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# THE BERNSTEIN PROBLEM FOR $(X, Y)$ -LIPSCHITZ SURFACES IN THREE-DIMENSIONAL SUB-FINSLER HEISENBERG GROUPS

GIANMARCO GIOVANNARDI AND MANUEL RITORÉ

**ABSTRACT.** We prove that in the Heisenberg group  $\mathbb{H}^1$  with a sub-Finsler structure, an  $(X, Y)$ -Lipschitz surface which is complete, oriented, connected and stable must be a vertical plane. In particular, the result holds for entire intrinsic graphs of Euclidean Lipschitz functions.

## 1. INTRODUCTION

Variational problems related to the sub-Riemannian perimeter introduced by Capogna, Danielli and Garofalo [6] (see also Garofalo and Nhieu [28] and Franchi, Serapioni and Serra Cassano [20]) have received great interest recently, specially in the Heisenberg groups  $\mathbb{H}^n$ . In particular, Bernstein type problems, either for stable intrinsic graphs or for stable surfaces without singular points, have been specially considered. We refer the reader to the introduction in [26] for an account of recent results, including [8, 37, 13, 2, 14, 31, 23, 38, 24, 35, 11, 10, 32, 5]. The monograph [7] provides a quite complete survey of progress on the subject.

In the last years, a left-invariant sub-Finsler perimeter has been considered on the Heisenberg groups, see [39, 34, 19, 29, 33]. A quite natural question is whether Bernstein type results similar to the sub-Riemannian ones hold for the sub-Finsler perimeter.

The main result in this paper is Theorem 6.3, where we prove that in the Heisenberg group  $\mathbb{H}^1$  with a sub-Finsler structure, a complete, oriented, connected and stable  $(X, Y)$ -Lipschitz surface is a vertical plane. Roughly speaking an  $(X, Y)$ -Lipschitz surface is locally the intrinsic graph of a Euclidean Lipschitz function. Theorem 6.3 is a generalization of the corresponding sub-Riemannian result for graphs obtained by Nicolussi and Serra-Cassano in [32]. Recently, R. Young [43] proved that a *ruled* area-minimizing entire intrinsic graph in  $\mathbb{H}^1$  is a vertical plane by introducing a family of deformations of graphical strips based on variations of a vertical curve.

A sub-Finsler structure is obtained from a left-invariant asymmetric norm  $\|\cdot\|$  in the horizontal distribution of  $\mathbb{H}^1$ . Such a norm can be obtained from a convex set  $K$  contained in the horizontal plane at the origin in  $\mathbb{H}^1$ . The associated  $K$ -perimeter is defined by

$$|\partial E|_K(V) = \sup \left\{ \int_E \operatorname{div}(U) d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), \|U\|_{K,\infty} \leq 1 \right\} < +\infty,$$

where  $\mathcal{H}_0^1(V)$  is the space of horizontal vector fields of class  $C^1$  with compact support in the open set  $V$ , and  $\|U\|_{K,\infty} = \sup_{p \in V} \|U_p\|_K$ . The integral is computed with respect to

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the Riemannian measure  $d\mathbb{H}^1$  of a fixed left-invariant Riemannian metric  $g$  in  $\mathbb{H}^1$ , while the divergence is the one associated to this Riemannian metric. When  $K = D$ , the closed unit disk centered at the origin of  $\mathbb{R}^2$ , the  $K$ -perimeter coincides with the classical sub-Riemannian perimeter.

The first variation formula of the sub-Finsler perimeter was computed in [34, § 3] for surfaces of class  $C^2$  without singular points under the hypothesis that  $K$  is a convex set of class  $C^2_+$ . This means that  $\partial K$  is of class  $C^2$  and has strictly positive sectional curvature. The following formula was obtained

$$(*) \quad \left. \frac{d}{ds} \right|_{s=0} A_K(\varphi_s(S)) = \int_S u H_K dS,$$

where  $\{\varphi_s\}_{s \in \mathbb{R}}$  is the flow (i.e., the one-parameter group of diffeomorphisms) associated to a vector field  $U$  with compact support in the regular part of  $S$ , the function  $u$  is equal to the normal component  $\langle U, N \rangle$  of the variation ( $N$  is a unit normal for the Riemannian metric  $g$ ), and  $dS$  is the Riemannian area element. The function  $H_K$  appearing in  $(*)$  is the  $K$ -mean curvature

$$H_K = \langle D_Z \pi_K(\nu_h), Z \rangle,$$

where  $Z$  is a unit horizontal vector field in  $S$ ,  $\nu_h$  is the horizontal unit normal obtained by rotating  $Z$  by ninety degrees, and  $\pi_K$  is the inverse of the normal map of  $\partial K$ . Hence formula  $(*)$  has sense whenever the horizontal curves in  $S$  are of class  $C^2$ . However, the computations in [34] require to take one derivative of the normal to the surface and so they are not valid for surfaces with less regularity.

In [29], also under the assumption that  $K \in C^2_+$ , the authors proved that a Euclidean Lipschitz and  $\mathbb{H}$ -regular surface with prescribed continuous mean curvature has horizontal (characteristic) curves of class  $C^2$ , extending the corresponding sub-Riemannian result in [25]. Hence the  $K$ -mean curvature can be computed along the characteristic curves in this type of surfaces. Our main task in Section 3 is to compute the first variation for  $(X, Y)$ -Lipschitz surfaces and to check that the first variation formula  $(*)$  also holds for these surfaces with lower regularity. Of course the proof is different from the one in [34] and makes use of a Jacobian of horizontal type introduced by Galli in his Ph.D. Thesis [22], see also [23]. In particular, for area-stationary surfaces we get  $H_K = 0$  on  $S$ . Following the arguments in [25, 29] we prove that an area-stationary surface  $S$  is foliated by horizontal straight lines and following [32] we show that  $S$  is  $\mathbb{H}$ -regular.

In Section 4 we show that for an area-stationary surface  $S$  the function  $y = \langle N, T \rangle / |N_h|$  satisfies the differential equation

$$y'' - 6y'y + 4y^3 = 0$$

along almost every horizontal curve in  $S$ . Here  $N$  is a Riemannian unit normal to  $S$ ,  $N_h$  the orthogonal projection to the horizontal distribution and  $T$  the Reeb vector field on  $\mathbb{H}^1$ . The function  $D = 1/y$  was proven to satisfy the equivalent equation

$$DD'' = 2(D' + 1)(D' + 2)$$

for  $C^1$  surfaces by Cheng, Hwang, Malchiodi and Yang [9].

Both equations play an important role in the study of the singular set for  $C^1$  surfaces. Moreover, the regularity of  $\langle N, T \rangle / |N_h|$  along the horizontal (characteristic) curves in  $S$  is crucial to compute the second variation formula. The function  $\langle N, T \rangle / |N_h|$  appears frequently in the sub-Riemannian theory of hypersurfaces in the Heisenberg groups  $\mathbb{H}^n$ . For instance, it is the curvature of a length-minimizing geodesic realizing the distance between a hypersurface to a given point [36].

In Section 5 we compute the second variation formula of the area for *horizontal* vector fields with compact support. The second variation formula, which is formally similar to the one obtained in the sub-Riemannian case, is given by

$$\left. \frac{d^2}{ds^2} \right|_{s=0} A_K(\varphi_s(S)) = \int_S (Z(u)^2 + qu^2) \frac{|N_h|}{\kappa(\pi_K(\nu_h))} dS,$$

where  $\{\varphi_s\}_{s \in \mathbb{R}}$  is the flow associated to a horizontal vector field  $U$  with compact support,  $u = \langle U, N \rangle$  is the normal component and  $q$  is the function defined by

$$\frac{q}{4} = Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) - \frac{\langle N, T \rangle^2}{|N_h|^2}.$$

The function  $\kappa$  is the geodesic curvature of  $\partial K$ . In the sub-Riemannian case, where  $K$  is the unit disc, we have  $\kappa = 1$ . In our case, the vector  $\nu_h$  is constant along horizontal lines in  $S$ , so that  $\kappa(\pi_K(\nu_h))$  is constant on horizontal lines. The computation of this second variation follows the lines of [26], where the second variation of the sub-Riemannian area was computed for stable  $C^1$  surfaces to solve the Bernstein problem. There is a slight difference in the definition of the function  $q$  with respect to [26] that is related to the choice of  $Z$  as  $J(\nu_h)$  or  $-J(\nu_h)$ . We also use some ideas from Nicolussi and Serra-Cassano [32], who proved Bernstein's Theorem in the sub-Riemannian setting when  $S$  is the intrinsic graph of a Euclidean Lipschitz function.

Finally, in Section 6 we prove in our main result, Theorem 6.3, that a complete, oriented, connected and stable  $(X, Y)$ -Lipschitz surface is a vertical plane. We emphasize that Nicolussi and Serra-Cassano showed that this result is optimal in the sub-Riemannian setting, exhibiting two counterexamples when the Euclidean Lipschitz regularity assumption is missing, see Theorems 7.1 and 8.1 in [32].

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## 2. PRELIMINARIES

**2.1. The Heisenberg group.** We denote by  $\mathbb{H}^1$  the *first Heisenberg group*, defined as the 3-dimensional Euclidean space  $\mathbb{R}^3$  with the product

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' + x'y - xy').$$

A basis of left invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

For  $p \in \mathbb{H}^1$ , the *left translation by  $p$*  is the diffeomorphism  $L_p(q) = p * q$ . The *horizontal distribution*  $\mathcal{H}$  is the planar distribution generated by  $X$  and  $Y$ , which coincides with the kernel of the contact one-form  $\omega = dt - ydx + xdy$ . The distribution  $\mathcal{H}$  is completely nonintegrable.

We shall consider on  $\mathbb{H}^1$  the auxiliary left-invariant Riemannian metric  $g = \langle \cdot, \cdot \rangle$ , so that  $\{X, Y, T\}$  is an orthonormal basis at every point. Let  $D$  be the Levi-Civita connection associated to the Riemannian metric  $g$ . The following relations can be easily computed

$$\begin{aligned} (2.1) \quad & D_X X = 0, & D_Y Y = 0, & D_T T = 0 \\ & D_X Y = -T, & D_X T = Y, & D_Y T = -X \\ & D_Y X = T, & D_T X = Y, & D_T Y = -X. \end{aligned}$$

Setting  $J(U) = D_U T$  for any vector field  $U$  in  $\mathbb{H}^1$  we get  $J(X) = Y$ ,  $J(Y) = -X$  and  $J(T) = 0$ . Therefore  $-J^2$  coincides with the identity when restricted to the horizontal

distribution. The Riemannian volume of a set  $E$  is, up to a constant, the Haar measure of the group and is denoted by  $|E|$ . The integral of a function  $f$  with respect to the Riemannian measure is denoted by  $\int f d\mathbb{H}^1$ .

**2.2. The pseudo-hermitian connection.** The pseudo-hermitian connection  $\nabla$  is the only affine connection satisfying the following properties:

1.  $\nabla$  is a metric connection, and
2.  $\text{Tor}(U, V) = 2\langle J(U), V \rangle T$  for all vector fields  $U, V \in \mathfrak{X}(\mathbb{H}^1)$ .

We recall that a metric connection must satisfy

$$U(g(V, W)) = g(\nabla_U V, W) + g(V, \nabla_U W)$$

for vector fields  $U, V, W \in \mathfrak{X}(\mathbb{H}^1)$ . The torsion tensor associated to  $\nabla$  is defined by

$$\text{Tor}(U, V) = \nabla_U V - \nabla_V U - [U, V]$$

for all  $U, V \in \mathfrak{X}(\mathbb{H}^1)$ . From this definition and Koszul formula, see formula (9) in the proof of Theorem 3.6 in [15], it follows easily that  $\nabla X = \nabla Y = 0$  and  $\nabla J = 0$ . For a general discussion about the pseudo-hermitian connection see for instance [16, § 1.2]. Given a curve  $\gamma : I \rightarrow \mathbb{H}^1$  we denote by  $\nabla/ds$  the covariant derivative induced by the pseudo-hermitian connection along  $\gamma$ .

**2.3. Sub-Finsler norms.** Given a convex set  $K \subset \mathbb{R}^2$  with  $0 \in \text{int}(K)$  and associated asymmetric norm  $\|\cdot\|$  in  $\mathbb{R}^2$ , we define a left-invariant norm  $\|\cdot\|_K$  on the horizontal distribution of  $\mathbb{H}^1$  by means of the equality

$$(\|fX + gY\|_K)(p) = \|(f(p), g(p))\|,$$

for any  $p \in \mathbb{H}^1$ . The dual norm is denoted by  $\|\cdot\|_{K,*}$ .

If the boundary of  $K$  is of class  $C^\ell$ ,  $\ell \geq 2$ , and the geodesic curvature of  $\partial K$  is strictly positive, we say that  $K$  is of class  $C_+^\ell$ . When  $K$  is of class  $C_+^2$ , the outer Gauss map  $N_K$  is a diffeomorphism from  $\partial K$  to  $\mathbb{S}^1$  and the map

$$\pi_K(fX + gY) = N_K^{-1} \left( \frac{(f, g)}{\sqrt{f^2 + g^2}} \right),$$

defined for nowhere vanishing horizontal vector fields  $U = fX + gY$ , satisfies

$$\|U\|_{K,*} = \langle U, \pi_K(U) \rangle.$$

See § 2.3 in [34].

**2.4. Sub-Finsler perimeter.** Here we summarize some of the results contained in subsection 2.4 in [34].

Given a compact convex set  $K \subset \mathbb{R}^2$  with  $0 \in \text{int}(K)$ , the norm  $\|\cdot\|_K$  defines a perimeter functional: given a measurable set  $E \subset \mathbb{H}^1$  and an open subset  $\Omega \subset \mathbb{H}^1$ , we say that  $E$  has locally finite  $K$ -perimeter in  $\Omega$  if for any relatively compact open set  $V \subset \Omega$  we have

$$|\partial E|_K(V) = \sup \left\{ \int_E \text{div}(U) d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), \|U\|_{K,\infty} \leq 1 \right\} < +\infty,$$

where  $\mathcal{H}_0^1(V)$  is the space of horizontal vector fields of class  $C^1$  with compact support in  $V$ , and  $\|U\|_{K,\infty} = \sup_{p \in V} \|U_p\|_K$ . The integral is computed with respect to the Riemannian measure  $d\mathbb{H}^1$  of the left-invariant Riemannian metric  $g$ , and the divergence is the one associated to  $g$ . When  $K = D$ , the closed unit disk centered at the origin of  $\mathbb{R}^2$ , the  $K$ -perimeter coincides with classical sub-Riemannian perimeter.

If  $K, K'$  are bounded convex bodies containing 0 in its interior then there exist constants  $\alpha, \beta > 0$  such that

$$\alpha \|x\|_{K'} \leq \|x\|_K \leq \beta \|x\|_{K'}, \quad \text{for all } x \in \mathbb{R}^2,$$

and it is not difficult to prove that

$$\beta^{-1} |\partial E|_{K'}(V) \leq |\partial E|_K(V) \leq \alpha^{-1} |\partial E|_{K'}(V).$$

Then  $E$  has locally finite  $K$ -perimeter if and only if it has locally finite  $K'$ -perimeter. In particular, any set with locally finite  $K$ -perimeter has locally finite sub-Riemannian perimeter.

Riesz Representation Theorem implies the existence of a  $|\partial E|_K$ -measurable vector field  $\nu_K$  so that for any horizontal vector field  $U$  with compact support of class  $C^1$  we have

$$\int_{\Omega} \operatorname{div}(U) d\mathbb{H}^1 = \int_{\Omega} \langle U, \nu_K \rangle d|\partial E|_K.$$

In addition,  $\nu_K$  satisfies  $|\partial E|_K$ -a.e. the equality  $\|\nu_K\|_{K,*} = 1$ , where  $\|\cdot\|_{K,*}$  is the dual norm of  $\|\cdot\|_K$ .

Given two convex sets  $K, K' \subset \mathbb{R}^2$  containing 0 in their interiors, we have the following representation formula for the sub-Finsler perimeter measure  $|\partial E|_K$  and the vector field  $\nu_K$

$$|\partial E|_K = \|\nu_{K'}\|_{K,*} |\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{\|\nu_{K'}\|_{K,*}}.$$

Indeed, for the closed unit disk  $D \subset \mathbb{R}^2$  centered at 0 we know that in the Euclidean Lipschitz case  $\nu_D = \nu_h$  and  $|N_h| = \|N_h\|_{D,*}$  where  $N$  is the *outer* unit normal. Hence we have

$$|\partial E|_K = \|\nu_h\|_{K,*} d|\partial E|_D, \quad \nu_K = \frac{\nu_h}{\|\nu_h\|_{K,*}}.$$

Here  $|\partial E|_D$  is the standard sub-Riemannian measure. Moreover,  $\nu_h = N_h/|N_h|$  and  $|N_h|^{-1} d|\partial E|_D = dS$ , where  $dS$  is the standard Riemannian measure on  $S$ . Hence we get, for a set  $E$  with Euclidean Lipschitz boundary  $S$

$$(2.2) \quad |\partial E|_K(\Omega) = \int_{S \cap \Omega} \|N_h\|_{K,*} dS,$$

where  $dS$  is the Riemannian measure on  $S$ , obtained from the area formula using a local Lipschitz parameterization of  $S$ , see Proposition 2.14 in [20]. It coincides with the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by  $g$ . We stress that here  $N$  is the *outer* unit normal. This choice is important because of the lack of symmetry of  $\|\cdot\|_K$  and  $\|\cdot\|_{K,*}$ . Moreover when  $S = \partial E \cap \Omega$  is a Euclidean Lipschitz surface the  $K$ -perimeter coincides with the area functional

$$A_K(S) = \int_S \|N_h\|_{K,*} dS.$$

**2.5. Surfaces in  $\mathbb{H}^1$ .** Following [1, 20] we provide the following definition.

**Definition 2.1** ( $\mathbb{H}$ -regular surface). A real continuous function  $f$  defined on an open set  $\Omega \subset \mathbb{H}^1$  is of class  $C_{\mathbb{H}}^1(\Omega)$  if the distributional derivative  $\nabla_{\mathbb{H}} f = (Xf, Yf)$  is represented by a continuous vector field on  $\Omega$ .

We say that  $S \subset \mathbb{H}^1$  is an  $\mathbb{H}$ -regular surface if for each  $p \in \mathbb{H}^1$  there exist an open set  $U$  containing  $p$  and a function  $f \in C_{\mathbb{H}}^1(U)$  such that  $\nabla_{\mathbb{H}} f \neq 0$  on  $U$  and  $S \cap U = \{f = 0\}$ . Under such conditions, the horizontal unit normal  $\nu_h$  on  $S \cap U$  is defined as the restriction of the non-vanishing continuous vector field

$$\frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|},$$

defined on all of  $U$ .

Given a vertical plane  $P \subset \mathbb{H}^1$ , and a function  $u$  defined on a domain  $D \subset P$ , we denote by  $\text{Gr}(u)$  the *intrinsic graph* of  $u$ , defined as the Riemannian normal graph of the function  $u$ . Since the Riemannian unit normal to  $P$  is the restriction of a unitary left-invariant vector field  $X_P$ , the intrinsic graph of  $u$  is given by

$$\text{Gr}(u) = \{ \exp_p(u(p) X_P(p)) : p \in D \}.$$

where  $\exp$  is the exponential map on the Riemannian manifold  $(\mathbb{H}^1, g)$ . Using Euclidean rotations about the vertical axis  $x = y = 0$ , that are isometries of the Riemannian metric  $g$ , we may assume that  $P$  is the plane  $\{y = 0\}$ . Since in this case  $X_P = Y$ , the intrinsic graph  $\text{Gr}(u)$  can be parameterized by the map

$$\Phi^u(x, t) = (x, u(x, t), t - xu(x, t)),$$

for  $(x, 0, t) \in D$ . Notice that  $\Phi^u(x, t) = (x, 0, t) * (0, u(x, t), 0)$ , where  $*$  is the Heisenberg product defined in 2.1. For further details, we refer the reader to [21]. Note also that  $u$  measures the signed distance of  $\Phi^u(x, t)$  to the plane  $P$ , see [36].

Given the intrinsic graph  $\text{Gr}(u)$  of a Euclidean Lipschitz function defined on some domain  $D$  of the vertical plane  $P$ , we know by Rademacher's Theorem that  $u$  is  $\mathcal{H}^2$ -a.e. differentiable on  $D$ , where  $\mathcal{H}^2$  is the 2-dimensional Euclidean Hausdorff measure on  $D$ . Assuming  $P = \{y = 0\}$ , and given a differentiability point  $(x_0, 0, t_0)$  of  $u$ , the tangent plane of  $\text{Gr}(u)$  is well defined at  $\Phi^u(x_0, t_0)$  and so it is the normal vector field  $N$ . Hence  $N$  is defined  $\mathcal{H}^2$ -a.e. on  $\text{Gr}(u)$ . Moreover,

$$(2.3) \quad N = \frac{(u_x + 2uu_t)X - Y + u_t T}{\sqrt{1 + u_t^2 + (u_x + 2uu_t)^2}},$$

see the computations in § 4 in [27]. Hence  $N$  is never vertical. At differentiability points of  $\text{Gr}(u)$  we define

$$\nu_h = \frac{N_h}{|N_h|} = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}},$$

and the vector field  $Z$  by

$$Z = -J(\nu_h),$$

which is tangent to  $S$  and horizontal. An orthonormal basis at the tangent space of  $\text{Gr}(u)$  at the differentiable point is obtained by adding to  $Z$  the vector

$$(2.4) \quad E = \langle N, T \rangle \nu_h - |N_h| T.$$

Following [42] we provide the following definition.

**Definition 2.2.** A set  $S \subset \mathbb{H}^1$  is a  $(X, Y)$ -Lipschitz surface if for each  $p \in S$  there exist an open neighborhood  $U_p \subset \mathbb{H}^1$  of  $p$ , and a Lipschitz function  $f : U \rightarrow \mathbb{R}$  such that

$$S \cap U = \{f = 0\}$$

and

$$Xf \geq l \quad \text{a.e. on } U \quad \text{or} \quad Yf \geq l \quad \text{a.e. on } U$$

for a suitable  $l > 0$ .

We stress that following result was previously obtained by [42, Theorem 3.2], using the notion of homogeneous cone. Here we provide a different proof.

**Theorem 2.3.** A set  $S \subset \mathbb{H}^1$  is a  $(X, Y)$ -Lipschitz surface if and only if  $S$  is locally the intrinsic graph of a Euclidean Lipschitz function.

*Proof.* Assume that  $S$  is a  $(X, Y)$ -Lipschitz surface. Given  $p \in S$  there exist an open ball  $B_r(p) \subset \mathbb{H}^1$  and a Euclidean Lipschitz function  $f$  defined on  $B_r(p)$  such that

$$S \cap B_r(p) = \{(x, y, t) : f(x, y, t) = 0\}.$$

Since  $S$  is  $(X, Y)$ -Lipschitz, after a rotation about the vertical axis we may assume the existence of  $l > 0$  such that  $Yf(q) \geq l > 0$  for every point of differentiability of  $f$  close enough to  $p$ . In particular the convex hull of

$$\left\{ \lim_{i \rightarrow \infty} Yf(q_i) : \lim_{i \rightarrow \infty} q_i = p, q_i \text{ differentiability point of } f \right\}$$

does not contain 0. Let us consider the  $C^\infty$  diffeomorphism  $H(x, y, t) = (x, y, t - xy)$  on  $\mathbb{H}^1$ . Then the function  $f \circ H$  is Lipschitz and

$$\frac{\partial(f \circ H)}{\partial y}(q) = \left( \frac{\partial f}{\partial y} - x \frac{\partial f}{\partial t} \right)(q) = Yf(q)$$

for each point  $q$  of differentiability of  $f$ . Therefore by the Implicit function Theorem for Lipschitz functions [12, p. 255] there exists an open neighborhood  $D \subset \{y = 0\}$  of the projection of  $p$  on  $\{y = 0\}$  and a Euclidean Lipschitz function  $u : D \rightarrow \mathbb{R}$  such that  $f(x, u(x, t), t - xu(x, t)) = 0$ . In other words, the surface  $S$  is locally an intrinsic graph of a Lipschitz function over the vertical plane  $\{y = 0\}$ .

Assume now that  $S$  is locally the intrinsic graph of a Euclidean Lipschitz function  $u$ . Let  $p$  in  $S$  and assume that  $S \cap B_r(p) = \Phi^u(D)$  where  $\Phi^u(x, t) = (x, u(x, t), t - xu(x, t))$  and  $u : D \rightarrow \mathbb{R}$  is a Euclidean Lipschitz function. Setting

$$f(x, y, t) = y - u(x, t + yx),$$

we clearly have that  $f$  is a Euclidean Lipschitz function defined in an open neighborhood of  $p$ . Eventually reducing the radius  $r > 0$  we get that  $S \cap B_r(p) = \{f = 0\}$  and  $Y(f) = 1 > 0$  a.e. on  $B_r(p)$ . Therefore  $S$  is a  $(X, Y)$ -Lipschitz surface.  $\square$

**Remark 2.4.** Notice that

1. An  $(X, Y)$ -Lipschitz surface is an embedded surface by Theorem 2.3.
2. If a Euclidean Lipschitz function  $f$  defined on an open domain of  $\mathbb{H}^1$  is  $C_{\mathbb{H}}^1$  then their level sets are  $(X, Y)$ -Lipschitz. Indeed, since the horizontal gradient  $\nabla_{\mathbb{H}} f = (Xf, Yf)$  is a never vanishing *continuous* vector field, we obtain that locally  $Xf \geq l > 0$  or  $Y(f) \geq l > 0$  eventually replacing  $f$  by  $-f$ .

**Definition 2.5.** Let  $S \subset \mathbb{H}^1$  be a  $C^1$  surface. We say that  $p \in S$  belongs to the *singular set*  $S_0$  of  $S$  if the tangent space  $T_p S$  coincides with the horizontal distribution  $\mathcal{H}_p$ .

The following result, whose proof can be found in [23], will be used to compute the first and second variation of a surface.

**Proposition 2.6.** *Let  $S$  be an oriented immersed  $C^2$  surface in  $\mathbb{H}^1$  with singular set  $S_0 = \emptyset$  and let  $f \in C^1(S)$ . Then*

$$\operatorname{div}_S(fZ) = Z(f) - (\langle N, T \rangle \theta(E) + 2\langle N, T \rangle |N_h|)f$$

and

$$\operatorname{div}_S(fE) = E(f) + \langle N, T \rangle \theta(Z)f,$$

where we have set  $\theta(W) = \langle \nabla_W \nu_h, Z \rangle$  for each vector field  $W$ .



## 3. THE FIRST VARIATION FORMULA

We start this section computing the first variation formula for Lipschitz surfaces which are twice differentiable in the horizontal directions. We start by proving some technical lemmas.

**Lemma 3.1.** *Let  $U$  be a smooth vector field on  $\mathbb{H}^1$  and  $\{\varphi_s\}_{s \in \mathbb{R}}$  be the flow associated to  $U$ . Let  $p \in \mathbb{H}^1$  and  $e \in T_p \mathbb{H}^1$ . Define the smooth curve  $\beta(s) = \varphi_s(p)$  and the smooth vector field  $E(s) = (d\varphi_s)_p(e)$  along  $\beta$ . Then we have*

$$(3.1) \quad \left. \frac{\nabla}{ds} \right|_{s=0} E(s) = \nabla_e U + 2 \langle J(U_p), e \rangle T_p.$$

*Proof.* Let us rename the standard coordinates  $(x, y, t)$  as  $(x_1, x_2, x_3)$ . Let  $\varphi_s = (\varphi_1, \varphi_2, \varphi_3)$ , and  $e = (e_1, e_2, e_3)$ , and  $U = \sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}$ . Then

$$(d\varphi_s)_p = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \frac{\partial \varphi_1}{\partial x_3} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \frac{\partial \varphi_2}{\partial x_3} \\ \frac{\partial \varphi_3}{\partial x_1} & \frac{\partial \varphi_3}{\partial x_2} & \frac{\partial \varphi_3}{\partial x_3} \end{pmatrix},$$

and

$$E(s) = (d\varphi_s)_p(e) = \sum_{i=1}^3 \sum_{j=1}^3 e_j \frac{\partial \varphi_i}{\partial x_j} \left( \frac{\partial}{\partial x_i} \right)_{\beta(s)} = \sum_{i=1}^3 g_i(s) \left( \frac{\partial}{\partial x_i} \right)_{\beta(s)},$$

where  $g_i(s) = \sum_{j=1}^3 e_j \frac{\partial \varphi_i}{\partial x_j}$ . Therefore

$$\left. \frac{\nabla}{ds} \right|_{s=0} E(s) = \sum_{i=1}^3 g'_i(0) \left( \frac{\partial}{\partial x_i} \right)_p + \sum_{i=1}^3 g_i(0) \nabla_{U_p} \frac{\partial}{\partial x_i}.$$

Since  $g_i(0) = e_i$  and  $g'_i(0) = e(f_i)$  we have

$$\left. \frac{\nabla}{ds} \right|_{s=0} E(s) = \sum_{i=1}^3 e(f_i) \left( \frac{\partial}{\partial x_i} \right)_p + e_i \nabla_{U_p} \frac{\partial}{\partial x_i}.$$

On the other hand

$$\nabla_e U = \sum_{i=1}^3 e(f_i) \left( \frac{\partial}{\partial x_i} \right)_p + f_i \nabla_e \frac{\partial}{\partial x_i}$$

Since

$$\begin{aligned} \nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y} &= \nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x} + \text{Tor} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x} + 2 \langle J \left( \frac{\partial}{\partial x} \right), \frac{\partial}{\partial y} \rangle T \\ &= \nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x} + 2 \langle J(X) - yJ(T), Y \rangle T + 2x \langle J(X), T \rangle T \\ &= \nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x} + 2T, \end{aligned}$$

and

$$\nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \nabla \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i},$$

for  $\{i, j\} \neq \{1, 2\}$  there follows, evaluating at  $p$ ,

$$\begin{aligned}
\sum_{i=1}^3 e_i \nabla_{U_p} \frac{\partial}{\partial x_i} &= \sum_{i,j=1}^3 e_i f_j \nabla \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} = e_1 f_2 \nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x} + e_2 f_1 \nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y} \\
&\quad + \sum_{i,j=1, \{i,j\} \neq \{1,2\}}^3 e_i f_j \nabla \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \\
&= e_1 f_2 (\nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y} - 2T) + e_2 f_1 (\nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x} + 2T) \\
&\quad + \sum_{i,j=1, \{i,j\} \neq \{1,2\}}^3 e_i f_j \nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \\
&= 2(e_2 f_1 - e_1 f_2) T + \sum_{j=1}^3 f_j \nabla_e \frac{\partial}{\partial x_j} \\
&= 2(e_2 f_1 - e_1 f_2) T + \sum_{i=1}^3 f_i \nabla_e \frac{\partial}{\partial x_i}.
\end{aligned}$$

Hence we obtain (3.1).  $\square$

Let  $S \subset \mathbb{H}^1$  be a  $(X, Y)$ -Lipschitz surface and  $p \in S$  a point of differentiability of  $S$ . Consider a  $C^1$  vector field  $U$  with compact support on  $\mathbb{H}^1$  and let  $\{\varphi_t\}_{t \in \mathbb{R}}$  be the associated flow. A basis of the tangent space to  $\varphi_s(S)$  at  $\varphi_s(p)$  is given by  $E_1(s) = (d\varphi_s)_p(Z_p)$  and  $E_2(s) = (d\varphi_s)_p(E_p)$ , where  $E$  is defined in (2.4). Let  $N_s$  be the unit normal to  $\varphi_s(S)$ . Then

$$[(N_s)_h]_{\varphi_s(p)} = \frac{[E_1(s) \times E_2(s)]_h}{|E_1(s) \times E_2(s)|}.$$

Observe that  $\text{Jac}(\varphi_s)(p) = |E_1(s) \times E_2(s)|$ . Setting  $V(s, p) = [E_1(s) \times E_2(s)]_h$  we have

$$(3.2) \quad V(s, p) = \langle E_1, T \rangle T \times E_2 + \langle E_2, T \rangle E_1 \times T.$$

Since  $\{Z, \nu_h, T\}$  is positively oriented, we have  $V(0, \cdot) = |N_h| \nu_h$ . The area functional  $A_K$  of  $\varphi_s(S)$  is then given by

$$\begin{aligned}
A_K(\varphi_s(S)) &= \int_S \langle \pi_K([E_1(s) \times E_2(s)]_h), \frac{[E_1(s) \times E_2(s)]_h}{|E_1(s) \times E_2(s)|} \rangle |E_1(s) \times E_2(s)| dS \\
(3.3) \quad &= \int_S \langle \pi_K(V(s, p)), V(s, p) \rangle dS(p) \\
&= \int_S \|V(s, p)\|_*^2 dS(p).
\end{aligned}$$

**Lemma 3.2.** *Let  $S$  be an  $(X, Y)$ -Lipschitz surface, and  $\{\varphi_s\}_{s \in \mathbb{R}}$  the flow associated to a compactly supported  $C^1$  vector field  $U$  in  $\mathbb{H}^1$ . Let  $L = \text{supp}(U) \cap S$ . Then there exist positive constants  $l, s_0$  such that  $|V(s, p)| \geq \frac{l}{2}$  for a.e.  $p \in L$  and any  $s \in (-s_0, s_0)$ .*

*Proof.* Theorem 2.3 implies that, after a rotation about the vertical axis, there exists  $r > 0$  and a Euclidean Lipschitz function  $u : D \rightarrow \mathbb{R}$ , defined on an open set  $D$  of the vertical plane  $y = 0$ , such that  $S \cap B_r(p) = \Phi^u(D)$ . Let  $B \subset D$  be the set of points of differentiability of  $u$ . Since  $u$  is Euclidean Lipschitz, equation (2.3) implies

$$|N_h| \geq \frac{1}{\sqrt{1 + (u_t)^2 + (u_x + 2uu_t)^2}} \geq l_{B_r(p)} > 0$$

for some constant  $l_{B_r(p)}$  depending on a Lipschitz constant of  $u$  on  $B_r(p)$ . Therefore by the compactness of  $\text{supp}(U)$ , there exists  $l > 0$  such that  $|N_h| \geq l > 0$  for a.e.  $q \in L$ .

Assume by contradiction the existence of a measurable set  $A \subset L$  such that  $\mathcal{H}^2(A) > 0$  and of a sequence  $\{s_j\}_{j \in \mathbb{N}}$  converging to 0 such that  $|V(s_j, q)| < \frac{l}{2}$  for each  $q \in A$ . Then

$$(3.4) \quad \lim_{j \rightarrow \infty} \int_A |V(s_j, q)| dS(q) \leq \frac{l}{2} \mathcal{H}^2(A).$$

On the other hand, since  $|E_1(s_j)| = |d\varphi_{s_j}(Z)| \leq C'$ ,  $|E_2(s_j)| = |d\varphi_{s_j}(E)| \leq C'$  for some  $C' > 0$ , we obtain from (3.2) the existence of  $C > 0$  such that  $|V(s_j)| \leq C$  on  $A$ . By the continuity of  $V(s, q)$  in  $s$  we have  $\lim_{j \rightarrow \infty} |V(s_j)| = |N_h|$  a.e. in  $A$ . By dominated convergence

$$\lim_{j \rightarrow \infty} \int_A |V(s_j, q)| dS(q) = \int_A |N_h| dS \geq l \mathcal{H}^2(A).$$

Therefore, since  $\mathcal{H}^2(A) > 0$  we get a contradiction to (3.4)  $\square$

Now we compute the first variation of the sub-Finsler area.

**Proposition 3.3.** *Let  $K \in C_+^2$  be a convex body with  $0 \in \text{int}(K)$ , and  $S \subset \mathbb{H}^1$  an oriented  $(X, Y)$ -Lipschitz surface. Then the first variation of the sub-Finsler area induced by a  $C^1$  vector field  $U$  with compact support in  $\mathbb{H}^1$ , with  $\partial S \cap \text{supp}(U) = \emptyset$ , is given by*

$$(3.5) \quad \left. \frac{d}{ds} \right|_{s=0} A_K(\varphi_s(S)) = \int_S \left[ -\langle N, T \rangle Z(\langle U, T \rangle) \pi_Z - E(\langle U, T \rangle) \pi_\nu - 2\langle N, T \rangle \langle J(U), \pi(\nu_h) \rangle - |N_h| \langle J(\pi(\nu_h)), \nabla_Z U \rangle \right] dS,$$

where  $\{\varphi_s\}_{s \in \mathbb{R}}$  is the flow associated to  $U$ .

Moreover, if we assume that the derivative in the  $Z$ -direction of  $\nu_h$  and  $Z$  exists and is continuous, then we have

$$(3.6) \quad \left. \frac{d}{ds} \right|_{s=0} A_K(\varphi_s(S)) = \int_S u \langle \nabla_Z \pi_K(\nu_h), Z \rangle dS,$$

where  $u = \langle N, U \rangle$ .

*Proof.* We denote  $V(s, \cdot)$  by  $V(s)$  for simplicity. Let us prove first that

$$(3.7) \quad \left. \frac{d}{ds} \right|_{s=0} A_K(\varphi_s(S)) = \int_S \left. \frac{d}{ds} \right|_{s=0} \|V(s)\|_* dS.$$

This means we can differentiate under the integral sign.

By Lemma 3.2 the norm of  $V(s)$  is strictly positive a.e. in  $\text{supp}(U) \cap S$ . So we have

$$(3.8) \quad \begin{aligned} \left. \frac{d}{ds} \right|_{s=0} \|V(s)\|_* &= \langle \pi_K(V(s)), \frac{\nabla}{ds} V(s) \rangle \leq \|\pi_K(V(s))\|_K \left\| \frac{\nabla}{ds} V(s) \right\|_{K,*} \\ &= \left\| \frac{\nabla}{ds} V(s) \right\|_* \leq \beta \left| \frac{\nabla}{ds} V(s) \right|, \end{aligned}$$

for a.e. in  $\text{supp } U \cap S$ , where  $\beta$  is the positive constant defined in § 2.4 taking  $K'$  equal to the Euclidean ball centered at zero. By Lemma 3.1 there holds

$$\begin{aligned} \frac{\nabla}{ds} V(s) &= \langle \nabla_{E_1(s)} U, T \rangle T \times E_2(s) + 2\langle J(U), E_1(s) \rangle T \times E_2(s) + \langle E_1(s), T \rangle T \times \nabla_{E_2(s)} U \\ &\quad + 2\langle J(U), E_2(s) \rangle E_1(s) \times T + \langle \nabla_{E_2(s)} U, T \rangle E_1(s) \times T + \langle E_2(s), T \rangle \nabla_{E_1(s)} U \times T. \end{aligned}$$

Since  $|E_1(s)| = |d\varphi_s(Z)| \leq C$  and  $|E_2(s)| = |d\varphi_s(E)| \leq C$  for  $s \in (-s_0, s_0)$  where  $C > 0$  is independent of  $s$ . Then, writing the covariant derivative  $\nabla_{E_i(s)} U$  in standard coordinates, we obtain

$$\left| \frac{\nabla}{ds} V(s) \right| \leq \tilde{C} \|U\|_{C^1}$$

a.e. in  $\text{supp } U \cap S$  for a suitable constant  $\tilde{C} > 0$ . Here  $\|U\|_{C^1}$  denotes the standard  $C^1$  norm of  $U$ . Since  $\text{supp } U \cap S$  is compact, dominated convergence implies (3.7).

Let us compute now

$$\frac{d}{ds}\Big|_{s=0} \|V(s)\|_* = \frac{d}{ds}\Big|_{s=0} \langle \pi_K(V(s)), V(s) \rangle$$

at a point  $p$  of differentiability of  $S$ . By Remark 3.3 in [34],

$$\frac{d}{ds}\Big|_{s=0} \langle \pi_K(V(s)), V(s) \rangle = \langle \pi_K(V(0)), \frac{\nabla}{ds}\Big|_{s=0} V(s) \rangle = \langle \pi_K((\nu_h)_p), \frac{\nabla}{ds}\Big|_{s=0} V(s) \rangle.$$

Since  $T$  is parallel with respect to the pseudo-hermitian connection  $\nabla$  and  $\langle Z, T \rangle = 0$ , we have

$$\begin{aligned} \frac{\nabla}{ds}\Big|_{s=0} V(s) &= \left\langle \frac{\nabla}{ds}\Big|_{s=0} E_1(s), T \right\rangle T \times S + \left\langle \frac{\nabla}{ds}\Big|_{s=0} E_2(s), T \right\rangle Z \times T \\ &\quad + \langle E, T \rangle \frac{\nabla}{ds}\Big|_{s=0} E_1(s) \times T. \end{aligned}$$

Lemma 3.1 implies

$$\frac{\nabla}{ds}\Big|_{s=0} E_1(s) = \nabla_Z U + 2\langle J(U), Z \rangle T$$

and

$$\frac{\nabla}{ds}\Big|_{s=0} E_2(s) = \nabla_E U + 2\langle J(U), E \rangle T.$$

Therefore, evaluating at  $p$  but omitting it for clarity,

$$\begin{aligned} \frac{\nabla}{ds}\Big|_{s=0} V(s) &= (\langle \nabla_Z U, T \rangle + 2\langle J(U), Z \rangle) T \times E \\ (3.9) \quad &\quad + (\langle \nabla_E U, T \rangle + 2\langle J(U), E \rangle) Z \times T + \langle E, T \rangle \nabla_Z U \times T \\ &= \langle N, T \rangle (Z(\langle U, T \rangle) + 2\langle J(U), Z \rangle) T \times \nu_h \\ &\quad + (E(\langle U, T \rangle) + 2\langle J(U), E \rangle) Z \times T - |N_h| \nabla_Z U \times T. \end{aligned}$$

We set  $\pi_K(\nu_h) = \pi_Z Z + \pi_\nu \nu_h$ , where  $\pi_Z = \langle \pi(\nu_h), Z \rangle$  and  $\pi_\nu = \langle \pi(\nu_h), \nu_h \rangle$ . Notice that  $T \times \nu_h = -Z$ ,  $Z \times T = -\nu_h$  and  $\langle \pi(\nu_h), \nabla_Z U \times T \rangle = \langle J(\pi(\nu_h)), \nabla_Z U \rangle$ , then we obtain

$$\begin{aligned} \langle \pi_K(\nu_h), \frac{\nabla}{ds}\Big|_{s=0} V(s) \rangle &= -\langle N, T \rangle (Z(\langle U, T \rangle) + 2\langle J(U), Z \rangle) \pi_Z \\ (3.10) \quad &\quad - (E(\langle U, T \rangle) + 2\langle J(U), E \rangle) \pi_\nu \\ &\quad - |N_h| \langle J(\pi_K(\nu_h)), \nabla_Z U \rangle \\ &= -\langle N, T \rangle Z(\langle U, T \rangle) \pi_Z - E(\langle U, T \rangle) \pi_\nu \\ &\quad - 2\langle N, T \rangle \langle J(U), \pi_K(\nu_h) \rangle \\ &\quad - |N_h| \langle J(\pi_K(\nu_h)), \nabla_Z U \rangle. \end{aligned}$$

This implies (3.5).

Let us finally prove (3.6). We show in Lemma 3.4 that (3.6) holds for  $C^2$  surfaces. The general result follows by approximation. Following Proposition 1.20 in [18] or Remark 6.1 in [23] we approximate the  $(X, Y)$ -Lipschitz surface  $S = \{p \in \mathbb{H}^1 : f(p) = 0\}$  by a family of smooth surfaces  $S_j = \{p \in \mathbb{H}^1 : f_j(p) = 0\}$ , where  $f_j = \rho_j * f$  and  $\rho_j$  are the standard Friedrichs' mollifiers. Since  $S$  is  $(X, Y)$ -Lipschitz we gain that  $(S_j)_0 = \emptyset$ . Hence  $S_j$  converges to  $S$  on compact subsets in  $S$ . Given  $j \geq 1$ , let  $Z^j$  be the characteristic vector field of the regular part of  $S_j$  and  $\nu_h^j$  be the horizontal unit norm to  $S_j$ . Then we have

$$Z = \lim_{j \rightarrow \infty} Z^j, \quad \nu_h = \lim_{j \rightarrow \infty} \nu_h^j, \quad \lim_{j \rightarrow \infty} \nabla_{Z^j} \pi_K(\nu_h^j) = \nabla_Z \pi_K(\nu).$$

Since by assumption all the terms are continuous we have that the convergence is uniformly on each compact set of  $S$ . On the other hand, since  $E$  and  $N$  are only  $L^\infty$  (thus in particular

$L^1_{\text{loc}}$ ), we have that  $E^j$  and  $N^j$  converge to  $E$  and  $N$  a.e. in  $S$ . Applying Lemma 3.4 to the smooth surface  $S_j$  and Lebesgue's dominated convergence theorem we obtain from (3.10)

$$\begin{aligned}
\int_S \langle \pi(\nu_h), \frac{\nabla}{ds} \Big|_{s=0} V(s) \rangle dS &= \int_S \left[ -\langle N, T \rangle Z(\langle U, T \rangle) \pi_Z - E(\langle U, T \rangle) \pi_\nu \right. \\
&\quad \left. - 2\langle N, T \rangle \langle J(U), \pi(\nu_h) \rangle - |N_h| \langle J(\pi(\nu_h)), \nabla_Z U \rangle \right] dS \\
&= \lim_{j \rightarrow \infty} \int_{S_j} \left[ -\langle N^j, T \rangle Z^j(\langle U, T \rangle) \pi_{Z^j} - E^j(\langle U, T \rangle) \pi_{\nu^j} \right. \\
&\quad \left. - 2\langle N^j, T \rangle \langle J(U), \pi(\nu_h^j) \rangle dS_j - |N_h^j| \langle J(\pi(\nu_h^j)), \nabla_{Z^j} U \rangle \right] dS_j \\
&= \lim_{j \rightarrow \infty} \int_{S_j} u \langle \nabla_{Z^j} \pi(\nu_h^j), Z^j \rangle dS_j \\
&= \int_S u \langle \nabla_Z \pi(\nu_h), Z \rangle dS,
\end{aligned}$$

since  $\pi_K$  is  $C^1$ . This concludes the proof.  $\square$

**Lemma 3.4.** *Let  $S \subset \mathbb{H}^1$  be an oriented  $C^2$  surface with  $S_0 = \emptyset$ . Let  $U$  be a  $C^1$  vector field with compact support and normal component  $u = \langle U, N \rangle \in C_0^1(S)$ . Then we have*

$$\begin{aligned}
(3.11) \quad \int_S u \langle \nabla_Z \pi(\nu_h), Z \rangle dS &= \int_S \left[ -\langle N, T \rangle Z(\langle U, T \rangle) \pi_Z - E(\langle U, T \rangle) \pi_\nu \right. \\
&\quad \left. - 2\langle N, T \rangle \langle J(U), \pi(\nu_h) \rangle - |N_h| \langle J(\pi(\nu_h)), \nabla_Z U \rangle \right] dS,
\end{aligned}$$

where  $\pi_Z = \langle \pi_K, Z \rangle$  and  $\pi_\nu = \langle \pi_K, \nu_h \rangle$ .

*Proof.* Since both sides of (3.11) depend linearly on  $U$ , we consider the horizontal  $U_h = \langle U, Z \rangle Z + \langle U, \nu_h \rangle \nu_h$  and vertical  $U_v = \langle U, T \rangle T$  components of  $U$ .

We start considering the horizontal component  $U_h$  and we denote by  $I_h$  the corresponding right hand side of (3.11). Thus we obtain

$$\begin{aligned}
I_h &= \int_S \left\{ -2\langle N, T \rangle \langle J(U_h), \pi(\nu_h) \rangle - |N_h| \langle J(\pi(\nu_h)), \nabla_Z U_h \rangle \right\} dS \\
&= \int_S -2\langle N, T \rangle \langle J(U_h), \pi(\nu_h) \rangle dS \\
&\quad + \int_S \left\{ |N_h| Z(\langle \pi(\nu_h), J(U_h) \rangle) + |N_h| \langle \nabla_Z \pi(\nu_h), J(U_h) \rangle \right\} dS,
\end{aligned}$$

since  $\nabla_W J(V) = J(\nabla_W V)$  (as  $\nabla_W J = 0$ ) for each pair of vector fields  $W, V$ . By Proposition 2.6 we have

$$\begin{aligned}
\int_S |N_h| Z(\langle \pi(\nu_h), J(U_h) \rangle) dS &= - \int_S \langle \pi(\nu_h), J(U_h) \rangle Z(|N_h|) dS \\
&\quad - \int_S |N_h| \langle \pi(\nu_h), J(U_h) \rangle \operatorname{div}_S(Z) dS,
\end{aligned}$$

where  $\operatorname{div}_S(Z) = \langle D_E Z, E \rangle = -\langle N, T \rangle \theta(E) - 2\langle N, T \rangle |N_h|$ . Moreover, we have

$$\langle \nabla_Z \pi(\nu_h), J(U_h) \rangle = -\langle U, \nu_h \rangle \langle \nabla_Z \pi(\nu_h), Z \rangle,$$

since  $\langle \nabla_Z \pi(\nu_h), \nu_h \rangle = 0$ , by Remark 3.3 in [34]. Hence we have

$$\begin{aligned}
(3.12) \quad I_h &= \langle J(U_h), \pi(\nu_h) \rangle (-2\langle N, T \rangle^3 - Z(|N_h|) + |N_h| \langle N, T \rangle \theta(E)) \\
&\quad + |N_h| \langle U, \nu_h \rangle \langle \nabla_Z \pi(\nu_h), Z \rangle.
\end{aligned}$$

Now we consider the vertical component of the variational vector field  $U_v = \langle U, T \rangle T$ . We denote by  $I_v$  the left hand side of (3.11). Thus

$$(3.13) \quad I_v = \int_S -\langle N, T \rangle \pi_Z Z(\langle U, T \rangle) - E(\langle U, T \rangle) \pi_\nu dS$$

By Proposition 2.6 we get

$$\begin{aligned} - \int_S \langle N, T \rangle \pi_Z Z(\langle U, T \rangle) dS &= \int_S (Z(\langle N, T \rangle) \pi_Z + \langle N, T \rangle Z(\pi_Z)) \langle U, T \rangle dS \\ &\quad - \int_S \langle U, T \rangle \langle N, T \rangle \pi_Z (\langle N, T \rangle \theta(E)) + 2 \langle N, T \rangle |N_h| dS \\ &= \int_S \langle U, T \rangle Z(\langle N, T \rangle) \pi_Z dS \\ &\quad + \int_S \langle N, T \rangle \langle U, T \rangle (\langle \nabla_Z \pi(\nu_h), Z \rangle - \pi_\nu \theta(Z)) dS \\ &\quad - \int_S \langle U, T \rangle \langle N, T \rangle \pi_Z (\langle N, T \rangle \theta(E)) + 2 \langle N, T \rangle |N_h| dS, \end{aligned}$$

and

$$\begin{aligned} - \int_S \pi_\nu E(\langle U, T \rangle) dS &= \int_S E(\pi_\nu) \langle U, T \rangle dS + \int_S \langle U, T \rangle \langle N, T \rangle \pi_\nu \theta(Z) dS \\ &= \int_S \pi_Z \theta(E) \langle U, T \rangle dS + \int_S \langle U, T \rangle \langle N, T \rangle \pi_\nu \theta(Z) dS \end{aligned}$$

Adding the previous terms we obtain

$$(3.14) \quad \begin{aligned} I_v &= \int_S \langle U, T \rangle Z(\langle N, T \rangle) \pi_Z dS + \int_S \langle N, T \rangle \langle U, T \rangle (\langle \nabla_Z \pi(\nu_h), Z \rangle) dS \\ &\quad - \int_S 2 \langle U, T \rangle \pi_Z \langle N, T \rangle^2 |N_h| dS + \int_S \pi_Z |N_h|^2 \theta(E) \langle U, T \rangle dS \end{aligned}$$

Since tangential variations do not change the first variation formula, we consider a normal variational vector field  $U = uN$  with  $u \in C_0^1(S)$  so that  $\langle U, Z \rangle = 0$ ,  $\langle U, \nu_h \rangle = u|N_h|$  and  $\langle U, T \rangle = u\langle N, T \rangle$ . Then adding the integral of the horizontal term (3.12) and the vertical term (3.14) we obtain

$$(3.15) \quad \begin{aligned} I_h + I_v &= \int_S -u \pi_Z |N_h| (-2 \langle N, T \rangle^3 - Z(|N_h|) + |N_h| \langle N, T \rangle \theta(E)) dS \\ &\quad + \int_S u \pi_Z (\langle N, T \rangle Z(\langle N, T \rangle) - 2|N_h| \langle N, T \rangle^3 + |N_h|^2 \langle N, T \rangle \theta(E)) dS \\ &\quad + \int_S u (\langle N, T \rangle^2 + |N_h|^2) \langle \nabla_Z \pi(\nu_h), Z \rangle. \end{aligned}$$

Since  $|N_h| Z(|N_h|) + \langle N, T \rangle Z(\langle N, T \rangle) = 0$  we obtain

$$I_h + I_v = \int_S u \langle \nabla_Z \pi(\nu_h), Z \rangle$$

thus proving (3.11).  $\square$

To obtain the first variation formula (3.6) we had to assume that the derivatives in the  $Z$ -direction of the vector fields  $\nu_h$  and  $Z$  exist and are continuous functions on  $S$ . Let us see that the area-stationary property of  $S$  implies this regularity property. We follow the arguments in [29] and [32]. This result was first proven in the sub-Riemannian case by Nicolussi and Serra-Cassano [32].

**Definition 3.5.** Let  $S \subset \mathbb{H}^1$  be a  $(X, Y)$ -Lipschitz surface with boundary  $\partial S$ . We say  $S$  is area-stationary if, for any  $C^1$  vector field  $U$  with compact support such that  $\text{supp}(U) \cap \partial S = \emptyset$ , and associated one-parameter group of diffeomorphisms  $\{\varphi_s\}_{s \in \mathbb{R}}$ , we have

$$\left. \frac{d}{ds} \right|_{s=0} A_K(\varphi_s(S)) = 0.$$

**Remark 3.6.** Let  $D$  be a domain in the vertical plane  $\{y = 0\}$  and let  $u : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  be a Lipschitz function. Since the vector field  $Y$  is a unit normal to the plane, the intrinsic graph  $\text{Gr}(u)$  is parametrized by

$$(x, t) \rightarrow (x, u(x, t), t - xu(x, t)).$$

Let  $\gamma(s) = (x, t)(s)$  be a Lipschitz curve in  $D$ . Its lifting

$$\Gamma(s) = (x, u(x, t), t - xu(x, t))(s) \subset \text{Gr}(u)$$

is also Lipschitz and

$$\Gamma'(s) = x'X + (x'u_x + t'u_t)Y + (t' - 2ux')T$$

a.e. in  $s$ . In particular horizontal curves in  $\text{Gr}(u)$  satisfy the ordinary differential equation

$$(3.16) \quad t' = 2u(x, t)x'.$$

**Theorem 3.7.** Let  $K \in C_+^2$  be a convex body with  $0 \in \text{int}(K)$ . Let  $S \subset \mathbb{H}^1$  be an area-stationary  $(X, Y)$ -Lipschitz surface. Then  $S$  is an  $\mathbb{H}$ -regular surface foliated by horizontal straight lines.

*Proof.* Let  $p$  in  $S$ . Since  $S$  is  $(X, Y)$ -Lipschitz, by Theorem 2.3, there exist an open ball  $B_r(p)$  and a Lipschitz function  $u : D \rightarrow \mathbb{R}$  such that  $S \cap B_r(p) = \text{Gr}(u)$  where  $\text{Gr}(u) = \{(x, u(x, y), t - xu(x, t)) \in \mathbb{H}^1 : (x, t) \in D\}$ . Setting  $\pi_K = (\pi_1, \pi_2)$  the area functional is given by

$$A(\text{Gr}(u)) = \int_D (u_x + 2uu_t)\pi_1(u_x + 2uu_t, -1) - \pi_2(u_x + 2uu_t, -1) dxdt.$$

Given  $v \in C_0^\infty(D)$ , a straightforward computation shows that

$$(3.17) \quad \left. \frac{d}{ds} \right|_{s=0} A(\text{Gr}(u + sv)) = \int_D (v_x + 2vu_t + 2uv_t)M dxdt,$$

where

$$(3.18) \quad M = F(u_x + 2uu_t),$$

and  $F$  is the function

$$(3.19) \quad F(x) = \pi_1(x, -1) + x \frac{\partial \pi_1}{\partial x}(x, -1) - \frac{\partial \pi_2}{\partial x}(x, -1).$$

Let  $\Gamma(s)$  be a characteristic curve passing through  $p$  in  $\text{Gr}(u)$ . Let  $\gamma(s)$  be the projection of  $\Gamma(s)$  onto the  $xt$ -plane. By composition with a left-translation we may assume that  $(0, 0) \in D$  is the projection of  $p$  to the  $xt$ -plane. We parameterize  $\gamma$  by  $s \rightarrow (s, t(s))$ . By Remark 3.6 the curve  $s \rightarrow (s, t(s))$  satisfies the ordinary differential equation  $t' = 2u$ . For  $\varepsilon$  small enough, Picard-Lindelöf's theorem implies the existence of  $r > 0$  and a solution  $t_\varepsilon : ]-r, r[ \rightarrow \mathbb{R}$  of the Cauchy problem

$$(3.20) \quad \begin{cases} t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)), \\ t_\varepsilon(0) = \varepsilon. \end{cases}$$

We define  $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$  so that  $\gamma_0 = \gamma$ . By Lemma 3.9 we obtain that  $G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi))$  is a biLipschitz homeomorphisms where the determinant of the Jacobian of  $G$  is given by  $\partial t_\varepsilon(s)/\partial \varepsilon > C > 0$  for each  $s \in ]-r, r[$  and a.e. in  $\varepsilon$ .

Any function  $\varphi$  defined on  $D$  can be considered as a function of the variables  $(\xi, \varepsilon)$  by making  $\tilde{\varphi}(\xi, \varepsilon) = \varphi(\xi, t_\varepsilon(\xi))$ . Since the function  $G$  is  $C^1$  with respect to  $\xi$  we have

$$\frac{\partial \tilde{\varphi}}{\partial \xi} = \varphi_x + t'_\varepsilon \varphi_t = \varphi_x + 2u\varphi_t.$$

Furthermore, by [17, Theorem 2 in Section 3.3.3] or [30, Theorem 3], we may apply the change of variables formula for Lipschitz maps. Assuming that the support of  $v$  is contained in a sufficiently small neighborhood of  $(0, 0)$ , we can express the integral (3.17) as

$$(3.21) \quad \int_I \left( \int_{-r}^r \left( \frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v} \tilde{u}_t \right) \tilde{M} d\xi \right) d\varepsilon = 0,$$

where  $I$  is a small interval containing 0. In equation (3.21) we used the area stationary assumption. Instead of  $\tilde{v}$  in (3.21) we consider the function  $\tilde{v}h/(t_{\varepsilon+h} - t_\varepsilon)$ , where  $h$  is a small enough parameter. Then we obtain

$$\begin{aligned} \frac{\partial}{\partial \xi} \left( \frac{\tilde{v}h}{t_{\varepsilon+h} - t_\varepsilon} \right) &= \frac{\partial \tilde{v}}{\partial \xi} \frac{h}{t_{\varepsilon+h} - t_\varepsilon} - \tilde{v}h \frac{t'_{\varepsilon+h} - t'_\varepsilon}{(t_{\varepsilon+h} - t_\varepsilon)^2} \\ &= \frac{\partial \tilde{v}}{\partial \xi} \frac{h}{t_{\varepsilon+h} - t_\varepsilon} - 2\tilde{v}h \frac{u(\xi, t_{\varepsilon+h}(\xi)) - u(\xi, t_\varepsilon(\xi))}{(t_{\varepsilon+h} - t_\varepsilon)^2}, \end{aligned}$$

that tends to

$$\left( \frac{\partial t_\varepsilon}{\partial \varepsilon} \right)^{-1} \left( \frac{\partial \tilde{v}}{\partial \xi} - 2\tilde{v} \tilde{u}_t \right) \quad a.e. \text{ in } \varepsilon,$$

when  $h$  goes to 0. Putting  $\tilde{v}h/(t_{\varepsilon+h} - t_\varepsilon)$  in (3.21) instead of  $\tilde{v}$  we get

$$\int_I \left( \int_{-r}^r \frac{h \frac{\partial t_\varepsilon}{\partial \varepsilon}}{t_{\varepsilon+h} - t_\varepsilon} \left( \frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v} \left( \tilde{u}_t - \frac{\tilde{u}(\xi, \varepsilon+h) - \tilde{u}(\xi, \varepsilon)}{t_{\varepsilon+h} - t_\varepsilon} \right) \right) \tilde{M} d\xi \right) d\varepsilon = 0.$$

Using Lebesgue's dominated convergence theorem and letting  $h \rightarrow 0$  we have

$$(3.22) \quad \int_I \left( \int_{-r}^r \frac{\partial \tilde{v}}{\partial \xi} \tilde{M} d\xi \right) d\varepsilon = 0.$$

Let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  be a positive function compactly supported in  $I$  and for  $\rho > 0$  we consider the family  $\eta_\rho(x) = \rho^{-1} \eta(\frac{x - \varepsilon_0}{\rho})$ , that weakly converge to the Dirac delta distribution. Putting the test functions  $\eta_\rho(\varepsilon) \psi(\xi)$  in (3.22) and letting  $\rho \rightarrow 0$  we get

$$(3.23) \quad \int_{-r}^r \psi'(\xi) \tilde{M}(\xi, \varepsilon_0) d\xi = 0,$$

for each  $\psi \in C_0^\infty((-r, r))$  for a.e.  $\varepsilon_0$  in  $I$ . Since  $F$  is  $C^1$  and the distributional derivatives of a Lipschitz function belongs  $L^\infty$  we gain that  $M$  defined in (3.18) is  $L^\infty(D)$ . In particular we have that  $M$  belongs  $L^1_{\text{loc}}(D)$ , thus by Fubini's Theorem also  $\tilde{M}(\cdot, \varepsilon_0)$  belongs to  $L^1_{\text{loc}}((-r, r))$  for a.e.  $\varepsilon_0$  in  $I$ . By equation (3.23) we gain that  $\tilde{M}(\cdot, \varepsilon_0)$  belongs to  $W^{1,1}((-r, r))$  with  $\partial_\xi \tilde{M} = 0$  a.e. in  $(-r, r)$ . Then by [3, Theorem 8.2] we gain that  $\tilde{M}(\cdot, \varepsilon_0)$  is absolutely continuous and  $\partial_\xi \tilde{M} = 0$  a.e. in  $(-r, r)$  thus  $\tilde{M}(\cdot, \varepsilon_0)$  is constant in  $\xi$  for a.e.  $\varepsilon_0 \in I$ . Therefore  $M$  is constant along  $\gamma_{\varepsilon_0}(s) = (s, t_{\varepsilon_0}(s))$  for a.e.  $\varepsilon_0$  in  $I$ . By Lemma 3.2 in [29]  $F$  is a  $C^1$  invertible function, therefore also  $g(s) = (u_x + 2uu_t)_{\gamma_{\varepsilon_0}(s)} = F^{-1}(M)$  is constant in  $s$  for a.e.  $\varepsilon_0$  in  $I$ . This shows that horizontal normal given by

$$(3.24) \quad \nu_h = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}}$$

is constant along the characteristic curves, thus also  $Z = -J(\nu_h)$  is constant. Hence the characteristic curves of  $S$  are straight lines. Here we follow the approach developed by [32].



Moreover, since  $2g(s) = 2(u_x + 2uu_t)_{\gamma_{\varepsilon_0}(s)} = t_\varepsilon''(s)$  is constant in  $s$  we have that  $t_\varepsilon(s)$  is a polynomial of the second order given by

$$t_\varepsilon(s) = \varepsilon + a(\varepsilon)s + b(\varepsilon)s^2$$

where  $a(\varepsilon) = 2u(0, \varepsilon)$  that is Lipschitz continuous and  $b(\varepsilon) = (u_x + 2uu_t)(0, \varepsilon) = (u_x + 2uu_t)(s, \varepsilon)$ . Furthermore, choosing  $s > 0$  we can easily prove that  $b(\varepsilon)$  is also a Lipschitz function in  $\varepsilon$ . Hence in particular the horizontal normal  $\nu_h$  given by (3.24) is continuous, then the surface is an  $\mathbb{H}$ -regular surface.  $\square$

**Remark 3.8.** Notice that a  $(X, Y)$ -Lipschitz area-stationary surface is  $\mathbb{H}$ -regular and its horizontal normal  $\nu_h$  is  $C^\infty$  in the  $Z$ -direction.

**Lemma 3.9.** *With the previous notation, there exists a bi-Lipschitz homeomorphism  $G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi))$ . Moreover, there exists a constant  $C > 0$  such that  $\partial t_\varepsilon(s)/\partial \varepsilon > C$  for each  $s \in ]-r, r[$  and a.e. in  $\varepsilon$ .*

*Proof.* Here we exploit an argument similar to the one developed in [32]. By Theorem 2.8 in [41] we gain that  $t_\varepsilon$  is Lipschitz with respect to  $\varepsilon$  with Lipschitz constant less than or equal to  $e^{Lr}$ . Fix  $s \in ]-r, r[$ , the inverse of the function  $\varepsilon \rightarrow t_\varepsilon(s)$  is given by  $\bar{\chi}_t(-s) = \chi_t(-s)$  where  $\chi_t$  is the unique solution of the following Cauchy problem

$$(3.25) \quad \begin{cases} \chi_t'(\tau) = 2u(\tau, \chi_t(\tau)) \\ \chi_t(s) = t. \end{cases}$$

Again by Theorem 2.8 in [41] we have that  $\bar{\chi}_t$  is Lipschitz continuous with respect to  $t$ , thus the function  $\varepsilon \rightarrow t_\varepsilon$  is a locally biLipschitz homeomorphisms.

We consider the following Lipschitz coordinates

$$(3.26) \quad G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi)) = (s, t)$$

around the characteristic curve passing through  $(0, 0)$ . Notice that, by the uniqueness result for (4.5),  $G$  is injective. Given  $(s, t)$  in the image of  $G$  and using the inverse function  $\bar{\chi}_t$  defined in (3.25) we find  $\varepsilon$  such that  $t_\varepsilon(s) = t$ , therefore  $G$  is surjective. By the Invariance of Domain Theorem [4],  $G$  is a homeomorphism. By the uniqueness result of the Cauchy problem (4.5) we get that the map  $\varepsilon \rightarrow t_\varepsilon(s)$  is not decreasing in  $\varepsilon$ , then we have

$$(3.27) \quad \frac{\partial t_\varepsilon(s)}{\partial \varepsilon} \geq 0$$

for a.e. in  $\varepsilon$ . The differential of  $G$  is defined by

$$(3.28) \quad DG = \begin{pmatrix} 1 & 0 \\ t'_\varepsilon & \frac{\partial t_\varepsilon}{\partial \varepsilon} \end{pmatrix}$$

almost everywhere in  $\varepsilon$  and the differential of  $G^{-1}$  is given by

$$(3.29) \quad DG^{-1} = \left( \frac{\partial t_\varepsilon}{\partial \varepsilon} \right)^{-1} \begin{pmatrix} \frac{\partial t_\varepsilon}{\partial \varepsilon} & 0 \\ -t'_\varepsilon & 1 \end{pmatrix}$$

almost everywhere in  $\varepsilon$ . Since  $G^{-1}$  is Lipschitz we gain that there exists a constant  $C > 0$  such that

$$|\mathbf{J}_{G^{-1}}| = |\det(DG^{-1})| = \left| \frac{\partial t_\varepsilon}{\partial \varepsilon} \right|^{-1} \leq \frac{1}{C}$$

a.e. in  $\varepsilon$ . Thus, by (3.27) we deduce that  $\partial t_\varepsilon/\partial \varepsilon > C > 0$  a.e. in  $\varepsilon$ .  $\square$

4. A CODAZZI-LIKE EQUATION FOR  $(X, Y)$ -LIPSCHITZ MINIMAL SURFACES

In this section we shall show that, given an area-stationary surface  $S$ , the function  $\langle N, T \rangle / |N_h|$  satisfies a differential equation along almost every characteristic curve on  $S$ .

We first prove a technical result similar to Lemma 4.2 in [26]. We include the proof, with only slight differences, for completeness.

**Lemma 4.1.** *Given  $a, b \in \mathbb{R}$ , the only solution of equation*

$$(4.1) \quad y'' - 6y'y + 4y^3 = 0$$

*about the origin with initial conditions  $y(0) = a$ ,  $y'(0) = b$ , is*

$$(4.2) \quad y_{a,b}(s) = \frac{a - (2a^2 - b)s}{1 - 2as + (2a^2 - b)s^2}.$$

*Moreover, we have*

$$(4.3) \quad y_{a,b}^2(s) - y'_{a,b}(s) = \frac{a^2 - b}{(1 - 2as + (2a^2 - b)s^2)^2}$$

*If  $y_{a,b}$  is defined for every  $s \in \mathbb{R}$  then either  $a^2 - b > 0$  or  $y_{a,b} \equiv 0$ .*

*Proof.* By the uniqueness of solutions for ordinary differential equations we know that there exists a unique solution (4.1). Since we have

$$y'_{a,b} = \frac{b - 2as(2a^2 - b) + (2a^2 - b)^2 s^2}{(1 - 2as + (2a^2 - b)s^2)^2}$$

and

$$y''_{a,b} = \frac{2(a - (2a^2 - b)s)(3b - 2as(2a^2 - b) + (2a^2 - b)^2 s^2 - 2a^2)}{(1 - 2as + (2a^2 - b)s^2)^3},$$

a straightforward computation shows that  $y_{a,b}(s)$  solves (4.1) and satisfies (4.3).

Let us write  $y_{a,b} = p(s)/r(s)$  where  $p(s) = a - (2a^2 - b)s$  and  $r(s) = 1 - 2as + (2a^2 - b)s^2$ . If  $y_{a,b}$  is defined for every  $s \in \mathbb{R}$ , then there are two possibilities:  $r(s)$  has no real zeroes or  $r(s)$  has at least a zero at  $s_0 \in \mathbb{R}$ . In the first case the discriminant is  $4(b - a^2)$  is negative. In the second case  $r(s_0) = 0$  we must also have  $p(s_0) = 0$  in order to have  $y_{a,b}(s)$  well defined at  $s_0$ . Hence  $y_{a,b}(s)$  can be expressed as the quotient of a constant over a degree one polynomial. Then by (4.2) we get that  $y_{a,b} = a(1 - 2as)^{-1}$  which has a pole unless  $a = 0$ , hence  $y_{a,b}(s) \equiv 0$ .  $\square$

**Remark 4.2.** If  $f(s)$  is a solution of (4.1), then for each positive constant  $\lambda$  the function  $f_\lambda(s) = \lambda^{-1}f(\frac{s}{\lambda})$  is still a solution of (4.1).

**Remark 4.3.** Let  $f$  be a solution of (4.1), then  $f$  belongs to  $C^\infty$  class. Indeed setting  $y_1 = f$  and  $y_2 = y'_1$  we have that (4.1) is equivalent to

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = F(y_1, y_2) = \begin{pmatrix} y_2 \\ -6y_1y_2 - 4y_1^3 \end{pmatrix}.$$

Since  $F$  is  $C^\infty$  we obtain that  $y_1 = f$  is smooth.

**Proposition 4.4.** *Let  $S$  be a complete oriented area-stationary  $(X, Y)$ -Lipschitz surface. Then along any arc-length parameterized geodesic  $\bar{\gamma}_\varepsilon(s)$  in  $S$ , the function  $\langle N, T \rangle / |N_h|(\bar{\gamma}_\varepsilon(s))$  satisfies the ordinary differential equation (4.1) for a.e.  $\varepsilon$ . Furthermore,  $\langle N, T \rangle / |N_h|(\bar{\gamma}_\varepsilon(s))$  is smooth in  $s$  for a.e.  $\varepsilon$ .*

*Proof.* Let  $p$  in  $S$ . Since  $S$  is  $(X, Y)$ -Lipschitz, Theorem 2.3 implies the existence of an open ball  $B_r(p)$  and of a Lipschitz function  $u : D \rightarrow \mathbb{R}$  such that  $S \cap B_r(p) = \text{Gr}(u)$ . Let  $\Gamma(s)$  be a characteristic curve passing through  $p$  in  $\text{Gr}(u)$ . Let  $\gamma(s)$  be the projection of  $\Gamma(s)$  onto the  $xt$ -plane, and  $(0, 0) \in D$  the projection of  $p$  to the  $xt$ -plane. We parameterize  $\gamma$  by  $s \rightarrow (s, t(s))$ . By Remark 3.6 (see also [29, Remark 2.5]) the curve  $s \rightarrow (s, t(s))$  satisfies the ordinary differential equation  $t' = 2u$  and

$$\Gamma'(s) = X + (u_x + 2uu_t)Y.$$

As computed in § 2.6 in [29], at smooth points of the graph of  $u$ , the unit normal can be computed as  $N = \tilde{N}/|\tilde{N}|$ , where

$$\tilde{N} = (u_x + 2uu_t) - Y + u_t T.$$

The quantity  $|\tilde{N}|$  is the Riemannian Jacobian of the parameterization of the intrinsic graph of  $u$  by coordinates  $(x, t)$ . So we have

$$(4.4) \quad |N_h| dS = \sqrt{1 + (u_x + 2uu_t)^2} dx dt.$$

Since we have

$$\nu_h = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}} \quad \text{and} \quad Z = -J(\nu_h)$$

we get  $Z = -\Gamma'(s)/|\Gamma(s)|$ . For  $\varepsilon$  small enough, Picard-Lindelöf's theorem implies the existence of  $r > 0$  and a solution  $t_\varepsilon : ]-r, r[ \rightarrow \mathbb{R}$  of the Cauchy problem

$$(4.5) \quad \begin{cases} t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)), \\ t_\varepsilon(0) = \varepsilon. \end{cases}$$

We define  $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$  so that  $\gamma_0 = \gamma$ . Since  $S$  is area-stationary we have that  $(u_x + 2uu_t)$  is constant along  $\gamma_\varepsilon(s)$ . Moreover

$$t''_\varepsilon(s) = 2(u_x + 2uu_t)(\gamma_\varepsilon(s)) = 2b(\varepsilon) = 2(u_x + 2uu_t)(0, \varepsilon)$$

is constant as a function of  $s$ . Thus we have

$$(4.6) \quad t_\varepsilon(s) = \varepsilon + a(\varepsilon)s + b(\varepsilon)s^2,$$

where  $a(\varepsilon) = 2u(0, \varepsilon)$ . Choosing  $s > 0$  in (4.6) we can easily prove that  $b(\varepsilon)$ , that a priori is only continuous, is also a Lipschitz function. By equation (7) in [32, Theorem 3.7] we have

$$(4.7) \quad \frac{\partial}{\partial \varepsilon} \frac{\partial}{\partial s} t_\varepsilon(s) = \frac{\partial}{\partial s} \frac{\partial}{\partial \varepsilon} t_\varepsilon(s)$$

a.e. in  $\varepsilon$ , where the equality has to be interpreted in the sense of distributions. Putting (4.5) in the left hand side of (4.7) and applying the chain rule for Lipschitz functions (see [32, Remark 3.6]) we get

$$2u_t(s, t_\varepsilon(s))(1 + a'(\varepsilon)s + b'(\varepsilon)s^2) = (a'(\varepsilon) + 2b'(\varepsilon)s)$$

a.e. in  $\varepsilon$ . Therefore we get

$$u_t(s, t_\varepsilon(s)) = \frac{\frac{a'(\varepsilon)}{2} + b'(\varepsilon)s}{(1 + a'(\varepsilon)s + b'(\varepsilon)s^2)},$$

a.e. in  $\varepsilon$ , since by Lemma 3.9 we have  $\partial t_\varepsilon / \partial \varepsilon > 0$  a.e. in  $\varepsilon$ . Since we have  $Z = -\Gamma'(s)/|\Gamma(s)|$  we consider  $\tilde{\gamma}_\varepsilon(s) = \gamma_\varepsilon(-s)$ . Then we have that

$$u_t(\tilde{\gamma}_\varepsilon(s)) = \frac{\frac{a'(\varepsilon)}{2} - b'(\varepsilon)s}{(1 - a'(\varepsilon)s + b'(\varepsilon)s^2)}$$

solves the equation (4.1) with initial condition  $y(0) = a'(\varepsilon)/2$  and  $y'(0) = \frac{a'(\varepsilon)^2}{2} - b'(\varepsilon)$  for a.e.  $\varepsilon$ . Moreover we have

$$t_\varepsilon(-s) = \varepsilon - a(\varepsilon)s + b(\varepsilon)s^2$$

For each  $\varepsilon$  fixed we have  $b(\varepsilon) = (u_x + 2uu_t)(\tilde{\gamma}_\varepsilon)$  is constant, let

$$\bar{\gamma}_\varepsilon(s) = \tilde{\gamma}_\varepsilon\left(s/\sqrt{1+b(\varepsilon)^2}\right)$$

be an arc-length parametrization of  $\tilde{\gamma}_\varepsilon$ . Then Remark 4.2 shows that also

$$\langle N, T \rangle / |N_h|(\bar{\gamma}_\varepsilon) = \frac{u_t}{\sqrt{1+(u_x+2uu_t)^2}}(\bar{\gamma}_\varepsilon)$$

is a solution of (4.1) a.e. in  $\varepsilon$ .  $\square$

## 5. SECOND VARIATION FORMULA

In this section, we compute the second variation formula. First of all we need the following Lemma

**Lemma 5.1.** *Let  $U$  be a  $C^2$  horizontal vector field in  $\mathbb{H}^1$  with associated flow  $\{\varphi_s\}_{s \in \mathbb{R}}$ . Let  $p \in \mathbb{H}^1$  and  $e \in T_p \mathbb{H}^1$ . Define the smooth curve  $\beta(s) = \varphi_s(p)$  and the smooth vector field  $E(s) = (d\varphi_s)_p(e)$  along  $\beta$ . Then we have*

$$\frac{\nabla^2}{ds^2}\Big|_{s=0} E(s) = \nabla_e \nabla_U U + 2 \langle J(\nabla_{U_p} U), e \rangle T_p + 2 \langle J(U_p), \nabla_e U \rangle T_p.$$

*Proof.* From (3.1) we get

$$\frac{\nabla^2}{ds^2}\Big|_{s=0} E(s) = \frac{\nabla}{ds}\Big|_{s=0} (\nabla_{E(s)} U_s + 2 \langle J(U_s), E(s) \rangle T).$$

On the one hand we have

$$\begin{aligned} \frac{\nabla}{ds}\Big|_{s=0} \nabla_{E(s)} U_s &= \frac{\nabla}{ds}\Big|_{s=0} \sum_{i=1}^3 g_i(s) \nabla_{(\frac{\partial}{\partial x_i})_{\beta(s)}} U_s \\ &= \sum_{i=1}^3 g'_i(0) \nabla_{(\frac{\partial}{\partial x_i})_p} U + g_i(0) \frac{\nabla}{ds}\Big|_{s=0} \nabla_{(\frac{\partial}{\partial x_i})_p} U \\ &= \sum_{i=1}^3 e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + e_i \nabla_U \nabla_{(\frac{\partial}{\partial x_i})_p} U. \end{aligned}$$

Notice that

$$\begin{aligned} \nabla_e (\nabla_U U) &= \nabla_e \nabla_{\sum_{i=1}^3 f_i \frac{\partial}{\partial x_i}} U = \sum_{i=1}^3 e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + f_i(p) \nabla_e \nabla_{(\frac{\partial}{\partial x_i})_p} U \\ &= \sum_{i=1}^3 e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + \sum_{i,j=1}^3 f_i(p) e_j \nabla_{(\frac{\partial}{\partial x_j})_p} \nabla_{(\frac{\partial}{\partial x_i})_p} U \\ &= \sum_{i=1}^3 e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + \sum_{i,j=1}^3 f_i(p) e_j \nabla_{(\frac{\partial}{\partial x_i})_p} \nabla_{(\frac{\partial}{\partial x_j})_p} U \\ &= \sum_{i=1}^3 e(f_i) \nabla_{(\frac{\partial}{\partial x_i})_p} U + \sum_{i=1}^3 e_i \nabla_U \nabla_{(\frac{\partial}{\partial x_i})_p} U, \end{aligned}$$

where we use that the Riemann tensor of  $\nabla$  vanishes

$$0 = R\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right) U = \nabla_{(\frac{\partial}{\partial x_i})_p} \nabla_{(\frac{\partial}{\partial x_j})_p} U - \nabla_{(\frac{\partial}{\partial x_j})_p} \nabla_{(\frac{\partial}{\partial x_i})_p} U$$

Therefore

$$(5.1) \quad \frac{\nabla}{ds} \Big|_{s=0} \nabla_{E(s)} U_s = \nabla_e(\nabla_U U) + \nabla_e U'$$

On the other hand, since  $\nabla J = 0$  we have

$$(5.2) \quad \begin{aligned} \frac{\nabla}{ds} \Big|_{s=0} 2\langle J(U_s), E(s) \rangle T &= 2\langle J(\nabla_U U), e \rangle T + 2\langle J(U), \frac{\nabla}{ds} \Big|_{s=0} E(s) \rangle T \\ &= 2\langle J(\nabla_U U), e \rangle T + 2\langle J(U), \nabla_e U \rangle T, \end{aligned}$$

where we have used once again Lemma 3.1 and the fact that  $J(U)$  is horizontal. Finally adding (5.1) and (5.2) we get the result.  $\square$

Now we compute the second variation formula

**Theorem 5.2.** *Let  $K \in C_+^2$  be a convex body with  $0 \in \text{int}(K)$ . Let  $S \subset \mathbb{H}^1$  be an area-stationary  $(X, Y)$ -Lipschitz surface. Let  $U$  be an horizontal  $C^2$  vector field compactly supported in  $\mathbb{H}^1$ , with  $\partial S \cap \text{supp}(U) = \emptyset$ , and associated flow  $\{\varphi_s\}_{s \in \mathbb{R}}$ . Then the second variation of the sub-Finsler area induced by  $U$  is given by*

$$(5.3) \quad \frac{d^2}{ds^2} \Big|_{s=0} A_K(\varphi_s(S)) = \int_S (Z(f)^2 + qf^2) \frac{|N_h|}{\kappa(\pi_K(\nu_h))} dS,$$

where

$$q = 4 \left\{ Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) - \frac{\langle N, T \rangle^2}{|N_h|^2} \right\},$$

$\kappa$  is the positive curvature of the boundary  $\partial K$  and  $f = \langle U, \nu_h \rangle$ .

**Remark 5.3.** As noticed in the introduction, there is a slight difference in this second variation formula with respect to the sub-Riemannian one computed in [26], due to the definition of  $Z$  as  $-J(\nu_h)$  in this paper, instead of  $J(\nu_h)$  as in [26]. This change was introduced in [34] as the most convenient way of dealing with the lack of symmetry of the sub-Finsler norm.

*Proof of Theorem 5.2.* First of all we notice that  $\langle N, T \rangle / |N_h|$  is smooth in the  $Z$ -direction by Proposition 4.4, and so  $q$  is well defined. Moreover, by Theorem 3.7 an area-stationary  $(X, Y)$ -Lipschitz surface is  $\mathbb{H}$ -regular. The area functional is given by

$$A_K(\varphi_s(S)) = \int_S \langle \pi(V(s)), V(s) \rangle dS,$$

where

$$V(s) = \langle E_1(s), T \rangle T \times E_2(s) + \langle E_2(s), T \rangle E_1(s) \times T,$$

and  $dS$  is the Riemannian area element. At a regular point  $p \in S$ , a basis of tangent vectors to  $\varphi_s(S)$  at  $\varphi_s(p)$  is given by  $E_1(s) = (d\varphi_s)_p(Z_p)$  and  $E_2(s) = (d\varphi_s)_p(E_p)$ .

Since by Lemma 3.2 the norm of  $V(s)$  is strictly positive a.e. in  $\text{supp}(U) \cap S$ , we have

$$(5.4) \quad \begin{aligned} \frac{d^2}{ds^2} \|V(s)\|_* &= \frac{d}{ds} \langle \pi_K(V(s)), \frac{\nabla}{ds} V(s) \rangle \\ &= \langle d\pi(V(s)) \frac{\nabla}{ds} V(s), \frac{\nabla}{ds} V(s) \rangle + \langle \pi_K(V(s)), \frac{\nabla^2}{ds^2} V(s) \rangle \\ &\leq \frac{1}{\bar{\kappa}} \left| \frac{\nabla}{ds} V(s) \right|^2 + \beta \left| \frac{\nabla^2}{ds^2} V(s) \right|, \end{aligned}$$

a.e. in  $\text{supp}(U) \cap S$ , where  $\bar{\kappa} = \min_{\|v\|_* = 1} \kappa(v)$  and  $\beta$  is the positive constant defined in 2.4 with  $K'$  equal to the Euclidean ball centered at the origin. Setting  $\tilde{N}(s) = E_1(s) \times E_2(s)$  we get

$$\frac{\nabla^2}{ds^2} \tilde{N}(s) = \frac{\nabla^2}{ds^2} E_1(s) \times E_2(s) + 2 \frac{\nabla}{ds} E_1(s) \times \frac{\nabla}{ds} E_2(s) + E_1(s) \times \frac{\nabla^2}{ds^2} E_2(s).$$

Then Lemmas 3.1 and 5.1 imply

$$\begin{aligned}\frac{\nabla}{ds}E_i(s) &= \nabla_{E_i(s)}U + 2\langle J(U_p), E_i(s) \rangle T, \\ \frac{\nabla^2}{ds^2}E_i(s) &= \nabla_{E_i(s)}\nabla_U U + 2\langle J(\nabla_U U), E_i(s) \rangle T + 2\langle J(U_p), \nabla_{E_i(s)}U \rangle T.\end{aligned}$$

and  $|E_1(s)| = |d\varphi_s(Z)| \leq C'$  and  $|E_1(s)| = |d\varphi_s(E)| \leq C'$  for  $s \in (-s_0, s_0)$ , where the constant  $C'$  is independent of  $s$ . Then, writing the covariant derivative  $\nabla_{E_i(s)}U$  and  $\nabla_{E_i(s)}\nabla_U U$  in standard coordinates, we obtain

$$\left| \frac{\nabla^2}{ds^2}\tilde{N}(s) \right| \leq \tilde{C} \|U\|_{C^2}$$

a.e. in  $\text{supp}(U) \cap S$  for a suitable constant  $\tilde{C} > 0$  and where  $\|U\|_{C^2}$  denotes the standard  $C^2$  norm of  $U$ . Then, since  $V(s) = \tilde{N}(s) - \langle \tilde{N}(s), T \rangle T$  and thus  $\frac{\nabla^2}{ds^2}V(s) = \frac{\nabla^2}{ds^2}\tilde{N}(s) - \langle \frac{\nabla^2}{ds^2}\tilde{N}(s), T \rangle T$ , we have

$$\left| \frac{\nabla^2}{ds^2}V(s) \right| \leq 2 \left| \frac{\nabla^2}{ds^2}\tilde{N}(s) \right| \leq 2\tilde{C} \|U\|_{C^2}$$

a.e. in  $\text{supp}(U) \cap S$ . Then, since  $\text{supp}(U) \cap S$  is compact, Lebesgue's dominated convergence theorem yields

$$\frac{d^2}{ds^2}\Big|_{s=0} A_K(\varphi_s(S)) = \int_S \frac{d^2}{ds^2}\Big|_{s=0} \langle \pi(V(s)), V(s) \rangle dS.$$

By Remark 3.3 in [34] we get

$$(5.5) \quad \frac{d}{ds} \langle \pi(V(s)), \frac{\nabla}{ds}V(s) \rangle = \langle \frac{\nabla}{ds}\pi(V(s)), \frac{\nabla}{ds}V(s) \rangle + \langle \pi(V(s)), \frac{\nabla}{ds}\frac{\nabla}{ds}V(s) \rangle$$

Again by Remark 3.3 in [34] we have

$$0 = \frac{d}{ds} \langle \frac{\nabla}{ds}\pi(V(s)), V(s) \rangle = \langle \frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)), V(s) \rangle + \langle \frac{\nabla}{ds}\pi(V(s)), \frac{\nabla}{ds}V(s) \rangle.$$

Then we gain

$$(5.6) \quad \langle \frac{\nabla}{ds}\pi(V(s)), \frac{\nabla}{ds}V(s) \rangle = -\langle \frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)), V(s) \rangle.$$

Therefore substituting (5.6) in (5.5) we obtain

$$\frac{d}{ds} \langle \pi(V(s)), \frac{\nabla}{ds}V(s) \rangle = \langle \pi(V(s)), \frac{\nabla}{ds}\frac{\nabla}{ds}V(s) \rangle - \langle \frac{\nabla}{ds}\frac{\nabla}{ds}\pi(V(s)), V(s) \rangle$$

Evaluating the previous equality at  $s = 0$  we get

$$(5.7) \quad \frac{d^2}{ds^2}\Big|_{s=0} \langle \pi(V(s)), V(s) \rangle = \langle \pi(\nu_h), \frac{\nabla^2}{ds^2}\Big|_{s=0} V(s) \rangle - \langle \frac{\nabla^2}{ds^2}\Big|_{s=0} \pi(V(s)), \nu_h \rangle = \mathbf{I} + \mathbf{II},$$

since  $V(0) = |N_h|\nu_h$ .

As

$$\begin{aligned}\frac{\nabla}{ds}V(s) &= \langle \frac{\nabla}{ds}E_1(s), T \rangle T \times E_2(s) + \langle E_1(s), T \rangle T \times \frac{\nabla}{ds}E_2(s) \\ &\quad + \langle \frac{\nabla}{ds}E_2(s), T \rangle E_1(s) \times T + \langle E_2(s), T \rangle \frac{\nabla}{ds}E_1(s) \times T.\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\nabla^2}{ds^2}\Big|_{s=0} V(s) &= \langle \frac{\nabla^2}{ds^2}\Big|_{s=0} E_1(s), T \rangle T \times E_2(0) + 2\langle \frac{\nabla}{ds}\Big|_{s=0} E_1(s), T \rangle T \times \frac{\nabla}{ds}\Big|_{s=0} E_2(s) \\ &\quad + \langle E_1(0), T \rangle T \times \frac{\nabla^2}{ds^2}\Big|_{s=0} E_2(s) + \langle \frac{\nabla^2}{ds^2}\Big|_{s=0} E_2(s), T \rangle E_1(0) \times T \\ &\quad + 2\langle \frac{\nabla}{ds}\Big|_{s=0} E_2(s), T \rangle \frac{\nabla}{ds}\Big|_{s=0} E_1(s) \times T + \langle E_2(0), T \rangle \frac{\nabla^2}{ds^2}\Big|_{s=0} E_1(s) \times T.\end{aligned}$$

By Lemma 3.1

$$\frac{\nabla}{ds}\Big|_{s=0} E_i(s) = \nabla_{E_i(0)} U + 2\langle J(U), E_i(0) \rangle T,$$

for  $i = 1, 2$ . By Lemma 5.1 we gain

$$\frac{\nabla^2}{ds^2}\Big|_{s=0} E_i(s) = \nabla_{E_i(0)}(\nabla_U U) + 2\langle J(\nabla_U U), E_i(0) \rangle T + 2\langle J(U), \nabla_{E_i(0)} U \rangle T.$$

Noticing that  $\nabla_Z U$  is horizontal we get

$$\begin{aligned} \frac{\nabla^2}{ds^2}\Big|_{s=0} V(s) &= (\langle \nabla_Z(\nabla_U U), T \rangle + 2\langle J(\nabla_U U), Z \rangle + 2\langle J(U), \nabla_Z U \rangle) T \times E \\ &\quad + 4\langle J(U), Z \rangle T \times \nabla_E U \\ &\quad + (\langle \nabla_E(\nabla_U U), T \rangle + 2\langle J(\nabla_U U), E \rangle + 2\langle J(U), \nabla_E U \rangle) Z \times T \\ &\quad + 2(\langle \nabla_E U, T \rangle + 2\langle J(U), E \rangle) \nabla_Z U \times T - |N_h| \nabla_Z(\nabla_U U) \times T. \end{aligned}$$

We set  $\pi_K(\nu_h) = \pi_Z Z + \pi_\nu \nu_h$ , where  $\pi_Z = \langle \pi(\nu_h), Z \rangle$  and  $\pi_\nu = \langle \pi(\nu_h), \nu_h \rangle$ . Notice that  $T \times \nu_h = -Z$ ,  $Z \times T = -\nu_h$  and  $\langle \pi(\nu_h), W \times T \rangle = \langle J(\pi(\nu_h)), W \rangle$  for each vector field  $W$ , then a straightforward computation shows that

$$(5.8) \quad \mathbf{I} = \langle \pi(\nu_h), \frac{\nabla^2}{ds^2}\Big|_{s=0} V(s) \rangle = \mathbf{A} + \mathbf{B}$$

where

$$(5.9) \quad \begin{aligned} \mathbf{A} &= -2\langle N, T \rangle \langle J(U), \nabla_Z U \rangle \pi_Z - 4\langle J(U), Z \rangle \langle J(\pi(\nu_h)), \nabla_E U \rangle \\ &\quad - 2\pi_\nu \langle J(U), \nabla_E U \rangle + 4\langle J(\pi(\nu_h)), \nabla_Z U \rangle \langle J(U), E \rangle \end{aligned}$$

and

$$(5.10) \quad \begin{aligned} \mathbf{B} &= -\langle N, T \rangle Z(\langle \nabla_U U, T \rangle) \pi_Z - E(\langle \nabla_U U, T \rangle) \pi_\nu \\ &\quad - 2\langle N, T \rangle \langle J(\nabla_U U), \pi(\nu_h) \rangle - |N_h| \langle J(\pi(\nu_h)), \nabla_Z \nabla_U U \rangle \end{aligned}$$

Since  $S$  is area-stationary, by equation (3.5) in Proposition 3.3 we have that

$$(5.11) \quad \begin{aligned} 0 = \frac{d}{ds}\Big|_{s=0} A_K(\varphi_s(S)) &= \int_S \left[ -\langle N, T \rangle Z(\langle U, T \rangle) \pi_Z - E(\langle U, T \rangle) \pi_\nu \right. \\ &\quad \left. - 2\langle N, T \rangle \langle J(U), \pi(\nu_h) \rangle - |N_h| \langle J(\pi(\nu_h)), \nabla_Z U \rangle \right] dS, \end{aligned}$$

for every  $U$  compactly supported  $C^1$  vector field. Taking into account the first variation formula (5.11) induced by the  $C^1$  horizontal vector field  $\nabla_U U$  we get

$$\int_S \mathbf{B} = 0$$

Thus we obtain

$$(5.12) \quad \int_S \mathbf{I} = \int_S \mathbf{A}.$$

On the other hand, by Lemma 5.4 and by equation (5.17) in Remark 5.5, we obtain

$$\mathbf{II} = -\left\langle \frac{\nabla^2}{ds^2}\Big|_{s=0} \pi(V(s)), \nu_h \right\rangle = \left\langle \frac{\nabla}{ds}\Big|_{s=0} \left( \frac{V(s)}{|V(s)|} \right), (d\pi)_{\nu_h}^* Z \right\rangle \left\langle \frac{\nabla}{ds}\Big|_{s=0} V(s), Z \right\rangle.$$

Then [29, Lemma 4.3] yields

$$\mathbf{II} = \frac{1}{\kappa |N_h|} \left\langle \frac{\nabla}{ds}\Big|_{s=0} V(s), Z \right\rangle^2,$$

where  $\kappa = \kappa(\pi_K(\nu_h))$  is the positive constant curvature of  $\partial K$  evaluated at  $\pi_K(\nu_h)$ , that is constant along the characteristic curves by Theorem 3.7. Thanks to (3.9) we have

$$\begin{aligned} \left\langle \frac{\nabla}{ds} \Big|_{s=0} V(s), Z \right\rangle &= -2\langle N, T \rangle \langle J(U), Z \rangle - |N_h| \langle \nabla_Z U, \nu_h \rangle \\ &= 2\langle N, T \rangle \langle U, \nu_h \rangle - |N_h| \langle \nabla_Z U, \nu_h \rangle. \end{aligned}$$

Setting  $f = \langle U, \nu_h \rangle$  and  $g = \langle U, Z \rangle$  we get that  $\mathbf{A} + \mathbf{II}$  is equal to

$$\begin{aligned} &2\langle N, T \rangle \pi_Z(gZ(f) + fZ(g)) + 4f\pi_Z(E(f) - g\theta(E)) - 2f\pi_\nu(E(g) + f\theta(E)) \\ &- 2g\pi_\nu(E(f) - g\theta(E)) - 4g\langle N, T \rangle \pi_\nu Z(g) \\ &+ \frac{1}{\kappa |N_h|} (2\langle N, T \rangle f - |N_h| Z(f))^2. \end{aligned}$$

Notice that

$$\begin{aligned} (5.13) \quad \mathbf{II} &= \frac{1}{\kappa |N_h|} (2\langle N, T \rangle f - |N_h| Z(f))^2 \\ &= \frac{1}{\kappa |N_h|} (4\langle N, T \rangle^2 f^2 - 4\langle N, T \rangle |N_h| fZ(f) + |N_h|^2 Z(f)^2). \end{aligned}$$

The second term of (5.13) can be written as

$$\begin{aligned} 4 \frac{\langle N, T \rangle |N_h|}{\kappa |N_h|} fZ(f) &= 2|N_h| \frac{\langle N, T \rangle}{\kappa |N_h|} Z(f^2) \\ &= 2|N_h| \left( Z \left( \frac{\langle N, T \rangle}{\kappa |N_h|} f^2 \right) - Z \left( \frac{\langle N, T \rangle}{\kappa |N_h|} \right) f^2 \right). \end{aligned}$$

Then, setting  $h = \frac{\langle N, T \rangle f^2}{\kappa |N_h|}$  in Lemma 5.6, we obtain that the integrals of the first and second term in (5.13) are equal to

$$\int_S \left( 4 \frac{\langle N, T \rangle^2}{\kappa |N_h|^2} f^2 - 2 \frac{\langle N, T \rangle}{\kappa |N_h|} Z(f^2) \right) |N_h| dS = 2 \int_S Z \left( \frac{\langle N, T \rangle}{\kappa |N_h|} \right) f^2 |N_h| dS.$$

The integral of the third summand in (5.13) is equal to

$$\int_S Z(f)^2 \frac{|N_h|}{\kappa} dS.$$

Hence we obtain

$$(5.14) \quad \int_S \mathbf{II} dS = \int_S Z(f)^2 \frac{|N_h|}{\kappa} dS + 2 \int_S Z \left( \frac{\langle N, T \rangle}{\kappa |N_h|} \right) f^2 |N_h| dS.$$

Finally we deal with

$$\begin{aligned} \mathbf{A} &= 2\langle N, T \rangle \pi_Z(gZ(f) + fZ(g)) + 4f\pi_Z(E(f) - g\theta(E)) - 2f\pi_\nu(E(g) + f\theta(E)) \\ &- 2g\pi_\nu(E(f) - g\theta(E)) - 4g\langle N, T \rangle \pi_\nu Z(g) \end{aligned}$$

By equation (5.19) in Lemma 5.6 we have that

$$\begin{aligned} &2\pi_\nu(g^2\theta(E) - 2\langle N, T \rangle gZ(g)) \\ &= 2\pi_\nu |N_h| \left( g^2 \left( 2 \frac{\langle N, T \rangle^2}{|N_h|^2} - Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) \right) - \frac{\langle N, T \rangle}{|N_h|} Z(g^2) \right) \\ &= 2||N_h||_* \left( 2g^2 \frac{\langle N, T \rangle^2}{|N_h|^2} - Z \left( \frac{\langle N, T \rangle}{|N_h|} g^2 \right) \right) \end{aligned}$$

a.e. in  $S$ . Then, by Lemma 5.7 the integral over  $S$  of the previous term is equal to zero. Therefore we have

$$\mathbf{A} = 2\langle N, T \rangle \pi_Z Z(gf) + 4f\pi_Z(E(f) - g\theta(E)) - 2f^2\pi_\nu\theta(E) - 2\pi_\nu E(gf)$$



On the one hand we have

$$\int_S \pi_\nu E(gf) + \pi_Z \theta(E) gf = 0,$$

by equation (5.21) in Lemma 5.7. On the other hand using equation (5.20) in Lemma 5.7 and equation (5.19) in Lemma 5.6 we have

$$\begin{aligned} 2 \int_S \pi_Z (\langle N, T \rangle Z(gf) - gf \theta(E)) &= \\ 2 \int_S \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \pi_Z gf \right) - 2 \frac{\langle N, T \rangle^2}{|N_h|^2} \pi_Z gf \right) |N_h| dS &= 0. \end{aligned}$$

Hence we gain

$$\begin{aligned} \mathbf{A} &= 2\pi_Z E(f^2) - 2f^2 \pi_\nu \theta(E) = -2f^2 (E(\pi_Z) + \pi_\nu \theta(E)) \\ &= -2f^2 (\langle \nabla_E \pi(\nu_h), Z \rangle - \pi_\nu \theta(E) + \pi_\nu \theta(E)) \\ (5.15) \quad &= -2f^2 \langle (d\pi)_{\nu_h}(\nabla_E \nu_h), Z \rangle = -2f^2 \frac{\theta(E)}{\kappa(\pi(\nu_h))}, \end{aligned}$$

a.e. in  $S$ , where the last inequality follows from Lemma 4.3 in [29]. Finally, by (5.19) in Lemma 5.6 and (5.12), (5.14) and (5.15) we obtain

$$(5.16) \quad \int_S \mathbf{I} + \mathbf{II} = \int_S \mathbf{A} + \mathbf{II} = \int_S \left( Z(f)^2 + 4 \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) - \frac{\langle N, T \rangle^2}{|N_h|^2} \right) f^2 \right) \frac{|N_h|}{\kappa(\pi(\nu_h))} dS.$$

In the last equation we use that  $Z(\pi(\nu_h)) = 0$ , therefore  $\kappa(\pi(\nu_h))$  is constant along the characteristic curves.  $\square$

**Lemma 5.4.** *Following the previous notation we have*

$$\left\langle \frac{\nabla}{ds} \frac{\nabla}{ds} \Big|_{s=0} \pi(V(s)), V(s) \right\rangle = - \left\langle \frac{\nabla}{ds} \Big|_{s=0} \pi(V(s)), Z \right\rangle \left\langle \frac{\nabla}{ds} \Big|_{s=0} V(s), Z \right\rangle.$$

*Proof.* By Remark 3.3 in [34] we have  $\left\langle \frac{\nabla}{ds} \pi(V(s)), V(s) \right\rangle = 0$ . Then

$$\frac{\nabla}{ds} \pi(V(s)) = f(s) J \left( \frac{V(s)}{|V(s)|} \right),$$

where  $f(s) = \left\langle \frac{\nabla}{ds} \pi(V(s)), J \left( \frac{V(s)}{|V(s)|} \right) \right\rangle$ . Since

$$\frac{\nabla}{ds} \frac{\nabla}{ds} \pi(V(s)) = \frac{d}{ds} f(s) J \left( \frac{V(s)}{|V(s)|} \right) + f(s) \frac{\nabla}{ds} J \left( \frac{V(s)}{|V(s)|} \right)$$

and  $\nabla J = 0$  we obtain

$$\left\langle \frac{\nabla}{ds} \frac{\nabla}{ds} \pi(V(s)), V(s) \right\rangle = f(s) \left\langle \frac{\nabla}{ds} J \left( \frac{V(s)}{|V(s)|} \right), V(s) \right\rangle = -f(s) \left\langle \frac{\nabla}{ds} \left( \frac{V(s)}{|V(s)|} \right), J(V(s)) \right\rangle.$$

Evaluating at  $s = 0$  we gain

$$\left\langle \frac{\nabla}{ds} \frac{\nabla}{ds} \Big|_{s=0} \pi(V(s)), V(s) \right\rangle = - \left\langle \frac{\nabla}{ds} \Big|_{s=0} \pi(V(s)), Z \right\rangle \left\langle \frac{\nabla}{ds} \Big|_{s=0} V(s), Z \right\rangle,$$

since  $V(0) = |N_h| \nu_h$ .  $\square$

**Remark 5.5.** Letting

$$\pi(V(s)) = \pi_1(V(s)) X_{\gamma(s)} + \pi_2(V(s)) Y_{\gamma(s)},$$

and noticing that  $\nabla X = \nabla Y = 0$  we gain

$$\frac{\nabla}{ds} \Big|_{s=0} \pi(V(s)) = \frac{d}{ds} \Big|_{s=0} \pi_1(V(s)) X_{\gamma(0)} + \frac{d}{ds} \Big|_{s=0} \pi_2(V(s)) Y_{\gamma(0)}.$$

Setting  $\nu_h = aX + bY$  we obtain

$$(5.17) \quad \frac{\nabla}{ds} \Big|_{s=0} \pi(V(s)) = (d\pi)_{(a,b)} \left( \frac{\nabla}{ds} \Big|_{s=0} \frac{V(s)}{|V(s)|} \right)$$

where

$$(d\pi)_{(a,b)} = \begin{pmatrix} \frac{\partial \pi_1}{\partial a}(a,b) & \frac{\partial \pi_1}{\partial b}(a,b) \\ \frac{\partial \pi_2}{\partial a}(a,b) & \frac{\partial \pi_2}{\partial b}(a,b) \end{pmatrix}.$$

By Corollary 1.7.3 in [40]  $\pi_K = \nabla h$  where  $h$  is a  $C^2$  function, thus by Schwarz's theorem the Hessian  $\text{Hess}_{(a,b)}(h) = (d\pi)_{(a,b)}$  is symmetric.

**Lemma 5.6.** *Let  $S \subset \mathbb{H}^1$  be a  $(X, Y)$ -Lipschitz surface. Let  $h$  be a compactly supported function on  $S$ , differentiable in the  $Z$ -direction. Then we have*

$$(5.18) \quad \int_S \left( Z(h) - 2 \frac{\langle N, T \rangle}{|N_h|} h \right) |N_h| dS = 0$$

and

$$(5.19) \quad \theta(E) = -|N_h| Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) + 2|N_h| \frac{\langle N, T \rangle^2}{|N_h|^2}$$

a.e. in  $S$ , where  $\theta(E) = \langle \nabla_E \nu_h, Z \rangle$ .

*Proof.* Following Proposition 1.20 in [18, ] or Remark 6.1 [23] we approximate the  $(X, Y)$ -Lipschitz surface  $S = \{p \in \mathbb{H}^1 : f(p) = 0\}$  by a family of smooth surfaces  $S_j = \{p \in \mathbb{H}^1 : f_j(p) = 0\}$ , where  $f_j = \rho_j * f$  and  $\rho_j$  are the standard Friedrichs' mollifiers, that converges to  $S$  on compact subsets of  $S$ . Let  $Z^j$ ,  $N^j$  and  $E^j$  relative to  $S_j$ . Then we have

$$\text{div}(|N_h^j| h Z^j) = |N_h^j| Z^j(h) - \langle N^j, T \rangle h \left( -\frac{Z^j(|N_h^j|)}{\langle N^j, T \rangle} + |N_h^j| \theta(E^j) + 2|N_h^j|^2 \right).$$

Using  $-|N_h^j|^{-1} Z^j(\langle N^j, T \rangle) = \langle N^j, T \rangle^{-1} Z^j(|N_h^j|)$ ,  $|N_h^j|^{-1} Z^j(\langle N^j, T \rangle) - 2\langle N^j, T \rangle^2 + |N_h^j| \theta(E^j) = 0$  and the divergence theorem we get

$$\int_{S_j} |N_h^j| Z^j(h) - 2\langle N^j, T \rangle h dS_j = 0.$$

Then, passing to the limit when  $j \rightarrow +\infty$  we obtain (5.18). Since  $S_j$  are smooth a straightforward computation shows that

$$\theta(E^j) = -|N_h^j| Z^j \left( \frac{\langle N^j, T \rangle}{|N_h^j|} \right) + 2|N_h^j| \frac{\langle N^j, T \rangle^2}{|N_h^j|^2}.$$

Passing to the limit when  $j \rightarrow +\infty$  we have  $E^j \rightarrow E$  a.e. in  $S$ , since  $S$  is Euclidean Lipschitz. Therefore we obtain that (5.19) holds a.e. in  $S$ .  $\square$

**Lemma 5.7.** *Let  $S \subset \mathbb{H}^1$  be an area-stationary  $(X, Y)$ -Lipschitz surface. Let  $h$  be a compactly supported function in  $S$ , differentiable in the  $Z$  direction then we have*

$$(5.20) \quad \int_S \left( Z(h) - 2 \frac{\langle N, T \rangle}{|N_h|} h \right) \|N_h\|_* dS = 0.$$

Moreover, there holds

$$(5.21) \quad \int_S \pi_\nu E(h) + \pi_Z \theta(E) h = 0.$$

*Proof.* Let  $\pi_\nu = \langle \pi(\nu_h), \nu_h \rangle = \|\nu_h\|_*$ , then  $\|N_h\|_* = |N_h|\pi_\nu$ . Since  $S$  is an area-stationary surface, Theorem 3.7 implies that  $\nu_h$  is constant in the  $Z$  direction, thus in particular we have  $Z(\pi_\nu) = 0$ . Therefore, applying the same divergent argument of the proof of Lemma 5.6 we obtain

$$\int_S \left( Z(h) - 2 \frac{\langle N, T \rangle}{|N_h|} h \right) \|N_h\|_* dS = 0.$$

Always following [18, Proposition 1.20] or [23, Remark 6.1] we approximate the  $(X, Y)$ -Lipschitz surface  $S = \{p \in \mathbb{H}^1 : f(p) = 0\}$  by a family of smooth surfaces  $S_j = \{p \in \mathbb{H}^1 : f_j(p) = 0\}$ , where  $f_j = \rho_j * f$  and  $\rho_j$  are the standard Friedrichs' mollifiers, that converges to  $S$  on compact subsets of  $S$ . Let  $Z^j$ ,  $N^j$  and  $E^j$  relative to  $S_j$ . Using [34, Remark 3.3] it is easy to prove that  $E(\pi_{\nu^j}) = \pi_{Z^j} \theta(E^j)$ . Thus, by Proposition 2.6 we gain

$$(5.22) \quad \int_{S_j} \pi_{\nu^j} E^j(h) + \pi_{Z^j} \theta(E^j) h \, dS_j = - \int_{S_j} \langle N^j, T \rangle \theta(Z^j) h \, dS_j.$$

Since  $S$  is area-stationary, we get  $H_K = 0$  and Proposition 4.2 in [29] implies  $H_D = \langle \nabla_Z \nu_h, Z \rangle = 0$ . Then  $\theta(Z^j) = \langle \nabla_{Z^j} \nu_h^j, Z^j \rangle \rightarrow \langle \nabla_Z \nu_h, Z \rangle = 0$  and, passing to the limit in (5.22) when  $j \rightarrow +\infty$ , we obtain (5.21).  $\square$

## 6. THE BERNSTEIN'S PROBLEM FOR $(X, Y)$ -LIPSCHITZ SURFACES

**Definition 6.1.** We say that a complete oriented area-stationary  $(X, Y)$ -Lipschitz surface  $S \subset \mathbb{H}^1$  is stable if inequality

$$(6.1) \quad \int_S \left( Z(f)^2 + 4 \left( Z \left( \frac{\langle N, T \rangle}{|N_h|} \right) - \frac{\langle N, T \rangle^2}{|N_h|^2} \right) f^2 \right) \frac{|N_h|}{\kappa(\pi(\nu_h))} dS \geq 0$$

holds for any continuous function  $f$  on  $S$  with compact support such that  $Z(f)$  exists and is continuous.

The following lemma is proven in [2, page 45].

**Lemma 6.2.** Let  $A, B \in \mathbb{R}$  be such that  $A^2 \leq 2B$  and set  $h(s) := 1 + As + Bs^2/2$ . If

$$\int_{\mathbb{R}} \phi'(s)^2 h(s) ds \geq (2B - A^2) \int_{\mathbb{R}} \phi(s)^2 \frac{1}{h(s)} ds$$

for each  $\phi \in C_0^1(\mathbb{R})$  then  $2B = A^2$ .

**Theorem 6.3** (Bernstein's theorem). Let  $K \in C_+^2$  be a convex body with  $0 \in \text{int}(K)$ . Let  $S \subset \mathbb{H}^1$  be a complete, connected and stable  $(X, Y)$ -Lipschitz surface. Then  $S$  is a vertical plane.

*Proof.* First of all we have that  $S$  is an  $\mathbb{H}$ -regular surface by Theorem 3.7. Let  $p$  in  $S$ . Since  $S$  is  $(X, Y)$ -Lipschitz, by Theorem 2.3, there exist an open ball  $B_r(p)$  and a Lipschitz function  $u : D \rightarrow \mathbb{R}$  such that  $S \cap B_r(p) = \text{Gr}(u)$  where  $\text{Gr}(u) = \{(x, u(x, y), t - xu(x, t)) \in \mathbb{H}^1 : (x, t) \in D\}$ . Let  $(0, 0) \in D$  be the projection of  $p$  to the  $xt$ -plane. On  $D$  we consider the coordinates around  $(0, 0)$  furnished by  $G(s, \varepsilon)$  defined in Lemma 3.9. Let  $I$  be a small interval containing 0, then  $\varepsilon \in I$  and  $s \in ]-r, r[$ . Since  $S$  is complete by the Hopf-Rinow Theorem each geodesic (in particular the straight lines in the  $Z$ -direction) can be indefinitely extended along any direction, thus the open interval  $] -r, r[$  extend to  $\mathbb{R}$ . Notice that  $\bar{\gamma}_\varepsilon(s)$  is the integral curve of  $Z$ , thus  $Z(f) = \partial_s(f)$ . Hence, taking into account that  $(u_x + 2uu_t)(s)$  is constant along  $\bar{\gamma}_\varepsilon$  and equal to  $b(\varepsilon)$ , the stability condition (6.1) is equivalent to

$$(6.2) \quad \int_I \int_{\mathbb{R}} \left( (\partial_s f)^2 - 4 \left( \frac{\langle N, T \rangle^2}{|N_h|^2} - \partial_s \left( \frac{\langle N, T \rangle}{|N_h|} \right) \right) f^2 \right) \frac{\partial t_\varepsilon}{\partial \varepsilon} \frac{\sqrt{1 + b(\varepsilon)^2}}{\kappa(\pi(\nu_h))} ds d\varepsilon \geq 0,$$

for any continuous function  $f$  on  $S$  with compact support such that  $Z(f)$  exists and is continuous.

Since  $\langle N, T \rangle / |N_h|$  solves the equation (4.1) with initial condition  $y(0) = a'(\varepsilon)/2$  and  $y'(0) = a'(\varepsilon)^2/2 - b'(\varepsilon)$ , by (4.3) we get

$$\frac{\langle N, T \rangle^2}{|N_h|^2} - \left( \frac{\langle N, T \rangle}{|N_h|} \right)' = \frac{b'(\varepsilon) - \frac{a'(\varepsilon)^2}{4}}{(1 - a'(\varepsilon)s + b'(\varepsilon)s^2)^2}.$$

Therefore, computing  $\partial t_\varepsilon / \partial \varepsilon$  from (4.6), we obtain that (6.2) is equivalent to

$$\int_I \int_{\mathbb{R}} \left( (1 - a'(\varepsilon)s + b'(\varepsilon)s^2)(\partial_s f)^2 - \frac{4b'(\varepsilon) - a'(\varepsilon)^2}{(1 - a'(\varepsilon)s + b'(\varepsilon)s^2)} f^2 \right) \frac{\sqrt{1 + b(\varepsilon)^2}}{\kappa(\pi(\nu_h))} ds d\varepsilon \geq 0.$$

Let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  be a positive function compactly supported in  $\mathbb{R}$  and for  $\rho > 0$  we consider the family  $\eta_\rho(x) = \rho^{-1}\eta(x/\rho)$ , that weakly converge to the Dirac delta distribution. Putting the test functions  $\eta_\rho(x - \varepsilon)\psi(s)$ , where  $\psi \in C_0^1(\mathbb{R})$ , in the previous equation and letting  $\rho \rightarrow 0$  we get

$$\int_{\mathbb{R}} (1 - a'(\varepsilon)s + b'(\varepsilon)s^2)(\psi'(s))^2 ds \geq (4b'(\varepsilon) - a'(\varepsilon)^2) \int_{\mathbb{R}} \frac{\psi(s)^2}{(1 - a'(\varepsilon)s + b'(\varepsilon)s^2)} ds,$$

for a.e.  $\varepsilon$  since  $\kappa(\pi(\nu_h))$  is a positive constant along the horizontal straight lines for each  $\varepsilon$  (since  $\nu_h$  is constant along such horizontal straight lines) and  $\sqrt{1 + b(\varepsilon)^2}$  is a positive constant on  $\bar{\gamma}_\varepsilon$ .

Setting  $A = -a'(\varepsilon)$ ,  $B = 2b'(\varepsilon)$  and  $h(s) := 1 + As + Bs^2/2$ , we obtain

$$\int_{\mathbb{R}} h(s)\psi'(s)^2 ds \geq (2B - A^2) \int_{\mathbb{R}} \frac{\psi^2(s)}{h(s)} ds$$

for each  $\psi \in C_0^1(\mathbb{R})$ . Assume that  $2B - A^2 \geq 0$  then by Lemma 6.2 we get that  $2B = A^2$ , then  $4b'(\varepsilon) - a'(\varepsilon)^2 = 0$ . Therefore by Lemma 4.1 we obtain  $\langle N, T \rangle \equiv 0$ ,  $a'(\varepsilon) = b'(\varepsilon) = 0$  a.e. in  $\varepsilon$ . On the other hand, if  $2B - A^2 < 0$  then directly by Lemma 4.1 we obtain  $\langle N, T \rangle \equiv 0$ ,  $a'(\varepsilon) = b'(\varepsilon) = 0$  a.e. in  $\varepsilon$ . Hence  $a(\varepsilon)$  and  $b(\varepsilon)$  are constant functions in  $\varepsilon$  and

$$t_\varepsilon(s) = \varepsilon + as + bs^2,$$

for some constant  $a, b \in \mathbb{R}$ . Since  $t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)) = 2\tilde{u}(s, \varepsilon)$  we get  $\tilde{u}(s, \varepsilon) = a/2 + bs$ , thus  $\tilde{u}$  is an affine function. Hence  $S$  is locally a strip contained in a vertical plane. A standard connectedness argument implies that each connected component of  $S$  is a vertical plane.  $\square$

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