Nonlinear Anal. RWA 79 (2024) 104121

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Self-similar solutions for the heat equation with a positive non-Lipschitz continuous, semilinear source term

A. Farina *, R. Gianni

Università degli Studi di Firenze, Dipartimento di Matematica e Informatica "U. Dini", Firenze, Italy

ARTICLE INFO

Keywords:
Semilinear heat equation
Nonlinear ODE
Non-uniqueness
Self-similar solutions

ABSTRACT

We investigate the existence of self-similar solutions for the parabolic equation $u_t = \Delta u + u^m H(u)$, with $0 \le m < 1$ and H the Heaviside graph, coupled with the initial datum $u(\boldsymbol{x}, 0) = -c\left(|\boldsymbol{x}|^2\right)^{\frac{1}{1-m}}$, with $c > 0$. We analyze two cases: the problem in $\mathbb{R}^n$, $n > 1$, with $m = 0$ and the problem in $\mathbb{R}$ when $0 \le m < 1$. In the first case we extend the result of Gianni and Hulshof (1992) and show that there exist only two self-similar solutions changing sign, provided $0 < c < c_{cr}$, with c_{cr} obtained solving a specific algebraic equation depending on n. In the second case we prove that there exist at least two self-similar solutions of problem $u_t = u_{xx} + u^m H(u)$, $u(x, 0) = -c\left(x^2\right)^{\frac{1}{1-m}}$, changing sign and evolving region where $u > 0$. These solutions are of great interest. Indeed, on one hand they prove that the problem does not admit uniqueness and on the other they prove that a single point where $u(x, 0) = 0$, for an initial datum which is otherwise negative, can generate a region where $u(x, t)$ is positive.

1. Introduction

We consider the initial value problem

$$\begin{cases} u_t = \Delta u + u^m H(u), & \boldsymbol{x} \in \mathbb{R}^n, \quad 0 < t, \\ u(\boldsymbol{x}, 0) = -c|\boldsymbol{x}|^{\frac{2}{1-m}}, & \boldsymbol{x} \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $c > 0$, $m \in [0, 1)$, and where H is the Heaviside graph

$$\begin{cases} H(\xi) = 1, & \text{if } \xi > 0, \\ H(\xi) \in [0, 1], & \text{if } \xi = 0, \\ H(\xi) = 0, & \text{if } \xi < 0. \end{cases}$$

In $\mathbb{R}^3$ Eq. $(1)_1$ arises, when $m = 0$, in simple models for combustion in porous media and in other physical problems. For more references on the physical background we refer to [1–3]. Problem (1) has some similarity with the Fujita equation [4], which occurs in problems of nonlinear heat transfer and combustion theory.

The main feature of problem (1) is the presence of the non-Lipschitz continuous semilinear term $u^m H(u)$ which gives rise to non-standard behavior of the solution. In particular, the Heaviside graph does not appear in the Fujita equation and the exponent m in the Fujita model is larger than 1.

* Corresponding author.
E-mail addresses: angiolo.farina@unifi.it (A. Farina), roberto.gianni@unifi.it (R. Gianni).

https://doi.org/10.1016/j.nonrwa.2024.104121
Received 1 June 2023; Received in revised form 18 March 2024; Accepted 20 March 2024
Available online 27 March 2024

On the other hand if $m > 1$ the semilinear source term, $u^m H(u)$, is Lipschitz continuous which immediately implies uniqueness for Eq. $(1)_1$.

Referring to physical models, our study is motivated by the question as to whether a reaction diffusion process modeled by (1) can be switched on if, at $t = 0$, the initial datum $u(\boldsymbol{x}, 0)$ is everywhere negative (and therefore the source term initially vanishes), except at one single point, $\boldsymbol{x} = 0$, where its value is zero. Indeed, the essential feature of $(1)_1$ is that the reaction term (i.e. the source term) is switched on for $u > 0$. From this point of view, solutions of (1) with change sign are of particular interest.

In this paper we show that for any $0 < c < c_{cr}$, where c_{cr} depends on m and on the dimension of the space, problem (1) admits, besides the "classical" solution of the heat equation (always negative), two self-similar solutions changing sign even if the initial datum is always negative except at the point $\boldsymbol{x} = 0$.

The mathematical results obtained in this paper are, in our opinion, of some interest. Indeed, they allow to investigate the qualitative behavior of the non-unique solutions to problem (1). In this regard it is worthwhile mentioning that problem (1), in $\mathbb{R}$ and under different initial data, does indeed satisfy a uniqueness theorem. For example this is true if $u(x, 0) \in C^1(\mathbb{R})$, $u(x, 0)\, sign\left(x - x_0\right) > 0$, $\forall\, x \neq x_0$, $u_x(x, 0) > 0$ in a neighborhood of x_0 (see [5,6]).

The question of uniqueness of solutions to problem similar to (1), i.e. to the Fujita equation, has been addressed in [7], where the authors illustrate some examples of non-uniqueness of positive solutions (not of self-similar type). Still in the context of the Fujita equation we refer the readers to [8] and to the numerous references therein for the question of uniqueness of the solution to semilinear parabolic equations.

Concerning problem (1), we conjecture that it admits infinite solutions divided into two classes: (i) the solutions connecting the "unstable" self-similar to the solution of the heat equation (with the same initial datum), (ii) the solutions connecting the "unstable" to the "stable" self-similar. In this regard the adjectives "stable/unstable" are understood in the classical sense while the word "connecting" means that for $t \to 0$, the solutions "behave" like one of the two connected self-similar solutions, while for $t \to \infty$ are asymptotic to the other. In proving this conjecture the results of this paper will be important and this fact increases their mathematical relevance. Also thanks to the results of this paper it is possible, working in $\mathbb{R}$ and considering suitable assumptions, to prove that Eq. $(1)_1$ complemented with the initial condition $u(x, 0) = \varphi(x)$, where $\varphi(x) < 0$ in $\mathbb{R}\setminus\{0\}$, $\varphi(0) = 0$, $\varphi(x) \geq -c\,|x|^{\frac{2}{1-m}}$, with $0 < c < c_{cr}$, in a neighborhood of $x = 0$, do not admit uniqueness. Indeed, besides the solution which fulfills the heat equation (always negative) there is also a solution taking positive values.

The objective and novelty of this study are twofold. First, considering $m = 0$, we investigate problem (1) in $\mathbb{R}^n$ thus extending the results of [5] whose authors analyze problem (1) in $\mathbb{R}$ with $m = 0$. In particular, we prove the existence of two and only two similarity solutions, which take both negative and positive values, in the general n-dimensional context, provided c is positive and less than a critical threshold. The second target of this paper is to investigate problem (1) in $\mathbb{R}$, considering $0 \leq m < 1$. In analyzing this case we are led to a non-linear ODE and we will proceed to its qualitative study. Indeed, the case $0 \leq m < 1$ is far more complex than the case $m = 0$ (for this reason our study is limited to the study of the one-dimensional problem). We will prove that (1) in $\mathbb{R}$ admits at least two self-similar solutions changing sign. The exception is the case $m = 0$, where, as just stated, we prove that the self-similar solutions changing sign are exactly two (thus improving the results obtained in [5]).

We believe that the results obtained in the one-dimensional case for $m \in [0, 1)$, can be extended also to the n-dimensional case and this could be the subject of a future paper.

The paper is organized as follows: in Section 2 we analyze problem (1) in $\mathbb{R}^n$ taking $m = 0$. We rewrite the equation considering spherical symmetry and look for a self-similar solution. We thus end up with a linear ODE whose analytical solution is "almost" explicit. Imposing then the initial condition $(1)_2$ we find that there exist two self-similar solutions compatible with the initial datum if c is less than a critical threshold, denoted as c_{cr}. Section 3 is devoted to the case $n = 1$ and $0 \leq m < 1$. Following the same procedure of the previous section, we obtain a non-linear ODE. We then proceed to a purely qualitative study and prove the existence of at least two self-similar solutions compatible with the initial datum. In Section 3.4 we show that there exists a solution which is positive for some (x, t) even if the negative initial datum is not such as to admit self-similar solutions. Some concluding remarks are summarized in the last Section.

2. Case $\mathbb{R}^n$ and $m = 0$

We focus on the case $m = 0$ and $\boldsymbol{x} \in \mathbb{R}^n$. Setting $r = |\boldsymbol{x}|$, we rewrite problem (1) considering spherical symmetry

$$\begin{cases} u_t = \dfrac{1}{r^{n-1}}\left(r^{n-1}u_r\right)_r + H(u), & r \geq 0, \quad 0 < t, \\ u(\boldsymbol{x}, 0) = -cr^2\,, & r \geq 0. \end{cases} \tag{2}$$

In a 1D setting, i.e. $n = 1$, problem (2) has been analyzed by Gianni and Hulshof in [5], who showed the existence a positive critical value of c, i.e. the constant appearing in the initial datum denoted as c_{cr}, such that:

- For $0 < c < c_{cr}$ there are at least two similarity solutions which take both negative and positive values, besides the negative self-similar solution,

$$u(x, t) = -c(2t + x^2) = -ct\left(2 + \left(\frac{x}{\sqrt{t}}\right)^2\right).$$

- For $c \geq c_{cr}$ the heat equation solution, e.g. $u(x,t) = -c(2t + x^2)$, is the unique solution of the problem but it is negative.

In this paper we extend the results of [5], proving the existence of only two similarity solutions, which take both negative and positive values, problem (2), for sufficiently "small" values of the constant c appearing in the initial datum $(2)_2$. This implies, from the physical point of view, that the reaction is triggered, i.e. u becomes positive, by those initial data $(2)_2$ which, although negative, are characterized by suitable values of c. As far as we know, this is a rather surprising and unexpected phenomenon whose physical consequences are not yet properly understood.

2.1. Similarity solutions to (2)

We focus on the similarity solutions to $(2)_1$ which, as we shall see, become positive for $t > 0$ and $r \in \left(0, \sqrt{\frac{2n\alpha}{1-\alpha} t}\right)$, with α defined in formula (8) below, while they are negative outside this interval. Referring to [5], we search a self-similar solution of $(2)_1$ in the form

$$u(r,t) = t^{\mu} f\left(rt^{\nu}\right), \quad \Leftrightarrow \quad \frac{u(r,t)}{t^{\mu}} = f\left(rt^{\nu}\right). \tag{3}$$

Writing $\eta = rt^{\nu}$, upon substitution in $(2)_1$, we find[1]

$$t^{\mu-1}\left(\mu f + \nu \eta f'\right) = t^{\mu+2\nu}\left(f'' + \frac{n-1}{\eta} f'\right) + H(f). \tag{4}$$

For Eq. (4) to make sense we need $\mu - 1 = 0$, $\mu + 2\nu = 0$, yielding $\mu = 1$ and $\nu = -\frac{1}{2}$, namely

$$\eta = \frac{r}{\sqrt{t}}, \quad \text{and} \quad u(r,t) = t f\left(\frac{r}{\sqrt{t}}\right), \tag{5}$$

so that (4) acquires this form

$$f'' + \frac{\eta}{2} f' + \frac{n-1}{\eta} f' - f = -H(f), \tag{6}$$

which, when $n = 1$, becomes (see also [5])

$$f'' + \frac{\eta}{2} f' - f = -H(f). \tag{7}$$

Because of symmetry, we solve (6) for $\eta > 0$. As for the "initial" conditions to be coupled to (7), we take

$$f'(0) = 0, \quad f(0) = \alpha > 0. \tag{8}$$

The first one is due to the symmetry of problem (1). The second one is motivated by the fact that we look for a solution u which is positive in a neighborhood of $x = 0$.

In this Section we show that:

(a) System (6), (8) uniquely determine $f(\eta)$, for $\eta > 0$. The function $f(\eta)$ is positive only in an interval $[0, L)$, with L that depending on[2] α and f is decreasing for $\eta > 0$.

(b) There exists $\lim_{t\to 0^+}\left(-\frac{u(r,t)}{r^2}\right) = \lim_{\eta\to+\infty}\left(-\frac{f(\eta)}{\eta^2}\right)$, and it defines a smooth positive function of $\alpha \in [0,1)$, denoted as[3] $F(\alpha)$, which has an isolated maximum and tends to 0 as $\alpha \to 0$, or $\alpha \to 1$. This means that the initial condition $(2)_2$ can be fulfilled only if c is less than the maximum of F. Not only that, but for $c < c_{cr}$, with

$$c_{cr} = F(\tilde{\alpha}) = \max_{\alpha\in(0,1)} F(\alpha). \tag{9}$$

the equation $F(\alpha) = c$ admits two solutions and therefore there exist two self-similar solutions changing sign.

The proof of point **(a)** can be found in [5] when $n = 1$. Here we consider the more general case $n \geq 1$. We look for solution to (6) in the form $f(\eta) = a\eta^2 + b\eta + c$, positive in a neighborhood of $\eta = 0$. Upon substitution in (6) where $H = 1$, we find

$$f = a\eta^2 + 2an + 1,$$

which fulfills the initial conditions (8) provided $a = \dfrac{\alpha - 1}{2n}$, namely

$$f(\eta) = \alpha - \frac{1-\alpha}{2n}\eta^2, \tag{10}$$

---

[1] $(\,\cdot\,)'$ denotes the differentiation with respect to η.

[2] See formula (14) in the sequel.

[3] See formula (22) in the sequel.

which, in case $n = 1$, becomes

$$f(\eta) = \alpha - \frac{1-\alpha}{2}\eta^2 , \tag{11}$$

i.e. formula (A.2) of [5]. We have that $f > 0$ as long as

$$0 \le \eta < L(\alpha), \tag{12}$$

where

$$L(\alpha) = \sqrt{\frac{2n\alpha}{1-\alpha}}, \quad \alpha \in [0,1), \tag{13}$$

reducing to (A.3) of [5] when $n = 1$, namely

$$L(\alpha) = \sqrt{\frac{2\alpha}{1-\alpha}}, \quad \alpha \in [0,1). \tag{14}$$

We remark that condition (13) reported to the (r,t) plane, means that u is positive when $0 \le r < L\sqrt{t}$. Indeed, $\eta = L$, in the (r,t) plane, corresponds to the curve

$$r = \zeta(t), \quad \text{with} \quad \zeta(t) = L(\alpha)\sqrt{t} = \sqrt{\frac{2n\alpha}{1-\alpha}t},$$

where f vanishes. We further remark that on the curve $\zeta(t)$ we have $f = 0$, and (6) is still fulfilled.

We introduce

$$f'(\eta)\big|_{\eta=L} = -\delta(\alpha), \quad \text{with} \quad \delta(\alpha) = \sqrt{\frac{2\alpha(1-\alpha)}{n}}, \tag{15}$$

and note that $L(\alpha)$ and $\delta(\alpha)$, defined for $\alpha \in [0,1)$, have the following properties

$$\begin{array}{lll} L(\alpha) \text{ continuous and monotone increasing,} & L(\alpha) \underset{\alpha\to 0}{\longrightarrow} 0, & L(\alpha) \underset{\alpha\to 1}{\longrightarrow} \infty, \\ \delta(\alpha) \text{ continuous and } \|\delta\|_{C^0([0,1])} \text{ bounded,} & \delta(\alpha) \underset{\alpha\to 0}{\longrightarrow} 0. & \end{array} \tag{16}$$

If $\eta > L$, f given by (10) is no more solution to (7) since $f < 0$. So, the Cauchy problem (6), (8) has to be replaced with

$$\begin{cases} f'' + \dfrac{\eta}{2}f' + \dfrac{n-1}{\eta}f' - f = 0, & \eta > L, \\ f(L) = 0, \quad f'(L) = -\delta , \end{cases} \tag{17}$$

with L and δ given by (14) and (15), respectively. Following [5], we look for a solution to (17) in the form

$$f(\eta) = \left(\eta^2 + 2n\right) z(\eta),$$

so that $(17)_1$ reduce to

$$z''\left(\eta^2 + 2n\right) + z'\left[4\eta + \left(\eta^2 + 2n\right)\left(\frac{\eta}{2} + \frac{n-1}{\eta}\right)\right] = 0,$$

namely

$$\frac{z''}{z'} = -\left(\frac{4\eta}{\eta^2 + 2n} + \frac{\eta}{2} + \frac{n-1}{\eta}\right),$$

and so

$$z = A\int_L^\eta \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2}ds + B, \quad \Rightarrow \quad f(\eta) = \left(\eta^2 + 2n\right)\left[A\int_L^\eta \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2}ds + B\right].$$

Imposing the boundary conditions $(17)_2$ we obtain

$$B = 0,$$

$$A = -\underbrace{\sqrt{\frac{2\alpha(1-\alpha)}{n}}}_{\delta(\alpha)} e^{L^2/4} L^{n-1}\left(L^2 + 2n\right),$$

yielding

$$f(\eta) = -\delta(\alpha)\left(\eta^2 + 2n\right) w(\eta), \tag{18}$$

where

$$w(\eta) = \left(L^2 + 2n\right) L^{n-1} \exp\left\{\frac{L^2}{4}\right\} \int_L^\eta \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2}ds.$$

We notice that $f < 0$ for $\eta > L$, and that both conditions at $\eta = L$, i.e. (17)$_2$, are fulfilled.

Rewriting (18) more explicitly we have

$$f(\eta) = -\delta\left(\eta^2 + 2n\right)\left(L^2 + 2n\right) L^{n-1} \exp\left\{\frac{L^2}{4}\right\} \int_L^\eta \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2} ds. \tag{19}$$

If $n = 1$, the integral on the r.h.s. can be explicitly computed and we have

$$f(\eta) = -\frac{\delta}{8}\left(\eta^2 + 2\right)\left(L^2 + 2\right) \exp\left\{\frac{L^2}{4}\right\} \left[\sqrt{\pi}\,\mathrm{erf}\left(\frac{s}{2}\right) + \frac{2s}{s^2+2}\exp\left\{-\frac{s^2}{4}\right\}\right]_L^\eta. \tag{20}$$

We now focus on point (**b**). Using (19) we write $u(r,t) = tf\left(r/\sqrt{t}\right)$

$$u(r,t) = -t\delta\left(\frac{r^2}{t} + 2n\right)\left(L^2 + 2n\right) L^{n-1} \exp\left\{\frac{L^2}{4}\right\} \int_L^{r/\sqrt{t}} \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2} ds,$$

and define

$$F(\alpha) = -\lim_{t\to 0}\frac{u(r,t)}{r^2} = \delta(\alpha)\left(L^2(\alpha) + 2n\right) L^{n-1}(\alpha) \exp\left\{\frac{L^2(\alpha)}{4}\right\} \int_{L(\alpha)}^\infty \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2} ds, \tag{21}$$

which, exploiting (14) and (15), assumes this form

$$F(L(\alpha)) = 2L^n(\alpha) \exp\left\{\frac{L^2(\alpha)}{4}\right\} \int_{L(\alpha)}^\infty \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2} ds, \qquad \alpha \in [0,1). \tag{22}$$

In order to prove the properties of F listed in point (**b**), it is convenient to consider F as function of L. Hence, exploiting (14), we have

$$F(L) = 2L^n \exp\left\{\frac{L^2}{4}\right\} \int_L^\infty \frac{e^{-s^2/4}}{s^{n-1}\left(s^2+2n\right)^2} ds, \qquad L \in [0,\infty). \tag{23}$$

It can be easily realized that $F > 0$, for $L > 0$, and that $\lim_{L\to 0} F(0) = 0$ and, using de l'Hôpital, $\lim_{L\to\infty} F(L) = 0$.

We consider

$$g(L) = \frac{4L^2}{\left(L^2+2n\right)^3}, \qquad L \geq 0,$$

which is a positive function and has an isolated maximum at $L = \sqrt{n}$, where $g\left(\sqrt{n}\right) = 4/27n^2$. Moreover $g \to 0$ as $L \to \infty$, $g(0) = g'(0) = 0$ and $g''(0) = 1/n^3$. We then observe that $\forall\, n$ we have $F(L) > g(L)$, for $L \in (0,\sigma)$, provided σ is sufficiently small (this fact follows from Taylor expansions of $(F-g)$ around $L = 0$). So, taking $L_o \in (0,\sigma)$, $F(L)$ fulfills this Cauchy problem

$$\begin{cases} F' = \dfrac{L^2+2n}{2L}\left(F - \dfrac{4L^2}{\left(L^2+2n\right)^3}\right) = \dfrac{L^2+2n}{2L}(F-g), \\ F|_{L_o} = F\left(L_o\right) \gneqq g\left(L_o\right). \end{cases} \tag{24}$$

We remark that, if $F(L) \gneqq g(L)\ \forall\ L \geq L_o$, then we would have $F'(L) > 0$, $\forall\ L \geq L_o$ and that would contradict $\lim_{L\to\infty} F(L) = 0$. Hence there exists $\bar{L}$ such that $F\left(\bar{L}\right) = g\left(\bar{L}\right)$ (and therefore $F'\left(\bar{L}\right) = 0$), and $g'\left(\bar{L}\right) > 0$. Since $g' < 0$ for $L > \sqrt{n}$ (recall that in $L = \sqrt{n}$, g has an isolated maximum) it must be $\bar{L} \leq \sqrt{n}$. We now prove that $\bar{L}$ is strictly less than $\sqrt{n}$, so that $g'\left(\bar{L}\right) > 0$. We proceed by assuming that $\bar{L} = \sqrt{n}$ and show that we run into a contradiction. If $\bar{L} = \sqrt{n}$, then $F\left(\sqrt{n}\right) = g\left(\sqrt{n}\right)$, $F'\left(\sqrt{n}\right) = g'\left(\sqrt{n}\right)$, because (24)$_1$ and $F''\left(\sqrt{n}\right) = 0$ while $g''\left(\sqrt{n}\right) = -\dfrac{32}{81n^3}$. So taking $L \in \left(\sqrt{n}, \sqrt{n}+\varepsilon\right]$, for some sufficiently small $\varepsilon > 0$, we have $(F-g)(L) > 0$ entailing $F'(L) > 0$. In particular, if F' is positive in a neighborhood of $\sqrt{n}$, then $F' > 0$ for any $L > \sqrt{n}$, because g is monotone decreasing for $L > \sqrt{n}$. We thus get a contradiction since $\lim_{L\to\infty} F(L) = 0$.

Thus far, we have proved that there exists $0 < \bar{L} \lneqq \sqrt{n}$, such that $F\left(\bar{L}\right) = g\left(\bar{L}\right)$ and so $F'\left(\bar{L}\right) = 0$, because of (24)$_1$. For $L \in \left(\bar{L}, \sqrt{n}\right]$, F is decreasing while g is increasing and therefore $(F-g) \lneqq 0$. For $L \geq \sqrt{n}$, $(F-g)$ is definitely not positive because if $(F-g) > 0$ for some $\widetilde{L} > \sqrt{n}$, then F would be increasing $\forall\ L > \widetilde{L}$, contradicting the fact that $\lim_{L\to\infty} F(L) = 0$. We thus have proved that $F(L)$ has only one critical point (maximum) attained at $\bar{L} < \sqrt{n}$. In particular, since F is decreasing as $L > \bar{L}$, while g increases again until it reaches its peak at $L = \sqrt{n}$, we have an upper estimate for $\max_L F$

$$\max_{L\geq 0} F = F\left(\bar{L}\right) < g\left(\sqrt{n}\right) = \frac{4}{27n^2}.$$

The above properties, proved for $F(L)$, are reflected to $F(\alpha)$ since $L(\alpha)$, given by (13), is continuous and monotone increasing. Indeed, recalling (13), we define

$$\bar{\alpha} = \frac{\bar{L}^2}{\bar{L}^2 + 2n}, \tag{25}$$

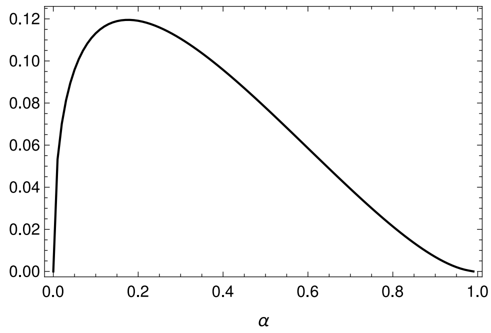

Fig. 1. Behavior of $F(\alpha)$ defined by (22) for $0 \leq \alpha < 1$ and $n = 1$.

and so, recalling (9), we obtain an upper estimate for c_{cr}

$$c_{cr} = F(\bar{\alpha}) = \max_{\alpha\in(0,1)} F(\alpha) = \max_{L\geq 0} F < \frac{4}{27n^2}.$$

Next, since $\bar{L} < \sqrt{n}$, we can estimate $\bar{\alpha}$, i.e. the value at which $F(\alpha)$ attains its maximum. Indeed, exploiting (25), we have $\bar{\alpha} < 1/3$.

We can therefore state that there exist only two similarity solutions (which take both negative and positive values) to problem (2) provided the equation $F(\alpha) = c$ has solution, i.e. provided $0 < c < c_{cr}$.

For readers' convenience, the behavior of $F(\alpha)$, for $0 \leq \alpha < 1$ and $n = 1$, is displayed in Fig. 1. In this case $\max_{\alpha\in(0,1)} F(\alpha) \approx 0.12 < 4/27$.

We emphasize that the proof of the existence of only one maximum for $F(\alpha)$ for any $n \geq 1$, enriches the results[4] of [5].

3. Case $\mathbb{R}$ and $0 \leq m < 1$

We now take $m \in [0, 1)$ and investigate the existence of self-similar solution to (1) in the one-dimensional setting, i.e.

$$\begin{cases} u_t = u_{xx} + H(u), & x \in \mathbb{R}, \quad 0 < t, \\ u(x,0) = -cx^2, & x \in \mathbb{R}. \end{cases} \tag{26}$$

In the same spirit of [5], we consider

$$u = t^{\beta} f(\eta), \quad \text{with} \quad \eta = xt^{\theta}.$$

Inserting such an ansatz in $(26)_1$ we obtain

$$\beta t^{\beta-1} f + \theta t^{\beta-1} \eta f' = t^{2\theta+\beta} f'' + t^{m\beta} f^m H(f),$$

which entails

$$\begin{cases} 2\theta + \beta = \beta - 1, \\ m\beta = \beta - 1, \end{cases} \quad \Longrightarrow \quad \begin{cases} \theta = -\dfrac{1}{2}, \\ \beta = \dfrac{1}{1-m}, \end{cases} \tag{27}$$

with $\beta \in [1, +\infty)$ for $0 \leq m < 1$.

Hence the self-similar solutions must have this form[5]

$$u(x,t) = t^{\frac{1}{1-m}} f\left(\frac{x}{\sqrt{t}}\right). \tag{28}$$

As in the previous section u is positive when $0 \leq x < L\sqrt{t}$. Indeed, $\eta = L$, in the (x,t) plane, corresponds to the curve

$$x = L\sqrt{t}, \tag{29}$$

where f vanishes. Fig. 2 displays the behavior of the curve (29) for three values of L computed when $m = 0.25$, 0.5, 0.75.

The initial condition $(26)_2$ can be rewritten as

$$u(x,0) = -cx^{2\beta}, \tag{30}$$

[4] We recall that in [5] the authors, working in $\mathbb{R}$, i.e. $n = 1$, prove the existence of at least two similarity solutions as they only prove that F is positive and that tends to 0 as $\alpha \to 0$, or $\alpha \to 1$.

[5] We note that (3) is recovered when $m = 0$.

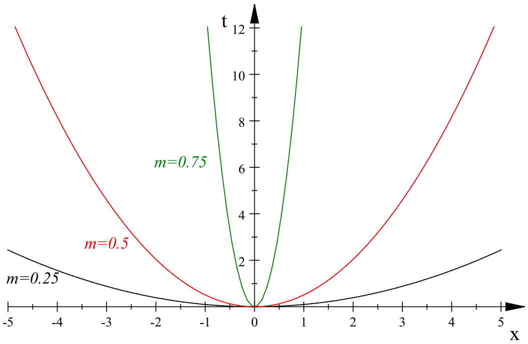

Fig. 2. Behavior of the curve (29) for three values of L computed when $m = 0.25$, 0.5, 0.75.

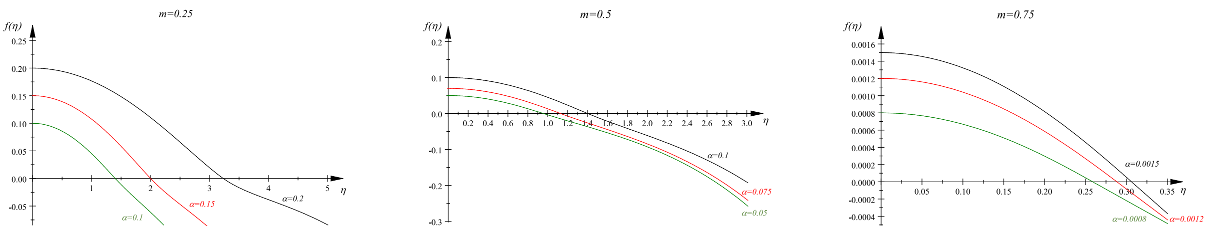

Fig. 3. Behavior of $f(\eta)$ solution to the Cauchy problem (34) when $m = 0.25$, 0.5, 0.75, for three values of $\alpha < \alpha_m$.

which is fulfilled by the similarity solutions if

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} t^{\beta} f\left(\frac{x}{\sqrt{t}}\right) = -c x^{2\beta}, \quad \Longrightarrow \quad \lim_{t\to 0}\left(\frac{\sqrt{t}}{x}\right)^{2\beta} f\left(\frac{x}{\sqrt{t}}\right) = -c. \tag{31}$$

So, following a similar path to the one illustrated in Section 2.1, we will show that there exists $\lim_{\eta\to+\infty}\left(-\frac{f(\eta)}{\eta^{2\beta}}\right)$, where, as usual, $\eta = x/\sqrt{t}$, and that it defines a smooth positive function of the parameter α, namely

$$F(\alpha) = -\lim_{\eta\to\infty}\frac{f(\eta)}{\eta^{2\beta}}, \tag{32}$$

which replaces definition (21).

Imposing condition (31) we obtain

$$\underbrace{\lim_{\eta\to\infty}\frac{f(\eta)}{\eta^{2\beta}}}_{-F(\alpha)} = -c\ , \quad \Longrightarrow \quad F(\alpha) = c. \tag{33}$$

In particular, we prove that the equation $F(\alpha) = c$ has at least two solutions for sufficiently "small" values of c, provided $0 < \alpha < \alpha_m$, with α_m defined by (35) a few lines below. We thus will show that there exist at least two similarity solutions (which take both negative and positive values) to problem (26).

Since we look for a even function, we have to solve this Cauchy problem

$$\begin{cases} f'' = \dfrac{1}{1-m} f - \dfrac{\eta}{2} f' - f^m H(f), & \eta > 0 \\ f'(0) = 0, \qquad f(0) = \alpha \in \mathbb{R}. \end{cases} \tag{34}$$

Actually, the solution of (34) depends also on the parameter α and therefore should be denoted by $f_\alpha(\eta)$. However, in order not to make the notation too heavy, we will omit to write the subscript α unless it is necessary to highlight this dependence.

This Section develops as follows. In Section 3.1 we show that, for each $\alpha \in (0, \alpha_m)$, with α_m defined by (35), there exists a unique L such that $f(L) = 0$ and that $f'(L) \lneqq 0$. We also show that both L and the derivative of f at $\eta = L$ are continuous functions of α and that L tends to 0 if $\alpha \to 0$ and that it tends to ∞ when $\alpha \to \alpha_m$. In Section 3.2 we analyze problem (34) for $\eta > L$. Section 3.3 is devoted to the existence of solutions to Eq. (33).

Fig. 3 displays $f(\eta)$ solution to the Cauchy problem (34) when $m = 0.25$, 0.5, 0.75, for three values of α, smaller than α_m.

3.1. Solution to problem (34) for $\eta < L$

We are interested in those solutions to problem (34) which change sign. Hence, resuming the notation already introduced in Section 2.1, we have

Proposition 1. *Problem (34) has a solution which changes sign provided $0 < \alpha < \alpha_m$, with*

$$\alpha_m = (1-m)^{\frac{1}{1-m}}. \tag{35}$$

In particular, there exists $L > 0$ such that $f > 0$ for $0 \le \eta < L$, $f(L) = 0$, $f'(L) < 0$, and $f < 0$ if $\eta > L$.

Moreover, f tends uniformly to 0 in any compact set, as $\alpha \to 0$.

Proof. First we will show that if $\alpha < 0$ or $\alpha \ge \alpha_m$ then the solution f to problem (34) does not change sign (and consequently the $u(x,t)$ does not change sign either).

Let us first consider the case $\alpha < 0$, so that (34) reduces to

$$\begin{cases} f'' = \dfrac{1}{1-m} f - \dfrac{\eta}{2} f', & \eta > 0 \\ f(0) = \alpha < 0, & f'(0) = 0, \end{cases} \tag{36}$$

at least for η sufficiently small. If we take $\eta \in [0, \eta_o]$, with η_o sufficiently small then $f'' < 0$ and so $f' < 0$ and $f < \alpha$ in $(0, \eta_o)$. Let us assume that $(0, \ell)$ is the maximal interval where $f' \le 0$. If $\ell < \infty$, we have $f'(\ell) = 0$. As a consequence, $f \le \alpha < 0$ in $[0, \ell]$, and so, exploiting the equation, we have $f''(\ell) < 0$, entailing that $f' \le 0$ in $\left(0, \hat{\ell}\right)$ with $\hat{\ell} > \ell$. Hence $(0, \ell)$ is not maximal and this yields $\ell = \infty$. We thus conclude that $f < \alpha < 0$, for $\eta \in (0, \infty)$ when $\alpha < 0$. Therefore, in this case f is always negative and consequently $u(x,t) < 0$, i.e. u being the solution of the heat equation.

Let us now focus on the case $\alpha \ge \alpha_m$. We begin by observing that if $\alpha > 0$, there exists some positive (possibly small) η_o, such that for $\eta \in [0, \eta_o]$ problem (34) has to be written as

$$\begin{cases} f'' = -\dfrac{\eta}{2} f' + \dfrac{1}{1-m} f - f^m, & \eta > 0, \\ f(0) = \alpha, & f'(0) = 0. \end{cases} \tag{37}$$

Next, let us consider the case $\alpha = \alpha_m$, with α_m given by (35). In this case the solution to problem (37) is $f \equiv \alpha_m$, i.e. always positive, since $\frac{1}{1-m}\alpha_m - \left(\alpha_m\right)^m = 0$.

We now consider the case $\alpha > \alpha_m$. First of all we observe that

$$\frac{1}{1-m} f - f^m = \frac{f}{1-m}\left(1 - \frac{1-m}{f^{1-m}}\right) \underset{(35)}{=} \frac{f}{1-m}\left[1 - \left(\frac{\alpha_m}{f}\right)^{1-m}\right].$$

Hence $\left(\dfrac{f}{1-m} - f^m\right) \ge 0$, if $f \ge \alpha_m = (1-m)^{\frac{1}{1-m}}$. So, if $\alpha > \alpha_m$, there exists $\eta_o > 0$, such that, if $\eta \in [0, \eta_o]$, $f''(\eta) > 0$ and so $f'(\eta) > 0$ in $(0, \eta_o)$ since $f'(0) = 0$. Hence $f > \alpha$ in $(0, \eta_o)$. Therefore, proceeding as in the case $\alpha < 0$, we prove that, if $\alpha > \alpha_m$, $f > \alpha > \alpha_m$, for $\eta \in [0, \infty)$, and so f is always positive (i.e. u does not change sign).

We now take $0 < \alpha < \alpha_m$ (which is the most delicate situation) and show that now $f(\eta)$, solution to the Cauchy problem (37), changes sign. We first show that $f' < 0$, for $\eta > 0$ as long as $f > 0$. As before, $f'' < 0$ for $\eta \in [0, \eta_o]$, for some η_o sufficiently small and so $f'(\eta) < 0$, in $(0, \eta_o]$. So, let $(0, L)$ be the maximal interval where $f' \le 0$, and $f > 0$. Three cases can occur:

Case 1.a. $L < +\infty$, with $0 < f(L) < f(0) = \alpha < \alpha_m$ and $f'(L) = 0$.

Case 1.b. $L < +\infty$, with $f(L) = 0$, and $f' \le 0$ in $[0, L]$ (we will prove that this is the only possible case).
Or $L = +\infty$, i.e.

Case 2. $f(\eta) > 0$, $f'(\eta) \le 0$ for $\eta > 0$ and consequently $\lim_{\eta\to\infty} f(\eta) = f_o \ge 0$, i.e. f does not change sign.

Let us first show that **Case 1.a** cannot occur. Indeed $0 < f(L) < \alpha < \alpha_m$ and $f'(L) = 0$, entail that $f''(L) \ge 0$ (otherwise $f''(L) < 0$ and so $f' \le 0$ and $f > 0$, beyond L, contradicting its maximality). On the other hand using $(37)_1$

$$f''(L) = \underbrace{\frac{f(L)}{1-m}\left[1 - \left(\frac{\alpha_m}{f(L)}\right)^{1-m}\right]}_{<0} - \underbrace{\frac{L}{2} f'(L)}_{=0}, \implies f''(L) < 0,$$

thus obtaining a contradiction.

In the sequel we show that also **Case 2** is impossible.

We proceed assuming that **Case 2** holds true, showing that in this way we obtain a contradiction. We first show that f' can have at most one local minimum. This fact entails two cases: (i) $\lim_{\eta\to\infty} f' = 0$; or (ii) $f' \le k < 0$, for η sufficiently large. Obviously option

(ii) trivially implies **Case 1.b**. Let us start the proof that f' has at most one local minimum. Assume that $\eta = p$ is a local minimum for $f'(p)$. We have $f(p) > 0$, $f'(p) < 0$, $f''(p) = 0$, and $f'''(p) \geq 0$. On the other hand, differentiating Eq. (37)$_1$, we have

$$f'''(p) = f'(p)\left[\frac{1}{1-m} - \frac{1}{2} - \frac{m}{f^{1-m}(p)}\right] \geq 0,$$

and so

$$\frac{1}{1-m} - \frac{1}{2} - \frac{m}{f^{1-m}(p)} \leq 0, \quad \Longrightarrow \quad f(p) \leq \underbrace{(1-m)^{\frac{1}{1-m}}}_{\alpha_m}\left(\frac{2m}{1+m}\right)^{\frac{1}{1-m}} < \alpha_m.$$

Now we suppose that at $\eta = s > p$, f' has a local maximum. Since, $f'(s) < 0$, $f''(s) = 0$, and $f'''(s) \leq 0$, by repeating the same procedure we obtain

$$f(s) \geq \alpha_m\left(\frac{2m}{1-m}\right)^{\frac{1}{1-m}},$$

which is an obvious contradiction since f is decreasing for $\eta > 0$. Hence, exploiting what we have just obtained, we conclude that, if f' is not monotone decreasing, i.e. if **Case 1.b** does not occur, then f' is monotone increasing for $\eta > p$, and, since $f' \leq 0$, we have that $\lim_{\eta\to\infty} f' = q \leq 0$. But if $q \lneqq 0$, we are again in case **Case 1.b**. Hence if **Case 1.b** does not occur we have $\lim_{\eta\to\infty} f' = 0$, and $\lim_{\eta\to\infty} f(\eta) = f_o \geq 0$, i.e. we are in **Case 2**. But we are going to prove that **Case 2** does not occur. Indeed, we have just seen that if f does not touch zero, then: (i) $\lim_{\eta\to\infty} f' = 0$; (ii) f' has only a local minimum attained at some point $\eta = p$. Hence, for $\eta \geq p$, we have $f''(\eta) \geq 0$, and so (37)$_1$ gives

$$\frac{\eta}{2}f' = \underbrace{\left(-f''\right)}_{\leq 0} + \left(\frac{1}{1-m}f - f^m\right) \leq \frac{1}{1-m}f - f^m = -\frac{f^m}{1-m}\underbrace{\left[(1-m) - f^{1-m}\right]}_{>0}, \quad \text{for} \quad \eta \geq p.$$

Recalling (35) we note that $\varphi(f) = (1-m) - f^{1-m} = \alpha_m^{1-m} - f^{1-m} > 0$, for $0 \leq f \leq \alpha < \alpha_m$, and that $\varphi(f)$ attains its minimum at $f = \alpha$, i.e. $\alpha_m^{1-m} - f^{1-m} \geq \alpha_m^{1-m} - \alpha^{1-m} > 0$, $\forall\ f \in [0, \alpha]$. Therefore, since $f = \alpha$ when $\eta = 0$ and f is decreasing, we have

$$f' \leq -\frac{2f^m}{\eta}\left[\ (1-m) - \underbrace{f^{1-m}(0)}_{\alpha^{1-m}}\right], \quad \text{for} \quad \eta \geq p,$$

namely

$$\frac{f'}{f^m} \leq -\frac{b}{\eta}, \tag{38}$$

with

$$b = 2\left(\alpha_m^{1-m} - \alpha^{1-m}\right) > 0. \tag{39}$$

Solving (38) for $\eta \geq p$, we obtain

$$f^{1-m}(\eta) \leq G(\eta),$$

with

$$G(\eta) = f^{1-m}(p) - (1-m)\,b\ln\left(\frac{\eta}{p}\right).$$

and b given by (39). But $G(\eta)$ vanishes for $\eta = \eta_p$, with

$$\eta_p = p\exp\left\{\frac{f^{1-m}(p)}{b(1-m)}\right\} \leq p\exp\left\{\frac{\alpha^{1-m}}{b(1-m)}\right\}, \tag{40}$$

and therefore also f vanishes at $\eta = L \leq \eta_p$. We thus have come to a contradiction since we were assuming to be in **Case 2**. We have therefore shown that **Case 2** is impossible and then only **Case 1.b** is feasible.

To conclude the proof of this proposition we have to show that: $f'(L) < 0$, and $f < 0$, for $\eta > L$.

So far we have shown that f reaches 0 at $\eta = L < +\infty$, i.e. that only **Case1.b** occurs. Now we prove that $f'(L) < 0$. To this purpose we proceed by assuming $f'(L) = 0$ and we show that this assumption implies a contradiction. Obviously if f' is always decreasing we have $f'(L) < 0$ and therefore the contradiction is obtained immediately. So, if f' is not always decreasing, f' has only one critical point which is a minimum (as just proved), attained at some point $p \lneqq L$. We thus consider $p < \eta < L$, with p minimum of f', that is $f'(p) < 0$ and $f''(p) \geq 0$. Hence, for $\eta \in (p, L)$, f' is increasing and therefore we will have[6] $\|f'\|_{C^0([\eta,L])} = -f'(\eta)$.

[6] Recall that $f' \leq 0$ in $(0, L)$.

Hence, exploiting (37)$_1$ for $\eta \in (p, L)$, we get[7]

$$\|f'\|_{C^0([\eta,L])} = -f'(\eta) = \underbrace{f'(L)}_{=0} - f'(\eta) = \int_\eta^L f'' ds = \frac{1}{1-m}\int_\eta^L f(s)\,ds + \int_\eta^L \frac{s}{2}(-f')\,ds - \underbrace{\left(\int_\eta^L f^m ds\right)}_{\geq 0}$$

$$\leq \frac{1}{1-m}\int_\eta^L \underbrace{\left(-\int_s^L f'(r)\,dr\right)}_{f(s)-\underbrace{f(L)}_{=0}} ds + (L-\eta)L\|f'\|_{C^0([\eta,L])} \tag{41}$$

We now remark that

$$\frac{1}{1-m}\int_\eta^L \left(-\int_s^L f'(r)\,dr\right) \leq \frac{1}{1-m}\int_\eta^L \|f'\|_{C^0([s,L])}(L-s)\,ds \leq \frac{L-\eta}{1-m}\int_\eta^L \|f'\|_{C^0([s,L])}\,ds,$$

and so (41) becomes

$$\|f'\|_{C^0([\eta,L])} \leq \frac{L-\eta}{1-m}\int_\eta^L \|f'\|_{C^0([s,L])}\,ds + (L-\eta)L\|f'\|_{C^0([\eta,L])}. \tag{42}$$

In particular, setting

$$g(\eta) = \|f'\|_{C^0([\eta,L])}, \tag{43}$$

we can write (42) as

$$\frac{g(\eta)}{L-\eta} \leq \frac{1}{1-m}\int_\eta^L g(s)\,ds + Lg(\eta), \quad \Rightarrow \quad g(\eta)\left(\frac{1}{L-\eta} - L\right) \leq \frac{1}{1-m}\int_\eta^L g(s)\,ds. \tag{44}$$

Hence, selecting $\eta > \max\left\{p, L - \frac{1}{2L}\right\} \lneqq L$, so that

$$\frac{1}{2(L-\eta)} \leq \frac{1}{L-\eta} - L,$$

(44) can be rewritten as

$$\frac{g(\eta)}{2(L-\eta)} \leq g(\eta)\left(\frac{1}{L-\eta} - L\right) \leq \frac{1}{1-m}\int_\eta^L g(s)\,ds. \tag{45}$$

We now increase the integral on the right hand side taking the maximum of $g(s)$ in the interval $[\eta, L]$. Recalling (43),

$$\max_{s\in[\eta,L]} g(s) = \max_{s\in[\eta,L]} \|f'\|_{C^0([s,L])} = \|f'\|_{C^0([\eta,L])} = g(\eta).$$

Thus, (45) becomes

$$\frac{g(\eta)}{2(L-\eta)} \leq \frac{1}{1-m}\int_\eta^L g(s)\,ds \leq \frac{(L-\eta)g(\eta)}{1-m}, \quad \Rightarrow \quad g(\eta) \leq \frac{2(L-\eta)^2}{1-m} g(\eta).$$

At this point it suffices to take η so close to L so that $\frac{2(L-\eta)^2}{1-m} < 1$, and we have that $g \equiv 0$, and so $f' \equiv 0$,thus entailing $f \equiv 0$ in (η, L), which contradicts the fact that L is the first zero of f (recall that f is decreasing). The inconsistency we have obtained is due to the fact that we hypothesized $f'(L) = 0$. Hence $f'(L) < 0$.

Let us now prove that $f(\eta) < 0$, for $\eta > L$, by proving that $f'(\eta) \leq 0$, for $\eta > L$. Let us consider the interval $[L, L+\rho]$, for some $\rho > 0$ (possibly "small"). Since f' is continuous, $f'(\eta) < 0$, for $\eta \in [L, L+\rho]$ and we have $f(\eta) < 0$, for $\eta \in (L, L+\rho]$. Hence Eq. (34)$_1$ becomes

$$f'' = \frac{1}{1-m} f - \frac{\eta}{2} f', \qquad \eta \in [L, L+\rho]. \tag{46}$$

Let (L, M) be the maximal interval where $f' \leq 0$. If $M < \infty$, we would have $f'(M) = 0$, and so, exploiting (46), we would obtain

$$f''(M) = \frac{1}{1-m} f(M) \lneqq 0,$$

contradicting the maximality of (L, M). Note that in (L, M) f is negative because of the previous considerations and therefore (46) holds true. We deduce that $f' \leq 0$ for any $\eta > L$ and so $f < 0$ for $\eta > L$.

As a last thing, we consider case $\alpha \to 0$ and show that[8] f_α tends uniformly to 0 in any compact set. We multiply (34)$_1$ by f'_α and, integrating in $[0, \eta]$, we get

$$\frac{f_\alpha'^{\,2}}{2} + \int_0^\eta \frac{s}{2} f_\alpha'^{\,2} ds + \frac{f_\alpha^{m+1}}{m+1} H(f) = \frac{f_\alpha^2}{2(1-m)} - \frac{\alpha^2}{2(1-m)} + \frac{\alpha^{m+1}}{m+1} \leq \frac{f_\alpha^2}{2(1-m)} + \frac{\alpha^{m+1}}{m+1},$$

[7] Recall we are assuming $f'(L) = 0$.

[8] In this part of the proof we use the notation f_α to explicitly indicate the dependence on α.

namely

$$\frac{f_\alpha'^{\,2}}{2} \leq \frac{f_\alpha^2}{2(1-m)} + \frac{\alpha^{m+1}}{m+1} \quad \Rightarrow \quad f_\alpha'^{\,2}(\eta) = \frac{2\alpha^{m+1}}{m+1} + \frac{1}{(1-m)} \underbrace{\left(\int_0^\eta f_a'(s)\,ds + \alpha\right)^2}_{f_\alpha^2(\eta)}.$$

In particular, we still increase the r.h.s. increasing the bracket

$$\begin{aligned} f_\alpha'^{\,2} &\leq \frac{2\alpha^{m+1}}{m+1} + \frac{1}{(1-m)}\left[2\alpha^2 + 2\left(\int_0^\eta f_\alpha'(s)\,ds\right)^2\right] \\ &= \underbrace{\frac{2\alpha^{m+1}}{m+1} + \frac{2\alpha^2}{(1-m)}}_{C(\alpha)} + \frac{2}{(1-m)}\left(\int_0^\eta f_\alpha'(s)\,ds\right)^2. \end{aligned} \tag{47}$$

We thus end up with

$$f_\alpha'^{\,2} \leq C(\alpha) + \frac{2}{(1-m)}\left(\int_0^\eta f_\alpha'(s)\,ds\right)^2,$$

and apply the Cauchy–Schwarz's inequality

$$f_\alpha'^{\,2} \leq C(\alpha) + \frac{2\eta}{(1-m)}\int_0^\eta f_\alpha'^{\,2}(s)\,ds,$$

and then the Gronwall inequality until we obtain

$$\left(f_\alpha'\right)^2 \leq C(\alpha)\exp\left\{\frac{\eta^2}{(1-m)}\right\},$$

with $C(\alpha)$ defined in the last line of (47). Hence, given a generic $0<\Omega<\infty$, if $\eta \in [0,\Omega]$, we have that f_α' tends uniformly to 0 in $[0,\Omega]$, as $\alpha \to 0$, since $\lim_{\alpha\to 0} C(\alpha) = 0$. This fact trivially implies that also f_a uniformly tends to 0 in $[0,\Omega]$ when $\alpha \to 0$.

Summarizing we have proved that:

(a) If $0<\alpha<\alpha_m$, with α_m given by (35), then $f(\eta)$ changes sign. In other words, f is monotone decreasing function for every $\eta \geq 0$; there exists $L>0$ such that $f(\eta)>0$ for $\eta \in [0,L)$, $f(L)=0$, $f'(L)<0$, and $f(\eta)<0$ for $\eta \in (L,\infty)$.
(b) If $\alpha \to 0$, then $f \to 0$, uniformly in any compact set. □

In the sequel we will find an estimate for L from above which depend only on "structural constants", i.e. on α, m.

Proposition 2. *For any $\alpha \in (0,\alpha_m)$ there exists a positive constant K, which depends only on α, and m, i.e. on "structural parameters", such that $L<K$.*

Proof. We start remarking that, $\forall\ 0 \leq f \leq \alpha \leq \alpha_m$

$$\frac{1}{1-m}f - f^m \leq 0, \quad \text{and} \quad \left|\frac{1}{1-m}f - f^m\right| \leq \theta, \tag{48}$$

where

$$\theta = \max_{f\in[0,\alpha_m]}\left|\frac{f}{1-m} - f^m\right| = \alpha_m^m\left|m^{\frac{1}{1-m}} - m^{\frac{m}{1-m}}\right| \underset{(35)}{=} (1-m)^{\frac{m}{1-m}}\left|m^{\frac{1}{1-m}} - m^{\frac{m}{1-m}}\right|. \tag{49}$$

We will carry out the proof by considering separately the two cases that can occur:
(i) f' is decreasing;
(ii) f' has a minimum attained at some point $\eta = p$.

Let us first focus on the first case (i.e. f' is decreasing) and, rewriting $(37)_1$ as

$$f'' = \frac{\eta}{2}\left|f'\right| - \left|\frac{1}{1-m}f - f^m\right|, \tag{50}$$

we have $f'' \geq -\theta$, with θ given by (49). Hence, integrating

$$f' \geq -\theta\eta, \quad \text{and} \quad f \geq \alpha - \theta\frac{\eta^2}{2}, \tag{51}$$

since $f'(0)=0$, and $f(0)=\alpha$. Thanks to (51) we have that $|f'| \leq 1$, when $\theta\eta \leq 1$, i.e. $0 \leq \eta \leq 1/\theta$, and that $f \geq \frac{\alpha}{2}$, when $\alpha - \theta\frac{\eta^2}{2} \geq \frac{\alpha}{2}$, i.e. $0 \leq \eta \leq \sqrt{\frac{\alpha}{\theta}}$. So taking $\eta \in [0,\varkappa]$, with

$$\varkappa = \min\left\{\frac{1}{\theta}, \sqrt{\frac{\alpha}{\theta}}\right\}, \tag{52}$$

we have

$$\frac{\alpha}{2} \leq f(\eta) \leq \alpha, \tag{53}$$

and $|f'(\eta)| \leq 1$. So, recalling (50) and taking $\eta \in [0, \varkappa]$ we can write

$$f'' = \frac{\eta}{2}|f'| - \left|\frac{1}{1-m}f - f^m\right| \leq \frac{\eta}{2} - \omega, \tag{54}$$

where

$$\omega = \min_{\frac{\alpha}{2} \leq f \leq \alpha}\left|\frac{1}{1-m}f - f^m\right| = \min\left\{\left|\frac{1}{1-m}\alpha - \alpha^m\right|, \left|\frac{1}{1-m}\frac{\alpha}{2} - \left(\frac{\alpha}{2}\right)^m\right|\right\} > 0. \tag{55}$$

Hence, setting

$$\bar{\eta} = \min\left\{\frac{\omega}{2}, \varkappa\right\} = \min\left\{\frac{\omega}{2}, \frac{1}{\theta}, \sqrt{\frac{\alpha}{\theta}}\right\}, \tag{56}$$

and taking $\eta \in [0, \bar{\eta}]$ we can rewrite (54) as

$$f'' \leq \frac{\eta}{2} - \omega \leq \bar{\eta} - \omega \leq \frac{\omega}{2} - \omega = -\frac{\omega}{2}.$$

which we integrate getting $f'(\eta) \leq -\frac{\omega}{2}\eta$, $\forall\, \eta \in [0, \bar{\eta}]$. Recalling that we are in case (i), i.e. f' decreasing, we have

$$f' \leq -\frac{\omega}{2}\bar{\eta}, \qquad \forall\, \eta \geq \bar{\eta},$$

and, consequently

$$f(\eta) \leq f(\bar{\eta}) - \frac{\omega}{2}\bar{\eta}(\eta - \bar{\eta}) \leq \alpha - \frac{\omega}{2}\bar{\eta}(\eta - \bar{\eta}), \qquad \forall\, \eta \geq \bar{\eta}.$$

We therefore conclude that L, the point at which f vanishes, can be estimated from above by K_1,

$$L \leq \hat{K}_1 \quad \text{with} \quad \hat{K}_1 = \bar{\eta} + \frac{2\alpha}{\omega\bar{\eta}},$$

with $\bar{\eta}$ and ω given by (56) and (55), respectively, provided $L > \varkappa$. Hence we can state that

$$L \leq K_1 \quad \text{with} \quad K_1 = \max\left\{\hat{K}_1, \varkappa\right\}.$$

We have therefore determined, at least in case (i), an upper estimate of L that depends only on the "structural parameters" α and m.

Let us now consider case (ii): f' has a minimum attained at $\eta = p$ (we recall that in Proposition 1 we have shown that p is the only local minimum of f'). In this situation two sub-cases can occur. If $K_1 < p$, then what has been done in (i) still holds true because f' is decreasing in the interval $(0, p)$ and therefore f vanishes in $L \leq K_1 < p$. In the opposite case, we have $p \leq K_1$. But the latter inequality can be read as an estimate for p. Indeed, we know from Proposition 1 that f vanishes at $L \leq \eta_p$, with η_p estimated by (40), namely

$$L \leq \eta_p \leq p \exp\left[\frac{\alpha^{1-m}}{b(1-m)}\right] \leq K_1 \exp\left[\frac{\alpha^{1-m}}{b(1-m)}\right],$$

with b given by (39). Hence, setting

$$K = \max\left\{K_1, K_1 \exp\left[\frac{\alpha^{1-m}}{b(1-m)}\right]\right\} \tag{57}$$

we have that

$$L \leq K, \tag{58}$$

regardless of whether case (i) or (ii) occurs. Moreover K, depends only on α and m, i.e. only on the "structural parameters". □

The previous proposition gives an upper bound for L. We now illustrate a result that gives a bound from below for $|f'(\eta)|$, which, evaluated at $\eta = L$, provides a quantitative estimate of the "transversality" of f at the point where it vanishes.

Proposition 3. *For any $\alpha \in (0, \alpha_m)$ and $\eta \in [0, L]$, we have*

$$|f'(\eta)| \geq \mathcal{H}, \quad \textit{with} \quad \mathcal{H} = e^{-\frac{K^2}{4}}\omega \min\{\eta, \varkappa\}, \tag{59}$$

where K, given by (57), is the constant that estimates L from above, ω is given by (55) and $\varkappa$ by (52).

Proof. First we note the consistency of the estimate (59). Indeed, if $\eta = 0$, we have $\min\{\eta, \varkappa\} = 0$ and consequently $\mathcal{H} = 0$, in agreement with the initial condition $(37)_2$. We begin the proof by taking $\alpha < \alpha_m$, $\eta \in [0, L]$ and considering $(37)_1$, which we rewrite as

$$\underbrace{\left(f'' + \frac{\eta}{2} f'\right) e^{\frac{\eta^2}{4}}}_{\left(f' e^{\frac{\eta^2}{4}}\right)'} = \left(\frac{f}{1-m} - f^m\right) e^{\frac{\eta^2}{4}}.$$

Integrating such an equation between 0 and η and exploiting the fact that $f'(0) = 0$, we have

$$f'(\eta) e^{\frac{\eta^2}{4}} = -\int_0^\eta \left(f^m - \frac{f}{1-m}\right) e^{\frac{t^2}{4}} dt, \quad \Rightarrow \quad |f'(\eta)| = e^{-\frac{\eta^2}{4}} \int_0^\eta \left(f^m - \frac{f}{1-m}\right) e^{\frac{t^2}{4}} dt. \tag{60}$$

In particular, recalling (48), we remark that the integrand is non-negative $\forall\ 0 \le f \le \alpha \le \alpha_m$. Exploiting (58), (60) gives rise to this inequality

$$|f'(\eta)| \ge e^{-\frac{L^2}{4}} \int_0^\eta \left(f^m - \frac{f}{1-m}\right) e^{\frac{t^2}{4}} dt \ge e^{-\frac{K^2}{4}} \int_0^\eta \left(f^m - \frac{f}{1-m}\right) dt. \tag{61}$$

We now focus on the integral and proceed as in the proof of Proposition 2. Indeed, since $f'(\eta) \le 0$, inequality (51) holds true from which estimate (53) derives, provided $\eta \in [0, \varkappa]$, with $\varkappa$ given by (52). Considering now the integral in the right hand side of (61), we have

$$\int_0^\eta \left(f^m(t) - \frac{f(t)}{1-m}\right) dt \ge \int_0^{\min(\eta,\varkappa)} \left(f^m(t) - \frac{f(t)}{1-m}\right) dt. \tag{62}$$

So, taking $t \in [0, \min(\eta, \varkappa)]$, it turns out that $\frac{\alpha}{2} \le f(t) \le \alpha$, and so, recalling (55), we obtain

$$f^m - \frac{f}{1-m} \ge \min_{f \in \left[\frac{\alpha}{2}, \alpha\right]} \left(f^m - \frac{f}{1-m}\right) = \omega.$$

Thus, from (62), we obtain this estimate from below of the integral

$$\int_0^\eta \left(f^m(t) - \frac{f(t)}{1-m}\right) dt \ge \omega \min(\eta, \varkappa),$$

which, once plugged into (61), gives (59). □

Using the same notation of Section 2.1, we set $\delta(\alpha) = -f'(L(\alpha))$, where, henceforth, $L(\alpha)$ denotes the abscissa at which f, solution to the Cauchy problem (34), vanishes.

Proposition 4. *For any $\alpha \in (0, \alpha_m)$, with α_m given by (35) and $m \in (0,1)$, $L(\alpha)$ and $\delta(\alpha)$ are continuous functions of α.*

Proof. We have to show that, fixed any $\alpha \in (0, \alpha_m)$ and $\varepsilon > 0$, there exists $\chi > 0$ (with χ possibly dependent on α) such that, denoting by $B_\chi(\alpha)$ the neighborhood centered in α and radius χ, we have

$$\left|L(\alpha_1) - L(\alpha_2)\right| \le \varepsilon \quad \text{and} \quad \left|\delta(\alpha_1) - \delta(\alpha_2)\right| \le \varepsilon, \quad \forall \ \alpha_1, \alpha_2 \in B_\chi(\alpha) \cap (0, \alpha_m),$$

where, according to the previous introduced notation, $f_{\alpha_i}(L(\alpha_i)) = 0$ and $\delta(\alpha_i) = -f'_{\alpha_i}(L(\alpha_i))$, $i = 1, 2$.

Before starting the proof, we introduce the notation that we will use in the sequel. We set

$$L_M = \max\left\{L(\alpha_1), L(\alpha_2)\right\}, \qquad L_m = \min\left\{L(\alpha_1), L(\alpha_2)\right\},$$

and assume $L_M > L_m$. The case $L_M = L_m$ will be analyzed afterwards. Next, the function f, solution to the Cauchy problem (37), depends on the selected value of α, and to highlight this dependence we use the notation f_α. Lastly, since we are only interested in the non-negative values of f_{α_i}, $i = 1, 2$, we set $f_{\alpha_i} = 0$ when it becomes negative, i.e.

$$f_{\alpha_1}(\eta) = 0, \quad \forall\ \eta \ge L(\alpha_1),$$
$$f_{\alpha_2}(\eta) = 0, \quad \forall\ \eta \ge L(\alpha_2).$$

We begin the proof by showing the continuous dependence of f_α on α. We select $\varepsilon > 0$ and denote by η_1 the first value of the abscissa η at which f_{α_1} or f_{α_2} is equal to $\varepsilon/10$, (see Fig. 4). Then, we exploit the continuous dependence theorem for differential equation in $[0, \eta_1]$ since f_{α_i}, $i = 1, 2$, are Lipschitz continuous functions because $f_{\alpha_i} \ge \varepsilon/10$, $i = 1, 2$. We therefore conclude that there exists $\chi(\varepsilon)$ such that, if $\alpha_1, \alpha_2 \in B_\chi(\alpha) \cap (0, \alpha_m)$, we have

$$\left\|f_{\alpha_1} - f_{\alpha_2}\right\|_{C^1([0,\eta_1])} \le \frac{\varepsilon}{10}. \tag{63}$$

In particular, the previous inequality entails $\left|f_{\alpha_1}(\eta_1) - f_{\alpha_2}(\eta_1)\right| \le \varepsilon/10$. Remembering that η_1 is the first value of η where f_{α_1} or f_{α_2} is equal to $\varepsilon/10$, it turns out that $\left|f_{\alpha_1}(\eta_1)\right|$; $\left|f_{\alpha_2}(\eta_1)\right| \le \varepsilon/10 + \varepsilon/10 = \varepsilon/5$, and therefore, since f is monotone decreasing, we have

$$f_{\alpha_1}(\eta) \le \frac{\varepsilon}{5}, \quad \text{and} \quad f_{\alpha_2}(\eta) \le \frac{\varepsilon}{5}, \quad \forall\ \eta \ge \eta_1. \tag{64}$$

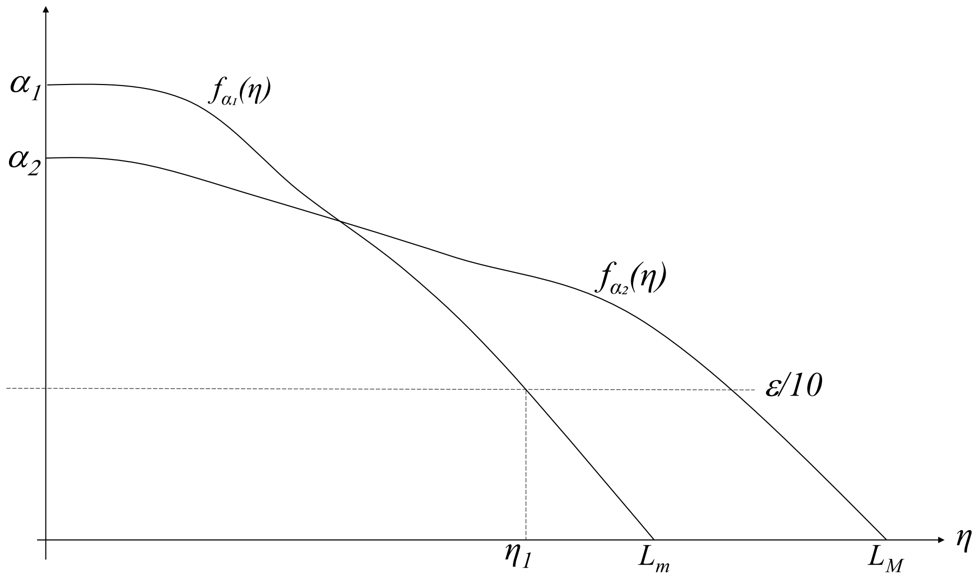

Fig. 4. Typical behavior of $f_{\alpha_1}(\eta)$ and $f_{\alpha_2}(\eta)$.

As a consequence

$$\left|f_{\alpha_1}(\eta)-f_{\alpha_2}(\eta)\right| \leq \left|f_{\alpha_1}(\eta)\right|+\left|f_{\alpha_2}(\eta)\right| \leq \frac{2\varepsilon}{5}, \quad \forall\, \eta \geq \eta_1 .$$

So, up to now we have obtained the following result, for any $\varepsilon>0$, we are able to determine χ such that

$$\left\|f_{\alpha_1}-f_{\alpha_2}\right\|_{C([0,L_M])} \leq \frac{2\varepsilon}{5}, \tag{65}$$

provided $|\alpha_1-\alpha_2| \leq \chi$.

Estimate (63) gives the C^1 norm of $\left(f_{\alpha_1}-f_{\alpha_2}\right)$ only for $\eta \in [0,\eta_1]$. Beyond η_1 the estimate (65) holds true, but it does not involve the derivatives of f_{α_i}, $i=1,2$. Now we determine an estimate of $\left|f'_{\alpha_1}-f'_{\alpha_2}\right|$ for $\eta \in [\eta_1, L_m]$. We therefore take $\eta \in [\eta_1, L_m]$, and define

$$V = f_{\alpha_1}-f_{\alpha_2},$$

which, recalling (34), fulfills this equation

$$\left(f_{\alpha_1}-f_{\alpha_2}\right)'' = \frac{f_{\alpha_1}-f_{\alpha_2}}{1-m} - \frac{\eta}{2}\left(f_{\alpha_1}-f_{\alpha_2}\right)' - f^m_{\alpha_1} - f^m_{\alpha 2}, \qquad \eta \in [\eta_1, L_m],$$

which we transform into the system

$$\begin{cases} V' = Z, \\ Z' + \dfrac{\eta}{2} Z = \dfrac{1}{1-m} V - \left(f^m_{\alpha_1} - f^m_{\alpha_2}\right). \end{cases} \tag{66}$$

In particular $(66)_2$ rewrites as

$$\left(Z(\eta)\, e^{\frac{\eta^2}{4}}\right)' = e^{\frac{\eta^2}{4}} \left[\frac{1}{1-m} V - \left(f^m_{\alpha_1} - f^m_{\alpha_2}\right)\right], \tag{67}$$

which we integrate between η_1 and η, with $\eta_1 < \eta \leq L_m$, getting

$$Z(\eta)\, e^{\frac{\eta^2}{4}} - Z(\eta_1)\, e^{\frac{\eta_1^2}{4}} = \int_{\eta_1}^{\eta} e^{\frac{t^2}{4}} \left[\frac{1}{1-m} V - \left(f^m_{\alpha_1} - f^m_{\alpha_2}\right)\right] dt, \tag{68}$$

yielding

$$\begin{aligned} \left|Z(\eta)\, e^{\frac{\eta^2}{4}}\right| &= \left|Z(\eta_1)\, e^{\frac{\eta_1^2}{4}} + \int_{\eta_1}^{\eta} e^{\frac{t^2}{4}} \left[\frac{V}{1-m} - \left(f^m_{\alpha_1} - f^m_{\alpha_2}\right)\right] dt\right| \\ &\leq \left|Z(\eta_1)\right| e^{\frac{\eta_1^2}{4}} + \int_{\eta_1}^{\eta} e^{\frac{t^2}{4}} \left[\frac{|V|}{1-m} + \left|f^m_{\alpha_1} - f^m_{\alpha_2}\right|\right] dt. \end{aligned} \tag{69}$$

Now, using estimate (65), which gives

$$\|V\|_{C([\eta_1,L_m])} \le \frac{2}{5}\varepsilon, \tag{70}$$

and the fact that there exists a positive $M \ge 1$, not depending on η, such that

$$\left|f_{\alpha_1}^m - f_{\alpha_2}^m\right| = \frac{\left|f_{\alpha_1}^m - f_{\alpha_2}^m\right|}{\left|f_{\alpha_1} - f_{\alpha_2}\right|^m}\left|f_{\alpha_1} - f_{\alpha_2}\right|^m \le M\,|V|^m,$$

from (69) we obtain

$$\begin{aligned}
|Z(\eta)| \le |Z(\eta)|\, e^{\frac{\eta^2}{4}} &\le \left|Z(\eta_1)\right| e^{\frac{\eta_1^2}{4}} + \int_{\eta_1}^{\eta} e^{\frac{t^2}{4}}\left[\frac{|V|}{1-m} + \left|f_{\alpha_1}^m - f_{\alpha_2}^m\right|\right] dt \\
&\le \left|Z(\eta_1)\right| e^{\frac{\eta_1^2}{4}} + \int_{\eta_1}^{\eta} e^{\frac{t^2}{4}}\left[\frac{\|V\|_{C([\eta_1,L_m])}}{(1-m)} + M\left(\|V\|_{C([\eta_1,L_m])}\right)^m\right] dt. \\
&\le \left|Z(\eta_1)\right| e^{\frac{\eta_1^2}{4}} + L_m e^{\frac{L_m^2}{4}}\left[\frac{2\varepsilon}{5(1-m)} + M\left(\frac{2}{5}\varepsilon\right)^m\right].
\end{aligned} \tag{71}$$

Now we take advantage of estimate (63), which entails

$$\left|Z(\eta_1)\right| = \left|f'_{\alpha_1}(\eta_1) - f'_{\alpha_2}(\eta_1)\right| \le \left\|f_{\alpha_1} - f_{\alpha_2}\right\|_{C^1([0,\eta_1])} \le \frac{\varepsilon}{10},$$

and of Proposition 2 which allows to estimate L_m from above with a constant[9] not depending on η. Since (71) holds true $\forall$ $\eta \in [\eta_1, L_m]$, we conclude that

$$\left\|f'_{\alpha_1} - f'_{\alpha_2}\right\|_{C([\eta_1,L_m])} = \|Z\|_{C([\eta_1,L_m])} \le A\varepsilon^m, \tag{72}$$

with A positive constant independent of η.

Formula (72) allows to extend (63) in the whole interval $[0, L_m]$, so that we can write

$$\left\|f_{\alpha_1} - f_{\alpha_2}\right\|_{C^1([0,L_m])} \le \frac{\varepsilon}{10} + \frac{2\varepsilon}{5} + A\varepsilon^m, \tag{73}$$

provided $\left|\alpha_1 - \alpha_2\right| \le \chi$.

For $\eta \ge L_m$, we introduce

$$f_M(\eta) = \max\left\{f_{\alpha_1}(\eta), f_{\alpha_2}(\eta)\right\}, \qquad f_m(\eta) = \min\left\{f_{\alpha_1}(\eta), f_{\alpha_2}(\eta)\right\}. \tag{74}$$

We have $f_m(L_m) = 0$, $f_M(L_m) > 0$, $f_M(L_M) = 0$. In particular, for $\eta \ge L_m$, we are left with $f_M(\eta)$, since $f_m(\eta) = 0$, $\forall\, \eta \in [L_m, L_M]$. We thus focus in the interval $[L_m, L_M]$, and, recalling (64), we have

$$f_M(L_m) \le \frac{2}{5}\varepsilon.$$

On the other hand

$$f_M(L_m) - f_M(L_M) \ge \min_{\eta\in[L_m,L_M]}\left|f'_M(\eta)\right|(L_M - L_m),$$

so that

$$\min_{\eta\in[L_m,L_M]}\left|f'_M(\eta)\right|(L_M - L_m) \le \frac{2}{5}\varepsilon. \tag{75}$$

We now exploit Proposition 3, getting an estimate from below for $\left|f'_M(\eta)\right|$ when $\eta \in [L_m, L_M]$. Indeed, recalling (59), for any $\eta \in [L_m, L_M]$ we have

$$\left|f'_M(\eta)\right| \ge \mathcal{H}_m, \quad \text{where} \quad \mathcal{H}_m = e^{-\frac{K^2}{4}}\omega \min\left\{L_m, \varkappa\right\} > 0.$$

Hence, from (75), we obtain $(L_M - L_m) \le \dfrac{2}{5\mathcal{H}_m}\varepsilon$, i.e.

$$\left|L(\alpha_1) - L(\alpha_2)\right| \le \frac{2}{5\mathcal{H}_m}\varepsilon, \tag{76}$$

if $\left|\alpha_1 - \alpha_2\right| \le \chi$. In this regard, recalling (53), we have an estimate from below of L_m and hence of $\mathcal{H}_m$. We thus have obtained the continuous dependence of $L(\alpha)$ with respect to α.

[9] The constant appearing in the estimate of L has been denoted K and is given by (57). Since we have two values of K, namely K_{α_1} and $K_{\alpha 2}$ corresponding to α_1 and α_2, we select $K = \max\left\{K_{\alpha_1}, K_{\alpha_2}\right\}$.

Let us now focus on $\left|\delta\left(\alpha_1\right)-\delta\left(\alpha_2\right)\right|=\left|f'_{\alpha_2}\left(L_{\alpha_2}\right)-f'_{\alpha_1}\left(L_{\alpha_1}\right)\right|$. Introducing

$$\delta_m=-f'_m\left(L_m\right),\quad \delta_M=-f'_M\left(L_M\right),$$

we have

$$\begin{aligned}\left|\delta\left(\alpha_1\right)-\delta\left(\alpha_2\right)\right| &= \left|\delta_M-\delta_m\right|=|\underbrace{f'_M\left(L_M\right)}_{\delta_M}-\underbrace{f'_m\left(L_m\right)}_{\delta_m}| \\ &= \left|f'_M\left(L_M\right)-f'_M\left(L_m\right)+f'_M\left(L_m\right)-f'_m\left(L_m\right)\right| \\ &\le \left|f'_M\left(L_m\right)-f'_m\left(L_m\right)\right|+\left|f'_M\left(L_M\right)-f'_M\left(L_m\right)\right|. \end{aligned}\tag{77}$$

The term $\left|f'_M\left(L_m\right)-f'_m\left(L_m\right)\right|$ can be estimated exploiting (73), while to estimate $\left|f'_M\left(L_M\right)-f'_M\left(L_m\right)\right|$ from above we proceed as follows. We apply a procedure similar to the one used to obtain (70). We rewrite Eq. $(34)_1$ as a system similar to (66), where now the role of Z is now played by f'_M and the one of V by f_M and the last term in the right hand side of $(66)_2$ is f_M^m. Hence, $\forall\,\eta\in\left[L_m,L_M\right]$ we rewrite (68) as

$$f'_M(\eta)\,e^{\frac{\eta^2}{4}}-f'_M\left(L_m\right)e^{\frac{L_m^2}{4}}=\int_{L_m}^{\eta}e^{\frac{t^2}{4}}\left[\frac{f_M(t)}{1-m}-f_M^m(t)\right]dt,$$

thus getting

$$\left[f'_M\left(L_M\right)-f'_M\left(L_m\right)\right]e^{\frac{L_M^2}{4}}=f'_M\left(L_m\right)\left(e^{\frac{L_m^2}{4}}-e^{\frac{L_M^2}{4}}\right)+\int_{L_m}^{L_M}e^{\frac{t^2}{4}}\left[\frac{f_M(t)}{1-m}-f_M^m(t)\right]dt.$$

Recalling (58) and (64) we have

$$\begin{aligned}\left|f'_M\left(L_M\right)-f'_M\left(L_m\right)\right| &\le \left|f'_M\left(L_M\right)-f'_M\left(L_m\right)\right|e^{\frac{L_M^2}{4}} \\ &\le \left|f'_M\left(L_m\right)\right|\left|e^{\frac{L_m^2}{4}}-e^{\frac{L_M^2}{4}}\right|+\int_{L_m}^{L_M}e^{\frac{t^2}{4}}\left[\frac{1}{1-m}\left|f_M\right|+\left|f_M^m\right|\right]dt \\ &\le \left|f'_M\left(L_m\right)\right|\left|e^{\frac{L_m^2}{4}}-e^{\frac{L_M^2}{4}}\right|+e^{\frac{L_M^2}{4}}\left(L_M-L_m\right)\left(\frac{\varepsilon}{5(1-m)}+\left(\frac{\varepsilon}{5}\right)^m\right). \end{aligned}\tag{78}$$

To estimate $\left|f'_M\left(L_m\right)\right|$ we suppose, with a view to simplifying, that $f'_M\left(L_m\right)$ coincides with $f'_{\alpha_2}\left(L_{\alpha_1}\right)$ (the estimate we present remains unchanged even if $f'_M\left(L_m\right)=f'_{\alpha_1}\left(L_{\alpha_2}\right)$). Starting from (60) we have

$$\begin{aligned}\left|f'_{\alpha_2}\left(L_{\alpha_1}\right)\right| &= e^{-\frac{L_{\alpha_1}^2}{4}}\int_0^{L_{\alpha_1}}\left(f_{\alpha_2}^m-\frac{f_{\alpha_2}}{1-m}\right)e^{\frac{t^2}{4}}dt\le\int_0^{L_{\alpha_1}}\left(f_{\alpha_2}^m-\frac{f_{\alpha_2}}{1-m}\right)e^{\frac{t^2}{4}}dt \\ &\le L_{\alpha_1}e^{\frac{L_{\alpha_1}^2}{4}}\max_{f\in[0,\alpha_m]}\left|\frac{f}{1-m}-f^m\right|\underset{(49)}{=}L_{\alpha_1}e^{\frac{L_{\alpha_1}^2}{4}}\theta.\end{aligned}$$

Hence, recalling (58), we get $\left|f'_{\alpha_2}\left(L_{\alpha_1}\right)\right|\le Ke^{\frac{K^2}{4}}\theta$, where K must be understood as the maximum between K_{α_1} and K_{α_2}. So, resuming from (78), using the Lagrange's Theorem for the term $\left|e^{\frac{L_m^2}{4}}-e^{\frac{L_M^2}{4}}\right|$ and exploiting again estimate (58) with K maximum between K_{α_1} and K_{α_2}, we obtain

$$\left|f'_M\left(L_M\right)-f'_M\left(L_m\right)\right|\le\left[K^2e^{2\frac{K^2}{4}}\theta+e^{\frac{K^2}{4}}\left(\frac{\varepsilon}{5(1-m)}+\left(\frac{\varepsilon}{5}\right)^m\right)\right]\left(L_M-L_m\right),$$

which, exploiting (76), can be rewritten as

$$\left|f'_M\left(L_M\right)-f'_M\left(L_m\right)\right|\le C\varepsilon,\tag{79}$$

with C positive constant (uniform with respect to η).

At this stage we are in position to estimate $\left|\delta\left(\alpha_1\right)-\delta\left(\alpha_2\right)\right|$. Resuming (77) and exploiting (73) and (79) we can write

$$\left|\delta\left(\alpha_1\right)-\delta\left(\alpha_2\right)\right|\le\frac{\varepsilon}{10}+\frac{2\varepsilon}{5}+A\varepsilon^m+C\varepsilon,$$

provided $\left|\alpha_1-\alpha_2\right|\le\chi$.

To conclude the proof we consider the case, previously excluded, in which $L_M=L_m$. In this situation it is sufficient to apply (73) to obtain an estimate of $\left|\delta\left(\alpha_1\right)-\delta\left(\alpha_2\right)\right|$ in terms of ε. □

We now show that $L(\alpha)$ and $\delta(a)$ fulfill the properties summarized in (16).

Proposition 5. *For any $m \in (0,1)$, we have:*
(i) $L(\alpha) \to 0$, and $\delta(\alpha) \to 0$ as $\alpha \to 0$.
(ii) $L(\alpha) \to \infty$, as $\alpha \to \alpha_m$, and $\delta(\alpha)$ remains bounded in a left neighborhood of α_m, with α_m given by (35).

Dim. We recall that the function f, solution to the Cauchy problem (37), depends on α. Therefore we proceed as in the proof of Proposition 4, by denoting f with f_α.

We start by proving the first part of point (i), i.e. $L(\alpha) \to 0$ as $\alpha \to 0$. By integrating $(37)_1$

$$f'_\alpha(\eta) = \int_0^\eta \left[\frac{f_\alpha(t)}{1-m} - f_\alpha^m(t)\right] dt - \frac{1}{2}\int_0^\eta t f'_\alpha(t)\, dt$$
$$= \int_0^\eta \left[\frac{f_\alpha(t)}{1-m} + \frac{f_\alpha(t)}{2} - f_\alpha^m(t)\right] dt - \frac{\eta}{2} f_\alpha(\eta),$$

and thus we obtain

$$f'_\alpha(\eta) \leq \int_0^\eta \left[f_\alpha(t)\left(\frac{1}{1-m} + \frac{1}{2}\right) - f_\alpha^m(t)\right] dt.$$

By analyzing the behavior of the integrand for f_α close to zero, that means considering α sufficiently small (recall $0 \leq f_\alpha \leq \alpha$), we have

$$f_\alpha\left(\frac{1}{1-m} + \frac{1}{2}\right) - f_\alpha^m \leq -\frac{f_\alpha^m}{2} \leq 0,$$

namely

$$f'_\alpha(\eta) \leq -\int_0^\eta \frac{f_\alpha^m(t)}{2} dt \leq -\frac{f_\alpha^m(\eta)}{2}\eta,$$

(recall that f_α is decreasing in $[0, L_\alpha]$). Hence we have

$$\frac{f'_\alpha(\eta)}{f_\alpha^m(\eta)} \leq -\frac{\eta}{2},$$

which, recalling $(37)_2$, gives

$$\int_0^\eta \frac{f'_\alpha(t)}{f_\alpha^m(t)} dt \leq -\int_0^\eta \frac{t}{2} dt. \tag{80}$$

i.e.

$$f_\alpha^{1-m}(\eta) - \alpha^{1-m} \leq -(1-m)\frac{\eta^2}{4},$$

which holds for α sufficiently small. Taking $\eta = L_\alpha$ we obtain

$$-\alpha^{1-m} \leq -(1-m)\frac{L(\alpha)^2}{4},$$

yielding

$$L(\alpha) \leq \frac{2}{\sqrt{1-m}}\alpha^{\frac{1-m}{2}}, \tag{81}$$

so that $L(\alpha) \to 0$ as $\alpha \to 0$.

To prove the second part of point (i) we essentially proceed as in (66), rewriting (37) as

$$\begin{cases} f'_\alpha = z_\alpha, \\ z'_\alpha + \dfrac{\eta}{2} z_\alpha = \dfrac{1}{1-m} f_\alpha - f_\alpha^m \\ f_\alpha(0) = \alpha, \quad z_\alpha(0) = 0, \end{cases}$$

getting

$$z_\alpha(\eta)\, e^{\frac{\eta^2}{4}} = \int_0^\eta e^{\frac{t^2}{4}} \left[\frac{f_\alpha}{1-m} - f_\alpha^m\right] dt = -\int_0^\eta e^{\frac{t^2}{4}} \left|\frac{f_\alpha}{1-m} - f_\alpha^m\right| dt,$$

which, when evaluated in $\eta = L(\alpha)$, gives

$$\underbrace{-\delta(\alpha)}_{z_\alpha(L(\alpha))} = e^{-\frac{L(\alpha)^2}{4}} \int_0^{L(\alpha)} e^{\frac{t^2}{4}} \left[\frac{f_\alpha}{1-m} - f_\alpha^m\right] dt = -e^{-\frac{L(\alpha)^2}{4}} \int_0^{L(\alpha)} e^{\frac{t^2}{4}} \left|\frac{f_\alpha}{1-m} - f_\alpha^m\right| dt. \tag{82}$$

So, recalling that $0 \le f_\alpha \le \alpha$ and that

$$0 \le \left| \frac{f_\alpha}{1-m} - f_\alpha^m \right| \le \left| \frac{\alpha}{1-m} - \alpha^m \right|,$$

provided α is close enough to zero, from (82) we obtain

$$\delta(\alpha) \le e^{-\frac{L(\alpha)^2}{4}} \int_0^{L(\alpha)} e^{\frac{t^2}{4}} \left| \frac{\alpha}{1-m} - \alpha^m \right| dt \le \left| \frac{\alpha}{1-m} - \alpha^m \right| L(\alpha).$$

Recalling now (81), we have that $\delta(\alpha)$ goes to 0 as $\alpha \to 0$.

Concerning point (ii) we start considering

$$V(f) = -\int_0^f \left(\frac{\xi}{1-m} - \xi^m \right) d\xi = -\frac{f^2}{2(1-m)} + \frac{f^{m+1}}{m+1} \ge 0, \quad \forall\, f \in (0, \alpha_m), \tag{83}$$

so that, after multiplying by f'_α Eq. $(37)_1$, we obtain

$$\frac{d}{d\eta}\left[\frac{f'^{\,2}_\alpha}{2} + V(f_\alpha) \right] = -\frac{\eta}{2} f'^{\,2}_\alpha \le 0. \tag{84}$$

Hence

$$\left[\frac{f'^{\,2}_\alpha}{2} + V(f_\alpha) \right] \le \underbrace{V(f_\alpha(0))}_{V(\alpha)},$$

since $f'_\alpha(0) = 0$. Thus, recalling that $0 \le f_\alpha \le \alpha$, we get

$$\frac{f'^{\,2}_\alpha}{2\left(V(\alpha) - V(f_\alpha)\right)} \le 1, \quad \Rightarrow \quad \frac{f'_\alpha}{\sqrt{2\left(V(\alpha) - V(f_\alpha)\right)}} \ge -1,$$

which, once integrated, gives rise to

$$\int_\alpha^{f_\alpha(\eta)} \frac{d\xi}{\sqrt{2(V(\alpha) - V(\xi))}} \ge -\eta, \quad \Longrightarrow \quad L(\alpha) \ge \int_0^\alpha \frac{d\xi}{\sqrt{2(V(\alpha) - V(\xi))}}.$$

We can easily prove that the integral diverges as $\alpha \to \alpha_m$, since α_m is a double root for V, namely $V'(\alpha_m) = 0$. We therefore conclude that $L(\alpha) \to \infty$ when $\alpha \to \alpha_m$.

Concerning the second part of point (ii), we multiply by f'_α $(37)_1$ and integrating in $[0, L(\alpha)]$, we get

$$\frac{f'^{\,2}_\alpha(\eta)}{2} = -V(f_\alpha) + V(\alpha) - \int_0^\eta \frac{s}{2} f'^2_\alpha\, ds \le V(\alpha), \quad \forall\, \eta \in [0, L(\alpha)], \tag{85}$$

since $V(f_\alpha) \ge 0$. Hence, exploiting (83) for writing explicitly V, (85) yields

$$\frac{f'^{\,2}_\alpha(\eta)}{2} \le -\frac{\alpha^2}{2(1-m)} + \frac{\alpha^{m+1}}{m+1} \le \alpha_m^{m+1}, \quad \Rightarrow \quad |f'_\alpha(\eta)| \le \sqrt{2\alpha_m^{m+1}} \quad \forall\, \eta \in [0, L(\alpha)],$$

and therefore

$$\delta(\alpha) = |f'_\alpha(L(\alpha))| \le \sqrt{2\alpha_m^{m+1}}. \quad \square$$

3.2. *Solution to problem (34) for $\eta \ge L(\alpha)$*

Let us now take $\alpha \in (0, \alpha_m)$ with α_m given by (35) and consider $\eta > L$. Proposition 1 guarantees that f is always negative for $\eta > L$. Therefore we consider this Cauchy problem

$$\begin{cases} f'' + \dfrac{\eta}{2} f' - \beta f = 0, & L < \eta, \\ f(L) = 0, \quad f'(L) = -\delta, & \text{with } \delta > 0, \end{cases} \tag{86}$$

with β defined in (27), i.e. $\beta = 1/(1-m)$. Setting

$$z = \eta^2/4, \quad \text{and} \quad f(\eta) = e^{-\frac{\eta^2}{4}} g(z(\eta)), \tag{87}$$

we transform Eq. $(86)_1$ into

$$z g_{zz} + \left(\frac{1}{2} - z\right) g_z - \left(\frac{1}{2} + \beta\right) g = 0, \tag{88}$$

which is the degenerate hypergeometric equation (see, for instance, formula (70), pag. 527 of [9]) whose general form is $zg'' + (b - z)g' - ag = 0$. Since b is not an integer, the general solution of Eq. (88) is expressed through the confluent hypergeometric functions of first and second kind[10] (see, e.g., Chapter 7 of [10] or Sec. 9.10 of [11])

$$g(z) = C_1 \Phi(a, b, z) + C_2 \Psi(a, b, z), \quad \text{with} \quad \begin{cases} a = \beta + \dfrac{1}{2}, \\ b = \dfrac{1}{2}, \end{cases} \tag{89}$$

where both solutions $\Phi(a, b, z)$, and $\Psi(a, b, z)$ are linearly independent of each other. Indeed, referring to formula (9.10.10), pag. 265 of [11], their Wronskian is

$$W = \det \begin{pmatrix} \Phi & \Psi \\ \Phi_z & \Psi_z \end{pmatrix} = -\frac{\Gamma(b)}{\Gamma(a)} z^{-b} e^{z} \underset{\substack{(87)\\(89)}}{=} -\frac{\Gamma(1/2)}{\Gamma(\beta + 1/2)} \frac{2}{\eta} e^{\eta^2/4}. \tag{90}$$

Recalling (87), the general solution to $(86)_1$ may be written in this form

$$f(\eta) = e^{-\frac{\eta^2}{4}} \left[C_1 \Phi\left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{\eta^2}{4}\right) + C_2 \Psi\left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{\eta^2}{4}\right) \right], \tag{91}$$

so that, enforcing $(86)_2$, we obtain

$$\begin{cases} C_1 \Phi\left(a, b, \dfrac{L^2}{4}\right) + C_2 \Psi\left(a, b, \dfrac{L^2}{4}\right) = 0, \\ C_1 \Phi_z\left(a, b, \dfrac{L^2}{4}\right) + C_2 \Psi_z\left(a, b, \dfrac{L^2}{4}\right) = -\dfrac{2\delta}{L} e^{\frac{L^2}{4}}, \end{cases}$$

from which, recalling (90), we obtain

$$C_1 = \frac{\det \begin{pmatrix} 0 & \Psi \\ -\dfrac{2\delta}{L} e^{\frac{L^2}{4}} & \Psi_z \end{pmatrix}}{W} = -\frac{\delta \Gamma(a)}{\Gamma(b)} \Psi\left(a, b, \frac{L^2}{4}\right), \tag{92}$$

$$C_2 = \frac{\det \begin{pmatrix} \Phi & 0 \\ \Phi_z & -\dfrac{2\delta}{L} e^{\frac{L^2}{4}} \end{pmatrix}}{W} = \frac{\delta \Gamma(a)}{\Gamma(b)} \Phi\left(a, b, \frac{L^2}{4}\right), \tag{93}$$

with a and b defined in formula (89). In other words, recalling that $z(\eta) = \eta^2/4$, we have

$$f(z) = \frac{\delta \Gamma(a) e^{-z}}{\Gamma(b)} \left[-\Psi\left(a, b, \frac{L^2}{4}\right) \Phi(a, b, z) + \Phi\left(a, b, \frac{L^2}{4}\right) \Psi(a, b, z) \right]. \tag{94}$$

We finally remark that, when $m = 0$, i.e. $\beta = 1$, (94) reduces (20).

3.3. Existence of similarity solutions

We now impose condition (33), with f given by (91) and C_1, C_2 by (92), (93), respectively, i.e.

$$-c = \lim_{\eta \to \infty} \frac{e^{-\frac{\eta^2}{4}}}{\eta^{2\beta}} \left[C_1 \Phi\left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{\eta^2}{4}\right) + C_2 \Psi\left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{\eta^2}{4}\right) \right].$$

In particular, recalling $z = \eta^2/4$ and (94), the above equation can be rewritten as

$$-c = \lim_{z \to \infty} \frac{f(\eta(z))}{4^{\beta} z^{\beta}} = \lim_{z \to \infty} \frac{\delta \Gamma(a) e^{-z}}{4^{\beta} z^{\beta} \Gamma(b)} \left[-\Psi\left(a, b, \frac{L^2}{4}\right) \Phi(a, b, z) + \Phi\left(a, b, \frac{L^2}{4}\right) \Psi(a, b, z) \right], \tag{95}$$

where a and b have been defined in formula (89).

Now, following [11] Sec. 9.12, we have that

$$\Phi(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{a-b} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \underset{(89)}{=} \frac{\Gamma(b)}{\Gamma(a)} e^{z} z^{\beta} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right), \tag{96}$$

$$\Psi(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} z^{-a} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \underset{(89)}{=} \frac{\Gamma(b)}{\Gamma(a)} z^{-\beta-1/2} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right), \tag{97}$$

[10] Also known as Kummer functions.

when $z \to +\infty$. Hence, recalling (94), we obtain

$$f = \delta \left[-\Psi \left(a, b, \frac{L^2}{4} \right) z^{\beta} + \Phi \left(a, b, \frac{L^2}{4} \right) e^{-z} z^{-\beta-1/2} \right] \left(1 + \mathcal{O}\left(\frac{1}{z} \right) \right), \quad \text{as } z \to \infty,$$

and so (95) yields

$$-c = \lim_{\eta \to \infty} \frac{f(\eta(z))}{4^{\beta} z^{\beta}} = -\frac{\delta}{4^{\beta}} \Psi \left(a, b, \frac{L^2}{4} \right),$$

i.e.

$$c = \frac{\delta}{4^{\beta}} \Psi \left(a, b, \frac{L^2}{4} \right) \underset{(89)}{=} \frac{\delta}{4^{\beta}} \Psi \left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{L^2}{4} \right). \tag{98}$$

In particular, recalling the definition of $F(\alpha)$ through (32), we now have

$$F(\alpha) = \frac{\delta(\alpha)}{4^{\beta}} \Psi \left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{L(\alpha)^2}{4} \right), \tag{99}$$

and (98) gives rise to the algebraic equation $F(\alpha) = c$. Therefore the existence and multiplicity of self-similar solutions depends on the behavior of $F(\alpha)$, for $\alpha \in (0, \alpha_m)$, with α_m given by ((35)). We have

Proposition 6. *The function $F(\alpha)$ given by (99) is a non-negative and continuous function for $\alpha \in (0, \alpha_m)$. Moreover, $\lim_{\alpha \to \alpha_m} F(\alpha) = 0$ and $\lim_{\alpha \to 0} F(\alpha) = 0$.*

Dim. Exploiting Proposition 4 and the fact that $\Psi\left(\beta + \frac{1}{2}, \frac{1}{2}, z\right)$ is a continuous function (see [9], Section S4.9.2) we have that $F(\alpha)$ is a continuous function for $\alpha \in (0, \alpha_m)$. Next, for $\alpha \in (0, \alpha_m)$, we have $\Psi\left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{L(\alpha)^2}{4}\right) > 0$ (see again [9], Section S4.9.2) and Proposition 1 entails that $\delta = |f'(L)| > 0$. So $F(\alpha) > 0$ for $\alpha \in (0, \alpha_m)$.

Concerning the behavior of F for $\alpha \to \alpha_m$, we exploit Proposition 5, which entails that $L \to +\infty$, and δ bounded, as $\alpha \to \alpha_m$. But, when $L \to +\infty$, recalling (97) we have that

$$\Psi \left(\beta + \frac{1}{2}, \frac{1}{2}, \frac{L^2}{4} \right) = \frac{1}{4^{\beta}} \frac{\Gamma(1/2)}{\Gamma(\beta + 1/2)} \left(\frac{L^2}{4} \right)^{-\beta-1/2} \left(1 + \mathcal{O}\left(\frac{1}{L^2} \right) \right).$$

We can therefore conclude

$$\lim_{\alpha \to \alpha_m} F(\alpha) = \lim_{\alpha \to \alpha_m} \frac{\delta(\alpha)}{4^{\beta}} \frac{\Gamma(1/2)}{\Gamma(\beta + 1/2)} \left(\frac{L^2}{4} \right)^{-\beta-1/2} \left(1 + \mathcal{O}\left(\frac{1}{L^2} \right) \right) = 0.$$

Let us now consider the case $\alpha \to 0$, which, still exploiting Proposition 5, yields $L \to 0$ and $\delta \to 0$. Again making use of the properties of Ψ, which has this expansion ([9], Section S4.9.2)

$$\Psi(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} \left(1 + \mathcal{O}\left(z^{1/2} \right) \right), \quad \text{if} \quad z \to 0,$$

we have that $\lim_{\alpha \to 0} F(\alpha) = 0$. $\square$

We thus have just proved that F is continuous non-negative and that tends to zero as α tends to zero or α_m. We can therefore conclude that F has a maximum in $(0, \alpha_m)$, and we define it as

$$F_{\max} = \max_{\alpha \in (0, \alpha_m)} F(\alpha) > 0.$$

So for small positive c the equation $F(\alpha) = c$ has at least two different solutions. Indeed, updating definition (9), we set $c_{cr} = F_{\max}$, and have the following

Proposition 7. *Problem (26) has at least two different self-similar solutions for $0 < c < c_{cr}$.*

3.4. Non-uniqueness of the solutions

Thanks to the results obtained in the previous sections it is possible to prove a more general result of non uniqueness for Eq. $(1)_1$ when $n = 1$ and $0 < m < 1$. We indeed prove that there may be solutions which change sign even if the initial datum is not $-c\,|x|^{\frac{2}{1-m}}$, i.e. solution of non self-similar type. This fact implies non-uniqueness of the solution to problem (1) in $\mathbb{R}$, because, in addition to the solution positive for some (x, t), there is the solution, always negative, of the heat equation (recall that the initial datum is assumed to be negative except in $x = 0$, where it vanishes).

Here we consider a simplified case under “strong” assumptions without delving too much into details since a more exhaustive and detailed coverage of this issue will be the object of a subsequent paper. Let us consider a more general version of problem (1) in $\mathbb{R}$, namely

$$\begin{cases} u_t = u_{xx} + u^m H(u), & x \in \mathbb{R}, \quad 0 < t, \\ u(x, 0) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{100}$$

under the assumptions:

H_1. $\varphi(x) \in C^{2+\gamma}(\mathbb{R})$, for some $\gamma \in (0,1)$ (this is a strong assumption that can be weakened).

H_2. $\varphi(x) < 0$, $\forall\, x \in \mathbb{R}\setminus\{0\}$, $\varphi(0) = 0$ and $\varphi(x) \geq -c\,|x|^{\frac{2}{1-m}}$, with $0 < c \leq c_{cr}$.

We remark that, when $\varphi(x) = -c\,|x|^{\frac{2}{1-m}}$, with $0 < c \leq c_{cr}$, problem (100) admits solutions changing sign besides the solution, always negative, that fulfills the heat equation. Hence, in this case, problem (100) does not admit uniqueness.

First we observe that Eq. $(100)_1$, complemented with the initial datum

$$u(x,0) = \psi(x), \tag{101}$$

fulfilling H_1 always admits solution $u \in W^{2,1}_{q,loc}(\mathbb{R}_T) \cap C^{1+\nu}(\mathbb{R}_T)$, $\forall\, q > 1$, $\forall\, \nu \in (0,1)$ if $m = 0$, or $u \in C^{2+\kappa}(\mathbb{R}_T)$, $\kappa = \min\{m,\gamma\}$, if $m \in (0,1)$. We do not enter into the details of the proof, here we limit to observe that this result can be obtained by relaxation, i.e. by replacing $u^m H(u)$, with $p_n(u) H_n(u)$, where

$$p_n(u) = \left(u^m - \frac{1}{n}\right)_+,$$

$$H_n(u) = \begin{cases} 0, & \text{if } u < 0, \\ nu, & \text{if } 0 \leq u \leq \dfrac{1}{n}, \\ 1, & \text{if } u > \dfrac{1}{n}, \end{cases}$$

and then using [12] to prove that the relaxed problem is well posed and admits a-priori estimates allowing to pass to the limit, thus obtaining a solution to $(100)_1$, (101). In this regard it is important to observe that $-\max|\psi| \leq u_n \leq e^t$, because of standard maximum and comparison principles (here u_n is the solution of the relaxed problem). As far as the needed a-priori estimate are concerned, they are obtained by using Theorem 5.2 pag. 320 and Theorem 9.1 pag. 341, with the subsequent corollary, of [12]. We leave the details of the proof to a forthcoming paper.

Next, it is important to note that, if $m = 0$, when passing to the limit we obtain a solution to the equation $u_t - u_{xx} \in H(u)$, in the sense of graph. However, in the simpler cases, it is easy to prove that we can write $u_t - u_{xx} = H(u)$. Assume, for instance, that $\psi(x)$ is an odd function and $\psi'(x)\,\mathrm{sign}(x) < 0$, $\forall\, x \neq 0$. In this case, setting $v = u_x$ we get $v_t - v_{xx} = H'(u)\,v$, in $\mathbb{R}^+ \times [0,T]$, $v(0,t) = 0$. The maximum principle assures that $u_x \neq 0$, in $\mathbb{R}\setminus\{x=0\}$, which implies that $\mathrm{meas}\left\{(x,t) \in \mathbb{R}_T,\ u(x,t) = 0\right\} = 0$, and hence $u_t - u_{xx} \in H(u)$ implies $u_t - u_{xx} = H(u)$.

We now illustrate the main idea of the proof of non uniqueness. Let us consider the following problem

$$\begin{cases} u_{n\,t} = u_{nxx} + u_n^m H\left(u_n\right), & x \in \mathbb{R}, \quad 0 < t, \\ u_n(x,0) = \varphi(x) + \dfrac{1}{n}, & x \in \mathbb{R}, \end{cases}$$

which admits solution because of the previous considerations. Clearly $\varphi(x) + \dfrac{1}{n} > -c\,|x|^{\frac{2}{1-m}} + \dfrac{1}{2n}$. Because of the uniform (with respect to $(x,t) \in \mathbb{R}_T$) Hölder continuity of u_n it is straightforward to prove that there exists $\tau_n > 0$ such that $u_n \geq \mathsf{Se}_c(x,t)$, in $\mathbb{R}_{\tau_n}$, where $\mathsf{Se}_c(x,t)$ is the self-similar solution to Eq. $(100)_1$ complemented with the initial datum $\mathsf{Se}_c(x,0) = -c\,|x|^{\frac{2}{1-m}}$. Let us now $\bar{\tau}_n$ the maximal time, $\bar{\tau}_n \leq T$, such that $u_n \geq \mathsf{Se}_c(x,t)$, in $\mathbb{R}_{\bar{\tau}_n}$. On the other hand, in $\mathbb{R}_{\bar{\tau}_n}$, we have $u_n^m H\left(u_n\right) \geq \mathsf{Se}_c^m H\left(\mathsf{Se}_c\right)$, and $u_n^m H\left(u_n\right) \not\equiv \mathsf{Se}_c^m H\left(\mathsf{Se}_c\right)$, hence $u_n\left(x,\bar{\tau}_n\right) \gneqq \mathsf{Se}_c\left(x,\bar{\tau}_n\right) + \dfrac{1}{2n}$. This implies that $\bar{\tau}_n = T$, otherwise, by continuity, we could contradict the maximality of $\bar{\tau}_n$. We have thus proved that $u_n(x,t) \geq \mathsf{Se}_c(x,t)$, in $\mathbb{R}_T$. Passing to the limit in n, we get that $u_n \to u$, with u solving problem (100) and $u \geq \mathsf{Se}_c(x,t)$ in $\mathbb{R}_T$. Since Se_c is positive in a region (evolving in time) also u is a solution to (100) which is positive for some (x,t). We thus have shown that problem (100) admits a solution, positive for some (x,t), even if the initial datum is not $-c\,|x|^{\frac{2}{1-m}}$. This implies non-uniqueness, keeping in mind that there exists a solution which is always negative (the solution of the heat equation).

4. Conclusion

The results of this paper indicate that the solutions of the problem (1) are not unique. In particular, if $m = 0$, we have extended the result of Gianni and Hulshof [5], showing that, for any $n \geq 1$, there exist only two self-similar solutions provided $c < c_{cr}$, with c_{cr} given by (9), and estimated from above by $4/27n^2$. If $0 < m < 1$, the situation is much more intricated and we have proved that there are at most two self-similar solutions since, unlike the case $m = 0$, we are unable to give a detailed description of the behavior of $F(\alpha)$. Indeed, when $m \in (0,1)$, $F(\alpha)$ is much more involved since it is defined by means of $\delta(\alpha)$ and $L(\alpha)$ which we do not know explicitly. Furthermore, it is very well possible that when $0 < m < 1$, $F(\alpha)$ given by (99) could have more relative maxima which would imply the existence of more than two self-similar solutions for suitable values of c. Nevertheless our conjecture is that, for any suitable values of c, there exist two and only two self-similar solutions changing sign. This will need closer investigation in future papers. Similarly it would be interesting to extend the results obtained in $\mathbb{R}$ for $m \in (0,1)$ to the n -dimensional case. The result obtained is closely linked to the question of the uniqueness of the solutions to Eq. $(1)_1$ complemented with “more general” initial data. In fact, while working in $\mathbb{R}$ and considering “strong” hypotheses on the initial datum, we have shown that, even if $u(x,0) \neq -cx^2$, there is still a solution which, for some (x,t), is positive, thus proving non-uniqueness.

Acknowledgments

The authors thank Prof. Antonio Fasano and Prof. Nicola Bellomo for their numerous and fruitful advices. This research was partially supported by GNFM of Italian INDAM, Italy.

References

[1] J. Norbury, A.M. Stuart, Parabolic free boundary problems arising in porous medium combustion, IMA J. Appl. Math. 39 (1987) 241–257.
[2] J. Norbury, A.M. Stuart, A model for porous medium combustion, Quart. J. Mech. Appl. Math. 42 (1989) 159–178.
[3] A. Friedman, A.E. Tzavaras, Combustion in a porous medium, Siam J. Math. Anal. 19 (1988) 509–519.
[4] H. Fujita, On the blowing up of solutions of the cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Faculty Sci., Univ. Tokyo. Sect. 1, Math., Astron., Phys., Chem. 13 (1966) 109–124.
[5] R. Gianni, J. Hulshof, The semilinear heat equation with a heaviside source term, European J. Appl. Math. 3 (1992) 367–379, http://dx.doi.org/10.1017/S0956792500000917.
[6] D. Andreucci, R. Gianni, Classical solutions to a multidimensional free boundary problem arising in combustion theory, Comm. Partial Differential Equations 19 (1994) 803–826, http://dx.doi.org/10.1080/03605309408821036.
[7] M. Fila, H. Matano, E. Yanagida, Non-uniqueness of solutions of a semilinear heat equation with singular initial data, in: P. Gurevich, J. Hell, B. Sandstede, A. Scheel (Eds.), Patterns of Dynamics, in: Springer Proceedings in Mathematics & Statistics, vol. 205, 2016.
[8] R. Laister, J.C. Robinson, M. Sierzega, A necessary and sufficient condition for uniqueness of the trivial solution in semilinear parabolic equations, J. Differential Equations 262 (2017) 4979–4987.
[9] A.D. Polyanin, V.F. Zaitsev, Handdbook of Exact Solutions for Ordinary Differential Equations, Second Edition, Chapman & Hall/CRC, Boca Raton, 2003.
[10] M.G. Smith, B. Spain, Functions of Mathematical Physics, Van Nostrand Reinhold, London, 1970.
[11] N.N. Lebedev, Special Fucntions and their Applications, Prentice-Hall, London, 1965.
[12] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Uarl'ceva, Linear and Quasi-Linear Equations of Parabolic Type, American Mathematical Society, Providence, 1968.