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Sobolev embeddings in Musielak-Orlicz spaces

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ABSTRACT

An embedding theorem for Sobolev spaces built upon general Musielak-Orlicz norms is offered. These norms are defined in terms of generalized Young functions which also depend on the x variable. Under minimal conditions on the latter dependence, a Sobolev conjugate is associated with any function of this type. Such a conjugate is sharp, in the sense that, for each fixed x, it agrees with the sharp Sobolev conjugate in classical Orlicz spaces. Both Sobolev inequalities in the whole $\mathbb{R}^n$ and Sobolev-Poincaré inequalities in domains are established. Compact Sobolev embeddings are also presented. In particular, optimal embeddings for standard Orlicz-Sobolev spaces, variable exponent Sobolev spaces, and double-phase Sobolev spaces are recovered and complemented in borderline cases. A key tool, of independent interest, in our approach is a new weak type inequality for Riesz potentials in Musielak-Orlicz spaces involving a sharp fractional-order Sobolev conjugate.

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1. Introduction

Among the diverse extensions of the classical Lebesgue spaces $L^p(\Omega)$ on a measurable set $\Omega \subset \mathbb{R}^n$, the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and the Orlicz spaces $L^A(\Omega)$ have a prominent position. Loosely speaking, the former are obtained by replacing the function t^p with a function $t^{p(x)}$, where the exponent $p(x)$ may depend on the space variable x. The latter surface when the role of the power t^p is played by a more general Young function $A(t)$, namely a nonnegative convex function vanishing at 0.

These two families of spaces are distinguished members of the class of Musielak-Orlicz spaces $L^{\varphi(\cdot)}(\Omega)$, also known as generalized Orlicz spaces. They are built upon generalized Young functions

$$\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty].$$

These functions are measurable in the variable $x \in \mathbb{R}^n$ for each fixed $t \in [0, \infty)$, and are Young functions in the variable $t \in [0, \infty)$ for a.e. fixed $x \in \mathbb{R}^n$.

Accordingly, when Ω is open, the homogeneous Musielak-Orlicz-Sobolev spaces $V^{1,\varphi(\cdot)}(\Omega)$ generalize the standard Sobolev spaces and consist of those functions whose weak derivatives belong to $L^{\varphi(\cdot)}(\Omega)$.

The theory of Musielak-Orlicz spaces and of their Sobolev counterparts has been considerably developed in recent years. They were systematically introduced in [35], although they already appear in the earlier contribution [36]. Recent comprehensive monographs on this topic are [26,33]. Special classes of Musielak-Orlicz spaces are the subject of the books [15,19]. The interest in these spaces is especially motivated by applications to the study of partial differential equations driven by nonstandard nonlinearities. A sample of results on the regularity of solutions to elliptic equations and variational problems in Musielak-Orlicz-Sobolev spaces can be collected from the papers [2–6,13,17,28,29] and the book [7].

Despite extensive investigations on Musielak-Orlicz spaces, the picture of Sobolev-type embeddings – a fundamental tool in the applications mentioned above – is still incomplete. A sharp embedding theorem is available for classical Orlicz-Sobolev spaces, associated with functions φ independent of x, and hence of the form

$$\varphi(x, t) = A(t) \tag{1.1}$$

for some Young function A. This embedding is established in [8,9] and is of use in the analysis of partial differential equations whose nonlinearities are not polynomial.

An optimal embedding for functions φ of plain variable exponent type, namely functions obeying

$$\varphi(x, t) = t^{p(x)}, \tag{1.2}$$

is also well-known. Such an embedding was first proved in [18] and extended in [21,22]. Its most general form is contained in [24], and can also be found in [19, Section 8.3]. This embedding is known to hold if the function $p(\cdot)$ satisfies a local log-Hölder continuity property, as well as a logarithmic decay assumption near infinity. On the other hand, it may fail if these assumptions are dropped, as demonstrated by examples from [19, Proposition 8.3.7]. Variable exponent Sobolev spaces enter various mathematical models of physical phenomena, such as the theory of electrorheological fluids.

Sobolev embeddings for Musielak-Orlicz spaces associated with generalized Young functions defined as

$$\varphi(x,t) = t^p + a(x)t^q \tag{1.3}$$

are obtained in [29] under suitable hypotheses on the function $a(\cdot)$. Functions of this kind have a role in the formulation of so-called double-phase variational problems, which have been the object of renewed attention in recent years.

Embeddings for Musielak-Orlicz-Sobolev spaces built upon generalized Young functions φ of other specific forms have been considered in various contributions. Results for functions φ of general type can also be found in the literature – see e.g. [16,23–26,34]. However, they require unessential restrictions on φ and need not yield optimal conclusions. In particular, the result of [23] necessitates a Lipschitz dependence of φ in the x variable, and, more importantly, in the case of functions φ independent of x it just reproduces the non-sharp Sobolev embedding from [20].

The purpose of the present paper is to offer an embedding theorem for general Musielak-Orlicz-Sobolev spaces. The Sobolev conjugate φ_n of any generalized Young function φ satisfying some natural assumptions on the dependence on the x-variable is exhibited. Such a Sobolev conjugate is sharp, in the sense that the receipt that yields $\varphi_n(x,\cdot)$ from $\varphi(x,\cdot)$, for each fixed x, is exactly the same that produces the optimal Sobolev conjugate of a classical Young function, independent of x, in the Orlicz-Sobolev embedding. The function φ_n is a generalized Young function such that

$$V_d^{1,\varphi(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\varphi_n(\cdot)}(\mathbb{R}^n).$$

Namely, the inequality

$$\|u\|_{L^{\varphi_n(\cdot)}(\mathbb{R}^n)} \le c\|\nabla u\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}$$

holds for some constant c and for every $u \in V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$. Here, the arrow "$\hookrightarrow$" stands for continuous embedding, and $V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$ denotes the space of those functions $u \in V^{1,\varphi(\cdot)}(\mathbb{R}^n)$ which decay near infinity, in the sense that the Lebesgue measure of the level set $\{|u| > t\}$ is finite for every $t > 0$. The assumptions to be imposed on the function φ are standard in the current theory of Musielak-Orlicz-Sobolev spaces, as recorded in [26]. They amount to the non-degeneracy and non-singularity in the x variable for

$t = 1$, to quantitative information on local oscillation in the x variable, and to a suitable asymptotic behavior as $|x| \to \infty$. These assumptions are a natural extension of those customarily requested in the basic case of variable exponent Sobolev spaces.

Sobolev-Poincaré type inequalities, and ensuing Sobolev embeddings, on open sets $\Omega \subset \mathbb{R}^n$ with finite measure are also established. Because of the finiteness of the ground domain, only the behavior for large t of the generalized Young functions defining the Musielak-Orlicz norms is relevant in these inequalities. The Sobolev conjugate of φ can thus be constructed after modifying φ for t near 0. This enables one to avoid unnecessary decay conditions on φ. The resultant Sobolev conjugate is denoted by $\varphi_{n,\diamond}$. The Sobolev-Poincaré inequality in $V_0^{1,\varphi(\cdot)}(\Omega)$, the subspace of those functions from $V^{1,\varphi(\cdot)}(\Omega)$ which vanish, in a suitable sense, on $\partial\Omega$, reads

$$\|u\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)} \le c\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}$$

for some constant c and every $u \in V_0^{1,\varphi(\cdot)}(\Omega)$.

A parallel mean-value Sobolev-Poincaré inequality of the form

$$\|u - u_\Omega\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)} \le c\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}$$

holds in sufficiently regular bounded domains Ω, for some constant c and every $u \in V^{1,\varphi(\cdot)}(\Omega)$. Here, u_Ω stands for the mean value of u over Ω. Bounded John domains Ω are admissible, for instance. Recall that the class of John domains includes, in particular, Lipschitz domains and domains with the interior cone property.

Compact embeddings for the spaces $V_0^{1,\varphi(\cdot)}(\Omega)$ and $V^{1,\varphi(\cdot)}(\Omega)$ are discussed as well.

Our embeddings recover the optimal known results for specific choices of the function φ. In particular, they reproduce the Sobolev embeddings for functions φ as in (1.1), (1.2), and (1.3), and also allow for critical and supercritical values of parameters associated with these functions.

A major dissimilarity between Musielak-Orlicz and classical Orlicz spaces is that, unlike the latter, the former are not rearrangement-invariant. This drawback makes some powerful tools in dealing with Sobolev type inequalities in Orlicz spaces, such as symmetrizations and rearrangement-based interpolation techniques, useless in the Musielak-Orlicz environment. As in most of the available proofs of Sobolev embeddings for spaces from the Musielak-Orlicz class, we resort to an approach relying on a weak-type estimate for the Riesz potential operator I_α in the same spaces. A critical step in our proof amounts to a new estimate of this kind, which takes the form:

$$\int_{\{|I_\alpha f|>t\}} \varphi_{\frac{n}{\alpha}}(x, ct)\,dx \le \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\,dx$$

for every (suitably normalized) generalized Young function φ, for some positive constant c and for every function f such that $\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le 1$. Here, $\alpha \in (0, n)$, and $\varphi_{\frac{n}{\alpha}}$ is the sharp Sobolev conjugate of φ of order α. The function $\varphi_{\frac{n}{\alpha}}$ agrees with φ_n if $\alpha = 1$, and reproduces, for each fixed x, the Sobolev conjugate for optimal weak-type inequalities for I_α in Orlicz spaces exhibited in [10]. Moreover, our estimate includes, as a special case, a well-known inequality in variable exponent Lebesgue spaces [18]. It also augments weak-type inequalities for Riesz potentials in Musielak-Orlicz spaces contained in [16,24,25].

Thanks to a representation formula for Sobolev functions in terms of Riesz potentials of order $\alpha = 1$, the weak-type inequality for these potentials implies a parallel weak-type Sobolev inequality in Musielak-Orlicz spaces. A nowadays standard discretization-truncation argument introduced by V. Maz'ya [31,32] then upgrades the weak-type Sobolev inequality to a strong one. The relevant representation formula, though classical for compactly supported functions, cannot apparently be found in the literature for functions decaying near infinity in the very weak sense specified above. Its proof requires some additional argument and it is hence stated in a separate lemma, of possible independent interest.

The paper is organized as follows. We begin by introducing notation, assumptions, and basic properties about generalized Young functions and the associated Musielak-Orlicz spaces. This is the content of Section 2, where the special instances of generalized Young functions mentioned above are also discussed in detail. Musielak-Orlicz-Sobolev spaces are introduced at the beginning of Section 3, which is devoted to the statements of our main results. Applications of these results to some classes of Musielak-Orlicz-Sobolev spaces are presented in the final part of this section. The weak-type inequality for Riesz potentials in Musielak-Orlicz spaces is the subject of Section 4. For ease of exposition, its technical proof is split into several lemmas. The proofs of the Sobolev and Sobolev-Poncaré inequalities are accomplished in the final Section 5, after stating and proving the representation formula for the admissible functions in the inequalities under consideration.

2. Generalized Young functions and Musielak-Orlicz spaces

This section is devoted to basic notions about generalized Young functions and the associated Musielak-Orlicz spaces. We rely upon the modern foundations of the theory of these spaces developed in the monographs [19,26], to which we refer the reader for more details.

A Young function is a function $A : [0, \infty) \to [0, \infty]$ which is convex, continuous, non-constant and such that $A(0) = 0$. The following representation formula holds

$$A(t) = \int_0^t a(\tau)\, d\tau,$$

where $a : [0, \infty) \to [0, \infty]$ is a non-decreasing function. By $A^{-1} : [0, \infty) \to [0, \infty)$ we denote the left-continuous inverse of A, given by

$$A^{-1}(t) = \inf \{\tau \geq 0 : A(\tau) \geq t\} \quad \text{for } t \geq 0.$$

A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ is called a generalized Young function if

(i) the function $\varphi(x,\cdot)$ is a Young function for a.e. $x \in \mathbb{R}^n$,
(ii) the function $\varphi(\cdot,t)$ is measurable for every $t \geq 0$.

Notice that

$$s\,\varphi(x,t) \leq \varphi(x,st) \quad \text{if } s \geq 1 \text{ and } t \geq 0, \tag{2.1}$$

for a.e. $x \in \mathbb{R}^n$.

For a.e. $x \in \mathbb{R}^n$, we denote by $\varphi^{-1}(x,\cdot)$ the left-continuous inverse of the Young function $\varphi(x,\cdot)$. One has that

$$\varphi(x,\varphi^{-1}(x,t)) = t \quad \text{and} \quad \varphi^{-1}(x,\varphi(x,t)) \leq t \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and } t \geq 0.$$

The Young conjugate of a generalized Young function φ is denoted by $\widetilde{\varphi}$ and defined as

$$\widetilde{\varphi}(x,t) = \sup\{\tau t - \varphi(x,\tau) : \tau \geq 0\} \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and } t \geq 0.$$

It is also a generalized Young function and

$$t \leq \varphi^{-1}(x,t)\widetilde{\varphi}^{-1}(x,t) \leq 2t \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{2.2}$$

The function φ is said to satisfy the Δ_2-condition if there exists a constant c such that

$$\varphi(x,2t) \leq c\,\varphi(x,t) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{2.3}$$

In what follows, we write

$$\varphi(x,t) \approx \psi(x,t) \tag{2.4}$$

to denote that there exist positive constants c_1 and c_2 such that $c_1\varphi(x,t) \leq \psi(x,t) \leq c_2\varphi(x,t)$ for a.e. $x \in \mathbb{R}^n$ and $t \geq 0$. This relation differs from the equivalence in the sense of Young functions, denoted by

$$\varphi(x,t) \simeq \psi(x,t), \tag{2.5}$$

which means that $\varphi(x,c_1t) \leq \psi(x,t) \leq \varphi(x,c_2t)$. The relations (2.4) and (2.5) are said to hold near zero or near infinity if the pertaining inequalities just hold, uniformly for a.e. x, for small or large values of t, respectively. These relations will also be employed between functions that are not generalized Young functions. Observe that, thanks to the property (2.1), if φ and ψ are generalized Young functions such that $\varphi \approx \psi$, then $\varphi \simeq \psi$ as well.

We say that the function ϑ grows essentially more slowly than φ near infinity in a set $\Omega \subset \mathbb{R}^n$ if

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} \frac{\vartheta(x,ct)}{\varphi(x,t)} = 0 \qquad \text{for every } c>0. \tag{2.6}$$

Let Ω be a measurable set in $\mathbb{R}^n$. Denote by $L^0(\Omega)$ the set of measurable functions in Ω. The Musielak-Orlicz space $L^{\varphi(\cdot)}(\Omega)$, associated with a generalized Young function φ, is defined as

$$L^{\varphi(\cdot)}(\Omega) = \left\{ u \in L^0(\Omega) \,:\, \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The space $L^{\varphi(\cdot)}(\Omega)$ is a Banach space when equipped with the norm

$$\|u\|_{L^{\varphi(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0 \,:\, \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \le 1 \right\}. \tag{2.7}$$

The following Hölder type inequality holds:

$$\int_\Omega |u(x)v(x)|\, dx \le 2\|u\|_{L^{\varphi(\cdot)}(\Omega)} \|v\|_{L^{\widetilde{\varphi}(\cdot)}(\Omega)} \tag{2.8}$$

for $u \in L^{\varphi(\cdot)}(\Omega)$ and $v \in L^{\widetilde{\varphi}(\cdot)}(\Omega)$. Furthermore,

$$\sup_{u\in L^{\varphi(\cdot)}(\Omega)} \frac{\int_\Omega |u(x)v(x)|\, dx}{\|u\|_{L^{\varphi(\cdot)}(\Omega)}} \ge \|v\|_{L^{\widetilde{\varphi}(\cdot)}(\Omega)} \tag{2.9}$$

for $v \in L^{\widetilde{\varphi}(\cdot)}(\Omega)$.

Musielak-Orlicz spaces are, in particular, normed function lattices on Ω. Recall that a normed function lattice $X(\Omega)$ is a normed linear space of functions in $L^0(\Omega)$, endowed with a norm $\|\cdot\|_{X(\Omega)}$ such that

$$\text{if } |u| \le |v| \text{ a.e. in } \Omega, \text{ then } \|u\|_{X(\Omega)} \le \|v\|_{X(\Omega)}. \tag{2.10}$$

Additional assumptions are needed on generalized Young functions for several classical results from Real and Harmonic Analysis to hold in the associated Musielak-Orlicz spaces. The requirements to be imposed in the model case of variable exponent spaces are thoroughly exposed in the monograph [19]. Their extension to the general Musielak-Orlicz space depends upon the following definitions, which were introduced in [26].

Definition 2.1. A generalized Young function φ is said to satisfy the condition:

(A0) if there exists $\beta \in (0,1]$ such that

$$\beta \le \varphi^{-1}(x,1) \le \frac{1}{\beta}$$

for a.e. $x \in \mathbb{R}^n$;

(A1) if there exists $\beta \in (0,1]$ such that

$$\beta\varphi^{-1}(x,t) \le \varphi^{-1}(y,t)$$

for every $t \in [1, \frac{1}{|B|}]$, for a.e. $x, y \in B$, and every ball $B \subset \mathbb{R}^n$ with $|B| \le 1$;

(A2) if for every $s > 0$ there exist $\beta \in (0,1]$ and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$\beta\varphi^{-1}(x,t) \le \varphi^{-1}(y,t)$$

for a.e. $x, y \in \mathbb{R}^n$ and every $t \in [h(x) + h(y), s]$.

The condition (A0) is usually called the weight condition, (A1) the local continuity condition, and (A2) the decay condition. Note that, in spite of its name, the condition (A1) just provides information on the local oscillation of φ in the x variable, but does not entail its continuity. This terminology is inspired by the special setting of variable exponent functions.

The condition (A0) also ensures that, if E is any set in $\mathbb{R}^n$ with finite Lebesgue measure $|E|$, then $\|\chi_E\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} < \infty$. Here, χ_E denotes the characteristic function of E. Thanks to the inequalities (2.2), the condition (A0), with $\frac{1}{\beta}$ replaced with $\frac{2}{\beta}$, is inherited by the function $\widetilde{\varphi}$. Thus, as a consequence of the inequality (2.8),

$$\int_E |u|\, dx < \infty \quad \text{ if } \quad \|u\chi_E\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} < \infty. \tag{2.11}$$

In particular, if φ satisfies (A0) and $|\Omega| < \infty$, then

$$L^{\varphi(\cdot)}(\Omega) \hookrightarrow L^1(\Omega). \tag{2.12}$$

The function $\varphi_\infty : [0, \infty) \to [0, \infty]$, associated with a generalized Young function φ by

$$\varphi_\infty(t) = \limsup_{|x|\to\infty} \varphi(x,t) \qquad \text{for } t \ge 0, \tag{2.13}$$

plays a role in equivalent formulations of the condition (A2). If φ_∞ is non-degenerate, in the sense that it is neither constantly equal to 0 nor to ∞, then it is a Young function. This is certainly the case when φ satisfies the condition (A0). As shown in [26, Lemma 4.2.9] and [27], for a generalized Young function φ satisfying the condition (A0), the following conditions are equivalent to (A2).

Definition 2.2. A generalized Young function φ is said to satisfy the condition:

(A2') if

$$L^{\varphi(\cdot)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) = L^{\varphi_\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

with equivalence of norms, where φ_∞ is the Young function defined by (2.13);

(A2") if there exist $\beta \in (0,1]$ and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that for a.e. $x \in \mathbb{R}^n$

$$\varphi(x, \beta t) \le \varphi_\infty(t) + h(x) \text{ if } \varphi_\infty(t) \in [0,1] \quad \text{and}$$
$$\varphi_\infty(\beta t) \le \varphi(x,t) + h(x) \text{ if } \varphi(x,t) \in [0,1].$$

(A2"') if for every $\sigma > 0$ there exist $\beta \in (0,1]$ and $h \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, with $h \ge 0$, such for a.e. $x, y \in \mathbb{R}^n$

$$\beta\varphi^{-1}(x,t) \le \varphi^{-1}(y, t + h(x) + h(y)) \quad \text{if } t \in [0,\sigma].$$

Let us briefly discuss the customary generalized Young functions defined as in (1.2) and (1.3) in connection with the properties (A0), (A1) and (A2) introduced above. We refer the reader to [26] for additional examples.

Example 2.3 *(Variable exponents).* The best known instances of generalized Young functions, depending on the x-variable in a non trivial way, are provided by variable powers, namely functions given by

$$\varphi(x,t) = t^{p(x)} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \ge 0, \tag{2.14}$$

where

$$p : \mathbb{R}^n \to [1, \infty).$$

They are at the basis of the definition of the variable exponent Lebesgue spaces, whose systematic analysis was initiated in the paper [30].

If the function p satisfies a log-Hölder continuity property of the form

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \le \frac{c}{\log(e + \frac{1}{|x-y|})} \quad \text{for } x, y \in \mathbb{R}^n, \tag{2.15}$$

for some constant c, then φ is a generalized Young function fulfilling the properties (A0) and (A1) – see [26, Section 7.1].

If, in addition, the logarithmic decay condition near infinity

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \le \frac{c}{\log(e + |x|)} \quad \text{for } x \in \mathbb{R}^n \tag{2.16}$$

holds for some constants $p_\infty \in [1,\infty)$ and $c>0$, then the function φ also satisfies the property (A2). In particular, under the assumptions (2.15) and (2.16), we have that $p \in L^\infty(\mathbb{R}^n)$.

Observe that the condition (2.16) implies Nekvinda's condition introduced in [37].

The same assertions are true for the variant given by

$$\varphi(x,t) = \tfrac{1}{p(x)} t^{p(x)} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0,$$

which is of common use in the formulation of variational problems associated with variable exponent functionals.

Example 2.4 *(Double-phase).* Another popular class of generalized Young functions consists of so-called double-phase Young functions. They have the form

$$\varphi(x,t) = t^p + a(x)t^q \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{2.17}$$

for some exponents $q > p \geq 1$ and for a nonnegative function $a \in L^\infty(\mathbb{R}^n)$. As shown in [26, Section 7.2], a function φ of this kind satisfies the properties (A0) and (A2), with $h = 0$. It also fulfills the property (A1) if and only if

$$a(x) \leq c\Big(a(y) + |x-y|^{\frac{n}{p}(q-p)}\Big) \quad \text{for } x,y \in \mathbb{R}^n, \tag{2.18}$$

for some constant $c > 0$. In particular, the condition (2.18) holds, provided that

$$q \leq \frac{np}{n-1}, \tag{2.19}$$

and

$$a \in C^{0,\frac{n}{p}(q-p)}(\mathbb{R}^n). \tag{2.20}$$

An alternate version of the function (2.17), which still enjoys the properties (A0), (A1), and (A2) under the same assumptions, is obtained by replacing the sum with the maximum on the right-hand side of (2.17). This results in the function

$$\varphi(x,t) = \max\{t^p, a(x)t^q\} \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{2.21}$$

A combination of the functions from (2.14) and (2.17) yields variable exponent double-phase functions given by

$$\varphi(x,t) = t^{p(x)} + a(x)t^{q(x)} \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{2.22}$$

They have been examined, also in connection with the conditions (A0), (A1), and (A2), in the papers [14] and [29].

A handy subclass of generalized Young functions consists of normalized functions according to the next definition patterned on [25].

Definition 2.5. A generalized Young function φ will be called normalized if:
(i)

$$\varphi^{-1}(x,1)=1 \quad \text{for a.e. } x\in\mathbb{R}^n; \tag{2.23}$$

(ii) there exists $\beta\in(0,1]$ such that

$$\beta\varphi^{-1}(x,t)\leq\varphi^{-1}(y,t) \tag{2.24}$$

for every $t\in[0,\frac{1}{|B|}]$, for a.e. $x,y\in B$, and every ball $B\subset\mathbb{R}^n$;
(iii)

$$\varphi(x,t)=\varphi_\infty(t) \quad \text{for } t\in[0,1]. \tag{2.25}$$

The following properties of normalized generalized Young functions follow from their definition, via the inequalities (2.2).

Proposition 2.6. *Assume that φ is a normalized generalized Young function, and let $\beta\in(0,1]$ be the constant appearing in Definition 2.5. Then:*
(i)

$$\varphi(x,t)\leq\varphi(y,\tfrac{t}{\beta}) \tag{2.26}$$

for every $t\in[0,\beta\varphi^{-1}(y,|B|^{-1})]$, for a.e. $x,y\in B$, and every ball $B\subset\mathbb{R}^n$;
(ii)

$$\tfrac{\beta}{2}\widetilde{\varphi}^{-1}(x,t)\leq\widetilde{\varphi}^{-1}(y,t) \tag{2.27}$$

for every $t\in[0,|B|^{-1}]$, for a.e. $x,y\in B$, and every ball $B\subset\mathbb{R}^n$;
(iii)

$$\widetilde{\varphi}(x,t)\leq\widetilde{\varphi}(y,\tfrac{2}{\beta}t) \tag{2.28}$$

for every $t\in[0,\frac{\beta}{2}\widetilde{\varphi}^{-1}(y,|B|^{-1})]$, for a.e. $x,y\in B$, and every ball $B\subset\mathbb{R}^n$.

A useful result from [25] ensures that any generalized Young function φ satisfying the conditions (A0), (A1), and (A2) can be replaced with a normalized generalized Young function $\overline{\varphi}$ without altering the associated Musielak-Orlicz space, up to equivalent norms. The recipe is the following. Define the generalized Young function φ_0 as

$$\varphi_0(x,t) = \max\left\{\varphi\big(x, \varphi^{-1}(x,1)t\big), 2t-1\right\} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{2.29}$$

Then, the function $\overline{\varphi}$ is given by

$$\overline{\varphi}(x,t) = \begin{cases} 2\varphi_0(x,t) - 1 & \text{if } t \geq 1 \\ \limsup_{|x|\to\infty} \varphi_0(x,t) & \text{if } 0 \leq t < 1, \end{cases} \tag{2.30}$$

for $x \in \mathbb{R}^n$.

Proposition 2.7 *([25, Proposition 4.2]). Let φ be a generalized Young function satisfying the conditions (A0), (A1) and (A2). Then, the function $\overline{\varphi}$ defined by (2.30) is a normalized generalized Young function, which satisfies the condition (2.24) with the same β as in the condition (A1) for φ. Moreover,*

$$L^{\varphi(\cdot)}(\mathbb{R}^n) = L^{\overline{\varphi}(\cdot)}(\mathbb{R}^n), \tag{2.31}$$

up to equivalent norms, with equivalence constants depending on the constant β from the conditions (A0), (A1), and on the function h from the condition (A2).

Notice that $\overline{\varphi}$ is a generalized Young function fulfilling (A0) even if φ does not satisfy (A0), (A1), and (A2). These are needed for $\overline{\varphi}$ to be normalized and for equation (2.31) to hold.

As pointed out in [25], despite the property (2.31), the functions φ and $\overline{\varphi}$ need not be globally equivalent, in general. However, as shown in the same paper, one has that

$$\varphi(x, \beta t) \leq \overline{\varphi}(x,t) \leq \varphi\big(x, \tfrac{4}{\beta}t\big) \qquad \text{if } t \geq 1, \tag{2.32}$$

and

$$\varphi_\infty\big(\beta t\big) \leq \overline{\varphi}(x,t) \leq \varphi_\infty\big(\tfrac{2t}{\beta}\big) \qquad \text{if } 0 \leq t < 1, \tag{2.33}$$

for $x \in \mathbb{R}^n$, where β is the constant appearing in the property (A0) for the function φ.

Consequently, if φ fulfills both the condition (A0) and the Δ_2-condition, then the function $\varphi^\circ : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ obeying

$$\varphi^\circ(x,t) = \begin{cases} \varphi(x,t) & \text{if } t \geq 1 \\ \varphi_\infty(t) & \text{if } 0 \leq t < 1, \end{cases} \tag{2.34}$$

for $x \in \mathbb{R}^n$, is such that

$$\varphi^\circ \approx \overline{\varphi}. \tag{2.35}$$

When dealing with Musielak-Orlicz spaces defined on sets of finite measure, the assumption (A2) on the generating function φ is immaterial. The last result of this section shows that, if φ just fulfills the assumptions (A0) and (A1), then there exists a normalized generalized Young function $\widehat{\varphi}$ such that $L^{\widehat{\varphi}(\cdot)}(\Omega)$ and $L^{\varphi(\cdot)}(\Omega)$ agree, up to equivalent norms, for every measurable set $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$. The function $\widehat{\varphi}$ is defined as

$$\widehat{\varphi}(x,t) = \begin{cases} 2\varphi_0(x,t) - 1 & \text{if } t \geq 1 \\ t & \text{if } 0 \leq t < 1, \end{cases} \tag{2.36}$$

for $x \in \mathbb{R}^n$, where φ_0 is given by equation (2.29).

Proposition 2.8. *Let φ be a generalized Young function satisfying the properties (A0) and (A1). Then the function $\widehat{\varphi}$ given by* (2.36) *is a normalized generalized Young function, which satisfies the condition* (2.24) *with the same β as in the condition (A1) for φ. Moreover,*

$$L^{\widehat{\varphi}(\cdot)}(\Omega) = L^{\varphi(\cdot)}(\Omega), \tag{2.37}$$

up to equivalent norms, for every measurable set $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$.

Proof. Observe that

$$\widehat{\varphi}(x,t) = \overline{\varphi}(x,t) \qquad \text{if } t \geq 1 \tag{2.38}$$

and $x \in \mathbb{R}^n$. Since $\overline{\varphi}$ is a normalized generalized Young function satisfying the condition (2.24) with the same β as in the condition (A1) for φ, the fact that $\widehat{\varphi}$ enjoys the same property can be verified as a consequence of the identities (2.38) and

$$\widehat{\varphi}(x,t) = t \qquad \text{if } 0 \leq t < 1$$

and $x \in \mathbb{R}^n$.

As far as the property (2.37) is concerned, the identity (2.38) and the inequalities (2.32) yield:

$$\widehat{\varphi}(x,t) \leq \varphi\big(x, \tfrac{4}{\beta}t\big) \qquad \text{if } t \geq 1, \tag{2.39}$$

and

$$\varphi(x,t) \leq \widehat{\varphi}(x, \tfrac{t}{\beta}) \qquad \text{if } t \geq \beta, \tag{2.40}$$

for $x \in \mathbb{R}^n$. On the other hand,

$$\widehat{\varphi}(x,t) \leq \widehat{\varphi}(x,1) = 1 \qquad \text{if } 0 \leq t < 1 \tag{2.41}$$

and

$$\varphi(x,t) \le \varphi(x,\beta) \le 1 \qquad \text{if } 0 \le t < \beta, \tag{2.42}$$

for $x \in \mathbb{R}^n$, where the last inequality holds thanks to the property (A0) of φ. Combining (2.39) and (2.41) tells us that

$$\widehat{\varphi}(x,t) \le \varphi\big(x, \tfrac{4}{\beta}t\big) + 1 \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \ge 0 \tag{2.43}$$

whereas (2.40) and (2.42) yield

$$\varphi(x,t) \le \widehat{\varphi}(x, \tfrac{t}{\beta}) + 1 \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \ge 0, \tag{2.44}$$

Equation (2.37) follows from the inequalities (2.43) and (2.44), via [19, Theorem 2.8.1]. □

Under the assumption that φ fulfils both the condition (A0) and the Δ_2-condition, consider the function $\varphi^\bullet : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ defined as

$$\varphi^\bullet(x,t) = \begin{cases} \varphi(x,t) & \text{if } t \ge 1 \\ t & \text{if } 0 \le t < 1, \end{cases} \tag{2.45}$$

for $x \in \mathbb{R}^n$. Although, like φ°, the function $\varphi^\bullet$ may not even be continuous at $t = 1$, one has that

$$\varphi^\bullet \approx \widehat{\varphi}. \tag{2.46}$$

This is a consequence of equations (2.32) and (2.33), and of the fact that $\widehat{\varphi}(x,t) = \overline{\varphi}(x,t)$ if $t \ge 1$ and $x \in \mathbb{R}^n$.

3. Main results

Assume that Ω is an open set in $\mathbb{R}^n$ and φ is a generalized Young function. The homogeneous Musielak-Orlicz-Sobolev space $V^{1,\varphi(\cdot)}(\Omega)$ is defined as

$$V^{1,\varphi(\cdot)}(\Omega) = \{u \in W^{1,1}_{\mathrm{loc}}(\Omega) : |\nabla u| \in L^{\varphi(\cdot)}(\Omega)\}. \tag{3.1}$$

If Ω is connected and G is a bounded open set such that $\overline{G} \subset \Omega$, then the functional

$$\|u\|_{L^1(G)} + \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \tag{3.2}$$

defines a norm on $V^{1,\varphi(\cdot)}(\Omega)$. One can show that different choices of the set G result in equivalent norms.

The subspace of functions which vanish on $\partial\Omega$ is suitably given by

$$V_0^{1,\varphi(\cdot)}(\Omega) = \{u \in W^{1,1}_{\rm loc}(\Omega) : \text{the extension to } \mathbb{R}^n \text{ of } u \text{ by 0 outside } \Omega \text{ belongs to } V^{1,\varphi(\cdot)}(\mathbb{R}^n)\}, \tag{3.3}$$

and can equipped with the norm

$$\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)}. \tag{3.4}$$

When $\Omega = \mathbb{R}^n$, we also define the space

$$V_d^{1,\varphi(\cdot)}(\mathbb{R}^n) = \{u \in V^{1,\varphi(\cdot)}(\mathbb{R}^n) : |\{|u| > t\}| < \infty \text{ for every } t > 0\}. \tag{3.5}$$

Heuristically speaking, $V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$ is the subspace of $V^{1,\varphi(\cdot)}(\mathbb{R}^n)$ of those functions decaying near infinity in a weakest possible sense for a homogeneous Sobolev inequality in $\mathbb{R}^n$ to hold. The functional (3.4), with $\Omega = \mathbb{R}^n$, defines a norm in $V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$.

When $\varphi(x,t) = t^p$ for some constant $p \in [1,\infty)$, we shall simply write $V^{1,p}$ instead of $V^{1,\varphi(\cdot)}$ in the above notation.

The present section is split into three subsections. Subsections 3.1 and 3.2 are dedicated to inequalities in $\mathbb{R}^n$ and in bounded domains, respectively. Proofs of the results stated in these subsections can be found in Section 5. Subsection 3.3 concerns applications of these results to particular instances of Musielak-Orlicz-Sobolev spaces.

3.1. The inequality in $\mathbb{R}^n$

Our embedding theorem in $\mathbb{R}^n$ holds for any Musielak-Orlicz-Sobolev space associated with a generalized Young function φ satisfying the conditions (A0), (A1), (A2), and

$$\int_0 \left(\frac{t}{\varphi_\infty(t)}\right)^{\frac{1}{n-1}} dt < \infty. \tag{3.6}$$

As shown in Proposition 3.4 below, the assumption (3.6) is indispensable when considering Sobolev inequalities in $V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$.

The Sobolev conjugate of φ is the generalized Young function φ_n defined as

$$\varphi_n(x,t) = \overline{\varphi}(x, H_n^{-1}(x,t)) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{3.7}$$

where $H_n : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is the function given by

$$H_n(x,t) = \left(\int_0^t \left(\frac{\tau}{\overline{\varphi}(x,\tau)}\right)^{\frac{1}{n-1}} d\tau\right)^{\frac{1}{n'}} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{3.8}$$

Here, $\overline{\varphi}$ denotes the normalized generalized Young function associated with φ as in (2.30) and $n' = \frac{n}{n-1}$, the Hölder conjugate of n.

Thanks to equation (2.33), the assumption (3.6) ensures that the function H_n is finite-valued for $x \in \mathbb{R}^n$ and $t \geq 0$. Here, $H_n^{-1}(x, \cdot)$ denotes the generalized left-continuous inverse of $H_n(x, \cdot)$ for $x \in \mathbb{R}^n$. Note that, if

$$\int^{\infty} \left(\frac{\tau}{\varphi(x, \tau)} \right)^{\frac{1}{n-1}} d\tau < \infty \tag{3.9}$$

for some $x \in \mathbb{R}^n$, then, by equation (2.32), $\lim_{t\to\infty} H_n(x,t) < \infty$. Hence, $H_n^{-1}(x,t) = \infty$ for t exceeding this limit. The right-hand side of equation (3.7) has then to be interpreted as ∞ for these values of t.

Theorem 3.1 *(Sobolev inequality in $\mathbb{R}^n$). Assume that φ is a generalized Young function satisfying the conditions (A0), (A1), (A2) and (3.6). Then,*

$$V_d^{1,\varphi(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{\varphi_n(\cdot)}(\mathbb{R}^n),$$

and there exists a constant $c = c(n, \beta, h)$ such that

$$\|u\|_{L^{\varphi_n(\cdot)}(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \tag{3.10}$$

for every $u \in V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$. Here, β denotes the constant from the conditions (A0) and (A1), and h the function from the condition (A2). In particular, if $\varphi = \overline{\varphi}$, then the constant c in the inequality (3.10) only depends on n and β.

Remark 3.2. One can verify that replacing $\overline{\varphi}$ with any equivalent (in the sense of the relation $\simeq$) generalized Young function in the formulas (3.7) and (3.8) results in a generalized Young function equivalent to φ_n. For customary choices of φ, such an equivalent function is immediately obtained from the expression of φ. These replacements just affect the constant c appearing in the inequality (3.10). In fact, the same conclusion holds even if φ is replaced with an equivalent function, which is not necessarily a generalized Young function. The norm $\|\cdot\|_{L^{\varphi_n(\cdot)}(\mathbb{R}^n)}$ is in fact equivalent to the functional obtained after this replacement. Thus, for instance, if φ satisfies the Δ_2-condition, then, thanks to equation (2.35), the norm $\|\cdot\|_{L^{\varphi_n(\cdot)}(\mathbb{R}^n)}$ in the inequality (3.10) can be replaced with the functional generated by the use of the function φ° given by (2.34), instead of $\overline{\varphi}$, in (3.7) and (3.8).

Remark 3.3. An inspection of the proof of Theorem 3.1 will reveal that, under the same assumptions, the following Sobolev inequality in modular form holds:

$$\int_{\mathbb{R}^n} \varphi_n(x, c|u|)\, dx \leq \int_{\mathbb{R}^n} \overline{\varphi}(x, |\nabla u|)\, dx \tag{3.11}$$

for some constant $c = c(n, \beta)$ and for every u such that $\int_{\mathbb{R}^n} \overline{\varphi}(x, |\nabla u|)\, dx \leq 1$. The inequality (3.11) can be of use in applications to the analysis of solutions to partial differential equations. Parallel conclusions about possible replacements of the function $\overline{\varphi}$ in the definition of the function φ_n as in Remark 3.2 hold with regard to the inequality (3.11).

The following result demonstrates the necessity of the condition (3.6) for the embedding of Theorem 3.1, and even for more general embeddings of the space $V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$ into normed function lattices.

Proposition 3.4 *(Necessity of condition (3.6)). Let φ be as in Theorem 3.1. Assume that there exists a normed function lattice $Y(\mathbb{R}^n)$ such that*

$$\|u\|_{Y(\mathbb{R}^n)} \leq c \|\nabla u\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \tag{3.12}$$

for every $u \in V_d^{1,\varphi(\cdot)}(\mathbb{R}^n)$. Then, the condition (3.6) must be fulfilled.

3.2. Inequalities in domains

Here, we offer Sobolev-Poincaré type inequalities in open sets $\Omega \subset \mathbb{R}^n$ with finite Lebesgue measure. Of course, they imply corresponding Sobolev embeddings. Thanks to the finiteness of the measure of Ω, the condition (A2) on the function φ defining the Musielak-Orlicz-Sobolev space is no longer required. For the same reason, the assumption (3.6) can also be dropped. Namely, the Sobolev-Poincaré inequality holds whenever φ satisfies the conditions (A0) and (A1). Indeed, owing to Proposition 2.8, if φ fulfills these conditions, then the space $V^{1,\varphi(\cdot)}(\Omega)$ remains unchanged, up to equivalent norms, after replacing φ with the normalized generalized Young function $\widehat{\varphi}$ given by (2.36), which plainly satisfies the assumption (3.6).

The Sobolev conjugate of φ on domains with finite measure can thus be defined as the generalized Young function $\varphi_{n,\diamond}$ obeying

$$\varphi_{n,\diamond}(x,t) = \widehat{\varphi}(x, H_{n,\diamond}^{-1}(x,t)) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{3.13}$$

where $H_{n,\diamond} : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is the function given by

$$H_{n,\diamond}(x,t) = \left(\int_0^t \left(\frac{\tau}{\widehat{\varphi}(x,\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{1}{n'}} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{3.14}$$

We begin with a Sobolev-Poincaré inequality for functions in $V_0^{1,\varphi(\cdot)}(\Omega)$. This only requires that Ω has a finite measure.

Theorem 3.5 *(Poincaré-Sobolev inequality with zero boundary values). Let* Ω *be an open set in* $\mathbb{R}^n$ *such that* $|\Omega| < \infty$ *and let* φ *be a generalized Young function satisfying the conditions (A0) and (A1). Then,*

$$V_0^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^{\varphi_n(\cdot)}(\Omega),$$

and there exists a constant $c = c(\beta, n, |\Omega|)$ *such that*

$$\|u\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)} \le c\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \tag{3.15}$$

for every $u \in V_0^{1,\varphi(\cdot)}(\Omega)$*. Here,* β *denotes the constant appearing in the conditions (A0) and (A1).*

Like any other mean-value Sobolev-Poincaré inequality, this kind of inequalities in Musielak-Orlicz spaces requires that the domain Ω be regular enough. Any bounded John domain is admissible. Recall that a bounded open set $\Omega \subset \mathbb{R}^n$ is called a bounded John domain if there exist positive constants c and ℓ, and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\gamma : [0, \ell] \to \Omega$, parametrized by arclenght, such that $\gamma(0) = x$, $\gamma(\ell) = x_0$, and

$$\operatorname{dist}(\gamma(r), \partial\Omega) \ge cr \qquad \text{for } r \in [0, \ell].$$

Any connected bounded open set satisfying the uniform interior cone condition is a bounded John domain. In particular, connected bounded Lipschitz domains are bounded John domains. A bounded John domain may admit inward cusps, whereas outward cusps are forbidden. The class of bounded John domains also includes certain domains with highly irregular boundaries. This is the case of the interior of Koch's snowflake, for instance.

Theorem 3.6 *(Mean-value Poincaré-Sobolev inequality). Let* Ω *be a bounded John domain in* $\mathbb{R}^n$ *and let* φ *be a generalized Young function satisfying the conditions (A0) and (A1). Then,*

$$V^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^{\varphi_n(\cdot)}(\Omega),$$

and there exists a constant $c = c(\beta, \Omega)$ *such that*

$$\|u - u_\Omega\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)} \le c\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \tag{3.16}$$

for every $u \in V^{1,\varphi(\cdot)}(\Omega)$*. Here,* β *denotes the constant appearing in the conditions (A0) and (A1).*

We conclude our discussion about Musielal-Orlicz-Sobolev embeddings with a condition for compactness.

Theorem 3.7 *(Compact embeddings). Let Ω be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$ and let φ be a generalized Young function satisfying the conditions (A0) and (A1). Let ϑ be a generalized Young function growing essentially more slowly than $\varphi_{n,\diamond}$ near infinity and such that*

$$\int_\Omega \vartheta(x,t)\,dx < \infty \qquad \textit{for } t > 0. \tag{3.17}$$

(i) The embedding

$$V_0^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^{\vartheta(\cdot)}(\Omega) \tag{3.18}$$

is compact.
(ii) Assume, in addition, that Ω is a bounded John domain. Then, the embedding

$$V^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^{\vartheta(\cdot)}(\Omega) \tag{3.19}$$

is compact.

Remark 3.8. Observe that, if the function φ fulfills the condition (3.6), and hence the function φ_n is well defined, then $\varphi_n \simeq \varphi_{n,\diamond}$ near infinity. Therefore, since $|\Omega| < \infty$, the inequalities (3.15) and (3.16) can be equivalently formulated with the norm $\|\cdot\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)}$ replaced with $\|\cdot\|_{L^{\varphi_n(\cdot)}(\Omega)}$. An equivalent inequality is also obtained whenever $\varphi_{n,\diamond}$ is replaced with the function defined as in (3.13)–(3.14), with $\widehat{\varphi}$ replaced with any generalized Young function ψ such that $\psi(x,t) \simeq \widehat{\varphi}(x,t)$ for a.e. $x \in \Omega$ and for $t \geq 0$. Moreover, under the additional assumption that φ satisfies the Δ_2-condition, owing to equation (2.46) the norm $\|\cdot\|_{L^{\varphi_{n,\diamond}(\cdot)}(\mathbb{R}^n)}$ in the inequalities (3.15) and (3.16) can be replaced with the functional obtained via the function $\varphi^\bullet$ defined by (2.45), instead of $\widehat{\varphi}$, in (3.13) and (3.14).

3.3. Special instances

The remaining part of this section is devoted to applications of Theorems 3.1, 3.5 and 3.6 in the case of Musielak-Orlicz-Sobolev spaces built upon generalized Young functions of the specific forms presented in Section 2.

Example 3.9 *(Classical Orlicz-Sobolev conjugate).* Assume that φ is independent of x, namely

$$\varphi(x,t) = A(t)$$

for some classical Young function A. By replacing, if necessary, A with an equivalent Young function, we may assume, without loss of generality, that $A(1) = 1$. Consequently,

φ is normalized and $\varphi = \overline{\varphi}$. The Sobolev conjugate φ_n is hence also independent of x. Thus, with abuse of notation, we may drop the dependence on x, and the formulas (3.7) and (3.8) yield

$$\varphi_n(t) = A(H_n^{-1}(t)) \qquad \text{for } t \geq 0, \tag{3.20}$$

where $H_n : [0, \infty) \to [0, \infty)$ is the function given by

$$H_n(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{1}{n'}} \quad \text{for } t \geq 0. \tag{3.21}$$

The function φ_n reproduces the sharp Sobolev conjugate of A, in the form exhibited in [9].

Example 3.10 *(Variable exponent Sobolev conjugate).* Let φ be the variable exponent function as in Example 2.3. Namely,

$$\varphi(x, t) = t^{p(x)} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0.$$

Clearly,

$$\varphi_\infty(t) = t^{p_\infty} \quad \text{for } t \geq 0.$$

Consider inequalities in $\mathbb{R}^n$ and suppose that

$$p_\infty < n, \tag{3.22}$$

an assumption indispensable in view of Proposition 3.4. Since $p \in L^\infty(\mathbb{R}^n)$, the function φ satisfies the Δ_2-condition. Thus, in view of Remark 3.2, using the function φ°, given by (2.34), instead of $\overline{\varphi}$ in the definition of φ_n, results in an equivalent function. Let us still denote, with abuse of notation, by H_n and φ_n the functions obtained after this replacement. One has that

$$\varphi^\circ(x, t) = \begin{cases} t^{p(x)} & \text{if } t \geq 1 \\ t^{p_\infty} & \text{if } 0 \leq t < 1, \end{cases}$$

for $x \in \mathbb{R}^n$. Thus,

$$H(x, t) = t_0 \, t^{\frac{n - p_\infty}{n}} \qquad \text{if } 0 \leq t < 1, \tag{3.23}$$

and

$$H^{-1}(x, t) = t_0^{-\frac{n}{n - p_\infty}} t^{\frac{n}{n - p_\infty}} \qquad \text{if } t < t_0, \tag{3.24}$$

where we have set $t_0 = \big(\frac{n-1}{n-p_\infty}\big)^{\frac{1}{n'}}$. Consequently,

$$\varphi_n(x,t) = t_0^{-\frac{np_\infty}{n-p_\infty}} t^{\frac{np_\infty}{n-p_\infty}} \qquad \text{if } t < t_0. \tag{3.25}$$

The behavior of the function $\varphi_n(x,t)$ for $t \geq t_0$ takes a different form, depending on whether $1 \leq p(x) < n$, $p(x) = n$, or $p(x) > n$.

Assume first that $1 \leq p(x) < n$. Then,

$$H(x,t) = \Big(\frac{n-1}{n-p(x)}\Big)^{\frac{1}{n'}} \Big(\frac{p_\infty - p(x)}{n-p_\infty} + t^{\frac{n-p(x)}{n-1}}\Big)^{\frac{1}{n'}} \qquad \text{if } t \geq 1, \tag{3.26}$$

whence

$$H^{-1}(x,t) = \bigg(\Big(\frac{n-p(x)}{n-1}\Big)^{\frac{1}{n'}} t^{n'} - \frac{p_\infty - p(x)}{n-p_\infty}\bigg)^{\frac{n-1}{n-p(x)}} \qquad \text{if } t \geq t_0. \tag{3.27}$$

Therefore,

$$\varphi_n(x,t) = \bigg(\Big(\frac{n-p(x)}{n-1}\Big)^{\frac{1}{n'}} t^{n'} - \frac{p_\infty - p(x)}{n-p_\infty}\bigg)^{\frac{(n-1)p(x)}{n-p(x)}} \qquad \text{if } t \geq t_0. \tag{3.28}$$

In particular,

$$\varphi_n(x,t) \approx (n-p(x))^{\frac{1}{n-p(x)}} t^{\frac{np(x)}{n-p(x)}} \quad \text{if } t \geq t_1, \tag{3.29}$$

for a suitable $t_1 > t_0$ and with equivalence constants independent of x.

Next, assume that $p(x) = n$. Since

$$H(x,t) = \Big(\frac{n-1}{n-p_\infty} + \log t\Big)^{\frac{1}{n'}} \qquad \text{if } t \geq 1, \tag{3.30}$$

we obtain that

$$\varphi_n(x,t) = e^{\frac{n(1-n)}{n-p_\infty}} e^{nt^{n'}} \qquad \text{if } t \geq t_0. \tag{3.31}$$

Finally, if $p(x) > n$, then

$$H(x,t) = \Big(\frac{n-1}{p(x)-n}\Big)^{\frac{1}{n'}} \Big(\frac{p(x)-p_\infty}{n-p_\infty} - t^{\frac{n-p(x)}{n-1}}\Big)^{\frac{1}{n'}} \qquad \text{if } t \geq 1. \tag{3.32}$$

Thereby,

$$H^{-1}(x,t) = \begin{cases} \infty & \text{if } t \geq t_\infty(x) \\ \Big(\dfrac{p(x)-p_\infty}{n-p_\infty} - \dfrac{p(x)-n}{n-1} t^{n'}\Big)^{\frac{n-1}{n-p(x)}} & \text{if } t_0 \leq t < t_\infty(x), \end{cases} \tag{3.33}$$

where we have set $t_\infty(x) = \big(\frac{(p(x)-p_\infty)(n-1)}{(n-p_\infty)(p(x)-n)}\big)^{\frac{1}{n'}}$. Hence,

$$\varphi_n(x,t) = \begin{cases} \infty & \text{if } t \geq t_\infty(x) \\ \left(\dfrac{p(x)-p_\infty}{n-p_\infty} - \dfrac{p(x)-n}{n-1} t^{n'}\right)^{\frac{(n-1)p(x)}{n-p(x)}} & \text{if } t_0 \leq t < t_\infty(x). \end{cases} \tag{3.34}$$

Altogether, by Theorem 3.1, the inequality (3.10) follows with φ_n given by (3.25) and (3.28).

As far as inequalities on domains are concerned, the inequalities (3.15) and (3.16) hold with $\varphi_{n,\diamond}$ equivalent, for large values of t, to the function φ_n defined as above, even if the assumption (3.22) is dropped. These conclusions recover results from [18] and [24], and also extend them to the case when $p(x) > n$.

Example 3.11 *(Double-phase Sobolev conjugate).* Consider the double-phase function

$$\varphi(x,t) = t^p + a(x)t^q \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0,$$

from Example 2.4. In particular, $1 \leq p < q$ and the function a belongs to $L^\infty(\mathbb{R}^n)$. Hence, the function φ satisfies the Δ_2-condition (2.3). Since $0 \leq a(x) \leq \|a\|_{L^\infty(\mathbb{R}^n)}$, one can verify that

$$\varphi(x, \varphi^{-1}(x,1)t) \approx \varphi(x,t) \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0.$$

Moreover, inasmuch as

$$t^p \leq \varphi(x,t) \leq \big(1 + \|a\|_{L^\infty(\mathbb{R}^n)}\big)\, t^p \qquad \text{if } 0 \leq t \leq 1, \tag{3.35}$$

for $x \in \mathbb{R}^n$, we have that

$$\varphi_0(x,t) \approx \varphi(x,t) \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0,$$

where φ_0 is the function defined as in (2.29), and also

$$2\varphi_0(x,t) - 1 \approx \varphi(x,t) \quad \text{if } t \geq 1,$$

for $x \in \mathbb{R}^n$. Altogether, we infer that

$$\overline{\varphi}(x,t) \approx \varphi(x,t) \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0.$$

Let us focus on inequalities in $\mathbb{R}^n$. By equation (3.35),

$$\varphi_\infty(t) \approx t^p \qquad \text{if } 0 \leq t \leq 1.$$

Hence, the condition (3.6) entails that $p < n$. In view of Remark 3.2, we can define the function φ_n via φ itself instead of $\overline{\varphi}$.

If $a(x) = 0$, then $\varphi(x,t) = t^p$, whence

$$\varphi_n(x,t) \approx t^{\frac{np}{n-p}} \quad \text{for } t \geq 0.$$

Let us consider the nontrivial case when $a(x) \neq 0$. Therefore,

$$\begin{aligned} H_n(x,t) &= \left(\int_0^t \left(\frac{\tau}{\tau^p + a(x)\tau^q} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{1}{n'}} \\ &= a(x)^{\frac{n-p}{(p-q)n}} \left(\int_0^{ta(x)^{\frac{1}{q-p}}} \frac{ds}{(s^{p-1} + s^{q-1})^{\frac{1}{n-1}}} \right)^{\frac{1}{n'}} \end{aligned} \tag{3.36}$$

for $x \in \mathbb{R}^n$ and $t \geq 0$. Define the function $F : [0,\infty) \to [0,\infty)$ as

$$F(r) = \left(\int_0^r \frac{ds}{(s^{p-1} + s^{q-1})^{\frac{1}{n-1}}} \right)^{\frac{1}{n'}} \quad \text{for } r \geq 0. \tag{3.37}$$

Hence,

$$H_n^{-1}(x,t) = a(x)^{\frac{1}{p-q}} F^{-1}\big(ta(x)^{\frac{n-p}{n(q-p)}}\big) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{3.38}$$

Assume first that $q < n$. Then the function F obeys

$$F^{-1}(t) \approx \begin{cases} t^{\frac{n}{n-q}} & \text{if } t \geq 1 \\ t^{\frac{n}{n-p}} & \text{if } 0 \leq t < 1, \end{cases} \tag{3.39}$$

with equivalence constants depending on p, q and n. From equations (3.38) and (3.39), one can deduce that

$$\varphi_n(x,t) = \varphi(x, H_n^{-1}(x,t)) \approx t^{\frac{np}{n-p}} + a(x)^{\frac{n}{n-q}} t^{\frac{nq}{n-q}} \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{3.40}$$

This recovers [29, Proposition 3.4] in the case of constant exponents.

Assume next that $q = n$. Let F be defined as in (3.37), with $q = n$. Then,

$$F^{-1}(t) \approx \begin{cases} e^{t^{n'}} & \text{if } t \geq 1 \\ t^{\frac{n}{n-p}} & \text{if } 0 \leq t < 1. \end{cases} \tag{3.41}$$

Hence,

$$H_n^{-1}(x,t) = a(x)^{\frac{1}{p-n}} F^{-1}\big(ta(x)^{\frac{1}{n}}\big) \approx \begin{cases} a(x)^{\frac{1}{p-n}} e^{a(x)^{\frac{1}{n-1}} t^{n'}} & \text{if } ta(x)^{\frac{1}{n}} \geq 1 \\ t^{\frac{n}{n-p}} & \text{if } ta(x)^{\frac{1}{n}} < 1, \end{cases} \tag{3.42}$$

and

$$\varphi_n(x,t) = \varphi(x, H_n^{-1}(x,t)) \approx \begin{cases} a(x)^{\frac{p}{p-n}} e^{na(x)^{\frac{1}{n-1}} t^{n'}} & \text{if } ta(x)^{\frac{1}{n}} \geq 1 \\ t^{\frac{np}{n-p}} & \text{if } ta(x)^{\frac{1}{n}} < 1. \end{cases} \tag{3.43}$$

Therefore,

$$\varphi_n(x,t) \simeq t^{\frac{np}{n-p}} + a(x)^{\frac{p}{p-n}} \left(e^{a(x)^{\frac{1}{n-1}} t^{n'}} - \sum_{k=0}^{\lceil \frac{p(n-1)}{n(n-p)} \rceil} \frac{(a(x)^{\frac{1}{n}} t)^{n'k}}{k!} \right) \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{3.44}$$

where $\lceil \cdot \rceil$ stands for integer part.

Finally, suppose that $q > n$. Set

$$t_\infty = \left(\int_0^\infty \frac{ds}{(s^{p-1} + s^{q-1})^{\frac{1}{n-1}}} \right)^{\frac{1}{n'}}.$$

One can verify that

$$F^{-1}(t) \approx \begin{cases} \infty & \text{if } t \geq t_\infty \\ (t_\infty - t)^{-\frac{n-1}{q-n}} & \text{if } t_\infty/2 \leq t < t_\infty \\ t^{\frac{n}{n-p}} & \text{if } 0 \leq t < t_\infty/2 \end{cases} \approx \begin{cases} \infty & \text{if } t \geq t_\infty \\ t^{\frac{n}{n-p}} (t_\infty - t)^{-\frac{n-1}{q-n}} & \text{if } 0 \leq t < t_\infty. \end{cases} \tag{3.45}$$

Hence, by equation (3.38),

$$H^{-1}(x,t) \approx \begin{cases} \infty & \text{if } ta(x)^{\frac{n-p}{n(q-p)}} \geq t_\infty \\ a^{\frac{1}{p-q}} \left(ta(x)^{\frac{n-p}{n(q-p)}} \right)^{\frac{n}{n-p}} (t_\infty - ta(x)^{\frac{n-p}{n(q-p)}})^{-\frac{n-1}{q-n}} & \text{if } 0 \leq ta(x)^{\frac{n-p}{n(q-p)}} < t_\infty \end{cases} \tag{3.46}$$

$$= \begin{cases} \infty & \text{if } ta(x)^{\frac{n-p}{n(q-p)}} \geq t_\infty \\ t^{\frac{n}{n-p}} (t_\infty - ta(x)^{\frac{n-p}{n(q-p)}})^{-\frac{n-1}{q-n}} & \text{if } 0 \leq ta(x)^{\frac{n-p}{n(q-p)}} < t_\infty. \end{cases}$$

Thereby,

$$\varphi_n(x,t) = \varphi(x, H^{-1}(x,t))$$

$$\approx \begin{cases} \infty & \text{if } t \geq t_\infty a(x)^{\frac{n-p}{n(p-q)}} \\ \dfrac{a^{\frac{p}{p-q}}\Big(ta(x)^{\frac{n-p}{n(q-p)}}\Big)^{\frac{np}{n-p}}}{\big(t_\infty - ta(x)^{\frac{n-p}{n(q-p)}}\big)^{\frac{(n-1)p}{q-n}}} + \dfrac{a^{\frac{p}{p-q}}\Big(ta(x)^{\frac{n-p}{n(q-p)}}\Big)^{\frac{nq}{n-p}}}{\big(t_\infty - ta(x)^{\frac{n-p}{n(q-p)}}\big)^{\frac{(n-1)q}{q-n}}} & \text{if } 0 \leq t < t_\infty a(x)^{\frac{n-p}{n(p-q)}}, \end{cases} \tag{3.47}$$

$$\approx \begin{cases} \infty & \text{if } t \geq t_\infty a(x)^{\frac{n-p}{n(p-q)}} \\ \dfrac{t^{\frac{np}{n-p}}}{\big(t_\infty - ta(x)^{\frac{n-p}{n(q-p)}}\big)^{\frac{(n-1)q}{q-n}}} & \text{if } 0 \leq t < t_\infty a(x)^{\frac{n-p}{n(p-q)}}. \end{cases}$$

4. Weak-type estimates for Riesz potentials

This section is devoted to a weak-type inequality for Riesz potentials in Musielak-Orlicz spaces, which plays a crucial role in the proof of our Sobolev inequalities.

Given $\alpha \in (0, n)$, we denote by I_α the Riesz potential operator, defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \qquad \text{for } x \in \mathbb{R}^n, \tag{4.1}$$

for every function $f \in L^0(\mathbb{R}^n)$ which makes the integral converge.

The definition of the generalized Young function $\varphi_{\frac{n}{\alpha}}$ that enters the weak-type inequality for I_α in the space $L^{\varphi(\cdot)}(\mathbb{R}^n)$ extends that of φ_n to all $\alpha \in (0, n)$. Assume that φ is a generalized Young function satisfying the conditions (A0), (A1) and (A2), and such that

$$\int_0 \left(\frac{t}{\varphi_\infty(t)}\right)^{\frac{\alpha}{n-\alpha}} dt < \infty, \tag{4.2}$$

a condition which will be shown to be necessary for any weak-type inequality for I_α to hold in $L^{\varphi(\cdot)}(\mathbb{R}^n)$.

The Sobolev conjugate of order α of φ is defined as

$$\varphi_{\frac{n}{\alpha}}(x, t) = \overline{\varphi}\big(x, H_{\frac{n}{\alpha}}^{-1}(x, t)\big) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{4.3}$$

where $\overline{\varphi}$ is the normalized version of φ given by the formula (2.30), and the function $H_{\frac{n}{\alpha}} : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ obeys:

$$H_{\frac{n}{\alpha}}(x, t) = \left(\int_0^t \left(\frac{\tau}{\overline{\varphi}(x, \tau)}\right)^{\frac{\alpha}{n-\alpha}} d\tau\right)^{\frac{n-\alpha}{n}} \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{4.4}$$

Owing to the assumption (4.2) and the property (2.25) of normalized generalized Young functions, the function $H_{\frac{n}{\alpha}}$ is finite-valued for $x \in \mathbb{R}^n$ and $t \geq 0$. The definition of the inverse $H_{\frac{n}{\alpha}}^{-1}(x, \cdot)$ is analogous to that of $H_n^{-1}(x, \cdot)$, explained in the previous section. Plainly, choosing $\alpha = 1$ in (4.3) and (4.4) reproduces the function φ_n given by (3.7).

Theorem 4.1 *(Weak-type inequality for Riesz potentials). Let $\alpha \in (0, n)$ and let φ be a generalized Young function satisfying the conditions (A0), (A1), (A2), and (4.2). Then, there exists a positive constant $c = c(n, \alpha, \beta, h)$ such that*

$$\int\limits_{\{|I_\alpha f| > t\}} \varphi_{\frac{n}{\alpha}}(x, ct)\, dx \leq \int\limits_{\mathbb{R}^n} \overline{\varphi}(x, |f(x)|)\, dx \quad \text{for } t \geq 0, \tag{4.5}$$

if $\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leq 1$. Here, β denotes the constant from (A0) and (A1), and h the function from (A2). In particular, if $\varphi = \overline{\varphi}$, then the constant c in the inequality (4.5) only depends on n, α, and β.

Remark 4.2. It will be clear from its proof that the inequality (4.5) also holds if $\overline{\varphi}$ is replaced with any normalized generalized Young function satisfying the condition (4.2), and $\varphi_{\frac{n}{\alpha}}$ is directly defined by the use of φ, instead of $\overline{\varphi}$, in (4.3) and (4.4).

The necessity of the assumption (4.2) in Theorem 4.1 is shown by the next proposition.

Proposition 4.3 *(Necessity of condition (4.2)). Let α and φ be as in Theorem 4.1. Assume that there exist a generalized Young function ϑ and a positive constant c such that*

$$\int\limits_{\{|I_\alpha f| > t\}} \vartheta(x, ct)\, dx \leq \int\limits_{\mathbb{R}^n} \overline{\varphi}(x, |f(x)|)\, dx \quad \text{for } t \geq 0, \tag{4.6}$$

if $\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leq 1$. Then, the condition (4.2) must be fulfilled.

For convenience, we split the main steps of the proof of Theorem 4.1 into separate lemmas. Unless otherwise stated, in these lemmas and the in whole remaining part of this section,

φ denotes a normalized generalized Young function fulfilling the condition (4.2),

and β is the constant from Definition 2.5. The lemmas will then be applied with $\varphi = \overline{\varphi}$ in accomplishing the proof of Theorem 4.1.

We begin by introducing some notation and definitions. Let $k \in \mathbb{R}$ be such that

$$k \geq \frac{4}{\beta}, \tag{4.7}$$

and let $\sigma \in \mathbb{R}$ be such that

$$\sigma \geq \omega_n \max\left\{\frac{1}{2^n}, \frac{n}{n-\alpha}\right\}, \tag{4.8}$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Define the generalized Young function ψ as

$$\psi(x,t) = \sigma\, t^{\frac{n}{n-\alpha}} \int_0^t \frac{\widetilde{\varphi}(x,k\tau)}{\tau^{1+\frac{n}{n-\alpha}}}\, d\tau \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{4.9}$$

and the function $\lambda : \mathbb{R}^n \times [0,\infty) \to [0,\infty]$ as

$$\lambda(x,\delta) = \frac{1}{\delta^{n-\alpha}\psi^{-1}(x,\delta^{-n})} \quad \text{if } \delta > 0, \tag{4.10}$$

and $\lambda(x,0) = \lim_{\delta\to 0^+} \lambda(x,\delta)$ for $x \in \mathbb{R}^n$. Notice that the latter limit exists and equals $\left(\sigma \int_0^\infty \frac{\widetilde{\varphi}(x,k\tau)}{\tau^{1+\frac{n}{n-\alpha}}}\, d\tau\right)^{\frac{n-\alpha}{n}}$.

In what follows, we shall make use of the fact that, if $h : [0,\infty) \to [0,\infty)$ is a concave function such that $h(0) = 0$, then

$$\min\{1,c\}h(t) \leq h(ct) \leq \max\{1,c\}h(t) \quad \text{for } c,t \geq 0. \tag{4.11}$$

Lemma 4.4. *Let $x \in \mathbb{R}^n$ and let $\delta > 0$. If*

$$0 \leq t \leq \frac{1}{\lambda(x,\delta)\delta^{n-\alpha}}, \tag{4.12}$$

then

$$\widetilde{\varphi}\Big(x + \frac{\nu}{(\lambda(x,\delta)t)^{\frac{1}{n-\alpha}}}, t\Big) \leq \widetilde{\varphi}\Big(x, \frac{2t}{\beta}\Big) \quad \textit{for } \nu \in \mathbb{S}^{n-1}. \tag{4.13}$$

Proof. Fix $x \in \mathbb{R}^n$ and $\delta > 0$, and, for ease of notation, denote $\lambda(x,\delta)$ simply by λ throughout this proof. Since

$$\frac{1}{\lambda\delta^{n-\alpha}} = \psi^{-1}(x,\delta^{-n}), \tag{4.14}$$

the inequality (4.12) implies that

$$t \leq \psi^{-1}(x,\delta^{-n}). \tag{4.15}$$

Thus, owing to (2.28), the inequality (4.13) will follow if we show that

$$t \leq \frac{\beta}{2}\widetilde{\varphi}^{-1}\Big(x, \frac{2^n(\lambda t)^{\frac{n}{n-\alpha}}}{\omega_n}\Big) \tag{4.16}$$

for t as in (4.15). We have that

$$\psi(x,t) \geq \sigma t^{\frac{n}{n-\alpha}} \int_{\frac{t}{2}}^{t} \frac{\widetilde{\varphi}(x,k\tau)}{\tau^{1+\frac{n}{n-\alpha}}}\, d\tau \geq \sigma \widetilde{\varphi}(x,\tfrac{k}{2}t) t^{\frac{n}{n-\alpha}} \int_{\frac{t}{2}}^{t} \frac{1}{\tau^{1+\frac{n}{n-\alpha}}}\, d\tau \tag{4.17}$$
$$= \sigma \widetilde{\varphi}(x,\tfrac{k}{2}t) \frac{2^{\frac{n}{n-\alpha}}-1}{\frac{n}{n-\alpha}} \geq \sigma \widetilde{\varphi}(x,\tfrac{k}{2}t).$$

Hence, by the inequalities (4.15) and (4.7),

$$\widetilde{\varphi}(x,\tfrac{2}{\beta}t) \leq \frac{1}{\sigma}\psi(x,\tfrac{4}{k\beta}t) = \Big(\frac{4t}{k\beta}\Big)^{\frac{n}{n-\alpha}} \int_0^{\frac{4t}{k\beta}} \frac{\widetilde{\varphi}(x,k\tau)}{\tau^{1+\frac{n}{n-\alpha}}}\, d\tau \tag{4.18}$$
$$\leq \Big(\frac{4}{k\beta}\Big)^{\frac{n}{n-\alpha}} t^{\frac{n}{n-\alpha}} \int_0^{\frac{4}{k\beta}\psi^{-1}(x,\delta^{-n})} \frac{\widetilde{\varphi}(x,k\tau)}{\tau^{1+\frac{n}{n-\alpha}}}\, d\tau$$
$$\leq t^{\frac{n}{n-\alpha}} \int_0^{\psi^{-1}(x,\delta^{-n})} \frac{\widetilde{\varphi}(x,k\tau)}{\tau^{1+\frac{n}{n-\alpha}}}\, d\tau = \frac{t^{\frac{n}{n-\alpha}}}{\sigma\delta^n \psi^{-1}(x,\delta^{-n})^{\frac{n}{n-\alpha}}} = \frac{1}{\sigma}(\lambda t)^{\frac{n}{n-\alpha}}.$$

Thus,

$$t \leq \frac{\beta}{2}\widetilde{\varphi}^{-1}\Big(x, \frac{(\lambda t)^{\frac{n}{n-\alpha}}}{\sigma}\Big). \tag{4.19}$$

Coupling the latter inequality with (4.8) yields (4.16). $\square$

In what follows, we denote by $B(x,r)$ the ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$, with radius $r > 0$.

Lemma 4.5. *Let $x \in \mathbb{R}^n$ and $\delta \geq 0$. Then,*

$$\Big\| \frac{\chi_{\mathbb{R}^n\setminus B(x,\delta)}(\cdot)}{|x-\cdot|^{n-\alpha}} \Big\|_{L^{\widetilde{\varphi}(\cdot)}(\mathbb{R})} \leq \lambda(x,\delta). \tag{4.20}$$

Proof. Fix $x \in \mathbb{R}^n$. Recall that

$$\Big\| \frac{\chi_{\mathbb{R}^n\setminus B(x,\delta)}(\cdot)}{|x-\cdot|^{n-\alpha}} \Big\|_{L^{\widetilde{\varphi}(\cdot)}(\mathbb{R}^n)} = \inf\Big\{\lambda>0 : \int_{\mathbb{R}^n\setminus B(x,\delta)} \widetilde{\varphi}\Big(y, \frac{1}{\lambda|x-y|^{n-\alpha}}\Big)\, dy \leq 1\Big\}. \tag{4.21}$$

Choose $\lambda = \lambda(x,\delta)$ given by (4.10). Assume first that $\delta > 0$, and hence $\lambda(x,\delta) < \infty$. By the inequality (4.13) and the assumptions (4.7) and (4.8),

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n\setminus B(x,\delta)} \widetilde{\varphi}\Big(y, \frac{1}{\lambda|x-y|^{n-\alpha}}\Big)\, dy &= \int\limits_{\delta}^{\infty}\int\limits_{\mathbb{S}^{n-1}} \widetilde{\varphi}\Big(x+\nu r, \frac{1}{\lambda r^{n-\alpha}}\Big) r^{n-1}\, d\mathcal{H}^{n-1}(\nu)\, dr \qquad (4.22)\\
&= \frac{1}{(n-\alpha)\lambda^{\frac{n}{n-\alpha}}} \int\limits_0^{\frac{1}{\lambda\delta^{n-\alpha}}}\int\limits_{\mathbb{S}^{n-1}} \widetilde{\varphi}\Big(x+\frac{\nu}{(\lambda t)^{\frac{1}{n-\alpha}}}, t\Big)\, d\mathcal{H}^{n-1}(\nu)\, \frac{dt}{t^{1+\frac{n}{n-\alpha}}}\\
&\le \frac{1}{(n-\alpha)\lambda^{\frac{n}{n-\alpha}}} \int\limits_0^{\frac{1}{\lambda\delta^{n-\alpha}}}\int\limits_{\mathbb{S}^{n-1}} \widetilde{\varphi}\Big(x, \tfrac{2}{\beta}t\Big)\, d\mathcal{H}^{n-1}(\nu)\, \frac{dt}{t^{1+\frac{n}{n-\alpha}}}\\
&= \frac{n\omega_n}{(n-\alpha)\lambda^{\frac{n}{n-\alpha}}} \int\limits_0^{\frac{1}{\lambda\delta^{n-\alpha}}} \frac{\widetilde{\varphi}(x, \frac{2}{\beta}t)}{t^{1+\frac{n}{n-\alpha}}}\, dt\\
&\le \frac{n\omega_n}{(n-\alpha)\lambda^{\frac{n}{n-\alpha}}} \int\limits_0^{\frac{1}{\lambda\delta^{n-\alpha}}} \frac{\widetilde{\varphi}(x, kt)}{t^{1+\frac{n}{n-\alpha}}}\, dt.
\end{aligned}
$$

Here, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Since

$$
\frac{n\omega_n}{(n-\alpha)\lambda^{\frac{n}{n-\alpha}}} \int\limits_0^{\frac{1}{\lambda\delta^{n-\alpha}}} \frac{\widetilde{\varphi}(x, kt)}{t^{1+\frac{n}{n-\alpha}}}\, dt = \frac{n\omega_n}{\sigma(n-\alpha)}\delta^n\psi\Big(x, \frac{1}{\lambda\delta^{n-\alpha}}\Big) = \frac{n\omega_n}{\sigma(n-\alpha)} \le 1, \tag{4.23}
$$

the inequality (4.20) follows from (4.22) and (4.23).

If $\delta = 0$ and $\lambda(x,0) < \infty$, then

$$
\frac{n\omega_n}{(n-\alpha)\lambda^{\frac{n}{n-\alpha}}} \int\limits_0^{\frac{1}{\lambda\delta^{n-\alpha}}} \frac{\widetilde{\varphi}(x, kt)}{t^{1+\frac{n}{n-\alpha}}}\, dt = \frac{n\omega_n}{\sigma(n-\alpha)} \le 1, \tag{4.24}
$$

and the inequality (4.20) is a consequence of (4.22) and (4.24). Finally, if $\lambda(x,0) = \infty$, then (4.20) holds trivially. $\square$

Lemma 4.6. *Let B be any ball in $\mathbb{R}^n$ and let $x \in B$. Then,*

$$
\|\chi_B\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le \frac{1}{\beta\varphi^{-1}(x, |B|^{-1})} \tag{4.25}
$$

and

$$
\|\chi_B\|_{L^{\widetilde{\varphi}(\cdot)}(\mathbb{R}^n)} \le \frac{2}{\beta\widetilde{\varphi}^{-1}(x, |B|^{-1})}. \tag{4.26}
$$

The inequality (4.25) is proved in [25, Proposition 4.7], as a consequence of (2.24). The inequality (4.26) follows analogously, thanks to (2.27).

Lemma 4.7. *There exists a constant $c = c(n, \alpha, \beta)$ such that*

$$\delta^\alpha \leq c \frac{\lambda(x, \delta)}{\varphi^{-1}(x, \delta^{-n})} \quad \text{for } x \in \mathbb{R}^n \text{ and } \delta > 0. \tag{4.27}$$

Proof. Fix $x \in \mathbb{R}^n$. By an elementary computation using polar coordinates and the inequality (2.8),

$$\delta^\alpha \leq c \int\limits_{\mathbb{R}^n \setminus B(x,\delta)} \frac{\chi_{B(x,2\delta)}(y)}{|x-y|^{n-\alpha}}\, dy \leq 2c \|\chi_{B(x,2\delta)}(\cdot)\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \left\| \frac{\chi_{\mathbb{R}^n \setminus B(x,\delta)}(\cdot)}{|x - \cdot|^{n-\alpha}} \right\|_{L^{\widetilde{\varphi}(\cdot)}(\mathbb{R}^n)} \tag{4.28}$$

for some constant $c = c(n, \alpha)$. The inequality (4.27) hence follows, via (4.20) and (4.25). Here, one has also to make use of (4.11). □

Given a ball $B \subset \mathbb{R}^n$ and a function $f \in L^0(\mathbb{R}^n)$, we set

$$M_B f = \fint_B |f(y)|\, dy,$$

where $\fint$ denotes a mean-value integral. The centered Hardy-Littlewood maximal function $Mf : \mathbb{R}^n \to [0, \infty]$ is then defined as

$$Mf(x) = \sup_{r>0} M_{B(x,r)} f \qquad \text{for } x \in \mathbb{R}^n.$$

Thanks to the inequalities (2.8) and (4.26), we have that $L^{\varphi(\cdot)}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$. Therefore,

$$M_{B(x,r)} f < \infty \qquad \text{for } x \in \mathbb{R}^n \text{ and } r > 0,$$

if $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$.

The following property of the averaging operator M_B is a consequence of [26, Theorem 4.3.3].

Lemma 4.8. *[26] There exists a positive constant $\gamma = \gamma(n, \beta)$ such that*

$$\varphi\big(x, \gamma M_B f\big) \leq M_B\big(\varphi(\cdot, f(\cdot))\big) \tag{4.29}$$

for every ball $B \subset \mathbb{R}^n$, $x \in B$ and $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{\varphi(\cdot)} \leq 1$.

Define the function $\omega : \mathbb{R}^n \times [0,\infty) \to (0,\infty]$ as

$$\omega(x,t) = \lambda\Big(x, \frac{1}{\varphi(x,t)^{1/n}}\Big) \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{4.30}$$

where λ is given by (4.10).

Lemma 4.9. *There exists a constant $c = c(n,\alpha,\beta)$ such that*

$$|I_\alpha f(x)| \leq c\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}\, \omega\bigg(x, \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}\bigg) \quad \textit{for a.e. } x \in \mathbb{R}^n, \tag{4.31}$$

for $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$.

Proof. It suffices to prove the inequality (4.31) under the assumption that $Mf(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Indeed, since, as observed above, $L^{\varphi(\cdot)}(\mathbb{R}^n) \subset L^1_{\rm loc}(\mathbb{R}^n)$, if $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$ then $f\chi_{B(0,R)} \in L^1(\mathbb{R}^n)$ for every $R > 0$. Thus, $M\big(f\chi_{B(0,R)}\big) < \infty$ a.e. in $\mathbb{R}^n$. An application of the inequality (4.31) with f replaced with $f\chi_{B(0,R)}$ and a passage to the limit as $R \to \infty$ then yield the result in the general case.
Fix $\delta > 0$ and any $x \in \mathbb{R}^n$ such that $Mf(x) < \infty$. The following estimate is well-known:

$$\int\limits_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \leq \frac{n}{\alpha}\delta^\alpha Mf(x), \tag{4.32}$$

see e.g. [1, Inequality (3.1.1)]. Hence, by the inequalities (2.8) and (4.20),

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy &= \int\limits_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy + \int\limits_{\mathbb{R}^n\setminus B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \\
&\leq \frac{n}{\alpha}\delta^\alpha Mf(x) + 2\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \bigg\| \frac{\chi_{\mathbb{R}^n\setminus B(x,\delta)}(\cdot)}{|x-\cdot|^{n-\alpha}} \bigg\|_{L^{\widetilde{\varphi}(\cdot)}(\mathbb{R}^n)} \\
&\leq \frac{n}{\alpha}\delta^\alpha Mf(x) + 2\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}\lambda(x,\delta).
\end{aligned} \tag{4.33}$$

The choice

$$\delta = \varphi\bigg(x, \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}\bigg)^{-\frac{1}{n}}$$

in equation (4.33) yields:

$$|I_\alpha f(x)| \leq \frac{n}{\alpha} \frac{Mf(x)}{\varphi\Big(x, \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}\Big)^{\frac{\alpha}{n}}} + 2\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}\omega\bigg(x, \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}\bigg). \tag{4.34}$$

Now, observe that

$$\frac{t}{\varphi(x,t)^{\frac{\alpha}{n}}} \leq c\,\omega(x,t) \quad \text{for } t > 0, \tag{4.35}$$

where c is the constant from the inequality (4.27). Indeed, owing to (4.27),

$$\varphi^{-1}(x,s^{-1})s^{\frac{\alpha}{n}} \leq c\,\lambda\big(x,s^{\frac{1}{n}}\big) \quad \text{for } s > 0,$$

whence the inequality (4.35) follows, on setting $s = \frac{1}{\varphi(x,t)}$.

Exploiting (4.35), with

$$t = \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}},$$

to estimate the first addend on the right-hand side of the inequality (4.34) tells us that

$$|I_\alpha f(x)| \leq c\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}\omega\left(x, \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}\right) + 2\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}\omega\left(x, \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}\right) \tag{4.36}$$

for some constant $c = c(n,\alpha,\beta)$, namely (4.31). $\square$

Lemma 4.10. *There exist positive constants $c_1 = c_1(n,\alpha,\sigma)$ and $c_2 = c_2(n,\alpha,\sigma)$ such that*

$$c_1 k\, H_{\frac{n}{\alpha}}(x,t) \leq \omega(x,t) \leq c_2 k\, H_{\frac{n}{\alpha}}(x,t) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{4.37}$$

Here, σ and k denote the constants appearing in equation (4.9).

Proof. Define the generalized Young function φ_k as

$$\varphi_k(x,t) = \varphi\Big(x, \frac{t}{k}\Big) \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{4.38}$$

Moreover, let $H_{\frac{n}{\alpha},k} : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be the function defined as in (4.4), with φ replaced with φ_k. Namely,

$$H_{\frac{n}{\alpha},k}(x,t) = \left(\int_0^t \left(\frac{\tau}{\varphi_k(x,\tau)}\right)^{\frac{\alpha}{n-\alpha}} d\tau\right)^{\frac{n-\alpha}{n}} \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{4.39}$$

Observe that

$$\widetilde{\varphi_k}(x,t) = \widetilde{\varphi}(x,tk) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{4.40}$$

where $\widetilde{\varphi_k}$ denotes the Young conjugate of the function φ_k. Also,

$$H_{\frac{n}{\alpha},k}(x,t) = kH_{\frac{n}{\alpha}}\Big(x,\frac{t}{k}\Big) \quad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{4.41}$$

By [12, Lemma 2], applied to the function $\varphi_k(x,\cdot)$ for each fixed $x \in \mathbb{R}^n$, and equation (4.11), there exist positive constants $c_1 = c_1(n,\alpha,\sigma)$ and $c_2 = c_2(n,\alpha,\sigma)$ such that

$$c_1 H_{\frac{n}{\alpha},k}(x,t) \leq \omega\Big(x,\frac{t}{k}\Big) \leq c_2 H_{\frac{n}{\alpha},k}(x,t) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0.$$

Hence, equation (4.37) follows, via the inequalities (4.41). □

Lemma 4.11. *Let σ and k be the constants from equation (4.9), and let $b > 0$. There exists a constant $c = c(n,\alpha,\beta,\sigma)$ such that, if*

$$\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leq b, \tag{4.42}$$

then

$$|I_\alpha f(x)| \leq ck\max\{b,1\}H_{\frac{n}{\alpha}}\big(x, Mf(x)\big) \quad \textit{for a.e. } x \in \mathbb{R}^n. \tag{4.43}$$

Proof. The inequalities (4.31) and (4.37) ensure that there exists a constant $c = c(n,\alpha,\beta,\sigma)$ such that

$$|I_\alpha f(x)| \leq ck\,\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} H_{\frac{n}{\alpha}}\bigg(x, \frac{Mf(x)}{\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}\bigg) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{4.44}$$

Since, for each fixed $x \in \mathbb{R}^n$, the function

$$t \mapsto H_{\frac{n}{\alpha}}(x,t) \text{ is concave and } H_{\frac{n}{\alpha}}(x,0) = 0, \tag{4.45}$$

the function

$$t \mapsto \frac{1}{t}H_{\frac{n}{\alpha}}(x,t) \qquad \text{is non-increasing.} \tag{4.46}$$

By the assumption (4.42) and the properties (4.46) and (4.11), the inequality (4.44) entails that

$$|I_\alpha f(x)| \leq ckbH_{\frac{n}{\alpha}}\Big(x, \frac{Mf(x)}{b}\Big) \leq ckb\max\Big\{1, \frac{1}{b}\Big\}H_{\frac{n}{\alpha}}\big(x, Mf(x)\big) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Hence, the inequality (4.43) follows. □

Lemma 4.12. *Let $z \in \mathbb{R}^n$ and $t \geq 0$. Assume that $f \in L^{\varphi(\cdot)}(\mathbb{R}^n)$. Let B_z be a ball such that $z \in B_z$ and*

$$H_{\frac{n}{\alpha}}\big(z, M_{B_z} f\big) > t. \tag{4.47}$$

Assume that

$$0 < \eta \le \min\{\tfrac{\beta}{4}, \tfrac{\gamma}{\beta}\}, \tag{4.48}$$

where γ is the constant from Lemma 4.8. If $x \in B_z$ and

$$\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le \eta, \tag{4.49}$$

then

$$\varphi\big(x, H^{-1}_{\frac{n}{\alpha}}(x, \beta t)\big) \le M_{B_z}\big(\varphi(\cdot, \tfrac{\beta}{\gamma} f(\cdot)\big). \tag{4.50}$$

Proof. Assume that the function f fulfills the condition (4.49), with η to be chosen later. Fix $x \in B_z$. If

$$0 \le s \le \beta \varphi^{-1}(z, |B_z|^{-1}),$$

then, by the inequality (2.26),

$$\begin{aligned} H_{\frac{n}{\alpha}}(x, s) &= \bigg(\int_0^s \Big(\frac{t}{\varphi(x,t)}\Big)^{\frac{\alpha}{n-\alpha}} dt\bigg)^{\frac{n-\alpha}{n}} \ge \bigg(\int_0^s \Big(\frac{t}{\varphi(z,t/\beta)}\Big)^{\frac{\alpha}{n-\alpha}} dt\bigg)^{\frac{n-\alpha}{n}} \\ &= \beta\bigg(\int_0^{\frac{s}{\beta}} \Big(\frac{t}{\varphi(z,t)}\Big)^{\frac{\alpha}{n-\alpha}} dt\bigg)^{\frac{n-\alpha}{n}} = \beta H_{\frac{n}{\alpha}}\Big(z, \frac{s}{\beta}\Big). \end{aligned} \tag{4.51}$$

Hence,

$$H^{-1}_{\frac{n}{\alpha}}(x, \beta t) \le \beta H^{-1}_{\frac{n}{\alpha}}(z, t) \tag{4.52}$$

if

$$0 \le t \le H_{\frac{n}{\alpha}}\big(z, \varphi^{-1}(z, |B_z|^{-1})\big). \tag{4.53}$$

Thanks to the assumption (4.49), the inequalities (2.8), (4.26) and (2.2) imply that

$$\begin{aligned} M_{B_z} f &= \frac{1}{|B_z|}\int_{B_z} |f(y)|\, dy \le \frac{2}{|B_z|}\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}\|\chi_{B_z}\|_{L^{\widetilde{\varphi}(\cdot)}(\mathbb{R}^n)} \\ &\le \frac{4\eta}{\beta|B_z|\widetilde{\varphi}^{-1}(z, |B_z|^{-1})} \le \frac{4\eta}{\beta}\varphi^{-1}(z, |B_z|^{-1}). \end{aligned} \tag{4.54}$$

By (4.47) and (4.54),

$$t \le H_{\frac{n}{\alpha}}(z, M_{B_z}f) \le H_{\frac{n}{\alpha}}\Big(z, \frac{4\eta}{\beta}\varphi^{-1}(z, |B_z|^{-1})\Big). \tag{4.55}$$

Thus, the condition (4.53), and hence (4.52), hold provided that

$$H_{\frac{n}{\alpha}}\Big(z, \frac{4\eta}{\beta}\varphi^{-1}(z, |B_z|^{-1})\Big) \le H_{\frac{n}{\alpha}}\big(z, \varphi^{-1}(z, |B_z|^{-1})\big). \tag{4.56}$$

The latter inequality is fulfilled if

$$\eta \le \frac{\beta}{4}. \tag{4.57}$$

The inequalities (4.52) and (4.47) tell us that

$$H_{\frac{n}{\alpha}}^{-1}(x, \beta t) \le \beta H_{\frac{n}{\alpha}}^{-1}(z, t) \le \beta M_{B_z}f. \tag{4.58}$$

Hence, owing to Proposition 4.8,

$$\varphi\big(x, H_{\frac{n}{\alpha}}^{-1}(x, \beta t)\big) \le \varphi\big(x, \beta M_{B_z}f\big) = \varphi\big(x, \gamma M_{B_z}\big(\tfrac{\beta}{\gamma}f\big)\big) \le M_{B_z}\big(\varphi(\cdot, \tfrac{\beta}{\gamma}f(\cdot)\big), \tag{4.59}$$

if η satisfies both the inequality (4.57) and

$$\eta \le \frac{\gamma}{\beta}. \tag{4.60}$$

Hence, the inequality (4.50) follows. □

Proof of Theorem 4.1. Thanks to Proposition 2.7, it suffices to prove the inequality (3.10) with φ replaced with the normalized function $\overline{\varphi}$ given by (2.30). For ease of notation, we denote $\overline{\varphi}$ simply by φ.

Fix $t \ge 0$. Let η be as in (4.48), let k be the constant in the definition (4.9) of the function ψ, and let c be the constant from (4.43). Assume that

$$\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le \eta. \tag{4.61}$$

By Lemma 4.11, applied with $b = \eta$,

$$\{|I_\alpha(x)| > c't\} \subset \{H_{\frac{n}{\alpha}}(x, Mf(x)) > t\}, \tag{4.62}$$

where $c' = ck\max\{\eta, 1\}$.

For each $z \in \{H_{\frac{n}{\alpha}}(x, Mf(x)) > t\}$, let B_z be a ball such that $z \in B_z$ and the inequality (4.47) holds. By Besicovitch's covering theorem, there exists a countable sub-covering $\{B_i\}$ of the covering $\{B_z\}$ enjoying the bounded overlap property. Thus, by Lemma 4.12, the inequality (4.50) and this property of the covering, there exists a constant $c'' = c''(n)$ such that

$$\begin{aligned}\int\limits_{\{|I_\alpha(x)|>c't\}} \varphi(x, H_{\frac{n}{\alpha}}^{-1}(x,\beta t))\,dx &\le \int\limits_{\{H_{\frac{n}{\alpha}}(x,Mf(x))>t\}} \varphi(x, H_{\frac{n}{\alpha}}^{-1}(x,\beta t))\,dx \\ &\le \sum_{i=1}^\infty \int\limits_{B_i} \varphi(x, H_{\frac{n}{\alpha}}^{-1}(x,\beta t))\,dx \\ &\le \sum_{i=1}^\infty \int\limits_{B_i} \fint\limits_{B_i} \varphi(y, \tfrac{\beta}{\gamma}|f(y)|)\,dy\,dx \\ &= \sum_{i=1}^\infty \int\limits_{B_i} \varphi(y, \tfrac{\beta}{\gamma}|f(y)|)\,dy \le c'' \int\limits_{\mathbb{R}^n} \varphi(y, \tfrac{\beta}{\gamma}|f(y)|)\,dy \\ &\le \int\limits_{\mathbb{R}^n} \varphi(y, \tfrac{\beta}{\gamma}\max\{c'',1\}|f(y)|)\,dy \\ &\le \int\limits_{\mathbb{R}^n} \varphi(y, c'''|f(y)|)\,dy,\end{aligned} \tag{4.63}$$

provided that the constant c''' fulfills the inequality

$$c''' \ge \frac{\beta \max\{c'',1\}}{\gamma}.$$

An application of the inequality (4.63), with f replaced with $\frac{f}{c'''}$, yields

$$\int\limits_{\{|I_\alpha f(x)|>c'c''t\}} \varphi(x, H_{\frac{n}{\alpha}}^{-1}(x,\beta t))\,dx \le \int\limits_{\mathbb{R}^n} \varphi(y, |f(y)|)\,dy, \tag{4.64}$$

if

$$\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le \eta\, c'''. \tag{4.65}$$

Now, choose

$$c''' = \max\Big\{\frac{\beta \max\{c'',1\}}{\gamma}, \frac{1}{\eta}\Big\}.$$

Thereby, we deduce from (4.64) that

$$\int\limits_{\{|I_\alpha f(x)|>c'c'''t\}} \varphi(x, H_{\frac{n}{\alpha}}^{-1}(x,\beta t))\,dx \le \int\limits_{\mathbb{R}^n} \varphi(y, |f(y)|)\,dy, \tag{4.66}$$

if

$$\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leq 1.$$

The inequality (4.5) is thus established. □

Proof of Proposition 4.3. As in the proof of Theorem 4.1, we may assume that $\varphi = \overline{\varphi}$. Consider the Young function $A : [0, \infty) \to [0, \infty]$ defined as

$$A(t) = \sup_{x \in \mathbb{R}^n} \varphi(x, t) \qquad \text{for } t \geq 0. \tag{4.67}$$

Plainly,

$$A(t) \geq \varphi(x, t) \qquad \text{for } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{4.68}$$

and, since φ is normalized,

$$A(t) = \varphi(x, t) = \varphi_\infty(t) \qquad \text{if } 0 \leq t \leq 1, \tag{4.69}$$

for $x \in \mathbb{R}^n$. As a consequence of the latter identity, there exists $t_0 > 0$ such that

$$\widetilde{A}(t) = \widetilde{\varphi}(t) \qquad \text{if } 0 \leq t \leq t_0. \tag{4.70}$$

Choose trial functions f in the inequality (4.6) of the form $f(x) = g(\omega_n |x|^n)$, where $g : [0, \infty) \to [0, \infty)$ is a bounded measurable function with bounded support. Then

$$I_\alpha f(x) \geq 2^{\alpha - n} \int\limits_{\{|y| \geq |x|\}} \frac{f(y)}{|y|^{n-\alpha}}\, dy = \omega_n^{\frac{n-\alpha}{n}} 2^{\alpha - n} \int\limits_{\omega_n |x|^n}^{\infty} g(s) s^{-\frac{n-\alpha}{n}}\, ds$$

for $x \in \mathbb{R}^n$. Define the function $Tg : [0, \infty) \to [0, \infty]$ as

$$Tg(r) = \int\limits_{\omega_n r^n}^{\infty} g(s) s^{-\frac{n-\alpha}{n}}\, ds \quad \text{for } r \in [0, \infty).$$

Thus, the inequality (4.6) tells us that

$$\int\limits_{\{x : Tg(|x|) > t\}} \vartheta(x, ct)\, dx \leq \int\limits_{\mathbb{R}^n} \varphi(x, g(\omega_n |x|^n))\, dx \tag{4.71}$$

provided that

$$\|g(\omega_n |\cdot|^n)\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leq 1. \tag{4.72}$$

By the monotone convergence theorem for Lebesgue integrals and the Fatou property of Musielak-Orlicz norms, replacing, for $t > 0$, any measurable function $g : [0, \infty) \to [0, \infty)$ with the function $g_t : [0, \infty) \to [0, \infty)$ given by

$$g_t(s) = \min\{t, g(s)\}\chi_{(0,t)}(s) \qquad \text{for } s \geq 0, \tag{4.73}$$

and passing to the limit as $t \to \infty$, tell us that the inequality (4.71) continues to hold for every nonnegative measurable function g fulfilling (4.72).

Assume that $Tg(0) = \infty$, the case when $Tg(0) < \infty$ requiring just minor variants. Let $\{r_j\}$ be a non-increasing sequence in $[0, \infty)$ such that

$$Tg(r_j) = 2^j \qquad \text{for } j \in \mathbb{Z}.$$

Define the sequence of functions $\{g_j\}$ by

$$g_j = g\chi_{(\omega_n r_j^n, \omega_n r_{j-1}^n)} \tag{4.74}$$

for $j \in \mathbb{Z}$. Inasmuch as

$$Tg(r) \leq Tg(r_j) \leq 2^j \qquad \text{for } r \in (r_j, r_{j-1}), \tag{4.75}$$

one has that

$$\begin{aligned} \int_{\mathbb{R}^n} \vartheta\Big(x, \frac{cTg(|x|)}{4}\Big)\, dx &= \sum_{j\in\mathbb{Z}} \int_{\{x: r_j < |x| < r_{j-1}\}} \vartheta\Big(x, \frac{cTg(|x|)}{4}\Big)\, dx \\ &\leq \sum_{j\in\mathbb{Z}} \int_{\{x: r_j < |x| < r_{j-1}\}} \vartheta\Big(x, \frac{c2^j}{4}\Big)\, dx \\ &= \sum_{j\in\mathbb{Z}} \int_{\{x: r_j < |x| < r_{j-1}\}} \vartheta(x, c2^{j-2})\, dx. \end{aligned} \tag{4.76}$$

Assume that $r_j < |x| < r_{j-1}$ for some $j \in \mathbb{Z}$. Thus,

$$\begin{aligned} Tg_{j-1}(|x|) &\geq \int_{\omega_n r_{j-1}^n}^{\infty} g_{j-1}(s) s^{-\frac{n-\alpha}{n}}\, ds = \int_{\omega_n r_{j-1}^n}^{\infty} g(s) s^{-\frac{n-\alpha}{n}} \chi_{(\omega_n r_{j-1}^n, \omega_n r_{j-2}^n)}(s)\, ds \\ &= \int_{\omega_n r_{j-1}^n}^{\omega_n r_{j-2}^n} g(s) s^{-\frac{n-\alpha}{n}}\, ds = Tg(r_{j-1}) - Tg(r_{j-2}) = 2^{j-2}. \end{aligned} \tag{4.77}$$

Hence,

$$\{x : r_j < |x| < r_{j-1}\} \subset \{x : Tg_{j-1}(|x|) > 2^{j-2}\}. \tag{4.78}$$

Coupling the latter equation with the inequality (4.71), applied with g replaced with g_{j-1}, tells us that

$$\begin{aligned}\int\limits_{\{x:r_j<|x|<r_{j-1}\}} \vartheta(x, c2^{j-2})\, dx &\le \int\limits_{\{x:Tg_{j-1}(|x|)>2^{j-2}\}} \vartheta(x, c2^{j-2})\, dx \\ &\le \int\limits_{\mathbb{R}^n} \varphi(x, g_{j-1}(\omega_n|x|^n))\, dx.\end{aligned} \tag{4.79}$$

Observe that the application of the inequality (4.71) is legitimate, since $g_{j-1} \le g$, whence $\|g_{j-1}(\omega_n|\cdot|^n)\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le 1$ if g satisfies the condition (4.72). From the inequalities (4.76) and (4.79) we deduce that

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \vartheta\Big(x, \frac{cTg(|x|)}{4}\Big)\, dx &\le \sum_{j\in\mathbb{Z}} \int\limits_{\mathbb{R}^n} \varphi(x, g_{j-1}(\omega_n|x|^n))\, dx \\ &= \sum_{j\in\mathbb{Z}} \int\limits_{\{x:r_j<|x|<r_{j-1}\}} \varphi(x, g(\omega_n|x|^n))\, dx \\ &= \int\limits_{\mathbb{R}^n} \varphi(x, g(\omega_n|x|^n))\, dx\end{aligned} \tag{4.80}$$

for every g fulfilling the condition (4.72). Applying the latter inequality with $g(\omega_n|x|^n)$ replaced with $\frac{g(\omega_n|x|^n)}{\|g(\omega_n|\cdot|^n)\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}$ tells us that

$$\Big\| \int\limits_{\omega_n|x|^n}^{\infty} g(s)s^{-\frac{n-\alpha}{n}}\, ds \Big\|_{L^{\vartheta(\cdot)}(\mathbb{R}^n)} \le \frac{4}{c} \|g(\omega_n|x|^n)\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \tag{4.81}$$

for every g such that $g(\omega_n|x|^n) \in L^{\varphi(\cdot)}(\mathbb{R}^n)$. Owing to the inequality (4.68),

$$\|g(\omega_n|x|^n)\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le \|g(\omega_n|x|^n)\|_{L^A(\mathbb{R}^n)} = \|g\|_{L^A(0,\infty)}. \tag{4.82}$$

On the other hand, if g vanishes in $(0, \omega_n)$, then

$$\begin{aligned}\Big\| \int\limits_{\omega_n|x|^n}^{\infty} g(s)s^{-\frac{n-\alpha}{n}}\, ds \Big\|_{L^{\vartheta(\cdot)}(\mathbb{R}^n)} &\ge \Big\| \chi_{B(0,1)}(x) \int\limits_{\omega_n|x|^n}^{\infty} g(s)s^{-\frac{n-\alpha}{n}}\, ds \Big\|_{L^{\vartheta(\cdot)}(\mathbb{R}^n)} \\ &= \|\chi_{B(0,1)}\|_{L^{\vartheta(\cdot)}(\mathbb{R}^n)} \int\limits_{\omega_n}^{\infty} g(s)s^{-\frac{n-\alpha}{n}}\, ds.\end{aligned} \tag{4.83}$$

Combining the inequalities (4.81) – (4.83) implies that

$$\sup_{g} \frac{\int_{\omega_n}^{\infty} g(s) s^{-\frac{n-\alpha}{n}}\, ds}{\|g(s)\|_{L^A(\omega_n,\infty)}} \le \frac{c}{\|\chi_{B(0,1)}\|_{L^{\vartheta(\cdot)}(\mathbb{R}^n)}} \tag{4.84}$$

for some constant c and for every nonnegative function $g \in L^A(\omega_n, \infty)$. Thanks to the inequalities (4.84) and (2.9), applied with φ replaced with A,

$$\left\| s^{-\frac{n-\alpha}{n}} \right\|_{L^{\widetilde{A}}(\omega_n,\infty)} < \infty. \tag{4.85}$$

By a chain analogous to (and, in fact, simpler than) (4.22), the inequality (4.85) is equivalent to

$$\int_0 \frac{\widetilde{A}(t)}{t^{1+\frac{n}{n-\alpha}}}\, dt < \infty, \tag{4.86}$$

which, by [11, Lemma 2.3], is in turn equivalent to

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{\alpha}{n-\alpha}} dt < \infty. \tag{4.87}$$

The inequality (4.2) hence follows, owing to equation (4.69). □

5. Proof of the Sobolev inequalities

The link between the weak-type inequality for Riesz potentials from the previous section and the Sobolev inequalities of Theorems 3.1 and 3.6 is provided by the representation formula which is the subject of the next lemma. The formula is standard for compactly supported functions. The punctum of this lemma is the broader class of admissible functions.

Lemma 5.1. *Assume that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is such that*

$$|\{|u| > t\}| < \infty \qquad \textit{for } t > 0, \tag{5.1}$$

and

$$\int_{\mathbb{R}^n \setminus B(0,1)} \frac{|\nabla u(x)|}{|x|^{n-1}}\, dx < \infty. \tag{5.2}$$

Then,

$$u(x) = \frac{1}{n\omega_n} \int\limits_{\mathbb{R}^n} \frac{\nabla u(y) \cdot (x-y)}{|x-y|^n} \, dy \quad \textit{for a.e. } x \in \mathbb{R}^n. \tag{5.3}$$

Here, the dot "·" stands for scalar product in $\mathbb{R}^n$.

Proof. To begin with, recall that, since $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, for a.e. $x \in \mathbb{R}^n$ the function

$$[0, \infty) \ni t \mapsto u(x + \nu r) \tag{5.4}$$

is absolutely continuous, for $\mathcal{H}^{n-1}$-a.e. $\nu \in \mathbb{S}^{n-1}$, on every interval $[0, \ell]$ with $\ell > 0$. Furthermore,

$$\frac{d}{dr} u(x + \nu r) = \nabla u(x + \nu r) \cdot \nu \qquad \text{for a.e. } r > 0, \tag{5.5}$$

see e.g., [38, Exercise 3.15]. Fix any $x \in \mathbb{R}^n$ for which this property holds. We claim that

$$\liminf_{r \to \infty} |u(x + \nu r)| = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \nu \in \mathbb{S}^{n-1}. \tag{5.6}$$

To verify this claim, set

$$E = \{\nu \in \mathbb{S}^{n-1} : \liminf_{r \to \infty} |u(x + \nu r)| > 0\}, \tag{5.7}$$

and

$$E_j = \{\nu \in \mathbb{S}^{n-1} : \liminf_{r \to \infty} |u(x + \nu r)| > 1/j\}$$

for $j \in \mathbb{N}$. Assume, by contradiction, that $\mathcal{H}^{n-1}(E) > 0$. Since $E = \bigcup_{j \in \mathbb{N}} E_j$, one hence infers that

$$0 < \mathcal{H}^{n-1}(E) = \lim_{j \to \infty} \mathcal{H}^{n-1}(E_j). \tag{5.8}$$

Therefore, there exists $j_0 \in \mathbb{N}$ such that

$$\mathcal{H}^{n-1}(E_{j_0}) > 0. \tag{5.9}$$

In particular, this implies that

$$\int\limits_0^\infty \chi_{\{|u(x+\nu r)| > 1/j_0\}}(\nu, r) r^{n-1} \, dr = \infty \tag{5.10}$$

for every $\nu \in E_{j_0}$. Thus,

$$|\{|u| > 1/j_0\}| = \int_{\{|u|>1/j_0\}} dy = \int_{\mathbb{S}^{n-1}} \int_0^\infty \chi_{\{|u(x+\nu r)|>1/j_0\}}(\nu, r) r^{n-1}\, dr\, d\mathcal{H}^{n-1}(\nu)$$
$$\geq \int_{E_{j_0}} \int_0^\infty \chi_{\{|u(x+\nu r)|>1/j_0\}}(\nu, r) r^{n-1}\, dr\, d\mathcal{H}^{n-1}(\nu) = \infty, \tag{5.11}$$

and this contradicts the assumption (5.1). The property (5.6) is therefore established.

Next, observe that

$$\int_0^\infty |\nabla u(x+\nu r)\cdot \nu|\, dr < \infty \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \nu \in \mathbb{S}^{n-1}. \tag{5.12}$$

In order to prove (5.12), notice the following chain:

$$\int_{\mathbb{S}^{n-1}} \int_1^\infty |\nabla u(x+\nu r)\cdot \nu|\, dr\, d\mathcal{H}^{n-1}(\nu) \tag{5.13}$$
$$= \int_{\mathbb{S}^{n-1}} \int_1^\infty \frac{|\nabla u(x+\nu r)\cdot \nu|}{r^{n-1}} r^{n-1}\, dr\, d\mathcal{H}^{n-1}(\nu)$$
$$= \int_{\mathbb{R}^n\setminus B_1(x)} \frac{|\nabla u(y)\cdot (y-x)|}{|y-x|^n}\, dy \leq \int_{\mathbb{R}^n\setminus B_1(x)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy.$$

The last integral converges, thanks to the assumption (5.2). Thereby,

$$\int_1^\infty |\nabla u(x+\nu r)\cdot \nu|\, dr < \infty \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \nu \in \mathbb{S}^{n-1}. \tag{5.14}$$

On the other hand,

$$\int_0^1 |\nabla u(x+\nu r)\cdot \nu|\, dr < \infty \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \nu \in \mathbb{S}^{n-1}, \tag{5.15}$$

since the function in (5.4) is absolutely continuous on $[0,1]$ and the property (5.5) holds. Coupling the inequality (5.14) with (5.15) yields (5.12).

Now, fix any $\nu \in \mathbb{S}^{n-1}$ such that equations (5.6) and (5.12) hold. Therefore, there exists a sequence $\{r_j\}$ such that $\lim_{j\to\infty} r_j = \infty$ and $\lim_{j\to\infty} u(x+\nu r_j) = 0$. Inasmuch as

$$u(x+\nu r_j) - u(x) = \int_0^{r_j} \nabla u(x+\nu r)\cdot \nu \, dr, \tag{5.16}$$

passing to the limit as $j \to \infty$ in the latter equation yields

$$u(x) = -\int_0^{\infty} \nabla u(x+\nu r)\cdot \nu \, dr. \tag{5.17}$$

Observe that the passage to the limit in the integral is legitimate, thanks to the dominated convergence theorem and to equation (5.12). Integrating the identity (5.17), which holds for $\mathcal{H}^{n-1}$-a.e. $\nu \in \mathbb{S}^{n-1}$, with respect to ν over $\mathbb{S}^{n-1}$ tells us that

$$n\omega_n u(x) = -\int_{\mathbb{S}^{n-1}} \int_0^{\infty} \nabla u(x+\nu r)\cdot \nu \, dr \, d\mathcal{H}^{n-1}(\nu) = -\int_{\mathbb{R}^n} \frac{\nabla u(y)\cdot(y-x)}{|y-x|^n}\, dy. \tag{5.18}$$

Equation (5.3) is thus established. □

We are now in a position to prove our Sobolev inequalities. We begin with the inequality in $\mathbb{R}^n$.

Proof of Theorem 3.1. Proposition 2.7 enables us to assume, without loss of generality, that $\varphi = \overline{\varphi}$.

Our assumptions on φ and u ensure that the latter fulfills the hypotheses of Lemma 5.1. In particular, the condition (5.2) is satisfied owing to the Hölder inequality (2.8), the inequality (4.20) and the fact that $|\nabla u| \in L^{\varphi(\cdot)}(\mathbb{R}^n)$. From equation (5.3) one thus infers that

$$|u(x)| < \kappa I_1(|\nabla u|)(x) \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{5.19}$$

for some constant $\kappa = \kappa(n)$. Thereby, if, in addition,

$$\|\nabla u\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \le 1, \tag{5.20}$$

then Lemma 4.1 yields:

$$\begin{aligned}\int_{\{|u|\ge t\}} \varphi_n(x, c't)\, dx = \int_{\{|u|\ge t\}} \varphi(x, H_n^{-1}(x,c't))\, dx \le \int_{\{I_1(|\nabla u|)\ge t/\kappa\}} \varphi(x, H_n^{-1}(x,c't))\, dx \\ \le \int_{\mathbb{R}^n} \varphi(x, |\nabla u|)\, dx,\end{aligned} \tag{5.21}$$

where $c' = \frac{c}{\kappa}$, and c is the constant from (4.5). Define the function $u_j : \mathbb{R}^n \to [0,\infty)$ as

$$u_j = \max\{\min\{|u| - 2^j, 2^j\}, 0\},$$

and the set

$$U_j = \{x \in \mathbb{R}^n : 2^j < |u(x)| \leq 2^{j+1}\}$$

for $j \in \mathbb{Z}$. Observe that

$$U_j \subset \{x \in \mathbb{R}^n : u_{j-1}(x) = 2^{j-1}\}$$

for $j \in \mathbb{Z}$. An application of the inequality (5.21), with u replaced with u_{j-1}, enables one to deduce that

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \varphi_n(x, \tfrac{c'}{4}|u|)\,dx &= \sum_{j\in\mathbb{Z}} \int\limits_{U_j} \varphi_n(x, \tfrac{c'}{4}|u|)\,dx \leq \sum_{j\in\mathbb{Z}} \int\limits_{U_j} \varphi_n(x, c'2^{j-1})\,dx \\ &\leq \sum_{j\in\mathbb{Z}} \int\limits_{\{u_{j-1}\geq 2^{j-1}\}} \varphi_n(x, c'2^{j-1})\,dx \\ &\leq \sum_{j\in\mathbb{Z}} \int\limits_{\mathbb{R}^n} \varphi(x, |\nabla u_j|)\,dx = \int\limits_{\mathbb{R}^n} \varphi(x, |\nabla u|)\,dx \leq 1,\end{aligned} \tag{5.22}$$

where the last inequality holds thanks to the assumption (5.20). The inequality (3.10) follows via (5.22), applied with u replaced with $\frac{u}{\|\nabla u\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)}}$. $\square$

Proof of Proposition 3.4. We may asssume, as in the proof of Theorem 3.1, that $\varphi = \overline{\varphi}$. Let A be the Young function defined as in (4.67). Consider trial functions u in the inequality (3.12) of the form

$$u(x) = \int\limits_{\omega_n|x|^n}^{\infty} g(s)s^{-\frac{1}{n'}}\,ds \qquad \text{for } x \in \mathbb{R}^n,$$

where $g : [0,\infty) \to [0,\infty)$ is a measurable bounded function, with bounded support contained in $[\omega_n, \infty)$. Thus u is weakly differentiable, and

$$|\nabla u(x)| = cg(\omega_n|x|^n) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

for some constant $c = c(n)$.

The inequalities (3.12) and (4.68) yield:

$$\Bigg\| \int\limits_{\omega_n|x|^n}^{\infty} g(s)s^{-\frac{1}{n'}}\,ds \Bigg\|_{Y(\mathbb{R}^n)} \leq c\|g(\omega_n|x|^n)\|_{L^A(\mathbb{R}^n)} = c\|g(s)\|_{L^A(\omega_n,\infty)} \tag{5.23}$$

for some constant c. Thanks to the property (2.10),

$$\Big\| \int_{\omega_n|x|^n}^{\infty} g(s)s^{-\frac{1}{n'}}\,ds \Big\|_{Y(\mathbb{R}^n)} \geq \Big\| \chi_{B(0,1)}(x) \int_{\omega_n|x|^n}^{\infty} g(s)s^{-\frac{1}{n'}}\,ds \Big\|_{Y(\mathbb{R}^n)} = \|\chi_{B(0,1)}\|_{Y(\mathbb{R}^n)} \int_{\omega_n}^{\infty} g(s)s^{-\frac{1}{n'}}\,ds. \tag{5.24}$$

Coupling (5.23) with (5.24) implies that

$$\sup_g \frac{\int_{\omega_n}^{\infty} g(s)s^{-\frac{1}{n'}}\,ds}{\|g(s)\|_{L^A(\omega_n,\infty)}} \leq \frac{c}{\|\chi_{B(0,1)}\|_{Y(\mathbb{R}^n)}}. \tag{5.25}$$

By the Fatou property of Luxemburg norms, replacing, for $t > 0$, any measurable function $g : [0,\infty) \to [0,\infty)$ with the function g_t defined as in (4.73), and passing to the limit as $t \to \infty$, tell us that the inequality (5.25) continues to hold for every $g \in L^A(\omega_n,\infty)$. The left-hand side of the inequality (5.25) agrees with the left-hand side of the inequality (4.84), with $\alpha = 1$. The inequality (3.6) hence follows, since, as shown in the proof of Theorem 4.1, the inequality (4.84) implies (4.2) for every $\alpha \in (0,n)$. □

We conclude with our Sobolev inequalities on domains. Apart from the use of the simplified function $\varphi_{n,\diamond}$ instead of φ_n, Theorem 3.5 is just a consequence of Theorem 3.1. The justification of the replacement of $\varphi_{n,\diamond}$ by φ_n is the same as at the beginning of the proof of Theorem 3.6 below.

Proof of Theorem 3.6. Owing to equation (2.37) of Proposition 2.8, we may assume that $\varphi = \widehat{\varphi}$, the function associated with φ as in the formula (2.36). By the same proposition, the function φ, which we are assuming to agree with $\widehat{\varphi}$, is a normalized generalized Young function satisfying the assumption (3.6). Hence, by Theorem 4.1 and Remark 4.2, there exists a positive constant $c = c(n,\alpha,\beta)$ such that

$$\int_{\{|I_1 f|>t\}} \varphi_{n,\diamond}(x,ct)\,dx \leq \int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx \quad \text{for } t \geq 0, \tag{5.26}$$

if $\|f\|_{L^{\varphi(\cdot)}(\mathbb{R}^n)} \leq 1$.

As a preliminary, observe that, owing to the property (2.12),

$$V^{1,\varphi(\cdot)}(\Omega) \hookrightarrow V^{1,1}(\Omega). \tag{5.27}$$

Moreover, since Ω is a John domain, one classically has that

$$\|u\|_{L^1(\Omega)} \leq c\big(\|u\|_{L^1(G)} + \|\nabla u\|_{L^1(\Omega)}\big) \tag{5.28}$$

for any open set G such that $\overline{G} \subset \Omega$ and some constant $c = c(\Omega, G)$. Hence,

$$V^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^1(\Omega), \tag{5.29}$$

and the mean value u_Ω is well defined for every $u \in V^{1,\varphi(\cdot)}(\Omega)$.

In order to prove the inequality (3.16), it suffices to show that there exist positive constants $c_1 = c_1(\beta, \Omega)$ and $c_2 = c_(\beta, \Omega)$, such that

$$\int\limits_\Omega \varphi_{n,\diamond}(x, c_1|u - u_\Omega|)\, dx \le c_2, \tag{5.30}$$

if

$$\|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \le \frac{1}{1 + 2\|1\|_{L^{\widetilde{\varphi}(\cdot)}(\Omega)}}. \tag{5.31}$$

Let $u_j : \Omega \to [0, \infty)$ be the function defined by

$$u_j = \max\{\min\{|u - u_\Omega| - 2^j, 2^j\}, 0\},$$

and let

$$\Omega_j = \{x \in \Omega : 2^j < |u(x) - u_\Omega| \le 2^{j+1}\}$$

for $j \in \mathbb{Z}$. We have that

$$\Omega_j \subset \{x \in \Omega : u_{j-1}(x) = 2^{j-1}\} \tag{5.32}$$

for $j \in \mathbb{Z}$. By [19, Lemma 8.2.1], there exist a ball $B \subset \Omega$ and a constant $\kappa = \kappa(n)$ such that

$$|u_j(x) - (u_j)_B| < \kappa I_1(|\nabla u_j|)(x) \quad \text{for a.e. } x \in \Omega, \tag{5.33}$$

for $j \in \mathbb{Z}$. Therefore,

$$\begin{aligned} u_j(x) &\le |u_j(x) - (u_j)_B| + (u_j)_B < \kappa I_1(|\nabla u_j|)(x) + \fint\limits_B |u - u_\Omega|\, dy \\ &\le \kappa I_1(|\nabla u_j|)(x) + c' \int\limits_\Omega |u - u_\Omega|\, dy \le \kappa I_1(|\nabla u_j|)(x) + c'' \int\limits_\Omega |\nabla u|\, dy \\ &\le \kappa I_1(|\nabla u_j|)(x) + 2c''\|1\|_{L^{\widetilde{\varphi}(\cdot)}(\Omega)} \|\nabla u\|_{L^{\varphi(\cdot)}(\Omega)} \le c'''\big(I_1(|\nabla u_j|)(x) + 1\big) \\ &\quad \text{for a.e. } x \in \Omega, \end{aligned} \tag{5.34}$$

for some constants c', c'', c''' depending on Ω, and for $j \in \mathbb{Z}$. Notice that here we made use of the Poincaré inequality in $W^{1,1}(\Omega)$.

Denote by j_0 the largest index j such that $2^{j-2} \le c'''$. Let c be the constant from the inequality (5.26), and set

$$\overline{c} = \frac{1}{c'''} \max\{1, c\}. \tag{5.35}$$

Owing to equation (5.32) and the inequality (5.26) applied with $f = |\nabla u|$, the following chain holds:

$$\begin{aligned}
&\int_\Omega \varphi_{n,\diamond}(x, \tfrac{\overline{c}}{8}|u - u_\Omega|)\,dx \qquad (5.36)\\
&= \sum_{j\in\mathbb{Z}} \int_{\Omega_j} \varphi_{n,\diamond}(x, \tfrac{\overline{c}}{8}|u - u_\Omega|)\,dx \le \sum_{j\in\mathbb{Z}} \int_{\Omega_j} \varphi_{n,\diamond}(x, \overline{c}2^{j-2})\,dx\\
&\le \sum_{j\in\mathbb{Z}} \int_{\{u_{j-1}\ge 2^{j-1}\}} \varphi_{n,\diamond}(x, \overline{c}2^{j-2})\,dx \le \sum_{j\in\mathbb{Z}} \int_{\{c'''(I_1(|\nabla u_{j-1}|)+1)\ge 2^{j-1}\}} \varphi_{n,\diamond}(x, \overline{c}2^{j-2})\,dx\\
&\le \sum_{j\in\mathbb{Z}} \int_{\{c'''I_1(|\nabla u_{j-1}|)\ge 2^{j-2}\}} \varphi_{n,\diamond}(x, \overline{c}2^{j-2})\,dx + \sum_{j\le j_0} \int_\Omega \varphi_{n,\diamond}(x, \overline{c}2^{j-2})\,dx\\
&\le \sum_{j\in\mathbb{Z}} \int_\Omega \varphi(x, |\nabla u_{j-1}|)\,dx + \sum_{j\le j_0} \int_\Omega \varphi_{n,\diamond}(x, \overline{c}2^{j-2})\,dx.
\end{aligned}$$

Observe that the choice (5.35) plays a role in the application of the inequality (5.26). Plainly,

$$\sum_{j\in\mathbb{Z}} \int_\Omega \varphi(x, |\nabla u_{j-1}|)\,dx = \int_\Omega \varphi(x, |\nabla u|)\,dx \le 1. \tag{5.37}$$

On the other hand, since $\varphi(x, \cdot)$ is a Young function such that $\varphi(x, 1) = 1$,

$$\varphi(x, t) \le t \quad \text{if } 0 \le t \le 1,$$

for $x \in \mathbb{R}^n$. Thus, $H_{n,\diamond}^{-1}(x, t) \le t^{n'} \le 1$ if $0 \le t \le 1$. As a consequence,

$$\varphi_{n,\diamond}(x, t) = \varphi(x, H_{n,\diamond}^{-1}(x, t)) \le \varphi(x, t^{n'}) \le t^{n'} \qquad \text{if } 0 \le t \le 1,$$

for $x \in \mathbb{R}^n$. Hence, owing to equation (5.35)

$$\sum_{c''' \ge 2^{j-2}} \int_\Omega \varphi_{n,\diamond}(x, \overline{c}2^{j-2})\,dx \le \overline{c}^{n'} |\Omega| \sum_{j\le j_0} 2^{n'(j-2)} < \infty. \tag{5.38}$$

Combining the inequalities (5.36) – (5.38) yields (5.30).

Owing to (5.27), the embedding

$$V^{1,\varphi(\cdot)}(\Omega) \hookrightarrow L^{\varphi_{n,\diamond}}(\Omega)$$

is a consequence of the inequality (3.16) and of the embedding (5.29). □

Proof of Theorem 3.7. It suffices to show that, if $\{u_k\}$ is a sequence in $V^{1,\varphi(\cdot)}(\Omega)$ such that $\|u_k\|_{V^{1,\varphi(\cdot)}(\Omega)} \le 1$, then it admits a Cauchy subsequence in $L^{\vartheta(\cdot)}(\Omega)$. Owing to the embeddings (5.27) and (5.29), the sequence $\{u_k\}$ is also bounded in $W^{1,1}(\Omega)$. Since the embedding $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, there exists a subsequence, still denoted by $\{u_k\}$ which converges in $L^1(\Omega)$ and a.e. in Ω to some function $u \in L^1(\Omega)$. We claim that $u_k \to u$ also in $L^{\vartheta(\cdot)}(\Omega)$. This claim will follow if we show that, for every $\gamma > 0$,

$$\limsup_{k\to\infty} \int_\Omega \vartheta\left(x, \frac{|u_k - u|}{\gamma}\right) dx \le 1. \tag{5.39}$$

By Theorem 3.6, there exists a constant K such that $\sup_k \|u_k\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)} \le K$. Since $u_k \to u$ a.e. in Ω, [19, Theorem 2.3.17] ensures that $\|u\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)} \le K$ as well. The assumption that ϑ grows essentially more slowly than $\varphi_{n,\diamond}$ near infinity entails that there exists $t_0 > 0$ such that

$$\vartheta\left(x, \frac{t}{\gamma}\right) \le \varphi_{n,\diamond}\left(x, \frac{t}{2K}\right) \qquad \text{for a.e. } x \in \Omega \text{ and for } t \ge t_0.$$

Hence,

$$\begin{aligned} \int_\Omega \vartheta\left(x, \frac{|u_k - u|}{\gamma}\right) dx &\le \int_{\{|u_k-u|\le t_0\}} \vartheta\left(x, \frac{|u_k - u|}{\gamma}\right) dx \\ &\quad + \int_{\{|u_k-u|> t_0\}} \vartheta\left(x, \frac{|u_k - u|}{\gamma}\right) dx \\ &\le \int_{\{|u_k-u|\le t_0\}} \vartheta\left(x, \frac{|u_k - u|}{\gamma}\right) dx \\ &\quad + \int_{\{|u_k-u|> t_0\}} \varphi_{n,\diamond}\left(x, \frac{|u_k - u|}{2K}\right) dx \end{aligned} \tag{5.40}$$

for $k \in \mathbb{N}$. Inasmuch $\|u_k - u\|_{L^{\varphi_{n,\diamond}(\cdot)}(\Omega)} \le 2K$, we have that

$$\int_{\{|u_k-u|> t_0\}} \varphi_{n,\diamond}\left(x, \frac{|u_k - u|}{2K}\right) dx \le 1 \tag{5.41}$$

for $k \in \mathbb{N}$. On the other hand, $|u_k - u| \to 0$ a.e. in Ω and

$$\vartheta\left(x, \frac{|u_k - u|}{\gamma}\right) \le \vartheta(x, t_0) \quad \text{for a.e. } x \in \Omega,$$

for $k \in \mathbb{N}$. Moreover, owing to the assumption (3.17), $\vartheta(\cdot, t_0) \in L^1(\Omega)$. Therefore, by the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{\{|u_k - u| \le t_0\}} \vartheta\left(x, \frac{|u_k - u|}{\gamma}\right) dx = 0. \tag{5.42}$$

Equation (5.39) follows from (5.40)–(5.42). □

Compliance with ethical standards

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The authors declare that they have no conflict of interest.

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