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Forward lateral photovoltage scanning problem: Perturbation approach and existence-uniqueness analysis

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ABSTRACT

In this paper, we present analytical results for the so-called forward lateral photovoltage scanning (LPS) problem. The (inverse) LPS model predicts doping variations in crystal by measuring the current leaving the crystal generated by a laser at various positions. The forward model consists of a set of nonlinear elliptic equations coupled with a measuring device modeled by a resistance. Standard methods to ensure the existence and uniqueness of the forward model cannot be used in a straightforward manner due to the presence of an additional generation term modeling the effect of the laser on the crystal. Hence, we scale the original forward LPS problem and employ a perturbation approach to derive the leading order system and the correction up to the second order in an appropriate small parameter. While these simplifications pose no issues from a physical standpoint, they enable us to demonstrate the analytic existence and uniqueness of solutions for the simplified system using standard arguments from elliptic theory adapted to the coupling with the measuring device.

1. Introduction

Estimating crystal inhomogeneities in semiconductors noninvasively holds significance for various industrial applications. For example, understanding and avoiding doping fluctuations is important if very pure semiconductor crystal is required in industrial applications [25]. However, such fluctuations, often referred to as striations, may not always have a negative impact which may be the case for grain boundaries in solar cells [23].

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Doping variations lead to local electrical fields. In order to detect inhomogeneities, variations, and defects in the doping profile noninvasively, one may systematically generate electron-hole pairs through some form of electromagnetic radiation and attach an external circuit to measure the resulting current. Due to the local fields created by doping inhomogeneities, charge carriers tend to redistribute in the surrounding region of the excitation to minimize energy. The excess charge carriers then traverse the external circuit, generating a measurable current. Scanning the semiconductor sample with the electromagnetic source at different positions allows one to visualize the distribution of electrically active charge-separating defects and variations in the doping profile along the scan locations. Several different types of technologies rely on this physical principle such as Electron Beam Induced Current (EBIC) [24], Laser Beam Induced Current (LBIC) [4], scanning photovoltage (SPV) [10], and Lateral Photovoltage Scanning (LPS) [15,11,12,16]. While EBIC uses localized electron beams as the source of electromagnetic radiation, LBIC, SPV and LPS use laser beams.

In this paper, we focus on the LPS method. A nonlinear drift-diffusion model for the forward problem was mathematically formulated in [7], where the coupling between the charge transport in the crystal and the external circuit for the voltmeter is realized through an implicit boundary condition. This forward model computes current/voltages at the contacts for given doping variations. However, no analytical existence results were presented. The main difficulty in establishing the existence is due to the generation rate on the right-hand side of the continuity equations which impedes the direct use of standard arguments [3,17]. Known existence results in the literature for related PDE models require the generation term to be small [5]. However, the authors in [7] demonstrated through simulations based on the Voronoi finite-volume method [9,20,8,14,1] that, on the one hand, numerically the system also converges for higher laser powers and that, on the other hand, the electron density is several orders of magnitude larger than the hole density. In this paper, we make use of the second fact by deriving an appropriately scaled unipolar model and identifying the relevant dimensionless small parameters. However, setting the smallest parameter to zero, oversimplifies the physics in the sense that the coupling between the circuit and the charge transport model is ignored. Therefore, through a perturbation approach the leading order corrections up to the second order in the small parameter are derived. The resulting system is simpler than the original LPS model. However, for realistic applications, for example studied in [7,15,11], these simplifications are reasonable since the perturbation parameters for silicon and gallium arsenide are small. Thus, these simplifications are not problematic from a physical point of view. Moreover, from a mathematical point of view, we have the additional benefit that the leading order model is a decoupled nonlinear elliptic system and the second order correction is a linear elliptic system coupled with the external circuit. The existence and uniqueness of solutions for the simplified model can be shown by using arguments from elliptic PDE theory modified to include the coupling with the external circuit. While this paper focuses on the mathematical analysis of the forward problem, it is worth noting that numerical, data-driven strategies exist to address the more complicated inverse problem [21], that is, the prediction of the doping variations in a crystal by measuring the current generated by a laser beam impinging at various positions of the crystal.

The remainder of this paper is organized as follows: In Section 2, we state the original LPS model. The model is then scaled in Section 3 and the asymptotic simplified model is derived in Section 4. In Section 5, we show the existence and uniqueness of solutions for the asymptotic PDE system and draw some conclusions in Section 6.

2. Lateral photovoltage scanning method

In this section, we present the model for the LPS method. The description of the model strongly relies on [7].

2.1. Presentation of the model

We represent the silicon crystal as a confined domain $\Omega \subset \mathbb{R}^d$, where $d = 1, 2, 3$. The doping profile, denoted by $C(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega$, reflects the difference between donor and acceptor concentrations.

Within the crystal, two charge carriers exist: electrons with a negative elementary charge $-q$ and holes with a positive elementary charge q. Charge transport is described by the electrostatic potential $\psi(\boldsymbol{x})$ and quasi Fermi potentials $\varphi_n(\boldsymbol{x})$ and $\varphi_p(\boldsymbol{x})$ for electrons and holes, respectively, which are governed by the steady-state van Roosbroeck model:

$$\begin{aligned} -\nabla \cdot (\varepsilon \nabla \psi) &= q(p - n + C(\boldsymbol{x})) \\ -\nabla \cdot (\mu_n n \nabla \varphi_n) &= R - G(\boldsymbol{x}), \\ -\nabla \cdot (\mu_p p \nabla \varphi_p) &= G(\boldsymbol{x}) - R. \end{aligned} \tag{1}$$

The first equation describes a self-consistent electric field through a nonlinear Poisson equation. The subsequent continuity equations describe charge transport in the crystal, with constant permittivity ε and mobilities μ_n and μ_p. Assuming Boltzmann statistics, the relations between electron and hole densities n and p, and the quasi-Fermi potentials, are established by:

$$\begin{aligned} n(\psi, \varphi_n) &= N_c \exp\left(\frac{q(\psi - \varphi_n) - E_c}{k_B T}\right), \\ p(\psi, \varphi_p) &= N_v \exp\left(\frac{q(\varphi_p - \psi) + E_v}{k_B T}\right). \end{aligned} \tag{2}$$

The effective conduction and valence band densities of states are denoted by N_c and N_v, while the Boltzmann constant and temperature are represented by k_B and T. The constant conduction and valence band-edge energies are denoted as E_c and E_v.

The current densities for electrons and holes, $\boldsymbol{J}_n(\boldsymbol{x})$ and $\boldsymbol{J}_p(\boldsymbol{x})$, are expressed as $-q\mu_n n \nabla \varphi_n$ and $-q\mu_p p \nabla \varphi_p$, respectively. Utilizing the relations (2), their drift-diffusion form is

$$\boldsymbol{J}_n = -q\mu_n(n\nabla\psi - V_{\text{th}}\nabla n), \qquad \boldsymbol{J}_p = -q\mu_p(p\nabla\psi + V_{\text{th}}\nabla p), \tag{3}$$

where $V_{\text{th}} = k_B T/q$ is the thermal voltage.

Recombination R and generation rates G are elaborated in subsequent sections.

The system (1) is complemented by mixed boundary conditions. The boundary $\partial\Omega$ comprises two disjoint parts, Γ_N and Γ_D. Neumann boundary conditions on Γ_N are given by

$$\frac{\partial \psi}{\partial \boldsymbol{\nu}} = \frac{\partial \varphi_n}{\partial \boldsymbol{\nu}} = \frac{\partial \varphi_p}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \tag{4}$$

where $\partial/\partial\boldsymbol{\nu} = \boldsymbol{\nu} \cdot \nabla$ is the normal derivative along the external unit normal $\boldsymbol{\nu}$. On Γ_D, Dirichlet-type boundary conditions model ohmic contacts

$$\psi - \psi_0 = \varphi_n = \varphi_p = u_{D_i} - u_{\text{ref}} \quad \text{on} \quad \Gamma_{D_i}, i = 1, 2, \tag{5}$$

where ψ_0 is the local electroneutral potential (see [17] for details),

$$\psi_0 := u_{D_i} + \frac{E_c}{q} + \frac{k_B T}{q} \log\left(\frac{n_0}{N_c}\right), \quad n_0 := \frac{1}{2}\left(C + \sqrt{C^2 + 4n_{\text{i}}^2}\right). \tag{6}$$

The parameter n_{i} refers to the intrinsic carrier density defined by

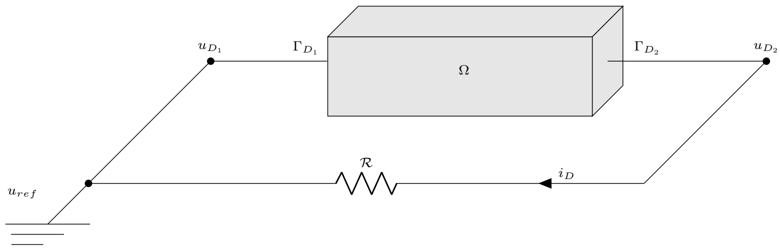

Fig. 1. A photo-sensitive semiconductor crystal Ω coupled to a voltmeter with resistance $\mathcal{R}$.

$$n_{\mathrm{i}}^2 = N_c N_v \exp\left(-\frac{E_c - E_v}{k_B T}\right). \tag{7}$$

The terms u_{D_i} represent contact voltages at the respective ohmic contacts, with a common practice of defining a reference potential u_{ref}. The total electric current j_{D_i} flowing through the i-th ohmic contact Γ_{D_i} is defined by the surface integral

$$j_{D_i} = -\int_{\Gamma_{D_i}} \boldsymbol{\nu} \cdot (\boldsymbol{J}_n(\boldsymbol{x}) + \boldsymbol{J}_p(\boldsymbol{x}))\mathrm{d}\sigma(\boldsymbol{x}), \qquad i = 1, 2. \tag{8}$$

Following charge conservation, the currents in (8) satisfy:

$$\sum_{i=1}^{2} j_{D_i} = 0 \qquad \Rightarrow \qquad j_{D_1} = -j_{D_2} =: i_D. \tag{9}$$

The voltage meter is simplified as a circuit with resistance $\mathcal{R}$, depicted in Fig. 1. The potential difference between nodes u_{D_1} and u_{D_2} appearing in (5), is given by

$$u_{D_2} - u_{D_1} = \mathcal{R}\, i_D(u_{D_2}), \tag{10}$$

where i_D is defined in (9). By assuming that one node of an electric circuit is grounded, we can freely assign $u_{D_1} = u_{\mathrm{ref}} = 0$, resulting in the simplification of (10) to:

$$u_{D_2} = \mathcal{R}\, i_D(u_{D_2}). \tag{11}$$

Equation (11) is an implicit equation for u_{D_2}, given that i_D depends implicitly on u_{D_2} via the van Roosbroeck system (1). After obtaining a solution $(\psi, \varphi_n, \varphi_p)$ to the van Roosbroeck system, where u_{D_2} influences the Dirichlet boundary condition (5), we compute the current i_D using (8). Further insights into this coupling can be found in [2] and [3]. To simplify the notation, we will use u_D instead of u_{D_2}.

The total recombination rate, denoted as R, includes three mechanisms:

$$R = R_{\mathrm{dir}} + R_{\mathrm{Aug}} + R_{\mathrm{SRH}}, \tag{12}$$

where R_{dir}, R_{Aug}, and R_{SRH} represent direct recombination, Auger recombination, and Shockley-Read-Hall (SRH) recombination, respectively, defined as:

$$\begin{aligned} R_{\mathrm{dir}} &= C_d(np - n_{\mathrm{i}}^2), \\ R_{\mathrm{Aug}} &= C_n n(np - n_{\mathrm{i}}^2) + C_p p(np - n_{\mathrm{i}}^2), \\ R_{\mathrm{SRH}} &= \frac{np - n_{\mathrm{i}}^2}{\tau_p(n + n_T) + \tau_n(p + p_T)}, \end{aligned} \tag{13}$$

where C_d, C_n and C_p are material-related constants. The SRH recombination process, which involves the trapping and release of charge carriers in semiconductors, differs from the Auger recombination process due to unintentional inclusion of elements during fabrication. This unintentional inclusion affects the properties of the semiconductor material, such as the lifetimes of charge carriers (τ_n for electrons, τ_p for holes) and the reference densities of charge carriers (n_T for electrons, p_T for holes). For later use we define the function

$$r(n,p) := C_d + C_n n + C_p p + \frac{1}{\tau_p(n+n_T) + \tau_n(p+p_T)} \tag{14}$$

so that $R = r(n,p)(np - n_{\mathrm{i}}^2)$.

The electromagnetic source is modeled by the generation term $G(\mathbf{x};\mathbf{x}_0)$. When the laser hits the crystal at the point $\mathbf{x}_0 := (x_0, y_0, z_0)^T$, some photons are *impinged* and create electron-hole pairs, resulting in a generation rate defined as follows

$$G(\mathbf{x};\mathbf{x}_0) = \kappa S(\mathbf{x} - \mathbf{x}_0), \tag{15}$$

where $S(\mathbf{x})$ is the shape function of the laser (normalized by $\int_{\mathbb{R}^3} S(\mathbf{x})\mathrm{d}\mathbf{x} = 1$), while κ is a constant given by $\kappa := \frac{P\lambda_L}{hc}(1-\rho)$, where c is the velocity of light in vacuum. Here, P is the laser power, λ_L is the wave length of the laser, h is the Planck constant, and ρ is the reflectivity rate of the crystal.

We assume that area of influence of the electromagnetic source decays exponentially fast from the incident point $\mathbf{x}_0$. In particular, we take a laser profile function S defined as

$$S(\mathbf{x}) := \frac{1}{2\pi\sigma_L^2 d_A} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma_L}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma_L}\right)^2\right] \exp\left[-\frac{|z|}{d_A}\right]. \tag{16}$$

Here σ_L is the laser spot radius, while d_A is the penetration depth (or the reciprocal of the absorption coefficient), which heavily depends on the laser wave length.

2.2. Numerical simulations

For the above model, we performed numerical simulation with `ddfermi` [6] of a donor-doped semiconductor crystal irradiated by a laser beam, both for silicon (see Fig. 2) and gallium arsenide (see Fig. 3). In both figures, we show the top and front view of the cubic 3D crystals, delimited by the black horizontal line, and apply the laser at the top center. We set the laser power $P = 2$ mW, laser spot position $(x_0, y_0)^T = (0 \text{ mm}, 0 \text{ mm})^T$, average doping concentration $N_{D0} = 1 \times 10^{16}$ cm^{-3} and $C(x,y,z) = N_{D0}\left(1 + 0.2\sin\left(2\pi\frac{x}{100\ \mu\text{m}}\right)\right)$. The additional parameters for both simulations can be found in Appendix A. The results show that the presence of the beam, modeled by the generation term G, greatly affects the minority charge carriers p, while the majority charge carriers n are not significantly impacted. As a matter of fact, the sinusoidal pattern of the doping is reflected in the electron density and the electrostatic potential alone.

These numerical findings are the main motivation for a unipolar scaling of the full model, also encompassing the electric network variables, which will be detailed in the following section.

3. Scaling of the model for unipolar devices

We consider a scaling of the model (1) which is suitable for unipolar devices. In this case, the density of the majority charge carriers is of the same order as the doping profile, while the density of the minority charge carriers is much smaller. Nevertheless, both charge carriers contribute to the total current, especially when the current is the result of an impinging laser beam.

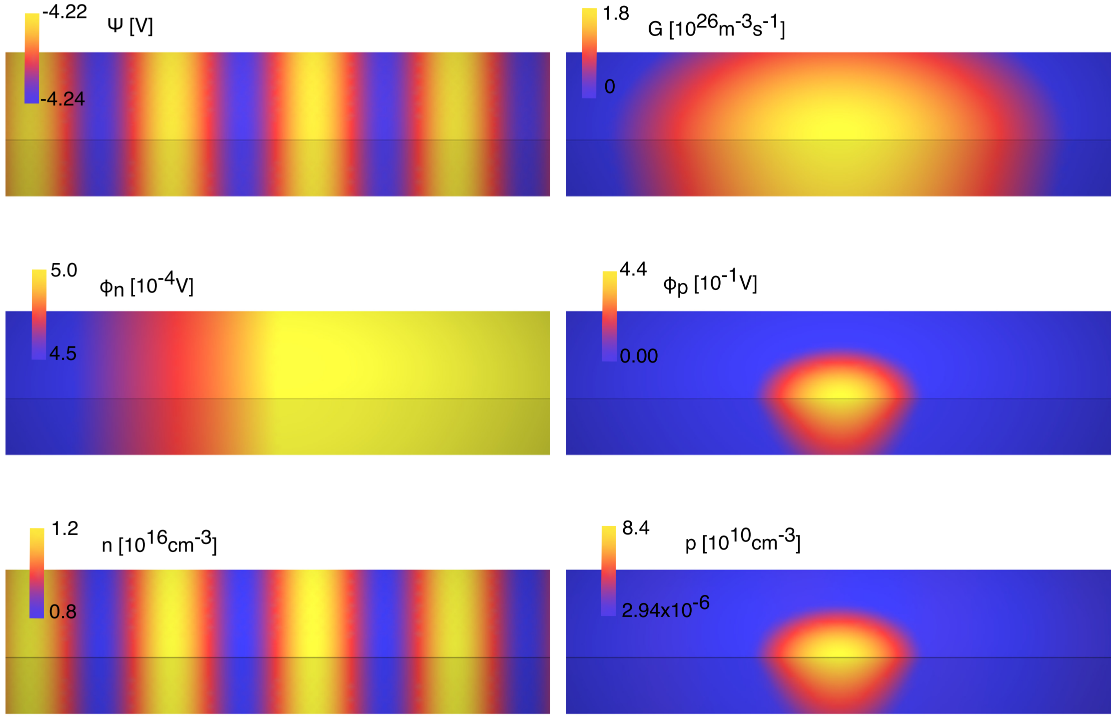

Fig. 2. For a 3D silicon crystal, we show the electrostatic potential ψ (upper left panel), the resulting generation rate G (upper right panel), the quasi Fermi potentials φ_n, φ_p for electrons and holes (middle panels). The related charge carrier densities n, p are shown in the bottom panels.

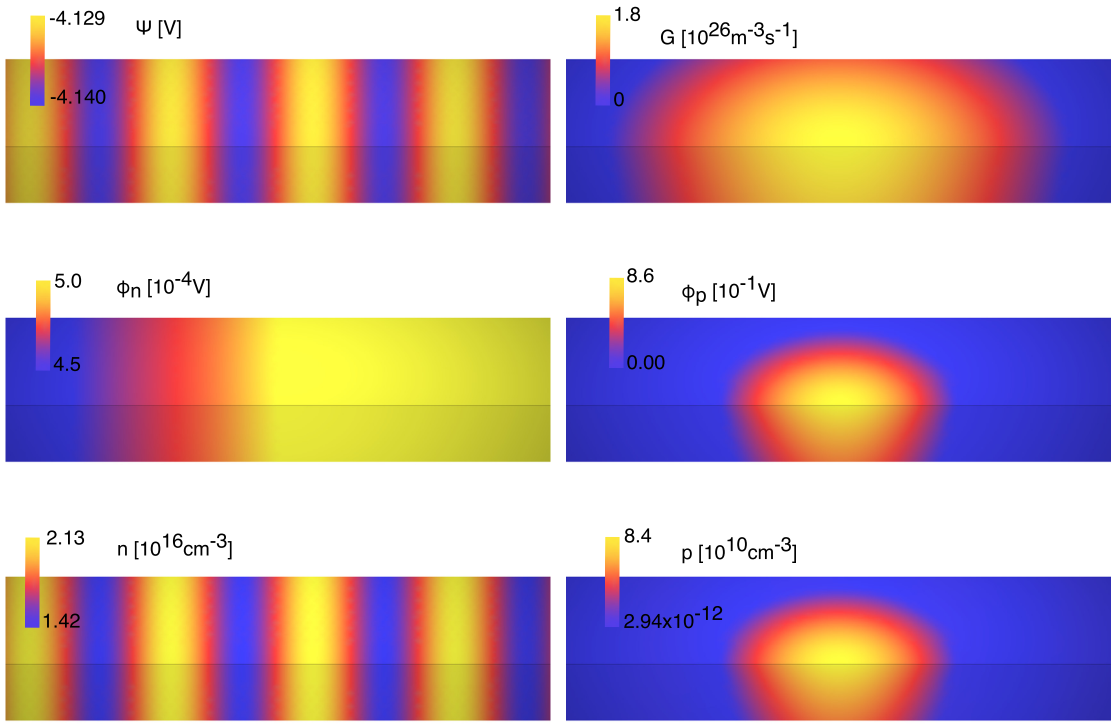

Fig. 3. For a 3D gallium arsenide crystal, we show the electrostatic potential ψ (upper left panel), the resulting generation rate G (upper right panel), the quasi Fermi potentials φ_n, φ_p for electrons and holes (middle panels). The related charge carrier densities n, p are shown in the bottom panels.

In the following we consider an n-doped device, so that

$$C(\boldsymbol{x}) > 0, \tag{17}$$

and electrons are the majority charge carriers. This condition ensures the well-posedness of the semiconductor equations (see, for instance, [17]). A similar analysis can be performed for a p-doped device, that is, with $C(\boldsymbol{x}) < 0$, reverting the role of electron and hole concentration. If $C(\boldsymbol{x})$ changes sign, there will be a non negligible region where both carrier concentrations have the same order of magnitude, so that the following asymptotics would not be valid. We write

$$\begin{gathered}\varphi_n = \bar{\psi}\hat{\varphi}_n - \varphi_0, \quad \varphi_p = \bar{\psi}\hat{\varphi}_p - \varphi_0, \quad \psi = \bar{\psi}\hat{\psi}, \quad \boldsymbol{x} = \bar{x}\hat{\boldsymbol{x}}, \\ \mu_n = \bar{\mu}\hat{\mu}_n, \quad \mu_p = \bar{\mu}\hat{\mu}_p, \quad \varepsilon(\boldsymbol{x}) = \bar{\varepsilon}\hat{\varepsilon}(\hat{\boldsymbol{x}}), \quad C(\boldsymbol{x}) = \bar{C}\hat{C}(\hat{\boldsymbol{x}}),\end{gathered}$$

where $\bar{\psi}$, φ_0, $\bar{x}$, $\bar{\mu}$, $\bar{\varepsilon}$, $\bar{C}$ are reference values, while $\hat{\varphi}_n$, $\hat{\varphi}_p$, $\hat{\psi}$, $\hat{\boldsymbol{x}}$, $\hat{\mu}_n$, $\hat{\mu}_p$, $\hat{\varepsilon}$, $\hat{C}$, are scaled quantities. We choose

$$\begin{aligned}&\bar{C} = \sup_{\boldsymbol{x}\in\Omega} C(\boldsymbol{x}), \quad \bar{\psi} = V_{\text{th}} := \frac{k_B T}{q}, \quad \varphi_0 = \frac{E_c}{q} - V_{\text{th}} \log \frac{N_c}{\bar{C}}, \quad \bar{x} = \operatorname{diam}\Omega, \\ &\tau = \frac{\bar{x}^2}{\bar{\mu} V_{\text{th}}}, \quad \bar{\mu} = \sup_{\boldsymbol{x}\in\Omega} \max\{\mu_n(\boldsymbol{x}), \mu_p(\boldsymbol{x})\}, \quad \bar{\varepsilon} = \sup_{\boldsymbol{x}\in\Omega} \varepsilon(\boldsymbol{x}).\end{aligned}$$

We notice explicitly that from (2), the scaling implies

$$\hat{n}(\hat{\varphi}_n, \hat{\varphi}_p) = \exp(\hat{\psi} - \hat{\varphi}_n), \quad \hat{p}(\hat{\varphi}_n, \hat{\varphi}_p) = \frac{n_{\mathrm{i}}^2}{\bar{C}^2} \exp(\hat{\varphi}_p - \hat{\psi}). \tag{18}$$

We scale the generation term G and the recombination term R as follows

$$G(\boldsymbol{x}) = \bar{G}\hat{G}(\hat{\boldsymbol{x}}), \quad R(n,p) = \bar{R}\hat{R}(\hat{n}, \hat{p}),$$

with

$$\bar{R} = \frac{\bar{C}}{\tau}, \quad \bar{G} = \frac{n_{\mathrm{i}}^2}{\bar{C}\tau},$$

and

$$\begin{aligned}&\hat{R}(\hat{n}, \hat{p}) = \hat{r}(\hat{n}, \hat{p})(\hat{n}\hat{p} - n_{\mathrm{i}}^2/\bar{C}^2) \\ &\hat{r}(\hat{n}, \hat{p}) = \hat{C}_d + \hat{C}_n\hat{n} + \hat{C}_p\hat{p} + \frac{1}{\hat{\tau}_p(\hat{n} + \frac{n_{\mathrm{i}}}{C}\hat{n}_T) + \hat{\tau}_n(\hat{p} + \frac{n_{\mathrm{i}}}{C}\hat{p}_T)}\end{aligned} \tag{19}$$

where

$$\begin{gathered}\hat{C}_d = \tau\bar{C}C_d, \quad \hat{C}_n = \tau\bar{C}^2 C_n, \quad \hat{C}_p = \tau\bar{C}^2 C_p, \\ \hat{\tau}_n = \frac{\tau_n}{\tau}, \quad \hat{\tau}_p = \frac{\tau_p}{\tau}, \quad \hat{n}_T = \frac{n_T}{n_{\mathrm{i}}}, \quad \hat{p}_T = \frac{p_T}{n_{\mathrm{i}}}.\end{gathered}$$

The system (1) becomes

$$\begin{cases} -\frac{\bar{\varepsilon}V_{\rm th}}{q\bar{C}\bar{x}^2}\hat{\nabla}\cdot(\hat{\varepsilon}\hat{\nabla}\hat{\psi}) & = \hat{p}-\hat{n}+\hat{C}(\hat{\boldsymbol{x}}) \\ -\hat{\nabla}\cdot(\hat{\mu}_n\hat{n}\hat{\nabla}\hat{\varphi}_n) & = \hat{R}(\hat{n},\hat{p})-\frac{n_{\rm i}^2}{\bar{C}^2}\hat{G}(\hat{\boldsymbol{x}}), \\ -\hat{\nabla}\cdot(\hat{\mu}_p\hat{p}\hat{\nabla}\hat{\varphi}_p) & = \frac{n_{\rm i}^2}{\bar{C}^2}\hat{G}(\hat{\boldsymbol{x}})-\hat{R}(\hat{n},\hat{p}), \end{cases} \tag{20}$$

The current densities scale according to:

$$\boldsymbol{J}_n = q\frac{\bar{\mu}\bar{C}V_{\rm th}}{\bar{x}}\hat{\boldsymbol{J}}_n, \qquad \boldsymbol{J}_p = q\frac{\bar{\mu}\bar{C}V_{\rm th}}{\bar{x}}\hat{\boldsymbol{J}}_p. \tag{21}$$

The boundary conditions become:

$$\frac{\partial\hat{\psi}}{\partial\boldsymbol{\nu}} = \frac{\partial\hat{\varphi}_n}{\partial\boldsymbol{\nu}} = \frac{\partial\hat{\varphi}_p}{\partial\boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \tag{22}$$

$$\begin{aligned} \hat{\psi}-\hat{\psi}_0 = \hat{\varphi}_n-\hat{\varphi}_0 = \hat{\varphi}_p-\hat{\varphi}_0 = 0 \quad \text{on} \quad \Gamma_{D_1}, \\ \hat{\psi}-\hat{\psi}_0 = \hat{\varphi}_n-\hat{\varphi}_0 = \hat{\varphi}_p-\hat{\varphi}_0 = \hat{u}_D \quad \text{on} \quad \Gamma_{D_2}, \end{aligned} \tag{23}$$

where $\psi_0 = V_{\rm th}\hat{\psi}_0$, $\varphi_0 = V_{\rm th}\hat{\varphi}_0$, $u_D = \bar{u}_D\hat{u}_D = V_{\rm th}\hat{u}_D$. Using (8), (9) and (21) we have

$$\begin{aligned} i_D(u_D) &= \int_{\Gamma_{D_2}} \boldsymbol{\nu}\cdot(\boldsymbol{J}_n(\boldsymbol{x})+\boldsymbol{J}_p(\boldsymbol{x}))\mathrm{d}\sigma(\boldsymbol{x}) \\ &= \bar{i}_D\int_{\hat{\Gamma}_{D_2}} \boldsymbol{\nu}\cdot(\hat{\boldsymbol{J}}_n+\hat{\boldsymbol{J}}_p)\mathrm{d}\hat{\sigma}(\boldsymbol{x}) = \bar{i}_D\hat{i}_D(\hat{u}_D), \end{aligned} \tag{24}$$

with $\bar{i}_D = q\bar{\mu}\bar{C}V_{\rm th}\bar{x}$, and

$$\hat{i}_D(\hat{u}_D) = \int_{\hat{\Gamma}_{D_2}} \boldsymbol{\nu}\cdot(\hat{\boldsymbol{J}}_n+\hat{\boldsymbol{J}}_p)\mathrm{d}\hat{\sigma}(\boldsymbol{x}).$$

The coupling equation (11) becomes

$$\hat{u}_D = \hat{\mathcal{R}}\,\hat{i}_D(\hat{u}_D), \quad \text{with } \hat{\mathcal{R}} = q\bar{\mu}\bar{C}\bar{x}\mathcal{R}. \tag{25}$$

Omitting the ˆ and introducing the nondimensional parameters

$$\lambda = \sqrt{\frac{\bar{\varepsilon}V_{\rm th}}{q\bar{C}\bar{x}^2}}, \quad \delta = \frac{n_{\rm i}}{\bar{C}}, \tag{26}$$

we obtain

$$\begin{cases} -\lambda^2\nabla\cdot(\varepsilon\nabla\psi) & = \delta^2 e^{\varphi_p-\psi}-e^{\psi-\varphi_n}+C(\boldsymbol{x}), \\ -\nabla\cdot(\mu_n e^{\psi-\varphi_n}\nabla\varphi_n) & = \delta^2 r_\delta(n,p)(e^{\varphi_p-\varphi_n}-1)-\delta^2 G(\boldsymbol{x}), \\ -\nabla\cdot(\mu_p e^{\varphi_p-\psi}\nabla\varphi_p) & = G(\boldsymbol{x})-r_\delta(n,p)(e^{\varphi_p-\varphi_n}-1), \end{cases} \tag{27}$$

in which we have used $n = e^{\psi-\varphi_n}$, $p = \delta^2 e^{\varphi_p-\psi}$ and set

$$r_\delta(n,p) = C_d + C_n n + C_p p + \frac{1}{\tau_p(n+\delta n_T) + \tau_n(p+\delta p_T)}. \tag{28}$$

The boundary and coupling conditions are

$$\frac{\partial\psi}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi_n}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi_p}{\partial\boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \tag{29}$$

$$\begin{aligned} &\psi-\psi_0 = \varphi_n-\varphi_0 = \varphi_p-\varphi_0 = 0 \quad \text{on} \quad \Gamma_{D_1}, \\ &\psi-\psi_0 = \varphi_n-\varphi_0 = \varphi_p-\varphi_0 = u_D \quad \text{on} \quad \Gamma_{D_2}, \end{aligned} \tag{30}$$

$$u_D = \mathcal{R} i_D(u_D), \qquad i_D(u_D) = \int_{\Gamma_{D_2}} \boldsymbol{\nu}\cdot(\boldsymbol{J}_n+\boldsymbol{J}_p)\mathrm{d}\sigma(\boldsymbol{x}). \tag{31}$$

The dimensionless parameters in (26) are physically meaningful: λ is the scaled Debye length, while δ^2 is the ratio between the basic magnitude estimates for minority concentration, $p \approx n_\mathrm{i}^2/\bar{C}$, and majority concentration, $n \approx \bar{C}$. We compute the dimensionless parameters in (26) for two standard semiconductors, namely silicon and gallium arsenide. The corresponding parameters can be found in Appendix A. The laser power P is chosen for both cases to be 20 mW and the conduction band-edge energy $E_c = k_B T\log(N_c/\bar{C})$. For a silicon crystal, as described in [7,15,11], the parameters in (26) become

$$\lambda \approx 1.249382\times10^{-5}, \quad \delta \approx 5.528936\times10^{-7}. \tag{32}$$

For a gallium arsenide crystal, as described in [13], the parameters in (26) become

$$\lambda \approx 1.306319\times10^{-6}, \quad \delta \approx 2.154036\times10^{-12}. \tag{33}$$

These calculations show that for two standard semiconductor crystals the parameter δ is indeed a small parameter, and it is appropriate to perform an asymptotic analysis for δ tending to zero. In this limit, the system (27) becomes

$$\begin{cases} -\lambda^2\nabla\cdot(\varepsilon\nabla\psi) = -e^{\psi-\varphi_n} + C(\boldsymbol{x}), \\ -\nabla\cdot(\mu_n e^{\psi-\varphi_n}\nabla\varphi_n) = 0, \\ -\nabla\cdot(\mu_p e^{\varphi_p-\psi}\nabla\varphi_p) = G(\boldsymbol{x}) - r_0(e^{\psi-\varphi_n})(e^{\varphi_p-\varphi_n}-1), \\ \dfrac{\partial\psi}{\partial\boldsymbol{\nu}} = \dfrac{\partial\varphi_n}{\partial\boldsymbol{\nu}} = \dfrac{\partial\varphi_p}{\partial\boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\ \psi-\psi_0 = \varphi_n-\varphi_0 = \varphi_p-\varphi_0 = 0, \quad \text{on} \quad \Gamma_{D_1}, \\ \psi-\psi_0 = \varphi_n-\varphi_0 = \varphi_p-\varphi_0 = u_D, \quad \text{on} \quad \Gamma_{D_2}, \\ u_D = \mathcal{R} i_D(u_D), \qquad i_D(u_D) = \displaystyle\int_{\Gamma_{D_2}} \boldsymbol{\nu}\cdot\boldsymbol{J}_n d\sigma(\boldsymbol{x}). \end{cases} \tag{34}$$

Here we have used the notation r_0 to indicate that we put $\delta = 0$ in (28) so that

$$r_0(n) = C_d + C_n n + \frac{1}{\tau_p n}. \tag{35}$$

We notice that the hole current for $\delta = 0$ vanishes since

$$\boldsymbol{J}_p = -\delta^2 \mu_p e^{\varphi_p - \psi} \nabla \varphi_p, \tag{36}$$

and from the second equation in (34) the electron current density $\boldsymbol{J}_n$ is constant, this implies that $i_D(u_D) = 0$ which means there is no coupling between the circuit and the drift diffusion system at the lowest order. For this reason, we perform an asymptotic expansion in the next section to find leading-order models which capture the most important characteristics from (27).

4. Asymptotic expansion

In this section, we perform an asymptotic analysis, using the standard procedure as codified, for instance, in [18,19]. We consider the following asymptotic expansions:

$$\psi = \sum_{k\geq 0} \psi^{(k)} \delta^k, \quad \varphi_n = \sum_{k\geq 0} \varphi_n^{(k)} \delta^k, \quad \varphi_p = \sum_{k\geq 0} \varphi_p^{(k)} \delta^k, \quad u_D = \sum_{k\geq 0} u_D^{(k)} \delta^k, \tag{37}$$

which in turns yield expansions for $n = \exp(\psi - \varphi_n)$, $p = \delta^2 \exp(\varphi_p - \psi)$, $R = \delta^2 r_\delta \left(\frac{np}{\delta^2} - 1\right)$.

Lemma 1. *Assuming expansions (37), we have*

$$n(\psi, \varphi_n) = \sum_{k\geq 0} n^{(k)} \delta^k, \qquad p(\psi, \varphi_p) = \delta^2 \sum_{k\geq 0} p^{(k)} \delta^k, \qquad R = \delta^2 \sum_{k\geq 0} R^{(k)} \delta^k, \tag{38}$$

with

$$\begin{aligned}
n^{(0)} &= \exp(\psi^{(0)} - \varphi_n^{(0)}), \quad n^{(1)} = n^{(0)}(\psi^{(1)} - \varphi_n^{(1)}), \\
n^{(k)} &= n^{(0)}(\psi^{(k)} - \varphi_n^{(k)}) + n^{(0)} F^{(k)} \left(\frac{n^{(1)}}{n^{(0)}}, \dots, \frac{n^{(k-1)}}{n^{(0)}}\right), \quad k \geq 2, \\
p^{(0)} &= \exp(\varphi_p^{(0)} - \psi^{(0)}), \quad p^{(1)} = p^{(0)}(\varphi_p^{(1)} - \psi^{(1)}), \\
p^{(k)} &= p^{(0)}(\varphi_p^{(k)} - \psi^{(k)}) + p^{(0)} F^{(k)} \left(\frac{p^{(1)}}{p^{(0)}}, \dots, \frac{p^{(k-1)}}{p^{(0)}}\right), \quad k \geq 2,
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
R^{(0)} &= r_0(n^{(0)})(n^{(0)} p^{(0)} - 1), \\
R^{(1)} &= s_n(n^{(0)}, p^{(0)}) n^{(1)} + s_p(n^{(0)}) p^{(1)} - \frac{\tau_n p_T + \tau_p n_T}{\tau_p^2 n^{(0)2}} (n^{(0)} p^{(0)} - 1), \\
R^{(k)} &= s_n(n^{(0)}, p^{(0)}) n^{(k)} + s_p(n^{(0)}) p^{(k)} + F_R^{(k)}(n^{(0)}, \dots, n^{(k-1)}, p^{(0)}, \dots, p^{(k-1)}), \quad k \geq 2,
\end{aligned} \tag{40}$$

in which

$$s_n(n, p) = C_d p + 2 C_n n p - C_n + \frac{1}{\tau_p n^2}, \quad s_p(n) = C_d n + C_n n^2 + \frac{1}{\tau_p},$$

and $F^{(k)}$, and $F_R^{(k)}$ are some multivariate polynomial functions.

Proof. Let us consider an analytic function $f(z)$, and a function defined by a power series $g(\delta) = \sum_{k\geq 0} a_k \delta^k$. We can expand $f(g(\delta))$ in power series of δ,

$$f(g(\delta)) = f(g(0)) + \sum_{k\geq 1} \left[\frac{\mathrm{d}^k}{\mathrm{d}\delta^k} f(g(\delta))\right]_{\delta=0} \frac{\delta^k}{k!}.$$

Using Faà di Bruno's formula, we can write the derivatives as

$$\frac{\mathrm{d}^k}{\mathrm{d}\delta^k} f(g(\delta)) = \sum_{j_1+2j_2+\cdots kj_k=k} \frac{k! f^{(j)}(g(\delta))}{j_1! j_2! \cdots j_k!} \left(\frac{g'(\delta)}{1!}\right)^{j_1} \left(\frac{g''(\delta)}{2!}\right)^{j_2} \cdots \left(\frac{g^{(k)}(\delta)}{k!}\right)^{j_k},$$

where the sum runs over all nonnegative integers $j_1, j_2, \ldots, j_k$ which satisfies the Diophantine equation written below the summation, and $j = j_1 + j_2 + \cdots + j_k$. Recalling the definition of $g(\delta)$, it is simple to obtain

$$g^{(i)}(\delta) = \sum_{k \geq i} k(k-1)\cdots(k-i+1) a_k \delta^{k-i}, \quad g^{(i)}(0) = i!\, a_i,$$

and thus we have

$$f(g(\delta)) = f(a_0) + \sum_{k\geq 1} \sum_{j_1+2j_2+\cdots kj_k=k} \frac{f^{(j)}(a_0)}{j_1! j_2! \cdots j_k!} a_1^{j_1} a_2^{j_2} \cdots a_k^{j_k} \delta^k. \tag{41}$$

Applying (41) to n, with $f(z) = e^z$, $a_k = \psi^{(k)} - \varphi_n^{(k)}$, and observing that $f^{(k)}(z) = f(z)$ for all $k \geq 0$, we get

$$n = \exp\left(\psi^{(0)} - \varphi_n^{(0)}\right) \left(1 + \sum_{k\geq 1} \sum_{j_1+\cdots+kj_k=k} \frac{\left(\psi^{(1)} - \varphi_n^{(1)}\right)^{j_1} \cdots \left(\psi^{(k)} - \varphi_n^{(k)}\right)^{j_k}}{j_1! \cdots j_k!} \delta^k\right). \tag{42}$$

Thus, we have

$$\begin{aligned} n^{(0)} &= \exp\left(\psi^{(0)} - \varphi_n^{(0)}\right), \\ n^{(k)} &= \sum_{j_1+\cdots+kj_k=k} \frac{n^{(0)}}{j_1! \cdots j_k!} \left(\psi^{(1)} - \varphi_n^{(1)}\right)^{j_1} \cdots \left(\psi^{(k)} - \varphi_n^{(k)}\right)^{j_k}. \end{aligned}$$

In particular, for $k = 1$ we have

$$n^{(1)} = n^{(0)} \left(\psi^{(1)} - \varphi_n^{(1)}\right),$$

which yields $f^{(1)} = 0$, and

$$\psi^{(1)} - \varphi_n^{(1)} = \frac{n^{(1)}}{n^{(0)}}.$$

For $k = 2$, the condition $j_1 + 2j_2 = 2$ is satisfied only by $(j_1, j_2) = (0, 1)$, $(2, 0)$, and we have

$$n^{(2)} = n^{(0)} \left(\psi^{(2)} - \varphi_n^{(2)}\right) + \frac{n^{(0)}}{2} \left(\psi^{(1)} - \varphi_n^{(1)}\right)^2 = n^{(0)} \left(\psi^{(2)} - \varphi_n^{(2)})\right) + \frac{n^{(0)}}{2} \left(\frac{n^{(1)}}{n^{(0)}}\right)^2,$$

that is

$$n^{(2)} = n^{(0)} \left(\psi^{(2)} - \varphi_n^{(2)}\right) + n^{(0)} F^{(2)} \left(\frac{n^{(1)}}{n^{(0)}}\right),$$

with $F^{(2)}(r) = \frac{1}{2} r^2$. Proceeding by iteration, we assume that

$$n^{(j)} = n^{(0)}\left(\psi^{(j)} - \varphi_n^{(j)}\right) + n^{(0)}F^{(j)}\left(\frac{n^{(1)}}{n^{(0)}}, \dots, \frac{n^{(j-1)}}{n^{(0)}}\right), \quad 1 \le j < k.$$

We observe that the only solution to $j_1 + 2j_2 + \cdots + kj_k = k$ with $j_k > 0$ is $(j_1, j_2, \dots, j_k) = (0, 0, \dots, 1)$. Then we can write

$$n^{(k)} = n^{(0)}\left(\psi^{(k)} - \varphi_n^{(k)}\right) + \sum_{j_1+\cdots+(k-1)j_{k-1}=k} \frac{n^{(0)}}{j_1!\cdots j_{k-1}!}\left(\psi^{(1)} - \varphi_n^{(1)}\right)^{j_1}\cdots\left(\psi^{(k-1)} - \varphi_n^{(k-1)}\right)^{j_{k-1}},$$

where the summation is intended over all indices, bearing in mind that $j_k = 0$. Hence, $j_k! = 1$ and $a_k^{j_k} = 1$. From the induction hypothesis we have

$$\psi^{(j)} - \varphi_n^{(j)} = \frac{n^{(j)}}{n^{(0)}} - F^{(j)}\left(\frac{n^{(1)}}{n^{(0)}}, \dots, \frac{n^{(j-1)}}{n^{(0)}}\right), \quad 1 \le j < k,$$

which yields

$$n^{(k)} = n^{(0)}\left(\psi^{(k)} - \varphi_n^{(k)}\right) + n^{(0)}F^{(k)}\left(\frac{n^{(1)}}{n^{(0)}}, \dots, \frac{n^{(k-1)}}{n^{(0)}}\right),$$

with

$$F^{(k)}(z_1, \dots, z_{k-1}) = \sum_{j_1+\cdots+(k-1)j_{k-1}=k} \frac{1}{j_1!\cdots j_{k-1}!} z_1^{j_1}\cdots\left(z_{k-1} - f_n^{(k-1)}(z_1, \dots, z_{k-2})\right)^{j_{k-1}}.$$

Similarly, we can prove the expansion of p. For the expansion of R, first we consider the expansion of $r_\delta(n, p)$. We apply (41) to the function $f(z) = z^{-1}$, with $a_0 = \tau_p n^{(0)}$, $a_1 = \tau_p n^{(1)} + \tau_p n_T + \tau_n p_T$, $a_k = \tau_p n^{(k)} + \tau_n p^{(k-2)}$, $k \ge 2$, and observe that $f^{(k)}(z) = (-1)^k k! z^{-k-1}$. Then

$$\frac{1}{\tau_p(n + \delta n_T) + \tau_n(p + \delta p_T)} = a_0^{-1}\left(1 + \sum_{k\ge1}\sum_{j_1+2j_2+\cdots kj_k=k} \frac{(-1)^j j!}{j_1! j_2! \cdots j_k!} \frac{a_1^{j_1} a_2^{j_2}\cdots a_k^{j_k}}{a_0^j}\delta^k\right),$$

and we find

$$r_\delta\left(\sum_k n^{(k)}\delta^k, \delta^2\sum_k p^{(k)}\delta^k\right) = \sum_{k\ge0} r^{(k)}\delta^k,$$

with

$$r^{(0)} = r_0(n^{(0)}), \quad r^{(k)} = c(n^{(0)})n^{(k)} + F_r^{(k)}, \quad k \ge 0,$$

where

$$r_0(n) := C_d + C_n n + \frac{1}{\tau_p n}, \quad c(n) := C_n - \frac{1}{\tau_p n^2}, \quad F_r^{(1)} := -\frac{\tau_p n_T + \tau_n p_T}{\tau_p^2 n^{(0)2}},$$

$$F_r^{(k)} := \left(C_p - \frac{\tau_n}{\tau_p n^{(0)2}}\right)p^{(k-2)} + \sum_{j_1+2j_2+\cdots+(k-1)j_{k-1}=k} \frac{(-1)^j j!}{j_1! j_2! \cdots j_{k-1}!}\frac{a_1^{j_1} a_2^{j_2}\cdots a_{k-1}^{j_{k-1}}}{a_0^j}, \quad k \ge 2.$$

Using the Cauchy product

$$\frac{np}{\delta^2} = \sum_{k\geq 0}\sum_{j=0}^{k} n^{(j)} p^{(k-j)} \delta^k,$$

and combining it with the expansion of r_δ, we get

$$R = \delta^2 \sum_{k\geq 0} R^{(k)}\delta^k, \quad R^{(k)} := \sum_{j=0}^{k} r^{(j)} \sum_{i=0}^{k-j} n^{(i)} p^{(k-j-i)} - r^{(k)},$$

which, thanks to the identities $s_n(n,p) = r_0(n)p + c(n)(np-1)$, $s_p(n) = r_0(n)n$, gives

$$R^{(0)} = r_0(n^{(0)})(n^{(0)}p^{(0)} - 1), \quad R^{(k)} = s_n(n^{(0)}, p^{(0)})n^{(k)} + s_p(n^{(0)})p^{(k)} + F_R^{(k)}, \quad k \geq 1,$$

where

$$\begin{aligned} F_R^{(1)} &:= F_r^{(1)}(n^{(0)}p^{(0)} - 1), \\ F_R^{(k)} &:= F_r^{(k)}(n^{(0)}p^{(0)} - 1) + r^{(0)} \sum_{i=1}^{k-1} n^{(i)} p^{(k-i)} + \sum_{j=1}^{k-1} r^{(j)} \sum_{i=0}^{k-j} n^{(i)} p^{(k-j-i)}. \end{aligned}$$

This proves the final part of the lemma. □

Using the above expansions in (27) we find:

$$\begin{cases} -\lambda^2 \nabla\cdot(\varepsilon\nabla\psi^{(k)}) = F_\psi^{(k)}, \\ -\nabla\cdot(\mu_n n^{(0)}\nabla\varphi_n^{(k)} + \mu_n n^{(k)}\nabla\varphi_n^{(0)}) = F_{\varphi_n}^{(k)}, \\ -\nabla\cdot(\mu_p p^{(0)}\nabla\varphi_p^{(k)} + \mu_p p^{(k)}\nabla\varphi_p^{(0)}) = F_{\varphi_p}^{(k)}, \end{cases} \tag{43}$$

where

$$\begin{gathered} F_\psi^{(0)} = -n^{(0)} + C(\boldsymbol{x}), \qquad F_\psi^{(1)} = -n^{(1)}, \qquad F_\psi^{(k)} = p^{(k-2)} - n^{(k)}, \qquad k > 1, \\ F_{\varphi_n}^{(0)} = F_{\varphi_n}^{(1)} = 0, \qquad F_{\varphi_n}^{(2)} = R^{(0)} - G(\boldsymbol{x}) + \nabla\cdot(\mu_n n^{(1)}\nabla\varphi_n^{(1)}), \\ F_{\varphi_n}^{(k)} = R^{(k)} + \sum_{j=1}^{k-1} \nabla\cdot(\mu_n n^{(j)}\nabla\varphi_n^{(k-j)}), \qquad k > 2, \\ F_{\varphi_p}^{(0)} = 2(G(\boldsymbol{x}) - R^{(0)}), \qquad F_{\varphi_p}^{(k)} = -R^{(k)} + \sum_{j=1}^{k-1} \nabla\cdot(\mu_p p^{(j)}\nabla\varphi_p^{(k-j)}), \qquad k > 0, \end{gathered}$$

and applying the expansion to the coupling condition (31) we find:

$$u_D^{(k)} = \begin{cases} -\mathcal{R}\displaystyle\int_{\Gamma_{D_2}} \mu_n n^{(0)} \frac{\partial\varphi_n^{(0)}}{\partial\boldsymbol{\nu}} \mathrm{d}\sigma(\boldsymbol{x}), \quad k = 0, \\ -\mathcal{R}\displaystyle\int_{\Gamma_{D_2}} \left(\mu_n n^{(0)} \frac{\partial\varphi_n^{(1)}}{\partial\boldsymbol{\nu}} + \mu_n n^{(1)} \frac{\partial\varphi_n^{(0)}}{\partial\boldsymbol{\nu}}\right) \mathrm{d}\sigma(\boldsymbol{x}), \quad k = 1, \\ -\mathcal{R}\displaystyle\int_{\Gamma_{D_2}} \left(\mu_n \sum_{j=0}^{k} n^{(j)} \frac{\partial\varphi_n^{(k-j)}}{\partial\boldsymbol{\nu}} + \mu_p \sum_{j=0}^{k-2} p^{(j)} \frac{\partial\varphi_p^{(k-2-j)}}{\partial\boldsymbol{\nu}}\right) \mathrm{d}\sigma(\boldsymbol{x}), \quad k > 1. \end{cases} \tag{44}$$

We have

$$\frac{\partial \psi^{(k)}}{\partial \boldsymbol{\nu}} = \frac{\partial \varphi_n^{(k)}}{\partial \boldsymbol{\nu}} = \frac{\partial \varphi_p^{(k)}}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \quad k \geq 0, \tag{45}$$

$$\begin{aligned} &\psi^{(0)} - \psi_0 = \varphi_n^{(0)} - \varphi_0 = \varphi_p^{(0)} - \varphi_0 = 0, \quad \text{on} \quad \Gamma_{D_1}, \\ &\psi^{(0)} - \psi_0 = \varphi_n^{(0)} - \varphi_0 = \varphi_p^{(0)} - \varphi_0 = u_D^{(0)}, \quad \text{on} \quad \Gamma_{D_2}, \end{aligned} \tag{46}$$

$$\begin{aligned} &\psi^{(k)} = \varphi_n^{(k)} = \varphi_p^{(k)} = 0, \quad \text{on} \quad \Gamma_{D_1}, \\ &\psi^{(k)} = \varphi_n^{(k)} = \varphi_p^{(k)} = u_D^{(k)}, \quad \text{on} \quad \Gamma_{D_2}, \quad k \geq 1. \end{aligned} \tag{47}$$

We focus our attention on the leading order terms for the expansion of the carrier concentrations n, p and of the potential u_D. In the next section, we will show that

$$\varphi_n^{(0)} - \varphi_0 = \psi^{(1)} = \varphi_n^{(1)} = 0, \qquad u_D^{(0)} = u_D^{(1)} = 0. \tag{48}$$

Thus, we concentrate on the corresponding leading order terms for the potentials

$$\psi = \psi^{(0)} + \delta^2 \psi^{(2)} + O(\delta^3), \quad \varphi_n = \varphi_0 + \delta^2 \varphi_n^{(2)} + O(\delta^3), \quad \varphi_p = \varphi_p^{(0)} + O(\delta),$$

which lead to

$$\begin{aligned} &n = e^{\psi^{(0)} - \varphi_0} \left(1 + \delta^2 (\psi^{(2)} - \varphi_n^{(2)})\right) + O(\delta^3), \quad p = \delta^2 e^{\varphi_p^{(0)} - \psi^{(0)}} + O(\delta^3), \\ &u_D = \delta^2 u_D^{(2)} + O(\delta^3). \end{aligned}$$

In the following section, we will show that the resulting simplified model depends only on the five unknowns $\psi^{(0)}$, $\varphi_p^{(0)}$, $\psi^{(2)}$, $\varphi_n^{(2)}$ and $u_D^{(2)}$ which satisfy the following equations

$$\begin{cases} -\lambda^2 \nabla \cdot (\varepsilon \nabla \psi^{(0)}) & = -e^{\psi^{(0)} - \varphi_0} + C(\boldsymbol{x}), \\ -\nabla \cdot (\mu_p e^{\varphi_p^{(0)} - \psi^{(0)}} \nabla \varphi_p^{(0)}) & = -r_0 (e^{\psi^{(0)} - \varphi_0})(e^{\varphi_p^{(0)} - \varphi_0} - 1) + G(\boldsymbol{x}), \\ -\lambda^2 \nabla \cdot (\varepsilon \nabla \psi^{(2)}) & = e^{\varphi_p^{(0)} - \psi^{(0)}} - e^{\psi^{(0)} - \varphi_0} (\psi^{(2)} - \varphi_n^{(2)}), \\ -\nabla \cdot (\mu_n e^{\psi^{(0)} - \varphi_0} \nabla \varphi_n^{(2)}) & = r_0 (e^{\psi^{(0)} - \varphi_0})(e^{\varphi_p^{(0)} - \varphi_0} - 1) - G(\boldsymbol{x}), \end{cases} \tag{49}$$

with coupling condition

$$u_D^{(2)} = -\mathcal{R} \int_{\Gamma_{D_2}} \left(\mu_n e^{\psi^{(0)} - \varphi_0} \frac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} + \mu_p e^{\varphi_p^{(0)} - \psi^{(0)}} \frac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}} \right) \mathrm{d}\sigma(\boldsymbol{x}), \tag{50}$$

and boundary conditions

$$\frac{\partial \psi^{(0)}}{\partial \boldsymbol{\nu}} = \frac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}} = \frac{\partial \psi^{(2)}}{\partial \boldsymbol{\nu}} = \frac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \tag{51}$$

$$\begin{aligned} &\psi^{(0)} - \psi_0 = \varphi_p^{(0)} - \varphi_0 = \psi^{(2)} = \varphi_n^{(2)} = 0, \quad \text{on} \quad \Gamma_{D_1}, \\ &\psi^{(0)} - \psi_0 = \varphi_p^{(0)} - \varphi_0 = 0, \quad \psi^{(2)} = \varphi_n^{(2)} = u_D^{(2)}, \quad \text{on} \quad \Gamma_{D_2}. \end{aligned} \tag{52}$$

We observe that $(49)_1$ and the corresponding boundary conditions result into a nonlinear elliptic boundary value problem for $\psi^{(0)}$. Once $\psi^{(0)}$ is known, equation $(49)_2$ and the corresponding boundary conditions become a nonlinear elliptic boundary value problem for $\varphi_p^{(0)}$. Then, equations $(49)_4$ and (50) and the corresponding boundary conditions are a coupled problem for $\varphi_n^{(2)}$ and $u_D^{(2)}$. Finally, equation $(49)_3$ and the corresponding boundary conditions constitute a linear boundary value problem for $\psi^{(2)}$.

In the following section, we prove the validity of (48) and the well-posedness of (49)–(52).

5. Existence and uniqueness of solutions

Based on the previous discussion we need to keep only the zeroth order term for φ_p in order to have second order expansion for the carrier densities. Hence, we focus our attention on

$$\begin{aligned}
\psi &= \psi^{(0)} + \delta\psi^{(1)} + \delta^2\psi^{(2)} + O(\delta^3),\\
\varphi_n &= \varphi_n^{(0)} + \delta\varphi_n^{(1)} + \delta^2\varphi_n^{(2)} + O(\delta^3),\\
\varphi_p &= \varphi_p^{(0)} + O(\delta),\\
u_D &= u_D^{(0)} + \delta u_D^{(1)} + \delta^2 u_D^{(2)} + O(\delta^3),
\end{aligned}$$

and consider the resulting equations for each order separately in order to prove the validity of (48) and show existence and uniqueness for the system (49)–(52).

5.1. Existence and uniqueness of solutions for the zero order problem

At order 0, the nonlinear equations for $\psi^{(0)}$, $\varphi_n^{(0)}$, $\varphi_p^{(0)}$ and $u_D^{(0)}$, spelled out in full detail, are:

$$\begin{cases}
-\lambda^2\nabla\cdot\left(\varepsilon\nabla\psi^{(0)}\right) & = -e^{\psi^{(0)}-\varphi_n^{(0)}} + C(\boldsymbol{x}),\\
-\nabla\cdot\left(\mu_n e^{\psi^{(0)}-\varphi_n^{(0)}}\nabla\varphi_n^{(0)}\right) & = 0,\\
-\nabla\cdot\left(\mu_p e^{\varphi_p^{(0)}-\psi^{(0)}}\nabla\varphi_p^{(0)}\right) & = G(\boldsymbol{x}) - r_0\left(e^{\psi^{(0)}-\varphi_n^{(0)}}\right)\left(e^{\varphi_p^{(0)}-\varphi_n^{(0)}}-1\right),
\end{cases} \tag{53}$$

$$\frac{\partial\psi^{(0)}}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi_n^{(0)}}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi_p^{(0)}}{\partial\boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \tag{54}$$

$$\begin{aligned}
&\psi^{(0)} - \psi_0 = \varphi_n^{(0)} - \varphi_0 = \varphi_p^{(0)} - \varphi_0 = 0, \quad \text{on} \quad \Gamma_{D_1},\\
&\psi^{(0)} - \psi_0 = \varphi_n^{(0)} - \varphi_0 = \varphi_p^{(0)} - \varphi_0 = u_D^{(0)}, \quad \text{on} \quad \Gamma_{D_2},
\end{aligned} \tag{55}$$

$$u_D^{(0)} = -\mathcal{R}\int_{\Gamma_{D_2}} \mu_n e^{\psi^{(0)}-\varphi_n^{(0)}}\frac{\partial\varphi_n^{(0)}}{\partial\boldsymbol{\nu}}\mathrm{d}\sigma(\boldsymbol{x}). \tag{56}$$

Theorem 1. *Assuming that C, $G \in L^\infty(\Omega)$, the zero-order equations (53) with boundary conditions (54)–(55) and coupling condition (56) admit a unique solution in $H^1(\Omega)\cap L^\infty(\Omega)$ for $\psi^{(0)}$, $\varphi_n^{(0)}$, $\varphi_p^{(0)}$, with $u_D^{(0)} \in \mathbb{R}$. Moreover this solution satisfies the identities*

$$u_D^{(0)} = 0, \quad \varphi_n^{(0)} = \varphi_0, \tag{57}$$

and the estimates

$$\min\left\{\inf_{\Gamma_D}\psi_0,\, \varphi_0 + \ln\underline{C}\right\} =: \underline{\psi}^{(0)} \le \psi^{(0)} \le \overline{\psi}^{(0)} := \max\left\{\sup_{\Gamma_D}\psi_0,\, \varphi_0 + \ln\overline{C}\right\}, \tag{58}$$

$$\varphi_0 + \underline{\varphi}_p \le \varphi_p^{(0)} \le \varphi_0 + \overline{\varphi}_p \tag{59}$$

where

$$\underline{\varphi}_p := \ln(\underline{r}/\overline{r}), \quad \overline{\varphi}_p := \ln \frac{\overline{r}+\overline{G}}{\underline{r}}, \quad \underline{C} := \inf_{\Omega} C(x),$$
$$\overline{C} := \sup_{\Omega} C(x) \equiv 1, \quad \overline{G} := \sup_{\Omega} G(x), \tag{60}$$

in which

$$\underline{r} = C_d + C_n e^{\underline{\psi}^{(0)}-\varphi_0} + \frac{1}{\tau_p} e^{\varphi_0 - \overline{\psi}^{(0)}}, \qquad \overline{r} = C_d + C_n e^{\overline{\psi}^{(0)}-\varphi_0} + \frac{1}{\tau_p} e^{\varphi_0 - \underline{\psi}^{(0)}}. \tag{61}$$

Proof. First, we prove that any solution of (53)–(56) satisfies (57). We multiply $(53)_2$ by $\varphi_n^{(0)} - \varphi_0$, integrate over Ω and use integration by parts to obtain

$$\begin{aligned} 0 &= -\int_{\Omega} (\varphi_n^{(0)} - \varphi_0) \nabla \cdot \left(\mu_n e^{\psi^{(0)} - \varphi_n^{(0)}} \nabla \varphi_n^{(0)} \right) \mathrm{d}\boldsymbol{x} \\ &= -\int_{\partial\Omega} (\varphi_n^{(0)} - \varphi_0) \mu_n e^{\psi^{(0)} - \varphi_n^{(0)}} \frac{\partial \varphi_n^{(0)}}{\partial \boldsymbol{\nu}} \mathrm{d}\sigma(\boldsymbol{x}) + \int_{\Omega} \mu_n e^{\psi^{(0)} - \varphi_n^{(0)}} |\nabla \varphi_n^{(0)}|^2 \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{\mathcal{R}} |u_D^{(0)}|^2 + \int_{\Omega} \mu_n e^{\psi^{(0)} - \varphi_n^{(0)}} |\nabla \varphi_n^{(0)}|^2 \mathrm{d}\boldsymbol{x}, \end{aligned}$$

which implies (57). Then $\psi^{(0)}$ must solve

$$\begin{cases} -\lambda^2 \nabla \cdot \left(\varepsilon \nabla \psi^{(0)} \right) & = -e^{\psi^{(0)} - \varphi_0} + C(\boldsymbol{x}), \\ \frac{\partial \psi^{(0)}}{\partial \boldsymbol{\nu}} & = 0, \quad \text{on } \Gamma_N, \\ \psi^{(0)} - \psi_0 & = 0, \quad \text{on } \quad \Gamma_{D_1} \cup \Gamma_{D_2}, \end{cases} \tag{62}$$

and, with standard methods it is simple to prove that a solution exists uniquely and satisfies the bounds (58).

Finally, $\varphi_p^{(0)}$ must solve

$$\begin{cases} -\nabla \cdot \left(\mu_p e^{\varphi_p^{(0)} - \psi^{(0)}} \nabla \varphi_p^{(0)} \right) & = G(\boldsymbol{x}) - r_0 \left(e^{\psi^{(0)} - \varphi_0} \right) \left(e^{\varphi_p^{(0)} - \varphi_0} - 1 \right), \\ \frac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}} & = 0, \quad \text{on } \Gamma_N, \\ \varphi_p^{(0)} - \varphi_0 & = 0, \quad \text{on} \quad \Gamma_{D_1} \cup \Gamma_{D_2}. \end{cases} \tag{63}$$

Equation (63) can be written in the form

$$-\nabla \cdot \left(\mu_p e^{\varphi_p^{(0)} - \psi^{(0)}} \nabla \varphi_p^{(0)} \right) = f(\varphi_p^{(0)}, \boldsymbol{x}), \tag{64}$$

with

$$f(\varphi, \boldsymbol{x}) := G(\boldsymbol{x}) + r_0 \left(e^{\psi^{(0)}(\boldsymbol{x}) - \varphi_0} \right) - r_0 \left(e^{\psi^{(0)}(\boldsymbol{x}) - \varphi_0} \right) e^{\varphi - \varphi_0}.$$

It is simple to verify that

$$\underline{r} \le r_0 \left(e^{\psi^{(0)}(\boldsymbol{x}) - \varphi_0} \right) \le \overline{r},$$

thus, we have

$$-\overline{r}\exp\varphi + \underline{r} =: \underline{f}(\varphi) \le f(\varphi, x) \le \overline{f}(\varphi) := -\underline{r}\exp\varphi + \overline{r} + \overline{G}. \tag{65}$$

The function $\underline{f}(\varphi)$ vanishes if and only if $\varphi - \varphi_0 = \underline{\varphi}_p := \ln(\underline{r}/\overline{r})$. Similarly, the function $\overline{f}(\overline{\varphi})$ vanishes if and only if $\varphi - \varphi_0 = \overline{\varphi}_p := \ln \frac{\overline{r}+\overline{G}}{\underline{r}}$. It follows that $\varphi_p^{(0)} = \varphi_0 + \underline{\varphi}_p$ is a lower solution and $\varphi_p^{(0)} = \varphi_0 + \overline{\varphi}_p$ is an upper solution for (64). Then, it follows that $\varphi_p^{(0)}$ satisfies the bounds (59). □

5.2. Existence and uniqueness of solutions for the first order problem

Next we consider equations at order 1, taking into account the results of Theorem 1. The resulting linear equations for $\psi^{(1)}$, $\varphi_n^{(1)}$, $u_D^{(1)}$ are:

$$\begin{cases} -\lambda^2\nabla\cdot(\varepsilon\nabla\psi^{(1)}) = -n^{(0)}(\psi^{(1)} - \varphi_n^{(1)}), \\ -\nabla\cdot(\mu_n n^{(0)}\nabla\varphi_n^{(1)}) = 0, \\ \frac{\partial\psi^{(1)}}{\partial\boldsymbol{\nu}} = \frac{\partial\varphi_n^{(1)}}{\partial\boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\ \psi^{(1)} = \varphi_n^{(1)} = 0, \quad \text{on} \quad \Gamma_{D_1}, \qquad \psi^{(1)} = \varphi_n^{(1)} = u_D^{(1)}, \quad \text{on} \quad \Gamma_{D_2}, \\ u_D^{(1)} = -\mathcal{R}\displaystyle\int\limits_{\Gamma_{D_2}} \mu_n e^{\psi^{(0)}}\frac{\partial\varphi_n^{(1)}}{\partial\boldsymbol{\nu}}\,\mathrm{d}\sigma(\boldsymbol{x}), \end{cases} \tag{66}$$

with $n^{(0)}$, $p^{(0)}$ given by Lemma 1, that is,

$$n^{(0)} = e^{\psi^{(0)}-\varphi_0}, \quad p^{(0)} = e^{\varphi_p^{(0)}-\psi^{(0)}}. \tag{67}$$

Theorem 2. *Assuming that $\psi^{(0)}$, $\varphi_p^{(0)} \in H^1(\Omega)\cap L^\infty(\Omega)$, $\varphi_n^{(0)} = \varphi_0$, $u_D^{(0)} = 0$ is the solution to the zero-order equations (53)–(56) given by Theorem 1, the first order system of equations (66) admits a unique solution in $H^1(\Omega)\cap L^\infty(\Omega)$ for $\psi^{(1)}$, $\varphi_n^{(1)}$, with $u_D^{(1)} \in \mathbb{R}$. Moreover, this solution is trivial and satisfies the identities*

$$u_D^{(1)} = 0, \quad \varphi_n^{(1)} = 0, \quad \psi^{(1)} = 0. \tag{68}$$

Proof. Using the second and the last equations in $(66)_2$, together with the boundary conditions, with a similar argument to the one used in Theorem 1, it is possible to show that $u_D^{(1)} = 0$ and $\varphi_n^{(1)} = 0$. Then the equation for the potential $\psi^{(1)}$ reduces to

$$\begin{cases} -\lambda^2\nabla\cdot(\varepsilon\nabla\psi^{(1)}) = -n^{(0)}\psi^{(1)}, \\ \frac{\partial\psi^{(1)}}{\partial\boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \quad \psi^{(1)} = 0, \quad \text{on} \quad \Gamma_{D_1}\cup\Gamma_{D_2}, \end{cases}$$

which is a linear elliptic homogeneous problem with unique solution $\psi^{(1)} = 0$. Thus, (68) is proven. □

5.3. Existence and uniqueness of solutions for the second order problem

Finally, taking into account the results of Theorem 1 and Theorem 2, the equations at order 2 decouple in two systems for $\varphi_n^{(2)}$ and $\psi^{(2)}$, namely

$$\begin{cases} -\nabla \cdot (\mu_n n^{(0)} \nabla \varphi_n^{(2)}) = r_0(n^{(0)})(n^{(0)} p^{(0)} - 1) - G(\boldsymbol{x}), & \text{in } \Omega, \\ \frac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} = 0 \quad \text{on } \Gamma_N, \\ \varphi_n^{(2)} = 0 \quad \text{on } \Gamma_{D,1}, \qquad \varphi_n^{(2)} = u_D^{(2)}, \quad \text{on } \Gamma_{D,2}, \\ u_D^{(2)} = -\mathcal{R} \displaystyle\int\limits_{\Gamma_{D_2}} \left(\mu_n n^{(0)} \frac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} + \mu_p p^{(0)} \frac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}} \right) \mathrm{d}\sigma(\boldsymbol{x}), \end{cases} \tag{69}$$

and

$$\begin{cases} -\lambda^2 \nabla \cdot (\varepsilon \nabla \psi^{(2)}) = p^{(0)} - n^{(0)}(\psi^{(2)} - \varphi_n^{(2)}), \\ \frac{\partial \psi^{(2)}}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\ \psi^{(2)} = 0, \quad \text{on} \quad \Gamma_{D_1}, \qquad \psi^{(2)} = u_D^{(2)}, \quad \text{on} \quad \Gamma_{D_2}, \end{cases} \tag{70}$$

where $n^{(0)}$ and $p^{(0)}$ are given by (67). Once $\varphi_p^{(0)}$ is known, equation (69) is a linear elliptic equation for $\varphi_n^{(2)}$, coupled with the condition $(69)_4$ for $u_D^{(2)}$. Then, equation (70) becomes a linear equation for $\psi^{(2)}$.

Theorem 3. *Assuming that $\varphi_n^{(0)} = \varphi_0$, $\psi^{(0)}$ and $\varphi_p^{(0)} \in H^1(\Omega) \cap L^\infty(\Omega)$, $u_D^{(0)} = 0$ is the solution to the zero-order equations (53)–(56) given by Theorem 1, the second-order system of equations (69) admits a unique solution $\varphi_n^{(2)} \in H^1(\Omega) \cap L^\infty(\Omega)$, $u_D^{(2)} \in \mathbb{R}$, and the boundary value problem (70) admits a unique solution $\psi^{(2)} \in H^1(\Omega) \cap L^\infty(\Omega)$. Moreover this solution satisfies the estimates*

$$\min\{0, \underline{r} - \overline{G}\} - \overline{u}_D \le \varphi_n^{(2)} \le \overline{r} + \overline{u}_D, \tag{71}$$

$$\underline{\psi}^{(2)} \le \psi^{(2)} \le \overline{\psi}^{(2)}, \tag{72}$$

and

$$|u_D^{(2)}| \le \overline{u}_D, \tag{73}$$

with

$$\begin{aligned} \overline{u}_D &:= \mathcal{R} \left(\|\mu_n n^{(0)}\|_{L^\infty(\Omega)} \|\varphi_n^*\|_{H^1(\Omega)} + \|\mu_p p^{(0)}\|_{L^\infty(\Omega)} \|\varphi_p^{(0)}\|_{H^1(\Omega)} \right) \|w\|_{H^1(\Omega)}, \\ \underline{\psi}^{(2)} &:= \min\left\{ -\overline{u}_D, \inf \frac{p^{(0)}}{n^{(0)}} \right\} + \min\{0, \underline{r} - \overline{G}\} - \overline{u}_D, \\ \overline{\psi}^{(2)} &:= \max\left\{ \overline{u}_D, \sup \frac{p^{(0)}}{n^{(0)}} \right\} + \overline{r} + \overline{u}_D, \end{aligned} \tag{74}$$

where w is the solution of the elliptic boundary value problem:

$$\begin{cases} -\nabla \cdot (\mu_n n^{(0)} \nabla w) = 0, & \text{in } \Omega, \\ \frac{\partial w}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\ w = w_D = \begin{cases} 0 & \text{on } \Gamma_{D,1}, \\ 1 & \text{on } \Gamma_{D,2}. \end{cases} \end{cases} \tag{75}$$

In (71) and (72), $\underline{r}, \overline{r}$ and $\overline{G}$ are defined in (60) and (61).

Proof. By standard results, see for example [22], there exists a unique solution $w \in H^1(\Omega) \cap L^\infty(\Omega)$ of problem (75) such that

$$0 \le w \le 1.$$

This allows to rewrite the last equation in (69). Assuming that w satisfies problem (75) and using integration py parts, we can write

$$\begin{aligned}\int_{\Gamma_{D,2}} w_D \left(\mu_n n^{(0)} \frac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} + \mu_p p^{(0)} \frac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}}\right) \mathrm{d}\sigma(\boldsymbol{x}) &= \int_{\partial\Omega} w \left(\mu_n n^{(0)} \frac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} + \mu_p p^{(0)} \frac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}}\right) \mathrm{d}\sigma(\boldsymbol{x}) \\ &= \int_{\Omega} \nabla \cdot \left(w(\mu_n n^{(0)} \nabla \varphi_n^{(2)} + \mu_p p^{(0)} \nabla \varphi_p^{(0)}\right) \mathrm{d}\boldsymbol{x} \\ &= \int_{\Omega} \nabla w \cdot (\mu_n n^{(0)} \nabla \varphi_n^{(2)} + \mu_p p^{(0)} \nabla \varphi_p^{(0)}) \mathrm{d}\boldsymbol{x}\end{aligned}$$

in which we have considered the first equation in (69) and (63). This implies that

$$u_D^{(2)} = -\mathcal{R} \left(\int_{\Omega} \mu_n n^{(0)} \nabla \varphi_n^{(2)} \cdot \nabla w \, \mathrm{d}\boldsymbol{x} + \int_{\Omega} \mu_p p^{(0)} \nabla \varphi_p^{(0)} \cdot \nabla w \, \mathrm{d}\boldsymbol{x}\right). \tag{76}$$

We define the function

$$\varphi_n^* = \varphi_n^{(2)} - u_D^{(2)} w, \tag{77}$$

which is the solution of the problem

$$\begin{cases} -\nabla \cdot (\mu_n n^{(0)} \nabla \varphi_n^*) = r_0(n^{(0)})(n^{(0)} p^{(0)} - 1) - G(\boldsymbol{x}), & \text{in } \Omega, \\ \frac{\partial \varphi_n^*}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\ \varphi_n^* = 0, \quad \text{on } \Gamma_D, \end{cases} \tag{78}$$

since $u_D \nabla \cdot (\mu_n n^{(0)} \nabla w) = 0$ on the domain Ω, on $\Gamma_{D,1}$ both $\varphi_n^{(2)}$ and w_D vanish, while on $\Gamma_{D,2}$ the right hand side vanishes since $\varphi_n^{(2)} = u_D^{(2)}$ and $w_D = 1$. Again, by standard results it can be proved that there exists a unique solution $\varphi_n^* \in H^1(\Omega) \cap L^\infty(\Omega)$ of problem (78) such that

$$\min\{0, \underline{r} - \overline{G}\} \le \varphi_n^* \le \overline{r}, \tag{79}$$

where $\underline{r}$ and $\overline{r}$ are defined in (61).

Using (76) and (77), we can estimate the function $u_D^{(2)}$ in the following way

$$\begin{aligned} u_D^{(2)} &= -\mathcal{R} \left(\int_{\Omega} \mu_n n^{(0)} \nabla (\varphi_n^* + u_D^{(2)} w) \cdot \nabla w \, \mathrm{d}\boldsymbol{x} + \int_{\Omega} \mu_p p^{(0)} \nabla \varphi_p^{(0)} \cdot \nabla w \, \mathrm{d}\boldsymbol{x}\right) \\ &= -\left(\mathcal{R} \int_{\Omega} \mu_n n^{(0)} |\nabla w|^2 \, \mathrm{d}\boldsymbol{x}\right) u_D^{(2)} - \mathcal{R} \int_{\Omega} (\mu_n n^{(0)} \nabla \varphi_n^* + \mu_p p^{(0)} \nabla \varphi_p^{(0)}) \cdot \nabla w \, \mathrm{d}\boldsymbol{x} \end{aligned} \tag{80}$$

which implies

$$u_D^{(2)} = -\frac{\mathcal{R}\displaystyle\int_\Omega (\mu_n n^{(0)} \nabla \varphi_n^* + \mu_p p^{(0)} \nabla \varphi_p^{(0)}) \cdot \nabla w \,\mathrm{d}\boldsymbol{x}}{1 + \mathcal{R}\displaystyle\int_\Omega \mu_n n^{(0)} |\nabla w|^2 \,\mathrm{d}\boldsymbol{x}}$$

thus, estimate (74) holds. We can conclude that $\varphi_n^{(2)} = \varphi_n^* + u_D^{(2)} w \in H^1(\Omega) \cap L^\infty(\Omega)$ and

$$\min\{0, \underline{r} - \overline{G} - \overline{u}_D\} \le \varphi_n^{(2)} \le \overline{r} + \overline{u}_D, \tag{81}$$

with $\overline{u}_D$ defined in (74).

Finally, we can conclude that the linear problem (70) admits a unique solutions $\psi^{(2)} \in H^1(\Omega) \cap L^\infty(\Omega)$, this solution satisfies the estimate in (72). □

6. Conclusion

Summarizing the results in the previous section, namely, equations (57) and (68), the expansion in (37) reduces to

$$\begin{aligned}
\psi &= \psi^{(0)} + \delta^2 \psi^{(2)} + O(\delta^3), \\
\varphi_n &= \varphi_0 + \delta^2 \varphi_n^{(2)} + O(\delta^3), \\
\varphi_p &= \varphi_p^{(0)} + O(\delta), \\
u_D &= \delta^2 u_D^{(2)} + O(\delta^3),
\end{aligned}$$

and the system (27) is asymptotically equivalent to the following decoupled boundary value problems for the leading order functions $\psi^{(0)}$ and $\varphi_p^{(0)}$:

$$\begin{cases}
-\lambda^2 \nabla \cdot \left(\varepsilon \nabla \psi^{(0)}\right) = -e^{\psi^{(0)} - \varphi_0} + C(\boldsymbol{x}), \\
\dfrac{\partial \psi^{(0)}}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\
\psi^{(0)} = \psi_0, \quad \text{on} \quad \Gamma_{D_1} \cup \Gamma_{D_2},
\end{cases} \tag{82}$$

$$\begin{cases}
-\nabla \cdot \left(\mu_p e^{\varphi_p^{(0)} - \psi^{(0)}} \nabla \varphi_p^{(0)}\right) = G(\boldsymbol{x}) - r_0 \left(e^{\psi^{(0)} - \varphi_0}\right)\left(e^{\varphi_p^{(0)} - \varphi_0} - 1\right), \\
\dfrac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\
\varphi_p^{(0)} = \varphi_0, \quad \text{on} \quad \Gamma_{D_1} \cup \Gamma_{D_2},
\end{cases} \tag{83}$$

which establish the leading order carrier densities $n^{(0)} = \exp(\psi^{(0)} - \varphi_0)$, $p^{(0)} = \exp(\varphi_p^{(0)} - \psi^{(0)})$; to the following coupled problem for the second order correction of the electron quasi-Fermi potential $\varphi_n^{(2)}$ and the current $u_D^{(2)}$:

$$\begin{cases}
-\nabla \cdot (\mu_n n^{(0)} \nabla \varphi_n^{(2)}) = r_0(n^{(0)})(n^{(0)} p^{(0)} - 1) - G(\boldsymbol{x}), \quad \text{in } \Omega, \\
\dfrac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\
\varphi_n^{(2)} = 0, \quad \text{on } \Gamma_{D,1}, \qquad \varphi_n^{(2)} = u_D^{(2)}, \quad \text{on } \Gamma_{D,2}, \\
u_D^{(2)} = -\mathcal{R} \displaystyle\int_{\Gamma_{D_2}} \left(\mu_n n^{(0)} \frac{\partial \varphi_n^{(2)}}{\partial \boldsymbol{\nu}} + \mu_p p^{(0)} \frac{\partial \varphi_p^{(0)}}{\partial \boldsymbol{\nu}}\right) \mathrm{d}\sigma(\boldsymbol{x}),
\end{cases} \tag{84}$$

and the following boundary-value problem for the second order correction of the potential $\psi^{(2)}$:

$$\begin{cases} -\lambda^2\nabla\cdot(\varepsilon\nabla\psi^{(2)}) = p^{(0)} - n^{(0)}(\psi^{(2)} - \varphi_n^{(2)}), \\ \dfrac{\partial\psi^{(2)}}{\partial\boldsymbol{\nu}} = 0, \quad \text{on } \Gamma_N, \\ \psi^{(2)} = 0, \quad \text{on} \quad \Gamma_{D_1}, \qquad \psi^{(2)} = u_D^{(2)}, \quad \text{on} \quad \Gamma_{D_2}, \end{cases} \tag{85}$$

which provides the second order correction to the electron carrier density $n^{(2)} = n^{(0)}(\psi^{(2)} - \varphi_n^{(2)})$ — recall that $n = n^{(0)} + \delta^2 n^{(2)}$, $p = \delta^2 p^{(0)}$.

A final comment on the effectiveness of the final reduced model. It comprises two decoupled nonlinear equations (82) and (83) and two linear equations. The coupling to the circuit appears in the linear equation (84). For this model, we were able to show existence and uniqueness without having to rely on a smallness assumption for the laser generation term G. We expect the reduced model to be valuable for a theoretical study of the inverse problem, which is the natural setting for the LPS method.

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Appendix A. Parameter lists

We briefly list the physical parameters for silicon and gallium arsenide. More details can be found in [7] and [12], respectively. The following optical parameters agree in both cases.

Physical Quantity	Symbol	Value	Units
Reference temperature	T	300	K
Thermal voltage	V_{th}	300	K
Laser power	P_{L}	0 – 20	mW
Laser wave length	λ_{L}	685	nm
Laser penetration depth	d_A	4.8	µm
Laser spot radius	σ_{L}	≥ 0.02585	V

A.1. Parameters for silicon

Physical Quantity	Symbol	Value	Units
Band gap	E_g	1.12	eV
Density of states in the conduction band	N_c	1.04×10^{19}	$1/\mathrm{cm}^3$
Density of states in the valence band	N_v	2.8×10^{19}	$1/\mathrm{cm}^3$
Relative permittivity	$\varepsilon_{\mathrm{Si}}$	11.8	-
Reference mobility value	$\bar{\mu} = \mu_{\mathrm{n},0}^{\mathrm{ref}}$	1323	$\mathrm{cm}^2/\mathrm{Vs}$
Reference doping profile value	$\bar{C}$	1.2×10^{16}	$1/\mathrm{cm}^3$

A.2. Parameters for gallium arsenide

Physical Quantity	Symbol	Value	Units
Band gap	E_g	1.424	eV
Density of states in the conduction band	N_c	4.7×10^{17}	$1/\mathrm{cm}^3$
Density of states in the valence band	N_v	9×10^{18}	$1/\mathrm{cm}^3$
Relative permittivity	$\varepsilon_{\mathrm{SiGa}}$	12.9	-
Reference mobility value	$\bar{\mu} = \mu_{\mathrm{n},0}^{\mathrm{ref}}$	9400	$\mathrm{cm}^2/\mathrm{Vs}$
Reference doping profile value	$\bar{C}$	1.2×10^{18}	$1/\mathrm{cm}^3$

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