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Convergence results in Orlicz spaces for sequences of max-product sampling Kantorovich operators

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ABSTRACT

In this paper, we provide a unifying theory concerning the convergence properties of the so-called max-product sampling Kantorovich operators based upon generalized kernels in the setting of Orlicz spaces. The approximation of functions defined on both bounded intervals and on the whole real axis has been considered. Here, under suitable assumptions on the kernels, considered in order to define the operators, we are able to establish a modular convergence theorem for these sampling-type operators. As a direct consequence of the main theorem of this paper, we obtain that the involved operators can be successfully used for approximation processes in a wide variety of functional spaces, including the well-known interpolation and exponential spaces. This makes the Kantorovich variant of max-product sampling operators suitable for reconstructing not necessarily continuous functions (signals) belonging to a wide range of functional spaces. Finally, several examples of Orlicz spaces and of kernels for which the above theory can be applied, are presented.

1. Introduction

In view of their many applications in signal and image processing, sampling-type operators represent one of the most challenging topics in Approximation Theory. Beginning in the first half of the 1900s with the classical sampling theorem (see, e.g., [1,2]), with the major contributions of Whittaker, Kotelnikov and Shannon, their possible use for signal reconstruction or approximation have attracted strong interest in the scientific community. Since the required assumptions on the signal to be approximated limit its applicability to solving real-world problems, possible approximate versions of the above theorem have been studied since the 1960s. A first rigorous theory in this direction has been developed by the German mathematician P. L. Butzer and his school in Aachen with the introduction of the so-called generalized sampling operators (series) which are characterized by good approximating properties (see, e.g., [3–8]). In the following years, many authors have been concerned with the study of alternative versions of the above operators acting in a wide variety of functional spaces. Recently, in many papers (see, e.g., [9–12]), the max-product approach applied to several approximation operators has been studied by Coroianu and Gal. More in detail, by replacing in the definition of the discrete approximation operators the series (or sum in the case of finite terms) by the symbol $\bigvee$, denoting the supremum (or maximum, in the finite case) of a given set of real numbers, it is possible to obtain the nonlinear (more precisely, sub-additive) version of any family of linear operators. In addition to preserving the approximation capability of the corresponding linear counterparts, max-product operators often return better orders of approximation (for more details, see, e.g., [13–15] or Remark 5.1 of Section 5). For this reason the max-product method is widely used for applications to various theoretical fields of Approximation Theory,

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such as function approximation (see, e.g., [16–20]), theory of neural networks (see, e.g., [21,22]) and also fuzzy sets theory (see, e.g., [23,24]). Within the framework of sampling theory, in [25], the authors introduced the max-product version of sampling Kantorovich operators based on generalized kernels for the approximation of univariate signals defined on both bounded intervals and on the whole real axis. We recall here that the linear version of the sampling Kantorovich operators (series) have been introduced and studied by Bardaro et al. [26] in univariate form, where approximation results in Orlicz spaces have been established. In the case of a uniform sampling-scheme, these series are defined (in the univariate setting) as follows:

$$S_w^\chi(f)(x) := \sum_{k\in\mathbb{Z}} \chi(wx-k)\left[w\int_{k/w}^{(k+1)/w} f(t)\,dt\right], \quad x\in\mathbb{R}, \quad w>0,$$

where $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function (signal) such that the above series is convergent for every $x\in\mathbb{R}$. Here, $\chi:\mathbb{R}\to\mathbb{R}$ is a kernel function satisfying the usual conditions of the discrete approximate identities (see, e.g., [27,28]). It is well known that, in view of their averaged nature, sampling operators of Kantorovich-type represent a useful tool to obtain an approximate version of the celebrated Whittaker–Kotelnikov–Shannon sampling theorem, proving particularly suitable for the reconstruction of signals that are not necessarily continuous (see, e.g., [26,29–32]). In general, Kantorovich-variants of sampling operators have been extensively studied in the last fifteen years from both a theoretical and applied point of view, returning a wide variety of applications to image processing in several fields, such as biomedical and engineering (see, e.g., [33–37]).

As mentioned above, sampling Kantorovich operators of max-product type have been introduced by Coroianu et al. [25] and they are formally expressed by:

$$K_n^\chi(f)(x) := \frac{\displaystyle\bigvee_{k\in\mathbb{Z}} \chi(nx-k)\left[n\int_{k/n}^{(k+1)/n} f(t)\,dt\right]}{\displaystyle\bigvee_{k\in\mathbb{Z}} \chi(nx-k)}, \quad x\in\mathbb{R}, \quad \text{(I)}$$

where $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function and $\chi:\mathbb{R}\to\mathbb{R}$ is a generalized kernel satisfying suitable assumptions. We will also consider in the paper the above operators K_n^χ in the case of $f:[a,b]\to\mathbb{R}$. By adopting the so-called moment-type approach in its max-product variant, obtained by replacing the series (or the sum in the case of bounded intervals) in the usual definition of the discrete absolute moments of a given kernel by the sup (or the max) (see, e.g., [3]), it has been possible to investigate the approximation properties of max-product sampling Kantorovich operators (I) in several functional spaces, including the L^p-spaces, $1\le p<+\infty$, and spaces of continuous functions. In both cases, qualitative and quantitative convergence properties have been obtained. In order to prove these results, unlike the corresponding linear counterpart, it is not necessary to assume that χ is an approximate identity. Indeed, the theory developed in [25] still holds if we consider measurable and bounded kernels with in addition a finiteness assumption on the so-called generalized absolute moment for a given order $\beta>0$.

The goal of this paper is to show that, keeping these assumptions on χ, we can extend the results proved in [25,38,39] in a twofold sense, on the one hand considering kernels not necessarily having compact support or generated by sigmoidal functions (as, e.g., in [9]) and on the other hand approximating functions which belong to the Orlicz space L^φ generated by a convex φ-function. As it is well-known from the literature (see, e.g., [40,41]), these general spaces include the usual L^p-spaces and several other well-known functional spaces, such as the interpolation and the exponential spaces. In particular, we will prove a converge result for sequences of max-product sampling Kantorovich operators (I) with respect to the most natural notion of convergence which can be defined in L^φ, i.e., the so-called modular convergence. To reach this, after recalling in Section 2 the definition of the involved operators and all the auxiliary results used in this paper, we will establish a modular inequality for the same family of non-linear operators. Next, as often happens in the case of approximation processes in Orlicz spaces (whose theory is briefly recalled in Section 3), we will need to investigate the modular convergence for continuous functions on compact intervals (or continuous functions with compact support if f is defined on $\mathbb{R}$) so that a density approach can be applied, exploiting the closure of $C_c^+(\mathbb{R})$ on $L_+^\varphi(\mathbb{R})$ with respect to the modular topology. In this way, the main theorem of the present paper given in Section 4 follows. In Section 5, we will consider some particular cases of Orlicz spaces, obtaining the convergence results with respect to the L^p-norm, already proved in [25]. Then, we will provide several examples of kernels for which the above theory can be applied, together with the corresponding graphical representations.

2. Preliminary results and notations

From now on, in the whole paper, by the notation $f:\Omega\to\mathbb{R}$ we will denote real-valued functions defined on $\Omega\subseteq\mathbb{R}$. If Ω is a compact interval $[a,b]$, then $C(\Omega)$ denotes the set of all continuous functions defined on Ω, while in the case of $\Omega=\mathbb{R}$, $C(\Omega)$ denotes the set of all uniformly continuous and bounded functions defined on Ω. In both cases, $C(\Omega)$ is a normed linear space with the usual sup-norm, that we will denote by $\|\cdot\|_\infty$. Moreover, let us denote by $C^+(\Omega)$ the subspace of $C(\Omega)$ of the non-negative functions and by $C_c^+(\Omega)\subset C^+(\Omega)$ the subspace whose elements have compact support.

Now, we recall that the symbol $\bigvee$, in the literature (see, e.g., [10,11,14,18,19,42,43]), denotes the supremum of any set of real numbers $\{A_k : k\in\mathcal{J}\}$ with respect to a given $\mathcal{J}\subseteq\mathbb{Z}$ set of indexes, i.e.,

$$\bigvee_{k\in\mathcal{J}} A_k := \sup\{A_k\in\mathbb{R}, k\in\mathcal{J}\}.$$

Obviously, if $\mathcal{J}$ has finite cardinality, the supremum reduces to a maximum. As we will see later, the above definition is crucial in order to define the type of sampling operators that we will study in this paper in the general context of Orlicz spaces. Such operators have been firstly introduced and studied in [25], where the approximation of (non-negative) continuous and L^p-integrable functions has been considered.

In order to recall the definition of the above-mentioned operators, we firstly give the notion of generalized kernels, i.e., any bounded and measurable function $\chi : \mathbb{R} \to \mathbb{R}$ such that the following properties are satisfied:

(χ_1) the generalized absolute moment of order β of χ is finite for some $\beta \geq 0$, i.e.,

$$m_\beta(\chi) := \sup_{x\in\mathbb{R}} \bigvee_{k\in\mathbb{Z}} |\chi(x-k)| \cdot |x-k|^\beta \;=\; \sup_{y\in\mathbb{R}} |\chi(y)|\, |y|^\beta \;<\; +\infty;$$

(χ_2) For a suitable positive constant $a_\chi > 0$, we have:

$$\inf_{x\in[-2,2]} \chi(x) =: a_\chi > 0, \text{ if } \Omega = [a,b],$$

or (χ_2')

$$\inf_{x\in[-1/2,1/2]} \chi(x) =: a_\chi > 0, \text{ if } \Omega = \mathbb{R}.$$

Conditions (χ_1) and (χ_2') have been firstly introduced in [44] in order to establish a pointwise and uniform convergence result and quantitative estimates for generalized sampling operators of max-product type. Similarly to above, as it was shown in [25], keeping the same assumptions on χ, generates similar convergence properties for the max-product sampling operators of Kantorovich-type, with respect to both the norm $\|\cdot\|_\infty$ and the usual L^p-norm $\|\cdot\|_p$, $1 \leq p < +\infty$, except for the bounded case $\Omega = [a,b]$, where it is necessary to replace condition (χ_2') with the slightly stronger assumption (χ_2).

In Section 4, we will prove that, considering kernels of the same type, it is possible to provide an extension of the results obtained in [25] in the general framework of Orlicz spaces generated by convex φ-functions, which include the L^p-spaces as some special instances.

We recall here some useful results concerning the kernel χ in the max-product setting.

Lemma 2.1. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be a bounded function such that $\chi(x) = \mathcal{O}(|x|^{-\alpha})$, as $|x| \to +\infty$, for some $\alpha > 0$. Then it turns out that:*

$$m_\beta(\chi) < +\infty,$$

for every $0 \leq \beta \leq \alpha$.

Lemma 2.2. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be a bounded function which satisfies (χ_1) for some $\beta > 0$. Then there holds:*

$$m_\upsilon(\chi) < +\infty,$$

for every $0 \leq \upsilon \leq \beta$. In particular, we have $m_0(\chi) \leq \|\chi\|_\infty$.

Lemma 2.3. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be a given function which satisfies condition (χ_2'). Then the following inequality:*

$$\bigvee_{k\in\mathbb{Z}} \chi(nx-k) \geq a_\chi > 0, \tag{1}$$

holds for all $x \in \mathbb{R}$ and for every $n \in \mathbb{N}$.

In addition, let $[a,b]$ be a compact interval. If χ satisfies assumption (χ_2), then for all $x \in [a,b]$ there holds:

$$\bigvee_{k\in\mathcal{J}_n} \chi(nx-k) \geq a_\chi > 0, \tag{2}$$

for every $n \in \mathbb{N}$ sufficiently large, where $\mathcal{J}_n := \{k \in \mathbb{Z} : \lceil na \rceil \leq k \leq \lfloor nb \rfloor - 1\}$. Here, $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote respectively, the "ceiling" and the "integer part" of a given number. Finally, a_χ is the constant from condition (χ_2') and (χ_2), respectively.

Proof. We prove only the case $\Omega = [a,b]$ since the proof of (1) is similar. Let $x \in [a,b]$ and $n \in \mathbb{N}$ be fixed. Then there exists at least a $\tilde{k} \in \mathcal{J}_n$ such that $|nx - \tilde{k}| \leq 2$, so by using (χ_2) we immediately get:

$$\bigvee_{k\in\mathcal{J}_n} \chi(nx-k) \geq \chi(nx-\tilde{k}) \geq a_\chi. \quad \square$$

Remark 2.1. As noted in [25] too, it is possible to weaken the assumptions on the kernel χ by directly assuming that both the inequalities (1) and (2) hold, together with condition (χ_1), instead of conditions (χ_2) and (χ_2'). Indeed, in this setting, the approximation results which we will prove in this paper still hold. It follows that we can also consider kernels which are not necessarily bounded from below by a strictly positive constant on $[-1/2, 1/2]$ or $[-2, 2]$.

We are now able to recall the definition of the operators considered in this paper. In the following definition $\mathcal{J}_n = \mathbb{Z}$, if $\Omega = \mathbb{R}$ and $\mathcal{J}_n = \{k \in \mathbb{Z} : \lceil na \rceil \leq k \leq \lfloor nb \rfloor - 1\}$, if $\Omega = [a,b]$.

Definition 2.1. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a fixed generalized kernel. We define the family of max-product sampling Kantorovich operators $(K_n^{\chi})_{n\in\mathbb{N}}$ based upon χ as follows:

$$K_n^{\chi}(f)(x) := \frac{\displaystyle\bigvee_{k\in\mathcal{J}_n} \chi(nx-k)\left[n\int_{k/n}^{(k+1)/n} f(t)\,dt\right]}{\displaystyle\bigvee_{k\in\mathcal{J}_n} \chi(nx-k)}, \quad x \in \Omega,$$

where $f : \Omega \to \mathbb{R}$ is a locally integrable function.

From now on, in the whole paper, we will always assume that the kernel χ satisfies (χ_1) and (χ_2) or (χ_2') for $\Omega = [a,b]$ and $\Omega = \mathbb{R}$, respectively. Therefore, since in this setting Lemma 2.3 holds, we always have $\bigvee_{k\in\mathcal{J}_n} \chi(nx-k) > 0$, for all $x \in \Omega$. Moreover, it is easy to see that $K_n^{\chi}(f)$ is well-defined, for example, if f is bounded. Indeed, in this case, by Lemma 2.2, we have:

$$\left|K_n^{\chi}(f)(x)\right| \le \frac{\|f\|_{\infty}}{a_{\chi}} m_0(\chi) < +\infty, \tag{3}$$

for every $x \in \Omega$.

We summarize some important proprieties for K_n^{χ} in the following lemma.

Lemma 2.4 ([25], Lemma 2.6)**.** *Let χ be a fixed generalized kernel. Let $f, g : \Omega \to [0,+\infty)$ be non-negative and bounded functions on Ω. The following properties hold for all $n \in \mathbb{N}$:*
(i) if $f \le g$, then $K_n^{\chi}(f) \le K_n^{\chi}(g)$;
(ii) $K_n^{\chi}(f+g) \le K_n^{\chi}(f) + K_n^{\chi}(g)$, i.e., K_n^{χ} is sub-additive;
(iii) $|K_n^{\chi}(f) - K_n^{\chi}(g)| \le K_n^{\chi}(|f-g|)$;
(iv) $K_n^{\chi}(\lambda f) = \lambda K_n^{\chi}(f)$ for each $\lambda \ge 0$, i.e., K_n^{χ} is positive homogeneous.

Let us now recall the following convergence theorem for the operators $(K_n^{\chi})_{n\in\mathbb{N}}$ in the classical setting, i.e., pointwise and uniform convergence in spaces of continuous functions.

Theorem 2.1 (*[25]*, Theorem 3.1)**.** *Let χ be a given generalized kernel. Let $f : \Omega \to [0,+\infty)$ be a non-negative and bounded function, continuous at $x \in \Omega$. Then*

$$\lim_{n\to+\infty} K_n^{\chi}(f)(x) = f(x).$$

In addition, if $f \in C^{+}(\Omega)$, then

$$\lim_{n\to+\infty} \|K_n^{\chi}(f) - f\|_{\infty} = 0.$$

Now, at the end of this section, we recall the statement of the well-known Vitali convergence theorem, that will be very useful later.

Theorem 2.2 (*Vitali Convergence Theorem*)**.** *Let $(f_w)_{w>0}$ be a net of functions in $L^p(\Omega,\Sigma,\mu)$, with $1 \le p < +\infty$, and where Ω is a measure space, Σ is a σ-algebra, and μ is a non-negative complete measure.*

The net $(f_w)_{w>0}$ converges to f in $L^p(\Omega)$ if and only if the following properties are satisfied:

(i) *$(f_w)_{w>0}$ converges in measure to f;*

(ii) *for every $\varepsilon > 0$, there exists a set $E_{\varepsilon} \in \Sigma$ with $\mu(E_{\varepsilon}) < +\infty$ for which, for every $F \in \Sigma$ with $F \cap E_{\varepsilon} = \emptyset$, we have:*

$$\int_F |f_w|^p \, d\mu < \varepsilon^p, \; for\ every\ w > 0;$$

(iii) *$(f_w)_{w>0}$ have equi-absolutely continuous integrals, i.e., for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every $E \in \Sigma$ with $\mu(E) < \delta(\varepsilon)$, we have:*

$$\int_E |f_w|^p \, d\mu < \varepsilon^p, \; for\ every\ w > 0.$$

3. The Orlicz spaces

In this section, we recall some basic notions concerning Orlicz spaces, which have been introduced by the Polish mathematician W. Orlicz as a natural extension of Lebesgue spaces (see, e.g., [41,45,46]). Let us denote by $M(\Omega)$ the set of all (Lebesgue-)measurable real functions defined on Ω, where Ω can be (again) a bounded interval or the whole real axis.

In what follows, we consider a function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which satisfies the following assumptions:

(φ1) $\varphi(0) = 0$, $\varphi(u) > 0$ for every $u > 0$;

(φ2) φ is continuous and non-decreasing on $\mathbb{R}_0^+$;

(φ3) $\lim_{u\to+\infty} \varphi(u) = +\infty$.

One can call this function as φ-function. For a fixed φ-function φ, it is possible to define the functional $I^\varphi : M(\Omega) \to [0, +\infty]$ as follows:

$$I^\varphi[f] := \int_\Omega \varphi\big(|f(x)|\big)\, dx, \quad f \in M(\Omega).$$

As it is well known, I^φ is a modular functional on $M(\Omega)$. Now, we can introduce the Orlicz space generated by φ as the set:

$$L^\varphi(\Omega) = \{f \in M(\Omega) : \exists\ \lambda > 0 \text{ such that } I^\varphi[\lambda f] < +\infty\}.$$

From the literature, if φ is convex, it turns out that $L^\varphi(\Omega)$ is a normed linear space with norm $\|\cdot\|_\varphi$, that is known as Luxemburg norm, and it is defined by:

$$\|f\|_\varphi := \inf\{\lambda > 0 : I^\varphi[f/\lambda] \le 1\}, \quad f \in L^\varphi(\Omega).$$

An important and useful subspace of $L^\varphi(\Omega)$ is

$$E^\varphi(\Omega) := \{f \in M(\Omega) : I^\varphi[\lambda f] < +\infty, \text{ for every } \lambda > 0\},$$

i.e., the so-called space of all finite elements of $L^\varphi(\Omega)$. In general, $E^\varphi(\Omega)$ is a proper subspace of $L^\varphi(\Omega)$ but these two spaces coincide if and only if the inequality:

$$\varphi(2u) \le M\varphi(u), \quad u \in \mathbb{R}_0^+,$$

holds for some $M > 0$. This condition on φ is known in the literature as the Δ_2-condition.

Examples of φ-functions for which the Δ_2-condition is satisfied are $\varphi(u) = u^p$, $1 \le p < +\infty$, or $\varphi_{\alpha,\beta}(u) = u^\alpha \log^\beta(u+e)$, for $\alpha \ge 1$ and $\beta > 0$, which generate the usual L^p-spaces and the well-known interpolation spaces $L^\alpha \log^\beta L(\Omega)$, respectively. On the other hand, it is easy to prove that, for example, for $\varphi_\gamma(u) = e^{u^\gamma} - 1$, $\gamma > 0$, the Δ_2-condition is not fulfilled. Hence, each of the corresponding Orlicz spaces $L^{\varphi_\gamma}(\Omega)$, known as the exponential spaces, properly contains its subspace of the finite elements $E^{\varphi_\gamma}(\Omega)$.

Now, we recall that it is possible to introduce in $L^\varphi(\Omega)$ a natural notion of convergence based on the definition of the modular functional I^φ, i.e., the so-called modular convergence. From now on, we will say that a net of functions $(f_w)_{w>0} \subset L^\varphi(\Omega)$ is modularly convergent to a function $f \in L^\varphi(\Omega)$ if:

$$\lim_{w\to+\infty} I^\varphi\big[\lambda\big(f_w - f\big)\big] = 0,$$

for some $\lambda > 0$. Moreover, the Luxemburg norm $\|\cdot\|_\varphi$ defines itself a notion of convergence in $L^\varphi(\Omega)$ which is called the (Luxemburg) norm-convergence. Namely, a net of functions $(f_w)_{w>0} \subset L^\varphi(\Omega)$ is said norm convergent to a function $f \in L^\varphi(\Omega)$, i.e., $\|f_w - f\|_\varphi \to 0$ for $w \to +\infty$, if:

$$\lim_{w\to+\infty} I^\varphi\big[\lambda\big(f_w - f\big)\big] = 0,$$

for every $\lambda > 0$. From the above definitions, it is clear that the Luxemburg norm convergence is stronger than the modular convergence. Further, it is possible to prove that, assuming that φ satisfies the Δ_2-condition, these two notions of convergence are equivalent. Moreover, the vice versa also holds.

Moreover, in order to reach the purpose of this paper, we need to recall the following density properties of the Orlicz spaces. Namely, it is easy to see that $C([a,b]) \subset L^\varphi([a,b])$ and $C_c(\mathbb{R}) \subset L^\varphi(\mathbb{R})$, and moreover, it turns out that $C([a,b])$ and $C_c(\mathbb{R})$ are dense in $L^\varphi([a,b])$ and $L^\varphi(\mathbb{R})$, respectively, with respect to the topology induced by the modular convergence. Similarly to above, we can prove that $C^+([a,b])$ and $C_c^+(\mathbb{R})$ are modularly dense in $L_+^\varphi(\Omega)$, with $\Omega = [a,b]$ or $\Omega = \mathbb{R}$, respectively, where $L_+^\varphi(\Omega)$ denotes the subspace of $L^\varphi(\Omega)$ of the non-negative functions.

For further results and details concerning the Orlicz spaces, one can see, e.g., [40,47–50].

Finally, before to consider the operators $(K_n^\chi)_{n\in\mathbb{N}}$ in the Orlicz spaces, we need to establish the following proposition in which the max-product symbol $\bigvee$ is related to the φ-function φ.

Proposition 3.1. *Let φ be a φ-function. Then the following inequality:*

$$\varphi\left(\bigvee_{k\in\mathcal{J}} A_k\right) \le \bigvee_{k\in\mathcal{J}} \varphi\big(2A_k\big),$$

holds for any set of indexes $\mathcal{J} \subseteq \mathbb{Z}$, and $A_k \ge 0$, $k \in \mathcal{J}$.

Proof. If $\mathcal{J}$ is finite, it is easy to observe that the following equality:

$$\varphi\left(\bigvee_{k\in\mathcal{J}} A_k\right) = \bigvee_{k\in\mathcal{J}} \varphi(A_k),$$

holds, since φ is non-decreasing. Thus, the above inequality is obvious. Now, let us consider the case $\mathcal{J} = \mathbb{Z}$ (if $\mathcal{J}$ is any infinite set, the proof is similar). Without loss of generality, we can assume that $\bigvee_{k\in\mathbb{Z}} A_k > 0$, otherwise the proof becomes trivial. From the definition of the symbol $\bigvee$, taking $\varepsilon = \frac{1}{2}\bigvee_{k\in\mathbb{Z}} A_k$, it follows that there exists $\bar{k} \in \mathbb{Z}$ such that $\frac{1}{2}\bigvee_{k\in\mathbb{Z}} A_k < A_{\bar{k}}$. Therefore, by the monotonicity of φ, we can write what follows:

$$\varphi\left(\bigvee_{k\in\mathbb{Z}} A_k\right) \le \varphi\left(A_{\bar{k}} + \frac{1}{2}\bigvee_{k\in\mathbb{Z}} A_k\right) \le \varphi(2A_{\bar{k}}) \le \bigvee_{k\in\mathbb{Z}} \varphi(2A_k). \qquad \square$$

4. Convergence results in Orlicz spaces

In this section, we study the convergence properties of the max-product sampling Kantorovich operators in the general setting of Orlicz spaces.

First of all, we need to prove an inequality for these non-linear operators with respect to the modular functional I^{φ}. We will see that such inequality is crucial in order to prove the desired modular convergence result for the family $(K_n^{\chi})_{n\in\mathbb{N}}$ in $L_+^{\varphi}(\Omega)$, with $\Omega = [a,b]$ or $\Omega = \mathbb{R}$.

In all that follows in this section, if otherwise not stated, the φ-function φ will be convex and the kernel χ from the definition of K_n^{χ} will belong to $L^1(\mathbb{R})$.

Theorem 4.1. *For every $f, g \in L_+^{\varphi}(\Omega)$ and $\lambda > 0$, it turns out that:*

$$I^{\varphi}\left[\lambda\left(K_n^{\chi}(f) - K_n^{\chi}(g)\right)\right] \le \frac{\|\chi\|_1}{m_0(\chi)} I^{\varphi}\left[\frac{m_0(\chi)}{a_{\chi}} 2\lambda(f-g)\right],$$

for $n \in \mathbb{N}$ sufficiently large.

Proof. By the property (iii) of the operator K_n^{χ} established in Lemma 2.4 and the inequalities of Lemma 2.3, we can write what follows for sufficiently large $n \in \mathbb{N}$:

$$\begin{aligned} I^{\varphi}\left[\lambda\left(K_n^{\chi}(f) - K_n^{\chi}(g)\right)\right] &= \int_{\Omega} \varphi\left(\lambda|K_n^{\chi}(f)(x) - K_n^{\chi}(g)(x)|\right)\, dx \le \int_{\Omega} \varphi\left(\lambda K_n^{\chi}(|f-g|)(x)\right)\, dx \\ &\le \int_{\Omega} \varphi\left(\frac{\lambda}{a_{\chi}} \bigvee_{k\in\mathcal{J}_n} |\chi(nx-k)| \left[n\int_{k/n}^{(k+1)/n} |f(t)-g(t)|\, dt\right]\right) dx \end{aligned}$$

Now, from Proposition 3.1, we immediately get that:

$$\varphi\left(\bigvee_{k\in\mathcal{J}} A_k\right) \le \bigvee_{k\in\mathcal{J}} \varphi(2A_k), \tag{4}$$

for any set of indexes $\mathcal{J}$, and $A_k \ge 0$, $k \in \mathcal{J}$. Thus, by using (4) with $A_k := \frac{\lambda}{a_{\chi}}|\chi(nx-k)|\left[n\int_{k/n}^{(k+1)/n}|f(t)-g(t)|\, dt\right]$, the convexity of φ with the fact that $|\chi(nx-k)| \le m_0(\chi)$, for every $k \in \mathcal{J}_n$, and the Jensen inequality (see, e.g., [51]), we obtain:

$$\begin{aligned} I^{\varphi}\left[\lambda\left(K_n^{\chi}(f) - K_n^{\chi}(g)\right)\right] &\le \int_{\Omega} \bigvee_{k\in\mathcal{J}_n} \varphi\left(2\frac{\lambda}{a_{\chi}}|\chi(nx-k)| \left[n\int_{k/n}^{(k+1)/n} |f(t)-g(t)|\, dt\right]\right) dx \\ &\le \int_{\Omega} \bigvee_{k\in\mathcal{J}_n} \frac{|\chi(nx-k)|}{m_0(\chi)}\, \varphi\left(2\frac{m_0(\chi)}{a_{\chi}}\lambda\left[n\int_{k/n}^{(k+1)/n} |f(t)-g(t)|\, dt\right]\right) dx \\ &\le \int_{\Omega} dx \left\{\bigvee_{k\in\mathcal{J}_n} \frac{|\chi(nx-k)|}{m_0(\chi)} \left[n\int_{k/n}^{(k+1)/n} \varphi\left(2\frac{m_0(\chi)}{a_{\chi}}\lambda|f(t)-g(t)|\right) dt\right]\right\} \\ &\le (m_0(\chi))^{-1} \int_{\Omega} \left[n\sum_{k\in\mathcal{J}_n} |\chi(nx-k)| \int_{k/n}^{(k+1)/n} \varphi\left(2\frac{m_0(\chi)}{a_{\chi}}\lambda|f(t)-g(t)|\right) dt\right] dx \end{aligned}$$

Now, it is easy to observe that in the case $\Omega = \mathbb{R}$ we can use the Fubini–Tonelli theorem in order to interchange the integral with the series in the above computations. Thus, we get:

$$\begin{aligned} I^{\varphi}\left[\lambda\left(K_n^{\chi}(f) - K_n^{\chi}(g)\right)\right] &\le (m_0(\chi))^{-1} \int_{\Omega} \left[n\sum_{k\in\mathcal{J}_n} |\chi(nx-k)| \int_{k/n}^{(k+1)/n} \varphi\left(2\frac{m_0(\chi)}{a_{\chi}}\lambda|f(t)-g(t)|\right) dt\right] dx \\ &= (m_0(\chi))^{-1} \sum_{k\in\mathcal{J}_n} \int_{k/n}^{(k+1)/n} \varphi\left(2\frac{m_0(\chi)}{a_{\chi}}\lambda|f(t)-g(t)|\right) dt \int_{\Omega} n|\chi(nx-k)|\, dx \end{aligned} \tag{5}$$

Moreover, for every $k \in \mathcal{J}_n$, by using the change of variable $y = nx - k$, we have:

$$\int_\Omega n|\chi(nx-k)|\, dx \leq \int_\mathbb{R} |\chi(y)|\, dy = \|\chi\|_1.$$

Hence, using this result in (5), we finally obtain:

$$\begin{aligned} I^\varphi\left[\lambda\left(K_n^\chi(f) - K_n^\chi(g)\right)\right] &\leq (m_0(\chi))^{-1}\|\chi\|_1 \sum_{k\in\mathcal{J}_n} \int_{k/n}^{(k+1)/n} \varphi\left(2\frac{m_0(\chi)}{a_\chi}\lambda|f(t)-g(t)|\right) dt \\ &\leq \frac{\|\chi\|_1}{m_0(\chi)} \int_\Omega \varphi\left(2\frac{m_0(\chi)}{a_\chi}\lambda|f(t)-g(t)|\right) dt \\ &= \frac{\|\chi\|_1}{m_0(\chi)} I^\varphi\left[\frac{m_0(\chi)}{a_\chi}2\lambda(f-g)\right], \end{aligned}$$

for every $n \in \mathbb{N}$ sufficiently large. This completes the proof. □

Now, in order to establish a modular convergence theorem for the max-product sampling Kantorovich operators in $L^\varphi_+(\Omega)$, we need to test the (Luxemburg) norm-convergence in case of functions belonging to $C^+(\Omega)$, when $\Omega = [a,b]$, and in case of functions $f \in C^+_c(\Omega)$, when $\Omega = \mathbb{R}$. Hence, we can give the proof of the following.

Theorem 4.2. *Let $f \in C^+(\Omega)$, $\Omega = [a,b]$, be fixed. Then for every $\lambda > 0$, it turns out that:*

$$\lim_{n\to+\infty} I^\varphi\left[\lambda\left(K_n^\chi(f) - f\right)\right] = 0.$$

In addition, in case of $f \in C^+_c(\Omega)$, with $\Omega = \mathbb{R}$, we also have:

$$\lim_{n\to+\infty} I^\varphi\left[\lambda\left(K_n^\chi(f) - f\right)\right] = 0,$$

for every $\lambda > 0$.

Proof. Let us begin by considering the case $\Omega = [a,b]$. First of all, it is easy to see that $K_n^\chi(f)$ belongs to $E^\varphi([a,b]) \subseteq L^\varphi([a,b])$, for every $n \in \mathbb{N}$ sufficiently large. In fact, for every $\lambda > 0$, by using (2) and the inequality established in Proposition 3.1, we have:

$$\begin{aligned} I^\varphi\left[\lambda K_n^\chi(f)\right] &= \int_a^b \varphi\left(\lambda|K_n^\chi(f)(x)|\right) dx \leq \int_a^b \varphi\left(\frac{\lambda}{a_\chi}\bigvee_{k\in\mathcal{J}_n}|\chi(nx-k)|\left[n\int_{k/n}^{(k+1)/n} f(t)\, dt\right]\right) dx \\ &\leq \int_a^b \bigvee_{k\in\mathcal{J}_n} \varphi\left(2\frac{\lambda}{a_\chi}|\chi(nx-k)|\left[n\int_{k/n}^{(k+1)/n} f(t)\, dt\right]\right) dx \leq \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty\|\chi\|_\infty\right)(b-a) < +\infty, \end{aligned}$$

for every $n \in \mathbb{N}$ sufficiently large. Now, let $\varepsilon > 0$ be fixed. Thus, for every fixed $\lambda > 0$, since the φ-function φ is convex and Theorem 2.1 holds, it follows that:

$$\begin{aligned} I^\varphi\left[\lambda\left(K_n^\chi(f) - f\right)\right] &= \int_a^b \varphi\left(\lambda|K_n^\chi(f)(x) - f(x)|\right) dx \leq \int_a^b \varphi\left(\lambda\|K_n^\chi(f) - f\|_\infty\right) dx \\ &\leq \int_a^b \varphi\left(\lambda\varepsilon\right) dx \leq \varepsilon\, \varphi(\lambda)(b-a), \end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large. Thus, by the arbitrariness of $\varepsilon > 0$ we get the thesis.

Now, we can consider the case $\Omega = \mathbb{R}$. Let γ and $\bar\gamma$ be two positive constants such that $supp\ f \subseteq [-\bar\gamma, \bar\gamma]$ and $\gamma > \bar\gamma + 1$. Thus, whenever $k/n \notin [-\gamma, \gamma]$, $n \in \mathbb{N}$, we have $[k/n, (k+1)/n] \cap [-\bar\gamma, \bar\gamma] = \emptyset$ and therefore:

$$\int_{k/n}^{(k+1)/n} f(t)\, dt = 0.$$

Now, we can prove that $K_n^\chi(f) \in E^\varphi(\mathbb{R}) \subseteq L^\varphi(\mathbb{R})$, for every $n \in \mathbb{N}$. Indeed, for every $\lambda > 0$, we can write what follows:

$$I^\varphi\left[\lambda K_n^\chi(f)\right] = \int_\mathbb{R} \varphi\left(\lambda|K_n^\chi(f)(x)|\right) dx = \int_\mathbb{R} \varphi\left(\lambda\left|\frac{\bigvee_{k\in\mathcal{J}_n^\gamma} \chi(nx-k)n\int_{k/n}^{(k+1)/n} f(t)\, dt}{\bigvee_{k\in\mathbb{Z}} \chi(nx-k)}\right|\right) dx,$$

where $\mathcal{J}_n^{\gamma} := \{k \in \mathbb{Z} : |k/n| \leq \gamma\}$. Hence, by inequalities (1), and Proposition 3.1, and by using the convexity of φ with the fact that $|\chi(nx-k)| \leq \|\chi\|_\infty$, for every $k \in \mathcal{J}_n^{\gamma}$, we obtain for every $\lambda > 0$:

$$\begin{aligned}
\int_{\mathbb{R}} \varphi\left(\lambda |K_n^{\chi}(f)(x)|\right) dx &\leq \int_{\mathbb{R}} \varphi\left(\frac{\lambda}{a_\chi}\left|\bigvee_{k\in\mathcal{J}_n^{\gamma}} \chi(nx-k) n \int_{k/n}^{(k+1)/n} f(t)\, dt\right|\right) dx \\
&\leq \int_{\mathbb{R}} \bigvee_{k\in\mathcal{J}_n^{\gamma}} \varphi\left(2\frac{\lambda}{a_\chi}|\chi(nx-k)| n \int_{k/n}^{(k+1)/n} f(t)\, dt\right) dx \\
&\leq \int_{\mathbb{R}} \bigvee_{k\in\mathcal{J}_n^{\gamma}} \frac{|\chi(nx-k)|}{\|\chi\|_\infty} \varphi\left(2\frac{\lambda}{a_\chi}\|\chi\|_\infty n \int_{k/n}^{(k+1)/n} f(t)\, dt\right) dx \\
&\leq \int_{\mathbb{R}} \bigvee_{k\in\mathcal{J}_n^{\gamma}} \frac{|\chi(nx-k)|}{\|\chi\|_\infty} \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty \|\chi\|_\infty\right) dx \\
&\leq \|\chi\|_\infty^{-1} \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty \|\chi\|_\infty\right) \sum_{k\in\mathcal{J}_n^{\gamma}} \int_{\mathbb{R}} |\chi(nx-k)|\, dx \\
&= n^{-1}\|\chi\|_\infty^{-1} \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty \|\chi\|_\infty\right) \sum_{k\in\mathcal{J}_n^{\gamma}} \|\chi\|_1 \\
&= n^{-1}\|\chi\|_\infty^{-1} \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty \|\chi\|_\infty\right) \|\chi\|_1 \cdot L < +\infty,
\end{aligned}$$

where in the last integral, we made the change of variable $y = nx - k$ and $L \leq 2n\gamma + 1$ represents the number of terms in the above finite sum.

Now, the goal of the last part of the proof is to prove that:

$$\lim_{n\to+\infty} I^{\varphi}\left[\lambda\left(K_n^{\chi}(f) - f\right)\right] = \lim_{n\to+\infty} \int_{\mathbb{R}} \varphi\left(\lambda |K_n^{\chi}(f)(x) - f(x)|\right) dx = 0,$$

for every $\lambda > 0$, which is equivalent to proving that the family $\left(\varphi(\lambda|K_n^{\chi}(f) - f|)\right)_{n\in\mathbb{N}}$ converges to zero with respect to the L^1-norm, for every $\lambda > 0$. In order to establish this, we will exploit the Vitali convergence theorem (see Theorem 2.2). Let $\lambda > 0$ be fixed. First, by Theorem 2.1 and the properties of φ, we have that for every $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ sufficiently large such that:

$$\varphi\left(\lambda |K_n^{\chi}(f)(x) - f(x)|\right) \leq \varphi\left(\lambda \|K_n^{\chi}(f) - f\|_\infty\right) \leq \varepsilon,$$

for every $n \geq \bar{n}$ and for all $x \in \mathbb{R}$. Therefore,

$$|\{x \in \mathbb{R} : \varphi(\lambda \left|K_n^{\chi}(f)(x) - f(x)\right|) > \varepsilon\}| = |\emptyset| = 0,$$

for every $n \geq \bar{n}$, where $|\cdot|$ denotes the (Lebesgue) measure of the corresponding set. Hence, we proved that the family $\left(\varphi(\lambda|K_n^{\chi}(f) - f|)\right)_{n\in\mathbb{N}}$ converges in measure to zero (i.e., (i) of Theorem 2.2 is satisfied).

Now, let $\varepsilon > 0$ be fixed. Since χ is absolutely integrable, it is easy to prove that corresponding to

$$T(\varepsilon) := \frac{\varepsilon \|\chi\|_\infty}{\varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty \|\chi\|_\infty\right)(2\gamma + 1)},$$

there exists a constant $M > 0$ (we can assume $M > \bar{\gamma}$), such that

$$n \int_{|x|>M} |\chi(nx-k)|\, dx < T(\varepsilon),$$

for every $k \in [-\gamma n, \gamma n]$, $n \in \mathbb{N}$. Hence, proceeding as in the previous estimates, we have:

$$\begin{aligned}
\int_{|x|>M} \varphi\big(\lambda |K_n^\chi(f)(x)|\big)\, dx &\le \int_{|x|>M} \bigvee_{k\in\mathcal{J}_n^\gamma} \varphi\left(2\frac{\lambda}{a_\chi}|\chi(nx-k)| n \int_{k/n}^{(k+1)/n} f(t)\, dt\right) dx \\
&\le \int_{|x|>M} \bigvee_{k\in\mathcal{J}_n^\gamma} \frac{|\chi(nx-k)|}{\|\chi\|_\infty} \varphi\left(2\frac{\lambda}{a_\chi}\|\chi\|_\infty n \int_{k/n}^{(k+1)/n} f(t)\, dt\right) dx \\
&\le \int_{|x|>M} \bigvee_{k\in\mathcal{J}_n^\gamma} \frac{|\chi(nx-k)|}{\|\chi\|_\infty} \varphi\left(2\frac{\lambda}{a_\chi}\|\chi\|_\infty \|f\|_\infty\right) dx \\
&\le n^{-1}\|\chi\|_\infty^{-1} \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty\|\chi\|_\infty\right) \sum_{k\in\mathcal{J}_n^\gamma} n \int_{|x|>M} |\chi(nx-k)|\, dx \\
&\le n^{-1}\|\chi\|_\infty^{-1} \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty\|\chi\|_\infty\right) \sum_{k\in\mathcal{J}_n^\gamma} T(\varepsilon) \\
&\le \|\chi\|_\infty^{-1} \varphi\left(2\frac{\lambda}{a_\chi}\|f\|_\infty\|\chi\|_\infty\right) (2\gamma+1) T(\varepsilon) = \varepsilon,
\end{aligned}$$

where the last estimate follows from the fact that the sum in the previous inequality consists of at most $2n\gamma + 1$ terms, corresponding to the number of intervals $[k/n, (k+1)/n]$, $n \in \mathbb{N}$, having non-empty intersection with $[-\bar{\gamma}, \bar{\gamma}]$. Therefore, we proved that for every $\varepsilon > 0$ there exists a set $E_\varepsilon := [-M, M] \subset \mathbb{R}$ for which, for every measurable set F, with $F \cap E_\varepsilon = \emptyset$, we get:

$$\int_F \varphi\big(\lambda|K_n^\chi(f)(x) - f(x)|\big)\, dx = \int_F \varphi\big(\lambda|K_n^\chi(f)(x)|\big)\, dx \le \int_{|x|>M} \varphi\big(\lambda|K_n^\chi(f)(x)|\big)\, dx < \varepsilon,$$

for every $n \in \mathbb{N}$ (i.e., (ii) of Theorem 2.2 is satisfied).

Finally, let $\varepsilon > 0$ and $B \subset \mathbb{R}$ be a measurable set such that:

$$|B| < \delta(\varepsilon) := \frac{\varepsilon}{\varphi\left(2\lambda \frac{m_0(\chi)}{a_\chi}\|f\|_\infty\right)}.$$

Then, by the convexity of φ and observing that

$$m_0(\chi) \ge a_\chi, \tag{6}$$

we obtain:

$$\begin{aligned}
&\int_B \varphi\big(\lambda|K_n^\chi(f)(x) - f(x)|\big)\, dx \le \int_B \varphi\left(\frac{1}{2}2\lambda|K_n^\chi(f)(x)| + \frac{1}{2}2\lambda f(x)\right) dx \\
&\le \frac{1}{2}\int_B \varphi\big(2\lambda|K_n^\chi(f)(x)|\big)\, dx + \frac{1}{2}\int_B \varphi\big(2\lambda f(x)\big)\, dx \\
&\le \frac{1}{2}\int_B \varphi\left(2\lambda\frac{m_0(\chi)}{a_\chi}\|f\|_\infty\right) dx + \frac{1}{2}\int_B \varphi\big(2\lambda\|f\|_\infty\big)\, dx \\
&= \frac{|B|}{2}\left[\varphi\left(2\lambda\frac{m_0(\chi)}{a_\chi}\|f\|_\infty\right) + \varphi\big(2\lambda\|f\|_\infty\big)\right] \\
&\le |B|\varphi\left(2\lambda\frac{m_0(\chi)}{a_\chi}\|f\|_\infty\right) < \varepsilon,
\end{aligned}$$

for every $n \in \mathbb{N}$. Thus, it turns out that the integrals

$$\int_{(\cdot)} \varphi\big(\lambda|K_n^\chi(f)(x) - f(x)|\big)\, dx$$

are equi-absolutely continuous (i.e., (iii) of Theorem 2.2 holds). Then, the proof follows by the Vitali convergence theorem, together with the arbitrariness of $\lambda > 0$. $\square$

Now, we can finally prove the main theorem of this paper, i.e., a modular convergence theorem for the max-product sampling Kantorovich operators.

Theorem 4.3. *Let $f \in L_+^\varphi(\Omega)$ be fixed. Then there exists $\lambda > 0$ such that:*

$$\lim_{n\to+\infty} I^\varphi\left[\lambda\big(K_n^\chi(f) - f\big)\right] = 0.$$

Proof. Let $\varepsilon > 0$ be fixed. First of all, we recall that the spaces $C^+([a,b])$ and $C_c^+(\mathbb{R})$ are modularly dense in $L_+^\varphi([a,b])$ and $L_+^\varphi(\mathbb{R})$, respectively. Thus, corresponding to

$$T(\varepsilon) := \frac{\varepsilon}{2\left(\frac{\|\chi\|_1}{m_0(\chi)} + 1\right)},$$

there exists $\bar{\lambda} > 0$ and a function $g \in C^+([a,b])$, if $\Omega = [a,b]$, or $g \in C_c^+(\mathbb{R})$, in case of $\Omega = \mathbb{R}$, such that:

$$I^\varphi\left[\bar{\lambda}\left(f-g\right)\right] < T(\varepsilon). \tag{7}$$

Now, let us choose $\lambda > 0$ such that:

$$\frac{m_0(\chi)}{a_\chi} 6\lambda \le \bar{\lambda}.$$

By the properties of modular I^φ, the convexity of φ, recalling (6) and exploiting Theorem 4.1, we can write what follows:

$$\begin{aligned} I^\varphi\left[\lambda\left(K_n^\chi(f)-f\right)\right] &\le \frac{1}{3}\left\{I^\varphi\left[3\lambda\left(K_n^\chi(f)-K_n^\chi(g)\right)\right] + I^\varphi\left[3\lambda\left(K_n^\chi(g)-g\right)\right] + I^\varphi\left[3\lambda\left(g-f\right)\right]\right\} \\ &\le \frac{\|\chi\|_1}{m_0(\chi)} I^\varphi\left[\frac{m_0(\chi)}{a_\chi} 6\lambda\left(f-g\right)\right] + I^\varphi\left[3\lambda\left(K_n^\chi(g)-g\right)\right] + I^\varphi\left[3\lambda\left(f-g\right)\right] \\ &\le \left(\frac{\|\chi\|_1}{m_0(\chi)}+1\right) I^\varphi\left[\bar{\lambda}\left(f-g\right)\right] + I^\varphi\left[\bar{\lambda}\left(K_n^\chi(g)-g\right)\right], \end{aligned}$$

for every $n \in \mathbb{N}$ sufficiently large. Now, by using (7), we obtain:

$$I^\varphi\left[\lambda\left(K_n^\chi(f)-f\right)\right] \le \frac{\varepsilon}{2} + I^\varphi\left[\bar{\lambda}\left(K_n^\chi(g)-g\right)\right],$$

and in view of Theorem 4.2, we have for every $n \in \mathbb{N}$ sufficiently large:

$$I^\varphi\left[\bar{\lambda}\left(K_n^\chi(g)-g\right)\right] < \frac{\varepsilon}{2}.$$

Thus, the proof follows by the arbitrariness of $\varepsilon > 0$. □

Remark 4.1. As often happens when one deals with max-product type operators, all the convergence results established in this section can be extended to the case of not-necessarily non-negative functions that are bounded from below. Indeed, for any fixed $f : \Omega \to \mathbb{R}$ belonging to suitable spaces, denoting $\inf_{x\in\Omega} f(x) =: c$, it turns out that the sequences $(K_n^\chi(f(\cdot)-c)+c)_{n\in\mathbb{N}}$ converges to f in all the above considered senses.

5. Applications to particular cases, examples with special kernels and graphical representations

In this section, we want to show how the convergence results obtained in the previous section can be applied to some special instances of Orlicz spaces. Then, in the last part of the section, we will provide several specific examples of kernels for the max-product sampling Kantorovich operators $(K_n^\chi)_{n\in\mathbb{N}}$.

Let us begin considering the case that occurs when choosing the convex φ-function $\varphi(u) := u^p$, $u \ge 0$, $1 \le p < +\infty$, which generates the corresponding Orlicz space $L^\varphi(\Omega) = L^p(\Omega)$. Indeed, it results $I^\varphi[f] = \|f\|_p^p$, i.e., the modular functional I^φ coincides with the usual L^p-norm. Further, it is immediate to prove that the Δ_2-condition is fulfilled. Hence, it follows that $L^p(\Omega)$ coincides with the space of its finite elements and the modular convergence and the usual (Luxemburg) norm-convergence are equivalent. From the theory established in the previous section, we can state the following corollaries.

Corollary 5.1. *Let $\chi \in L^1(\mathbb{R})$ be a fixed kernel. Then for every non-negative f, $g \in L^p(\Omega)$, $1 \le p < +\infty$, we have:*

$$\|K_n^\chi(f)-K_n^\chi(g)\|_p \le \frac{2\left((m_0(\chi))^{p-1}\|\chi\|_1\right)^{1/p}}{a_\chi} \cdot \|f-g\|_p,$$

for $n \in \mathbb{N}$ sufficiently large.

Proof. Since Theorem 4.1 holds with $I^\varphi[f] = \|f\|_p^p$, we obtain:

$$\|K_n^\chi(f)-K_n^\chi(g)\|_p^p \le 2^p a_\chi^{-p}\left(m_0(\chi)\right)^{p-1}\|\chi\|_1\|f-g\|_p^p,$$

from which the assertion follows. □

In addition, as a direct consequence of Theorem 4.3, we immediately obtain the following L^p-convergence result.

Corollary 5.2. *Let $\chi \in L^1(\mathbb{R})$ be a given kernel. For any non-negative function $f \in L^p(\Omega)$, $1 \le p < +\infty$, there holds:*

$$\lim_{n\to+\infty} \|K_n^\chi(f)-f\|_p = 0.$$

Other useful examples of Orlicz spaces are the so-called interpolation spaces $L^\alpha \log^\beta L(\Omega)$. The Zygmund spaces (another name for the interpolation spaces) can be obtained by φ-functions $\varphi_{\alpha,\beta}(u) := u^\alpha \log^\beta(u+e)$, $u \ge 0$, for $\alpha \ge 1$ and $\beta > 0$. The corresponding modular functionals

$$I^{\varphi_{\alpha,\beta}}[f] := \int_\Omega |f(x)|^\alpha \log^\beta\left(|f(x)|+e\right)\, dx, \quad f \in M(\Omega),$$

are convex, and $L^\alpha \log^\beta L(\Omega)$ is defined as the set of all (Lebesgue-)measurable functions for which $I^{\varphi_{\alpha,\beta}}[\lambda f] < +\infty$ for some $\lambda > 0$, for any α and β as above. Applying the modular inequality established in Theorem 4.1 with $\varphi_{\alpha,\beta}(u) = u^\alpha \log^\beta(u + e)$, we obtain the following corollary, which for simplicity we provide for $\alpha = \beta = 1$.

Corollary 5.3. *Let $\chi \in L^1(\mathbb{R})$ be a fixed kernel. Then for every non-negative $f, g \in L\log L(\Omega)$, we have:*

$$\int_\Omega \left|K_n^\chi(f)(x) - K_n^\chi(g)(x)\right| \log\left(\lambda \left|K_n^\chi(f)(x) - K_n^\chi(g)(x)\right| + e\right) dx$$
$$\leq \frac{2\|\chi\|_1}{a_\chi} \int_\Omega |f(x) - g(x)| \log\left(\frac{m_0(\chi)}{a_\chi} 2\lambda\, |f(x) - g(x)| + e\right) dx,$$

for $\lambda > 0$ and $n \in \mathbb{N}$ sufficiently large.

In addition, we have $L^\alpha \log^\beta L(\Omega) = E^{\varphi_{\alpha,\beta}}(\Omega)$, since also $\varphi_{\alpha,\beta}$ satisfies the Δ_2-condition. This implies that the modular convergence and norm convergence are equivalent, and the following convergence theorem holds for $\alpha = \beta = 1$.

Corollary 5.4. *Let $\chi \in L^1(\mathbb{R})$ be a given kernel. For any non-negative function $f \in L\log L(\Omega)$, there holds:*

$$\lim_{n\to+\infty} \int_\Omega \left|K_n^\chi(f)(x) - f(x)\right| \log\left(\lambda \left|K_n^\chi(f)(x) - f(x)\right| + e\right) dx = 0,$$

for every $\lambda > 0$ or, equivalently,

$$\lim_{n\to+\infty} \|K_n^\chi(f) - f\|_{\varphi_{1,1}} = 0,$$

where $\|\cdot\|_{\varphi_{1,1}}$ is the Luxemburg norm associated to $I^{\varphi_{1,1}}$.

As a final example, as mentioned in Section 3, an important case in which modular and norm convergence are not equivalent occurs at so-called exponential spaces, generated by $\varphi_\gamma(u) = e^{u^\gamma} - 1$, $u \geq 0$, for $\gamma > 0$. Indeed, it is easy to prove that in this case the Δ_2-condition is not satisfied. Hence, the corresponding space of the finite elements $E^{\varphi_\gamma}(\Omega)$ is properly contained in $L^{\varphi_\gamma}(\Omega)$. Now, the modular functional corresponding to φ_γ is given by:

$$I^{\varphi_\gamma}[f] := \int_\Omega \left(e^{|f(x)|^\gamma} - 1\right) dx,$$

for any (Lebesgue-)measurable function $f : \Omega \to \mathbb{R}$. As a consequence of Theorem 4.1, we can establish the following.

Corollary 5.5. *Let $\chi \in L^1(\mathbb{R})$ be a fixed kernel. Then for every non negative $f, g \in L^{\varphi_\gamma}(\Omega)$, we have:*

$$\int_\Omega \left(\exp\left(\lambda\left|K_n^\chi(f)(x) - K_n^\chi(g)(x)\right|\right)^\gamma - 1\right) dx$$
$$\leq \frac{\|\chi\|_1}{m_0(\chi)} \int_\Omega \left(\exp\left(\frac{m_0(\chi)}{a_\chi} 2\lambda\, |f(x) - g(x)|\right)^\gamma - 1\right) dx,$$

for $\lambda > 0$ and $n \in \mathbb{N}$ sufficiently large.

In contrast to previous cases, in the latter setting there is no analogous convergence result with respect to the Luxemburg norm, but only a modular convergence theorem.

Corollary 5.6. *Let $\chi \in L^1(\mathbb{R})$ be a given kernel. For any non-negative function $f \in L^{\varphi_\gamma}(\Omega)$, there exists $\lambda > 0$ such that:*

$$\lim_{n\to+\infty} \int_\Omega \left(\exp\left(\lambda \left|K_n^\chi(f)(x) - f(x)\right|\right)^\gamma - 1\right) dx = 0.$$

Now, we give some examples of specific kernels for which the results proved in this paper hold.

First, we consider the Fejér kernel $\mathcal{F}(x) := \frac{1}{2}\mathrm{sinc}^2(x/2)$, $x \in \mathbb{R}$, (see Fig. 1) where the well-known sinc-function is given by:

$$sinc(x) := \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \in \mathbb{R} \setminus \{0\}, \\ 1, & x = 0. \end{cases}$$

Obviously, $\mathcal{F}$ is a non-negative bounded function, and it belongs to $L^1(\mathbb{R})$, since it turns out that $\mathcal{F}(x) = \mathcal{O}(|x|^{-2})$ as $|x| \to +\infty$ (see, e.g., [38]). In addition, by Lemma 2.1, condition (χ_1) holds with $\beta = 2$. Then, since $a_{\mathcal{F}} = \mathcal{F}(1/2) > 0$ (but $\mathcal{F}(2) = 0$), it follows that (χ_2') is satisfied (but (χ_2) is not satisfied). Therefore, for the max-product sampling Kantorovich operators based upon $\mathcal{F}$, all the corollaries of this section hold.

The so-called de la Vallée-Poussin kernel $\mathcal{P}$ (see Fig. 2) provides another example of band-limited kernels and it is defined by:

$$\mathcal{P}(x) := \begin{cases} \dfrac{\sin(x/2)\sin(3x/2)}{9x^2/4}, & x \in \mathbb{R} \setminus \{0\}, \\ \dfrac{1}{3}, & x = 0. \end{cases}$$

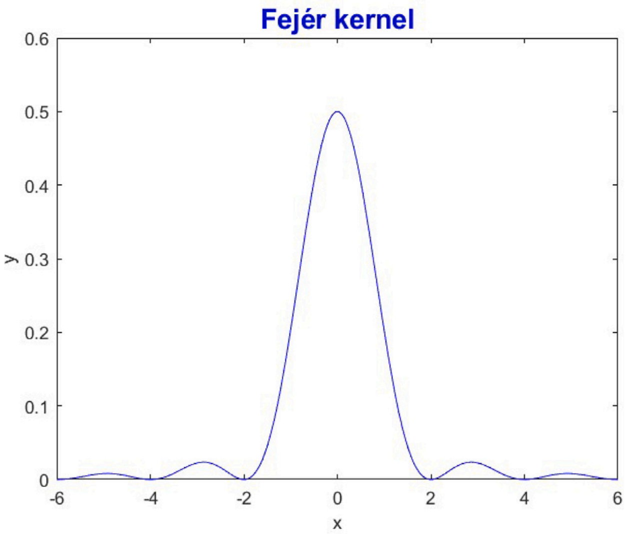

Fig. 1. The Fejér kernel $\mathcal{F}$.

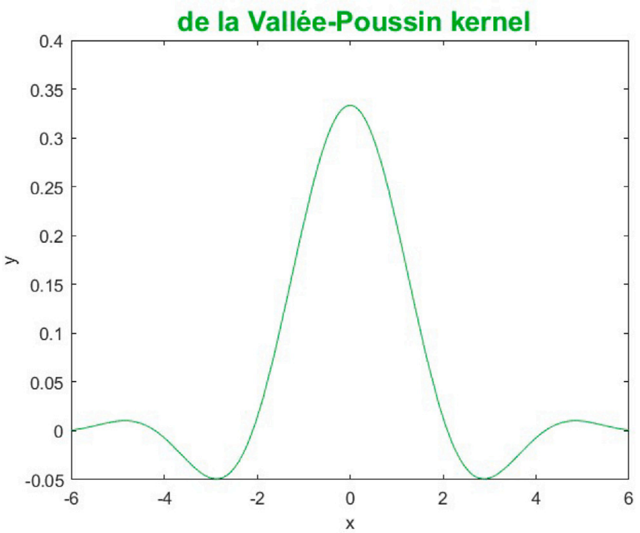

Fig. 2. The de la Vallée-Poussin kernel $\mathcal{P}$.

We recall that $\mathcal{P}(x) = \mathcal{O}(|x|^{-2})$ as $|x| \to +\infty$, which means that $\mathcal{P}$ is absolutely integrable on $\mathbb{R}$ and $m_\beta(\mathcal{P}) < +\infty$, i.e., (χ_1) holds with $\beta = 2$ by Lemma 2.1. Then, it is easy to check that (χ_2) is satisfied with $a_{\mathcal{P}} = \mathcal{P}(2) > 0$, and also (χ_2') is fulfilled (see, e.g., [25]). It follows that the de la Vallée-Poussin kernel $\mathcal{P}$ can be used to define the max-product sampling Kantorovich operators and to obtain the previous corollaries of the section.

Remark 5.1. Let $\Omega = \mathbb{R}$. We recall that, under suitable assumptions on the kernel χ, the linear version of the sampling Kantorovich operators can be defined as follows:

$$S_w^\chi(f)(x) = \sum_{k\in\mathbb{Z}} \chi(wx-k) \left[w \int_{k/w}^{(k+1)/w} f(t)\, dt \right], \quad x \in \mathbb{R}, \quad w > 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function such that the above series is convergent for every $x \in \mathbb{R}$. In various papers (see, e.g., [30,52]), quantitative estimates for the order of approximation of $(S_w^\chi)_{w>0}$ have been established in case of uniformly continuous and bounded functions, by exploiting the modulus of continuity of the function to be approximated. From the literature,

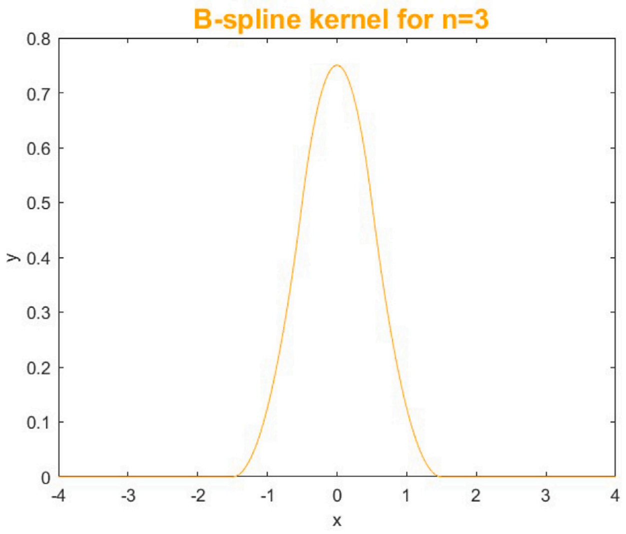

Fig. 3. The B-spline kernel M_3.

if $f \in C(\mathbb{R})$, one can define the modulus of continuity of f as follows:

$$\omega(f,\delta) := \sup\left\{|f(x) - f(y)| \,:\, x, y \in \mathbb{R}, |x-y| \leq \delta\right\}, \quad \delta > 0.$$

In particular, in Theorem 3.2 in [30], the authors proved that, if $\chi = \mathcal{F}$ or $\chi = \mathcal{P}$, then for any $f \in C(\mathbb{R})$, there exist two suitable positive constants C_1, $C_2 > 0$ such that:

$$\|S_w^\chi(f) - f\|_\infty \leq C_1 \omega(f, w^{-\beta}) + C_2 w^{-\beta}, \tag{8}$$

for some $0 < \beta < 1$ and $w > 0$ sufficiently large. On the other hand, if $f \in C^+(\mathbb{R})$ and $\chi = \mathcal{F}$ or $\chi = \mathcal{P}$, for the max-product version of the same sampling-type operators, the following estimate (see Theorem 3.2 in [25]):

$$\|K_n^\chi(f) - f\|_\infty \leq \frac{2m_0(\chi) + m_1(\chi)}{a_\chi} \omega(f, n^{-1}),$$

holds for sufficiently large $n \in \mathbb{N}$. We stress that, by Lemma 2.1, $m_0(\chi)$ and $m_1(\chi)$ are both finite. Comparing the above results, it is clear that the Jackson-type estimate for the max-product sampling Kantorovich operators $(K_n^\chi)_{n\in\mathbb{N}}$ is essentially better than that given in (8) for their linear counterparts and this explains the importance of studying the max-product version of the mentioned operators.

Among the kernels with compact support, also known as duration limited kernels, we consider the case generated by the well-known central B-spline of order $n \in \mathbb{N}$, defined by:

$$M_n(x) := \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(\frac{n}{2} + x - i\right)_+^{n-1}, \quad x \in \mathbb{R},$$

where $(x)_+ := \max\{x, 0\}$ denotes the "positive part" of $x \in \mathbb{R}$ (see, e.g., [53–55]). $M_n(x)$ is a non-negative continuous function, and it has support coincident with the interval $[-n/2; n/2]$. This means that, in order to obtain all the convergence results proved in this paper, the central B-spline $M_n(x)$ can be used as the kernel of the max-product sampling Kantorovich operators only for $n > 4$. Indeed, otherwise (χ_2) is not fulfilled. For example, let us now consider the particular case with $n = 3$ in detail, for which only (χ_2') is satisfied. For a sake of completeness, it is easy to prove that $M_3(x)$ (see Fig. 3) can be formally expressed by:

$$M_3(x) := \begin{cases} \frac{3}{4} - x^2, & |x| \leq \frac{1}{2}, \\ \frac{1}{2}\left(\frac{3}{2} - |x|\right)^2, & \frac{1}{2} < |x| \leq \frac{3}{2}, \\ 0, & |x| > \frac{3}{2}. \end{cases}$$

For more details about the above examples and other useful examples of kernels, one can see, e.g., [27,29,56].

Data availability

No data was used for the research described in the article.

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