Meccanica
https://doi.org/10.1007/s11012-024-01755-9

Nonlinear nonholonomic systems: a simple approach and various examples

Federico Talamucci

Received: 15 March 2023 / Accepted: 9 January 2024

Abstract The main theme of the article is the study of discrete systems of material points subjected to constraints not only of a geometric type (holonomic constraints) but also of a kinematical type (anolonomic constraints). The setting up of the equations of motion follows a simple principle which generalizes the holonomic case. Furthermore, attention is paid to the fact that the kinematical variables retain their velocity meaning, without resorting to the pseudo-velocity technique. Particular situations are examined in which the modeling of the constraints can be carried out in several ways to evaluate their effective equivalence. Numerous examples, many of which taken from the most recurring ones in the literature, are provided in order to illustrate the proposed theory.

Keywords Nonholonomic mechanical systems · Linear and nonlinear kinematical constraints · Lagrangian equations of motion · Voronec's equations of motion

1 Introduction

1.1 The object of the work

The definition of holonomic or geometric constraint is well known as a position constraint, i.e. one analytically characterized by the annulment of a function of the coordinates alone. In turn, a system subject to holonomic constraints is classified as scleronomous if the constraints do not explicitly depend on time or reonomic if they do not. The definition of a nonholonomic system is more complex, if in this term we want to gather all the complementary situations with respect to the case of geometric restrictions: in fact the term nonholonomic refers to all those restrictions which cannot be expressed solely through a relationship between the spatial coordinates that fix the positions of the system. In the broad scenario of restrictions of this type we focus our attention on the constraints expressible through relations involving the coordinates of the points and their velocities: this category of nonholonomic constraints certainly concerns a wide field of applications in mechanical and engineering problems (for example in robotics), beyond than to be relevant from a point of view of the theory of motion.

The main purpose of the work is to provide a simple method to address the formulation of the mathematical model of a nonholonomic system, mainly having in mind two aspects:

(*a*) to extend as much as possible the fundamental features of the known and consolidated theory of holonomic systems,

(*b*) to mantain the real velocities as the kinetic variables, placing in second order the need to adopt a set of pseudo–velocities for the analytical resolution of the problem.

F. Talamucci (✉)
DIMAI, University of Florence, Via di S. Marta, 3, 50139 Florence, Italy
e-mail: federico.talamucci@unifi.it

In regard to the second aspect, there is no doubt that there is a considerable series of problems associated with kinematical constraints whose natural formulation occurs through Cartesian coordinates or in any case through variables which correspond directly to the velocities, without any transformation (we denote this situation by true coordinates systems). Moreover, the fact of using true coordinates makes, in our opinion, some standard operations on the equations of the Lagrangian type easier to interpret (we think, for example, of the energy-type balance which comes from multiplying the equations of motion by the generalized velocities). On the other hand, we are aware that there exists a remarkable of models where the introduction of appropriate combinations of velocities (these variables may no longer represent velocities and are therefore called pseudo–velocities) simplifies the mathematical discussion of the problem; notable examples in this sense are the pure rolling of disks or spheres on the plane or a surface (we quote [6, 7] and the cited literature). Such as category, which anyway refers mainly to linear kinematic constraints, does not fit into our scheme.

1.2 A synthetic historical note

As far as point (*a*) is concerned, a very brief review is needed to frame the historical development of the study of nonholonomic systems.

It is well known that the theory of holonomic systems has its foundation in the analytic fromalism of the end of the eighteenth century by Euler and Lagrange. In the following years a series of simple examples (such as the pure rolling of a rigid body on a plane) brought attention to the fact that, if it is true that a geometrical restriction entails a precise restriction on the speed of the system, the converse it is not necessarily true: a constraint on the possible speeds does not necessarily imply a restriction on the possible configurations. This has led to consider a new category of constraints which consists in formulating a relationship between the coordinates and the velocity variables. If this relation cannot be reported in terms of the coordinates only (in this case the constraint is said to be integrable), then we are in the presence of an anholonomic constraint.

The study of nonholonomic systems initially concerned mainly linear constraints in the velocities, even if there are examples of nonlinear systems [1] already at the beginning of the last century and of affine constraints in the textbook [41] during the 50 s. The main and historical proposals of formal equations, edited by Čaplygin, Hamel, Maggi, Appell, Voronec and others, are exhaustively reported and commented in the fundamental work by Neĭmark and Fufaev on this subject [37]; the book contains a large and accurate bibliography on nonholonomic systems, up to the 60 s. A historical review on nonholonomic systems from the beginning up to the latest developments (mainly via differential geometry) has been carried out in [19]; the report [33] equally places the theory of nonlinear systems in the historical path of the various approaches.

Recently a certain number of authors dealt with nonholonomic systems subject to nonlinear constraints: a list of significant references is [3, 4, 18, 26, 29], the aforementioned report [33, 36, 48, 55], and the textbooks [8, 13].

It must also be said that many aspects of the classical theoretical mechanics – as symmetries, energy conservation have been formulated and extended to the more general context of nonholonomic systems, even in the nonlinear case: a non–exhaustive but remarkable collection of references divided by topics concerns reduction and simmetries [10, 34, 35, 45], variational principles [38, 42], energy conservation and conservation laws [9, 20, 24, 28], Hamiltonian formalism [54], Hamilton–Jacobi equation [27, 50].

From the geometric point of view, the condition of a nonholonomic system with linear constraints is strongly analogous to that of a holonomic system: the space of the configurations, established by the Lagrangian coordinates, remains the same, while the space of the admitted velocities, instead of being the entire vector space of dimension equal to the degrees of freedom, will be a linear subspace of it. In this way, writing Newton's law along the directions admitted by the constraints leads to equations of the Lagrangian type which are the obvious extension of the holonomic case.

The situation regarding nonlinear kinematical constraints is decidedly different: we can trace the first example of realization of a mechanical system with nonlinear kinematical constraints to the publication [1] and at a later time analyzed at least by [25, 37, 56]. Among the elements that animate the debate on the implementation of nonlinear constraints there is the frequent situation of multiple possibilities to give rise to the same constraint constraints with nonlinear expressions or with a set of equivalent linear expressions. It must also be said that from a theoretical point of view there are not many texts on analytical mechanics that deal with the theoretical formulation of nonlinear nonholonomic systems: even the most notable texts on nonholonomic systems (as for instance [32, 40]) provide an exhaustive theory essentially for the linear case. A notable exception is the text [39], which offers a valuable overview of the study of nonlinear nonholonomic systems and formulates the theory of motion for them, using several methods.

In general terms we can identify at least three distinct methods for formulating the equations of motion (even if they can evidently interact):

(*i*) a procedure based on the analysis of the displacements admitted to the system by the constraints and on the generalization of the d'Alembert's principle,

(*ii*) methods based on a variational principle or on the use of multipliers,

(*iii*) an approach based on the formalism of differential geometry.

As for the latter, we can trace the geometric treatment of constrained mechanical systems in [49, 53] the pioneering works and indicate, among others, in [17, 30, 44] the formal complexity of this theoretical sector which has promoted considerable progress in differential geometry. An excellent publication that strikes a balance between the formal presentation of theory and the development of practical examples and problem solving is [46]. Regarding (*ii*), a significant reference that contains, among other things, the main bibliography on the subject is [15]. As for the question of making the equations of motion of a nonlinear nonholonomic system derive from a variational principle, the problem is still open under various aspects; an extensive and timely discussion of these issues is in a significant reference that contains the main bibliography about it is [22]. A further recent contribution which highlights critical and unresolved points of the question is [14].

1.3 Plan of the work

In the present work we turn to a method of type (*i*), basing ourselves on the possible displacements compatible with the constrained restrictions and generalizing the well–known equation formulation procedure through the so-called "virtual work" principle.

The structure of the work is very simple: in Sect. 2 we deal with the geometric and kinematical aspects of constrained systems, admitting a very generic class of constraints. Various examples proposed develop the theory presented. In Sect. 3 the equations of motion for nonholonomic systems with nonlinear constraints are formulated and various comments and observations are added regarding particular cases or specific hypotheses. Also in this part examples of systems with associated equations of motion are proposed.

At the end of this Introduction, let us be clearer in the motivation behind the work and in the choice and construction of the examples that support the formulation of the equations of motion.

The intention to make a contribution through this work starts from the following impression: the study of nonholonomic systems is undoubtedly vast and supported by an extensive literature in the case where the constraints are linear (that is the velocities appear linearly in the equations that define the restrictions).

On the other hand, the nonlinear case is less frequently analysed and probably suffers from the disadvantage of being examined either in a single isolated example or following complex formal treatments, involving complicated geometry or uncertain variational theories. Another unusual aspect in the literature that we have preferred to follow is that of dedicating a large part at the beginning to the formal writing of the constraints and to the examples, rather than directly from the equations of motion.

In our opinion the approach to nonlinear nonholonomic systems with the simple method of admissible displacements (which is a dated and consolidated theory for holonomic systems and linear nonholonomic systems) is uncommon and for this reason Sect. 3 is dedicated to how to obtain the equations of motion simply by declaring which the admissibile displacements are and posing the ordinary differential equations that express the constraints in an explicit form. The method leads to the generalization of the Voronec equations, formulated in the linear case.

We anticipate here that, although the use of the terms “virtual displacement”, virtual velocity” are frequent and considered classic notions in the literature, it just as often does not refer to single clear concepts. For this reason we prefere to use the terminology “admissible displacements” (as above) and the first part of Sect. 3 is dedicated to the explicit definition of those quantities.

In support of the fact that we are presenting a simple method from the point of view of theory and aimed at applications, it is appropriate and necessary to associate the theory with the presentation of significant examples of nonholonomic systems. We believe that this review of examples, which are rarely stated in the mathematical community, can be considered a novelty contribution to the literature dedicated to the topic. As far as we know, the authors which combine the formulation of the theory with a large part dedicated to the discussion of examples of nonlinear constraints (not only as isolated and short episodes) and have inspired the construction of our examples are [4, 5, 46, 59].

The selection of the examples is guided by the double need to show some situations, even simple ones, which in our opinion are absent in the literature and also to recover some existing models to provide an improvement contribution, in our opinion, in the formal theory or in the generalization. The models, presented in Sect. 2 dedicated to the formulation of the constraints, concern nonlinear nonholonomic systems (the first four), systems in which the writing of the constraints can take place through linear or nonlinear kinematical conditions (examples from 5 to 8), in the end a system (example 9) with a time-dependent linear constraint.

After having discussed in Sect. 3 our point of view regarding the introduction of the equations of motion, in the two formulations Lagrangian and Newtonian, the dynamical aspect of some models is completed by writing the corresponding equations of motion; in our opinion this aspect is also non-trivial and somehow infrequent. Almost all the models of Sect. 2 are retraced from the point of view of the equations of motion (examples from 10 to 16), not in the same order of presentation, but according to the occurrence of the context.

From our theoretical point of view we have not dealt with the question of the physical realization of the models that implement the constraint.

2 The mathematical model

2.1 A general class of constraints

Let us consider a discrete system of N particles of mass M_i which are located in an inertial frame of reference by the coordinates $\mathbf{x}_i = (x_i, y_i, z_i)$, $i = 1, \dots, N$. The Newton’s equation for each particle $M_i \ddot{\mathbf{x}}_i = \mathbf{F}_i + \mathbf{\Phi}_i$, $i = 1, \dots, N$, where M_i is the mass of the i–th point and $\mathbf{F}_i$, $\mathbf{\Phi}_i$ are respectively the active and the constraint forces on the i-th particle, can be written in $\mathbb{R}^{3N}$ in the compact form

$$\dot{\mathbf{Q}} = \mathbf{F} + \mathbf{\Phi} \tag{1}$$

where $\mathbf{Q}$ is the representative vector in $\mathbb{R}^{3N}$ of linear momenta:

$$\mathbf{Q} = \dot{\mathbf{X}}^{(\mathrm{M})},\ \mathbf{X}^{(\mathrm{M})} = (M_1 x_1, M_1 y_1, M_1, z_1, \dots, M_N x_N, M_N y_N, M_N z_N) \tag{2}$$

and $\mathbf{F} = (\mathbf{F}_1, \dots, \mathbf{F}_N) \in \mathbb{R}^{3N}$, $\mathbf{\Phi} = (\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_N) \in \mathbb{R}^{3N}$.

The unknown forces $\mathbf{\Phi}$ in (1) are due to restrictions enforced to the positions and to the velocities of the system, so that the constraints can be formulated by the $r < 3N$ equations:

$$\begin{cases} f_1(\mathbf{X}, \dot{\mathbf{X}}, t) = 0, \\ \quad \dots \quad \dots \\ f_r(\mathbf{X}, \dot{\mathbf{X}}, t) = 0. \end{cases} \qquad \mathbf{X} = (\mathbf{x_1}, \dots, \mathbf{x}_N) \in \mathbb{R}^{3N} \tag{3}$$

As it is known, the presence of t in ate least one of (3) makes the system a moving or rheonomic constraints), otherwise it is said fixed or scleronomic.

The conditions (3) are assumed to be independent with respect to the kinematical variables $\dot{\mathbf{X}}$, in the sense that the r vectors in $\mathbb{R}^{3N}$ $\nabla_{\dot{\mathbf{X}}} f_i = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \dots, \frac{\partial f_i}{\partial \dot{z}_N} \right)$, $i = 1, \dots, r$ are linearly independent or, equivalently, the jacobian matrix formed by the r vectors has the rank:

$$rank\ J_{\dot{\mathbf{X}}}(f_1, \dots, f_r) = r. \tag{4}$$

We don’t rule out the possiblity that one or more of the constraints are actually geometrical: this occurs if in correspondence of f_j it exists a function $\psi_j(\mathbf{X}, t)$ such that

$$\frac{d\psi_j}{dt} = \nabla_{\mathbf{X}}\psi_j \cdot \dot{\mathbf{X}} + \frac{\partial \psi_j}{\partial t} = f_j(\mathbf{X}, \dot{\mathbf{X}}, t). \tag{5}$$

We settle such an occurrence by assuming that the first h conditions in (3) are indeed integer conditions (holonomic constraints), whereas the last $k = r - h$ are purely kinematical (nonintegrable) constraints and entail the nonholonomic state of the system.

The validity of (5) for $j = 1, \dots, h$ and the regularity assumption (4), which in turn implicates that the vectors $\nabla_{\mathbf{X}}\psi_j = \nabla_{\dot{\mathbf{X}}} f_j$, $j = 1, \dots, h$ are linearly independent, allow us to acquire the $n = 3N - h$ lagrangian coordinates $q_1, \dots, q_n$ from the system $\psi_1(\mathbf{X}, t) = 0, \dots, \psi_h(\mathbf{X}, t) = 0$ by achieving the $3N$ relations

$$\mathbf{X} = \mathbf{X}(\mathbf{q}, t), \qquad \mathbf{q} = (q_1, \dots, q_n). \tag{6}$$

The well known formula of the velocity of the system

$$\dot{\mathbf{X}} = \sum_{j=1}^{n} \frac{\partial \mathbf{X}}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{X}}{\partial t} = \dot{\mathbf{X}}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{7}$$

which is pertinent to holonomic systems has to be considered in the present case together with the additional kinematical conditions $f_{h+1} = 0, \dots, f_r = 0$ of (3). In light of this, it is suitable to define

$$\begin{aligned} &\phi_j(\mathbf{q}, \dot{\mathbf{q}}, t) = f_{h+j}(\mathbf{X}(\mathbf{q}, t), \dot{\mathbf{X}}(\mathbf{q}, \dot{\mathbf{q}}, t), t) \\ &\text{for each} \quad j = 1, \dots, k = r - h \end{aligned} \tag{8}$$

and to rewrite the last $k = r - h$ conditions in (3) as

$$\begin{cases} \phi_1(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = 0, \\ \quad \dots \quad \dots \quad \dots \\ \phi_k(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = 0. \end{cases} \tag{9}$$

It is worth noting that the starting point of the problem might will be any mechanical system identified by the n Lagrangian free coordinates $q_1, \dots, q_n$ and subject to the kinematical conditions (9): the same conclusions we will present in the following analysis apply also in this general case.

It is also significant to observe that the constraints are independent without any extra assumption:

$$rank\, J_{(\dot{q}_1, \dots, \dot{q}_n)}(\phi_1, \dots, \phi_k) = k. \tag{10}$$

Actually, one has owing to (7) and (8):

$$J_{\dot{\mathbf{q}}}(\phi_1, \dots, \phi_k) = J_{\dot{\mathbf{X}}}(f_{h+1}, \dots, f_r)\, J_{\dot{\mathbf{q}}}\dot{\mathbf{X}} = J_{\dot{\mathbf{X}}}(f_{h+1}, \dots, f_r)\, J_{\mathbf{q}}\mathbf{X}.$$

The first jacobian matrix has full rank k (because of (4)) and the second one has rank n (because of the indipendence of the vectors $\frac{\partial \mathbf{X}}{\partial q_i}$, $i = 1, \dots, n$), so that (10) is fulfilled.

The crucial point in our analysis is the implementation of the explicit writing of (9) with respect to the velocities (as it occurs when passing from Lagrangian to Cartesian coordinates in geometric constraints). Indeed, condition (10) ensures that (9) can be locally written explicitly with respect to a selection of $n - k = m$ variables, which we can assume with no loss in generality to be $(\dot{q}_1, \dots, \dot{q}_m)$, so that (9) can be locally written as

$$\begin{cases} \dot{q}_{m+1} = \alpha_1(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t), \\ \quad \dots \quad \dots \quad \dots \\ \dot{q}_n = \alpha_k(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t). \end{cases} \tag{11}$$

We underline that this construction is strictly local and it holds only where the functions α_ν, $\nu = 1, \dots, k$, are defined; at the end of Paragraph 3.2 we will examine how the equations modify if we change the subset of m indepependent velocities.

The parameters $(\dot{q}_1, \dots, \dot{q}_m)$ are now playing the role of the basic and the independent velocities: in any position $(q_1, \dots, q_n)$ an arbitrary m–uple $(\dot{q}_1, \dots, \dot{q}_m) \in \mathbb{R}^m$ defines a possible (inthe sense of consistent with the restrictions) kinematical state of the system, by means of (7) through (11):

$$\dot{\mathbf{X}} = \sum_{r=1}^{m} \frac{\partial \mathbf{X}}{\partial q_r}\dot{q}_r + \sum_{\nu=1}^{k} \frac{\partial \mathbf{X}}{\partial q_{m+\nu}}\alpha_\nu + \frac{\partial \mathbf{X}}{\partial t} = \dot{\mathbf{X}}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t). \tag{12}$$

In simple terms, each of the h integer constraints in (3) subctracts a degree of freedom from the $3N$ coordinates $\mathbf{X}$ leading to $n = 3N - h$ independent coordinates $q_1, \dots, q_n$; on the other hand, each of the k kinematical constraints removes one of the velocities $\dot{q}_1, \dots, \dot{q}_n$, so that only $m = n - k$ have to be considered independent.

2.2 Some examples of nonlinear nonholonomic systems

We find it convenient at this juncture to introduce some concrete models. The first one is a simple nonlinear constraint recurrent in literature.

Example 1 A particle P moves in the space $\mathbb{R}^3$ in respect of the following condition on the velocity:

$$|\dot{P}| = C(t) \tag{13}$$

with C nonnegative function of time. The case C constant is frequently debated in literature (a non-exhaustive list includes [20, 30, 46, 51]); the case $C(t) = 1/\sqrt{t}$, which simulates the decelerated motion of a free particle, is analyzed in [30, 46]. In a coordinate system (13) writes $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - C(t) = 0$ and according to (3) it is $h = 0$ (no geometric restriction), $k = r = 1$ and (6) is simply $(x, y, z) = (q_1, q_2, q_3)$, so that in (9) $n = 3$ and (11) is ($m = 1$)

$$\dot{q}_3 = \pm\sqrt{C^2(t) - (\dot{q}_1^2 + \dot{q}_2^2)}.$$

A condition on the cartesian components of the velocity which generalizes (13) is the following:

$$a\dot{x}^2 + b\dot{y}^2 + c\dot{z}^2 = C^2(t).$$

It is evident that $a = b = c = 1$ corresponds to (13), while the case $a = b$, $c = -1$, $C(t) = 0$ is examined in [20].

The next three Examples (number 2, 3 and 4) concern nonlinear nonholonomic constraints related to kinematical conditions very natural to figure out (same magnitude, parallelism, orthogonality of the velocities). While not tracing in the literature directly the formulation given here (namely a set of various points that verify the same kinematical condition), the types of constraints are the basis of many other models of nonholonomic constraints, as we can imagine and as we will find in some subsequent examples (Fig. 1).

Example 2 Let us consider N points moving in the space without geometric constraints while keeping the same magnitude of the velocity:

$$|\dot{P}_1| = |\dot{P}_2| = \cdots = |\dot{P}_N|.$$

The case $N = 2$ is studied in [51]; for a general integer $N \geq 2$ equations (3) correspond to the $N - 1$ kinematical conditions

$$\begin{cases} \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 - \left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right) = 0, \\ \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 - \left(\dot{x}_3^2 + \dot{y}_3^2 + \dot{z}_3^2\right) = 0, \\ \quad \dots \quad \dots \quad \dots \quad \dots \\ \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 - \left(\dot{x}_N^2 + \dot{y}_N^2 + \dot{z}_N^2\right) = 0 \end{cases} \tag{14}$$

which are independent (in the sense of (4)) whenever the velocity in common is not null. In this problem $k = r = N - 1$ and $n = 3N$, thus $m = n - k = 2N + 1$, which is the number of independent kinetic variables (selected among $\dot{x}_1, \dot{y}_1, \dots, \dot{z}_N$) by which the remaining $N - 1$ variables can be expressed; regarding the selection of the $2N + 1$ independent velocities, one possibility which simplifies the transition from (9) to (11) is

$$(\overbrace{\dot{q}_1, \dot{q}_2, \dot{q}_3}^{3}, \overbrace{\dot{q}_4, \dot{q}_5, \dot{q}_6, \dot{q}_7, \dots, \dot{q}_{2N}, \dot{q}_{2N+1}}^{2(N-1)}) = (\overbrace{\dot{x}_1, \dot{y}_1, \dot{z}_1}^{3}, \overbrace{\dot{x}_2, \dot{y}_2, \dot{x}_3, \dot{y}_3, \dots, \dot{x}_N, \dot{y}_N}^{2(N-1)})$$

and the $N - 1$ explicit equations (11) for the dependent velocities $(\dot{q}_{2N+2}, \dot{q}_{2N+3}, \dots, \dot{q}_{3N}) = (\dot{z}_2, \dot{z}_3, \dots, \dot{z}_N)$ are

$$\begin{cases} \dot{z}_2 = \dot{q}_{2N+2} = \alpha_1 = \pm\sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - \dot{q}_4^2 - \dot{q}_5^2}, \\ \dot{z}_3 = \dot{q}_{2N+3} = \alpha_2 = \pm\sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - \dot{q}_6^2 - \dot{y}_7^2}, \\ \quad \dots \quad \dots \quad \dots \quad \dots \\ \dot{z}_N = \dot{q}_{3N} = \alpha_{N-1} = \pm\sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - \dot{q}_{2N}^2 - \dot{y}_{2N+1}^2}. \end{cases} \tag{15}$$

Example 3 The circumstance of N points with parallel velocities can be formulated by means of the restrictions

$$\dot{P}_1 \wedge \dot{P}_2 = \dot{P}_1 \wedge \dot{P}_3 = \dots = \dot{P}_1 \wedge \dot{P}_N = \mathbf{0} \tag{16}$$

which are equivalent to the $2(N - 1)$ conditions

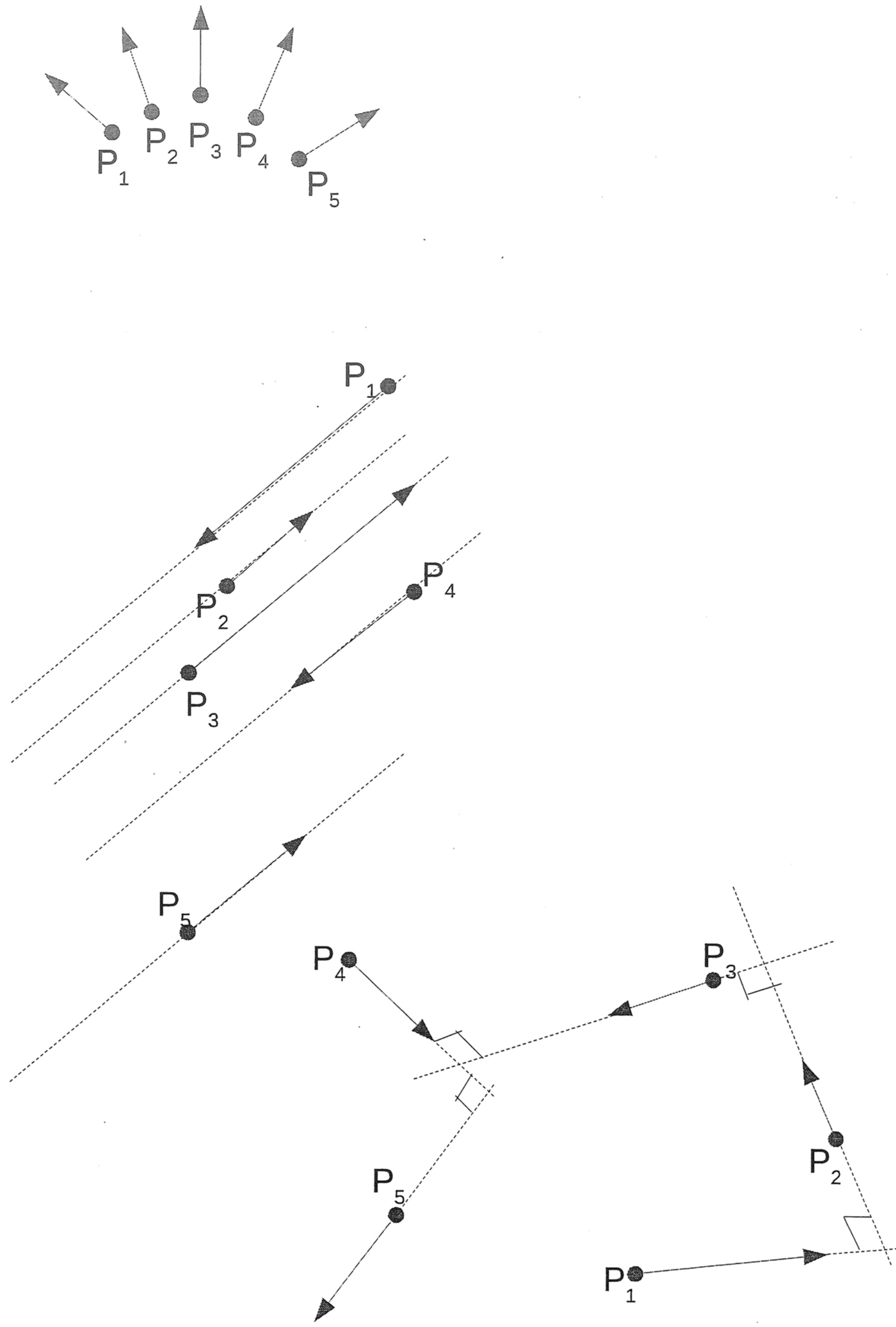

Fig. 1 Schematic representation of Example 2, top (same magnitude of velocities), Example 3, centre (parallel velocities) and Example 4, bottom (perpendicular velocities), sketched for $N = 5$

$$\begin{cases} \dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1 = 0, & \dot{x}_1\dot{z}_2 - \dot{x}_2\dot{z}_1 = 0, \\ \quad\dots & \quad\dots \\ \quad\dots & \quad\dots \\ \dot{x}_1\dot{y}_N - \dot{x}_N\dot{y}_1 = 0, & \dot{x}_1\dot{z}_N - \dot{x}_N\dot{z}_1 = 0. \end{cases} \tag{17}$$

Since $n = 3N$ and $k = 2(N-2)$ we expect $m = 3N - 2(N-2) = N + 2$: a choice for the $N + 2$ independent velocities which facilitates the explicit formulation can be

$$(\overbrace{\dot{q}_1, \dot{q}_2, \ldots \ldots, \dot{q}_N}^{N}, \overbrace{\dot{q}_{N+1}, \dot{q}_{N+2}}^{2}) = (\overbrace{\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N}^{N}, \overbrace{\dot{y}_1, \dot{z}_1}^{2}) \tag{18}$$

so that (11) for the $k = n - m = 2(N-1)$ remaining velocities $(\dot{q}_{N+3}, \dot{q}_{N+4}, \ldots, \dot{q}_{3N}) = (\dot{y}_2, \dot{y}_3, \ldots, \dot{y}_N, \dot{z}_2, \dot{z}_3, \ldots, \dot{z}_N)$ is (we use two columns to save space)

$$\begin{cases} \dot{y}_2 = \dot{q}_{N+3} = \alpha_1 = \dfrac{\dot{q}_{N+1}}{\dot{q}_1}\dot{q}_2, & \dot{z}_2 = \dot{q}_{2N+2} = \alpha_N = \dfrac{\dot{q}_{N+2}}{\dot{q}_1}\dot{q}_2, \\ \dot{y}_3 = \dot{q}_{N+4} = \alpha_2 = \dfrac{\dot{q}_{N+1}}{\dot{q}_1}\dot{q}_3, & \dot{z}_3 = \dot{q}_{2N+3} = \alpha_{N+1} = \dfrac{\dot{q}_{N+2}}{\dot{q}_1}\dot{q}_3, \\ \quad \ldots \quad \ldots & \quad \ldots \quad \ldots \\ \dot{y}_N = \dot{q}_{2N+1} = \alpha_{N-1} = \dfrac{\dot{q}_{N+1}}{\dot{q}_1}\dot{q}_N, & \dot{z}_N = \dot{q}_{3N} = \alpha_{2N-2} = \dfrac{\dot{q}_{N+2}}{\dot{q}_1}\dot{q}_N. \end{cases} \tag{19}$$

Example 4 A different system consists in N points in the space moving in a way that the velocity of each of them is perpendicular to the velocity of the previous one:

$$\dot{P}_1 \cdot \dot{P}_2 = \dot{P}_2 \cdot \dot{P}_3 = \cdots = \dot{P}_{N-1} \cdot \dot{P}_N. \tag{20}$$

The constraints (3) correspond to the $N-1$ conditions

$$\begin{cases} \dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 + \dot{z}_1\dot{z}_2 = 0, \\ \dot{x}_2\dot{x}_3 + \dot{y}_2\dot{y}_3 + \dot{z}_2\dot{z}_3 = 0, \\ \quad \ldots \quad \ldots \\ \dot{x}_{N-2}\dot{x}_{N-1} + \dot{y}_{N-2}\dot{y}_{N-1} + \dot{z}_{N-2}\dot{z}_{N-1} = 0, \\ \dot{x}_{N-1}\dot{x}_N + \dot{y}_{N-1}\dot{y}_N + \dot{z}_{N-1}\dot{z}_N = 0. \end{cases} \tag{21}$$

Whenever the velocities $\dot{P}_1$, $\dot{P}_2, \ldots, \dot{P}_{N-1}$ are not simultaneously null, the conditions (21) are independent, that is the rank of the jacobian matrix with respect to $(\dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots, \dot{x}_N, \dot{y}_N, \dot{z}_N)$ takes its full value N_1. With respect to (9) we have $n = 3N$ (no holonomic constraint, we leave the cartesian notation instead of $q_1, \ldots, q_n$ for the sake of simplicity), $r = k = N-1$ and $m = n - k = 2N + 1$. Concerning the calculation of (11), an efficient choice of the $2N+1$ independent velocities can be the one inspired by the particular structure of the conditions (21), which are combined in pairs: assuming at first N odd, each of

$$\begin{cases} \dot{x}_{2j-1}\dot{x}_{2j} + \dot{y}_{2j-1}\dot{y}_{2j} + \dot{z}_{2j-1}\dot{z}_{2j} = 0, \\ \dot{x}_{2j}\dot{x}_{2j+1} + \dot{y}_{2j}\dot{y}_{2j+1} + \dot{z}_{2j}\dot{z}_{2j+1} = 0, \end{cases} \qquad j = 1, \ldots, \tfrac{1}{2}(N-1) \tag{22}$$

can be solved whenever $\dot{P}_{2j-1} \wedge \dot{P}_{2j+1} \neq \mathbf{0}$, taking as dependent two of the variables $(\dot{x}_{2j}, \dot{y}_{2j}, \dot{z}_{2j})$: if the last two is the case, one has, for $j = 1, \ldots, \frac{1}{2}(N-1)$:

$$\dot{y}_{2j} = \dot{x}_{2j}\frac{\dot{z}_{2j-1}\dot{x}_{2j+1} - \dot{x}_{2j-1}\dot{z}_{2j+1}}{\dot{y}_{2j-1}\dot{z}_{2j+1} - \dot{z}_{2j-1}\dot{y}_{2j+1}}, \qquad \dot{z}_{2j} = \dot{x}_{2j}\frac{\dot{x}_{2j-1}\dot{y}_{2j+1} - \dot{y}_{2j-1}\dot{x}_{2j+1}}{\dot{y}_{2j-1}\dot{z}_{2j+1} - \dot{z}_{2j-1}\dot{y}_{2j+1}} \tag{23}$$

so that $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N$, $\dot{y}_1, \dot{y}_3, \ldots, \dot{y}_N$, $\dot{z}_1, \dot{z}_3, \ldots, \dot{z}_N$ (odd index) are the $2N+1$ independent parameters expressing the $N-1$ variables $\dot{y}_2, \dot{y}_4, \ldots, \dot{y}_{N-1}$, $\dot{z}_2, \dot{z}_4, \ldots, \dot{z}_{N-1}$ (even index).

If N is even, equations (22), (23) write the first $N-2$ conditions of (21) for $j = 1, \ldots, \frac{N}{2} - 1$, while the last equation provides one of the variables with index N, for instance

$$\dot{z}_N = -\frac{1}{\dot{z}_{N-1}}(\dot{x}_{N-1}\dot{x}_N + \dot{y}_{N-1}\dot{y}_N) \tag{24}$$

in a way that $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N$, $\dot{y}_1, \dot{y}_3, \ldots, \dot{y}_{N-1}$ (odd index), $\dot{y}_N$, $\dot{z}_1, \dot{z}_3, \ldots, \dot{z}_{N-1}$ (odd index) are the $2N+1$ independent parameters expressing the $N-1$ variables $\dot{y}_2, \dot{y}_4, \ldots, \dot{y}_{N-2}, \dot{z}_2, \dot{z}_4, \ldots, \dot{z}_{N-2}, \dot{z}_N$ (even index).

When the additional condition of closure

$$\dot{P}_1 \cdot \dot{P}_N = 0 \tag{25}$$

is considered, then the constraint

$$\dot{x}_1\dot{x}_N + \dot{y}_1\dot{y}_N + \dot{z}_1\dot{z}_N = 0 \tag{26}$$

has to be added to (21). If N is even (apart from the trivial case $N = 2$, where (21) counts only one condition which is identical to (26)), equation (24) combined with (26) gives

$$\dot{y}_N = \dot{x}_N\frac{\dot{z}_1\dot{x}_{N-1} - \dot{x}_1\dot{z}_{N-1}}{\dot{y}_1\dot{z}_{N-1} - \dot{y}_N\dot{z}_{N-1}}, \qquad \dot{z}_N = \dot{x}_N\frac{\dot{y}_1\dot{x}_{N-1} - \dot{x}_1\dot{y}_{N-1}}{\dot{y}_1\dot{z}_{N-1} - \dot{y}_N\dot{z}_{N-1}}$$

and $\dot{y}_N$ switches to the group of dependent velocities ($k = N$, $m = 2N$).

In the case of N odd, the velocities in (26) are all independent and it suffices to make explicit one of them, such as

$$\dot{z}_N = -\frac{1}{\dot{z}_1}(\dot{x}_1\dot{x}_N + \dot{y}_1\dot{y}_N)$$

which diminishes also in this case of one unity the number of independent velocities.

The procedure based on (23) is favored by the presence of a small number of independent velocities for each expression, however it fails when condition $\dot{P}_{2j-1} \wedge \dot{P}_{2j+1} \neq \mathbf{0}$ is not valid: this occurs in the planar case, which is after all the most encountered case in literature ("nonholonomic chains"). As a matter of fact, in this case all the velocities $\dot{P}_i$ with even index are parallel, the same for the velocities with odd index; it follows that the closure condition (26) ("closed chain") is automatically fulfilled for N even, is unfeasible for N odd (we still assume that the velocities are not null). Assuming that $y = 0$ is the plane which the points belong to, the constraints (21) reduce to $\dot{x}_i \dot{x}_{i+1} + \dot{z}_i \dot{z}_{i+1} = 0$ for $i = 1, \dots, N-1$ and can be written explicitly by means of

$$\begin{cases} \dot{z}_2 = -\dot{x}_2 \dfrac{\dot{x}_1}{\dot{z}_1} \\ \dot{z}_3 = \dot{x}_3 \dfrac{\dot{z}_1}{\dot{x}_1} \\ \dot{z}_4 = -\dot{x}_4 \dfrac{\dot{x}_1}{\dot{z}_1} \\ \dots \quad \dots \quad \dots \\ \dot{z}_N = (-1)^{N+1} \dot{x}_N \left(\dfrac{\dot{x}_1}{\dot{z}_1} \right)^{(-1)^N} \end{cases}$$

so that $k = N - 1$ once again, $n = 2N$ (actually the N holonomic conditions $y_i = 0$, $i = 1 \dots, N$ are present) and the $m = n - k = N + 1$ independent parameters are $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_N$ $\dot{y}_1$. With respect to the general case (21), it must be said that the analogous procedure in the space, that is achieving

$$\dot{z}_2 = -\frac{1}{\dot{z}_1}(\dot{x}_1 \dot{x}_2 + \dot{y}_1 \dot{y}_2) \quad \dots \quad \dots \quad \dot{z}_N = -\frac{1}{\dot{z}_{N-1}}(\dot{x}_{N-1} \dot{x}_N + \dot{y}_{N-1} \dot{y}_N)$$

then expressing each of $\dot{z}_j$, $j = 2, \dots, N$ in favour of the $2N + 1$ independent velocities by choosing $\dot{x}_1, \dots, \dot{x}_N$, $\dot{y}_1, \dots, \dot{y}_N$, $\dot{z}_N$, leads to very complex expressions presenting also problematic aspects when the case of closure (26) is contemplated.

The next example refers to the model proposed in [5] as a model of a physically feasible non-holonomic non-linear system. In the work just mentioned an extensive analysis dedicated to the writing of the equations and to the mathematical study of them in various cases is also carried out. From a formal point of view, we have slightly generalized the model by admitting that the intersection point of the straight lines orthogonal to the velocities (see below) can belong to any curve, not necessarily a horizontal axis (Fig. 2).

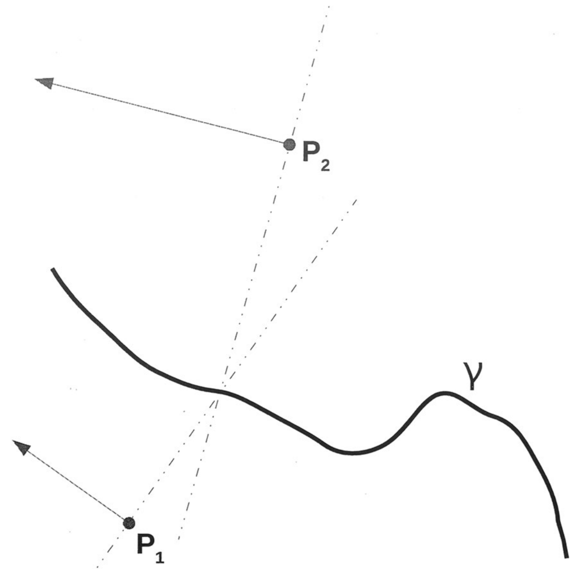

Fig. 2 Sketch of the nonholonomic model presented in Example 5

Example 5 Two points P_1 and P_2 are constrained on a plane π and the straight lines orthogonal to the velocities $\dot{P}_1$, $\dot{P}_2$ intersect in a point of a given curve γ lying on the plane. In other words, any point P verifying $(P - P_1) \cdot \dot{P}_1 = 0$ and $(P - P_2) \cdot \dot{P}_2 = 0$ has to be a point of γ.

Let us settle the carthesian frame of reference such that the plane π is $z = 0$: the equations of two straight lines are

$$(x - x_1)\dot{x}_1 + (y - y_1)\dot{y}_1 = 0, \qquad (x - x_2)\dot{x}_2 + (y - y_2)\dot{y}_2 = 0 \tag{27}$$

where $(x_i, y_i, 0)$ are the coordinates of P_i, $i = 1, 2$. The coordinates of the intersection point, definable if $\dot{P}_1 \wedge \dot{P}_2 \neq \mathbf{0}$ that is $\dot{x}_1 \dot{y}_2 \neq \dot{x}_2 \dot{y}_1$, are

$$\bar{x} = \frac{\dot{y}_1\dot{y}_2(y_1 - y_2) - x_2\dot{y}_1\dot{x}_2 + x_1\dot{x}_1\dot{y}_2}{\dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1},$$
$$\bar{y} = \frac{\dot{x}_1\dot{x}_2(x_2 - x_1) - y_1\dot{y}_1\dot{x}_2 + y_2\dot{x}_1\dot{y}_2}{\dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1}.$$

Giving the curve γ as the graph of $y = g(x)$, the kinematical constraint is then formulated in the following way:

$$\bar{y}(x_1, y_1, x_2, y_2, \dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2) = g(\bar{x}(x_1, y_1, x_2, y_2, \dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2)).$$

If γ is the straight line $ax + by = 0$, the constraint is

$$a\big(\dot{y}_1\dot{y}_2(y_1 - y_2) - x_2\dot{y}_1\dot{x}_2 + x_1\dot{x}_1\dot{y}_2\big) + b\big(\dot{x}_1\dot{x}_2(x_2 - x_1) - y_1\dot{y}_1\dot{x}_2 + y_2\dot{x}_1\dot{y}_2\big) = 0. \tag{28}$$

In particular, γ as the x–axis (nonholonomic pendulum, studied in [5]) yields $a = 0$ and the kinematical condition

$$\dot{x}_1(x_2\dot{x}_2 + y_2\dot{y}_2) - \dot{x}_2(x_1\dot{x}_1 + y_1\dot{y}_1) = 0. \tag{29}$$

Regarding the explicit form (11), the two holonomic conditions of restriction on the plane make simply leave the two coordinates z_1 and z_2: defining (6) as $x_1 = q_1$, $y_1 = q_2$, $z_1 = 0$, $x_2 = q_3$, $y_2 = q_4$, $z_2 = 0$, the explicit form of (28) is

$$\dot{q}_4 = \frac{b(q_1 - q_3)\dot{q}_1 + (bq_2 + aq_3)\dot{q}_2}{(aq_1 + bq_4)\dot{q}_1 + a(q_2 - q_4)\dot{q}_2}\dot{q}_3 = \alpha_1(q_1, q_2, q_3, q_4, \dot{q}_1, \dot{q}_2, \dot{q}_3) \tag{30}$$

showing $n = 4$ and $m = 3$.

We point out that the constraints (14), (17), (21), (28) are represented by homogeneous quadratic kinematical functions. In terms of lagrangian coordinates and referring to (9), if the geometric constraints are absent or independent of time (see (6)) we can outline this category by

$$\sum_{i,j=1}^{n} a_{i,j}^{(\nu)}(q_1, \dots, q_n)\dot{q}_i\dot{q}_j = 0, \qquad \nu = 1, \dots, k. \tag{31}$$

In particular, for (14), (17) and (21) the coefficients $a_{i,j}^{(\nu)}$ are constant.

2.3 Linear kinematical constraints

A special case which covers a wide framework of models and applications concerns the linear dependence of the kinematical conditions with respect to the velocities: this corresponds to have the last k conditions in (3) of the form

$$f_{h+1}(\mathbf{X}, \dot{\mathbf{X}}, t) = \boldsymbol{\eta}_j(\mathbf{X}, t) \cdot \dot{\mathbf{X}} + \theta_j(\mathbf{X}, t), \qquad j = 1, \dots, k$$

with $\boldsymbol{\eta}_j$ vector–valued function with values in $\mathbb{R}^{3N}$, θ_j real–value function for each $j = 1, \dots, k$. In turn, the functions (8) are linear with respect to the kinetic variables $\dot{\mathbf{q}}$, so that (9) becomes the linear system

$$\begin{cases} \sum\limits_{i=1}^{n} \sigma_{1,i}(q_1, \dots, q_n, t)\dot{q}_i + \zeta_1(q_1, \dots, \dots, q_n, t) = 0, \\ \qquad \dots\dots\dots \\ \qquad \dots\dots\dots \\ \sum\limits_{i=1}^{n} \sigma_{k,i}(q_1, \dots, q_n, t)\dot{q}_i + \zeta_k(q_1, \dots, \dots, q_n, t) = 0 \end{cases} \tag{32}$$

where

$$\sigma_{\nu,i}(\mathbf{q}, t) = \boldsymbol{\eta}_\nu(\mathbf{X}(\mathbf{q}, t), t) \cdot \frac{\partial \mathbf{X}}{\partial q_i}(\mathbf{q}, t), \quad \nu = 1, \dots, k, \quad i = 1, \dots, n$$

$$\zeta_\mu(\mathbf{q}, t) = \boldsymbol{\eta}_\mu(\mathbf{X}(\mathbf{q}, t), t) \cdot \frac{\partial \mathbf{X}}{\partial t} + \theta_\mu(\mathbf{X}(\mathbf{q}, t), t), \quad \mu = 1, \dots, k.$$

If the $k \times n$ matrix $(\sigma)_{\nu,i}$ has full rank k, then conditions (32) provide (11) in the form

$$\begin{cases} \dot{q}_{m+1} = \sum\limits_{r=1}^{m} \alpha_{1,r}(q_1, \dots, q_n, t)\dot{q}_r + \beta_1(q_1, \dots, q_n, t), \\ \dots \\ \dots \\ \dot{q}_n = \sum\limits_{r=1}^{m} \alpha_{k,r}(q_1, \dots, q_n, t)\dot{q}_r + \beta_k(q_1, \dots, q_n, t) \end{cases} \tag{33}$$

for suitable coefficients $\alpha_{\nu,r}$ and β_ν, $\nu = 1, \dots, k$, $r = 1, \dots, m$. Among the treatises dealing with linear nonholonomic systems, in our opinion the main point of reference is the mentioned [37].

2.4 Examples of nonholonomic systems containing linear and nonlinear constraints

It is worth to mention that not a few examples of systems with linear constraints originate from nonlinear conditions (e. g. parallelism, orthogonality,..) by adding some specific request: an evident instance is the pair of constraints (27), which turn into linear if the intersection point $P \equiv (x, y)$ becomes part of the system.

As a second instance, if in (16) one specifies that the velocities are parallel to the same vector $\mathbf{v} = (\alpha, \beta, \gamma)$, the set (17) is revised as

$$\beta \dot{x}_i - \alpha \dot{y}_i = 0, \quad \gamma \dot{x}_i - \alpha \dot{z}_i = 0$$

forming $2N$ linear kinematical conditions.

Or else, if in (20), $N = 2$, it is required that the velocity of P_2 is also perpendicular to the straight line joining the two points, the kinematical constraints are

$$\dot{P}_2 \cdot \overrightarrow{P_1P_2} = 0, \quad \dot{P}_1 \wedge \overrightarrow{P_1P_2} = \mathbf{0}$$

which are linear (we will discuss deeper the question in Example 8).

For the most part, nonholonomic systems considered in literature deal with linear kinematical constraints: the rolling disc or the rolling sphere on a fixed or mobile surface are largely studied in textbooks as [8, 13, 16, 23, 32, 37, 39, 40] and papers articles ranging from pioneering studies like [11] up to several more recent articles, as for instance [6, 7]. We incidentally remark that in the same mentioned examples of linear constraints the question of introducing pseudo–velocities in order to simplify the resolution of the problem is very often present.

Our selection of nonholonomic systems where linear constraints are present aims to present some examples (namely n. 6, 7 and 8 just below) where the same modellistic situation can be implemented by either a list of linear constraints, or a mixed set where also nonlinear conditions take part. In doing this it is necessary to point out which are the critical configurations where the equivalence between one group of restrictions and another is lost. By the selected examples hereafter, insipired by some models proposed in [56, 57], we investigate on various possibilities of obtaining the same restrictions and we seek legibility (sometimes lacking) regarding the independence among the stated conditions.

Example 6 A simple category of models taken as basic examples in most texts considers two points P_1 and P_2 moving on a plane and combines two (or more) of the following constraints:

(a) the distance between the points is constant: $|\overrightarrow{P_1P_2}| = \ell > 0$,

(b) the velocities have the same magnitude: $|\dot{P}_1| = |\dot{P}_2|$,

(c) the velocity of the midpoint B of $\overrightarrow{P_1P_2}$ is parallel to the straight line joining the two points: $\dot{B} \wedge \overrightarrow{P_1P_2} = \mathbf{0}$.

It is worth to make plain the inferences among them:

I (a), (c) $\Rightarrow$ (b),
II (a), (b) $\Rightarrow$ (c) whenever $\dot{P}_1 \neq \dot{P}_2$,
III (b), (c) $\Rightarrow$ (a) whenever $\dot{B} \neq \mathbf{0}$.

As a matter of fact, the formulation (3) of the three constraints (a), (b) and (c) is

$$\begin{cases} (x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + (y_1 - y_2)(\dot{y}_1 - \dot{y}_2) = 0, & (a) \\ (\dot{x}_1 + \dot{x}_2)(\dot{x}_1 - \dot{x}_2) + (\dot{y}_1 + \dot{y}_2)(\dot{y}_1 - \dot{y}_2) = 0, & (b) \\ (x_1 - x_2)(\dot{y}_1 + \dot{y}_2) - (y_1 - y_2)(\dot{x}_1 + \dot{x}_2) = 0. & (c) \end{cases}$$

Regarding statement I, conditions (a), (c) can be considered as a linear system for the two quantities $(x_1 - x_2)$ and $(y_1 - y_2)$; since $P_1 \not\equiv P_2$, the existence of not null solutions entails that the determinant $-(\dot{x}_1 - \dot{x}_2)(\dot{x}_1 + \dot{x}_2) - (\dot{y}_1 - \dot{y}_2)(\dot{y}_1 + \dot{y}_2)$ vanishes, so that (b) is valid. For II one argues analogously: (a), (b) is a system for $(\dot{x}_1 - \dot{x}_2)$ and $(\dot{y}_1 - \dot{y}_2)$; any not null solution (the null solution corresponds to $\dot{P}_1 = \dot{P}_2$) requires $(x_1 - x_2)(\dot{y}_1 + \dot{y}_2) = (y_1 - y_2)(\dot{x}_1 + \dot{x}_2)$, that is condition ($c$). Lastly, ($b$), ($c$) is a system for $\dot{x}_1 + \dot{x}_2$ and $\dot{y}_1 + \dot{y}_2$; the null solution corresponds to $\dot{B} = \mathbf{0}$

and nullifying the determinant is equivalent to condition (a).

Remark 1 Introducing standard coordinates for such as problems makes evident the geometrical or kinematical meaning of the previous scheme. Actually, if (a) is in effect, the holonomic constraint let us write (6) in the form

$$\begin{aligned} x_1 &= q_1 + \frac{\ell}{2}\cos q_3, \quad y_1 = q_2 + \frac{\ell}{2}\cos q_3, \\ x_2 &= q_1 - \frac{\ell}{2}\cos q_3, \quad y_2 = q_2 - \frac{\ell}{2}\sin q_3 \end{aligned} \tag{34}$$

(q_1 and q_2 are the coordinates of the midpoint, q_3 is the angle that $\overrightarrow{P_1P_2}$ forms with the x–axis, increasing anticlockwise). The form (9) in the lagrangian coordinates (34) of the kinematical constraints is

$$\begin{cases} \dot q_3(\dot q_1 \sin q_3 - \dot q_2 \cos q_3) = 0, & (b) \\ \dot q_1 \sin q_3 - \dot q_2 \cos q_3 = 0 & (c) \end{cases}$$

which makes evident that if the velocity of the midpoint is along the conjoining line then the endpoints must have the same magnitude of the velocity (statement *I*); the opposite sense is not necessarily true, since in any translational motion (corresponding to $\dot q_3 = 0$) not parallel to $\overrightarrow{P_1P_2}$, the velocities of the ends are equal but $\dot B$ does not have the direction $\overrightarrow{P_1P_2}$. For this reason statement *II* is valid for $\dot P_1 \neq \dot P_2$, in such a way that a non–null angular velocity is present ($\dot q_3 \neq 0$) and (b), (c) turn out to be equivalent. An anologous explication holds for statment *III*: if $\dot B$ is zero, a rotational motion around B of the segment P_1P_2 shows the same magnitude of the velocities of the ends even if the lenght of the segment changes (so that (a) fails).

Example 7 A second example with similar points concerns the following conditions:

(a) the distance between the points is constant: $|\overrightarrow{P_1P_2}| = \ell > 0$,

(b_1) the velocities are parallel: $\dot P_1 \wedge \dot P_2 = \mathbf{0}$,

(c_1) the velocity of the midpoint B is orthogonal to the straight line joining the two points: $\dot B \cdot \overrightarrow{P_1P_2} = \mathbf{0}$,

(d_1) the velocities are orthogonal to the joining straight line: $\dot P_1 \cdot \overrightarrow{P_1P_2} = 0$, $\dot P_2 \cdot \overrightarrow{P_1P_2} = 0$.

Analogously to the previous example, we can prove the statements:

I	(d_1)	$\Rightarrow$ (a), (b_1), (c_1),	
II	(a), (c_1)	$\Rightarrow$ (b_1), (d_1),	
III	(b_1), (c_1)	$\Rightarrow$ (d_1)	whenever $\dot B \neq \mathbf{0}$,
IV	(a), (b_1)	$\Rightarrow$ (d_1)	whenever $\dot P_1 \neq \dot P_2$.

The cartesian expressions of the four conditions are

$$\begin{cases} (x_1 - x_2)(\dot x_1 - \dot x_2) + (y_1 - y_2)(\dot y_1 - \dot y_2) = 0, & (a) \\ \dot x_1 \dot y_2 - \dot x_2 \dot y_1 = 0, & (b_1) \\ (x_1 - x_2)(\dot x_1 + \dot x_2) + (y_1 - y_2)(\dot y_1 + \dot y_2) = 0, & (c_1) \\ (x_1 - x_2)\dot x_1 + (y_1 - y_2)\dot y_1 = 0, \ (x_1 - x_2)\dot x_2 + (y_1 - y_2)\dot y_2 = 0. & (d_1) \end{cases}$$

Property *I* is evident: (a) and (c_1) are the sum and the difference of the conditions (d_1), (b_1) comes from the condition of existence of non–vanishing solutions of system (d_1), which is linear with respect to $x_1 - x_2$, $y_1 - y_2$ (actually $P_1 \not\equiv P_2$). Arguing in the same way, system (a), (c_1) entails (b_1) (as the condition of null determinant) and (d_1) (by summing and subtracting), hence *II* is valid. As for *III*, a simple calculation on (b_1) and (c_1) leads to $(\dot x_1 + \dot x_2)[(x_1 - x_2)\dot x_1 + (y_1 - y_2)\dot y_1] = 0$, $(\dot y_1 + \dot y_2)[(x_1 - x_2)\dot x_2 + (y_1 - y_2)\dot y_2] = 0$; it follows that, if $\dot B \neq 0$, at least one of the two conditions (d_1) (which appear in square brackets in the previous expressions) must be true and the other one follows from (c_1). Finally, statement *IV* is obtained in a similar way, by rearranging (a) and (b_1) and by assuming $(\dot x_1 - \dot x_2)^2 + (\dot y_1 - \dot y_2)^2 \neq 0$ in order to get (d_1).

We remark that in statement *III* the assumption $\dot B \neq \mathbf{0}$ is necessary, since in a rotational motion around B the velocities of the ends are not orthogonal to the joining segment, if the lenght of the latter is not constant. In the same way, statement *III* fails in a translational motion along any direction not parallel to $\overrightarrow{P_1P_2}$: the critical case $\dot P_1 = \dot P_2$ allows the velocities to be not orthogonal to the joining line (Fig. 3).

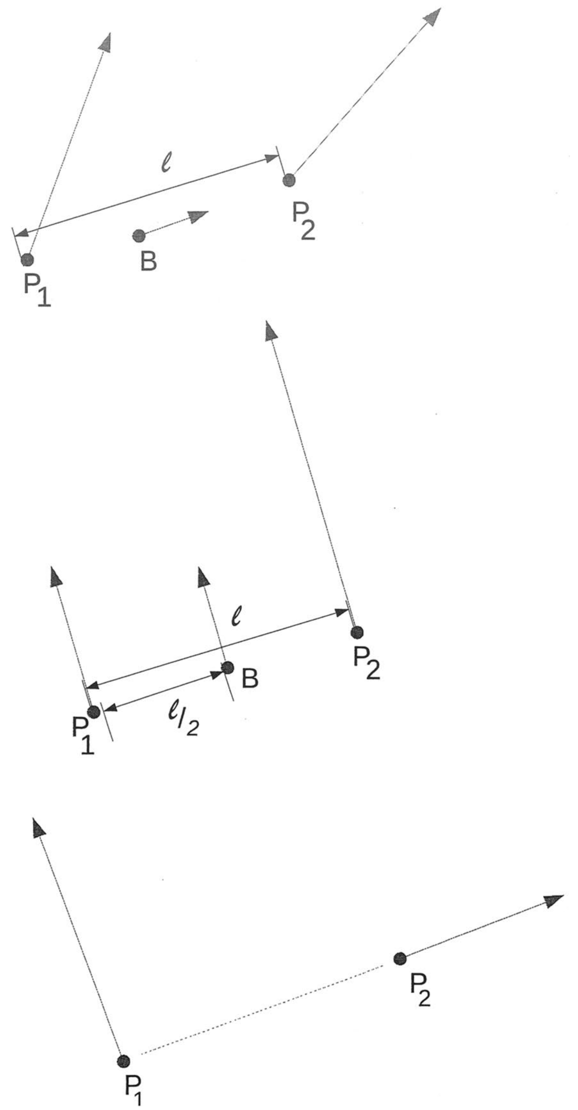

Fig. 3 Schematic sketches of Example 6 (on top), Example 7 (in the middle) and Example 8 (below)

Example 8 Let us introduce a further example, examined in [58]: two points move on a plane and

(a_2) the velocities are perpendicular: $\dot{P}_1 \cdot \dot{P}_2 = 0$,

(b_2) the velocity of one of them is perpendicular to the straight line joining the points: $\dot{P}_1 \cdot \overrightarrow{P_1P_2} = 0$,

(c_2) the velocity of the other point is parallel to the joining line: $\dot{P}_2 \wedge \overrightarrow{P_1P_2} = \mathbf{0}$.

The setting (9) is

$$\begin{cases} \dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 = 0, & (a_2) \\ (x_1 - x_2)\dot{x}_1 + (y_1 - y_2)\dot{y}_1 = 0, & (b_2) \\ (x_1 - x_2)\dot{y}_2 - (y_1 - y_2)\dot{x}_2 = 0. & (c_2) \end{cases}$$

However, the three conditions are not independent: actually, it can be show that

I $(a_2), (b_2) \Rightarrow (c_2)$ whenever $\dot{P}_1 \neq \mathbf{0}$,
II $(a_2), (c_2) \Rightarrow (b_2)$ whenever $\dot{P}_2 \neq \mathbf{0}$,
III $(b_2), (c_2) \Rightarrow (a_2)$ whenever $P_1 \not\equiv P_2$.

In order to prove that, it suffices to argue as in the previous examples; moreover, the critical configurations to exclude have an evident meaning.

Remark 2 The examples presented so far in this Paragraph show that some system can be treated either with linear kinematical constraints or with nonlinear kinematical constraints (or a mixture of them): a crucial point is to know whether the two (or more) approaches are equivalent or they are incoherent in any critical configuration. In the Examples 7,8,9 we discussed the issue of equivalence highlighting the critical states (always null velocities, apart from III of Example 9) where the implications decay.

The last example of the Section is a linear kinematical constraint depending explicitly on time. Although it does not fall into the category of nonlinear systems (main theme of the work) we decided to include it because the generalization in three dimensions of the well–known "pursuing" problem on the plane (for which we refer to [46]) appears interesting and because the case is a good test for writing at the later time the equations of motion, when specified (in our theoretical frame) to the linear case.

Example 9 Let us consider in the three-dimensional space a reference point $Q \equiv (x_Q, y_Q, z_Q)$ moving according to the given relations $x_Q = x_Q(t)$, $y_Q = y_Q(t)$, $z_Q = z_Q(t)$. A point $P \equiv (x, y, z)$ "pursues" Q in a way such that its velocity is at any time parallel to the straight line joining P with Q: this means $(\dot{x}, \dot{y}, \dot{z}) \wedge (x_Q(t) - x, y_Q(t) - y, z_Q(t) - z) = \mathbf{0}$ giving the two independent conditions

$$\begin{cases} (y_Q(t) - y)\dot{x} - (x_Q(t) - x)\dot{y} = 0, \\ (z_Q(t) - z)\dot{x} - (x_Q(t) - x)\dot{z} = 0 \end{cases} \tag{35}$$

which represent the constraints equations (3), both of kinematical type ($r = k = 2$).

Since no holonomic condition is present (that is $h = 0$), the lagrangian coordinates are $q_1 = x$, $q_2 = y$, $q_3 = z$ ($n = 3$) and (6) is $\mathbf{X} = \mathbf{q}$. Moreover, $k = r = 2$ (only kinematical constraints), $m = 1$ (one independent velocity) and the explicit form (33) is, wherever $x \neq x_Q$,

$$\begin{cases} \dot{q}_2 = \dfrac{y_Q(t) - q_2}{x_Q(t) - q_1}\dot{q}_1 = \alpha_{1,1}(q_1, q_2, t)\dot{q}_1, \\ \dot{q}_3 = \dfrac{z_Q(t) - q_3}{x_Q(t) - q_1}\dot{q}_1 = \alpha_{2,1}(q_1, q_3, t)\dot{q}_1. \end{cases} \tag{36}$$

In (35) the point P running after the moving object Q can get around the whole space: a modification can be done by considering the chasing point P constrained to a regular surface $f(x, y, z) = 0$ but the moving object $Q(t)$ not necessarily lying on the surface; the kinematical condition enforces the velocity to have the direction, at any time t, of the projection of $\overrightarrow{PQ(t)}$ on the tangent plane to the surface at P. In simple terms, P runs after $Q(t)$ by choosing on the tangent plane the direction the closest to the joining straight line $PQ(t)$ as possible (Fig. 4).

Calling Q_π the orthogonal projection of Q on the tangent plane to the surface at P, the kinematical constraint is

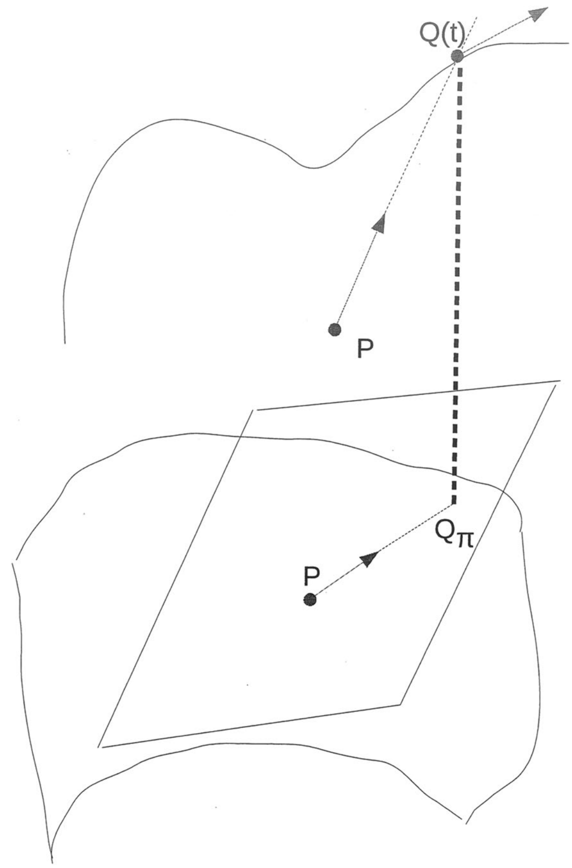

Fig. 4 The pursuing model of Example 9, when P is free in the space (above) and when P is constrained on a surface (below.)

$$\dot{P} \wedge \overrightarrow{PQ_\pi} = \mathbf{0}. \tag{37}$$

On the other hand, the vector $\overrightarrow{Q_\pi Q}$ is aligned with the direction $\nabla_{\mathbf{x}} f\big|_P = (f_x, f_y, f_z)\big|_P$ perpendicular to the tangent plane at P:

$$\overrightarrow{Q_\pi Q} = \frac{\overrightarrow{PQ} \cdot \nabla_{\mathbf{x}} f}{|\nabla_{\mathbf{x}} f|^2} \nabla_{\mathbf{x}} f$$

(it is assumed $|\nabla_{\mathbf{x}} f| \neq 0$ anywhere). Having in mind that $\overrightarrow{PQ_\pi} = \overrightarrow{PQ} - \overrightarrow{Q_\pi Q}$, the constraint (37) is equivalent to the conditions

$$\begin{cases} B(x,y,z,t)\dot{x} - A(x,y,z,t)\dot{y} = 0, \\ C(x,y,z,t)\dot{x} - A(x,y,z,t)\dot{z} = 0 \end{cases} \tag{38}$$

where

$$A(x,y,z,t) = (f_y^2 + f_z^2)(x_Q(t) - x) - f_x(f_y(y_Q(t) - y) + f_z(z_Q(t) - z)),$$

$$B(x,y,z,t) = (f_x^2 + f_z^2)(y_Q(t) - y) - f_y(f_x(x_Q(t) - x) + f_z(z_Q(t) - z)),$$

$$C(x,y,z,t) = (f_x^2 + f_y^2)(z_Q(t) - z) - f_z(f_x(x_Q(t) - x) + f_y(y_Q(t) - y))$$

being $(x,\ y,\ z)$, (x_Q, y_Q, z_Q) the coordinates of P and $Q(t)$ respectively and all the derivatives f_x, f_y, f_z calculated at P. The holonomic constraint $f(x,y,z) = 0$ appears according to (3) as

$$f_x\dot{x} + f_y\dot{y} + f_z\dot{z} = 0.$$

The latter condition and the two conditions (38) are actually not independent: as a matter of fact, the rank of (4) is not full, according to

$$det\begin{pmatrix} C & 0 & -A \\ B & -A & 0 \\ f_x & f_y & f_z \end{pmatrix} = -A(Af_x + Bf_y + Cf_z) = 0$$

since the quantity in round brackets is null: indeed $(A,\ B,\ C)$ is parallel to $(\dot{x}, \dot{y}, \dot{z})$), hence perpendicular to (f_x, f_y, f_z).

Hence, the problem can be treated as $h = 1$ making explicit $f(x,y,z) = 0$ by introducing two lagrangian parameters q_1, q_2, so that (6) is $x = x(q_1, q_2)$, $y = y(q_1, q_2)$, $z = z(q_1, q_2)$. Moreover, only one of (38) is independent ($k = 1$): the explicit expression (33) for $\dot{q}_2$ is achieved from one of the two kinematical constraints and writing (7) as $\dot{x} = x_{q_1}\dot{q}_1 + x_{q_2}\dot{q}_2$ and similarly for $\dot{y}$, $\dot{z}$.

A special case recurring in literature considers the flat surface $z = 0$ and the trajectory $Q(t)$ belonging to the plane: the parametrization is simply $x = q_1$, $y = q_2$, $z = 0$ and the kinematical conditions drastically reduce to $(y_Q - y)\dot{x} - (x_Q - x)\dot{y} = 0$, $\dot{z} = 0$ (see [46]).

Remark 3 The constraints (32) ar linear affine functions of degree 1. More broadly, for a positive integer p the set of conditions

$$\sigma_i(q_1, \dots, q_n, t) + \sum_{j=1}^{n} \zeta_{i,j}(q_1, \dots, q_n, t)\dot{q}_j^p = 0, \qquad i = 1, \dots, k \tag{39}$$

refers to affine nonholonomic constraints of degree p. The explicit form (11) corresponding to (33) when $p = 1$ is

$$\dot{q}_{m+\nu}^p = (\pm 1)^{p+1}\left(\sum_{j=1}^{m} \alpha_{\nu,j}(q_1, \dots, q_n, t)\dot{q}_j^p + \beta_\nu(q_1, \dots, q_n, t)\right)^{1/p} \tag{40}$$

for any $\nu = 1, \dots, k$ and for suitable coefficients $\alpha_{\nu,j}$ and β_ν. The special case (13) can be seen as an affine constraint of degree $p = 2$.

3 The equations of motion

We now come back to (1), having in mind to write explicitly the right equations of motion. Our theoretical context is the d'Alembert's principle (for the theoretical aspects we refer, for example, to [40]) which we introduce by considering $\hat{\hat{\mathbf{X}}} \in \mathbb{R}^{3N}$ as any admissible displacement (often referred to as "virtual displacement", but the term virtual is here deliberatery avoided, as anticipated in the Introduction): by "admissible" we simply mean any displacement which does not violate the constraints conditions, namely which is consistent with the instantaneous configuration of the system at a blocked time t (as is known, the time dependence of a mobile constraint must be eliminated to evaluate the permitted movements).

Taking the product with (1), we write

$$\left(\dot{\mathbf{Q}} - \mathbf{F} - \mathbf{\Phi}\right) \cdot \hat{\hat{\mathbf{X}}} = \mathbf{0} \tag{41}$$

where $\mathbf{Q}$, $\mathbf{F}$ and $\mathbf{\Phi}$ are the $\mathbb{R}^{3N}$–vectors already introduced in Sect. 2.1. The ideal or smooth constraints are those which release (41) from the presence of constraint forces:

$$\mathbf{\Phi} \cdot \hat{\hat{\mathbf{X}}} = 0 \qquad \text{for any admissible displacement } \hat{\hat{\mathbf{X}}}. \tag{42}$$

The d'Alembert's principle, here summarized by (41) joined with (42), has to be made operational by expressing the latter conditions in terms of the constraints (3). Such a procedure is clear and consolidated for holonomic systems (HC): the system's velocity formula (7) clearly identifies the admissible displacements with linear combinations of the vectors $\frac{\partial \mathbf{X}}{\partial q_j}$, $j=1,\dots,n$, namely $\hat{\dot{\mathbf{X}}} = \sum\limits_{j=1}^{n} \frac{\partial \mathbf{X}}{\partial q_j}\dot{q}_j$ for arbitrary $\dot{q}_1,\dots,\dot{q}_n$. As it is known, in the (HC) case there is a simple geometric reading: the zero–set (3), where $\dot{\mathbf{X}}$ does not appear among the variables, gives rise to a manifold in $\mathbb{R}^{3N}$ and $\hat{\dot{\mathbf{X}}}$ is any vector of the tangent space to the manifold at each position $\mathbf{X}$, that is the linear space generated by the vectors $\frac{\partial \mathbf{X}}{\partial q_j}$, $j=1,\dots,n$.

The situation of linear nonholonomic constraints (LNC) changes the situation in a non–substantial way: it is enough to take into account the expression (12) to realize that the velocities in question form a linear subspace (with respect to the full holonomic case); actually, (12) in the linear case (33) is

$$\dot{\mathbf{X}} = \sum_{r=1}^{m}\left(\frac{\partial \mathbf{X}}{\partial q_r} + \sum_{\nu=1}^{k}\alpha_{\nu,r}\frac{\partial \mathbf{X}}{\partial q_{m+\nu}}\right)\dot{q}_r + \boldsymbol{\tau}$$

where $\boldsymbol{\tau} = \frac{\partial \mathbf{X}}{\partial t} + \sum\limits_{\nu=1}^{k}\frac{\partial \mathbf{X}}{\partial q_{m+\nu}}\beta_\nu$ has to be ascribed to the non–stationarity of the constraints. Hence, in the (LNC) case the admissible displacements are expressed by means of

$$\hat{\dot{\mathbf{X}}} = \sum_{r=1}^{m}\left(\frac{\partial \mathbf{X}}{\partial q_r} + \sum_{\nu=1}^{k}\alpha_{\nu,r}\frac{\partial \mathbf{X}}{\partial q_{m+\nu}}\right)\dot{q}_r \qquad (LNC)\text{ systems.} \tag{43}$$

(the formula includes the holonomic case for $m=n$). We see that also in the (LNC) case the set of the vectors $\hat{\dot{\mathbf{X}}}$ forms at each position $\mathbf{X}$ a linear space generated by the linear combinations (via the independent generalized velocities $\dot{q}_1,\dots,\dot{q}_m$) of the m vectors in brackets.

In both cases (HC) and (LNC) the claim for the vectors $\hat{\dot{\mathbf{X}}}$ in the way we mentioned (see the corresponding expressions in Table 1) is fully justified by the fact that the velocity vector $\dot{\mathbf{X}}$ deprived of the component due to the possible motion of the constraints $\frac{\partial \mathbf{X}}{\partial t}$ (which must be left out the investigation on the movements permitted by the restrictions themselves) belongs to the tangent space (more precisely to a linear subspace in the case of linear kinematic constraints).

The nonlinear nonholonomic case (NNC) is decidedly different, since there is no geometric space (at least without increasing the degree of complexity of the geometric approach used) in which to set the velocity vectors. The situation is in a certain sense reversed: it is the same formulas of the velocity of the system that suggest which admissible displacements $\hat{\dot{\mathbf{X}}}$ should conveniently appear in (42) in order to make the constraint forces smooth. Following this perspective for treating (NNC) systems, let us start from the observation that the quantity in brackets in (43) is equal to $\frac{\partial \dot{\mathbf{X}}}{\partial \dot{q}_r}\dot{q}_r$ for each $r=1,\dots,m$, so that in both cases (HC) and (LNC) the admissible displacements can equivalently be defined via the statement

$$\hat{\dot{\mathbf{X}}} = \sum_{r=1}^{m}\frac{\partial \dot{\mathbf{X}}}{\partial \dot{q}_r}\dot{q}_r \tag{44}$$

(the holonomic case regards $m=n$). Although (43) and (44) define the same set of admissible displacements, the advantage of the latter formula consists in the fact that it is not necessarily confined to the linear case: this prepares the formula for the possibility of use even in a more general context. On that basis, the working hypothesis is to consider (44) as the assumption which defines the admissible displacements even in the nonlinear case. Taking into account (12), which applies to the nonlinear case (11), the assumption corresponds to declare for the case (NNC) the following class for admissible displacements $\hat{\dot{\mathbf{X}}}$, to be used in (42):

Table 1 Virtual displacements $\hat{\dot{\mathbf{X}}}$

(HC)	(LNC)	(NNC)
$\sum\limits_{j=1}^{n}\dot{q}_j\frac{\partial \mathbf{X}}{\partial q_j}$	$\sum\limits_{r=1}^{m}\dot{q}_r\left(\frac{\partial \mathbf{X}}{\partial q_r} + \sum\limits_{\nu=1}^{k}\alpha_{\nu,r}\frac{\partial \mathbf{X}}{\partial q_{m+\nu}}\right)$	$\sum\limits_{r=1}^{m}\dot{q}_r\left(\frac{\partial \mathbf{X}}{\partial q_r} + \sum\limits_{\nu=1}^{k}\frac{\partial \alpha_\nu}{\partial \dot{q}_r}\frac{\partial \mathbf{X}}{\partial q_{m+\nu}}\right)$
for arbitrary $\dot{q}_1,\dots,\dot{q}_n$	for arbitrary $\dot{q}_1,\dots,\dot{q}_m$	for arbitrary $\dot{q}_1,\dots,\dot{q}_m$

$$\hat{\mathbf{X}} = \sum_{r=1}^{m} \frac{\partial \dot{\mathbf{X}}}{\partial \dot{q}_r}\dot{q}_r = \sum_{r=1}^{m} \left(\frac{\partial \mathbf{X}}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \mathbf{X}}{\partial q_{m+\nu}} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \right) \dot{q}_r \quad (NNC) \text{ systems.} \tag{45}$$

From our point of view, the main reason and advantage of declaring the statement (44) consists in the possibility of accessing the equations of motion very easily, since this simple argument applied to the velocities allows us to identify a finite set of principal directions even in the nonlinear case. Nevertheless, it is significant to remark that (44), despite the very simple arguments we adduced, leads to the same conclusions as many different theoretical approaches to (NNC) offer.

Remark 4 Following our intention of taking care above all of examples we do not dwell on the theoretical aspects of the equations and principles, however it is worth at least noting that (44) can be traced back to the so–called Četaev condition (let us use the notation δ for admissible displacements in this Remark)

$$\sum_{i=1}^{n} \frac{\partial \phi_\nu}{\partial \dot{q}_i} \delta q_i = 0, \qquad \nu = 1, \dots, k \tag{46}$$

where the displacements δq_i render the constraints forces activated by the restrictions (9) as smooth. Historical references of (46) are [12] and [43], while a recent discussion on the topic can be found in [31] and in the bibliography cited there. The condition (46) corresponds to

$$\delta \mathbf{X} = \sum_{r=1}^{m} \frac{\partial \dot{\mathbf{X}}}{\partial \dot{q}_r} \delta q_r \tag{47}$$

which entails the same information as (44) for what concerns the admissible displacements.

We summarize in Table 1 the three sets of admissible displacements $\hat{\mathbf{X}}$ for holonomic constraints (HC), linear nonholonomic constraints (LNC) and nonlinear nonholonomic constraints systems (NNC), where it is also evident that one case generalizes the previous one.

Concerning (NNC), it is worth noting that the arbitrariness of the coefficients $(\dot{q}_1, \dots, \dot{q}_m) \in \mathbb{R}^m$ makes the set of vectors $\hat{\mathbf{X}}$ a linear space of dimension m and generated by the vectors enclosed in the round brackets, columuns (NNC).

Remark 5 For a (NNC) system, if we compare (44) with the "frozen" velocity (i. e. considered at a blocked time)

$$\hat{\mathbf{X}} = \sum_{r=1}^{m} \frac{\partial \mathbf{X}}{\partial q_r}\dot{q}_r + \sum_{\nu=1}^{k} \frac{\partial \mathbf{X}}{\partial q_{m+\nu}} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \tag{48}$$

(which can be deduced from (12) by ignoring the term due to the mobility of the constraints), we see that the two expressions coincide whenever the following hypothesis holds:

$$\sum_{i=1}^{m} \dot{q}_i \frac{\partial \alpha_\nu}{\partial \dot{q}_i} = \alpha_\nu \quad \text{for each} \quad \nu = 1, \dots, k \tag{49}$$

which is definitely verified if α_ν is a homogeneous function of degree one with respect to kinetic variables $\dot{q}_1, \dots, \dot{q}_m$.

Let us finally write the equations of motion: starting from (41), seeped through (42) and having in mind the arbitrariness of the displacements pertaining each case, we conclude that the holonomic constraints case

$$(HC) \left(\dot{\mathbf{Q}} - \mathbf{F} \right) \cdot \frac{\partial \mathbf{X}}{\partial q_j} = 0, \; j = 1, \dots, n$$

is generalized by claiming that the equations of motion appearing on the left of the following scheme must be combined with the vectors $\mathbf{X}_r$ (which represent the independent directions of ($\hat{\mathbf{X}}$, according to Table 1)) defined on the right of the same scheme:

$$\left(\dot{\mathbf{Q}} - \mathbf{F} \right) \cdot \mathbf{X}_r = 0, \; r = 1, \dots, m \begin{cases} (LNC) \;\; \mathbf{X}_r = \dfrac{\partial \mathbf{X}}{\partial q_r} + \displaystyle\sum_{\nu=1}^{k} \alpha_{\nu,r} \dfrac{\partial \mathbf{X}}{\partial q_{m+\nu}} \\ \qquad \text{jointly with } \dot{q}_\nu = \displaystyle\sum_{j=1}^{m} \alpha_{\nu,j}\dot{q}_j + \beta_\nu, \\ (NNC) \;\; \mathbf{X}_r = \dfrac{\partial \mathbf{X}}{\partial q_r} + \displaystyle\sum_{\nu=1}^{k} \dfrac{\partial \alpha_\nu}{\partial \dot{q}_r} \dfrac{\partial \mathbf{X}}{\partial q_{m+\nu}} \\ \qquad \text{jointly with } \dot{q}_\nu = \alpha_\nu \end{cases} \tag{50}$$

(we also recall the completing equations (11) and (33) in the case (LNC) and (NNC), respectively). In all three cases (HC), (LNC) and (NNC) the corresponding system makes the list of n equations in the n unknown functions $(q_1, \dots, q_n)$. In the nonholonomic cases (LNC) and (NNC) the m equations of motion (41) considered with assumption (42) and paired with the k equations (11) form a differential system of $m + k = n$ equations.

3.1 The equations of motions in the lagrangian form

Let us introduce the kinetic energy of the system $T = \sum\limits_{i=1}^{N} m_i \dot{\mathbf{x}}_i^2$, which can be written by means of the $\mathbb{R}^{3N}$–representative vectors (2) as

$$T = \frac{1}{2} \mathbf{Q} \cdot \dot{\mathbf{X}}. \tag{51}$$

In a (HC) system one has $T = T(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$, where the dependence on the listed variables is acquired via (7).

The well known property

$$\dot{\mathbf{Q}} \cdot \frac{\partial \mathbf{X}}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}, \qquad j = 1, \dots, n$$

is sufficient to formulate the equations of motion of a (HC) system in terms of the kinetic energy. The same property can be employed for nonholonomic systems: let us look after (NNC) systems and comment later the linear case (LNC) and write

$$\begin{aligned} \dot{\mathbf{Q}} \cdot \mathbf{X}_r =& \dot{\mathbf{Q}} \cdot \frac{\partial \mathbf{X}}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \dot{\mathbf{Q}} \cdot \frac{\partial \mathbf{X}}{\partial q_{m+\nu}} \\ =& \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{m+\nu}} - \frac{\partial T}{\partial q_{m+\nu}} \right) \end{aligned} \tag{52}$$

for each $r = 1, \dots, m$. At this point it is essential to refer to the independent kinetic variables: we then define

$$\begin{aligned} & T^*(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t) \\ & = T(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, \alpha_1, \dots, \alpha_k, t) \end{aligned} \tag{53}$$

where each α_ν, $\nu = 1, \dots, k$, depends on $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t$ according to (11), as the kinetic energy restricted to the independent kinetic parameters $(\dot{q}_1, \dots, \dot{q}_m)$. In (53) the functions α_ν are the same as in (9) (possibly linear in the case (33)).

Remark 6 Recalling (7), the explicit expression of (51) is

$$T(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = \frac{1}{2} \sum_{i,j=1}^{n} g_{i,j} \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} b_i \dot{q}_i + c \tag{54}$$

where

$$\begin{aligned} & g_{i,j}(q_1, \dots, q_n, t) = \frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial q_i} \cdot \frac{\partial \mathbf{X}}{\partial q_j}, \quad b_i(q_1, \dots, q_n, t) = \frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial q_i} \cdot \frac{\partial \mathbf{X}}{\partial t}, \\ & c(q_1, \dots, q_n, t) = \frac{1}{2} \frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial t} \cdot \frac{\partial \mathbf{X}}{\partial t} \end{aligned} \tag{55}$$

so that (53) writes

$$\begin{aligned} T^* =& \frac{1}{2} \left(\sum_{r,s=1}^{m} g_{r,s} \dot{q}_r \dot{q}_s + \sum_{\nu,\mu=1}^{k} g_{m+\nu,m+\mu} \alpha_\nu \alpha_\mu \right) + \sum_{r=1}^{m} \sum_{\nu=1}^{k} g_{r,m+\nu} \dot{q}_r \alpha_\nu \\ & + \sum_{r=1}^{m} b_r \dot{q}_r + \sum_{\nu=1}^{k} b_{m+\nu} \alpha_\nu + c. \end{aligned} \tag{56}$$

By virtue of the interrelations (easily inferable) among the derivatives of T and T^*

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_r} &= \frac{\partial T^*}{\partial \dot{q}_r} - \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \frac{\partial T}{\partial \dot{q}_{m+\nu}}, \quad r = 1, \dots, m \\ \frac{\partial T}{\partial q_j} &= \frac{\partial T^*}{\partial q_j} - \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial q_j} \frac{\partial T}{\partial \dot{q}_{m+\nu}}, \quad j = 1, \dots, n \end{aligned} \tag{57}$$

it is possible to express (52) in terms of T^* and of the derivatives of T with respect to the dependent velocities $\dot{q}_{m+\nu}$, $\nu = 1, \dots, k$:

$$\begin{aligned} \dot{\mathbf{Q}} \cdot \mathbf{X}_r =& \frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_r} - \frac{\partial T^*}{\partial q_r} - \sum_{\nu=1}^{k} \frac{\partial T^*}{\partial q_{m+\nu}} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \\ & - \sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{m+\nu}} \left(\frac{d}{dt}\left(\frac{\partial \alpha_\nu}{\partial \dot{q}_r} \right) - \frac{\partial \alpha_\nu}{\partial q_r} - \sum_{\mu=1}^{k} \frac{\partial \alpha_\mu}{\partial \dot{q}_r} \frac{\partial \alpha_\nu}{\partial q_{m+\mu}} \right) \end{aligned} \tag{58}$$

At this point we can state the following

Proposition 1 *The equations of motion* (50) *for (NNC) systems subject to the kinematical constraints* (11) *are, for each* $r = 1, \dots, m$,

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_r} - \frac{\partial T^*}{\partial q_r} - \sum_{\nu=1}^{k} \frac{\partial T^*}{\partial q_{m+\nu}} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} - \sum_{\nu=1}^{k} B_r^\nu \frac{\partial T}{\partial \dot{q}_{m+\nu}} = \mathcal{F}^{(q_r)} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \mathcal{F}^{(q_{m+\nu})} \tag{59}$$

where the coefficients B_r^ν *are calculated by*

$$B_r^\nu(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t) = \frac{d}{dt}\left(\frac{\partial \alpha_\nu}{\partial \dot{q}_r}\right) - \frac{\partial \alpha_\nu}{\partial q_r} - \sum_{\mu=1}^{k} \frac{\partial \alpha_\mu}{\partial \dot{q}_r} \frac{\partial \alpha_\nu}{\partial q_{m+\mu}} \tag{60}$$

and

$$\mathcal{F}^{(q_j)} = \mathbf{F} \cdot \frac{\partial \mathbf{X}}{\partial q_j}, \quad j = 1, \dots, n. \tag{61}$$

The variables $(\dot{q}_{k+1}, \dots, \dot{q}_n)$ appearing in the function $\frac{\partial T}{\partial \dot{q}_{m+\nu}}$ of (59) must be expressed in terms of $(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t)$ by means of (11). The same care has to be adopted for the explicit calculation of (60):

$$B_r^\nu = \sum_{j=1}^{m} \left(\frac{\partial^2 \alpha_\nu}{\partial \dot{q}_r \partial q_j} \dot{q}_j + \frac{\partial^2 \alpha_\nu}{\partial \dot{q}_r \partial \dot{q}_j} \ddot{q}_j \right) - \frac{\partial \alpha_\nu}{\partial q_r} + \sum_{\mu=1}^{k} \left(\frac{\partial^2 \alpha_\nu}{\partial \dot{q}_r \partial q_{m+\mu}} \alpha_\mu - \frac{\partial \alpha_\mu}{\partial \dot{q}_r} \frac{\partial \alpha_\nu}{\partial q_{m+\mu}} \right) + \frac{\partial^2 \alpha_\nu}{\partial \dot{q}_r \partial t}. \tag{62}$$

3.2 Some remarks on the equations of motion

The equations of motion (59) extend to the nonlinear case the Voronec equations appeared in [52] and proposed again in [37] for linear nonholonomic constraints (33). The generalized velocities of the Lagrangian coordinates appear directly in the equations, that is the set of the kinetic variables is not altered by linear transformations leading towards the type equations (generally known as Hamel–Boltzmann equations) which make use of pseudo–velocities.

The range of application of equations (59) is very vast and the major effort from the computational point of view is to obtain the explicit expressions (11). In many cases this is possible because the mathematical structure of the constraint is often linked to manageable formal expressions (as we have seen, a considerable category pertains to homogeneous functions). A limiting aspect can be seen in the strict use of real velocities (i. e. the generalized velocities) without involving the technique of pseudo–velocities. In [57] the same goal of deriving the most general form of the equations of motion is pursued, using more generically the pseudo–coordinates (Poincaré–Chetaev variables); actually, when the latter coincide with the real generalized velocities, the equations are the same as we wrote: we have found confirmation of this at least in [4, 39, 57]. Also in other cases in which the geometric approach is used for the description of nonlinear nonholonomic mechanical systems (Lagrangian systems on fibered manifolds) the final motion equations are the same as (59) (as we checked at least in confront of [30] and [46]).

We examine below some aspects regarding the equations of motion and we point some special cases of (59).

3.2.1 The linear case

The equations for linear systems (LNC) are achieved by setting $\frac{\partial \alpha_\nu}{\partial \dot{q}_r} = \alpha_{\nu,r}$ in (59) and the coefficients (62) as $B_r^\nu = \sum_{r=1}^{m} b_{r,j}^\nu 0n + \frac{\partial \alpha_{\nu,r}}{\partial t}$ with

$$b_{r,j}^\nu(q_1, \dots, q_n) = \frac{\partial \alpha_{\nu,r}}{\partial q_j} - \frac{\partial \alpha_{\nu,j}}{\partial q_r} + \sum_{\mu=1}^{k} \left(\frac{\partial \alpha_{\nu,r}}{\partial q_{m+\mu}} \alpha_{\mu,j} - \frac{\partial \alpha_{\nu,j}}{\partial q_{m+\mu}} \alpha_{\mu,r} \right) \tag{63}$$

so that equations (59) take the form, for each $r = 1, \dots, m$,

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_r} - \frac{\partial T^*}{\partial q_r} - \sum_{\nu=1}^{k} \alpha_{\nu,r} \frac{\partial T^*}{\partial q_{m+\nu}} - \sum_{\nu=1}^{k} \sum_{j=1}^{m} b_{r,j}^\nu \dot{q}_j \frac{\partial T}{\partial \dot{q}_{m+\nu}} = \mathcal{F}^{(q_r)} + \sum_{\nu=1}^{k} \alpha_{\nu,r} \mathcal{F}^{(q_{m+\nu})} \tag{64}$$

known as Voronec's equations (as stated in [37]).

The general instruction for the examples presented in Sect. 3 from here on (numbered from 10 to 16) is that we simply implement the equations of motion of almost all the models introduced in Sect. 2. The order of appearance is not the same, but the examples are placed in the context that the dynamical formulation offers. As stated in the Introduction, the phase of writing the equations of motion is anything but trivial, both due to the frequent complexity in nonholonomic systems of the calculations to be performed, and to the choice towards the most convenient typology of equations.

Example 10 We call to mind Example 8, concerning two points on a plane verifying $\dot{P}_1 \cdot \overrightarrow{P_1P_2} = 0$, $\dot{P}_2 \wedge \overrightarrow{P_1P_2} = \mathbf{0}$ which correspond to the linear kinematical constraints (b_2) and (c_2) of the same Example. Setting (6) as $x_1 = q_1$, $y_1 = q_3$, $x_2 = q_2$, $y_2 =_4$ the explicit formulation is

$$\dot{q}_3 = -\frac{q_1 - q_2}{q_3 - q_4}\dot{q}_1, \qquad \dot{q}_4 = \frac{q_3 - q_4}{q_1 - q_2}\dot{q}_2$$

so that with respect to (33) it is $k - 2$ and $m = 2$, $\alpha_{1,1} = -(q_1 - q_2)/(q_3 - q_4)$, $\alpha_{1,2} = \alpha_{2,1} = 0$, $\alpha_{2,2} = (q_3 - q_4)/(q_1 - q_2)$, $\beta_1 = \beta_2 = 0$. The function (53) is

$$T^*(q_1, q_2, q_3, q_4, \dot{q}_1, \dot{q}_2) = \frac{1}{2}M_1\left(1 + \alpha_{1,1}^2\right)\dot{q}_1^2 + \frac{1}{2}\left(1 + \alpha_{2,2}^2\right)\dot{q}_2^2$$

where M_1 and M_2 are the masses. The calculation of the coefficients (63) leads to

$$b_{1,1}^1 = b_{1,1}^2 = b_{1,2}^1 = 0, \quad b_{1,2}^2 = \frac{1}{q_3 - q_4}(1 + \alpha_{2,2}^2) \qquad \text{first equation } (r = 1),$$

$$b_{2,1}^1 = b_{2,2}^1 = b_{2,2}^2 = 0, \quad b_{2,1}^2 = -b_{1,2}^2 \qquad \text{second equation } (r = 2).$$

Assuming that the two points are connected by a spring exerting the force $-K(P_1 - P_2)$ on P_1 and the opposite one on P_2 (K positive constant) and including also the gravitational force directed in the direction of decreasing y, the equations of motion (64) are

$$\begin{cases} M_1(1 + \alpha_{1,1}^2)\left(\ddot{q}_1 + \dfrac{q_1 - q_2}{(q_3 - q_4)^2}\dot{q}_1^2\right) = -K[(q_1 - q_2) + \alpha_{1,1}(q_3 - q_4)] \\ -M_1\alpha_{1,1}g, \\ \\ M_2(1 + \alpha_{2,2}^2)\left(\ddot{q}_2 - M_2\dfrac{1}{q_1 - q_2}M_2\dot{q}_1\dot{q}_2\right) = K[(q_1 - q_2) + \alpha_{2,2}(q_3 - q_4)] \\ -M_2\alpha_{2,2}g. \end{cases}$$

In [58] the qualitative analysis of the model is extensively performed.

Remark 7 The nonholonomic device can be expanded by adding a third point P_3 (still on the same plane) which interplays with P_2 in the same way as the first pair of points, that is

$$\dot{P}_2 \cdot \overrightarrow{P_2P_3} = 0, \qquad \dot{P}_3 \wedge \overrightarrow{P_2P_3} = \mathbf{0}.$$

The construction can continue up to N points, giving rise to the so–called nonholonomic chains [56]: the complete scheme of constraints is

$$\dot{P}_i \cdot \overrightarrow{P_iP_{i+1}} = 0, \qquad \dot{P}_{i+1} \wedge \overrightarrow{P_iP_{i+1}} = \mathbf{0}, \qquad i = 1, \dots, N-1 \tag{65}$$

and the cartesian formulation (9) is given by the $2(N-1)$ conditions

$$\dot{x}_i(x_i - x_{i+1}) + \dot{y}_i(y_i - y_{i+1}) = 0,$$
$$\quad \dot{x}_{i+1}(y_i - y_{i+1}) - \dot{y}_{i+1}(x_i - x_{i-1}) = 0$$

for $i = 1, \dots, N-1$. Nevertheless, it must be said that the pair of constraints formed by the second of index i and the first of index $i+1$, namely

$$\begin{cases} \dot{x}_{i+1}(y_i - y_{i+1}) - \dot{y}_{i+1}(x_i - x_{i-1}) = 0, \\ \dot{x}_{i+1}(x_{i+1} - x_{i+2}) + \dot{y}_{i+1}(y_{i+1} - y_{i+2}) = 0 \end{cases}$$

entails the holonomic constraint $(x_i - x_{i+1})(x_{i+1} - x_{i+2}) + (y_i - y_{i+1})(y_{i+1} - y_{i+2}) = 0$, expressing the orthogonality condition

$$\overrightarrow{P_iP_{i+1}} \cdot \overrightarrow{P_{i+1}P_{i+2}} = 0 \quad \text{for each } i = 1, \dots, N-2. \tag{66}$$

Hence, the problem requires to be set up with $N+2$ lagrangian coordinates which can be, as an instance, (x_1, y_1), the angle φ that $\overrightarrow{P_1P_2}$ forms with the horizontal direction and the $N-1$ abscissae $\xi_{i,i+1}$, $i = 1, \dots, N-1$ on the straight lines $\overrightarrow{P_iP_{i+1}}$. The presence of the kinematical condition $\dot{P}_1 \cdot \overrightarrow{P_1P_2}$ makes the problem different from the merely holonomic problem (66): as a matter of facts, the kinematical restrictions generated by (66) are not (65), but

$$\dot{P}_{i+1} \cdot \overrightarrow{P_{i+1}P_{i+2}} - \dot{P}_i \cdot \overrightarrow{P_{i+1}P_{i+2}} + \dot{P}_{i+2} \cdot \overrightarrow{P_iP_{i+1}}$$
$$- \dot{P}_{i+1} \cdot \overrightarrow{P_iP_{i+1}} = 0, \qquad i = 1, \dots N-2.$$

3.2.2 *The stationary case*

The situation corresponding to the independence of conditions (3) from time t explicitly (fixed or scleronomic constraints, either holonomic or nonholonomic) entails that t is absent in (9), hence in α_ν, $\alpha_{\nu,j}$ and that (56) is the form

$$T^* = \frac{1}{2}\left(\sum_{r,s=1}^{m} g_{r,s}\dot{q}_r\dot{q}_s + \sum_{\nu,\mu=1}^{k} g_{m+\nu,m+\mu}\alpha_\nu\alpha_\mu\right) + \sum_{r=1}^{m}\sum_{\nu=1}^{k} g_{r,m+\nu}\dot{q}_r\alpha_\nu$$

(again, t does not appear in $g_{r,s}$, $r,s=1,\dots,n$). The only difference in the equations of motion is the lack in (62) of the term $\frac{\partial^2\alpha_\nu}{\partial\dot{q}_r\partial t}$ for (NNC) systems or of the term $\frac{\partial\alpha_{\nu,r}}{\partial t}$ for (LNC).

3.2.3 Čaplygin's systems

As mentioned in [37], Čaplygin pointed out that in many linear nonholonomic systems the generalized coordinates can be chosen in a way that

$$\begin{aligned} &\alpha_{\nu,j} = \alpha_{\nu,j}(q_1,\dots,q_m) \text{for each } \nu=1,\dots,k \text{ and } j=1,\dots,\\ &m \;\; T = T(q_1,\dots,q_m,\dot{q}_1,\dots,\dot{q}_n),\\ &\mathcal{F}_i = \mathcal{F}_i(q_1,\dots,q_m,\dot{q}_1,\dots,\dot{q}_n). \end{aligned} \tag{67}$$

This special case concerning special (LNC) is recurrent in literature and applications. Equations (59) reduce to, for each $r=1,\dots,m$,

$$\frac{d}{dt}\frac{\partial T^*}{\partial\dot{q}_r} - \frac{\partial T^*}{\partial q_r} - \sum_{\nu=1}^{k}\sum_{j=1}^{m}\left(\frac{\partial\alpha_{\nu,r}}{\partial q_j} - \frac{\partial\alpha_{\nu,j}}{\partial q_r}\right)\dot{q}_j\frac{\partial T}{\partial\dot{q}_{m+\nu}} = \mathcal{F}^{(q_r)} + \sum_{j=1}^{k}\alpha_{j,r}\mathcal{F}^{(q_{m+j})} \tag{68}$$

and they are called Čaplygin's equations (dating [11], see also [37]). The evident advantage is that (68) contains only the unknown functions $q_1,\dots,q_m$ and it is disentangled from the constraints equations (33). For a modern approach to such systems and for the question of reducibility (via geometrical methods) we refer to [21] and to the bibliography quoted there.

3.2.4 Further special forms

Let us underline the following aspects.

(a) If T is of the form $T=T(\dot{q}_1,\dots,\dot{q}_n)$, then the left part of (59) reduces to $\frac{d}{dt}\frac{\partial T^*}{\partial\dot{q}_r} - \frac{\partial T^*}{\partial q_r} - \sum_{\nu=1}^{k} B_r^\nu\frac{\partial T}{\partial\dot{q}_{m+\nu}}$.

(b) On the other hand, if the conditions (9) for a (NNC) system are $\phi_\nu(\dot{q}_1,\dots,\dot{q}_n,t)=0$, $\nu=1,\dots,k$, then (11) are of the form

$$\alpha_\nu = \alpha_\nu(\dot{q}_1,\dots,\dot{q}_m,t), \quad \nu=1,\dots,k. \tag{69}$$

In this case, the coefficients (60) are simply $B_r^\nu = \frac{d}{dt}\left(\frac{\partial\alpha_\nu}{\partial\dot{q}_r}\right)$.

Putting together (a) and (b), we deduce that if both the conditions on T and on ϕ_ν are verified, then the equations of motion (59) are simplified as

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T^*}{\partial\dot{q}_r} - \sum_{\nu=1}^{k}\frac{d}{dt}\left(\frac{\partial\alpha_\nu}{\partial\dot{q}_r}\right)\frac{\partial T}{\partial\dot{q}_{m+\nu}}\\ &\quad = \mathcal{F}^{(q_r)} + \sum_{\nu=1}^{k}\frac{\partial\alpha_\nu}{\partial\dot{q}_r}\mathcal{F}^{(q_{m+\nu})}, \qquad r=1,\dots,m. \end{aligned} \tag{70}$$

We also remark that the equality

$$\frac{d}{dt}\frac{\partial T^*}{\partial\dot{q}_r} - \sum_{\nu=1}^{k}\frac{d}{dt}\left(\frac{\partial\alpha_\nu}{\partial\dot{q}_r}\right)\frac{\partial T}{\partial\dot{q}_{m+\nu}} = \frac{d}{dt}\frac{\partial T}{\partial\dot{q}_r} + \sum_{\nu=1}^{k}\frac{\partial\alpha_\nu}{\partial\dot{q}_r}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_{m+\nu}}\right) \tag{71}$$

where the variables $\dot{q}_{k+1},\dots,\dot{q}_n$ in T have to be expressed according to (11) after differentiation, is valid for any $r=1,\dots,m$ by virtue of (57). Hence, when (70) are applied, one of the two expressions in (71) can be considered, on the basis of convenience. In particular, if T is the quadratic function

$$T = \frac{1}{2}\sum_{i=1}^{n} M^{(i)}\dot{q}_i^2, \qquad M^{(i)} > 0 \tag{72}$$

(for instance in the case of cartesian coordinates) where $M^{(i)}$ is the mass pertaining to the i–th coordinate, then (53) takes the form

$$T^* = \frac{1}{2}\sum_{i=1}^{m} M^{(i)}\dot{q}_i^2 + \frac{1}{2}\sum_{\nu=1}^{k} M^{(\nu)}\alpha_\nu^2 \tag{73}$$

hence $\frac{\partial T}{\partial\dot{q}_{m+\nu}} = M^{(m+\nu)}\dot{q}_{m+\nu} = M^{(m+\nu)}\alpha_\nu$ and (70) are, owing to (71),

$$M^{(r)}\ddot{q}_r + \sum_{\nu=1}^{k} M^{(m+\nu)} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \frac{d\alpha_\nu}{dt} = \mathcal{F}^{(q_r)} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \mathcal{F}^{(q_{m+\nu})}, \quad r = 1, \ldots, m. \tag{74}$$

Example 11 The equations of motion for the point P of mass M subject to the constraint (13) and to the weight force are immediately written by means of (74):

$$\begin{cases} (C^2 - \dot{q}_2^2)\ddot{q}_1 + \dot{q}_1(\dot{q}_2\ddot{q}_2 - C\dot{C}) = \pm g\dot{q}_1 R \\ (C^2 - \dot{q}_1^2)\ddot{q}_2 + \dot{q}_2(\dot{q}_1\ddot{q}_1 - C\dot{C}) = \pm g\dot{q}_2 R \end{cases}$$

where $R(\dot{q}_1, \dot{q}_2, t) = C^2(t) - \dot{q}_1^2 - \dot{q}_2^2$ (the equations are multiplied by R^2/M and the derivative $\frac{d}{dt}$ is calculated explicitly).

Example 12 The constraints (31) with $a_{i,j}^{(\nu)}$ constant are of the type (69) and the Examples 2 (same magnitude of velocities), 3 (parallel velocities) and 4 (perpendicular velocities) show the kinetic energy of statement (a): concerning Example 2 and referring to the formulation (19), the equations of motion (74) are

where we set $R_j = \sqrt{\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - \dot{q}_{2j}^2 - \dot{q}_{2j+1}^2}$ for $j = 2, \ldots, N$ and for $r = 4, 5, \ldots, 2N+1$:

$$M^{(r)} = \begin{cases} M_{r/2} & \textit{for } r \textit{ even}, \\ M_{(r-1)/2} & \textit{for } r \textit{ odd}, \end{cases} \quad R^{(r)} = \begin{cases} R_{r/2} & \textit{for } r \textit{ even}, \\ R_{(r-1)/2} & \textit{for } r \textit{ odd}. \end{cases}$$

Likewise, the case of N point with parallel velocities of Example 3 is suitable for the use of the form (74): the jacobian matrix $\frac{\partial \alpha_\nu}{\partial \dot{q}_j}$, $\nu = 1, \ldots, 2(N-1)$, $j = 1, \ldots, N+2$, according to the selection (18) of independent velocities (we recall $(\dot{q}_1, \ldots, \dot{q}_{N+2}) = (\dot{x}_1, \ldots, \dot{x}_N, \dot{y}_1, \dot{z}_1)$) and to the explicit form (19), is

$$J_{(\dot{q}_1,\ldots,\dot{q}_{N+2})} = \frac{1}{\dot{q}_1} \begin{pmatrix} -\frac{\dot{q}_{N+1}}{\dot{q}_1}\dot{\mathbf{q}}_{2,N} & \dot{q}_{N+1}\mathbb{I}_{N-1} & \dot{\mathbf{q}}_{2,N} & \mathbf{0}_{N-1} \\ -\frac{\dot{q}_{N+2}}{\dot{q}_1}\dot{\mathbf{q}}_{2,N} & \dot{q}_{N+2}\mathbb{I}_{N-1} & \mathbf{0}_{N-1} & \dot{\mathbf{q}}_{2,N} \end{pmatrix}$$

where $\mathbb{I}_{N-1}$ is the identity matrix of order $N-1$, $\dot{\mathbf{q}}_{2,N}$ the column vector $\begin{pmatrix} \dot{q}_2 \\ \cdots \\ \cdots \\ \dot{q}_N \end{pmatrix} \in \mathbb{R}^{N-1}$ and $\mathbf{0}_{N-1}$ the null column vector in $\mathbb{R}^{N-1}$. The corresponding $N+2$ equations of motion (74) are immediately available:

$$\begin{cases} M_1\ddot{q}_r + \dot{q}_r \sum_{j=2}^{N} \frac{M_j}{R_j}\frac{dR_j}{dt} = \mathcal{F}^{(q_r)} \pm \dot{q}_r \sum_{j=2}^{N} \frac{1}{R_j}\mathcal{F}^{(q_{2N+j})}, & r = 1, 2, 3 \\ M^{(r)}\ddot{q}_r - \frac{\dot{q}_r}{R^{(r)}}\frac{dR^{(r)}}{dt} = \mathcal{F}^{(q_r)} \mp \frac{\dot{q}_r}{R^{(r)}}\mathcal{F}^{(q_{2N+r})}, & r = 4, 5, \ldots, 2N+1 \end{cases} \tag{75}$$

$$
\left\{
\begin{array}{l}
M_1\ddot{q}_1 - \dfrac{1}{\dot{q}_1^2}\sum_{j=2}^{N} M_j\dot{q}_j\left(\dot{q}_{N+1}\dfrac{d}{dt}\left(\dfrac{\dot{q}_{N+1}}{\dot{q}_1}\dot{q}_j\right)+\dot{q}_{N+2}\dfrac{d}{dt}\left(\dfrac{\dot{q}_{N+2}}{\dot{q}_1}\dot{q}_j\right)\right) = \\
\quad \mathcal{F}^{(q_1)} - \dfrac{1}{\dot{q}_1^2}\sum_{j=2}^{N}\dot{q}_j\left(\dot{q}_{N+1}\mathcal{F}^{(q_{N+1+j})}+\dot{q}_{N+2}\mathcal{F}^{(q_{2N+j})}\right), \\
M_r\ddot{q}_r + M_r\left(\dfrac{\dot{q}_{N+1}}{\dot{q}_1}\dfrac{d}{dt}\left(\dfrac{\dot{q}_{N+1}}{\dot{q}_1}\dot{q}_r\right)+\dfrac{\dot{q}_{N+2}}{\dot{q}_1}\dfrac{d}{dt}\left(\dfrac{\dot{q}_{N+2}}{\dot{q}_1}\dot{q}_r\right)\right) = \\
\quad \mathcal{F}^{(q_r)} + \dfrac{1}{\dot{q}_1}\left(\dot{q}_{N+1}\mathcal{F}^{(q_{N+1+r})}+\dot{q}_{N+2}\mathcal{F}^{(q_{2N+r})}\right), \qquad r=2,3,\dots,N \\
M_1\ddot{q}_{N+1} + \sum_{j=2}^{N} M_j\dfrac{\dot{q}_j}{\dot{q}_1}\dfrac{d}{dt}\left(\dfrac{\dot{q}_j}{\dot{q}_1}\dot{q}_{N+1}\right) = \mathcal{F}^{(q_{N+1})} + \dfrac{1}{\dot{q}_1}\sum_{j=2}^{N}\dot{q}_j\mathcal{F}^{(qN+1+j)}, \\
m_1\ddot{q}_{N+2} + \sum_{j=2}^{N} M_j\dfrac{\dot{q}_j}{\dot{q}_1}\dfrac{d}{dt}\left(\dfrac{\dot{q}_j}{\dot{q}_1}\dot{q}_{N+2}\right) = \mathcal{F}^{(q_{N+2})} + \dfrac{1}{\dot{q}_1}\sum_{j=2}^{N}\dot{q}_j\mathcal{F}^{(q_{2N+j})}.
\end{array}
\right. \tag{76}
$$

Example 13 A similar scheme can be obtained in the case of Example 4 (perpendicular velocities), if one considers the contraints (21): for $N=2$ (hence $k=1$ and $m=5$), setting $(x_1,y_1,z_1,x_2,y_2,z_2)=(q_1,q_2,q_3,q_4,q_5,q_6)$ the only constraint can be put in the form (11) as $\dot{q}_6=\alpha_1(\dot{q}_1,\dot{q}_2,\dot{q}_3,\dot{q}_4,\dot{q}_5)=-\dfrac{\dot{q}_1\dot{q}_4+\dot{q}_2\dot{q}_5}{\dot{q}_3}$ and equations (74) are (the last two equations are written on the second column, in order to save space)

$$
\left\{
\begin{array}{l}
M_1\ddot{q}_1 - M_2\dfrac{\dot{q}_4}{\dot{q}_3}\dfrac{d\alpha_1}{dt} = \mathcal{F}^{(q_1)} - \dfrac{\dot{q}_4}{\dot{q}_3}\mathcal{F}^{(q_6)},\ \ M_2\ddot{q}_4 - M_2\dfrac{\dot{q}_1}{\dot{q}_3}\dfrac{d\alpha_1}{dt} = \mathcal{F}^{(q_1)} - \dfrac{\dot{q}_1}{\dot{q}_3}\mathcal{F}^{(q_6)} \\
M_1\ddot{q}_2 - M_2\dfrac{\dot{q}_5}{\dot{q}_3}\dfrac{d\alpha_1}{dt} = \mathcal{F}^{(q_1)} - \dfrac{\dot{q}_5}{\dot{q}_3}\mathcal{F}^{(q_6)},\ \ M_2\ddot{q}_5 - M_2\dfrac{\dot{q}_2}{\dot{q}_3}\dfrac{d\alpha_1}{dt} = \mathcal{F}^{(q_1)} - \dfrac{\dot{q}_2}{\dot{q}_3}\mathcal{F}^{(6)} \\
M_1\ddot{q}_3 + M_2\dfrac{\alpha_1}{\dot{q}_3}\dfrac{d\alpha_1}{dt} = \mathcal{F}^{(q_1)} - \dfrac{\alpha_1}{\dot{q}_3}\mathcal{F}^{(q_6)}
\end{array}
\right.
$$

where M_1 and M_2 are the masses of the two points.

3.2.5 *The equations of motion via the acceleration vector*

An alternative formal way to achieve the equations of motion (50) consists in calculating directly the acceleration of the system and the scalar product with $\mathbf{X}_r$: namely, recalling (7), (2) and taking into account (11), one has

$$
\begin{aligned}
\dot{\mathbf{Q}} = \sum_{r=1}^{m} \Bigg[& \left(\frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial q_r} + \sum_{\nu=1}^{k}\frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial q_{m+\nu}}\frac{\partial \alpha_\nu}{\partial \dot{q}_r}\right)\ddot{q}_r \\
& + \sum_{s=1}^{m}\frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial q_r \partial q_s}\dot{q}_r\dot{q}_s + \sum_{\nu=1}^{k}\left(2\frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial q_r \partial q_{m+\nu}}\alpha_\nu + \frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial q_{m+\nu}}\frac{\partial \alpha_\nu}{\partial q_r}\right) + 2\frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial q_r \partial t}\Bigg]\dot{q}_r \\
& + \sum_{\nu,\mu=1}^{k}\left(\frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial q_{m+\nu}\partial q_{m+\mu}}\alpha_\nu\alpha_\mu + \frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial q_{m+\nu}}\frac{\partial \alpha_\nu}{\partial q_{m+\mu}}\alpha_\mu\right) \\
& + \sum_{\nu=1}^{k}\left(2\frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial q_{m+\nu}\partial t}\alpha_\nu + \frac{\partial \mathbf{X}^{(\mathrm{M})}}{\partial q_{m+\nu}}\frac{\partial \alpha_\nu}{\partial t}\right) + \frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial t^2}
\end{aligned}
$$

and $(\dot{\mathbf{Q}} - \mathbf{F}) \cdot \mathbf{X}_r = 0$ for (NNC) systems corresponds to the m equations (details can be found in [47])

$$\sum_{s=1}^{m} \left(C_r^s \ddot{q}_s + \sum_{\tau=1}^{m} D_r^{s,\tau} \dot{q}_s \dot{q}_\tau + E_r^s \dot{q}_s \right) + G_r = \mathcal{F}^{(q_r)} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \mathcal{F}^{(q_{m+\nu})} \tag{77}$$

for each $r = 1, \ldots, m$ and where the coefficients C_r^s, $D_r^{s,\tau}$, E_r^s and G_r for $r, j, k = 1, \ldots, m$, depending on $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_m, t$, are defined by

$$\begin{aligned}
C_r^s =& g_{r,s} + \sum_{\nu,\mu=1}^{k} \left(g_{r,m+\nu} \frac{\partial \alpha_\nu}{\partial \dot{q}_s} + g_{m+\nu,j} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} + g_{m+\nu,m+\mu} \frac{\partial \alpha_\mu}{\partial \dot{q}_r} \frac{\partial \alpha_\nu}{\partial \dot{q}_s} \right), \\
D_r^{s,\tau} =& \xi_{s,\tau,r} + \sum_{\nu=1}^{k} \xi_{s,\tau,m+\nu} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}, \\
E_r^s =& \sum_{\nu,\mu=1}^{k} \left(2\xi_{s,m+\nu,m+\mu} \alpha_\nu + g_{m+\nu,m+\mu} \frac{\partial \alpha_\nu}{\partial q_s} + 2\eta_{s,m+\mu} \right) \frac{\partial \alpha_\mu}{\partial \dot{q}_r} + \sum_{\nu=1}^{k} \left(2\xi_{s,m+\nu,r} \alpha_\nu + g_{m+\nu,r} \frac{\partial \alpha_\nu}{\partial q_s} \right) + 2\eta_{s,r}, \\
G_r =& \sum_{\nu,\mu,p=1}^{k} \left(\alpha_\mu \left(\xi_{m+\nu,m+\mu,m+p} \alpha_\nu + g_{m+\nu,m+p} \frac{\partial \alpha_\nu}{\partial q_{m+\mu}} \right) + 2\eta_{m+\nu,m+p} \alpha_\nu + g_{m+\nu,m+p} \frac{\partial \alpha_\nu}{\partial t} + \zeta_{m+p} \right) \frac{\partial \alpha_p}{\partial \dot{q}_r} \\
&+ \sum_{\nu,\mu=1}^{k} \alpha_\mu \left(\xi_{m+\nu,m+\mu,r} \alpha_\nu + g_{m+\nu,r} \frac{\partial \alpha_\nu}{\partial q_{m+\mu}} \right) + \sum_{\nu=1}^{k} \left(2\eta_{m+\nu,r} \alpha_\nu + g_{m+\nu,r} \frac{\partial \alpha_\nu}{\partial t} \right) + \zeta_r.
\end{aligned} \tag{78}$$

where $g_{r,s}$, $g_{r,m+\nu}$, $g_{m+\nu,s}$, $g_{m+\nu,m+\mu}$ for $r, s = 1, \ldots, m$ and $\nu, \mu = 1, \ldots, k$ are the same coefficients as in (55) and where we set, for any $a, b, c = 1, \ldots, n$,

$$\xi_{a,b,c} = \frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial q_a \partial q_b} \cdot \frac{\partial \mathbf{X}}{\partial q_c}, \; \eta_{a,b} = \frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial q_a \partial t} \cdot \frac{\partial \mathbf{X}}{\partial q_b}, \; \zeta_a = \frac{\partial^2 \mathbf{X}^{(\mathrm{M})}}{\partial t^2} \cdot \frac{\partial \mathbf{X}}{\partial q_a}. \tag{79}$$

The equations of motion are given by the system of $m + k$ ordinary differential equations of the second order, formed by the m equations (77) coupled with the k restrictions (11). The $m + k = n$ unknown quantities of the system are the functions $q_1(t), \ldots, q_n(t)$. The explicit form (77) easily disclose the terms with the second derivatives $\ddot{q}_s$, $s = 1, \ldots, m$, which appear linearly via the coefficients C_r^s. The system of $m + k$ equations can be put in normal form: to prove that, it suffices to show the following

Lemma 1 *The $m \times m$ matrix C with entries C_i^r, $r = 1, \ldots, m$ appearing in (77) is positive definite.*

Proof The terms come from the calculus $C_r^s = (\frac{\partial \mathbf{X}^{(M)}}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \frac{\partial \mathbf{X}^{(M)}}{\partial q_{m+\nu}}) \cdot (\frac{\partial \mathbf{X}}{\partial q_s} + \sum_{\mu=1}^{k} \frac{\partial \alpha_\mu}{\partial \dot{q}_s} \frac{\partial \mathbf{X}}{\partial q_{m+j}})$. Setting $\mathbf{X}_r^{(\sqrt{M})} = \frac{\partial \mathbf{X}^{(\sqrt{M})}}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \frac{\partial \mathbf{X}^{(\sqrt{M})}}{\partial q_{m+\nu}}$, where we set, in analogy with (2) and (50), $\mathbf{X}^{(\sqrt{M})} = (\sqrt{M_1} x_1, \ldots, \sqrt{M_n} z_N)$, we see that $C_r^s = \mathbf{X}_r^{(\sqrt{M})} \cdot \mathbf{X}_s^{(\sqrt{M})}$, so that the matrix C is positive definite, since the vectors $\mathbf{X}_r^{(\sqrt{M})}$, $r = 1, \ldots, m$ are linearly independent, as it can be easily verified. □

Taking the cue from the previous proof, it is worth noting that the expression that multiplies $\ddot{q}_r$ in the calculation of $\dot{\mathbf{Q}}$ is exactly the vector $\mathbf{X}_r$ of (50) multiplied by the masses of the points; on the other hand this is the only term of the acceleration vector in which the second derivatives $\ddot{q}_r$ appear. This allows us to write, if we define $S = \frac{1}{2}\dot{\mathcal{Q}} \cdot \ddot{\mathbf{X}}$ as the acceleration energy or Gibbs–Appell function:

$$\frac{\partial S}{\partial \ddot{q}_r} = \dot{\mathcal{Q}} \cdot \frac{\partial \ddot{\mathbf{X}}}{\partial \ddot{q}_r} = \dot{\mathcal{Q}} \cdot \left(\frac{\partial \mathbf{X}}{\partial q_r} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \frac{\partial \mathbf{X}}{\partial q_{m+\nu}} \right).$$

This means that we can formally write the equations of motion (77) for (NNC) systems in the equivalent form

$$(\dot{\mathbf{Q}} - \mathbf{F}) \cdot \mathbf{X}_r = 0 \qquad \Leftrightarrow \qquad \frac{\partial S}{\partial \ddot{q}_r} = \mathcal{F}^{(q_r)} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \mathcal{F}^{(q_{m+\nu})}, \qquad r = 1, \dots, m.$$

The latter write is known as Gibbs–Appell equations. This extremely compact and general form to be linked to the Gauss principle appears in [2] and is developed in [23] and [39].

In the applications, the explicit form (77) is frequently more accessible compared to (59): actually, once the coefficients (79) are known, only the derivatives of the functions α_ν have to be calculated. Concerning again the computational point of view, it has to be said that equations (59) contain many redundant (in the sense of deleting each other) terms: it can be checked (see [47]) that all the terms of $-\sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{m+\nu}} B_r^\nu, = 1, \dots, m$, cancel out with part of the addends of $\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_r}$, of $-\frac{\partial T^*}{\partial q_r}$ and of $-\sum_{\nu=1}^{k} \frac{\partial T^*}{\partial q_{m+\nu}} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}$. However, despite the advantages we have mentioned for equations (77) compared to (59), it would not be exhaustive to present the equations only in the form (77) and to omit the formulation (59): actually, it is important from the theoretical point of view to clearly highlight the structure of equations of motions as a generalization of the lagrangian (i. e. referring to the kinetic energy and to the generalized forces) holonomic case, which is traceable by setting all the functions α_ν, $\nu = 1, \dots, k$, equal to zero in (59). Thus, the benefit of the latter formulation lies in the possibility of extending to more general contexts the lagrangian formalism of the simpler (say holonomic) cases.

We also notice that in case of scleronomic holonomic constraints $\mathbf{X}(\mathbf{q})$ (see (6)) we get $\eta_{a,b} = 0$, $\zeta_a = 0$ for any $a, b = 1, \dots, n$. Furthermore, in the absence of geometric constraints and using the $3N$ cartesian coordinates for $(q_1, \dots, q_n)$, it is $g_{r,r} = M^{(r)}$, with $M^{(r)}$ the mass of the point which the coordinate q_r refers to, $g_{r,j} = 0$ for $r \neq j$ and all the quantities in (79) are null. Hence the coefficients (78) are

$$\begin{aligned} &C_r^r = M^{(r)} + \sum_{\nu=1}^{k} M^{(m+\nu)} \left(\frac{\partial \alpha_\nu}{\partial \dot{q}_r} \right)^2, \quad C_r^s = \sum_{\nu=1}^{k} M^{(m+\nu)} \frac{\partial \alpha_\nu}{\partial \dot{q}_r} \frac{\partial \alpha_\nu}{\partial \dot{q}_s} \quad r \neq s \\ &D_r^{j,k} = 0, \quad E_r^j = \sum_{\nu=1}^{k} M^{(m+\nu)} \frac{\partial \alpha_\nu}{\partial q_j} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}, \quad G_r = \sum_{\nu,\mu=1}^{k} M^{(m+\nu)} \alpha_\mu \frac{\partial \alpha_\nu}{\partial q_{m+\mu}} \frac{\partial \alpha_\nu}{\partial \dot{q}_r}. \end{aligned} \tag{80}$$

Example 14 The case of the nonholonomic pendulum (Example 5) fits for the just described procedure: rewriting (30) as

$$\alpha_1(q_1,q_2,q_3,q_4,\dot q_1,\dot q_2,\dot q_3) = \frac{b(q_1\dot q_1+q_2\dot q_2)-q_3(bq_1-aq_2)}{a(q_1\dot q_1+q_2\dot q_2)+q_4(bq_1-aq_2)}\dot q_3 = \frac{N(q_1,q_2,q_3,\dot q_1,\dot q_2)}{D(q_1,q_2,q_4,\dot q_1,\dot q_2)}\dot q_3$$

and defining $P(q_1,q_2,q_3,q_4)=(aq_1+bq_2)(aq_3+bq_4)$, one has

$$\frac{\partial\alpha_1}{\partial\dot q_1} = -\dot q_2\dot q_3\frac{P}{D^2}, \qquad \frac{\partial\alpha_1}{\partial\dot q_2} = \dot q_1\dot q_3\frac{P}{D^2}, \qquad \frac{\partial\alpha_1}{\partial\dot q_3} = \frac{N}{D}$$

and the calculation of (80) allows to write the equations of motion (77) in the form

$$\begin{cases} M^{(1)}\ddot q_1 + \dot q_2\dot q_3\left(\Psi + \dfrac{P}{D^2}\mathcal{F}^{(q_4)}\right) = \mathcal{F}^{(q_1)} \\[2ex] M^{(2)}\ddot q_2 - \dot q_1\dot q_3\left(\Psi + \dfrac{P}{D^2}\mathcal{F}^{(q_4)}\right) = \mathcal{F}^{(q_2)} \\[2ex] M^{(3)}\ddot q_3 - \dfrac{ND}{P}\left(\Psi + \dfrac{P}{D^2}\mathcal{F}^{(q_4)}\right) = \mathcal{F}^{(q_3)} \end{cases}$$

where $\Psi(q_1,q_2,q_3,q_4,\dot q_1,\dot q_2,\dot q_3,\ddot q_1,\ddot q_2,\ddot q_3) =$

$$M^{(4)}\frac{P}{D^4}\Big[P\dot q_3(\dot q_2\ddot q_1 - \dot q_1\ddot q_2) - ND\ddot q_3 - (b\dot q_1 - a\dot q_2)\times$$

$$\times\left((aq_3+bq_4)(\dot q_1^2+\dot q_2^2) - \dot q_3\left(D - \frac{N}{D}\dot q_3\right)\right)\Big].$$

The equations of motion are simply and promptly obtained by means of (80) in comparison with other methods and they appear arranged in the correct way in order to search for solutions of specific type.

Example 15 A particular situation, analysed in [51], outlines the fact that equations (59) are still valid even though the function T^* defined in (56) degenerates w. r. t. the restricted variables $\dot q_1,\dots,\dot q_m$, in the sense that none of these variables are present in it: let P be a point of mass M and cartesian coordinates (x,y,z) and whose velocity is constant in module: $|\dot P| = C > 0$. In terms of $(q_1,q_2,q_3)=(x,y,z)$ we write (11) as $\dot q_3 = \pm\sqrt{C^2-\dot q_1^2-\dot q_2^2} = \alpha_1(\dot q_1,\dot q_2)$, where the sign depends on the initial conditions. Concerning (77), we use (80) with $g_{1,1}=g_{2,2}=g_{3,3}=m$, $g_{i,j}=0$ for $i\neq j$ and the only nonzero coefficients (since α_1 does not depend on q_1,q_2,q_3) are

$$C_i^i = M\left(1+\frac{M\dot q_i^2}{C^2-\dot q_1^2-\dot q_2^2}\right),$$

$$i=1,2, \qquad C_2^1 = C_1^2 = \frac{M\dot q_1\dot q_2}{C^2-\dot q_1^2-\dot q_2^2}.$$

Hence (77) are

$$\begin{cases} M\frac{C^2-\dot q_2^2}{C^2-\dot q_1^2-\dot q_2^2}\ddot q_1 + M\frac{\dot q_1\dot q_2}{C^2-\dot q_1^2-\dot q_2^2}\ddot q_2 = \mathcal{F}^{(q_1)} \mp \mathcal{F}^{(q_3)}\frac{\dot q_1}{\sqrt{C^2-\dot q_1^2-\dot q_2^2}}, \\ M\frac{\dot q_1\dot q_2}{C^2-\dot q_1^2-\dot q_2^2}\ddot q_1 + M\frac{C^2-\dot q_1^2}{C^2-\dot q_1^2-\dot q_2^2}\ddot q_2 = \mathcal{F}^{(q_1)} \mp \mathcal{F}^{(q_3)}\frac{\dot q_2}{\sqrt{C^2-\dot q_1^2-\dot q_2^2}}. \end{cases} \tag{81}$$

On the other hand, if the equations are written by means of (59), one has that (56) is $T^* = \frac{1}{2}mC^2$, hence the only contributions to the left–hand side terms of (59) are

$$-B_i^1\frac{\partial T}{\partial\dot q_3} = -m\left(\frac{\partial^2\alpha_1}{\partial\dot q_i^2}\ddot q_1 + \frac{\partial^2\alpha_1}{\partial\dot q_i\partial\dot q_2}\ddot q_2\right)\dot q_3(\dot q_1,\dot q_2), \quad i=1,2$$

Once the second derivatives are calculated, equations (81) are found again. We remark that the more general case $C=C(t)$ has been treated in Example 11: in the present example the attention is drawn to the comparison between (59) and (77).

Example 16 When the lagrangian parameters are the lagrangian parameters, the equations of motion (77) prefigure convenient in order to simplify calculations, even if the constraints depend explicitly on time: let us for instance recover the Example 9 of a point pursuing a moving particle where $m=1$ and $k=2$; using the same notation as in Example 9, straightforward complutations lead to the equation of motion (we assume that no external force is acting)

$$\left(1+\frac{(q_2-\eta)^2+(q_3-\zeta)^2}{(q_1-\xi)^2}\right)\ddot q_1$$

$$+\left(\dot\xi\frac{(q_2-\eta)^2+(q_3-\zeta)^2}{(q_1-\xi)^3} - \dot\eta\frac{q_2-\eta}{(q_1-\xi)^2} - \dot\zeta\frac{q_3-\zeta}{(q_1-\xi)^2}\right)\dot q_1 = 0$$

which generalizes in the three–dimensional space the problem in the plane that we find in [46].

3.2.6 On the selection of independent velocities

Let us finally investigate the question let us examine the question of how the choice of coordinates affects the presented construction. In a more complete way we have to consider the question in two steps, distinguishing between the use of specific lagrangian coordinates extracted from the holonomic constraints (as (6) shows) and the choice of a particular m–uple of independent velocities, as done in (11). The first aspect can be framed in the standard argument of switching from the coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ to the new set $\bar{\mathbf{q}} = (\bar{q}_1, \ldots, \bar{q}_n)$; the rank of (9) rewritten with the new coordinates is still k and without losing generality it can be assumed that the independent velocities are $(\dot{\bar{q}}_1, \ldots, \dot{\bar{q}}_k)$. There is a one–to–one correspondence between the two set of velocities $(\dot{q}_1, \ldots, \dot{q}_m)$ and $(\dot{\bar{q}}_1, \ldots, \dot{\bar{q}}_m)$; moreover, it is simple to check that the expression (48) does not change switching from one set of independent velocities to the other. We conclude that the construction presented in Sect. 3 does not depend on the choice of the lagrangian coordinates.

The second step can be focused on the following question: how the equations of motion modify if different m independent velocities are chosen, within the same set $\dot{q}_1, \ldots, \dot{q}_n$)? To this end we consider, besides a certain selection of the independent velocities $(\dot{q}_1, \ldots, \dot{q}_m)$, a second choice of m–uple deduced from (9):

$$\begin{cases} \dot{q}_{\sigma_{m+1}} = \alpha_1^{(\sigma)}(q_1, \ldots, q_n, \dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m}, t) \\ \cdots \\ \dot{q}_{\sigma_{m+k}} = \alpha_k^{(\sigma)}(q_1, \ldots, q_n, \dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m}, t) \end{cases} \tag{82}$$

where $(\dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m}, \dot{q}_{\sigma_{m+1}}, \ldots, \dot{q}_{\sigma_n})$ is a permutation of $(\dot{q}_1, \ldots, \dot{q}_n)$ fulfilling $det\, J_{(\dot{q}_{\sigma_{m+1}}, \ldots, \dot{q}_{\sigma_n})}(\Phi_1, \ldots, \Phi_k) \neq 0$.

The independent velocities $(\dot{q}_1, \ldots, \dot{q}_m)$ and the dependent kinetic variables (11) $\dot{q}_{m+1} = \alpha_1, \ldots, \dot{q}_n = \alpha_k$ can be splitted on the basis of:

- $(\dot{q}_1, \ldots, \dot{q}_\ell)$, $0 \leq \ell \leq m$, are the velocities among $(\dot{q}_1, \ldots, \dot{q}_m)$ which remain independent parameters, say $(\dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_\ell})$ in the same order, without loss of generality,
- $(\dot{q}_{\ell+1}, \ldots, \dot{q}_m)$ turn into the $m - \ell$ dependent variables $(\dot{q}_{\sigma_{m+h+1}}, \ldots, \dot{q}_{\sigma_{m+k}})$, where $k - h = m - \ell$,
- $(\dot{q}_{m+1}, \ldots, \dot{q}_{m+h})$ remain the dependent variables $(\dot{q}_{\sigma_{m+1}}, \ldots, \dot{q}_{\sigma_{m+h}})$,
- $(\dot{q}_{m+h+1}, \ldots, \dot{q}_{m+k})$ turn into the $k - h = m - \ell$ independent velocities $(\dot{q}_{\sigma_{\ell+1}}, \ldots, \dot{q}_{\sigma_m})$.

Consequently, the position $\dot{q}_i = \dot{q}_i(\dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m})$, $i = 1, \ldots, m$ is

$$\begin{array}{l} \dot{q}_1 = \dot{q}_{\sigma_1}, \quad \ldots \quad \dot{q}_\ell = \dot{q}_{\sigma_\ell}, \\ \dot{q}_{\ell+1} = \alpha_{\sigma_{h+1}}^{(\sigma)}(\dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m}), \quad \ldots \quad \dot{q}_m = \alpha_{\sigma_k}^{(\sigma)}(\dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m}). \end{array} \tag{83}$$

At the same time:

$$\begin{array}{l} \alpha_1 = \alpha_1^{(\sigma)} \quad \ldots \quad \alpha_h = \alpha_h^{(\sigma)}, \\ \dot{q}_{\sigma_{\ell+1}} = \alpha_{h+1}(\dot{q}_1, \ldots, \dot{q}_m), \quad \ldots \quad \dot{q}_{\sigma_m} = \alpha_k(\dot{q}_1, \ldots, \dot{q}_m). \end{array} \tag{84}$$

We refer to $\ell = 0$ as the case when all of the original kinetic variables become dependent; the case $\ell = m$ is trivial.

The relation between the two sets of equations can be expressed in terms of the transposed jacobian matrix

$$J^T_{(\dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m})}(\dot{q}_1, \ldots \dot{q}_m) = \left(\frac{\partial \dot{q}_j}{\partial \dot{q}_{\sigma_i}} \right)_{i,j=1,\ldots,m} = \begin{pmatrix} & \frac{\partial \alpha_{h+1}^{(\sigma)}}{\partial \dot{q}_{\sigma_1}} & \cdots & \frac{\partial \alpha_k^{(\sigma)}}{\partial \dot{q}_{\sigma_1}} \\ \mathbb{I}_\ell & \cdots & & \cdots \\ & & & \\ \mathbb{O}_{(m-\ell)\times\ell} & \frac{\partial \alpha_{h+1}^{(\sigma)}}{\partial \dot{q}_{\sigma_m}} & \cdots & \frac{\partial \alpha_k^{(\sigma)}}{\partial \dot{q}_{\sigma_m}} \end{pmatrix} \tag{85}$$

where $\mathbb{I}_\ell$ is the identity matrix of size ℓ, $\mathbb{O}_{(m-\ell)\times\ell}$ is the $(m - \ell) \times \ell$–null matrix and the functions appearing in the entries are those of (82). More precisely, it is not difficult to prove the following

Proposition 2 *The equations of motion written with the selection $(\dot{q}_{\sigma_1}, \ldots, \dot{q}_{\sigma_m})$ as independent velocities are obtained by left multiplying the m vector of equations of motion relative to $(\dot{q}_1, \ldots, \dot{q}_m)$ by the matrix* (85)*. That is, if $E_i = 0$, $i = 1, \ldots, m$, are the m equations* (59) (*with the force terms moved to the left side*), *thw equations of motion $E_{\sigma_i} = 0$ corresponding to the setting* (82) *verify*

$$E_{\sigma_i} = \sum_{r=1}^{m} \frac{\partial \dot{q}_r}{\partial \dot{q}_{\sigma_i}} E_r, \quad i = 1, \ldots, m. \tag{86}$$

4 Conclusion

The analysis of nonlinear kinematical constraints is certainly less debated than in the linear case, despite some very spontaneous constraint restrictions are naturally nonlinear (parallelism, perpendicularity, $\cdots$).

The present work aims to pursue a dual purpose:

(*i*) to give rise to a simple approach that generalizes the ordinary situation of the Eulero–Lagrange equations in the holonomic case, understood as Newton's equations projected along the directions of the possible speeds,

(*ii*) to provide a set of equations which can be used to formulate examples and applications, which maintain the real velocities as the kinetic variables, and exhibit a series of examples and applications for which this approach is congenial.

As regards the first point, a proposal has been made regarding the description in terms of vectors of admissible displacements, extending what is known in the standard cases. About point (*ii*) it must be said that the mere fact of writing the equations of motion for nonholonomic linear and nonlinear systems is anything but trivial, the procedures are almost always very complex. In our opinion, focussing on nonlinear systems which preserve rela velocities in their formulation is far from insignificant: the technical expedient of linearly transforming the velocities (by the definition of the pseudo–velocities) is often suggested by the typology of the kinematic constraints, therefore it is expected that it is more congenial in the linear case. For instance, in our opinion the remarkable category of nonlinear homogeneous constraints (which will be our next study) does not lend itself naturally to be treated with pseudo–velocities in order to have a mathematical improvement.

We have also set ourselves the goal of taking care of an aspect that is sometimes treated superficially in the literature: at least in some examples, the effective equivalence of a condition or of a group of conditions has been examined in order to formulate the same binding situation. In other cases we have taken care of writing the equations for decidedly recurring problems (the pursuing problem in the space, the nonholonomic pendulum, $\dots$). We have tried as much as possible to trace typologies of Lagrangians or constraints for which the calculation of the equations can be particularly shortened.

The approach chosen predisposes to at least two themes of deepening the problem, which will be the next subjects of study:

(*a*) to generalize the class of constraints, also admitting the presence of higher-order derivatives,

(*b*) to carry out the energy balance that follows from the equations, to examine the possibility of the presence of the integral of the energy, on the basis of certain hypotheses that the constraints and the applied forces must satisfy.

Declarations

Conflict of Interest The author has no conflicts of interest to declare that are relevant to the content of this article.

References

1. Appell P (1899) Les mouvements de roulement en dynamique. Paris Scientia Phys Math 4
2. Appell P (1911) Exemple de mouvement d'un point assujetti à une liason exprimée par une relation non linéaire entre les composantes de la vitesse. Rend Circ Mat Palermo 32:48–50
3. Benenti S (1996) Geometrical aspects of the dynamics of non-holonomic systems. Rend Semin Matem Univ Polit Torino 54(3):203–212
4. Benenti S (2008) A general method for writing the dynamical equations of nonholonomic systems with ideal constraints. Regular Chaotic Dyn 13(4):283–315

5. Benenti S (2011) The non-holonomic double pendulum, an example of non-linear non-holonomic system. Regular Chaotic Dyn 1(5):417–442
6. Bloch AM, Krishnaprasad PS, Mardsen JE, Murray R (1996) Nonholonomic mechanical systems with symmetry. Arch Ration Mech Anal 136:21–99
7. Bloch AM, Mardsen JE, Zenkov DV (2009) Quasivelocities and symmetries in non-holonomic systems. Dyn Syst 24:187–222
8. Bloch AM (2015) Nonholonomic mechanics and control, interdisciplinary applied mathematics 24, 2nd edn. Springer, New York
9. Borisov AV, Mamaev IS (2008) Conservation laws, hierarchy of dynamics and explicit integration of nonholonomic systems. Regular Chaotic Dyn 13:443–490
10. Cantrijn F, de Leon M, Marrero JC, de Diego DM (1997) Reduction of nonholonomic mechanical systems with symmetries. Rep Math Phys 42:25–45
11. Čaplygin SA (1897) On the motion of a heavy figure of revolution on a horizontal plane. Trudy Otd Fiz Nauk Obsš č Ljubitel Estest 9(1):10–16
12. Četaev NG (1962) On the Gauss principles, papers on analytical mechanics, vol 323. Science Academy
13. Cortés Monforte J (2002) Geometric, control and numerical aspects of nonholonomic systems, lecture notes in mathematics, vol 1793. Springer, Berlin
14. Cronström C (2010) On the compatibility of nonholonomic systems and related variational systems. Acta Physica Universitatis Comenianae L–LI(1 & 2): 25–36
15. Cariñena JF, Rañada MF (1993) Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers. J Phys A Math Gen 26:1335–1351
16. Cushman R, Hans Duistermaat H, Sniatycki J (2009) Geometry of nonholonomically constrained systems advanced series in nonlinear dynamics 26
17. de León M, de Diego DM (1996) On the geometry of nonholonomic Lagrangian systems. Mechanical systems with nonlinear constraints. J Math Phys 37:3389–3414
18. de León M, Marrero JC, de Diego DM (1997) Mechanical systems with nonlinear constraints. Int J Theor Phys 36:979–995
19. de León M (2012) A historical review on nonholomic mechanics. RACSAM 106:191–224
20. Fassò F, Sansonetto N (2015) Conservation of energy and momenta in nonholonomic systems with affine constraints. Regular Chaotic Dyn 20(4):449–462
21. Fernandez OE, Mestdag T, Bloch AM (2009) A generalization of Čhaplygin's reducibility theorem. Regular Chaotic Dyn 14(6):635–655
22. Flannery MR (2005) The enigma of nonholonomic constraints. Am J Phys 73:265
23. Gantmacher FR (1970) Lectures in analytical mechanics. Mir Publisher, Moskow
24. Gorni G, Zampieri G (2000) Time reversibility and energy conservation for Lagrangian systems with nonlinear nonholonomic constraints. Rep Math Phys 45:217–227
25. Hamel G (1904) Die Lagrange–Eulersche Gleichungen der Mechanik. Z Math Phys 50:1–57, Fortschritte 34:757
26. Ibort A, de León M, Marmo G, de Diego DM (1996) Nonholonomic constrained systems as implicit differential equations. Geometrical structures for physical theories, I (Vietri). Rend Sem Mat Univ Politec Torino 54:295–317
27. Iglesias-Ponte D, de Léon M, de Diego DM (2008) Towards a Hamilton–Jacobi theory for nonholonomic mechanical systems. J Phys A Math Theor 41(1):015205
28. Kobayashi MH, Oliva WM (2004) A note on the conservation of energy and volume in the setting of nonholonomic mechanical systems. Qual Theory Dyn Syst 4:383–411
29. Kupka I, Oliva WM (2001) The non-holonomic mechanics. J Differ Equ 169(1):169–189
30. Krupkovà O (1997) Mechanical systems with nonholonomic constraints. J Math Phys 38:5098–5126
31. Li SM, Berakadar J (2009) A generalization of the Chetaev condition for nonlinear nonholonomic constraints: the velocity-determined virtual displacement approach. Rep Math Phys 63(2):179–189
32. Lurie AI (2002) Analytical mechanics. Springer, Berlin
33. Marle CM (1998) Various approaches to conservative and nonconservative nonholonomic systems. Rep Math Phys 42(1–2):211–229
34. Marle CM (1995) Reduction of constrained mechanical systems and stability of relative equilibria. Commun Math Phys 174:295–318
35. Marle CM (2003) On symmetries and constants of motion in Hamiltonian systems with nonholonomic constraints. Banach Center Publ 59(1):223–242
36. Massa E, Pagani E (1991) Classical dynamics of nonholonomic systems: a geometric approach. Ann Inst Henri Poincaré 55(1):511–544
37. Neĭmark JuI, Fufaev NA (1972) Dynamics of nonholonomic systems. American Mathematical Society, Providence. Translations of Mathematical Monographs 33
38. Papastravidis JG (1997) Time-integral variational principles for nonlinear nonholonomic systems. J Appl Math 64:985–991
39. Papastavridis JG (2014) Analytical mechanics: a comprehensive treatise on the dynamics of constrained systems. World Scientific, Singapore
40. Pars LA (1965) A treatise on analytical dynamics. Heinemann Educational Books Ltd, London
41. Routh EG (1955) The advanced part of a treatise on the dynamics of a system of rigid bodies: Part II of a treatise on the whole subject, 6th edn. Dover, New York
42. Rumyantsev VV (2000) Forms of Hamilton's principle for nonholonomic systems. Mech Autom Robot. 2(10):1035–1048
43. Rumyantsev VV (1973) On the Chetaev principle (in Russian). Dokl Akad Nauk SSSR 210(4):787–790
44. Salehani MK (2019) A jet bundle approach to the variational structure of nonholonomic mechanical systems. Rep Math Phys 83(3):373–385
45. Śniatycki J (1998) Nonholonomic Noether theorem and reduction of symmetries. Rep Math Phys 42(1–2):5–23
46. Swaczyna M (2011) Several examples of nonholonomic mechanical systems. Commun Math 19:27–56
47. Talamucci F (2020) Rheonomic systems with nonlinear nonholonomic constraints: the Voronec equations. Regular Chaotic Dyn 25(6):662–673

48. Terra G, Kobayashi MH (2004) On classical mechanical systems with non-linear constraints. J Geom Phys 49:385–417
49. Vagner VV (1941) Geometric interpretation of the motion of nonholonomic systems. Trudy Sem Vektor Tenzor Anal 5:301–327 (Russian)
50. Van Dooren R (1975) The generalized Hamilton-Jacobi method for non-holonomic dynamical systems of Četaev type. Z Angew Math Mech 55:407–411
51. Virga E (1987) Un'osservazione sui vincoli anolonomi non perfetti. Riv Mat Univ Parma 13:379–384
52. Voronec PV (1901) On the equations of motion of a heavy rigid body rolling without sliding on a horizontal plane. Kiev Univ Izv 11:1–17
53. Vranceanu G (1929) Studio geometrico dei sistemi anolonomi. Ann Mat Pura Appl 4(6):9–43
54. Weber RW (1986) Hamiltonian systems with constraints and their meaning in mechanics. Arch Ration Mech Anal 91:309–335
55. Zampieri G (2000) Nonholonomic versus vakonomic mechanics. J Differ Equ 163:335–347
56. Zeković DN (2011) Dynamics of mechanical systems with nonlinear nonholonomic constraints—I The history of solving the problem of a material realization of a nonlinear nonholonomic constraint. Z Angew Math Mech 91(11):883–898
57. Zeković DN (2011) Dynamics of mechanical systems with nonlinear nonholonomic constraints—II Differential equations of motion. Z Angew Math Mech 91(11):899–922
58. Zeković DN (2011) Dynamics of mechanical systems with nonlinear nonholonomic constraints—III Analysis of motion. Z Angew Math Mech 93(8):550–574
59. Zeković DN (1991) Examples of nonlinear nonholonomic constraints in classical mechanics (in Russian). Vesnik Moskow Univ Ser 1 Mat Mekh 1:100–103