoretical and methodological considerations for designing a sequence of tasks based on the implementation of such digital interactive artifacts.

The interactiveness and the dynamism offered by environments, such as those used in the studies discussed above, positively influence the development of students' learning process in mathematics, fostering the creation of new images and semiotic tools that allow the construction of mathematical meanings (Antonini & Lisarelli, 2021; Baccaglini-Frank, 2021; Hollebrands et al., 2021).

4. The dynamic asymmetry potential (DAP) and its exploitation

The key functionality of the kind of dynamic geometry and algebra environment that we used to design the sequence, in our case the graphic view in GeoGebra, is what we call the *asymmetry* functionality (which is also present in other dynamic geometry and algebra environments). Such a functionality refers to this: a new point (or object in general) can be defined through properties that relate it to other digital objects previously constructed, and when the figure is acted on through dragging, the dependent point (or object) maintains the properties through which it was defined. The simplest case is, perhaps, a point that belongs to a line: if the line is dragged, the point remains on it; if the point is dragged, it moves, but stays linked to the line. Another example is a point P constructed as the symmetric image of a point A through line symmetry across the line *r*. If A or *r* are dragged, P will move remaining the image of A, while P cannot be acted on directly, since it is completely dependent on A and *r*. Therefore, we say that points A and P do not have a symmetric behavior for the person trying to act on them through dragging: this becomes particularly evident when the person attempting to change the figure clicks on P (or touches it in the case of a touch screen), attempting to drag it, but it does not move; however, they realize that P can be moved indirectly, by acting on A.

What we refer to here as the asymmetry functionality has been the focus of various studies in the literature, many of which in the domain of geometry. One of the most significant studies was, perhaps, that of Arzarello and colleagues (2002), who identified various dragging modalities used by secondary school students when asked to solve open problems by producing conjectures in geometry. One of these dragging modalities, then called *dummy locus dragging*, was used by students to maintain a certain geometrical property (not necessarily involving the dragged point directly) while the figure changed as an effect of dragging one of its points. Later, Baccaglini-Frank and Mariotti (2010) further explored this functionality implemented through *maintaining dragging,* now a way of dragging, explicitly taught to secondary school students, who were then asked to generate conjectures in open problem situations. Maintaining dragging exploits the asymmetry functionality in the following way: the figure's movement, and the invariant property induced indirectly, depends on the movement of the dragged point, that the student controls directly (e.g., Antonini & Baccaglini-Frank, 2016; Baccaglini-Frank, 2019).

The asymmetry functionality (though not referred to in these terms) has been used in some of the studies mentioned in the previous section of this chapter (e.g., Hazzan & Goldenberg; 1997; Falcade et al., 2007).

For such an asymmetry functionality to be considered a "potential" we need to relate it to specific learning goals and to ways in which it can be used by teachers and students. We will frame this discussion, leading to the introduction of what we will call the *dynamic asymmetry potential* (DAP), in the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008), that will also provide the theoretical perspective we use to introduce the sequence of activities.

4.1 The DAP as the semiotic potential of an artifact

The Theory of Semiotic Mediation has its roots in Vygotskian socio-constructivism, according to which students are guided by their teacher to construct mathematical knowledge by solving appropriately designed tasks with the use of appropriately designed and chosen artifacts (Bartolini Bussi & Mariotti, 2008). Students' activity with artifacts allows the unfolding of their *semiotic potential*, that expresses the relationship between the students' personal meanings emerging from the experience of acting with the artifact and the mathematical meanings recognizable by the expert in such actions.

As we have seen from the examples in the previous section, the asymmetry functionality in dynamic digital environments can be used to design digital artifacts and tasks to be carried out using such artifacts, with underlying educational goals concerning the construction of specific mathematical knowledge. In this case we will speak of the *DAP* of a digital interactive artifact (hence the attribute "dynamic") and a given task (or set of tasks), with respect to the mathematical knowledge of which the activity with the artifact is supposed to foster construction.

As an example of digital artifact designed to exploit the DAP, let us consider the classical example of DynaGraphs[1] (Goldenberg et al., 1992), dynamic representations of functions (from R to R) in which the relationship between the dependent and independent variables can be captured only thanks to the DAP. In a DynaGraph the input and output were represented separately, each on its own (horizontal) visualization of a segment of the real line. These horizontal segments were originally referred to as "the x Line" and "the f(x) Line". The independent variable was a draggable point on the x Line, and the dependent variable was a point that would move accordingly on the f(x) Line.

An example of task that can be assigned to secondary school (or younger) students with the DynaGraph is: "Explore the dynagraph and write down your observations" (this was assigned, for example, in Antonini et al., 2020). The student can obtain two possible types of movement: indirect and direct. Direct motion occurs when a basic element is dragged by acting directly on it; while indirect motion occurs when there is a construction of elements that depend on the basic elements. The independence of the x variable is realized by the possibility of freely dragging a point, bound to a line (the x Line), and the resulting movement visually mediates the variation of the point within a specific domain. The dependence of the f(x) variable is realized by an indirect motion: dragging the independent variable along its axis causes the motion of a point, bound to another line (the f(x) Line), that cannot be directly dragged. This concludes a general a priori analysis of the DAP with respect to the mathematical notion of functional dependency. In the following section we will give more specific examples of DynaGraphs and other digital artifacts used in our sequence to exploit the DAP with respect to specific mathematical properties of functions.

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[1] We use "DynaGraph" to refer to the original constructions described in the literature and used by Goldenberg, Lewis and O'Keefe, and "dynagraph" for the constructions used in this study.

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4.2 DAP in the designed sequence of activities

The analysis of the previous section revealed interesting aspects of the semiotic potential of dynagraphs for the evolution of students' signs towards mathematical meanings characterizing the concept of function, such as dependent and independent variable and covariation. A DynaGraph is generally made up of two lines, the x-line and the f(x)-line, that are parallel and positioned horizontally on the screen; the relationships between the DynaGraph and the classical representation (with a 'static' Cartesian graph) of real-valued functions of real variables are not particularly transparent. With the aim of promoting in students a dynamic view of standard Cartesian graphs, and of overcoming the difficulties described earlier (see Section 2), we will play with the position of the two lines in dynagraphs, thus constructing a set of different artifacts (see Antonini & Lisarelli, 2021).

The main features of our artifacts are the DAP, which is the main focus of this chapter, and which we have already analyzed in the previous section, and the Trace tool that allows displaying the trajectory of a point on the screen during the movement (for example, by activating the trace on x or on f(x) when x is dragged within a part of the x-line it will be marked [a,b], that is the set in which x varies, or f([a,b]), that is the set in which f(x) varies for x dragged within [a,b]).

We will use the notation[2] $DG_{\text{1-line}}$, $DG_{\text{II-lines}}$, $DG_{\perp\text{-lines}}$ referring to different graphs, where DG stands for "dynamic graph" and we indicate using subscript the positions of the lines (1-line: one–usually horizontal–line, II-lines: two parallel lines, ⊥-lines: two perpendicular lines). DGc stands for the dynamic Cartesian graph that completes the $DG_{\perp\text{-lines}}$ with two lines that are parallel to the x-line and to the f(x)-line, respectively, one passing through (0,f(x)) and the other passing through (x,0) and that intersect at the point (x,f(x)). Moreover, we denote by SGc the Cartesian graph that can also be drawn on paper, where SG stands for "static graph" and c for "Cartesian plane".

Table 1. The different graphs used in the activities.

Symbol	**Screenshot**

[2] These notations are different from those used in Antonini and Lisarelli, 2021.

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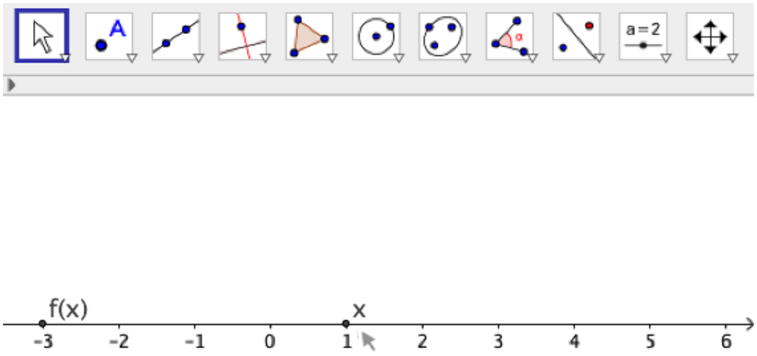
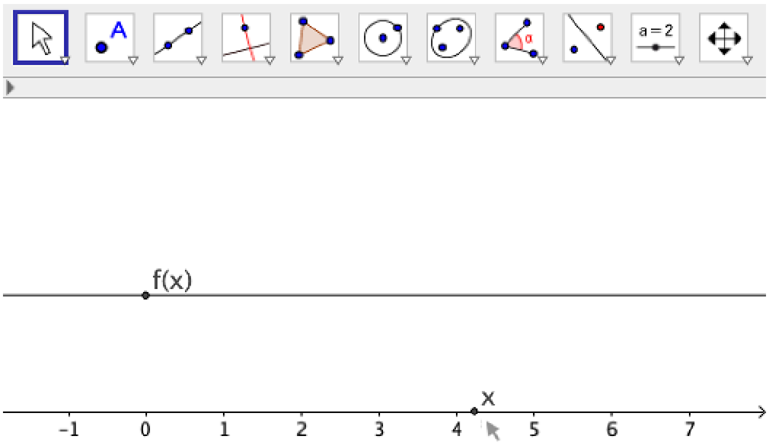
DG$_{\text{1-line}}$



DG$_{\text{II-lines}}$



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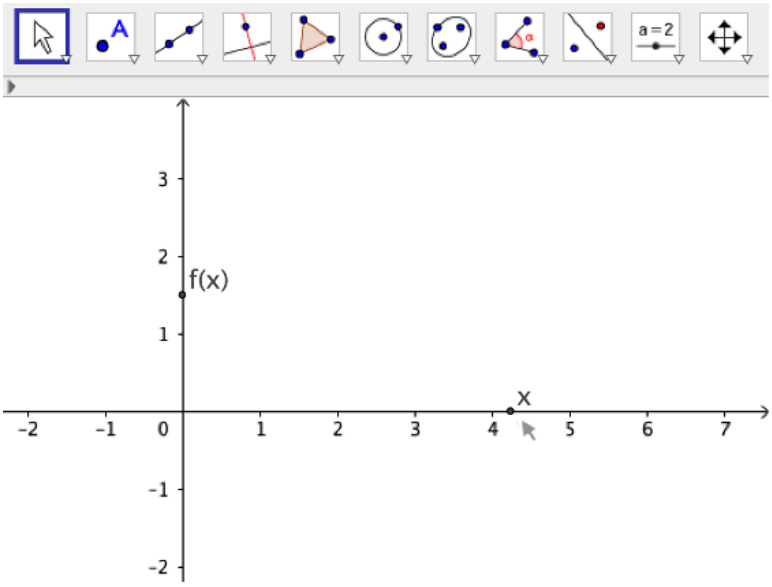
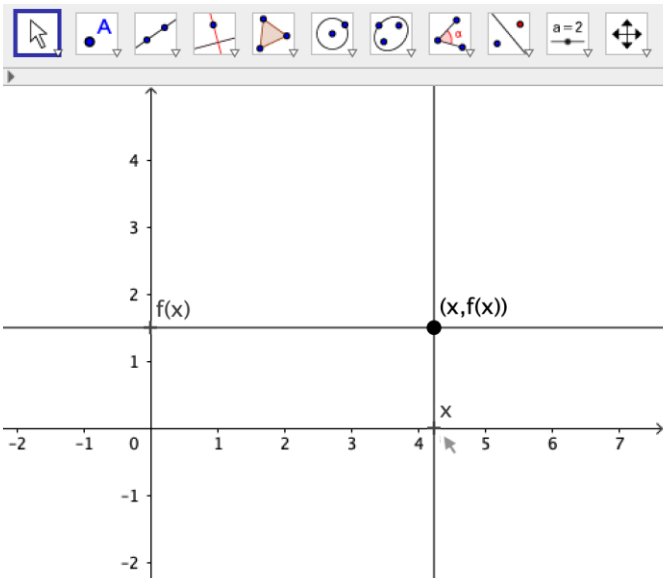
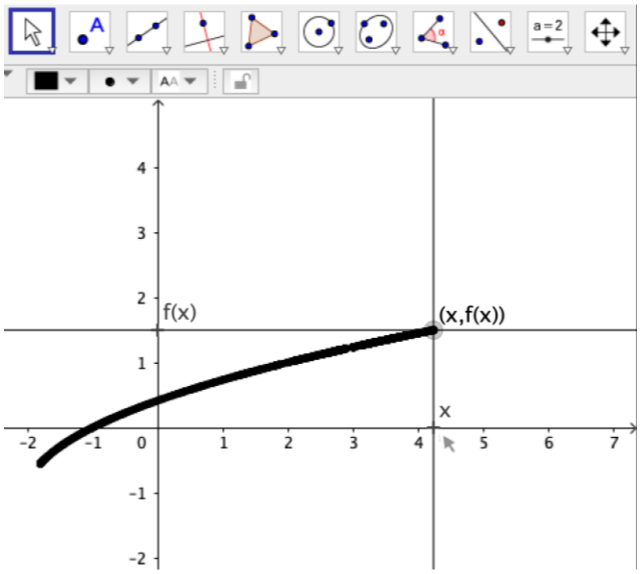
$DG_{\perp\text{-lines}}$	
DGc	
DGc with the Trace activated on (x,f(x))	



The semiotic potential of each of these artifacts, and in particular the DAP, expresses the relationship between students' personal meanings emerging from the experience of asymmetry of direct and indirect control during dragging and the mathematical meanings of relation between dependent and independent variables. Furthermore, in addition to the semiotic potential of each artifact, there is the potential of the whole set of artifacts in the specific didactical sequence in which students will use them. The sequence aims to promote processes leading to "seeing" covariation in the SGc, and to foster the evolution of students' personal signs towards mathematical meanings, that may be linked to the use of artifacts previously involved in the didactical sequence itself.

Now we delve into the sequence to deeper analyze how it exploits the DAP and the Trace tool functionalities.

$DG_{1\text{-line}}$ and $DG_{II\text{-line}}$ are designed to construct the meanings of dependency and of covariation, that are related to the concept of function, but these graphs are still far from the SGc. The two artifacts $DG_{\perp\text{-lines}}$ and DGc play the role of mediators between $DG_{II\text{-line}}$ and SGc, because they have close relations with both, not only from a mathematical point of view, but, above all, our hypothesis is that they can play a key role in mediating the evolution of students' personal signs. More specifically, our hypothesis is that the personal signs produced while working with the artifacts $DG_{1\text{-line}}$ and $DG_{II\text{-line}}$ are then transformed into personal signs related to the interaction with $DG_{\perp\text{-lines}}$ and DGc. Finally, the same signs will result in a "dynamic view" of the SGc. Indeed, in the SGc students should then be able to recognize the curve as the trajectory of the point (x, f(x)), since the movement of the point (x,f(x)) can now be seen as a combination of the movements of the two variables, x and f(x), moving along two different

lines in the previous dynagraphs. Therefore, the dynagraph $DG_{1\text{-line}}$ evolves, changing its shape, to become the SGc.
This process of change passes through various representations of function that can be seen as dynamic snapshots of the evolution of the artifact, and each of these representations is also manipulable. From covariation represented in the same direction of movement for the two variables, there is a transition to covariation in two orthogonal directions, which is more difficult to handle. The point (x, f(x)) incorporates both movements of the two variables, and the activation of the Trace tool on this point may be helpful for handling the covariation because, by showing the trajectory of the point in the Cartesian plane, it can make the coordination of the two variations easier to appreciate and manage.
From an expert's point of view, each artifact, as well as the entire set of artifacts, incorporates mathematical knowledge: each artifact represents a real-valued function of real variable; the various artifacts are certainly different, but they are related to one another, and they can represent the same thing (the same function).
In summary, the use of the $DG_{1\text{-line}}$ allows students to experience and describe in words what happens when moving the point x within the designed artifacts: students can experience and perceive what an expert "sees" in the SGc. The didactical sequence of activities guides the students, by transferring their personal signs from one artifact to another maintaining key meanings, and supports their development of a dynamic view and description of the SGc, in which "seeing" the covariation is traditionally particularly difficult.
In this proposal, we reverse the traditional teaching and learning process that starts from the introduction of Cartesian graphs, taking for granted a dynamic view, interpretation and reading of it. This inversion has the goal of firstly promoting a dynamic view and allowing students to experience the covariation of two variables, and secondly of making this view become the foundation for the Cartesian graph as a representation of function.
To investigate possible effects of our approach on the teaching and learning process of the concept of function, in the next section we focus on how the DAP can be exploited during students' learning processes, providing examples from our research.

5. Examples of how the DAP can be exploited

In this section we show how students can come to appreciate specific mathematical properties of functions, tightly related to the DAP of the digital artifacts we designed, and recognize them in the standard Cartesian graph representation. We present a short sequence of coherent personal signs that students construct and take along, with minor evolutions, from the dynagraphs to the Cartesian graph.
The designed sequence presented above can be proposed to both middle and high school students. Similar activities have been experimented with and analyzed through different

theoretical lenses (Antonini et al., 2020; Lisarelli, 2018; 2019a). Taking a semiotic mediation perspective, we now report on the emergence and evolution of personal signs about some properties of functions, produced by students who were introduced to functions for the first time with dynahgraphs, at least concerning their high school mathematics experience. Data in this section comes mainly from Lisarelli's doctoral work (Lisarelli, 2019b), who carried out the study in a 10th grade class of an Italian high school specialized in Mathematics and Science.
Here we focus on the DAP of our dynagraphs and present a selection of excerpts from the video-transcripts that contain instances where students express covariation. We focus on three students, Alessio, Nicco and Mati (these are their pseudonyms), who worked together during the activities. We chose the following mathematical meanings concerning aspects of functions from the larger set of meanings touched on during the sequence:

a) the distinction between the independent and the dependent variable;
b) the properties of a function of being increasing or decreasing in certain intervals;
c) the covariation between variables in the Cartesian graph.

Such meanings were those most frequently touched on by all the students in our study. Indeed, during the exploration of the digital interactive artifacts all students produced signs for describing meaning (a). Many signs also emerged concerning the relationship between the directions of movement of the two points, related to the meaning (b). Indeed, for an expert these signs can be seen as "seeds" for the mathematical meaning of "decreasing/increasing function". Moreover, almost all the students spoke about changes of speed of the point representing the dependent variable; this is related to the slope of the curve in the Cartesian plane, so to the meaning of "derivative function" of a function. Meaning (c) relates specifically to the starting point of this chapter.
In the following subsections we present excerpts from the activities conducted in the experimental classroom, to shed light onto the evolution of students' personal signs related to the mathematical meanings in focus. Section 5.1 contains a brief excerpt from a discussion between Nicco and Alessio that occurred during the first activity with the artifact $DG_{1\text{-line}}$. In section 5.2 we reconstruct a sequence of signs produced by students during various activities that, in the eyes of an expert, can be seen as related to the mathematical meaning (b) of "decreasing/increasing function". The excerpt in section 5.3 shows signs emerging in the context of a standard Cartesian graph, drawn out on paper.

5.1 Identifying the functional relationship

The students' very first exploitation of the DAP of the artifacts $DG_{1\text{-line}}$ and $DG_{\text{II-lines}}$ occurs when they realize that both points move (if the function is not constant), but only one can be directly dragged. During the first exploration of a $DG_{1\text{-line}}$ students discuss the asymmetry between the movements of the two points, looking for an agreement about "what moves". The excerpts below (Table 2 and Table 3) shows part of the interaction between the pair of students Nicco and Alessio, with the researcher (R), during an activity involving the $DG_{1\text{-line}}$ of the function f(x) = –x+5 (the students are not given the algebraic form) and the task: "Explore the construction, identify and describe possible movements by using the dragging tool and write down your own observations on a sheet of paper". The students' main focus seems to be to find suitable signs for expressing the asymmetry that they perceive.

Table 2: Excerpt 1 ($DG_{1\text{-line}}$)

	Who	**What is said**
1.1	Nicco	They move both… we move just one of them
1.2	Alessio	No they move both. They move both because, that is, with respect to the two fixed points that are zero and one,
1.3	Alessio	by moving maybe B [x] to the right, A [f(x)] moves to the left and then it goes below zero
1.4	Alessio	and by moving B [x] to the left, A [f(x)] goes to the right

Table 3: Excerpt 2 ($DG_{1\text{-line}}$)

	Who	**What is said**
2.1	Nicco	That one does not move
2.2	R	Let me see, do they move both or does it move just one of them?
2.3	Alessio	Aaah just one of them moves, it's true!
2.4	R	But what does it mean that just one of them moves?
2.5	Alessio	That is, they move both but one of them moves in function of the other one, of A [f(x)] and B [x], that is, B [x] can move while A [f(x)] moves in function of B [x]

Nicco's concerns are about distinguishing the different behaviors of the two points under dragging, as his words in line 1.1. Alessio seems confident in his description of the movements of one point, that he calls A, depending on the movements of the other point, that he calls B. This dependency is described through references to the passing of time (line 1.3) and to different degrees of freedom in the movements (lines 1.3, 1.4, 2.5).

5.2 Identifying the property of being increasing or decreasing

Now we present a short sequence of personal signs produced by the three students along the entire sequence of activities, concerning the property of a function of being increasing or decreasing. We have tried to provide give a little context by situating the signs within brief excerpts chosen from the complete dataset collected. Each table (Tables 4, 5, 6, and 7) contains a brief excerpt and reference to the digital interactive artifact the students were working with. We added emphasis to the emerging signs.

Table 4: Excerpt 3 ($DG_{1\text{-line}}$)

	Who	**What is said**
3.1	Alessio	They *move symmetrically* and *in opposite directions*, that is, *if this one moves some, the other one moves* of the same amount but in the *opposite direction*.
3.2	Alessio	They *move symmetrically* and *in opposite directions*, that is, *if this one moves some, the other one moves* of the same amount but in the *opposite direction*.
3.3	Mati	*If one decreases, so does the other*, *if one increases, so does the other*.
3.4	Alessio	No, *if one increases the other decreases*.
3.5	Alessio	I think they *move symmetrically*, because *if one goes farther away, the other goes farther away*, *if one gets closer, the other gets closer*.

Table 5: Excerpt 4 ($DG_{II\text{-line}}$)

	Who	**What is said**
4.1	Alessio	Between one and five, we wrote, in the time interval between one and five, *B moves in the opposite direction with respect to A*. A and B never meet each other.

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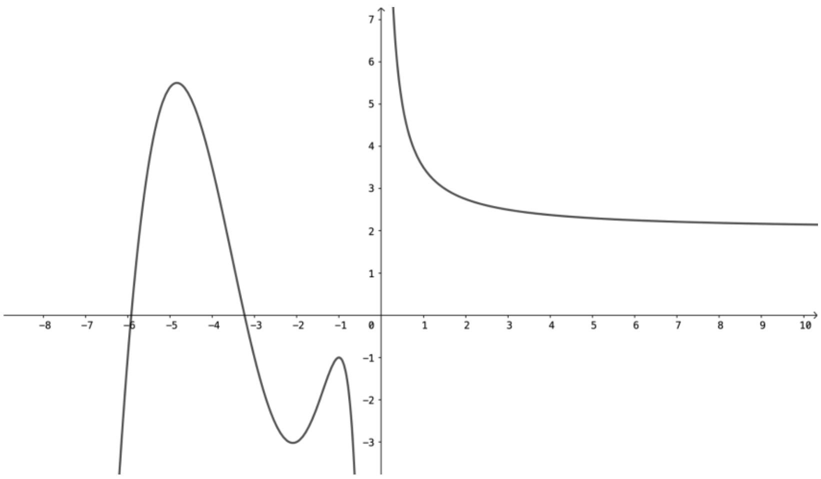
Table 6: Excerpt 5 ($DG_{\perp\text{-lines}}$)

	Who	What is said
5.1	Mati	Yes anyhow in the end it's like if they were, in quotes, inversely proportional, because *as one increases the other one decreases*. Indeed, when A increases, B starts moving, it enters also into the negatives, and also when A enters into the negatives, B [is] in the positives, and so *it increases and decreases*.
5.2	R	Where were you saying that you were surprised?
5.3	Mati	Over here? That it *went down and then up again*?
5.4	Nicco	Also here where it *goes down and does not come back up*, and the same going over there.
5.5	Nicco	This *goes back down*, this keeps *going forward*, eh, yes, you did it right.

The task assigned in Excerpt 6 started from the $DG_{\text{II-line}}$ of the function represented algebraically and graphically in Figure 2, and asked*:* "Draw on a sheet of paper the graph of this function on the Cartesian plane".
The students worked with a dynagraph: the static representations in Figure 2 are only to help the reader imagine what the students talk about.

$$f(x) = \begin{cases} \dfrac{3}{2x} + 2 & x > 0 \\ \dfrac{(x+1)^2(x+6)(x+3)}{x} - 1 & x \leq 0 \end{cases}$$

a)



b)

Figure 2: a) Algebraic representation and b) graphical representation of the function the students work with in the dynagraph $DG_{\text{II-line}}$ representation

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Table 7: Excerpt 6 (from $DG_{II\text{-}line}$ to SGc)

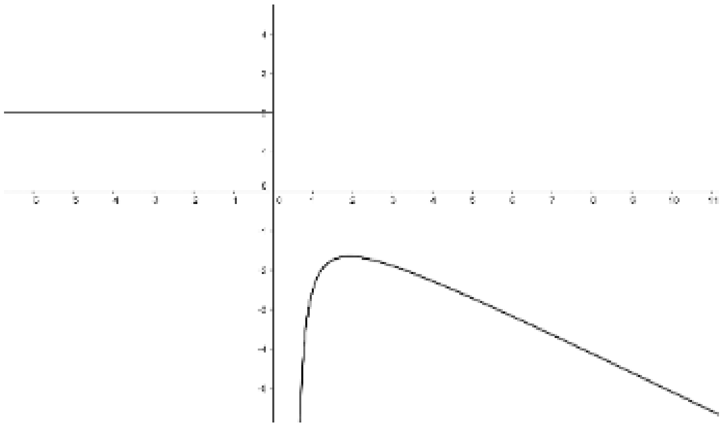
	Who	**What is said**
6.1	Mati	I don't think it's right because actually it *comes back* [f(x) in the $DG_{II\text{-}line}$], so it *goes down* [(x,f(x)) in the SGc] at a certain point.
6.2	Mati	Show me, here it *comes back*, because look at how it behaves, like here it goes forwards, I mean like the more I do like this, the more I do like this and *the more it goes up*,
6.3	Mati	so now I get, eh…*this does like this*, this goes up, wait a second, sorry, this goes up like this, this goes like this, and here there has to be a different connection…theoretically once, see here?
6.4	Mati	Look, eh, from negative five, where that is five and a half, *they decrease together*.

Notice that the students' signs developed to describe what for an expert concerns an increasing/decreasing behavior of a function remain almost the same through the manipulations of the various artifacts in the excerpts above. In the intervals in which the functions are increasing, the students refer to concordant movement (e.g., lines 6.1, 6.2, 6.4); while the intervals in which the functions are decreasing, they speak of "symmetric" movement or movement "in opposite directions" (e.g., lines 3.1, 3.2, 3.3, 3.4, 3.5, 4.1, 5.1, 5.5).
So, in this case, students seem to construct and carry with them, with minor evolutions, their personal signs from the DG1-line to the SGc. Indeed, an interesting finding is that we did not find evidence of significant changes in students' use of signs that can be related to the passage from one artifact to another one.

5.3 Seeing covariation between variables in the Cartesian graph

Such a result is consistent with findings from the passage from the dynagraph in one dimension ($DG_{II\text{-}line}$) to the standard graph in two dimensions (SGc), where students come "to see" what an expert in mathematics would do. In excerpt 7 two students are describing the SGc of the function in Figure 3. Table 8 also shows what signs appeared in the form of gestures.

$$f(x) = \begin{cases} 2, & x < 0 \\ -\frac{1}{3}\left(x + \frac{3}{x} + 2\right), & else \end{cases}$$

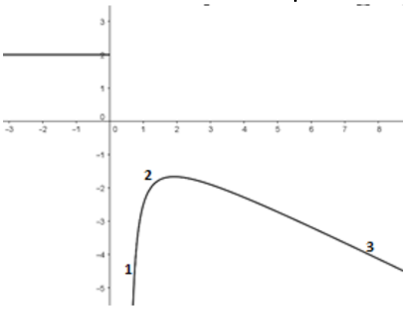


a) b)

Figure 3: a) Algebraic representation and b) graphical representation of the function the students are working with in Excerpt 7; they are given b) the Cartesian graph.

Table 8: Excerpt 7 (SGc)

	Who	**What is said**	**Gestures made**
7.1	M	This is okay, I mean this is y equals two	She points to the graph of the function for negative x-values
7.2	R	And how would y equals two be visualized with parallel lines? With parallel lines I mean in a GeoGebra file like the ones that we have seen with parallel axes	
7.3	M	Yes yes ehm there is one of them that stays still at two and y that moves	
7.4	N	No x moves, x moves and y stays still at two	He keeps the right hand fixed while moving the left hand horizontally:

			 **Figure 4**: Nicco’s gesture
7.5	M	While this one, you go forward with x and until it reaches one y goes down, I think, and then, going on, it comes up and then it goes down again…that is, you can drag x over here too [large positive x-values] and then y makes a sort of ...	After speaking she moves her hand down-up-down
		[…]	
7.11	M	But here [positive x-semiaxis] is x or y faster? No y, no no no, if you drag in this direction [to the right] ……..x is faster	
7.12	N	Yes yes, x is faster while y is slower I mean in this case it seems that here [1] y would move faster, ehm because from here to arrive up here [2] it takes like this[3], then from up here to go back to this point [3] it takes like this[4] and…it is the	He indicates different part of the graph: 

[3] In Italian he says “ci mette questo”, and he opens his fingers as measuring the distance between the points 1 and 2 on the curve.
[4] In Italian he says “ci mette così”, and he opens his fingers as measuring the distance between the points 2 and 3 on the curve in Figure 5.

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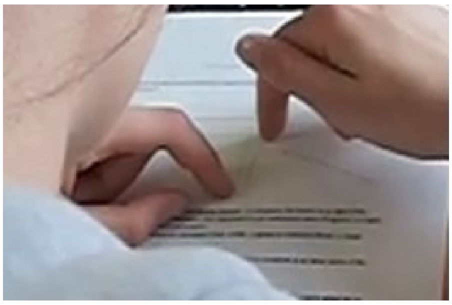
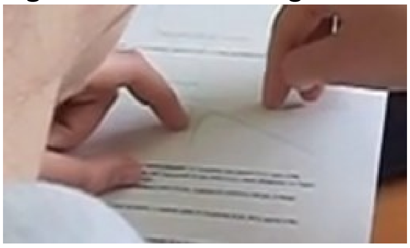
		same x, but faster, you drag one of them and so	**Figure 5**: Parts of the SGc indicated by Nicco as he speaks
7.13	M	Exactly, no x is always faster I think [Figure 6a]... [Figure 6b] Wait, no but, are we sure that this is faster? Wait! Why faster?	During the pause she moves the hands as follows:  **Figure 6a**: Mati’s first gesture  **Figure 6b**: Mati’s second gesture

Although dragging was not physically used by the students as they worked with paper and pencil, their expressions contained many references to the DAP of the dynagraphs. For example, Nicco moves his hands on the table describing the constant function y=2 (line 7.4) to show Mati a specific configuration of the dependent variable, which stays still, while x can be moved to the right and to the left.

Then the students discuss the relationship between the variations of the two variables in terms of their different speeds for positive x -values. They search for “*which one between x and y moves faster*” (line 7.11) and Nicco expresses his observations by pointing to different parts of the graph on the paper (line 7.12). Mati uses two gestures to mimic the movements of the two variables along the Cartesian axes (line 7.13).

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In summary, Mati and Nicco have come to dynamically see the properties of the function represented. Traces of the DAP in the signs produced by the students in previous environments persist, leading to a dynamic perception of covariation in the Cartesian graph.

6. Discussion

We have argued that the DAP of dynagraphs supports students in seeing how functions describe change and in developing a covariational view of functional dependency (e.g., Antonini et al., 2020; Antonini & Lisarelli, 2021). Thanks to the exploitation of the DAP, in the activities involving our digital interactive artifacts $DG_{1\text{-line}}$,$DG_{\text{II-lines}}$, $DG_{\perp\text{-lines}}$ and DGc, we have shown how students can come to perceive what experts in mathematics see in the traditional representation of functions through the Cartesian graph as we captured in experts' narratives. Indeed, as we have shown in Section 5, students' personal signs migrate from one artifact to another with minor changes, but augmenting the set of meanings to which they refer. Moreover, such signs can be interpreted by an expert as situated signs referring to very important mathematical properties of functions, such as monotonicity and derivative. Through the excerpts presented, we showed that our sequence of activities with the digital interactive artifacts helps accomplish the arduous task of fostering students' perceiving of covariation in the static Cartesian graph of a function
A key feature of the designed artifacts is the asymmetry functionality that can be exploited in the design of activities aimed at learning a variety of mathematical concepts–not only that of independent and dependent variables, and various properties characterizing the behavior of functions– such as the notions of premise and conclusion of conditional statements (e.g., Baccaglini-Frank & Mariotti, 2010; Antonini & Baccaglini-Frank, 2016; Baccaglini-Frank, 2019). In other words, dynagraphs provide easy entry venues into situated forms of mathematical notions that, with the mediation of the teacher, can evolve into more formal expressions of mathematical meanings (e.g., Lisarelli, 2019b; Antonini et al., 2020). Moreover, activities with dynagraphs, and dynamic interactive artifacts more in general, have been found to be quite engaging and motivational for students (e.g., Antonini et al., 2020; Baccaglini-Frank, 2021). These findings are in line with a great body of research suggesting that it is possible to design mathematical activities within dynamic interactive environments that have a positive impact on classroom practice, since they provide the ground for students to communicate in new, meaningful, and inclusive ways (e.g., Doorman et al., 2012; Lew, 2020).

Taking a different perspective, we appreciate a possible concern of some educators, who claim that introducing functions in this way is likely to improve students' understanding of certain aspects of such a mathematical concept, but at the same time it changes their "concept of function" (e.g., Günster & Weigand, 2020). In fact, in order to be incorporated into the mathematics curriculum, we think that our approach requires some reorganization of the curriculum itself.

Our proposed sequence of activities, involving the various artifacts for representing functions ($DG_{1\text{-line}}$,$DG_{\text{II-lines}}$, $DG_{\perp\text{-lines}}$, DGc and SGc), reverses the traditional teaching sequence of the notion of function, at least for what concerns the Italian mathematics high school curriculum presented in textbooks. Traditionally, students are introduced to functions through the presentation of the Cartesian graph of some examples of functions, while a dynamic view, interpretation and reading of this representation are by no means fostered, perhaps because taken for granted. In our sequence of activities, we actually do not change the mathematical concepts themselves, but we propose variations on the types of tasks and representations of functions used in such tasks, in order to promote functional thinking and highlight the covariational aspect that usually remains hidden from the eyes of the students. Instead of starting with a formal definition and with the Cartesian graph, we propose to first promote an informal dynamic view of functions and have students experience the covariation of two variables. These experiences can become the foundation on which to build the Cartesian graph as a representation of function.

We note that this kind of curricular shifts is somewhat in line with changes that were proposed over a century ago in the *Meraner Lehrplan*, by Felix Klein (Krüger, 2019). In particular, we share with him the objective of placing the concept of function at the center of teaching and learning mathematics in secondary schools, because it is a common and potentially unifying concept in various fields of mathematics. Indeed, "*Education in the habit of functional thinking* has to be considered a certain ability to perceive and analyse the variability of quantities and their functional dependencies" (Krüger, 2019, p. 49).

The educational approach suggested is a qualitative approach, as opposed to a quantitative one based on calculation (which is not necessarily "bad", just not what we are proposing). We see advantages in such a pathway, both in terms of addressing and overcoming common student difficulties with certain meanings characterizing functions, and in terms of mathematical learning in general.

Acknowledgments

We wish to kindly thank the anonymous experts whom we interviewed on what they "see" in graphs of functions. We also thank the high school students who participated in our sequence of activities with dynagraphs and hope that they also had fun, like we did.

Index terms

artifact

asymmetric relation

behavior (of a function)
Cartesian graph
covariation
covariational reasoning
coordinates
decreasing (behavior/function)
dependent variable
digital interactive environment
digital resources
drag/dragging
dynagraph
dynamic asymmetry potential (DAP)
function
functional dependency
graph
increasing (behavior/function)
independent variable
personal signs
representation (of function)

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