Existence of solutions for critical (p, q) -Laplacian equations in ℝ^N

EXISTENCE OF SOLUTIONS FOR CRITICAL (p, q)-LAPLACIAN EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this paper we are mainly interested in existence properties for a class of nonlinear PDEs driven by the (p, q)-Laplace operator where the reaction combines a power-type nonlinearity at critical level with a subcritical term. In addition nonnegative nontrivial weights and a positive parameter λ are included in the nonlinearity. An important role in the analysis developed is played by the two potentials. Precisely, under suitable conditions on the exponents of the nonlinearity, first a detailed proof of the tight convergence of a sequence of measures is given, then the existence of a nontrivial weak solution is obtained provided that the parameter λ is far from 0. Our proofs use concentration compactness principles by Lions and Mountain Pass Theorem by Ambrosetti and Rabinowitz.

1. Introduction

In this paper we consider the following critical nonlinear elliptic problem

$$(\mathcal{P}) \qquad -\Delta_p u - \Delta_q u = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N$$

where $\Delta_m u = \operatorname{div}(|Du|^{m-2}Du)$ is the m-Laplacian of u, the parameter λ is positive and the exponent k is p-superlinear and subcritical, namely

$$1 < q \le p < N, \quad p < k < p^* = \frac{Np}{N-p}. \tag{1}$$

Furthermore, we consider nonlinear weights such that

$$0 \le V \in L^r(\mathbb{R}^N), \qquad r = \frac{p^*}{p^*-k}, \tag{2}$$

and

$$0 \le K \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). \tag{3}$$

By using the Mountain Pass Lemma of Ambrosetti and Rabinowitz and concentration compactness principles of Lions, we obtain prove existence of nontrivial weak solutions of $(\mathcal{P})$ belonging to $X = D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$

Working on unbounded domains, the loss of compactness of the Sobolev embeddings shows up and this is underlined by the presence of the critical exponent in the nonlinearity. Thus, variational techniques are more delicate to apply.

The case $1 < q < k < p$ in problem $(\mathcal{P})$ has been treated in [4] where multiplicity results for solutions with negative energy are proved. Here, in dealing with the different range (1), new difficulties arise both in the proof of the tightness property and due also to the fact that the energy functional associated to the problem does not satisfy the $(\mathrm{PS})_c$ property.

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Indeed, the application of the second concentration compactness by Lions [27] in the entire $\mathbb{R}^N$ is rather complicated, since it requires the tight convergence of a certain sequence of measures. This latter property can be deduced almost easily if we consider sequences of measures in $D^{1,p}(\Omega)$, with Ω bounded, but when Ω is unbounded its proof requires a delicate analysis and tricky calculations. In addition, the presence of weights makes things quite delicate. For details in this direction we refer to Section 4 below.

Critical problems are widely studied in literature, starting with the pioneering paper by Brezis and Nirenberg [7] for the Laplacian. We refer for a detailed discussion on the topic to the monograph [32]. For Hardy type weights, in all of $\mathbb{R}^N$, we quote [44] in which non existence results, as well as existence results for radial solutions, are proved. In the recent paper [23], multiplicity and concentration results of nontrivial solutions are proved for a nonlinear magnetic Schrdinger equation with critical growth.

Later, the critical p-Laplacian case, that is the subcase $p = q$ in problem $(\mathcal{P})$ involving eventually different type of critical nonlinearities, was investigated by many authors, both in the entire $\mathbb{R}^N$ and in bounded domains. In the latter case, and without weights the picture is completely described for all parameters $\lambda > 0$ by Garcia Azorero and Peral in [19], where they obtain, among other results, two positive values λ_0, λ_1 such that existence of a nontrivial solution for the problem considered holds for $\lambda \geq \lambda_0 > 0$ if $1 < p < k < p^*$, while existence of infinitely many solution occurs if $1 < k < p$ for $\lambda \in (0, \lambda_1)$, see also [31] for concave-convex term depending on a parameter.

Among papers in all of $\mathbb{R}^N$, we mention [16] with double critical nonlinearities and no weights, while Swanson and Yu in [42] give a nice proof for the tight convergence in the presence of nonnegative weights, later extended in [4]. Drábek and Huang in [11] consider the case $k = p$ and weights which can change sign, then, Gonçalves and Alves in [20] show the existence of a weak nonnegative solution for $\lambda = 1$, $K \equiv 1$, $p \geq 2$ and for external domains we refer to [15], see also the references therein. For more general quasilinear elliptic problems where the superfluid film equation in plasma physics and fluid mechanics is investigated, we refer to [12] for $p = 2$, while to [45] for a perturbated version in the case $2 \leq p < N$.

The (p, q)-Laplacian operator, for instance appear in the study of stationary solutions of reaction diffusion systems, whose general form is

$$u_t = div[A(|Du|)Du] + c(x, u). \tag{4}$$

This equation has a wide range of applications in physical and related sciences, such as biophysics, plasma physics and chemical reaction design, where, in general, the function u describes a concentration, A denotes the diffusion coefficient $A(u)$ and $c(x, u)$ is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x, u)$ has a polynomial form with respect to the concentration u. In particular, in our case, we discuss stationary solutions of (4) with $A(|Du|) = |Du|^{p-2} + |Du|^{q-2}$ as diffusion coefficient and with $c(x, u) = \lambda V(x)|u|^{k-2}u + K(x)|u|^{p^*-2}u$ as reaction term.

Technical difficulties arise in applying usual elliptic methods for obtaining existence and regularity of weak solutions of problems involving this operator, due also to the fact it is not homogeneous.

For the case $q = 2$, that is $(p, 2)$-Laplacian, recently was studied by Papageorgiou et al. [34] and [35], where existence and multiplicity results are proved using a variational approach and Morse theory with $p > 2$ assuming that the reaction term satisfies particular conditions which imply the resonance at $\pm\infty$ and at $0^{\pm}$. For a general discussion, we refer to the recent book [36].

Problem $(\mathcal{P})$ is involved also in the study of solitary waves or solitons which are special solutions whose profile remains unchanged under the evolution in time, of the nonlinear Schrödinger equation see [10], [8] and [41] and the references therein, of the typical form

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + \Delta_q\psi - U(x)\psi + |\psi|^{k-1}\psi = 0, \quad 2 < k < 2^*, \tag{5}$$

where i is the imaginary unit and the function U is the potential. In particular, a function $\psi(x,t) = e^{-i\omega t}u(x)$ is a standing-wave solution of (5), where $\omega \in \mathbb{R}$ is the energy, if and only if the function u satisfies

$$-\Delta u - \Delta_q u + [U(x) - \omega]u = |u|^{k-1}u.$$

The subcase of (5) when $q = 2$ and a cubic nonlinearity, $k = 3$, is involved is called the Gross-Pitaevskii equation.

Critical Dirichlet problems for (p,q)-Laplacian on bounded domains under assumption (1) are treated in [22], in [47] (see also [14] for $p > q \geq 2$), while for singular nonlinearities we refer to [24].

Moving to the unbounded case, the situation is fairly delicate. In particular, problem $(\mathcal{P})$ is treated in [13] without weights and $p, q \geq 2$, while in [9], it is considered with $K \equiv 1$, where a detailed discussion on tight convergence seems to be missing. For multiplicity results of a class of superlinear (p,q) Laplacian type equations on $\mathbb{R}^N$ we refer to [5].

A classical approach to prove existence of solutions of $(\mathcal{P})$, is to construct them as critical points of the following functional

$$E_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{k}\int_{\mathbb{R}^N} V|u|^k dx - \frac{1}{p^*}\int_{\mathbb{R}^N} K|u|^{p^*} dx \tag{6}$$

via the Mountain Pass Lemma of Ambrosetti and Rabinowitz, see [2] and [39].

In our situation, we have to face with the loss of compactness, in the sense that $(PS)_c$ fails at certain levels and, hence, not all Palais Smale sequences contain some convergent subsequences, thus a careful analysis of the behavior of Palais Smale sequences is necessary to understand the consequences of spreading or concentration of mass. In particular, while in the case $1 < q < k < p$, treated in [4], one of the principal obstacles to this careful analysis can be found in proving tightness for $1 < k < p$, in this paper, in addition to the delicate proof of tightness, given in Lemma 7, a further difficulty relies on the fact that when $p < k < p^*$, it fails the validity of $(PS)_c$ condition for the functional E_λ defined in (6). This failure produces the strong L^p and L^q convergence of $(Du_n)_n$ only in a subset of $\mathbb{R}^N$, not in the entire $\mathbb{R}^N$. Thus, differently from [4] , only convergence a.e. of $(Du_n)_n$ in all of $\mathbb{R}^N$ occurs, nevertheless, this weaker condition guarantees the existence of a weak solution using Mountain Pass Theorem.

Furthermore, the presence of weights in problem $(\mathcal{P})$ produces another difficulty, indeed to prove that the solution obtained is non trivial, we cannot use results such as Lemma I.1 in [28], Proposition 2.5 in [13] and Lemma 2.8 in [17], which are classical tools in this framework when weights are not involved.

We mention here some papers in which the concentration compactness principle is combined with the Mountain Pass Lemma in order to obtain existence results. Precisely, [25] for the Laplacian, [19], [20], [42], [40] for the p Laplacian, [10] for the (p,q) Laplacian and [1] for more general operators.

We are now ready to state our main result, which completes, thanks to a detailed proof of the tightness property given in Lemma 7 in order to apply the second concentration compactness principle, and extends to the weighted case Theorem 1 in [9] .

Theorem 1. *Let $N \geq 3$. Assume* (1) *and that V satisfies* (2). *Furthermore, suppose $V > 0$ on some open subset $\Omega_V \subset \mathbb{R}^N$, with $|\Omega_V| > 0$. Let K verify* (3). *There exist $\lambda^* > 0$ such that, for any $\lambda > \lambda^*$, problem $(\mathcal{P})$ has at least one nontrivial nonnegative weak solution.*

As a standard procedure, the starting point for the proof of Theorem 1 is the boundedness of $(\mathrm{PS})_c$ sequences, $c \in \mathbb{R}$, for $E_\lambda(u)$, which holds for all k with $1 < k < p^*$ thanks to Lemma 3 in [4]. Next, we verify that the energy functional defined in (6) has a mountain pass geometry, bypassing the difficulty given by the non homogenuity of the (p, q)-Laplacian operator. Then, in Lemma 8, the lack of compactness is manifest. To solve this problem, as described before, in Lemma 7, we have to deal with tight convergence of $(|u_n|^{p^*})_n$ in order to apply the second concentration compactness principle, getting a strict upper bound for c_λ, the level of the Palais Smale sequence, valid for all λ large enough. Finally, using arguments derived from [9] and Lemma 7, we conclude that the nonnegative critical point for E_λ, obtained by the Mountain Pass Lemma, is not the trivial one.

Regularity results of solutions of problem $(\mathcal{P})$ are developed to [21], see [37] for the case of the p-Laplacian. Finally, for generalization to critical problems in the Heisenberg setting or to systems we refer to [17], [38], [33] and the references therein.

The paper is organized as follows. In Section 2, we recall some classical definitions, as well as some regularity results on $E_\lambda(u)$, some useful results, such as the Mountain Pass Lemma, while in Section 3 we prove some properties on Palais Smale sequences. In Section 4 we state the two concentration compactness principles due to Lions in [27] and [29], moreover the proof of Lemma 7, where tightness is proved, is performed. Section 5 is dedicated to preparatory lemmas to the the proof of Theorem 1 which is developed in Section 6.

2. Preliminaries

In this section we state some preliminary results, as well as some notations, useful in the proofs of the main theorem of the paper, given in the Sections 6.

In what follows, we denote with X the Banach space $D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, where $D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : Du \in L^p(\mathbb{R}^N)\}$, endowed with the norm

$$\|u\| := \|u\|_X = \|u\|_{D^{1,p}(\mathbb{R}^N)} + \|u\|_{D^{1,q}(\mathbb{R}^N)} = \|Du\|_p + \|Du\|_q$$

where $\|\cdot\|_p$ is the L^p norm in $\mathbb{R}^N$. Furthermore, we denote by S the Sobolev's constant, i.e

$$S = \inf\left\{\frac{\|Du\|_p^p}{\|u\|_{p^*}^p} : u \in D^{1,p}(\mathbb{R}^N), u \neq 0\right\}.$$

We recall that the value S is achieved in $D^{1,p}(\mathbb{R}^N)$, for details we refer to the celebrated papers [3] and [43] (cfr. also Appendix A in [16]).

Of course, the functional E_λ characterized in (6) is well defined in X, indeed if $u \in X$, by Hölder's inequality with the exponents $r = p^*/(p^* - k)$, $r' = p^*/k$, we have

$$E_\lambda(u) \leq \frac{1}{p}\|u\|^p + \frac{1}{q}\|u\|^q + \frac{\lambda}{k}\|V\|_r\|u\|_{p^*}^k + \frac{1}{p^*}\|K\|_\infty\|u\|_{p^*}^{p^*} < \infty,$$

thanks to (2) and (3).

The proof of the regularity of E_λ is almost standard, for instance we refer to [4]. In particular, $E_\lambda \in C^1(X)$, with $E'_\lambda : X \to X'$ and it results

$$
\begin{aligned}
E'_\lambda(u)\psi = & \int_{\mathbb{R}^N} |Du|^{p-2} Du D\psi dx + \int_{\mathbb{R}^N} |Du|^{q-2} Du D\psi dx \\
& - \lambda \int_{\mathbb{R}^N} V|u|^{k-2} u\psi dx - \int_{\mathbb{R}^N} K|u|^{p^*-2} u\psi dx.
\end{aligned} \tag{7}
$$

for all $\psi \in X$.

A *weak solution* of problem $(\mathcal{P})$ is a function $u \in X$ such that

$$E'_\lambda(u)\psi = 0 \quad \text{ for all } \psi \in X,$$

that is u is a critical point of the functional E_λ or equivalently, by (7), u satisfies the weak formulation of problem $(\mathcal{P})$, namely

$$\int_{\mathbb{R}^N} |Du|^{p-2} Du D\psi dx + \int_{\mathbb{R}^N} |Du|^{q-2} Du D\psi dx = \lambda \int_{\mathbb{R}^N} V|u|^{k-2} u\psi dx + \int_{\mathbb{R}^N} K|u|^{p^*-2} u\psi dx$$

for all $\psi \in X$.

Let us introduce the following version of the Mountain Pass Lemma (see [2]).

Theorem 2. *(Ambrosetti and Rabinowitz,* [2]*) Let $(V, \|.\|_V)$ be a Banach space and consider $F \in C^1(V)$. We assume that*

(i) $F(0) = 0$,

(ii) There exist $\alpha, R > 0$ such that $F(u) \geq \alpha$ for all $u \in V$, with $\|u\|_V = R$,

(iii) There exists $v_0 \in V$ such that $\limsup_{t\to\infty} F(tv_0) < 0$.

Let $t_0 > 0$ be such that $\|t_0 v_0\|_V > R$ and $F(t_0 v_0) < 0$ and let

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} F(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C^0([0,1], V) \,/\, \gamma(0) = 0 \text{ and } \gamma(1) = t_0 v_0\}.$$

Then, there exists a Palais Smale sequence at level c, that is a sequence $(u_n)_n \subset V$ such that

$$\lim_{n\to\infty} F(u_n) = c \quad \text{ and } \quad \lim_{n\to\infty} F'(u_n) = 0 \quad \text{strongly in } V'.$$

The next result follows from Brezis Lieb Lemma, that is Theorem 1 in [6], combined with the Banach Alaoglu's Theorem (see Remark (iii) of [6]).

Lemma 1. *Let $1 < p < \infty$ and let $(u_n)_n \subset L^p(\mathbb{R}^N)$ be a bounded sequence converging to $u \in L^p(\mathbb{R}^N)$ almost everywhere. Then, $u_n \rightharpoonup u$ (weakly) in $L^p(\mathbb{R}^N)$.*

Note that condition $u \in L^p(\mathbb{R}^N)$ follows by Fatou's Lemma.

Now we report a well-known property which will be useful in the following.

Remark 1. *For all $x, y \in \mathbb{R}^N$ with $|x| + |y| \neq 0$, there exists a constant $C(s) > 0$ such that*

$$< |x|^{s-2}x - |y|^{s-2}y, x - y > \geq C(s) \begin{cases} \frac{|x-y|^2}{(|x|+|y|)^{2-s}}, & \text{if } 1 \leq s < 2, \\ |x-y|^s, & \text{if } s \geq 2 \end{cases}$$

Finally, we present the following result about convergence, which is Lemma 2.7 in [26].

Lemma 2. *Let Ω be an open set in $\mathbb{R}^N$ and $m > 1$. Assume α, β positive numbers and $a(x,\xi)$ in $C(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$ such that*

(1) $\alpha|\xi|^m \leq a(x,\xi)\xi$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}^N$,
(2) $|a(x,\xi)| \leq \beta|\xi|^{m-1}$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}^N$,
(3) $(a(x,\xi) - a(x,\eta))(\xi,\eta) > 0$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}^N$ *with* $\xi \neq \eta$.

Consider $(u_n)_n, u \in W^{1,m}(\Omega)$*, then* $Du_n \to Du$ *in* $L^m(\Omega)$ *if and only if*

$$\lim_{n\to\infty} \int_\Omega < a(x, Du_n(x)) - a(x, Du(x)), Du_n(x) - Du(x) > dx = 0.$$

3. On Palais Smale sequences and Mountain Pass geometry

First, we briefly recall the basic definitions.

Definition 1. *Let* Y *be a Banach space and* $E : Y \to \mathbb{R}$ *be a differentiable functional. A sequence* $(u_n)_n \subset Y$ *is called a* $(PS)_c$ *sequence for the functional* E *if* $E(u_k) \to c$ *and* $E'(u_k) \to 0$ *as* $k \to \infty$*. Moreover, we say that* E *satisfies the* $(PS)_c$ *condition if every* $(PS)_c$ *sequence for* E *has a converging subsequence in* Y*.*

From now on, since the functional E_λ is even, without loss of generality, we may assume that every $(PS)_c$ sequence is nonnegative. We report the first main property for $(PS)_c$ sequences, proved in [4], for the functional $E_\lambda(u)$ defined in (6). We point out that here the value k does not satisfies (1), but simply $1 < k < p^*$.

Lemma 3. *(Lemma 4, [4]) Assume* $1 < k < p^*$*,* $1 < q \leq p$*. Let* (2) *and* (3) *be verified and let* $(u_n)_n \subset X$ *be a* $(PS)_c$ *sequence for* $E_\lambda(u)$ *defined in* (6) *for all* $c \in \mathbb{R}$*. Then* $(u_n)_n$ *is bounded in* X*.*

Next, we check that the functional E_λ satisfies a Mountain Pass geometry.

Lemma 4. *The functional* E_λ *verifies the hypotheses of the Mountain Pass Lemma.*

Proof. From the observations in Section 2, $E_\lambda \in C^1(X)$ and, clearly condition (i) of Theorem 2 is satisfied since $E_\lambda(0) = 0$. Using Hölder's inequality with exponents r and r', Sobolev's inequality, since $q < p$ and $K \in L^\infty(\mathbb{R}^N)$, we get

$$E_\lambda(u) \geq c_1\|u\|^p - \lambda c_2\|u\|^k - c_3\|u\|^{p^*} = \|u\|^p(c_1 - \lambda c_2\|u\|^{k-p} - c_3\|u\|^{p^*-p}),$$

where $c_1 = 1/p$, $c_2 = S^{-k/p}\|V\|_r/k$ and $c_3 = S^{-p^*/p}\|K\|_\infty/p^*$ are positive constants. Let us define $h(t) = t^p(c_1 - \lambda c_2 t^{k-p} - c_3 t^{p^*-p})$, for $t \geq 0$. It is easy to see that, since $p < k < p^*$, there exists $t_1 > 0$ such that $h(t) > 0$ for all $t \in (0, t_1]$. Therefore, there exist $\alpha, R > 0$ with R small enough, so that $E_\lambda(u) \geq \alpha > 0$ whenever $\|u\| = R$. Thus, condition (ii) of Theorem 2 is satisfied.

Now, let $u \in X \setminus \{0\}$ such that $u \geq 0$. Then, for any $t > 0$, we have

$$E_\lambda(tu) = \frac{t^p}{p}\|Du\|_p^p + \frac{t^q}{q}\|Du\|_q^q - \frac{t^k}{k}\int_{\mathbb{R}^N} V u^k dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} K u^{p^*} dx.$$

Since $p < k < p^*$, we get $E_\lambda(tu) \to -\infty$ as $t \to \infty$. Thus, let $t_u > 0$ be such that $E_\lambda(tu) < 0$ for all $t \geq t_u$ and $\|t_u u\| > R$. So the proof of (iii) is concluded.

Consider

$$\Gamma_u := \{\gamma \in C^0([0,1], X) \,/\, \gamma(0) = 0 \text{ and } \gamma(1) = t_u u\},$$

and

$$c_u := \inf_{\gamma \in \Gamma_u} \sup_{t \in [0,1]} E_\lambda(\gamma(t)). \tag{8}$$

Then the hypotheses of Theorem 2 are satisfied. This ends the proof. □

4. Concentration compactness

To establish arrangements under which the Palais Smale condition holds we need the concentration compactness lemmas due to Lions in [27] and [29].

Lemma 5. *(Lemma I.1, [29]) Let $1 \le p < N$ and $(u_n)_n$ be a bounded sequence in $D^{1,p}(\mathbb{R}^N)$ converging weakly to some u and such that $|Du_n|^p$ converges weakly to μ, while $|u_n|^{p^*}$ converges tightly to ν where μ, ν are bounded nonnegative measures on $\mathbb{R}^N$. Then*

(i) *There exist some at most countable set J and two families $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$ and $(\nu_j)_{j\in J}$ of a positive numbers such that*

$$\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}.$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$.

(ii) *In addition we have*

$$\mu \ge |Du|^p + \sum_{j\in J} \mu_j \delta_{x_j}$$

for some $\mu_j > 0$ satisfying $S\nu_j^{p/p^*} \le \mu_j$, for all j and with S the Sobolev's constant. Furthermore, it results $\sum_{j\in J} \nu_j^{p/p^*} < \infty$.

Usually, the tight convergence of measures, which is the weak star convergence respect to the space of real bounded functions $(C_b(Y))'$, is denoted with $\overset{*}{\rightharpoonup}$, and of course is stronger with respect to the weak convergence of measure, whose standard symbol is $\rightharpoonup$, which is the convergence respect to the space of all continuous functions that vanish at infinity $C_0(Y)$ or, equivalently, the completion of $C_c(Y)$ relative to the norm $\|\cdot\|_\infty$. Here, Y is any locally compact Hausdorff space.

On the other hand, if Ω is bounded, Lemma 5 holds without requiring the tight convergence but only the weak (or standard) convergence, since the two notions of convergence coincide.

In our context, since we work in the unbounded case, tightness is not trivial. From Proposition 1.202 in [18], any sequence of bounded measures admits a subsequence which converges in the weak (or standard) way. Thus, from Prokhorov Theorem (Theorem 1.208 in [18]), to obtain the tight convergence we need, in addition to the boundedness of the sequence, the compactness condition *(a)* of the following lemma.

Lemma 6. *(Lemma I.1, [27]) Let $(\rho_n)_n$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \ge 0 \text{ in } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N} \rho_n dx = \Lambda,$$

where $\Lambda > 0$ is fixed. Indeed, up to a subsequence, one of the following three situations hold:

(a) (Compactness) There exists a sequence $(y_n)_n$ in $\mathbb{R}^N$ such that $\rho_n(\cdot + y_n)$ is tight that is for any $\varepsilon > 0$ there exists $0 < R_\varepsilon < \infty$ for which

$$\int_{B_{R_\varepsilon}(y_n)} \rho_n(x)dx \ge \Lambda - \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ large.}$$

(b) (Vanishing) For all $R > 0$ there holds

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}\rho_n(x)dx\right)=0.$$

(c) (Dichotomy) There exists $\ell \in (0,\Lambda)$ such that for any $\varepsilon > 0$ there exists $k_0 \geq 1$ and $0 \leq \rho_n^1, \rho_n^2 \in L^1(\mathbb{R}^N)$ satisfying for $k \geq k_0$

$$\int_{\mathbb{R}^N}[\rho_n(x)-(\rho_n^1(x)+\rho_n^2(x))]dx \leq \varepsilon$$

$$\left|\int_{\mathbb{R}^N}\rho_n^1(x)dx-\ell\right| \leq \varepsilon$$

$$\left|\int_{\mathbb{R}^N}\rho_n^2(x)dx-(\Lambda-\ell)\right| \leq \varepsilon$$

$$dist\big(Supp(\rho_n^1), Supp(\rho_n^2)\big) \to \infty \quad as \quad n\to\infty.$$

The following lemma is crucial in the proof of Theorem 1, given in Section 6.

Lemma 7. *Assume* (1) *and let $\lambda \in (0,\infty)$. Suppose that V and K satisfy* (2) *and* (3) *respectively. If*

$$0 < c < \overline{c} := \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}}, \tag{9}$$

then every $(PS)_c$ sequence, $(u_n)_n \subset X$, for E_λ is such that, up to subsequences,

$$\nu_n = |u_n|^{p^*}dx \overset{*}{\rightharpoonup} \nu,$$

where ν is a bounded nonnegative measure.

Remark 2. *The case $c < 0$ of the above Lemma 7 is treated in Lemma 7 in* [4]*, whose proof is inspired on an argument based on Swanson and Yu* [42].

Proof of Lemma 7. We follow the proof of Lemma 7 in [4], but adapted to the new case. Let $(u_n)_n \subset X$ be a $(\mathrm{PS})_c$ sequence. Thus, as $n \to \infty$,

$$\frac{1}{p}\|Du_n\|_p^p+\frac{1}{q}\|Du_n\|_q^q-\frac{\lambda}{k}\int_{\mathbb{R}^N}V|u_n|^kdx-\frac{1}{p^*}\int_{\mathbb{R}^N}K|u_n|^{p^*}dx=c+o_n(1) \tag{10}$$

and

$$\|Du_n\|_p^p+\|Du_n\|_q^q-\lambda\int_{\mathbb{R}^N}V|u_n|^kdx-\int_{\mathbb{R}^N}K|u_n|^{p^*}dx=\|u_n\|o_n(1), \tag{11}$$

where $o_n(1) \to 0$ as $n \to \infty$. By Lemma 3, the sequence $(u_n)_n$ is bounded in X then, by Banach Alaoglu Theorem, since X is a reflexive space, there exists $u \in X$ such that, up to subsequence $u_n \rightharpoonup u$ in X, and the following hold

$$u_n \rightharpoonup u \text{ in } L^{p^*}(\mathbb{R}^N), \quad u_n \rightharpoonup u \text{ in } L^{q^*}(\mathbb{R}^N)$$
$$Du_n \rightharpoonup Du \text{ in } L^p(\mathbb{R}^N), \quad Du_n \rightharpoonup Du \text{ in } L^q(\mathbb{R}^N),$$
$$u_n \to u \text{ in } L^s(\omega), \quad \omega \Subset \mathbb{R}^N, \quad 1 \leq s < p^*.$$

Consequently, by using an increasing sequence of compact sets whose union is $\mathbb{R}^N$ and a diagonal argument, we also have

$$u_n(x) \to u(x) \text{ a. e. in } \mathbb{R}^N. \tag{12}$$

Consider the auxiliary sequence of functions $(z_n)_n$, $z_n(x) \geq 0$ in $\mathbb{R}^N$ for all $n \in \mathbb{N}$, given by

$$z_n(x) = |Du_n(x)|^p + |Du_n(x)|^q + |u_n(x)|^{p^*} + \lambda V(x)|u_n(x)|^k.$$

Define $\eta_n = z_n dx$ and we claim that η_n converges tightly to a bounded nonnegative measure η on $\mathbb{R}^N$, that is $z_n \overset{*}{\rightharpoonup} \eta$. First note that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} z_n dx = \Lambda > 0, \tag{13}$$

with Λ to be defined. Indeed, from the property of weak convergence, the sequences $(\|Du_n\|_p)_n$, $(\|u_n\|_{p^*})_n$, $(\|Du_n\|_q)_n$ are bounded so that, by Bolzano Weiestrass Theorem, up to subsequences, there exist L, M, $Q \geq 0$ such that

$$L = \lim_{n\to\infty} \|Du_n\|_p^p, \quad M = \lim_{n\to\infty} \|u_n\|_{p^*}^{p^*}, \quad Q = \lim_{n\to\infty} \|Du_n\|_q^q. \tag{14}$$

As a consequence of (10), being $c > 0$, necessarily $L + Q > 0$.

The continuity of the functional J in $L^{p^*}(\mathbb{R}^N)$, whose J is given by

$$J(u) = \int_{\mathbb{R}^N} V|u|^k dx,$$

implies the existence of the following limit

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^k dx =: H. \tag{15}$$

Clearly $H \geq 0$ by (2), we claim that $H > 0$. If $H = 0$, then (11) would imply,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K|u_n|^{p^*} dx = L + Q. \tag{16}$$

Consequently, from (10) being $q < p$, we get

$$c = \frac{L}{p} + \frac{Q}{q} - \frac{L+Q}{p^*} > \frac{L+Q}{N}, \tag{17}$$

equivalently, since c satisfies (9),

$$L + Q < cN < \frac{S^{N/p}}{\|K\|_\infty^{(N-p)/p}}. \tag{18}$$

Since K is non trivial and verifies (3), we have

$$\|u_n\|_{p^*}^{p^*} \geq \frac{1}{\|K\|_\infty} \int_{\mathbb{R}^N} K|u_n|^{p^*} dx,$$

so that

$$\|u_n\|_{p^*}^p \geq \|K\|_\infty^{-p/p^*} \left(\int_{\mathbb{R}^N} K|u_n|^{p^*} dx \right)^{p/p^*} \tag{19}$$

Furthermore, using Sobolev's inequality, (16) and (19), we gain

$$\begin{aligned} L + Q \geq L &\geq S \lim_{n\to\infty} \|u_n\|_{p^*}^p \\ &\geq S\|K\|_\infty^{-p/p^*} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} K|u_n|^{p^*} dx \right)^{p/p^*} = S\|K\|_\infty^{-p/p^*} (L+Q)^{p/p^*}, \end{aligned}$$

which is equivalent to $L + Q \geq S^{N/p}\|K\|_\infty^{-(N-p)/p}$ contradicting (18). Thus, $H > 0$.

Consequently, condition (13) holds with $\Lambda = L + Q + M + \lambda H > 0$, then we can apply Lemma 6 to the sequence $(\rho_n)_n = (z_n)_n$. Indeed, up to a subsequence, three situations can occur: *Compactness*, *Vanishing* or *Dichotomy*. In particular, *Compactness* is equivalent to tightness so that we have to exclude *Vanishing* and *Dichothomy* for the sequence $(z_n)_n$.

We immediately see that *Vanishing* cannot occur, indeed from (13), we can assume that there exists $R_1 \in (0, \infty)$ such that $\int_{B_{R_1}(0)} z_n dx \geq \Lambda/2 > 0$, in turn (b) in Lemma 6 fails.

To prove that even *Dichotomy* cannot hold, we argue by contradiction and we assume that there exists $\ell \in (0, \Lambda)$ such that for all $\varepsilon > 0$, there exist $R > 0$, $(R_n)_n$, with $R_n \to \infty$ (so that $R < R_n$ for n large) and $(y_n)_n \in \mathbb{R}^N$ such that

$$\begin{aligned} &\left| \int_{B_R(y_n)} z_n dx - \ell \right| < \varepsilon, \\ &\left| \int_{\mathbb{R}^N \setminus B_{R_n}(y_n)} z_n dx + \ell - \Lambda \right| < \varepsilon, \\ &\left| \int_{D_n} z_n dx \right| < \varepsilon, \quad D_n = B_{R_n}(y_n) \setminus B_R(y_n) \end{aligned} \tag{20}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi_{|B_1(0)} \equiv 1$ and $\varphi_{|B_2(0)^c} \equiv 0$. We define $u_n^1 = \varphi_n^1 u_n$ and $u_n^2 = (1 - \varphi_n^2) u_n$ where

$$\varphi_n^1(x) := \varphi\left(\frac{x - y_n}{R}\right), \qquad \varphi_n^2(x) := \varphi\left(\frac{x - y_n}{R_n}\right)$$

for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$. Then, $\mathrm{Supp}(u_n^1)$ and $\mathrm{Supp}(u_n^2)$ are disjoint sets for every $n \in \mathbb{N}$, in addition, $\mathrm{dist}\big(\mathrm{Supp}(u_n^1), \mathrm{Supp}(u_n^2)\big) \to \infty$ as $n \to \infty$. In particular, following [4], from (10), (11) and (20), we get

$$\sum_{i=1}^{2} \left(\frac{1}{p} \|Du_n^i\|_p^p + \frac{1}{q} \|Du_n^i\|_q^q - \frac{\lambda}{k} \int_{\mathbb{R}^N} V |u_n^i|^k dx - \frac{1}{p^*} \int_{\mathbb{R}^N} K |u_n^i|^{p^*} dx \right) = c + o_n(1) + o_\varepsilon(1),$$

and

$$\sum_{i=1}^{2} \left(\|Du_n^i\|_p^p + \|Du_n^i\|_q^q - \lambda \int_{\mathbb{R}^N} V |u_n^i|^k dx - \int_{\mathbb{R}^N} K |u_n^i|^{p^*} dx \right) = \sum_{i=1}^{2} \|u_n^i\| o_n(1) + o_\varepsilon(1) \tag{21}$$

where we first let $n \to \infty$ and then $\varepsilon \to 0$. Now we define nonnegative limits α_i, β_i, $i = 1, 2$, as follows

$$\alpha_i = \lim_{n \to \infty} \int_{\mathbb{R}^N} V |u_n^i|^k dx \quad \text{and} \quad \beta_i = \lim_{n \to \infty} \int_{\mathbb{R}^N} K |u_n^i|^{p^*} dx.$$

so we obtain the validity of both following conditions

$$c \leq \left(\frac{1}{q} - \frac{1}{p^*} \right) (\beta_1 + \beta_2) - \lambda \left(\frac{1}{k} - \frac{1}{q} \right) (\alpha_1 + \alpha_2),$$

and

$$c \geq \frac{\beta_1 + \beta_2}{N} - \lambda \left(\frac{1}{k} - \frac{1}{p} \right) (\alpha_1 + \alpha_2). \tag{22}$$

We claim that inequality (22) cannot occur. It results from (20) that either $\alpha_1 = 0$ or $\alpha_2 = 0$ depending whether$(y_n)_n$ is unbounded or not. Indeed, if $(y_n)_n$ is unbounded then

$\mathrm{Supp}(u_n^1)$ reduces to the empty set when $n \to \infty$, consequently, from

$$\int_{B_R(y_n)} V|u_n|^k dx \le \|V\|_{L^r(B_R(y_n))}\|u_n\|_{p^*}^k \le C\|V\|_{L^r(B_R(y_n))},$$

where C is the constant obtained from the boundedness of the $(\mathrm{PS})_c$ sequence and thanks to the continuity of the embedding of $D^{1,p}(\mathbb{R}^N)$ in $L^{p^*}(\mathbb{R}^N)$, then, thanks to (2) we can apply Lebesgue dominated convergence Theorem to the function $\chi_{B_R(y_n)}V^r$, obtaining

$$\begin{aligned}\alpha_1 &= \lim_{n\to\infty}\int_{\mathbb{R}^N} V|u_n^1|^k dx = \lim_{n\to\infty}\int_{B_R(y_n)} V|u_n|^k dx\\ &\le C\lim_{n\to\infty}\|V\|_{L^r(B_R(y_n))} = 0\end{aligned}$$

since $y_n \to \infty$ as $n \to \infty$ and $V \in L^r(\mathbb{R}^N)$.

On the other hand, if $(y_n)_n$ is bounded, then arguing as above and noting that in this case $\mathrm{Supp}(u_n^2)$ becomes the empty set for $n \to \infty$, we get $\alpha_2 = 0$. First, we consider the case $\alpha_2 = 0$, of course $\alpha_1 > 0$ since

$$\alpha_1 + \alpha_2 + o(1) = \lim_{n\to\infty}\int_{\mathbb{R}^N} V|u_n|^k dx = H > 0.$$

From (21) with $i = 2$, by the definition of β_2 and Sobolev's inequality we get, as in [42],

$$\begin{aligned}\beta_2 + o_\varepsilon(1) &= \lim_{n\to\infty}\left\{\|Du_n^2\|_p^p + \|Du_n^2\|_q^q\right\} \ge \lim_{n\to\infty}\|Du_n^2\|_p^p\\ &\ge S\lim_{n\to\infty}\|u_n^2\|_{p^*}^p \ge S\beta_2^{(N-p)/N}\|K\|_\infty^{-(N-p)/N},\end{aligned}$$

thus

$$\beta_2 \ge S^{N/p}\|K\|_\infty^{-(N-p)/p}. \tag{23}$$

By (22) and (23), we have

$$c \ge \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}} - \lambda\alpha_1\left(\frac{1}{k} - \frac{1}{p}\right) = \overline{c} - \lambda\alpha_1\left(\frac{1}{k} - \frac{1}{p}\right) > \overline{c}$$

which is a contradiction since $p < k < p^*$ and c satisfies (9). In the case $\alpha_1 = 0$ we can repeat exactly the argument above and analogously we reach an absurd.

This contradiction proves that there exists a sequence $(y_n)_n$ in $\mathbb{R}^N$ such that $z_n(\cdot + y_n)$ is tight, that is for arbitrary $\varepsilon > 0$ there exists $R = R(\varepsilon) \in (0,\infty)$ with

$$\int_{\mathbb{R}^N\setminus B_R(y_n)} z_n dx < \varepsilon, \tag{24}$$

so that

$$\int_{\mathbb{R}^N\setminus B_R(y_n)} V|u_n|^k dx < \varepsilon \tag{25}$$

from the definition of $(z_n)_n$. It must be that $(y_n)_n$ is a bounded sequence otherwise if $y_n \to \infty$, then

$$\lim_{n\to\infty}\int_{|x-y_n|<R} V|u_n|^k dx = 0,$$

thus, combining the above limit with (25), we arrive to

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V|u_n|^k dx = 0,$$

contradicting $H > 0$ in (15).

Hence, we can replace y_n by 0 in (24) to obtain the tightness of z_n. Moreover, since

$$\int_{\mathbb{R}^N\setminus B_R} |u_n|^{p^*} dx \le \int_{\mathbb{R}^N\setminus B_R} z_n dx < \varepsilon$$

we obtain the tightness of $|u_n|^{p^*}$. Finally, we define for all $n \in \mathbb{N}$ the measure $\nu_n = |u_n|^{p^*} dx$ on $\mathbb{R}^N$ which is nonnegative, bounded since $M \ge 0$. Thus, it admits a subsequence which converges tightly to ν, a bounded non negative measure on $\mathbb{R}^N$, that is $\nu_n \overset{*}{\rightharpoonup} \nu$ as claimed. The proof is complete.

□

5. Preparatory Lemmas

Before proving the main theorem let us give some preliminary results. The argument used in the following lemma appears in the proof of Lemma 8 in [9], see also Lemma 7 in [20].

Lemma 8. *Assume* (1). *Let* $(u_n)_n \subset X$ *be a* $(PS)_c$ *sequence with* c *satisfying* (9). *Then, there exists a nonnegative function* $u \in X$ *such that, up to subsequence,*

$$Du_n(x) \to Du(x) \quad a.e. \ in \ \mathbb{R}^N. \tag{26}$$

Proof. Let $(u_n)_n$ be a $(\text{PS})_c$ sequence with c satisfying (9), clearly $(u_n)_n$ is bounded in X by Lemma 3. Consequently, using also Lemma 7 there exists $u \in X$ such that, up to subsequences, we have

(I) $u_n \rightharpoonup u$ in X,

(II) Since $Du_n \rightharpoonup Du$ in $L^p(\mathbb{R}^N)$ and $Du_n \rightharpoonup Du$ in $L^q(\mathbb{R}^N)$, then the sequence of measures $(|Du_n|^p dx + |Du_n|^q dx)_n$ is bounded, thus we have

$$|Du_n|^p dx + |Du_n|^q dx \rightharpoonup \mu,$$

(III) $(|u_n|^{p^*} dx)_n$ converges tightly to ν,

where μ, ν are bounded nonnegative measures on $\mathbb{R}^N$. Applying (i) and (ii) of Lemma 5, there exist at most countable set J, a family $(x_j)_{j\in J}$ of distinct points in $\mathbb{R}^N$ and two families $(\nu_j)_{j\in J}$, $(\mu_j)_{j\in J} \in]0, \infty[$ such that

$$\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j} \tag{27}$$

$$\mu \ge |Du|^p + |Du|^q + \sum_{j\in J} \mu_j \delta_{x_j}$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$, with ν_j and μ_j satisfying

$$S\nu_j^{p/p^*} \le \mu_j, \qquad \sum_{j\in J} \nu_j^{p/p^*} < \infty. \tag{28}$$

Take a standard cut-off function $\psi \in C_c^\infty(\mathbb{R}^N)$, such that $0 \le \psi \le 1$ in $\mathbb{R}^N$, $\psi = 0$ for $|x| > 1$, $\psi = 1$ for $|x| \le 1/2$. For each index $j \in J$ and each $0 < \varepsilon < 1$, define

$$\psi_\varepsilon(x) := \psi\left(\frac{x - x_j}{\varepsilon}\right).$$

Since $E'_\lambda(u_n)\psi \to 0$ being $(u_n)_n$ a $(PS)_c$ sequence, choosing $\psi = \psi_\varepsilon u_n$ in (7) we have, as $n \to \infty$

$$\begin{aligned}\int_{\mathbb{R}^N} (|Du_n|^{p-2} + |Du_n|^{q-2}) Du_n D\psi_\varepsilon u_n dx &= \lambda \int_{\mathbb{R}^N} V|u_n|^k \psi_\varepsilon dx \\ &+ \int_{\mathbb{R}^N} K|u_n|^{p^*}\psi_\varepsilon dx - \int_{\mathbb{R}^N} (|Du_n|^p + |Du_n|^q)\,\psi_\varepsilon dx + o(1).\end{aligned} \tag{29}$$

Now, being $(u_n)_n$ bounded in $L^{p^*}(\mathbb{R}^N)$ then also $(|u_n|^k)_n$ is bounded in $L^{p^*/k}(\mathbb{R}^N)$. Furthermore, since $D^{1,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{p^*/k}(\omega)$ for $\omega \Subset \mathbb{R}^N$, being $p^*/k < p^*$, taking for instance $\omega = \overline{B}_\varepsilon(x_j)$ and using (12), we have, up to subsequences, $|u_n|^k \to |u|^k$ in $L^{p^*/k}(\omega)$, $|u_n(x)|^k \to |u(x)|^k$ a.e. in ω and there exists $w_1 \in L^{p^*/k}(\omega)$ such that $|u_n(x)|^k \le w_1(x)$ a.e. in ω. Thus, $V|u_n|^k\psi_\varepsilon \le Vw_1 \in L^1(\omega)$ a.e. in ω and, in turn by the Lebesgue dominated convergence Theorem, we get

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} V|u_n|^k\psi_\varepsilon dx\right) = \int_{\mathbb{R}^N} V|u|^k\psi_\varepsilon dx. \tag{30}$$

Consequently, using (II), (30) and (III) recalling the definition of tight convergence with $K \in L^\infty(\mathbb{R}^N)$, in (29), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\left(\int_{\mathbb{R}^N} (|Du_n|^{p-2} + |Du_n|^{q-2}) Du_n D\psi_\varepsilon u_n dx\right) \\ &\qquad = \lambda\int_{\mathbb{R}^N} V|u|^k\psi_\varepsilon dx + \int_{\mathbb{R}^N} K\psi_\varepsilon d\nu - \int_{\mathbb{R}^N} \psi_\varepsilon d\mu.\end{aligned} \tag{31}$$

From Hölder's inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} |Du_n|^{p-2} Du_n D\psi_\varepsilon u_n dx\right| \le \|Du_n\|_p^{p-1}\left(\int_{\mathbb{R}^N} |D\psi_\varepsilon|^p|u_n|^p dx\right)^{1/p} \\ &= \|Du_n\|_p^{p-1}\left(\int_{B_\varepsilon(x_j)\setminus B_{\varepsilon/2}(x_j)} |D\psi_\varepsilon|^p|u_n|^p dx\right)^{1/p} \\ &\le \|u_n\|^{p-1}\left(\int_{B_\varepsilon(x_j)} |D\psi_\varepsilon|^p|u_n|^p dx\right)^{1/p}.\end{aligned} \tag{32}$$

Furthermore, arguing as above and since $D^{1,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^p(\omega)$ for $\omega \Subset \mathbb{R}^N$, being $p < p^*$, then taking for instance $\omega = \overline{B}_\varepsilon(x_j)$, we have $u_n \to u$ in $L^p(\omega)$, $u_n(x) \to u(x)$ a.e. in ω, up to subsequences, and there exists $w_2 \in L^p(\omega)$ such that $|u_n(x)| \le w_2(x)$ a.e. in ω. Thus, $|u_n(x)D\psi_\varepsilon(x)| \le Cw_2(x)$ a.e. in ω, as well as in $\mathbb{R}^N$, and in turn, Lebesgue dominated convergence Theorem gives

$$|D\psi_\varepsilon u_n| \to |D\psi_\varepsilon u| \text{ in } L^p(\mathbb{R}^N). \tag{33}$$

Consequently, passing to the limit for $n \to \infty$ in (32), using the boundedness of $(u_n)_n$, Hölder's inequality with exponents $N/(N-p)$ and N/p, we obtain, thanks to (33),

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}|Du_n|^{p-2}Du_nD\psi_\varepsilon u_n dx\right| \le C\left(\int_{B_\varepsilon(x_j)}|D\psi_\varepsilon|^p|u|^p dx\right)^{1/p}$$
$$\le C\left(\int_{B_\varepsilon(x_j)}|D\psi_\varepsilon|^N dx\right)^{1/N}\left(\int_{B_\varepsilon(x_j)}|u|^{p^*}dx\right)^{1/p^*} \le C\left(\int_{B_\varepsilon(x_j)}|u|^{p^*}dx\right)^{1/p^*},$$

where in the last inequality we have used the properties of ψ_ε. Similarly, by replacing p with q, we gain

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}|Du_n|^{q-2}Du_nD\psi_\varepsilon u_n dx\right| \le C\left(\int_{B_\varepsilon(x_j)}|u|^{q^*}dx\right)^{1/q^*}.$$

In turn, by letting $\varepsilon \to 0$, being $u \in L^{p^*}(\mathbb{R}^N)\cap L^{q^*}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N}|Du_n|^{p-2}Du_nD\psi_\varepsilon u_n dx \to 0, \qquad \int_{\mathbb{R}^N}|Du_n|^{q-2}Du_nD\psi_\varepsilon u_n dx \to 0, \tag{34}$$

and, since $V|u|^k \in L^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}V|u_n|^k\psi_\varepsilon dx \le \int_{B_\varepsilon(x_j)}V|u_n|^k dx \to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Hence, from (31) and since $K \in C(\mathbb{R}^N)$, if $\varepsilon \to 0$ we deduce

$$K(x_j)\nu_j = \mu_j. \tag{35}$$

This equality establishes that the concentration of the measure μ cannot occur at points where $K(x_j)=0$, moreover, combining (35) and $(28)_1$, then $S\nu_j^{p/p^*-1} \le K(x_j)$, in particular, $\nu_j = 0$ if $K(x_j)=0$, so that the measure ν cannot concentrate in those points. Hence the set $X_J := \{x_j : j \in J\}$ does not contain the points x_j which are zeros for K.

Let $J_1 := \{j \in J : K(x_j) > 0\}$. Thanks to $(28)_1$ and (35), it follows,

$$\nu_j \ge \left(\frac{S}{K(x_j)}\right)^{N/p} \ge \left(\frac{S}{\|K\|_\infty}\right)^{N/p}, \qquad j \in J_1. \tag{36}$$

We first note that condition (36) forces that $|J_1| < \infty$ being ν a bounded measure, indeed, integrating (27), we get, thanks to (36),

$$\infty > \int_{\mathbb{R}^N}d\nu = \|u_n\|_{p^*}^{p^*} + \int_{\{x_j\}}\sum_{j\in J_2}\nu_j\delta_{x_j}dx \ge \|u_n\|_{p^*}^{p^*} + \left(\frac{S}{\|K\|_\infty}\right)^{N/p}|J_1|.$$

Thus, here exist a finite natural number $s = |J_1|$ of indices $i \in J_1$ such that (36) holds for all x_i with $i = 1,\dots,s$.

First, we consider the case in which $s > 0$. Take $\varepsilon_0 > 0$ such that

$$\{x_1,\dots,x_s\} \subset B_{1/2\varepsilon_0}, \quad B_{\varepsilon_0}(x_i)\cap B_{\varepsilon_0}(x_j) \quad \text{for} \quad i \ne j.$$

Define

$$\Psi_\varepsilon(x) := \psi(\varepsilon x) - \sum_{i=1}^s \psi\left(\frac{x-x_i}{\varepsilon}\right)$$

for all $0 < \varepsilon < \varepsilon_0$. Thus,

$$\Psi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in \bigcup_{i=1}^s B_{\varepsilon/2}(x_i), \\ 1, & \text{if } x \in A_\varepsilon, \end{cases}$$

where $A_\varepsilon := B_{1/2\varepsilon} \setminus \bigcup_{i=1}^s B_\varepsilon(x_i)$. Now, let

$$P_n := < |Du_n|^{p-2} Du_n - |Du|^{p-2} Du + |Du_n|^{q-2} Du_n - |Du|^{q-2} Du, Du_n - Du > .$$

Using Remark 1, we immediately get $P_n \geq 0$. Fix $\delta, \varepsilon > 0$ with $0 < \varepsilon < \delta < \varepsilon_0$. Then, since $A_\delta \subset A_\varepsilon$, we get $\Psi_\varepsilon \equiv 1$ in A_δ by the definition of Ψ_ε, so that

$$\int_{A_\delta} P_n dx = \int_{A_\delta} P_n \Psi_\varepsilon dx \leq \int_{A_\varepsilon} P_n \Psi_\varepsilon dx \leq \int_{\mathbb{R}^N} P_n \Psi_\varepsilon dx.$$

Thus, by (7), we gain

$$\begin{aligned}
\int_{A_\delta} P_n dx \leq& \int_{\mathbb{R}^N} (|Du_n|^p \Psi_\varepsilon - |Du_n|^{p-2} \Psi_\varepsilon Du_n Du - |Du|^{p-2} \Psi_\varepsilon Du_n Du + |Du|^p \Psi_\varepsilon) dx \\
&+ \int_{\mathbb{R}^N} (|Du_n|^q \Psi_\varepsilon - |Du_n|^{q-2} \Psi_\varepsilon Du_n Du - |Du|^{q-2} \Psi_\varepsilon Du_n Du + |Du|^q \Psi_\varepsilon) dx \\
&= E'_\lambda(u_n)(u_n \Psi_\varepsilon) - \int_{\mathbb{R}^N} |Du_n|^{p-2} Du_n D\Psi_\varepsilon u_n dx - \int_{\mathbb{R}^N} |Du_n|^{q-2} Du_n D\Psi_\varepsilon u_n dx \\
&- E'_\lambda(u_n)(u \Psi_\varepsilon) + \int_{\mathbb{R}^N} |Du_n|^{p-2} Du_n D\Psi_\varepsilon u \, dx + \int_{\mathbb{R}^N} |Du_n|^{q-2} Du_n D\Psi_\varepsilon u \, dx \\
&- \lambda \int_{\mathbb{R}^N} V |u_n|^{k-2} u_n u \Psi_\varepsilon dx - \int_{\mathbb{R}^N} K |u_n|^{p^*-2} u_n u \Psi_\varepsilon dx + \int_{\mathbb{R}^N} |Du|^p \Psi_\varepsilon dx \\
&+ \int_{\mathbb{R}^N} |Du|^q \Psi_\varepsilon dx - \int_{\mathbb{R}^N} |Du|^{p-2} \Psi_\varepsilon Du_n Du \, dx - \int_{\mathbb{R}^N} |Du|^{q-2} \Psi_\varepsilon Du_n Du \, dx \\
&+ \lambda \int_{\mathbb{R}^N} V |u_n|^k \Psi_\varepsilon dx + \int_{\mathbb{R}^N} K |u_n|^{p^*} \Psi_\varepsilon dx.
\end{aligned} \tag{37}$$

Being $(u_n)_n \subset X$ a $(PS)_c$ sequence, from Lemma 3 $(u_n)_n$ is bounded, thus we obtain

$$E'_\lambda(u_n)(u_n \Psi_\varepsilon),\ E'_\lambda(u_n)(u \Psi_\varepsilon) \to 0, \tag{38}$$

as $n \to \infty$. Then, similarly to the proof of (34), using the properties of Ψ_ε and the boundedness of $(u_n)_n$, we have

$$\int_{\mathbb{R}^N} |Du_n|^{p-2} Du_n D\Psi_\varepsilon u_n dx \to 0, \qquad \int_{\mathbb{R}^N} |Du_n|^{q-2} Du_n D\Psi_\varepsilon u_n dx \to 0, \tag{39}$$

and

$$\int_{\mathbb{R}^N} |Du_n|^{p-2} Du_n D\Psi_\varepsilon u \, dx \to 0, \qquad \int_{\mathbb{R}^N} |Du_n|^{q-2} Du_n D\Psi_\varepsilon u \, dx \to 0, \tag{40}$$

as $n \to \infty$ and $\varepsilon \to 0$. Thus, using (38)-(40), (37) becomes

$$\begin{aligned}\int_{A_\delta} P_n dx \le \lambda\bigg[&\int_{\mathbb{R}^N} V|u_n|^k \Psi_\varepsilon dx - \int_{\mathbb{R}^N} V|u_n|^{k-2}u_n u \Psi_\varepsilon dx\bigg] \\ &+ \int_{\mathbb{R}^N} K|u_n|^{p^*}\Psi_\varepsilon dx - \int_{\mathbb{R}^N} K|u_n|^{p^*-2}u_n u\Psi_\varepsilon dx \\ &+ \int_{\mathbb{R}^N} |Du|^p \Psi_\varepsilon dx - \int_{\mathbb{R}^N} |Du|^{p-2}\Psi_\varepsilon Du_n Du\, dx \\ &+ \int_{\mathbb{R}^N} |Du|^q \Psi_\varepsilon dx - \int_{\mathbb{R}^N} |Du|^{q-2}\Psi_\varepsilon Du_n Du\, dx + o_n(1),\end{aligned} \tag{41}$$

as $n \to \infty$. We proceed by defining the following functional

$$f(v) := \int_{\mathbb{R}^N} (|Du|^{p-2} + |Du|^{q-2}) Du Dv \Psi_\varepsilon dx$$

for every $v \in X$. Since f is clearly bounded in X, recalling that $u \in X$, we have

$$\int_{\mathbb{R}^N} (|Du|^{p-2} + |Du|^{q-2}) Du Du_n \Psi_\varepsilon dx \to \int_{\mathbb{R}^N} (|Du|^{p-2} + |Du|^{q-2})|Du|^2 \Psi_\varepsilon dx, \tag{42}$$

as $n \to \infty$. Hence, from (42), (41) turns into

$$\begin{aligned}\int_{A_\delta} P_n dx \le \lambda\bigg[&\int_{\mathbb{R}^N} V|u_n|^k \Psi_\varepsilon dx - \int_{\mathbb{R}^N} V|u_n|^{k-2}u_n u \Psi_\varepsilon dx\bigg] \\ &+ \int_{\mathbb{R}^N} K|u_n|^{p^*}\Psi_\varepsilon dx - \int_{\mathbb{R}^N} K|u_n|^{p^*-2}u_n u\Psi_\varepsilon dx + o_n(1).\end{aligned} \tag{43}$$

Again by the boundedness of $(u_n)_n$ in $L^{p^*}(\mathbb{R}^N)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$ by (12), thus, being

$$\||u_n|^{p^*-2}u_n\|_{(p^*)'} = \|u_n\|_{p^*}^{p^*-1}, \quad \||u_n|^k\|_{p^*/k} = \|u_n\|_{p^*}^k, \quad \||u_n|^{k-2}u_n\|_{p^*/(k-1)} = \|u_n\|_{p^*}^{k-1},$$

applying Lemma 1, we get

$$|u_n|^{p^*-2}u_n \rightharpoonup |u|^{p^*-2}u \text{ in } L^{(p^*)'}(\mathbb{R}^N), \qquad |u_n|^k \rightharpoonup |u|^k \text{ in } L^{p^*/k}(\mathbb{R}^N),$$
$$|u_n|^{k-2}u_n \rightharpoonup |u|^{k-2}u \text{ in } L^{p^*/(k-1)}(\mathbb{R}^N).$$

Therefore, we gain

$$\int_{\mathbb{R}^N} V|u_n|^k \Psi_\varepsilon dx, \int_{\mathbb{R}^N} V|u_n|^{k-2}u_n u \Psi_\varepsilon dx \to \int_{\mathbb{R}^N} V|u|^k \Psi_\varepsilon dx \tag{44}$$

as $n \to \infty$, where in the second integral we have used that $V \in L^r(\mathbb{R}^N)$, $|u_n|^{k-2}u_n \in L^{p^*/(k-1)}(\mathbb{R}^N)$ and $u \in L^{p^*}(\mathbb{R}^N)$. Moreover, we have

$$\int_{\mathbb{R}^N} K|u_n|^{p^*-2}u_n u \Psi_\varepsilon dx \to \int_{\mathbb{R}^N} K|u|^{p^*}\Psi_\varepsilon dx, \tag{45}$$

as $n \to \infty$. Moreover, from (I), it follows that

$$\int_{\mathbb{R}^N} K|u_n|^{p^*}\Psi_\varepsilon dx \to \int_{\mathbb{R}^N} K|u|^{p^*}\Psi_\varepsilon dx, \tag{46}$$

as $n \to \infty$. Now, replacing (44)-(46) in (43), we arrive to

$$\lim_{n\to\infty} \int_{A_\delta} P_n \le 0,$$

thus, since $P_n \geq 0$, we can see

$$\lim_{n\to\infty} \int_{A_\delta} P_n = 0,$$

for every $\delta > 0$ such that $\varepsilon < \delta < \varepsilon_0$. From Remark 1 applied first with p and then with q, we have that P_n can be divided in two nonnegative terms, thus, we get

$$\lim_{n\to\infty} \int_{A_\delta} < |Du_n|^{p-2} Du_n - |Du|^{p-2} Du, Du_n - Du > dx = 0, \tag{47}$$

and

$$\lim_{n\to\infty} \int_{A_\delta} < |Du_n|^{q-2} Du_n - |Du|^{q-2} Du, Du_n - Du > dx = 0. \tag{48}$$

Now we consider (47), since the same conclusions follow for (48) when p is replaced with q. By virtue of Lemma 2 applied with $a(x,\xi) = |\xi|^{m-2}\xi$ first with $m = p$ then with $m = q$, we obtain

$$Du_n \to Du \quad \text{in } L^p(A_\delta) \cap L^q(A_\delta).$$

Thus, up to subsequences, we get

$$Du_n(x) \to Du(x) \quad \text{a.e. in } A_\delta.$$

Taking $0 < \varepsilon_m < \delta_m < \varepsilon_0$ and letting $\varepsilon_m, \delta_m \to 0$ as $m \to \infty$, it holds $A_{\delta_m} \to \mathbb{R}^N$ as $m \to \infty$. Then (26) is satisfied in the case in which $s > 0$.

As regards the case where $s = 0$, that is $\nu_j = 0$ for all $j \in J$, it is enough to take $\Psi_\varepsilon(x) := \psi(\varepsilon x)$ and $A_\delta := B_{1/2\delta}$ and repeat the argument above. So the proof is complete. $\square$

Remark 3. *Differently from* [4], *condition* (1) *does not allow to prove that the functional* E_λ *satisfies the* $(PS)_c$ *condition. In particular, the main problem lies in the proof that the atomic part of the measure* ν *in* (27) *is zero, as it happens if* $1 < k < p$ *with* $c < 0$*. For this reason, in Lemma 8 we have to face both the case when the atomic part of* ν *is zero and when it is not zero. This latter case produces the strong* L^p *and* L^q *convergence of* $(Du_n)_n$ *only in a subset of* $\mathbb{R}^N$*, not in the entire* $\mathbb{R}^N$ *as it occurs in* [4]. *Consequently, only convergence a.e. of* $(Du_n)_n$ *occurs instead of strong convergence in* L^p *and* L^q*. Thus, we have solely to deal with* (26) *to prove the existence of a solution of* $(\mathcal{P})$.

From now on we denote, for each $\lambda > 0$,

$$c_\lambda := \inf_{u \in X\setminus\{0\}} \max_{t\geq 0} E_\lambda(tu). \tag{49}$$

Remark 4. *Obviously,* $c_\lambda \geq c_u$*, where* c_u *is defined in* (8)*, since* $E_\lambda(tu) < 0$ *for* $u \in X\setminus\{0\}$ *and* t *large by the structure of* E_λ*. Actually,* $c_u = c_\lambda$ *(see also Theorem 4.2 in* [46]*).*

Lemma 9. *Let* $\bar{c}$ *and* c_λ *defined in* (9) *and* (49)*, respectively. Then, there exists* $\lambda^* > 0$ *such that*

$$0 < c_\lambda < \bar{c}$$

for all $\lambda > \lambda^*$.

Proof. Clearly, from the definition of c_u given in (8), then $c_u \geq \alpha$ and, by Remark 4, we have $c_\lambda = c_u > 0$ for all $u \in X \setminus \{0\}$. We emphasize that α might depend on λ, but it is always positive. We take the open set Ω_V where V is positive. Let $u_0 \in X \setminus \{0\}$ with support in Ω_V such that $u_0 \geq 0$. Since

$$\begin{aligned}E_\lambda(tu_0) &= \frac{t^p}{p}\|Du_0\|_p^p + \frac{t^q}{q}\|Du_0\|_q^q - \lambda\frac{t^k}{k}\int_{\mathbb{R}^N} V u_0^k\,dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} K u_0^{p^*}\,dx \\ &\leq \frac{t^p}{p}\|Du_0\|_p^p + \frac{t^q}{q}\|Du_0\|_q^q - \lambda\frac{t^k}{k}\int_{\mathbb{R}^N} V u_0^k\,dx,\end{aligned}$$

for all $t \geq 0$, we have $E_\lambda(tu_0) \to -\infty$ as $t \to \infty$ and $E_\lambda(tu_0) \to 0^+$ as $t \to 0^+$. Thus, there exists $t_\lambda > 0$ such that

$$\max_{t\geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0).$$

In particular, we get

$$0 = \frac{d}{dt}\Big[E_\lambda(tu_0)\Big]_{t=t_\lambda} = t_\lambda^{p-1}\|Du_0\|_p^p + t_\lambda^{q-1}\|Du_0\|_q^q - \lambda t_\lambda^{k-1}\int_{\mathbb{R}^N} V u_0^k\,dx - t_\lambda^{p^*-1}\int_{\mathbb{R}^N} K u_0^{p^*}\,dx,$$

or, equivalently,

$$\lambda\int_{\mathbb{R}^N} V u_0^k\,dx = \frac{\|Du_0\|_p^p}{t_\lambda^{k-p}} + \frac{\|Du_0\|_q^q}{t_\lambda^{k-q}} - t_\lambda^{p^*-k}\int_{\mathbb{R}^N} K u_0^{p^*}dx, \tag{50}$$

for every $\lambda > 0$. Since the support of u_0 is contained in Ω_V, the left hand side of (50) is positive and it goes to ∞ if $\lambda \to \infty$. Thus, also the right hand side of (50) must go to ∞ if $\lambda \to \infty$. Hence, being $q < p < k < p^*$, necessarily $t_\lambda \to 0^+$ as $\lambda \to \infty$. From $E_\lambda(t_\lambda u_0) \to 0^+$ as $t_\lambda \to 0^+$ or equivalently when $\lambda \to \infty$, we can conclude that there exists $\lambda^* > 0$ such that

$$\max_{t\geq 0} E_\lambda(tu_0) = E_\lambda(t_\lambda u_0) < \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}} = \bar{c}.$$

By the definition of $\bar{c}$, we get $c_\lambda < \bar{c}$ for all $\lambda > \lambda^*$. Thus the proof is concluded. □

6. Proof of Theorem 1

Now, we are ready to prove our main result, that is the existence Theorem 1, whose statement is given in the Introduction.

Proof of Theorem 1. From Lemma 4 the energy functional E_λ satisfies the assumptions of Theorem 2, thus there exists a Palais Smale sequence $(u_n)_n \subset X$ of E_λ at level $c_u = c_\lambda$, as pointed out in Remark 4, that is

$$E_\lambda(u_n) \to c_\lambda, \qquad E'_\lambda(u_n)\varphi \to 0 \quad \text{as} \quad n \to \infty \tag{51}$$

for all $\varphi \in X$. By Lemma 9, there exists $\lambda^* > 0$ such that $0 < c_\lambda < \bar{c}$ for every $\lambda > \lambda^*$. Furthermore, according to Lemmas 3 and 8, there exists a nonnegative function $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{in } X, \qquad u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \qquad Du_n \to Du \quad \text{a.e. in } \mathbb{R}^N.$$

Using Lemma 1, we obtain the following convergences

$$\begin{aligned}&|Du_n|^{p-2}Du_n \rightharpoonup |Du|^{p-2}Du \quad \text{in } [L^{p'}(\mathbb{R}^N)]^N, \\ &|Du_n|^{q-2}Du_n \rightharpoonup |Du|^{q-2}Du \quad \text{in } [L^{q'}(\mathbb{R}^N)]^N, \\ &u_n^{p^*-1} \rightharpoonup u^{p^*-1} \quad \text{in } L^{(p^*)'}(\mathbb{R}^N), \qquad u_n^{k-1} \rightharpoonup u^{k-1} \quad \text{in } L^{p^*/(k-1)}(\mathbb{R}^N).\end{aligned} \tag{52}$$

Then, from (52), for every $\varphi \in X$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |Du_n|^{p-2} Du_n D\varphi \, dx \to \int_{\mathbb{R}^N} |Du|^{p-2} Du D\varphi \, dx \\
&\int_{\mathbb{R}^N} |Du_n|^{q-2} Du_n D\varphi \, dx \to \int_{\mathbb{R}^N} |Du|^{q-2} Du D\varphi \, dx \\
&\int_{\mathbb{R}^N} K u_n^{p^*-1} \varphi \, dx \to \int_{\mathbb{R}^N} K u^{p^*-1} \varphi \, dx, \ \int_{\mathbb{R}^N} V u_n^{k-1} \varphi \, dx \to \int_{\mathbb{R}^N} V u^{k-1} \varphi \, dx.
\end{aligned} \tag{53}$$

Thus, letting $n \to \infty$ in $(51)_2$ and by (53), we have $E'_\lambda(u)\varphi = 0$ for all $\varphi \in X$, that is u is a solution of $(\mathcal{P})$. We know that $u \geq 0$ by (12) and $u_n \geq 0$. We claim that $u \not\equiv 0$. Recalling the definition of L and Q in (14), as in (17), we get

$$c_\lambda = \frac{L}{p} + \frac{Q}{q} - \frac{L+Q}{p^*} = \frac{L}{N} + Q\left(\frac{1}{q} - \frac{1}{p^*}\right) \geq \frac{L+Q}{N}$$

that is

$$c_\lambda N \geq L + Q. \tag{54}$$

As in Lemma 7, being $(u_n)_n$ a $(PS)_{c_\lambda}$ sequence, necessarily $L + Q > 0$ and we obtain

$$S(L+Q)^{p/p^*} \leq \|K\|_\infty^{p/p^*}(L+Q),$$

that is

$$L + Q \geq \frac{S^{N/p}}{\|K\|_\infty^{(N-p)/p}}. \tag{55}$$

Thus, from (55) in (54), we gain

$$c_\lambda \geq \frac{S^{N/p}}{N\|K\|_\infty^{(N-p)/p}} = \overline{c}$$

which is a contradiction, since $c_\lambda < \overline{c}$. The proof is so concluded. □

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