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THE INTERIOR BACKUS PROBLEM: LOCAL RESOLUTION IN HÖLDER SPACES

TORU KAN, ROLANDO MAGNANINI, AND MICHIKI ONODERA

ABSTRACT. We prove an existence result for the interior Backus problem in the Euclidean ball. The problem consists in determining a harmonic function in the ball from the knowledge of the modulus of its gradient on the boundary. The problem is severely nonlinear. From a physical point of view, the problem can be interpreted as the determination of the velocity potential of an incompressible and irrotational fluid inside the ball from measurements of the velocity field's modulus on the boundary. The linearized problem is an irregular oblique derivative problem, for which a phenomenon of loss of derivatives occurs. As a consequence, a solution by linearization of the Backus problem becomes problematic. Here, we linearize the problem around the vertical height solution and show that the loss of derivatives does not occur for solutions which are either (vertically) axially symmetric or oddly symmetric in the vertical direction. A standard fixed point argument is then feasible, based on ad hoc weighted estimates in Hölder spaces.

1. INTRODUCTION

Let Ω be a bounded domain in the Euclidean space \mathbb{R}^N , $N \geq 2$, with boundary Γ . Let g be a positive continuous function on Γ . The *interior Backus problem* consists in determining a function $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$(1.1) \quad \Delta u = 0 \text{ in } \Omega, \quad |\nabla u| = g \text{ on } \Gamma.$$

This problem was first considered and completely solved in [2] (see also [12]), for $N = 2$. There, Backus was motivated by a problem in geophysics, which entails the reconstruction of the gravitational or geomagnetic terrestrial field from measurements of its intensity on the Earth's surface \mathcal{S} . In fact, if one models the Earth as the unit ball B , then the relevant geophysical problem amounts to determine solutions u of (1.1), with $\Omega = \mathbb{R}^N \setminus \bar{B}$ and $\Gamma = \mathcal{S}$, such that

$$(1.2) \quad u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

This is what we call the *exterior Backus problem*.

When $N = 2$, by the Riemann mapping theorem, we know that any simply connected (seen as a domain on the Riemann sphere) proper subdomain of the complex plane \mathbb{C} is conformally equivalent to the unit disk. Moreover, the harmonicity of functions is preserved by conformal mappings, while the modulus of their gradients changes just by a positive factor (which depends on the derivative of the conformal map). So, that is one reason why problem (1.1) has some interest.

Another physical motivation, which genuinely pertains the interior problem setting, has to do with the study of incompressible and irrotational fluid flows. Let

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\vec{V} be the velocity field of a fluid and let ρ be its density. Any fluid flow obeys the continuity equation:

$$\operatorname{div}(\rho \vec{V}) + \rho_t = 0.$$

If a fluid is incompressible, its density is constant. In particular, we have that $\rho_t = 0$, and hence the continuity equation reduces to $\operatorname{div}(\vec{V}) = 0$. If the fluid is irrotational, then $\operatorname{curl}(\vec{V}) = 0$, and hence there exists a harmonic velocity potential u such that $\nabla u = \vec{V}$. Thus, solving (1.1) can be interpreted as the determination of the velocity of the fluid inside the domain Ω from measurements of its modulus on the boundary.

It is worthwhile to clarify from this point of view the results obtained in [2] for planar domains. As already mentioned, in this case we can always assume that Ω is the unit disk. We shall describe the situation in the simplest case in which g is assumed to be constant, say $g \equiv 1$. As shown in [2], or by simply invoking *Weierstrass factorization theorem* (see [16]), in the complex variable $z = x + iy$, when $g \equiv 1$, the complex gradient $u_x - i u_y$ of u is uniquely determined by the *Blaschke product*

$$u_x - i u_y = e^{i\alpha} \prod_{j=1}^n \frac{z - z_j}{1 - \bar{z}_j z},$$

where $\alpha \in \mathbb{R}$ and $z_1, \dots, z_n \in \Omega$ are given parameters. In fact, each factor in the product has unitary modulus on the unit circle. Thus, up to a rotation of an angle α , the velocity field of the fluid can be uniquely determined from its modulus on the boundary if we know the position and the nature (the multiplicity, so as to speak) of its *stagnation points* z_1, \dots, z_n .

In order to conclude our motivations, it may be of interest to mention [6], in which it can be found a possible application to encephalography by magnetic means.

When $N \geq 3$, we can still use the Kelvin transformation to map the exterior of the ball to its interior (or an exterior domain to a bounded one) by preserving the harmonicity of functions. However, the boundary condition in (1.1) changes quite considerably. In fact, if

$$\mathcal{K}w(y) = |y|^{2-N} w\left(\frac{y}{|y|^2}\right), \quad y \neq 0,$$

is the standard Kelvin transformation of a function $w : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}$, we have that

$$\nabla \mathcal{K}w(y) = |x|^N \left\{ \nabla w(x) - \left[2x \cdot \nabla w(x) + (N-2)w(x) \right] \frac{x}{|x|^2} \right\},$$

with $y = x/|x|^2$. In particular, if we apply the transformation to the exterior of unit ball B centered at the origin, we obtain that the condition $|\nabla u| = g$ on \mathcal{S} changes into

$$|\nabla U + (N-2)U\nu| = g \quad \text{on } \mathcal{S},$$

with $U = \mathcal{K}u$. Here, ν denotes the exterior unit normal vector field on \mathcal{S} .

Another feature for which the interior and exterior problems differ from one another is that the latter admits solutions u whose normal derivative u_ν does not change sign on \mathcal{S} (e.g. the fundamental solution or, more in general, the capacity potential of a bounded domain), while in the former the divergence theorem tells us that the mean value on \mathcal{S} of u_ν must be equal to zero. The positivity of the normal derivative of the solution has been useful in [8] to obtain the local resolution of the exterior gravitational Backus problem for the Earth near the so-called *monopole* $\Phi(x) = 1/|x|$, based on a fixed-point argument. In other words, when $N = 3$ and

$\Omega = \mathbb{R}^3 \setminus \overline{B}$, the existence and uniqueness of a solution u of (1.1)-(1.2) is obtained as the perturbation

$$u = \Phi + v,$$

where v is harmonic in Ω and decays to zero at infinity. This result holds if the data g is sufficiently close to 1 in a Hölder norm. (Notice that 1 is the modulus of the gradient of the monopole on \mathcal{S} .)

We also mention that the positivity of u_ν is also used in [4] to construct a comparison principle for suitably defined viscosity solutions for the exterior Backus problem and hence develop a nonlinear approach to the problem.

When the positivity property is not available, the only existence result up to date is given in [10]. There, we consider, in physical dimension $N = 3$, the local resolution of the exterior Backus problem (1.1)-(1.2) near the so-called *dipole*:

$$d(x) = \frac{x_3}{|x|^3}.$$

The gradient ∇d models the terrestrial geomagnetic field.

The problem of finding solutions of (1.1)-(1.2) of the type

$$u = d + v,$$

with v harmonic in Ω , which decays to zero at infinity, has another level of difficulty, though. We can see this if we linearize (1.1)-(1.2) near d . Indeed, we obtain the boundary value problem:

$$(1.3) \quad \Delta v = 0 \text{ in } \Omega, \quad \nabla d \cdot \nabla v = \varphi \text{ on } \mathcal{S}, \quad v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

This can be classified as an *irregular* oblique derivative problem. In fact, in contrast with the monopole case, in which the vector field $\nabla \Phi$ governing the linearized problem is nothing else than the normal field on \mathcal{S} , in (1.3), instead, the field ∇d points outward to the Earth's surface on the southern hemisphere, becomes tangential on the equator $\mathcal{E} = \{x \in \mathcal{S} : x_N = 0\}$, and points inward on the northern hemisphere. For this reason, (1.3) suffers of two drawbacks. The first one is a severe lack of uniqueness, since its solutions can be uniquely determined only up to prescribing Dirichlet boundary values on \mathcal{E} . The second one is a *loss of regularity*: the expected solution v does not gain the desired regularity. In other words, the regularity of v does not improve that of the data φ by one order — it can be seen that it falls short of $1/2$. This inconvenience makes the perturbation approach more complicated, because the iterative scheme on which a fixed-point argument is based upon *loses derivatives* at each step.

In presence of a loss of derivatives, the Nash-Moser implicit function theorem has worked in other contexts (see the pioneering works [15, 13, 14, 7], for instance). Unfortunately, in attacking the Backus problem, this plan is so far out of reach. In fact, one would need sufficiently precise estimates for the relevant oblique derivative problems. Namely, it is necessary to have an accurate control not only for the solution of (1.3), but also for those of a class of oblique derivative problems obtained by perturbing ∇d .

Nevertheless, in [10] we showed that (1.1)-(1.2) is solvable near d for solutions which are *axially symmetric* around the Earth's axis $\mathcal{A} = \{\lambda(0, 0, 1) : \lambda \in \mathbb{R}\}$. In fact, we show that the solutions of the linearized problem (1.3) with this symmetry no longer lose derivatives in an appropriate scale of fractional Sobolev spaces on \mathcal{S} . This result is made possible by the use of spherical harmonics on \mathcal{S} . As a consequence, the relevant fixed-point scheme can be mended and the existence of a solution of (1.1)-(1.2) is obtained if g is sufficiently close to $|\nabla d|$ in some fractional Sobolev norm.

In the present paper, for $N \geq 3$, we turn our attention to the local resolution of the interior Backus problem (1.1) in the framework of Hölder spaces. This framework is that used in [8] for the gravitational case. The simplest instance in this case is to consider solutions of (1.1) in the form:

$$u(x) = x_N + v(x),$$

where v is harmonic in Ω . If we place the x_N -axis horizontally, in the fluidmechanical framework mentioned above, we want to determine the velocity of the fluid inside a domain from measurements of its modulus on the boundary as a perturbation of that of an horizontal laminar flow with potential

$$f(x) = x_N.$$

In this case, the associated linearized problem is simply:

$$(1.4) \quad \Delta v = 0 \quad \text{in } \Omega, \quad \partial_{x_N} v = \varphi \quad \text{on } \mathbb{S},$$

where φ is a given function. It is clear that the vector field governing (1.4) is $e_N = (0, \dots, 0, 1)$, that shows similar qualitative features to those of ∇d , in the sense that $-e_N$ points inward on the northern hemisphere, is tangential on \mathcal{E} , and points outward on the southern hemisphere. Also in this case, though, a loss of derivatives occurs for problem (1.4). Nevertheless, we shall see that the somewhat easier oblique derivative condition in (1.4) allows a treatment of (1.1) in the framework of Hölder spaces, in the cases where solutions are oddly symmetric with respect to the hyperplane $x_N = 0$ or axially symmetric around x_N -axis.

In order to describe the main result of this paper, we need to introduce some notation. For $k = 0, 1, 2, \dots$ and $\alpha \in (0, 1)$, we define the function spaces:

$$\begin{aligned} C_{\text{even}}^{k+\alpha}(\overline{B}) &= \{\varphi \in C^{k+\alpha}(\overline{B}) : \varphi(x', x_N) = \varphi(x', -x_N)\}, \\ C_{\text{odd}}^{k+\alpha}(\overline{B}) &= \{\varphi \in C^{k+\alpha}(\overline{B}) : \varphi(x', x_N) = -\varphi(x', -x_N)\}, \\ C_{\text{ax}}^{k+\alpha}(\overline{B}) &= \{\varphi \in C^{k+\alpha}(\overline{B}) : \varphi(x', x_N) = \varphi(|x'|e'_1, x_N)\}. \end{aligned}$$

Here $e'_1 = (1, 0, \dots, 0) \in \mathbb{R}^{N-1}$. We note that these are closed subspaces of the space $C^{k+\alpha}(\overline{B})$ of k -differentiable functions whose derivatives up to the order k are α -Hölder continuous. The usual Hölder seminorm and norm on $C^{k+\alpha}(\overline{B})$ are denoted by $[\cdot]_{\alpha, \overline{B}}$ and $|\cdot|_{k+\alpha, \overline{B}}$, respectively, and defined as:

$$\begin{aligned} [u]_{\alpha, \overline{B}} &= \sup_{x, y \in \overline{B}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \\ |u|_{k+\alpha, \overline{B}} &= \sum_{|\gamma| \leq k} |D^\gamma u|_{0, \overline{B}} + \sum_{|\gamma| = k} [D^\gamma u]_{\alpha, \overline{B}}. \end{aligned}$$

Here, $|\cdot|_{0, \overline{B}}$ stands for the standard (uniform) maximum norm.

With these premises, we can state the main result of this paper as follows.

Theorem 1.1. *Let $\alpha \in (0, 1)$ and set $\Omega = B$. Then there exist positive constants δ_0 and C with the following properties.*

(i) *If $g \in C_{\text{even}}^{1+\alpha}(\overline{B})$ and*

$$|g - 1|_{1+\alpha, \overline{B}} \leq \delta_0,$$

then problem (1.1) has a solution $u \in C^2(B) \cap C_{\text{odd}}^{1+\alpha}(\overline{B})$ satisfying

$$|u - f|_{1+\alpha, \overline{B}} \leq C |g - 1|_{1+\alpha, \overline{B}}.$$

(ii) *If $g \in C_{\text{ax}}^{1+\alpha}(\overline{B})$, $h \in \mathbb{R}$ and*

$$|g - 1|_{1+\alpha, \overline{B}} \leq \delta_0,$$

then problem (1.1) has a solution $u \in C^2(B) \cap C_{\text{ax}}^{1+\alpha}(\overline{B})$ satisfying

$$u = h \quad \text{on } \mathcal{E},$$

and

$$|u - h - f|_{1+\alpha, \overline{B}} \leq C \left(|g - 1|_{1+\alpha, \overline{B}} \right).$$

Note that the solution u obtained in (i) of Theorem 1.1 automatically satisfies the condition $u = 0$ on \mathcal{E} , since it is odd in the variable x_N ; while in (ii) we need to impose the additional boundary condition on \mathcal{E} , due to the fact that (1.1) is invariant under the addition of constants.

It is worthwhile noticing at this point that, even if the height f and the dipole d are the Kelvin transformations of one another, \mathcal{K} maps the linearized exterior problem (1.3) into the interior oblique derivative problem:

$$\Delta v = 0 \quad \text{in } B, \quad \partial_{x_N} v + (N-2)f \partial_\nu v + (N-1)(N-2)f v = \varphi \quad \text{on } \mathcal{S}.$$

This problem is more difficult to treat, even if the relevant vector field governing it has the same qualitative properties of e_N . A study of this problem will be the theme of future work.

The proof of Theorem 1.1 hinges on some a priori estimates for the solution of the linearized problem (1.4) subject to the Dirichlet-type condition

$$(1.5) \quad v = \psi \quad \text{on } \mathcal{E}.$$

Thus, in Section 2, we will first derive an explicit representation formula for the solutions of (1.4)-(1.5). We will also adapt a couple of lemmas in [1] to our purposes, by making explicit the dependence of the relevant norms of v on those of the data φ . Then, in Section 3, we shall derive crucial a priori estimates for the linearized problem (1.4)-(1.5) (see Theorem 3.1). Finally, in Section 4, based on these estimates, we will carry out the proof of Theorem 1.1.

2. AN EXPLICIT INTEGRAL REPRESENTATION FORMULA FOR THE OBLIQUE DERIVATIVE PROBLEM

In this section, we carry out explicit computations which lead to the construction of an integral representation formula for the linearized problem (1.4)-(1.5). We follow the scheme introduced in [1].

2.1. Uniqueness for problem (1.4)-(1.5). Let a function space $C_N^1(\overline{B})$ be defined by

$$C_N^1(\overline{B}) = \{v \in C(\overline{B}) : \partial_{x_N} v \text{ exists and } \partial_{x_N} v \in C(\overline{B})\}.$$

Then $C_N^1(\overline{B}) \cap C^2(B)$ is one of the natural spaces for solutions of (1.4)-(1.5). We show that a uniqueness result holds in this space. The argument follows the lines of one used in [9] and is based on Hopf's boundary lemma.

Proposition 2.1 (Uniqueness). *For any given $\varphi \in C(\mathcal{S})$ and $\psi \in C(\mathcal{E})$, the problem (1.4)-(1.5) has at most one solution of class $C_N^1(\overline{B}) \cap C^2(B)$.*

Proof. Let v_1 and v_2 be two solutions of class $C_N^1(\overline{B}) \cap C^2(B)$ of (1.4)-(1.5), and set $v = v_1 - v_2$. Then v solves (1.4)-(1.5) with $\varphi \equiv 0$ and $\psi \equiv 0$. If $v(x)$ is a (positive) maximum for v on \overline{B} , then we have that $x \in \mathcal{S}$, by the maximum principle, and $x \notin \mathcal{E}$, being as $v(x) > 0$.

Now, if x were in the upper hemisphere of \mathcal{S} , then we would have that $\partial_{x_N} v(x) > 0$, by Hopf's boundary lemma, since e_N is an outward direction on that hemisphere. This is a contradiction. By a similar argument, we infer that x cannot belong to the lower hemisphere of \mathcal{S} . Thus, we conclude that $v \leq 0$ on \overline{B} . By considering the minimum of v , we can infer that $v \geq 0$ on \overline{B} , and hence $v \equiv 0$ on \overline{B} . \square

We can easily use this proposition to infer that the solution v of (1.4)-(1.5) inherits possible symmetries of the data φ and ψ . For instance, if φ and ψ are axially symmetric around x_N -axis, then v is so.

2.2. Estimates for the Dirichlet problem for the Laplace equation. In the next result, we derive estimates for solutions of the Dirichlet problem in B :

$$(2.1) \quad \Delta w = 0 \text{ in } B, \quad w = \varphi \text{ on } \mathcal{S}.$$

It is well-known that if $\varphi \in C(\mathcal{S})$, this problem has the unique solution $w \in C(\overline{B}) \cap C^2(B)$ given by the *Poisson integral formula*:

$$(2.2) \quad w(x) = \int_{\mathcal{S}} P_B(x; y) \varphi(y) dS_y.$$

Here, P_Ω stands for the Poisson kernel for a bounded domain Ω . In particular, P_B is explicitly given by

$$P_B(x; y) = \frac{1}{N\omega_N} \frac{1 - |x|^2}{|x - y|^N} \text{ for } x \in B, y \in \mathcal{S},$$

where ω_N is the volume of B . For our aims, we need the following refinement of [1, Lemma 2.2].

Proposition 2.2. *Suppose that $\varphi \in C^{k+\alpha}(\overline{B})$ for some non-negative integer k and $\alpha \in [0, 1)$. Let β be a multi-index with $|\beta| > k + \alpha$.*

If w is the solution of (2.1), then there exists a positive constant $C = C(N, \beta, k, \alpha)$ such that

$$(2.3) \quad |D^\beta w(x)| \leq C |\varphi|_{k+\alpha, \overline{B}} (1 - |x|^2)^{-|\beta|+k+\alpha}$$

for all $x \in B$.

To prove Proposition 2.2, we first recall the following a priori estimate for the Laplace equation. For the proofs, see for instance [5, Theorem 6.6, Problem 6.2].

Lemma 2.3. *Suppose that φ is of class $C^{k+\alpha}(\overline{B})$ for some integer $k \geq 2$ and $\alpha \in (0, 1)$. Then the solution w of (2.1) satisfies the inequality*

$$|w|_{k+\alpha, \overline{B}} \leq C |\varphi|_{k+\alpha, \overline{B}}$$

for some positive constant $C = C(N, k, \alpha)$.

Next, we derive a pointwise estimate for the Poisson kernel.

Lemma 2.4. *For any multi-index β , it holds that*

$$D_x^\beta P_B(x; y) = \frac{a_\beta(x, y)}{|x - y|^{|\beta|+N-1}} \text{ for } x \in B, y \in \mathcal{S},$$

where $|a_\beta(x, y)| \leq C_$ for some positive constant C_* which only depends on N and β .*

Proof. We have that

$$P_B(x; y) = \frac{1}{N\omega_N} \frac{1 - |x|^2}{|x - y|^N} = \frac{1}{N\omega_N} \frac{|y|^2 - |x|^2}{|x - y|^N} = \frac{1}{N\omega_N} \frac{(x - y) \cdot (x + y)}{|x - y|^N}.$$

Hence, if we set $z = x - y$, we obtain that

$$P_B(z + y; y) = \frac{1}{N\omega_N} \frac{1}{|z|^{N-2}} + \frac{2}{N\omega_N} \frac{y \cdot z}{|z|^N}.$$

This function of z is the sum of a $(2 - N)$ -homogeneous and a $(1 - N)$ -homogeneous function. Thus, we infer that

$$D_z^\beta P_B(z + y; y) = A(z; y) + B(z; y),$$

where $A(z; y)$ and $B(z; y)$ are homogeneous of degree $2 - N - |\beta|$ and $1 - N - |\beta|$ in z . As a consequence, we get:

$$A(z; y) + B(z; y) = |z|^{1-N-|\beta|} [|z| A(z/|z|; y) + B(z/|z|; y)].$$

The function in the brackets is bounded since both $z/|z|$ and y have a unitary norm and $z \in 2B$. Therefore, we conclude by setting

$$a_\beta(x, y) = |x - y| A\left(\frac{x - y}{|x - y|}; y\right) + B\left(\frac{x - y}{|x - y|}; y\right),$$

for $x \in B$ and $y \in \mathcal{S}$. \square

We also need the following bound.

Lemma 2.5. *Let a multi-index β and a nonnegative number κ satisfy $|\beta| > \kappa$. Then, there exists a positive constant $C = C(N, \beta, \kappa)$ such that*

$$\int_{\mathcal{S}} |D_x^\beta P_B(x; y)| |y - x_0|^\kappa dS_y \leq C(1 - |x|)^{-|\beta|+\kappa},$$

for all $x \in B \setminus \{0\}$, where $x_0 = x/|x| \in \mathcal{S}$. When $x = 0$, we can choose x_0 to be any point in \mathcal{S} .

Proof. When $x = 0$, the bound easily follows from Lemma 2.4. Let $x \in B \setminus \{0\}$, $x_0 = x/|x|$, and set $r = 1 - |x|$. Then, for every $y \in \mathcal{S}$, we have that

$$\begin{aligned} |y - x| &= \frac{2}{3}|y - x| + \frac{1}{3}|(y - x_0) - (x - x_0)| \geq \\ &\frac{2}{3}(|y| - |x|) + \frac{1}{3}(|y - x_0| - |x - x_0|) = \frac{1}{3}r + \frac{1}{3}|y - x_0|. \end{aligned}$$

This with Lemma 2.4 shows that

$$\begin{aligned} \int_{\mathcal{S}} |D_x^\beta P_B(x; y)| |y - x_0|^\kappa dS_y &= \int_{\mathcal{S}} \frac{|a_\beta(x, y)| |y - x_0|^\kappa}{|y - x|^{|\beta|+N-1}} dS_y \leq \\ &3^{1-N-|\beta|} C_* \int_{\mathcal{S}} \frac{|y - x_0|^\kappa}{(r + |y - x_0|)^{|\beta|+N-1}} dS_y = \\ &3^{1-N-|\beta|} C_* r^{-|\beta|+\kappa} \int_{|rz+e_N|=1} \frac{|z|^\kappa}{(1 + |z|)^{|\beta|+N-1}} dS_z, \end{aligned}$$

where we have used the change of variables $y = x_0 + r\mathcal{R}z$ with the orthogonal matrix \mathcal{R} satisfying $\mathcal{R}^{-1}x_0 = e_N = (0, \dots, 0, 1)$. The last integral on the right-hand side of the above inequality is bounded with respect to r , because as $r \rightarrow 0$ it converges to the integral

$$\int_{\mathbb{R}^{N-1}} \frac{|z|^\kappa}{(1 + |z|)^{|\beta|+N-1}} dS_z,$$

which is finite, being as $|\beta| > \kappa$. We thus obtain the desired inequality. \square

Proof of Proposition 2.2. As usual, in this proof C will denote a generic constant possibly depending on N, β, k , and α .

Pick any point $x_0 \in \mathcal{S}$. Since $\varphi \in C^{k+\alpha}(\overline{B})$, we can write the following standard Taylor expansion for φ :

$$\varphi(y) = \sum_{j=0}^k \sum_{|\gamma|=j} \frac{D^\gamma \varphi(x_0)}{\gamma!} (y - x_0)^\gamma + \sum_{|\gamma|=k} \frac{D^\gamma \varphi(x_0 + \theta(y - x_0)) - D^\gamma \varphi(x_0)}{\gamma!} (y - x_0)^\gamma.$$

Here, we use the standard conventions on the multi-index notation. Thus, integrating $\varphi(y)$ for $y \in \mathcal{S}$ against $D_x^\beta P_B(x; y)$ gives that

$$(2.4) \quad D^\beta w(x) = \sum_{|\gamma| \leq k} \frac{D^\gamma \varphi(x_0)}{\gamma!} D^\beta h_\gamma(x) + R_k(x),$$

where h_γ is the solution of (2.1) with $\varphi = (\cdot - x_0)^\gamma$ and

$$R_k(x) = \sum_{|\gamma|=k} \int_{\mathcal{S}} D_x^\beta P_B(x; y) \frac{D^\gamma \varphi(x_0 + \theta(y - x_0)) - D^\gamma \varphi(x_0)}{\gamma!} (y - x_0)^\gamma dS_y.$$

Now, let x_* be any point in $B \setminus \{0\}$ such that $x_* = |x_*|x_0$. Then, Lemma 2.3 gives that

$$(2.5) \quad |D^\beta h_\gamma(x_*)| \leq |D^\beta h_\gamma|_{0, \overline{B}} \leq C.$$

Moreover, if $\alpha \in (0, 1)$, Lemma 2.5 shows that

$$(2.6) \quad |R_k(x_*)| \leq \sum_{|\gamma|=k} \frac{[D^\gamma \varphi]_{\alpha, \overline{B}}}{\gamma!} \int_{\mathcal{S}} |D_x^\beta P_B(x_*; y)| |y - x_0|^{k+\alpha} dS_y \leq C \sum_{|\gamma|=k} [D^\gamma \varphi]_{\alpha, \overline{B}} (1 - |x_*|)^{-|\beta|+k+\alpha}.$$

This inequality is also valid for $\alpha = 0$, if $[D^\gamma \varphi]_{\alpha, \overline{B}}$ is replaced by $|D^\gamma \varphi|_{0, \overline{B}}$. Plugging (2.5) and (2.6) into (2.4), then gives that

$$|D^\beta w(x_*)| \leq C |\varphi|_{k+\alpha, \overline{B}} (1 - |x_*|)^{-|\beta|+k+\alpha}.$$

which yields (2.3), after an update of the constant C . \square

2.3. Representation formulas for problem (1.4)-(1.5). In order to obtain a representation formula, we consider the Dirichlet problem

$$(2.7) \quad -\Delta_{x'} Z(x') = \partial_{x_N} w(x', 0) \quad \text{in } D, \quad Z = \psi \quad \text{on } \partial D,$$

where w is the solution of (2.1) and $D = \{x' \in \mathbb{R}^{N-1} : |x'| < 1\}$. We identify D and ∂D with the equatorial ball $\{x = (x', x_N) \in B : x_N = 0\}$ and the equator \mathcal{E} , respectively. From Lemma 2.5, we see that w satisfies

$$|\partial_{x_N} w(x', 0)| \leq |\varphi|_{0, \mathcal{S}} \int_{\mathcal{S}} |\partial_{x_N} P_B(x', 0; y)| dS_y \leq C |\varphi|_{0, \mathcal{S}} (1 - |x'|)^{-1}$$

for some constant C . Therefore, for any $\varphi \in C(\mathcal{S})$ and $\psi \in C(\mathcal{E})$, the existence and uniqueness of solutions of (2.7) in $C(\overline{D}) \cap C^2(D)$ are guaranteed by [5, Theorem 4.9].

Proposition 2.6 (Existence and representation formula). *Suppose that $\varphi \in C(\mathcal{S})$ and $\psi \in C(\mathcal{E})$. Let w and Z be the solutions of (2.1) and (2.7), respectively, and set*

$$(2.8) \quad W(x) = \int_0^{x_N} w(x', t) dt, \quad x = (x', x_N) \in \overline{B},$$

where w is defined in (2.2). Then the unique solution v of class $C_N^1(\overline{B}) \cap C^2(B)$ of (1.4)-(1.5) is given by

$$(2.9) \quad v(x) = W(x) + Z(x'), \quad x = (x', x_N) \in \overline{B}.$$

Proof. Let v be defined by (2.9). Then $v = W + Z \in C(\overline{B}) \cap C^2(B)$, since we know that $w \in C(\overline{B}) \cap C^2(B)$ and $Z \in C(\overline{D}) \cap C^2(D)$. Moreover, we have that $\partial_{x_N} v = w \in C(\overline{B})$, and therefore $v \in C_N^1(\overline{B}) \cap C^2(B)$.

Since $W(x', 0) = 0$, we see that

$$\partial_{x_N} v = w = \varphi \quad \text{on } \mathcal{S}, \quad v = Z = \psi \quad \text{on } \mathcal{E}.$$

Hence the assertion follows if we show that v is harmonic in B . By a direct calculation, we have that

$$\begin{aligned} \Delta W(x) &= \int_0^{x_N} \Delta_{x'} w(x', t) dt + \partial_{x_N} w(x', x_N) = \\ &\quad - \int_0^{x_N} \partial_{x_N x_N}^2 w(x', t) dt + \partial_{x_N} w(x', x_N) = \partial_{x_N} w(x', 0). \end{aligned}$$

We thus infer that

$$\Delta v(x) = \Delta W(x) + \Delta_{x'} Z(x') = \partial_{x_N} w(x', 0) - \partial_{x_N} w(x', 0) = 0,$$

as desired. \square

Even if it is not needed in the proof of Theorem 1.1, we also derive for future reference an explicit integral representation formula. The formula may be helpful for numerical approximations. To derive the formula, we recall that the fundamental solution Γ_d of the Laplace equation in a d -dimensional Euclidean space ($d \geq 2$) is given by

$$\Gamma_2(x) = \frac{1}{2\pi} \log \frac{1}{|x|}, \quad \Gamma_d(x) = \frac{1}{d(d-2)\omega_d} |x|^{2-d} \text{ if } d \geq 3.$$

Then, the Green's function for D is written as

$$G_D(x'; y') = \Gamma_{N-1}(x' - y') - \Gamma_{N-1}(|x'|(\mathcal{J}(x') - y')),$$

where \mathcal{J} denotes the inversion $\mathcal{J}(x') = x'/|x'|^2$ for $x' \neq 0$.

If we now define the kernel

$$K(x; y) = \int_0^{x_N} P_B(x', t; y) dt + \int_D G_D(x', z') \partial_{x_N} P_B(z', 0; y) dz',$$

for $x = (x', x_N) \in B$ and $y \in \mathcal{S}$, the representation formula is given as follows.

Proposition 2.7 (Integral representation formula). *Suppose that $\varphi \in C(\mathcal{S})$ and $\psi \in C(\mathcal{E})$. Then, the function defined by*

$$(2.10) \quad v(x) = \int_{\mathcal{S}} K(x; y) \varphi(y) dS_y + \int_{\mathcal{E}} P_D(x'; y') \psi(y') dS_{y'} \text{ for } x \in B$$

coincides with the unique solution of class $C_N^1(\overline{B}) \cap C^2(B)$ of the problem (1.4)-(1.5).

Proof. By the well-known representation formula for the Dirichlet problem for the Poisson equation, we know that

$$Z(x') = \int_D G_D(x'; z') \partial_{x_N} w(z', 0) dz' + \int_{\mathcal{E}} P_D(x'; z') \psi(z') dS_{z'}.$$

With the definition of w in mind, by the Fubini theorem we then infer that

$$\int_D G_D(x'; z') \partial_{x_N} w(z', 0) dz' = \int_{\mathcal{S}} \left[\int_D G_D(x'; z') \partial_{x_N} P_B(z', 0; y) dz' \right] \varphi(y) dS_y.$$

Being as $x' \in D$, the Fubini theorem is applicable in this formula, because the function F defined a.e. on $D \times \mathcal{S}$ by $F(z', y) = \varphi(y) \partial_{x_N} P_B(z', 0; y) G_D(x'; z')$ is in $L^1(D \times \mathcal{S})$. In fact, we have that

$$\int_{D \times \mathcal{S}} |F(z', y)| (dz' \times dS_y) = \int_D G_D(x'; z') \left[\int_{\mathcal{S}} |\partial_{x_N} P_B(z', 0; y)| |\varphi(y)| dS_y \right] dz'$$

(see [11, Theorem 1.12]). The right-hand side is finite thanks to the properties of G_D and Lemma 2.5 with $\kappa = 0$ and $|\beta| = 1$.

Finally, that

$$W(x) = \int_S \left[\int_0^{x_N} P_B(x', t; y) dt \right] \varphi(y) dS_y$$

follows from (2.8), again by a straightforward application of the Fubini theorem. We have thus proved that the right-hand side of (2.10) coincides with $W + Z$. \square

3. A PRIORI ESTIMATES FOR THE LINEARIZED PROBLEM

A crucial a priori bound we will use to prove Theorem 1.1 is contained in the following theorem.

Theorem 3.1. *Let $\alpha \in (0, 1)$ and suppose that $\varphi \in C^{1+\alpha}(\overline{B})$ and $\psi \in C^{3/2+\alpha}(\overline{D})$. Then a solution v of (1.4)-(1.5) has the properties*

$$v \in C^{1+\alpha}(\overline{B}), \quad \partial_{x_N} v \in C^{1+\alpha}(\overline{B}), \quad x_N D_{x'}^2 v \in C^\alpha(\overline{B}).$$

Moreover, the following inequality holds for some positive constant C independent of φ and ψ :

$$(3.1) \quad |v|_{1+\alpha, \overline{B}} + |\partial_{x_N} v|_{1+\alpha, \overline{B}} + |x_N D_{x'}^2 v|_{\alpha, \overline{B}} \leq C \left(|\varphi|_{1+\alpha, \overline{B}} + |\psi|_{3/2+\alpha, \overline{D}} \right).$$

We will use the following simple estimate, which is a refinement of [1, Lemma 3.2].

Lemma 3.2. *Let $\kappa > 0$. Then there exists a constant $C > 0$ such that*

$$|x_N| \int_0^{|x_N|} \frac{dt}{(1 - |x'|^2 - t^2)^{1+\kappa}} \leq \frac{C}{(1 - |x|^2)^\kappa}$$

for all $x \in B$.

Proof. Set $|x_N| = \sigma \sqrt{1 - |x'|^2}$; it holds that $0 \leq \sigma < 1$ for $x = (x', x_N) \in B$. By the change of variable $t = s \sqrt{1 - |x'|^2}$, we have that

$$\int_0^{|x_N|} \frac{dt}{(1 - |x'|^2 - t^2)^{1+\kappa}} = \frac{1}{(1 - |x'|^2)^{1/2+\kappa}} \int_0^\sigma \frac{ds}{(1 - s^2)^{1+\kappa}},$$

and hence

$$|x_N| (1 - |x|^2)^\kappa \int_0^{|x_N|} \frac{dt}{(1 - |x'|^2 - t^2)^{1+\kappa}} = \sigma (1 - \sigma^2)^\kappa \int_0^\sigma \frac{ds}{(1 - s^2)^{1+\kappa}}.$$

The right-hand side is bounded by some constant C , since L'Hôpital's rule shows that its limit as $\sigma \rightarrow 1^-$ is equal to $1/(2\kappa)$. Thus the lemma follows. \square

The following lemma is essentially shown in [1, Lemma 2.5]. For the sake of completeness, we give a proof.

Lemma 3.3. *Let $v \in C^1(B) \cap C(\overline{B})$. Suppose that there exist a positive constant M and an exponent $\alpha \in (0, 1)$ such that*

$$|\nabla v(x)| \leq M(1 - |x|^2)^{-1+\alpha} \quad \text{for all } x \in B.$$

Then $v \in C^\alpha(\overline{B})$ and it holds that

$$(3.2) \quad [v]_{\alpha, \overline{B}} \leq CM,$$

for some positive constant C only depending on α .

Proof. The assumption on v gives that

$$(3.3) \quad |\nabla v(x)| \leq M(1 - |x|)^{-1+\alpha} = M|x - \overline{x}|^{-1+\alpha} \quad \text{with } \overline{x} = x/|x|,$$

since $\alpha < 1$.

(i) Let $\vartheta = |x - \bar{x}|$ and $\ell = (x - \bar{x})/\vartheta = -\bar{x}$. Then, we have that

$$|v(x) - v(\bar{x})| = \left| \int_0^\vartheta \nabla v(\bar{x} + t\ell) \cdot \ell dt \right| \leq M \int_0^\vartheta t^{-1+\alpha} dt = \frac{M}{\alpha} \vartheta^\alpha.$$

(ii) Let \bar{x} and \bar{y} be arbitrary points on \mathcal{S} with $\vartheta = |\bar{x} - \bar{y}| < 1$. Take $x = (1 - \vartheta)\bar{x}$ and $y = (1 - \vartheta)\bar{y}$. Then, for an intermediate point ξ between x and y , we have that

$$\begin{aligned} |v(\bar{x}) - v(\bar{y})| &\leq |v(x) - v(\bar{x})| + |v(x) - v(y)| + |v(y) - v(\bar{y})| \leq \\ &\frac{2M}{\alpha} \vartheta^\alpha + |\nabla v(\xi)| |x - y| \leq \frac{2M}{\alpha} \vartheta^\alpha + M \vartheta^{-1+\alpha} |x - y| \leq \\ &\frac{2M}{\alpha} \vartheta^\alpha + M \vartheta^\alpha. \end{aligned}$$

Here, we have used (i), (3.3) and the fact that $|\xi - \bar{\xi}| \geq |x - \bar{x}| = \vartheta$.

If $\vartheta = |\bar{x} - \bar{y}| \geq 1$, then one can choose a finite number of points on \mathcal{S} , say $\bar{x} = \bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 = \bar{y}$ with $|\bar{x}_i - \bar{x}_{i+1}| < 1$ so that the previous estimate applies, and the combination of the estimates yields the desired inequality. Therefore, $|v(\bar{x}) - v(\bar{y})| \leq CM |\bar{x} - \bar{y}|^\alpha$, for some constant C .

(iii) Now, take $x, y \in B$, set $\vartheta = |x - y|$, and let \bar{x} and \bar{y} be the usual projections of x and y on \mathcal{S} . We can always assume that $|y - \bar{y}| \geq |x - \bar{x}|$. If $|x - \bar{x}| \geq \vartheta$, then, for an intermediate point ξ between x and y ,

$$|v(x) - v(y)| \leq |\nabla v(\xi)| |x - y| \leq M \vartheta^{-1+\alpha} |x - y| = M \vartheta^\alpha,$$

thanks to the inequalities $|\xi - \bar{\xi}| \geq |x - \bar{x}| \geq \vartheta$. If $|x - \bar{x}| < \vartheta$ instead, we first infer that

$$|y - \bar{y}| \leq |y - \bar{x}| \leq |y - x| + |x - \bar{x}| < 2\vartheta.$$

Thus, (i) gives that $|v(x) - v(\bar{x})| \leq M \vartheta^\alpha / \alpha$ and $|v(y) - v(\bar{y})| \leq 2^\alpha M \vartheta^\alpha / \alpha$, while (ii) yields:

$$|v(\bar{x}) - v(\bar{y})| \leq CM |\bar{x} - \bar{y}|^\alpha \leq 4^\alpha CM \vartheta^\alpha.$$

We then conclude thanks to the triangle inequality. The bound (3.2) then follows at once. \square

Proof of Theorem 3.1. Throughout the proof C will denote a generic positive constant only depending on N and α .

From Proposition 2.6, the unique solution $v \in C_N^1(\bar{B}) \cap C^2(B)$ of (1.4)-(1.5) is given by

$$v(x) = W(x) + Z(x') = \int_0^{x_N} w(x', t) dt + Z(x'), \quad x = (x', x_N) \in B,$$

w and Z being the solutions of (2.1) and (2.7), respectively. We note that Z is expressed as $Z = Z_1 + Z_2$, where Z_1 is the solution of (2.7) with $\partial_{x_N} w(x', 0)$ replaced by 0 and Z_2 is the solution of (2.7) with $\psi = 0$. Hence, it will be enough to prove the three estimates:

$$(3.4) \quad |W|_{1+\alpha, \bar{B}} + |\partial_{x_N} W|_{1+\alpha, \bar{B}} + |x_N D_{x'}^2 W|_{\alpha, \bar{B}} \leq C |\varphi|_{1+\alpha, \bar{B}},$$

$$(3.5) \quad |Z_1|_{1+\alpha, \bar{D}} + |x_N D_{x'}^2 Z_1|_{\alpha, \bar{B}} \leq C |\psi|_{3/2+\alpha, \bar{D}}.$$

$$(3.6) \quad |Z_2|_{1+\alpha, \bar{D}} + |x_N D_{x'}^2 Z_2|_{\alpha, \bar{B}} \leq C |\varphi|_{1+\alpha, \bar{B}}.$$

We first derive these inequalities under the additional assumptions $\varphi \in C^{2+\alpha}(\bar{B})$ and $\psi \in C^{2+\alpha}(\bar{D})$. We then have

$$(3.7) \quad w \in C^{2+\alpha}(\bar{B}), \quad W \in C^{2+\alpha}(\bar{B}), \quad Z_1 \in C^{2+\alpha}(\bar{D}).$$

We note that the following inequality holds:

$$(3.8) \quad |w|_{1+\alpha, \bar{B}} \leq C |\varphi|_{1+\alpha, \bar{B}}.$$

Indeed, this is shown as follows. Since

$$\int_S P_B(x; y) dS_y = 1, \quad \int_S P_B(x; y) y dS_y = x,$$

we have

$$\nabla w(x) = \nabla \varphi(x) + \int_S \nabla_x P_B(x; y) [\varphi(y) - \varphi(x) - \nabla \varphi(x) \cdot (y - x)] dS_y.$$

This with Lemma 2.4 shows that

$$|\nabla w(x)| \leq |\nabla \varphi(x)| + C[\nabla \varphi]_{\alpha, \overline{B}} \int_S |x - y|^{-N+1+\alpha} dS_y.$$

We easily find that the integral on the right is finite and is bounded by some constant independent of x , and hence $|\nabla w|_{0, \overline{B}} \leq C[\nabla \varphi]_{\alpha, \overline{B}}$. Since Proposition 2.2 gives the inequality $|D^2 w(x)| \leq C|\varphi|_{1+\alpha, \overline{B}}(1 - |x|^2)^{-1+\alpha}$, we have that $[\nabla w]_{\alpha, \overline{B}} \leq C|\varphi|_{1+\alpha, \overline{B}}$ by Lemma 3.3. Therefore (3.8) holds.

First, we observe that (3.6) easily follows from the Schauder estimates for the Poisson equation and (3.8). In fact, we have that

$$|Z_2|_{2+\alpha, \overline{D}} \leq C|\partial_{x_N} w(\cdot, 0)|_{\alpha, \overline{D}} \leq C|w|_{1+\alpha, \overline{B}} \leq C|\varphi|_{1+\alpha, \overline{B}}.$$

Next, we prove (3.4). It is clear that

$$|W|_{1+\alpha, \overline{B}} \leq C|w|_{1+\alpha, \overline{B}}.$$

This together with (3.8) and the fact that $\partial_{x_N} W = w$ then yields:

$$|W|_{1+\alpha, \overline{B}} + |\partial_{x_N} W|_{1+\alpha, \overline{B}} \leq C|w|_{1+\alpha, \overline{B}} \leq C|\varphi|_{1+\alpha, \overline{B}}.$$

Therefore, we only need to verify that

$$(3.9) \quad |x_N D_{x'}^2 W|_{\alpha, \overline{B}} \leq C|\varphi|_{1+\alpha, \overline{B}}.$$

To prove this inequality, we examine pointwise estimates of $D_{x'}^2 W$. Proposition 2.2 yields that

$$|D_{x'}^2 W(x)| = \left| \int_0^{x_N} D_{x'}^2 w(x', t) dt \right| \leq C|\varphi|_{1+\alpha, \overline{B}} \int_0^{|x_N|} \frac{dt}{(1 - |x'|^2 - t^2)^{1-\alpha}}.$$

We estimate the last integral in two ways. First, by monotonicity in t and $|x_N|$, we see that the integral can be bounded by $(1 - |x|^2)^{-1+\alpha}$. From this, we infer that

$$(3.10) \quad |D_{x'}^2 W(x)| \leq C|\varphi|_{1+\alpha, \overline{B}} (1 - |x|^2)^{-1+\alpha}.$$

Secondly, by the inequality

$$(3.11) \quad x_N^2 \leq 1 - |x'|^2 \quad \text{for } (x', x_N) \in B,$$

we get that

$$\int_0^{|x_N|} \frac{dt}{(1 - |x'|^2 - t^2)^{1-\alpha}} \leq \int_0^{|x_N|} \frac{dt}{(x_N^2 - t^2)^{1-\alpha}} = |x_N|^{-1+2\alpha} \int_0^1 \frac{ds}{(1 - s^2)^{1-\alpha}},$$

after the change of variable $t = |x_N|s$. This gives the bound:

$$(3.12) \quad |x_N D_{x'}^2 W|_{0, \overline{B}} \leq C|\varphi|_{1+\alpha, \overline{B}}.$$

In order to estimate the Hölder seminorm $[x_N D_{x'}^2 W]_{\alpha, \overline{B}}$, we consider the derivatives of $D_{x'}^2 W$. We use Proposition 2.2 to obtain

$$|D_{x'}^3 W(x)| = \left| \int_0^{x_N} D_{x'}^3 w(x', t) dt \right| \leq C|\varphi|_{1+\alpha, \overline{B}} \int_0^{|x_N|} \frac{dt}{(1 - |x'|^2 - t^2)^{2-\alpha}}.$$

Applying Lemma 3.2 to the right-hand side, we infer that

$$(3.13) \quad |x_N D_{x'}^3 W(x)| \leq C|\varphi|_{1+\alpha, \overline{B}} (1 - |x|^2)^{-1+\alpha}.$$

Furthermore, we see from Proposition 2.2 that

$$(3.14) \quad |\partial_{x_N} D_{x'}^2 W(x)| = |D_{x'}^2 w(x)| \leq C |\varphi|_{1+\alpha, \overline{B}} (1 - |x|^2)^{-1+\alpha}.$$

Combining (3.10), (3.13) and (3.14), we deduce that

$$|x_N D_{x'}^3 W(x)| + |\partial_{x_N} (x_N D_{x'}^2 W(x))| \leq C |\varphi|_{1+\alpha, \overline{B}} (1 - |x|^2)^{-1+\alpha}.$$

Thus, by (3.7) and Lemma 3.3, we obtain that $[x_N D_{x'}^2 W]_{\alpha, \overline{B}} \leq C |\varphi|_{1+\alpha, \overline{B}}$. This together with (3.12) shows that (3.9) holds.

Finally, we verify (3.5). Note that the same estimates as in Proposition 2.2 and (3.8) hold if w , φ and B are replaced by Z_1 , ψ and D , respectively. By (3.8), we see that

$$(3.15) \quad |Z_1|_{1+\alpha, \overline{D}} \leq C |\psi|_{1+\alpha, \overline{D}} \leq C |\psi|_{3/2+\alpha, \overline{D}}.$$

Proposition 2.2 and (3.11) show that

$$|x_N D_{x'}^2 Z_1(x')| \leq C |\psi|_{3/2, \overline{D}} |x_N| (1 - |x'|)^{-\frac{1}{2}} \leq C |\psi|_{3/2+\alpha, \overline{D}},$$

$$|x_N D_{x'}^3 Z_1(x')| \leq C |\psi|_{3/2+\alpha, \overline{D}} |x_N| (1 - |x'|)^{-\frac{3}{2}+\alpha} \leq C |\psi|_{3/2+\alpha, \overline{D}} (1 - |x|)^{-1+\alpha},$$

and

$$\begin{aligned} |\partial_{x_N} (x_N D_{x'}^2 Z_1(x'))| &= |D_{x'}^2 Z_1(x')| \leq \\ &C |\psi|_{1+\alpha, \overline{D}} (1 - |x'|)^{-1+\alpha} \leq C |\psi|_{3/2+\alpha, \overline{D}} (1 - |x|)^{-1+\alpha}. \end{aligned}$$

Hence it follows from (3.7) and Lemma 3.3 that

$$|x_N D_{x'}^2 Z_1|_{\alpha, \overline{B}} \leq C |\psi|_{3/2+\alpha, \overline{D}}.$$

This and (3.15) gives (3.5).

Now, if $\varphi \in C^{1+\alpha}(\overline{B})$ and $\psi \in C^{3/2+\alpha}(\overline{D})$, we take sequences $\{\varphi_j\} \subset C^{2+\alpha}(\overline{B})$ and $\{\psi_j\} \subset C^{2+\alpha}(\overline{D})$ such that

$$(3.16) \quad \varphi_j \rightarrow \varphi \text{ in } C(\overline{B}), \quad \psi_j \rightarrow \psi \text{ in } C(\overline{D}) \text{ as } j \rightarrow \infty,$$

$$(3.17) \quad |\varphi_j|_{1+\alpha, \overline{B}} \leq C |\varphi|_{1+\alpha, \overline{B}}, \quad |\psi_j|_{3/2+\alpha, \overline{D}} \leq C |\psi|_{3/2+\alpha, \overline{D}}, \quad j = 1, 2, \dots$$

Let w_j be a solution of (2.1) with $\varphi = \varphi_j$. Then, (3.16) and the Schauder interior estimates for Poisson's equation give that $w_j \rightarrow w$ in $C_{\text{loc}}^2(B)$ as $j \rightarrow \infty$. Hence $W_j(x) = \int_0^{x_N} w_j(x', t) dt$ converges to W in $C_{\text{loc}}^2(B)$. Since the inequality $|x_N D_{x'}^2 W_j|_{\alpha, \overline{B}} \leq C |\varphi_j|_{1+\alpha, \overline{B}}$ is valid, we obtain (3.9) by using (3.17) and letting $j \rightarrow \infty$. In a similar way, approximating ψ by ψ_j gives that (3.5) is still valid for $\psi \in C^{3/2+\alpha}(\overline{D})$. Thus the proof is complete. \square

4. LOCAL EXISTENCE FOR THE INTERIOR BACKUS PROBLEM WITH SYMMETRIC DATA

This section is devoted to the proof of Theorem 1.1.

4.1. The nonlinear operator \mathcal{T} . We define an operator \mathcal{T} by setting

$$\mathcal{T}[\varphi] = |\nabla v|^2,$$

where v satisfies (1.4)-(1.5) with $\psi = 0$. The next proposition shows that \mathcal{T} is locally bounded and locally Lipschitz continuous on bounded subsets of $C_{\text{even}}^{1+\alpha}(\overline{B})$.

Proposition 4.1. *We have that \mathcal{T} is a mapping from $C_{\text{even}}^{1+\alpha}(\overline{B})$ into itself. Furthermore, there are positive constants C_1 and C_2 such that*

$$(4.1) \quad |\mathcal{T}[\varphi]|_{1+\alpha, \overline{B}} \leq C_1 |\varphi|_{1+\alpha, \overline{B}}^2,$$

$$(4.2) \quad |\mathcal{T}[\varphi_1] - \mathcal{T}[\varphi_2]|_{1+\alpha, \overline{B}} \leq C_2 \left(|\varphi_1|_{1+\alpha, \overline{B}} + |\varphi_2|_{1+\alpha, \overline{B}} \right) |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}}$$

for all $\varphi, \varphi_1, \varphi_2 \in C_{\text{even}}^{1+\alpha}(\overline{B})$.

To prove this proposition, we need the following simple lemma.

Lemma 4.2. *Let k be a non-negative integer and take $\alpha \in (0, 1)$. Suppose that a function v defined on \overline{B} is such that $\partial_{x_N} v$ is in $C^{k+\alpha}(\overline{B})$ and is zero on $\overline{D} \times \{0\}$. Then the function defined by*

$$(4.3) \quad \omega(x) = \begin{cases} \frac{v(x)}{x_N} & \text{for } x_N \neq 0, \\ \partial_{x_N} v(x', 0) & \text{for } x_N = 0, \end{cases}$$

belongs to $C^{k+\alpha}(\overline{B})$ and satisfies the inequality:

$$(4.4) \quad |\omega|_{k+\alpha, \overline{B}} \leq |\partial_{x_N} v|_{k+\alpha, \overline{B}}.$$

Proof. By the fundamental theorem of calculus, we have that

$$v(x', x_N) = x_N \int_0^1 \partial_{x_N} v(x', x_N t) dt,$$

and hence ω can be written as

$$\omega(x) = \int_0^1 \partial_{x_N} v(x', x_N t) dt.$$

Thus, we see that the partial derivative $D_x^\beta \omega$ exists for any multi-index $\beta = (\beta_1, \dots, \beta_N)$ with $|\beta| \leq k$ and is given by

$$D_x^\beta \omega(x) = \int_0^1 t^{\beta_N} D_x^\beta \partial_{x_N} v(x', x_N t) dt.$$

The assertion then easily follows from this formula. \square

Proof of Proposition 4.1. Throughout the proof, i is any index in $\{1, 2, \dots, N-1\}$ and C is a generic positive constant only depending on N and α .

(i) Let $\varphi \in C_{\text{even}}^{1+\alpha}(\overline{B})$ and let v denote the solution of the problem (1.4)–(1.5) with $\psi = 0$. Since φ is even in x_N , we see that the function $-v(x', -x_N)$ is also a solution of (1.4)–(1.5) with $\psi = 0$. By uniqueness, we infer that v is odd in x_N . In particular, $v(x', 0) = 0$, and hence

$$(4.5) \quad \partial_{x_j} (\partial_{x_i} v)^2 = 2 \partial_{x_i} v \partial_{x_i x_j}^2 v = 2 (\partial_{x_i} \omega) (x_N \partial_{x_i x_j}^2 v) \quad \text{for } j = 1, \dots, N,$$

where ω is a function given by (4.3). From Theorem 3.1 and Lemma 4.2, we see that the right-hand side of this equality is in $C^\alpha(\overline{B})$. Therefore, we have that $(\partial_{x_i} v)^2 \in C^{1+\alpha}(\overline{B})$. This together with the fact that $(\partial_{x_N} v)^2 \in C^{1+\alpha}(\overline{B})$, which follows from Theorem 3.1, gives:

$$\mathcal{T}[\varphi] = |\nabla v|^2 \in C^{1+\alpha}(\overline{B}).$$

Since the fact that v is odd in x_N yields that $|\nabla v(x', x_N)| = |\nabla v(x', -x_N)|$, we conclude that \mathcal{T} is a mapping from $C_{\text{even}}^{1+\alpha}(\overline{B})$ to itself.

(ii) Let us derive (4.1). By (4.5), we have:

$$\begin{aligned} |(\partial_{x_i} v)^2|_{1+\alpha, \overline{B}} &= |(\partial_{x_i} v)^2|_{0, \overline{B}} + \sum_{j=1}^N |\partial_{x_j} (\partial_{x_i} v)^2|_{\alpha, \overline{B}} = \\ &= |(\partial_{x_i} v)^2|_{0, \overline{B}} + 2 \sum_{j=1}^{N-1} \left| (\partial_{x_i} \omega) (x_N \partial_{x_i x_j}^2 v) \right|_{\alpha, \overline{B}} + 2 |\partial_{x_i} v \partial_{x_i x_N}^2 v|_{\alpha, \overline{B}}. \end{aligned}$$

From (3.1), the first and third terms of the rightest-hand side are handled as

$$\begin{aligned} |(\partial_{x_i} v)^2|_{0,\overline{B}} &\leq |\partial_{x_i} v|_{0,\overline{B}}^2 \leq |v|_{1+\alpha,\overline{B}}^2 \leq C |\varphi|_{1+\alpha,\overline{B}}^2, \\ |\partial_{x_i} v \partial_{x_i x_N}^2 v|_{\alpha,\overline{B}} &\leq |\partial_{x_i} v|_{\alpha,\overline{B}} |\partial_{x_i x_N}^2 v|_{\alpha,\overline{B}} \leq |v|_{1+\alpha,\overline{B}} |\partial_{x_N} v|_{1+\alpha,\overline{B}} \leq C |\varphi|_{1+\alpha,\overline{B}}^2. \end{aligned}$$

Furthermore, (3.1) and (4.4) show that

$$\begin{aligned} |(\partial_{x_i} \omega) (x_N \partial_{x_i x_j}^2 v)|_{\alpha,\overline{B}} &\leq |\partial_{x_i} \omega|_{\alpha,\overline{B}} |x_N \partial_{x_i x_j}^2 v|_{\alpha,\overline{B}} \leq \\ &|\omega|_{1+\alpha,\overline{B}} |x_N \partial_{x_i x_j}^2 v|_{\alpha,\overline{B}} \leq |\partial_{x_N} v|_{1+\alpha,\overline{B}} |x_N \partial_{x_i x_j}^2 v|_{\alpha,\overline{B}} \leq C |\varphi|_{1+\alpha,\overline{B}}^2, \end{aligned}$$

for $j = 1, \dots, N-1$. From these estimates it follows that

$$(4.6) \quad |(\partial_{x_i} v)^2|_{1+\alpha,\overline{B}} \leq C |\varphi|_{1+\alpha,\overline{B}}^2.$$

In order to estimate $(\partial_{x_N} v)^2$, we use (3.1) to find that

$$(4.7) \quad |(\partial_{x_N} v)^2|_{1+\alpha,\overline{B}} \leq C |\partial_{x_N} v|_{1+\alpha,\overline{B}}^2 \leq C |\varphi|_{1+\alpha,\overline{B}}^2.$$

The combination of (4.6) and (4.7) then gives (4.1).

(iii) It remains to prove (4.2). For $m = 1, 2$, let v_m be the solution of the problem (1.4)-(1.5) with $\varphi = \varphi_m \in C_{\text{even}}^{1+\alpha}(\overline{B})$ and $\psi = 0$. Also, let ω_m be defined by (4.3) with $v = v_m$.

It is clear that

$$\begin{aligned} |(\partial_{x_i} v_1)^2 - (\partial_{x_i} v_2)^2|_{0,\overline{B}} &\leq \\ &(|v_1|_{0,\overline{B}} + |v_2|_{0,\overline{B}}) |v_1 - v_2|_{0,\overline{B}} \leq (|v_1|_{1+\alpha,\overline{B}} + |v_2|_{1+\alpha,\overline{B}}) |v_1 - v_2|_{1+\alpha,\overline{B}}. \end{aligned}$$

Hence, (3.1) easily gives that

$$(4.8) \quad |(\partial_{x_i} v_1)^2 - (\partial_{x_i} v_2)^2|_{0,\overline{B}} \leq C (|\varphi_1|_{1+\alpha,\overline{B}} + |\varphi_2|_{1+\alpha,\overline{B}}) |\varphi_1 - \varphi_2|_{1+\alpha,\overline{B}}.$$

Next, we write the differential identity:

$$\begin{aligned} \sum_{j=1}^N \partial_{x_j} [(\partial_{x_i} v_1)^2 - (\partial_{x_i} v_2)^2] &= 2 \sum_{j=1}^N [\partial_{x_i} v_1 \partial_{x_i x_j}^2 v_1 - \partial_{x_i} v_2 \partial_{x_i x_j}^2 v_2] = \\ &2 \sum_{j=1}^{N-1} [(\partial_{x_i} \omega_1)(x_N \partial_{x_i x_j}^2 v_1 - x_N \partial_{x_i x_j}^2 v_2) + x_N (\partial_{x_i x_j}^2 v_2)(\partial_{x_i} \omega_1 - \partial_{x_i} \omega_2)] + \\ &2 [(\partial_{x_i} v_1)(\partial_{x_i x_N}^2 v_1 - \partial_{x_i x_N}^2 v_2) + (\partial_{x_i x_N}^2 v_2)(\partial_{x_i} v_1 - \partial_{x_i} v_2)]. \end{aligned}$$

We then take care of the last summand:

$$\begin{aligned} |(\partial_{x_i} v_1)(\partial_{x_i x_N}^2 v_1 - \partial_{x_i x_N}^2 v_2) + (\partial_{x_i x_N}^2 v_2)(\partial_{x_i} v_1 - \partial_{x_i} v_2)|_{\alpha,\overline{B}} &\leq \\ |\partial_{x_i} v_1|_{\alpha,\overline{B}} |\partial_{x_i x_N}^2 v_1 - \partial_{x_i x_N}^2 v_2|_{\alpha,\overline{B}} + |\partial_{x_i x_N}^2 v_2|_{\alpha,\overline{B}} |\partial_{x_i} v_1 - \partial_{x_i} v_2|_{\alpha,\overline{B}} &\leq \\ |v_1|_{1+\alpha,\overline{B}} |\partial_{x_N} v_1 - \partial_{x_N} v_2|_{1+\alpha,\overline{B}} + |\partial_{x_N} v_2|_{1+\alpha,\overline{B}} |v_1 - v_2|_{1+\alpha,\overline{B}}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} |(\partial_{x_i} v_1)(\partial_{x_i x_N}^2 v_1 - \partial_{x_i x_N}^2 v_2) + (\partial_{x_i x_N}^2 v_2)(\partial_{x_i} v_1 - \partial_{x_i} v_2)|_{\alpha,\overline{B}} &\leq \\ C (|\varphi_1|_{1+\alpha,\overline{B}} + |\varphi_2|_{1+\alpha,\overline{B}}) |\varphi_1 - \varphi_2|_{1+\alpha,\overline{B}}. \end{aligned}$$

Moreover, for $j = 1, \dots, N-1$, we have:

$$\begin{aligned} & \left| (\partial_{x_i} \omega_1)(x_N \partial_{x_i x_j}^2 v_1 - x_N \partial_{x_i x_j}^2 v_2) + x_N (\partial_{x_i x_j}^2 v_2)(\partial_{x_i} \omega_1 - \partial_{x_i} \omega_2) \right|_{\alpha, \overline{B}} \leq \\ & |\omega_1|_{1+\alpha, \overline{B}} \left| x_N \partial_{x_i x_j}^2 v_1 - x_N \partial_{x_i x_j}^2 v_2 \right|_{\alpha, \overline{B}} + \left| x_N \partial_{x_i x_j}^2 v_2 \right|_{\alpha, \overline{B}} |\omega_1 - \omega_2|_{1+\alpha, \overline{B}} \leq \\ & |\partial_{x_N} v_1|_{1+\alpha, \overline{B}} \left| x_N \partial_{x_i x_j}^2 v_1 - x_N \partial_{x_i x_j}^2 v_2 \right|_{\alpha, \overline{B}} + \left| x_N \partial_{x_i x_j}^2 v_2 \right|_{\alpha, \overline{B}} |\partial_{x_N} v_1 - \partial_{x_N} v_2|_{1+\alpha, \overline{B}}. \end{aligned}$$

where we have used (4.4). Hence, (3.1) gives:

$$(4.9) \quad \left| (\partial_{x_i} \omega_1)(x_N \partial_{x_i x_j}^2 v_1 - x_N \partial_{x_i x_j}^2 v_2) + x_N (\partial_{x_i x_j}^2 v_2)(\partial_{x_i} \omega_1 - \partial_{x_i} \omega_2) \right|_{\alpha, \overline{B}} \leq C \left(|\varphi_1|_{1+\alpha, \overline{B}} + |\varphi_2|_{1+\alpha, \overline{B}} \right) |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}}.$$

All in all, by (4.8)–(4.9) we deduce that

$$|(\partial_{x_i} v_1)^2 - (\partial_{x_i} v_2)^2|_{1+\alpha, \overline{B}} \leq C \left(|\varphi_1|_{1+\alpha, \overline{B}} + |\varphi_2|_{1+\alpha, \overline{B}} \right) |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}},$$

for $i = 1, \dots, N-1$.

On the other hand, we can also use (3.1) to infer that

$$\begin{aligned} & |(\partial_{x_N} v_1)^2 - (\partial_{x_N} v_2)^2|_{1+\alpha, \overline{B}} \leq \\ & C \left(|\partial_{x_N} v_1|_{1+\alpha, \overline{B}} + |\partial_{x_N} v_2|_{1+\alpha, \overline{B}} \right) |\partial_{x_N} v_1 - \partial_{x_N} v_2|_{1+\alpha, \overline{B}} \leq \\ & C \left(|\varphi_1|_{1+\alpha, \overline{B}} + |\varphi_2|_{1+\alpha, \overline{B}} \right) |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}}. \end{aligned}$$

In conclusion, we obtain (4.2), and the lemma follows. \square

4.2. The nonlinear operator $\tilde{\mathcal{T}}_\psi$. We fix a cut-off function $\eta \in C^\infty(\mathbb{R})$ satisfying

$$\eta(t) = 1 \quad \text{if } |t| \leq \frac{1}{3}, \quad \eta(t) = 0 \quad \text{if } |t| \geq \frac{2}{3}.$$

For a function ϕ defined on \overline{B} , we set

$$\mathcal{J}[\phi](x) = \eta(x_N) \phi \left(\sqrt{1 - x_N^2} e'_1, x_N \right) + (1 - \eta(x_N)) \phi(x), \quad x = (x', x_N) \in \overline{B}.$$

Here $e'_1 = (1, 0, \dots, 0) \in \mathbb{R}^{N-1}$. Then, for fixed $\psi \in C^{3/2+\alpha}(\overline{D})$, we define an operator $\tilde{\mathcal{T}}_\psi$ by

$$\tilde{\mathcal{T}}_\psi[\varphi] = \mathcal{J} [|\nabla v|^2],$$

where v is the solution of (1.4)–(1.5). Our goal here is to obtain the properties of $\tilde{\mathcal{T}}_\psi$, which enables us to solve problem (1.1) in $C_{\text{ax}}^{1+\alpha}(\overline{B})$.

Proposition 4.3. *Let $\psi \in C^{3/2+\alpha}(\overline{D})$. Then the following hold.*

- (i) *The operator $\tilde{\mathcal{T}}_\psi$ is a mapping from $C^{1+\alpha}(\overline{B})$ into itself. Moreover, there exist positive constants C_3 and C_4 such that*

$$(4.10) \quad |\tilde{\mathcal{T}}_\psi[\varphi]|_{1+\alpha, \overline{B}} \leq C_3 \left(|\varphi|_{1+\alpha, \overline{B}}^2 + |\psi|_{3/2+\alpha, \overline{D}}^2 \right),$$

$$(4.11) \quad |\tilde{\mathcal{T}}_\psi[\varphi_1] - \tilde{\mathcal{T}}_\psi[\varphi_2]|_{1+\alpha, \overline{B}} \leq C_4 \left(|\varphi_1|_{1+\alpha, \overline{B}} + |\varphi_2|_{1+\alpha, \overline{B}} + |\psi|_{3/2+\alpha, \overline{D}} \right) |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}}$$

for all $\varphi, \varphi_1, \varphi_2 \in C^{1+\alpha}(\overline{B})$.

- (ii) If $\varphi \in C_{\text{ax}}^{1+\alpha}(\overline{B})$ and ψ is constant, then
- (4.12) $\tilde{\mathcal{I}}_\psi[\varphi] \in C_{\text{ax}}^{1+\alpha}(\overline{B})$ and $\tilde{\mathcal{I}}_\psi[\varphi] = |\nabla v|^2$ on \mathcal{S} .
- Here v is the solution of (1.4)-(1.5).

Before proving the proposition, we examine properties of \mathcal{J} .

Lemma 4.4. *If a function ϕ defined on \overline{B} satisfies $\phi(x', x_N) = \phi(|x'|e'_1, x_N)$, then*

$$\mathcal{J}[\phi] = \phi \text{ on } \mathcal{S}.$$

Proof. Let $x = (x', x_N) \in \mathcal{S}$. Then, since $|x'|^2 + x_N^2 = 1$, we have that

$$\phi\left(\sqrt{1-x_N^2}e'_1, x_N\right) = \phi(|x'|e'_1, x_N) = \phi(x).$$

This gives that $\mathcal{J}[\phi](x) = \eta(x_N)\phi(x) + (1 - \eta(x_N))\phi(x) = \phi(x)$, as desired. \square

Lemma 4.5. *Suppose that $\phi \in C^\alpha(\overline{B}) \cap C^1(B)$ satisfies $\partial_{x_N}\phi \in C^\alpha(\overline{B})$ and $x_N\nabla_{x'}\phi \in C^\alpha(\overline{B})$. Then $\mathcal{J}[\phi] \in C^{1+\alpha}(\overline{B})$ and*

$$|\mathcal{J}[\phi]|_{1+\alpha, \overline{B}} \leq C\left(|\phi|_{\alpha, \overline{B}} + |\partial_{x_N}\phi|_{\alpha, \overline{B}} + |x_N\nabla_{x'}\phi|_{\alpha, \overline{B}}\right),$$

where C is a positive constant independent of ϕ .

Proof. Let

$$B_1 = \left\{(x', x_N) \in B : |x_N| < \frac{2}{3}\right\}, \quad B_2 = \left\{(x', x_N) \in B : |x_N| > \frac{1}{3}\right\},$$

and define the mapping

$$\xi : \overline{B} \ni (x', x_N) \mapsto \xi(x) = \left(\sqrt{1-x_N^2}e'_1, x_N\right) \in \overline{B}.$$

We first show that

$$(4.13) \quad \phi \circ \xi \in C^{1+\alpha}(\overline{B_1}), \quad \phi \in C^{1+\alpha}(\overline{B_2}),$$

$$(4.14) \quad |\phi \circ \xi|_{1+\alpha, \overline{B_1}} + |\phi|_{1+\alpha, \overline{B_2}} \leq C\left(|\phi|_{\alpha, \overline{B}} + |\partial_{x_N}\phi|_{\alpha, \overline{B}} + |x_N\nabla_{x'}\phi|_{\alpha, \overline{B}}\right).$$

Here and subsequently, C denotes a positive constant independent of ϕ .

To prove $\phi \circ \xi \in C^{1+\alpha}(\overline{B_1})$, we observe that $\phi \circ \xi$ depends only on x_N and

$$\partial_{x_N}(\phi \circ \xi) = (\partial_{x_N}\phi) \circ \xi - \frac{1}{\sqrt{1-x_N^2}}(x_N\partial_{x_1}\phi) \circ \xi.$$

By assumption and the fact that ξ and $1/\sqrt{1-x_N^2}$ are smooth on $\overline{B_1}$, we see that the right-hand side of this equality is in $C^\alpha(\overline{B_1})$. Moreover,

$$\begin{aligned} |\partial_{x_N}(\phi \circ \xi)|_{\alpha, \overline{B_1}} &\leq |(\partial_{x_N}\phi) \circ \xi|_{\alpha, \overline{B_1}} + \left| \frac{1}{\sqrt{1-x_N^2}}(x_N\partial_{x_1}\phi) \circ \xi \right|_{\alpha, \overline{B_1}} \leq \\ &C|\partial_{x_N}\phi|_{\alpha, \overline{B}} + C|(x_N\partial_{x_1}\phi) \circ \xi|_{\alpha, \overline{B_1}} \leq C|\partial_{x_N}\phi|_{\alpha, \overline{B}} + C|x_N\partial_{x_1}\phi|_{\alpha, \overline{B}}. \end{aligned}$$

Therefore, we have that $\phi \circ \xi \in C^{1+\alpha}(\overline{B_1})$ and

$$(4.15) \quad |\phi \circ \xi|_{1+\alpha, \overline{B_1}} = |\phi \circ \xi|_{0, \overline{B_1}} + |\partial_{x_N}(\phi \circ \xi)|_{\alpha, \overline{B_1}} \leq |\phi|_{0, \overline{B}} + C|\partial_{x_N}\phi|_{\alpha, \overline{B}} + C|x_N\partial_{x_1}\phi|_{\alpha, \overline{B}}.$$

Since x_N^{-1} is smooth on $\overline{B_2}$, we deduce that $\phi = x_N^{-1} \cdot (x_N\phi) \in C^{1+\alpha}(\overline{B_2})$ and

$$(4.16) \quad |\phi|_{1+\alpha, \overline{B_2}} = |\phi|_{0, \overline{B_2}} + |\partial_{x_N}\phi|_{\alpha, \overline{B_2}} + |x_N^{-1} \cdot x_N\nabla_{x'}\phi|_{\alpha, \overline{B_2}} \leq |\phi|_{0, \overline{B}} + |\partial_{x_N}\phi|_{\alpha, \overline{B}} + C|x_N\nabla_{x'}\phi|_{\alpha, \overline{B}}.$$

By (4.15) and (4.16), we obtain (4.14).

We note that η and $1 - \eta$ vanish on $\overline{B} \setminus B_1$ and $\overline{B} \setminus B_2$, respectively. This with (4.13) shows that $\mathcal{J}[\phi] = \eta(\phi \circ \xi) + (1 - \eta)\phi \in C^{1+\alpha}(\overline{B})$. Furthermore,

$$|\mathcal{J}[\phi]|_{1+\alpha, \overline{B}} \leq C|\eta(\phi \circ \xi)|_{1+\alpha, \overline{B}_1} + C|(1 - \eta)\phi|_{1+\alpha, \overline{B}_2} \leq C|\phi \circ \xi|_{1+\alpha, \overline{B}_1} + C|\phi|_{1+\alpha, \overline{B}_2}.$$

Combining this and (4.14) proves the lemma. \square

Proof of Proposition 4.3. Let $\varphi, \varphi_1, \varphi_2 \in C^{1+\alpha}(\overline{B})$ and $\psi \in C^{3/2+\alpha}(\overline{D})$. In the proof, C stands for a generic positive constant only independent of these functions.

Let v stand for the solution of (1.4)-(1.5). Then Theorem 3.1 shows that the function $\phi = |\nabla v|^2$ satisfies $\phi \in C^\alpha(\overline{B}) \cap C^1(B)$, $\partial_{x_N} \phi = 2\nabla v \cdot \nabla \partial_{x_N} v \in C^\alpha(\overline{B})$ and $x_N \partial_{x_j} \phi = 2x_N \nabla v \cdot \nabla \partial_{x_j} v \in C^\alpha(\overline{B})$ for $j = 1, \dots, N-1$. Hence, using Lemma 4.5, we see that $\tilde{\mathcal{J}}_\psi[\varphi] = \mathcal{J}[\phi] \in C^{1+\alpha}(\overline{B})$ and

$$|\tilde{\mathcal{J}}_\psi[\varphi]|_{1+\alpha, \overline{B}} \leq C \left(|\nabla v|^2|_{\alpha, \overline{B}} + |\nabla v \cdot \nabla \partial_{x_N} v|_{\alpha, \overline{B}} + \sum_{j=1}^{N-1} |x_N \nabla v \cdot \nabla \partial_{x_j} v|_{\alpha, \overline{B}} \right).$$

Each term of the right-hand side is estimated as

$$|\nabla v|^2|_{\alpha, \overline{B}} \leq |v|_{1+\alpha, \overline{B}}^2, \quad |\nabla v \cdot \nabla \partial_{x_N} v|_{\alpha, \overline{B}} \leq C|v|_{1+\alpha, \overline{B}} |\partial_{x_N} v|_{1+\alpha, \overline{B}},$$

and

$$\begin{aligned} \sum_{j=1}^{N-1} |x_N \nabla v \cdot \nabla \partial_{x_j} v|_{\alpha, \overline{B}} &\leq C \sum_{j=1}^{N-1} |v|_{1+\alpha, \overline{B}} |x_N \nabla \partial_{x_j} v|_{\alpha, \overline{B}} \leq \\ &C|v|_{1+\alpha, \overline{B}} \left(|x_N D_x^2 v|_{\alpha, \overline{B}} + |\partial_{x_N} v|_{1+\alpha, \overline{B}} \right). \end{aligned}$$

Therefore, by (3.1), we conclude that $\tilde{\mathcal{J}}_\psi$ is a mapping from $C^{1+\alpha}(\overline{B})$ into itself and that (4.10) holds.

Inequality (4.11) is shown as follows. For $i = 1, 2$, we denote by v_i the solution of (1.4)-(1.5) with $\varphi = \varphi_i$ and $\psi = \psi_i$. We see from Lemma 4.5 that

$$\begin{aligned} (4.17) \quad |\tilde{\mathcal{J}}_\psi[\varphi_1] - \tilde{\mathcal{J}}_\psi[\varphi_2]|_{1+\alpha, \overline{B}} &= |\mathcal{J}[|\nabla v_1|^2 - |\nabla v_2|^2]|_{1+\alpha, \overline{B}} \leq \\ &C \left(|\nabla v_1|^2 - |\nabla v_2|^2|_{0, \overline{B}} + |\partial_{x_N} (|\nabla v_1|^2 - |\nabla v_2|^2)|_{\alpha, \overline{B}} + \right. \\ &\quad \left. \sum_{j=1}^{N-1} |x_N \partial_{x_j} (|\nabla v_1|^2 - |\nabla v_2|^2)|_{\alpha, \overline{B}} \right). \end{aligned}$$

For abbreviation, we write $w_1 = v_1 + v_2$ and $w_2 = v_1 - v_2$. Then the first and second terms on the right of the above inequality can be handled as

$$(4.18) \quad |\nabla v_1|^2 - |\nabla v_2|^2|_{0, \overline{B}} = |\nabla w_1 \cdot \nabla w_2|_{0, \overline{B}} \leq C|w_1|_{1+\alpha, \overline{B}} |w_2|_{1+\alpha, \overline{B}},$$

and

$$\begin{aligned} |\partial_{x_N} (|\nabla v_1|^2 - |\nabla v_2|^2)|_{\alpha, \overline{B}} &= |\nabla \partial_{x_N} w_1 \cdot \nabla w_2 + \nabla w_1 \cdot \nabla \partial_{x_N} w_2|_{\alpha, \overline{B}} \leq \\ &C|\partial_{x_N} w_1|_{1+\alpha, \overline{B}} |w_2|_{1+\alpha, \overline{B}} + C|w_1|_{1+\alpha, \overline{B}} |\partial_{x_N} w_2|_{1+\alpha, \overline{B}}. \end{aligned}$$

The other terms are estimated as

$$\begin{aligned} |x_N \partial_{x_j} (|\nabla v_1|^2 - |\nabla v_2|^2)|_{\alpha, \overline{B}} &= |x_N \nabla \partial_{x_j} w_1 \cdot \nabla w_2 + x_N \nabla w_1 \cdot \nabla \partial_{x_j} w_2|_{\alpha, \overline{B}} \leq \\ &|x_N \nabla \partial_{x_j} w_1|_{\alpha, \overline{B}} |w_2|_{1+\alpha, \overline{B}} + |w_1|_{1+\alpha, \overline{B}} |x_N \nabla \partial_{x_j} w_2|_{\alpha, \overline{B}}, \end{aligned}$$

and hence

$$\begin{aligned} \sum_{j=1}^{N-1} |x_N \partial_{x_j} (|\nabla v_1|^2 - |\nabla v_2|^2)|_{\alpha, \overline{B}} &\leq \\ &\left(|x_N D_{x'}^2 w_1|_{\alpha, \overline{B}} + |\partial_{x_N} w_1|_{1+\alpha, \overline{B}} \right) |w_2|_{1+\alpha, \overline{B}} + \\ &|w_1|_{1+\alpha, \overline{B}} \left(|x_N D_{x'}^2 w_2|_{\alpha, \overline{B}} + |\partial_{x_N} w_2|_{1+\alpha, \overline{B}} \right). \end{aligned}$$

We note that w_1 satisfies (1.4)-(1.5) with φ and ψ replaced by $\varphi_1 + \varphi_2$ and 2ψ , respectively. Hence it follows from Theorem 3.1 that

$$\begin{aligned} |w_1|_{1+\alpha, \overline{B}} + |\partial_{x_N} w_1|_{1+\alpha, \overline{B}} + |x_N D_{x'}^2 w_1|_{\alpha, \overline{B}} &\leq \\ &C \left(|\varphi_1|_{1+\alpha, \overline{B}} + |\varphi_2|_{1+\alpha, \overline{B}} + |\psi|_{3/2+\alpha, \overline{B}} \right). \end{aligned}$$

Moreover, since w_2 solves (1.4)-(1.5) with $\varphi = \varphi_1 - \varphi_2$ and $\psi = 0$, Theorem 3.1 shows that

$$(4.19) \quad |w_2|_{1+\alpha, \overline{B}} + |\partial_{x_N} w_2|_{1+\alpha, \overline{B}} + |x_N D_{x'}^2 w_2|_{\alpha, \overline{B}} \leq C |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}}.$$

We thus obtain (4.11) by plugging (4.18)-(4.19) into (4.17).

It remains to prove (ii). To this end, we assume that $\varphi \in C_{\text{ax}}^{1+\alpha}(\overline{B})$ and that ψ is constant. Then, we can directly check that for any $(N-1) \times (N-1)$ orthogonal matrix \mathcal{R} , the function $v(\mathcal{R}x', x_N)$ also satisfies (1.4)-(1.5). Hence Proposition 2.1 gives that $v(x', x_N) = v(\mathcal{R}x', x_N)$. In particular, we have that $|\nabla v(x', x_N)|^2 = |\nabla v(|x'|e_1', x_N)|^2$. Therefore (i) of Proposition 4.3 and Lemma 4.4 show that (4.12) holds, and the proof is complete. \square

4.3. Contraction mappings and the proof of Theorem 1.1. For $g \in C^{1+\alpha}(\overline{B})$ and $h \in \mathbb{R}$, we define operators Ψ_g and $\tilde{\Psi}_{g,h}$ by

$$\Psi_g[\varphi] = \frac{1}{2} (g^2 - 1 - \mathcal{T}[\varphi]), \quad \tilde{\Psi}_{g,h}[\varphi] = \frac{1}{2} (g^2 - 1 - \tilde{\mathcal{T}}_h[\varphi]).$$

Note that, by Propositions 4.1 and 4.3, Ψ_g is a mapping from $C_{\text{even}}^{1+\alpha}(\overline{B})$ into itself and $\tilde{\Psi}_{g,h}$ is a mapping from $C_{\text{ax}}^{1+\alpha}(\overline{B})$ into itself. We also define closed sets of $C^{1+\alpha}(\overline{B})$ by

$$\begin{aligned} X_g &= \{\varphi \in C_{\text{even}}^{1+\alpha}(\overline{B}) : |\varphi|_{1+\alpha, \overline{B}} \leq |g^2 - 1|_{1+\alpha, \overline{B}}\}, \\ \tilde{X}_{g,h} &= \{\varphi \in C_{\text{ax}}^{1+\alpha}(\overline{B}) : |\varphi|_{1+\alpha, \overline{B}} \leq |g^2 - 1|_{1+\alpha, \overline{B}} + |h|\}, \end{aligned}$$

and positive constants δ_1 and δ_2 by

$$\delta_1 = \min \left\{ \frac{1}{C_1}, \frac{\lambda}{C_2} \right\}, \quad \delta_2 = \min \left\{ \frac{1}{2C_3}, \frac{2\lambda}{3C_4} \right\}.$$

Here, $\lambda \in (0, 1)$ is an arbitrary fixed constant, and C_1, C_2, C_3, C_4 are the constants given in Propositions 4.1 and 4.3.

Lemma 4.6. *The following hold.*

- (i) *If $g \in C_{\text{even}}^{1+\alpha}(\overline{B})$ satisfies $|g^2 - 1|_{1+\alpha, \overline{B}} \leq \delta_1$, then Ψ_g has a unique fixed point in X_g .*
- (ii) *If $g \in C_{\text{ax}}^{1+\alpha}(\overline{B})$ and $h \in \mathbb{R}$ satisfy $|g^2 - 1|_{1+\alpha, \overline{B}} + |h| \leq \delta_2$, then $\tilde{\Psi}_{g,h}$ has a unique fixed point in $\tilde{X}_{g,h}$.*

Proof. We show (i). Set $\delta = |g^2 - 1|_{1+\alpha, \overline{B}}$, so that $\delta \leq \delta_1$. For $\varphi \in X_g$, we see from Proposition 4.1 that $\Psi_g[\varphi] \in C_{\text{even}}^{1+\alpha}(\overline{B})$ and

$$|\Psi_g[\varphi]|_{1+\alpha, \overline{B}} \leq \frac{1}{2} (\delta + |\mathcal{T}[\varphi]|_{1+\alpha, \overline{B}}) \leq \frac{1}{2} (\delta + C_1 |\varphi|_{1+\alpha, \overline{B}}^2) \leq \frac{1}{2} (1 + C_1 \delta_1) \delta \leq \delta.$$

Hence we have that $\Psi_g(X_g) \subseteq X_g$.

Furthermore, inequality (4.2) shows that for $\varphi_1, \varphi_2 \in X_g$,

$$\begin{aligned} |\Psi_g[\varphi_1] - \Psi_g[\varphi_2]|_{1+\alpha, \overline{B}} &= \frac{1}{2} |\mathcal{T}[\varphi_1] - \mathcal{T}[\varphi_2]|_{1+\alpha, \overline{B}} \leq \\ &\frac{1}{2} C_2 \left(|\varphi_1|_{1+\alpha, \overline{B}} + |\varphi_2|_{1+\alpha, \overline{B}} \right) |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}} \leq C_2 \delta_1 |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}}. \end{aligned}$$

Thus, we infer that

$$|\Psi_g[\varphi_1] - \Psi_g[\varphi_2]|_{1+\alpha, \overline{B}} \leq \lambda |\varphi_1 - \varphi_2|_{1+\alpha, \overline{B}},$$

i.e. we have proved that Ψ_g is a contraction mapping in X_g . The Banach fixed point theorem then gives the desired conclusion.

The assertion (ii) can be proved in the same way, by applying Proposition 4.3 instead of Proposition 4.1. \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We first prove (i). Note that

$$(4.20) \quad |g^2 - 1|_{1+\alpha, \overline{\Omega}} \leq C_0 |g + 1|_{1+\alpha, \overline{\Omega}} |g - 1|_{1+\alpha, \overline{\Omega}} \leq C_0 \left(|g - 1|_{1+\alpha, \overline{\Omega}} + 2 \right) |g - 1|_{1+\alpha, \overline{\Omega}}$$

for some constant $C_0 > 0$. Hence, if

$$|g - 1|_{1+\alpha, \overline{\Omega}} \leq \sqrt{1 + \frac{\delta_1}{C_0}} - 1,$$

we have that $|g^2 - 1|_{1+\alpha, \overline{\Omega}} \leq \delta_1$.

Suppose that the above condition is satisfied. Then, Ψ_g has a unique fixed point $\varphi \in X_g$, by Lemma 4.6. Let v be the solution of (1.4)-(1.5) with $\psi = 0$. Then $u = f + v$ is harmonic in B and

$$(4.21) \quad |\nabla u|^2 = |\nabla f|^2 + 2\nabla f \cdot \nabla v + |\nabla v|^2 = 1 + 2\partial_{x_N} v + \mathcal{T}[\varphi] = 1 + 2\varphi + (g^2 - 1 - 2\Psi_g[\varphi]) = g^2 \quad \text{on } \mathcal{S}.$$

This shows that u is a solution of (1.1). Moreover, Theorem 3.1 and the fact that $\varphi \in X_g$ give that u is in $C_{\text{odd}}^{1+\alpha}(\overline{B})$ and

$$|u - f|_{1+\alpha, \overline{B}} = |v|_{1+\alpha, \overline{B}} \leq C |\varphi|_{1+\alpha, \overline{B}} \leq C |g^2 - 1|_{1+\alpha, \overline{B}} \leq C |g - 1|_{1+\alpha, \overline{B}},$$

where $C > 0$ is a constant. The assertion (i) thus holds if we take δ_0 such that $\delta_0 \leq \sqrt{1 + \delta_1/C_0} - 1$.

For (ii), we may assume $h = 0$, by considering $\tilde{u} = u - h$ instead of u . Then, (ii) can be shown in a similar way. We see from (4.20) that $|g^2 - 1|_{1+\alpha, \overline{\Omega}} \leq \delta_2$ if

$$|g - 1|_{1+\alpha, \overline{\Omega}} \leq \sqrt{1 + \frac{\delta_2}{C_0}} - 1.$$

Under this condition, Lemma 4.6 shows that $\tilde{\Psi}_{g,0}$ has a fixed point $\tilde{\varphi} \in \tilde{X}_{g,0}$. We put $\tilde{u} = f + \tilde{v}$ for the solution \tilde{v} of (1.4)-(1.5) with $\varphi = \tilde{\varphi}$ and $\psi = 0$. Then \tilde{u} solves (1.1), since the same computation as in (4.21) is valid thanks to (4.12). The fact that $\tilde{u} \in C_{\text{ax}}^{1+\alpha}(\overline{B})$ and the inequality $|\tilde{u} - f|_{1+\alpha, \overline{B}} \leq C |g - 1|_{1+\alpha, \overline{B}}$ follow from Theorem 3.1. We have thus shown (ii), and the proof is complete. \square

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