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RESEARCH

Coupled nonlinear oscillators: dynamics with a period matching the forcing one

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Abstract We consider rather generic coupled nonlinear damped oscillators that are in some of their specifications frequently encountered in modeling mechanical and biological systems, subject to periodic external forcing. Combining a global co-bifurcation result based on topological degree theory with "local" properties, we establish sufficient conditions granting the emergence from equilibrium of nontrivial periodic solutions with a period exactly matching that of the forcing term. Also, under appropriate but still general structural assumptions, we derive quantitative estimates for the amplitude of such solutions. Recourse to topological degree theory allows us to leave aside explicit expressions of the nonlinear coupling. Particular choices of it have, indeed, different effects, which emerge, for example, by looking at the modulation of energy transfer across the system, as we show numerically, in order to clarify the matter. These aspects, however, do not avoid the possibility of a global bifurcation, as it is clear from the proof.

Keywords Nonlinear dynamics · Energy transfer · Periodic solutions · Topological degree

This work is dedicated to Fabrizio Vestroni.

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1 Introduction

The analysis of coupled nonlinear oscillators is a classical and widely explored topic. Pertinent literature is wide and it is difficult to construct a concise and significant list of references without being sure to have excluded deep and erudite works. Treatises as [1–5] provide a reasonable list of references after offering an extended view on the scenario. To make examples in a (perhaps too much) short list, we mention the Landau and Stuart's nonlinear oscillator describing the dynamics around a Hopf bifurcation [6,7] and, in addition to the traditional reference to finite degrees of freedom dynamics of mechanical structures, some models suggested by biology. An appreciably comprehensive overview on cases where the recourse to nonlinear oscillators appears to be appropriate if not necessary is in the multi-authored book [8]. Some prominent example in this panorama are the network-type scheme for brain firing, as the Hodgkin-Huxley model and its simplified form proposed by FitzHugh and Nagumo [9–11], the analysis of cortical waves [12], the Perskin model of pacemaker cells that create rhytmical impulses on the cardiac muscle [13–15], the description of genetic oscillations [16,17].

Here we look at a class of coupled nonlinear oscillators and are interested first in the role of nonlinearity in the branching of solutions when the system depend on a parameter. The topic has its roots in Euler's stability analysis of slender elastic columns and Poincaré's work on the three-body problem (this last aspect is particularly evident in Ziglin's complexification of the original Poincaré's approach to non-integrability [18]). Crucial in the development of branching theory was in 1927 a proof of the implicit function theorem in Banach spaces, provided by Hildebrandt and Graves [19], that was the catalyst for a great boost that the theory had, after the Second World War, especially thanks to Krasnosel'skii [20] and Veinberg [21]. Quoting from [22, p. 7], we can say that "reformulating the well-known words of Poincaré on periodic solutions, one may say that bifurcations, like torches, light the way from well-understood dynamical systems to unstudied ones".

In essence, one looks at equations of the type

$$F(\lambda, u) = 0\,,$$

where λ is a real parameter, u belongs to a Banach space $\mathfrak{B}$ and $F : \mathbb{R} \times \mathfrak{B} \longrightarrow \mathfrak{B}$ is a nonlinear map. We consider $F(0, 0) = 0$ and look for nontrivial solutions near $(0, 0)$ in an appropriate norm that emanate from $(0, 0)$. In spaces with finite or infinite dimension, the degree theory [23] is an appropriate tool for detecting the bifurcation generically after reduction by the Lyapunov-Schmidt method. The procedure gives a local result. However, when $F(\cdot, 0)$ admits a non-zero degree, the theory can be applied without any reduction. This is the case we tackle here by looking at the dynamics of two coupled dissipative oscillators in the plane. We consider a general nonlinear interaction presumed to be just a continuous function of the relative distance between the two oscillators; it is also differentiable at 0. The classical α and β interaction schemes for the Fermi-Pasta-Ulam lattice are special cases of what we consider, so are Duffing-type and any other physically significant continuous nonlinearity that is admissible. What we obtain is the existence of periodic solutions that emerge from the trivial one. Their period matches the forcing one. This is a co-bifurcation result in the sense discussed in [24–26] and is global, based on general theorems in [27]. It does not depend on specific choices of the nonlinear coupling, continuity apart, although specific choices influence the remaining aspects of the overall behavior. To evidence this aspect, we analyze numerically the energy modulation in the system under different choices of the nonlinear coupling. We choose this aspect among others because it characterizes various physically significant models that involve nonlinear oscillators, from Fermi-Pasta-Ulam chains [28] to network-based schemes of the brain behavior [29,30] but also it can be responsible for the insurgence of chaos, the annihilation of stable and/or unstable states, abrupt transition of energy localization from one oscillator to another, the coincidence with nonlinear Landau-Zener tunneling, the emergence of solitary waves, for example in trilinear-type constitutive response of the links, etc.

The paper is structured as follows: In Sect. 2 we describe the damped non-linearly coupled forced system under analysis. Our main result is proven in Sect. 3; the nonlinear coupling considered—we repeat—is only a general (in the sense of "not otherwise specified") continuous function that is differentiable at 0. In the same Sect. 3 we also discuss a partition of a parameter space distinguishing between sets of values that assure a sufficient condition for the emergence of periodic solutions with period matching the one of the forcing and those sets of values for which we cannot declare for sure the occurrence of the same phenomenon. After deriving our main result, we restrict the generality of the nonlinear coupling to integrable and monotonically non-decreasing continuous functions; in this special case and when the nonlinearity is small in a precise sense, we provide an exact estimate of the solution (Sect. 4). In Sect. 5 report a numerical analysis of the energy transfer between the two oscillators. In the Appendix, we recall a few general properties of finite degrees of freedom damped nonlinear systems.

As for the notation used, we shall adopt an interposed dot to indicate duality pairing; however, we will refer the analyses to orthonormal frames of reference, so that the interposed dot can be identified with a scalar product. A superposed dot will indicate time derivative, as usual.

2 A class of finite degrees of freedom dynamical systems

Let $\mathbf{q} \in \mathbb{R}^n$ be a time-dependent vector collecting n degrees of freedom. We assume that the map $t \longmapsto$

$\mathbf{q} = \tilde{\mathbf{q}}(t) \in \mathbb{R}^n$ is twice differentiable. We also indicate by $\mathbf{q}^\varsigma$ the *entire history of motion*.

We consider systems governed by balance equations of the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \partial_{\mathbf{q}} V(t, \mathbf{q}) + \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}; \mathbf{q}^\varsigma) = 0\ , \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are, respectively, mass, damping, and stiffness $n \times n$ matrices with real entries. In particular, $\mathbf{M}$ and $\mathbf{K}$ are positive definite. $V(t, \mathbf{q})$ is a possibly time dependent potential. The vector $\mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}; \mathbf{q}^\varsigma)$ is a non-potential dissipative force acting in/on the system. We assume that $\mathbf{f}$ is intrinsically dissipative, that is,

$$(\mathbf{C}\dot{\mathbf{q}} + \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}; \mathbf{q}^\varsigma)) \cdot \dot{\mathbf{q}} \leq 0\ , \quad (2)$$

and the identity holds if and only if $\dot{\mathbf{q}} = 0$.

We reduce the analysis to the case $n = 2$ and consider two sub-classes of systems. For the first class we choose

$$\begin{aligned} &\mathbf{q} = \tilde{\mathbf{q}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \\ &\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \\ &\mathbf{K} = \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \partial_{\mathbf{q}} V = \begin{bmatrix} -g(y-x) \\ g(y-x) \end{bmatrix}, \\ &\mathbf{f} = \begin{bmatrix} 0 \\ -\lambda f(t) \end{bmatrix}, \end{aligned} \quad (3)$$

where the entries of matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are constant, λ is a parameter, $f(t)$ is a smooth periodic function with period T, and g is a merely continuous function that is only required to be differentiable at $y - x = 0$.

The second class is characterized by $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{f}$ as above, while $\mathbf{K}$ changes into

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad (4)$$

where the entries are still constant.

3 Vibrations with period matching the forcing one

3.1 A few details on the systems considered

Explicitly, the system associated with the choices (3) reads

$$\begin{cases} \ddot{x} = -\dfrac{(c_1 + c_2)}{m_1}\dot{x} + \dfrac{c_2}{m_1}\dot{y} - \dfrac{k_1}{m_1}x + \dfrac{1}{m_1}g(y-x) \\ \ddot{y} = -\dfrac{c_2}{m_2}(\dot{y} - \dot{x}) - \dfrac{1}{m_2}g(y-x) + \lambda f(t) \end{cases} . \quad (5)$$

We look for the existence of periodic solutions with the same period T. To this aim, we compute the topological degree of the map $(x, y) \xrightarrow{\mathbf{H}} \mathbf{h}(x, y, 0, 0)$, with respect to 0 in a neighborhood of the origin [27], where

$$\mathbf{h}(x, y, \dot{x}, \dot{y}) := \begin{bmatrix} -\dfrac{(c_1 + c_2)}{m_1}\dot{x} + \dfrac{c_2}{m_1}\dot{y} - \dfrac{k_1}{m_1}x + \dfrac{1}{m_1}g(y-x) \\ -\dfrac{c_2}{m_2}(\dot{y} - \dot{x}) - \dfrac{1}{m_2}g(y-x) \end{bmatrix}.$$

In addition, the system associated with the choices (4) reads

$$\begin{cases} \ddot{x} = -\dfrac{(c_1 + c_2)}{m_1}\dot{x} + \dfrac{c_2}{m_1}\dot{y} - \dfrac{(k_1 + k_2)}{m_1}x \\ \quad + \dfrac{k_2}{m_1}y + \dfrac{1}{m_1}g(y-x) \\ \ddot{y} = -\dfrac{c_2}{m_2}(\dot{y} - \dot{x}) - \dfrac{k_2}{m_2}y + \dfrac{k_2}{m_2}x \\ \quad - \dfrac{1}{m_2}g(y-x) + \lambda f(t) \end{cases} . \quad (6)$$

Its analysis requires modifications associated with the different structure of the pertinent map with values $\mathbf{h}(x, y, \dot{x}, \dot{y})$, as it will be specified in the proof.

Elementary structures whose dynamics is described by the systems (5) and (6) are those in Figs. 1 and 2, chosen only for visualizing at least simple natural systems.

3.2 Basics on topological degree theory

A detailed exposition of the degree theory can be found in the treatises [23,31,32]. Here, we just recall basic notions.

Put simply, let z_0 be a regular value of a mapping $\ell \colon \mathbb{R}^n \to \mathbb{R}^n$ and U an open bounded subset of $\mathbb{R}^n$ such that $\ell^{-1}(z_0) \cap \partial U = \emptyset$. The degree of ℓ in U with respect to z_0, namely $\deg(\ell, U, z_0)$, is defined by

$$\deg(\ell, U, z_0) = \sum_{x_i \in \ell^{-1}(z_0) \cap U} \operatorname{sign} \det \left(\ell'(x_i)\right)\ ; \quad (7)$$

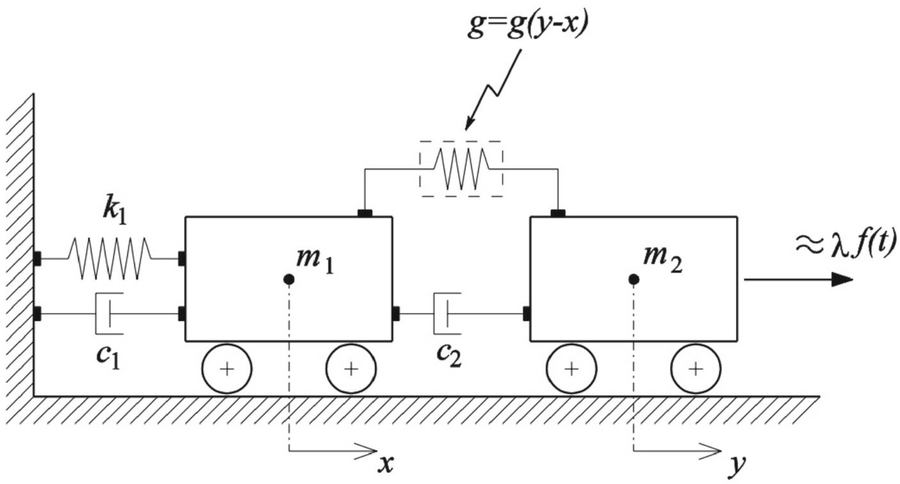

Fig. 1 Schematic representation of a system falling within the first sub-class considered

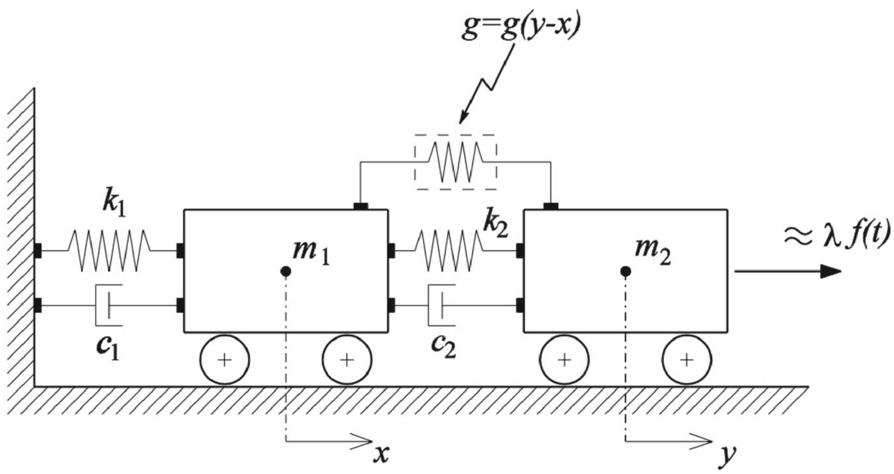

Fig. 2 Schematic representation of a system falling within the second sub-class considered

it is the sum of signs of the Jacobian taken over all points of the inverse image of z_0 that are contained in U. The degree of ℓ can be also defined, on the basis of approximations, when z_0 is not necessarily a regular value and U may be unbounded, a case in which $\ell^{-1}(z_0)$ is assumed to be compact (an extensive account of the notion of degree can be found in [31] where, in addition, other approaches are described). The general definition goes beyond the scope of this paper, so that we restrict our attention to regular values. We only mention that, when the mapping takes values in $\mathbb{R}^n$, as we consider here, it can be seen as a tangent vector field over U. Thus, taking the "target" point z_0 as the origin of a basis in $\mathbb{R}^n$, the degree $\deg(\ell, U, 0)$ coincides with the analogous notion (also called a *characteristic of rotation*) for the tangent vector field ℓ over U, and we will indicate it simply as $\deg(\ell, U)$, leaving understood $z_0 = 0$. This last way of considering the degree of a map can be introduced axiomatically as in [33].

Assume that $g(s) = 0$ if and only if $s = 0$, so that we have $\mathbf{H}(x, y) = \mathbf{h}(x, y, 0, 0) = 0$ only for $(x, y) = (0, 0)$. To compute the topological degree of $\mathbf{H}$, we look at its linear part at $(0, 0)$, that is,

$$D\mathbf{H}(0,0) = \begin{bmatrix} -\dfrac{k_1}{m_1} - \dfrac{1}{m_1} g'(0) & \dfrac{1}{m_1} g'(0) \\ \dfrac{1}{m_2} g'(0) & -\dfrac{1}{m_2} g'(0) \end{bmatrix},$$

where $D\mathbf{H}$ is the derivative of $\mathbf{H}$ and a prime indicates the one with respect to $y - x$. Since

$$\det(D\mathbf{H}(0,0)) = \frac{k_1}{m_1 m_2} g'(0),$$

if $g'(0) \neq 0$, the topological degree of $\mathbf{H}$ is nonzero in any neighborhood U of $(0, 0)$; in particular $\deg(\mathbf{H}) \neq 0$.

In these conditions, Furi and Spadini's structure theorem on the set of T-periodic solutions to T-forced ordinary differential equations [27, Th. 4.2] assures, informally speaking, that there exists a branch (that is, an unbounded connected set) of nontrivial solutions that emanates in some sense from the origin. A precise description of this concept stems from Definition 3.1 below and the following discussion. We point out that this result alone does not yield periodic solutions for small values of $\lambda > 0$; in fact the branch could be "vertical", namely, constituted by solutions that correspond to $\lambda = 0$. In order to exclude that this is the case, one needs some type of "local" conditions that we provide exactly.

3.3 The main result

Definition 3.1 With $C^1_T(\mathbb{R}^n)$ being the Banach space of $\mathbb{R}^n$-valued T-periodic C^1 functions, an element $(\lambda, \tilde{\mathbf{q}}) \in [0, \infty) \times C^1_T(\mathbb{R}^n)$ is a T-forced pair for (1) if $t \mapsto \mathbf{q} := \tilde{\mathbf{q}}(t)$ is a T-periodic solution of (1) that correspond to λ.

Definition 3.2 A T-forced pair $(\lambda, \tilde{\mathbf{q}})$ is called trivial when $\lambda = 0$ and $\tilde{\mathbf{q}}$ is constant.

Theorem 3.3 *When damping is present, both systems (5) and (6) admit solutions with period matching the forcing one for sufficiently small $\lambda > 0$. The same is true also in the case when damping is removed, provided that the forcing frequency is sufficiently high or the pertinent conditions (8) or (10) are satisfied, respectively.*

Proof For system (5), the unique trivial T-forced pair is $(\lambda, \mathbf{q}) = (\lambda, (x, y)) = (0, \mathbf{0})$, where $\mathbf{0}$ denotes the identically null function, namely $\tilde{\mathbf{q}} = 0$.

In our particular setting, since $\deg(\mathbf{H}) \neq 0$, Theorem 4.2 of [27] implies that there exists an unbounded connected set Γ of nontrivial T-forced pairs of (5) whose closure in $[0, \infty) \times C^1_T(\mathbb{R}^2)$ contains $(0, \mathbf{0})$.

In order to deduce from this result that there are T-periodic solutions of (5) for any sufficiently small $\lambda > 0$, it is enough to show that Γ cannot be entirely contained in the 'slice' $\{\lambda = 0\} \times C^1_T(\mathbb{R}^2)$.

In the presence of damping (at least when one between c_1 and c_2 does not vanish and is, of course, positive), we claim that actually $\Gamma \cap \big(\{0\} \times C^1_T(\mathbb{R}^2)\big) = \emptyset$. In other words, $\{0\} \times C^1_T(\mathbb{R}^2)$ contains only the trivial T-forced pair $(0, \mathbf{0})$. We should expect this from the dissipative nature of the process because for $\lambda = 0$ we have no external source that is able to restore the energy dissipated by the damping. However, to check formally the claim, consider the function

$$\Phi(\xi_1, \xi_2, \eta_1, \eta_2) = \frac{1}{2m_2}\dot{\xi}_2^2 + \frac{1}{2m_1}\dot{\eta}_2^2 + \frac{k_1}{2m_1 m_2}\xi_1^2 + \frac{1}{m_1 m_2}G(\eta_1 - \xi_1),$$

where $G(\alpha) = \int_0^\alpha g(t)\,\mathrm{d}t$. Assume by contradiction that there exists a non-constant T-periodic solution $t \mapsto \big(x(t), y(t)\big)$ of system (5) for $\lambda = 0$. By omitting the explicit dependence on t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(x, \dot{x}, y, \dot{y}) = -\frac{(c_1 + c_2)\dot{x}^2 - 2c_1 c_2 \dot{x}\dot{y} + c_2\dot{y}^2}{m_1 m_2} \leq 0,$$

where the inequality is strict when at least one of $\dot{x}$ or $\dot{y}$ is different from zero.

Since $\big(x(t), y(t)\big)$ is non-constant, there exists $\bar{t} \in [0, T]$ such that $\dot{x}(\bar{t})^2 + \dot{y}(\bar{t})^2 \neq 0$. By continuity, this relation holds in a neighborhood of $\bar{t}$. So, we conclude that

$$\Phi\big(x(T), \dot{x}(T), y(T), \dot{y}(T)\big) - \Phi\big(x(0), \dot{x}(0), y(0), \dot{y}(0)\big) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\Phi\big(x(t), \dot{x}(t), y(t), \dot{y}(t)\big)\,\mathrm{d}t < 0,$$

contradicting the periodicity and proving the claim.

In the absence of damping (that is, when $c_1 = c_2 = 0$), the situation is less straightforward. A sufficient circumstance to exclude that Γ intersects any sufficiently small neighborhood of $(0, \mathbf{0})$ in $\{0\} \times C^1_T(\mathbb{R}^2)$ is the non-T-resonance condition; a 'local' concept dating back to Poincaré's work (compare [34]; see also [35, Ch. 7] and [36]) that in practice boils down to analyzing the spectrum of the linear component of the system at isolated zeros.

For system (5), a result in [37, Section 4] prescribes that the non-T-resonance condition holds at $(0, 0)$ if, for all $\mu \in \mathbb{Z}$, we have

$$-\left(\frac{2\mu\pi}{T}\right)^2 \notin \operatorname{spectrum}(D\mathbf{H}(0, 0)) , \tag{8}$$

that is

$$\left(\frac{2\mu\pi}{T}\right)^2 \notin \left\{\pm\frac{\sqrt{a_1} \mp a_2}{2m_1m_2}\right\},$$

where $a_1 := m_2^2(k_1 + g'(0))^2 + 2m_1m_2g'(0)(g'(0) - k_1) + m_1^2g'(0)^2$ and $a_2 = m_2(k_1 + g'(0)) + m_1g'(0)$. Thus, if the above condition holds for all $\mu \in \mathbb{Z}$, Γ cannot be contained in $\{0\} \times C_T^1(\mathbb{R}^2)$.

More interesting is the undamped case, with reference to the condition (8). Figure 3 shows in the space $(g'(0), T, k_1)$ the surfaces where the condition of T resonance is verified for $\mu = 1, \ldots, 6$. For all points over that surfaces we cannot guarantee the existence of periodic solutions. We excluded the case $\mu = 0$ because it corresponds to $k_1 = 0$, which we want to discard.

Moreover, the graphs represented in Fig. 3 refer to $g'(0) > 0$, specifically $g'(0) < 5$ for graphical reasons, and $1/2 < T < 3\pi$, as a possible choice. The surfaces emerge from the formula

$$k_1 = \frac{\mu^2T^2(\mu^2T^2 - 8\,g'(0))}{4\pi^2(\mu^2T^2 - 4\,g'(0))},$$

when either $\mu^2T^2 > 8g'(0)$ or $\mu^2T^2 < 4g'(0)$. The expression follows from (8), solving with respect to k_1 and imposing $k_1 > 0$. All triples of parameters (so, all points) not belonging to the surfaces represent conditions that are sufficient for the emergence of periodic solutions.

Finally, an inequality proven by Yorke [38] implies that, when the frequency of the load f in system (5) is sufficiently high, Γ cannot be contained in $\{0\} \times C_T^1(\mathbb{R}^2)$. Specifically, in [38] Yorke proved that for $t \longmapsto \mathbf{q}(t) \in \mathbb{R}^n$ a T-periodic solution to the system $\dot{\mathbf{q}} = \Psi(\mathbf{q})$, with $\Psi : \Omega \longrightarrow \mathbb{R}^n$ a Lipschitz map with constant L with respect to the Euclidean norm, and $\Omega \subset \mathbb{R}^n$, we have $T \geq 2\pi/\mathsf{L}$, and the estimate cannot be improved. In our case, although the right-hand side of system (5), written in a first order normal form, may not be globally Lipschitz, it is locally so. Thus, Γ cannot intersect small neighborhoods of $(0, \mathbf{0})$ in $\{0\} \times C_T^1(\mathbb{R}^2)$ when the period is short enough, that is, when the frequency is sufficiently high.

We consider the Lipschitz constant of the vector field in the undamped case to be greater than the operatorial norm of the linearized system at 0. Let $\hat{\mathbf{B}}$ such a matrix. It is given by

$$\hat{\mathbf{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1 + g'(0)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ g'(0) & 0 & -g'(0) & 0 \end{bmatrix}.$$

For simplicity, we take here $m_1 = m_2 = 1$. The operatorial norm we consider is the highest eigenvalue of the matrix $\hat{\mathbf{B}}\hat{\mathbf{B}}^\top$, whose eigenvalues are 1 with multiplicity 2,

$$-\frac{\sqrt{k_1^4 + 4\,g'(0)k_1^3 + 6\,g'(0)^2k_1^2 + 4\,g'(0)^3k_1 + 5\,g'(0)^4} - k_1^2 - 2\,g'(0)k_1 - 3\,g'(0)^2}{2}$$

and

$$\frac{\sqrt{k_1^4 + 4\,g'(0)k_1^3 + 6\,g'(0)^2k_1^2 + 4\,g'(0)^3k_1 + 5\,g'(0)^4} + k_1^2 + 2\,g'(0)k_1 + 3\,g'(0)^2}{2}.$$

Let us indicate by $l(k_1, g'(0))$ the highest eigenvalue. Yorke's inequality tells us that, if the period T of the perturbation is lesser that $2\pi/\sqrt{l(k_1, g'(0))}$, there aren't T-periodic solutions for $\lambda = 0$. Thus, by Theorem 4.2 in [35], there must be T-periodic solutions for $\lambda > 0$ that is sufficiently small.

Figure 4 shows the surface $T = 2\pi/\sqrt{l(k_1, g'(0))}$.

Once the pair $(g'(0), k_1)$ is chosen, if T is below the surface in Fig. 4, the undamped system will have T-periodic solutions for small $\lambda > 0$.

Although the existence of a branch of periodic solutions is granted by the previous arguments, the form of this branch depends on the forcing. Also, when the forcing is a live load, that is, it depends on the motion, we can perform the same type of analysis with similar results. An example thus as follows. Take $m_1 = m_2 = 1$, $g(s) = s/4$, $k_1 = 1$ e $T = 2\pi$ in the undamped case. The system of balance equations thus reduces to

$$\begin{cases} \ddot{x} = -x + \dfrac{y - x}{4}, \\ \ddot{y} = -\dfrac{y - x}{4} + \lambda(\sin t - 2x) \end{cases}. \tag{9}$$

The previous results apply. In particular, we can compute explicitly both the norm of the solution and

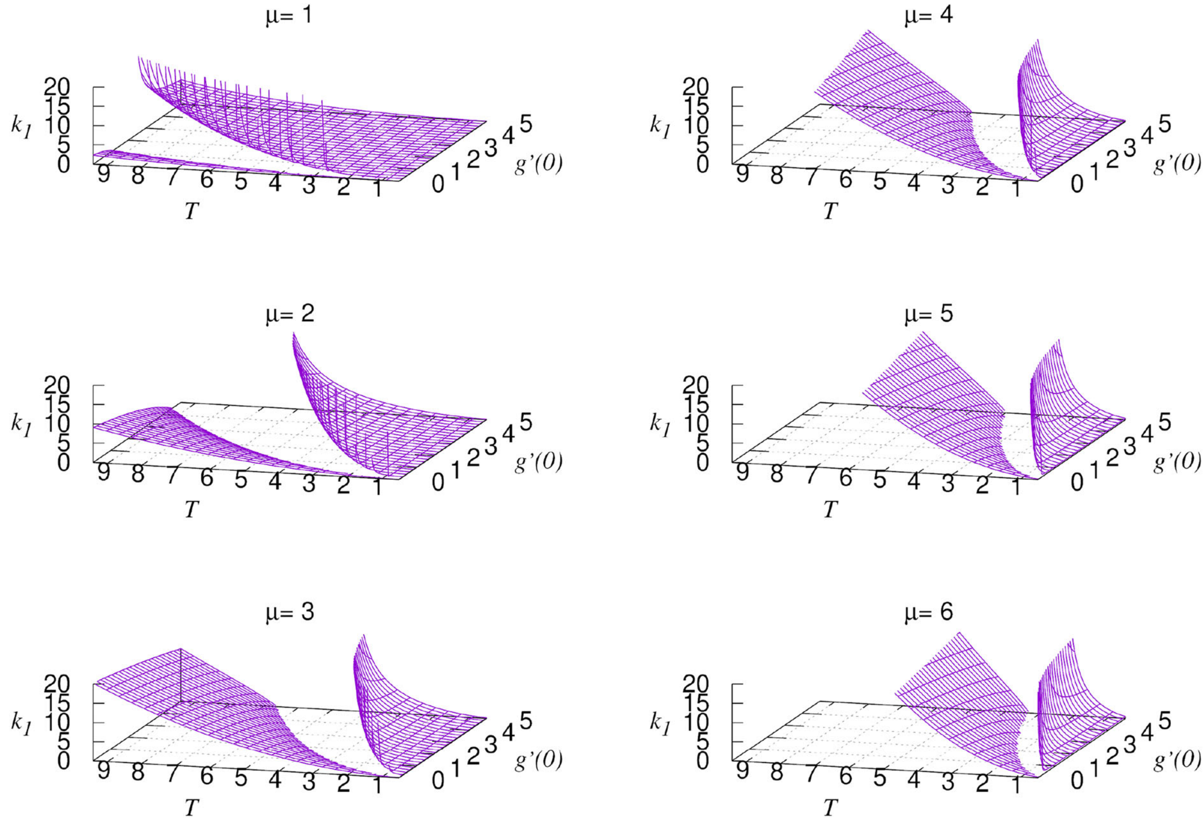

Fig. 3 Surfaces in the space $(g'(0), T, k_1)$ where the condition of T resonance is verified for $\mu = 1, \ldots, 6$

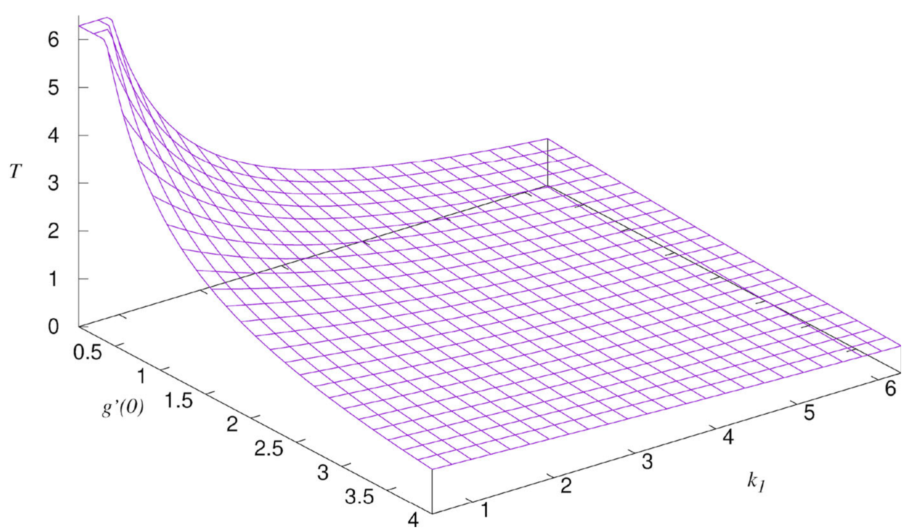

Fig. 4 The condition $T = 2\pi/\sqrt{l(k_1, g'(0))}$

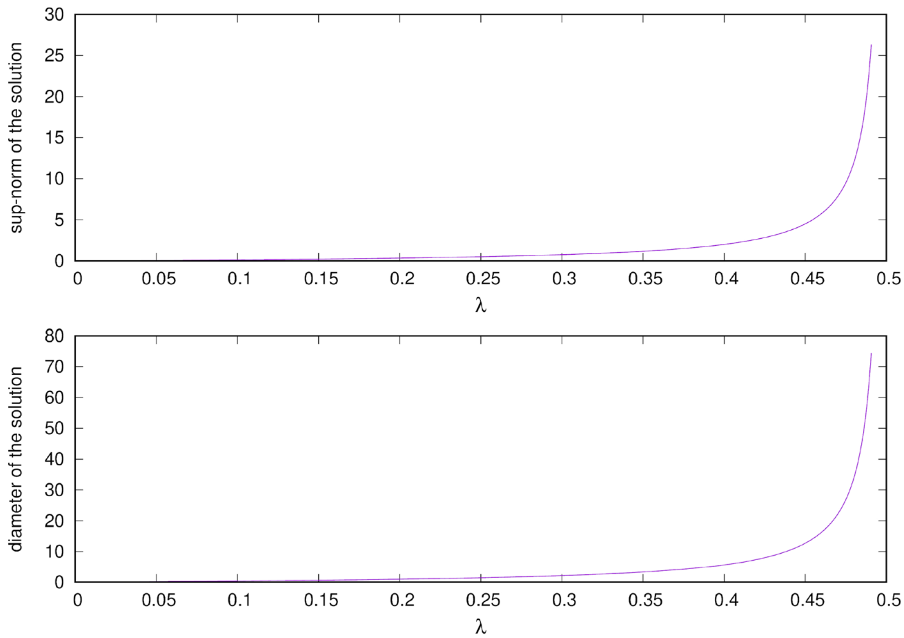

Fig. 5 Norm and radius of the periodic solutions of system (9), with period matching the one of the forcing, as λ varies

its radius, when λ varies. Figure 5 shows the result. We realize the presence of an asymptote: the branch of periodic solutions does not go beyond $\lambda = \frac{1}{2}$.

Consider now the system associated with the choices (4). By retracing the same path followed above, with reference to system (6), $\mathbf{H}$ changes now into

$$\mathbf{H}(x, y) = \begin{bmatrix} -\dfrac{(k_1 + k_2)}{m_1}x + \dfrac{k_2}{m_1}y + \dfrac{1}{m_1}g(y - x) \\ -\dfrac{k_2}{m_2}y + \dfrac{k_2}{m_2}x - \dfrac{1}{m_2}g(y - x) \end{bmatrix}.$$

In agreement with the assumptions on g adopted in the previous section, we assume $g(s) \neq -k_2 s$ for $s = y - x \neq 0$ and $g(0) = 0$. Thus, $(0, 0)$ is the unique zero of $\mathbf{H}$, as can be checked by imposing $\mathbf{H}(x, y) = 0$. Also, we have

$$D\mathbf{H}(0, 0) = \begin{bmatrix} -\dfrac{k_1 + k_2 + g'(0)}{m_1} & \dfrac{k_2 + g'(0)}{m_1} \\ \dfrac{k_2 + g'(0)}{m_2} & -\dfrac{k_2 + g'(0)}{m_2} \end{bmatrix}.$$

Its determinant, namely

$$\det(D\mathbf{H}(0, 0)) = \frac{k_1(k_2 + g'(0))}{m_1 m_2},$$

is different from zero when $g'(0) \neq -k_2$. Thus, as in the previous case, by [27, Th. 4.2] there exists an unbounded connected set Υ of nontrivial T-forced pairs for (6) whose closure in $[0, \infty) \times C^1_T(\mathbb{R}^2)$ contains $(0, \mathbf{0})$.

In the presence of damping, an argument similar to the one used for system (5) shows that $\Upsilon \cap \big(\{0\} \times C^1_T(\mathbb{R}^2)\big) = \emptyset$. Also, again as in the case of system (5), we see that the branch Υ is not entirely contained in $\{0\} \times C^1_T(\mathbb{R}^2)$ when

$$-\left(\frac{2\mu\pi}{T}\right)^2 \notin \operatorname{spectrum}(D\mathbf{H}(0, 0)), \tag{10}$$

for all $\mu \in \mathbb{Z}$ or, referring to Yorke's inequality, when the forcing frequency is sufficiently high. □

Notice that the same result can be derived when we consider a chain of oscillators, provided that the proof is appropriately adapted. An analysis along this direction is in [39] with reference to a generalization of the Fermi-Pasta-Ulam chain, so in the absence of damping.

Remark 3.4 Another way to explore the existence of (non-constant) periodic solutions of a dynamical system is singularity theory. In general, the qualitative analysis of a dynamical system

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = v(\gamma(t)),$$

with γ a 'point' on a manifold $\mathcal{M}$, refers to the global geometry of the space of flows of v. Two flows are regarded as equivalent if there is an homeomorphism of $\mathcal{M}$, which takes one flow into the other. The vector field can have singular points that are those where v vanishes; nature and type of these singularities are restricted by the topological properties of $\mathcal{M}$. Under perturbations these singularities can change into new ones (in this case they are called as *non-degenerate*) or not (and they are referred to as *degenerate*). The perturbation of a system with one or more degenerate singularity points is structurally unstable—it means that small perturbations induce large qualitative changes in the flow. Refined information can be obtained in the potential case, that is, when $v(\gamma(t)) = \nabla f(\gamma(t))$. The function f can be real or complex valued. In the latter case, the analysis of singular points involves factor algebras of power series algebras over the field of complex numbers [40]. We can even consider f to take values on a more generic manifold, that is, $f : \mathcal{M} \longrightarrow \mathcal{N}$, a differentiable map. If $\mathcal{N}$ is a real finite-dimensional manifold, the theory of Stiefel-Withney characteristic classes allows one to analyze singularities, while in the complex case the Chern classes need to be called upon. When $\mathcal{N}$ is infinite-dimensional, the Morse theory of critical points is required. The body of results in this landscape (an elegant survey is in the treatises [41,42]) can be profitably applied to dynamical systems in the search of periodic solutions to analyze how the structure of solutions changes near a singular point in the parameters of the system considered. This approach is local and focuses on the behavior at singular points. In contrast, the topological degree theory is a global view that allow us, besides existence, to count solutions across a space. Singularity theory allows us to determine how changes of the singularities (above all their sign) affect the possibility of periodic solutions. From its side, the topological degree approach essentially changes the search of a periodic solution into a fixed-point problem, where the topological degree can be computed. Moreover, and perhaps above all, with the approach that we have adopted here we are also allowed to account for fixed and distributed delays in the external forcing as discussed in more general situations in [39,43]. True, the two approaches can have a companion use. However, we do not deepen further the issue, which is outside the main focus of the present paper.

Remark 3.5 The stability of the found solutions depends on the choice of the function g and the forcing. Here we do not tackle the issue. We just mention that in terms of topological degree theory, little is done in general. A crucial result is due to Ortega and refers to a system of the type

$$\dot{\mathbf{w}} = \mathbf{f}(t, \mathbf{w})$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function that is periodic in t of period T and has continuous partial derivative with respect to $\mathbf{w}$. Write $\gamma(t, \hat{p})$ for its (possible) solution with condition $\gamma(0, \hat{p}) = \hat{p}$, $\hat{p} \in \mathbb{R}^n$, and consider the Poincaré-Andronov operator $\mathcal{P}\hat{p} = \gamma(T, \hat{p})$. Assume that there exists a constant symmetric $n \times n$ matrix $\mathbf{P}$ with exactly one negative eigenvalue and $n-1$ positive eigenvalues, and there are constants $\lambda, \epsilon > 0$ such that, with $\mathbf{f_w}$ the derivative of $\mathbf{f}$ with respect to $\mathbf{w}$,

$$\mathbf{P}\mathbf{f_w}(t, \mathbf{w}) + \mathbf{f_w}(t, \mathbf{w})^\top \mathbf{P} + 2\lambda \mathbf{P} \leq -2\epsilon \mathbf{I}_n \,, \quad \forall (t, \mathbf{f}) \in \mathbb{R} \times \mathbb{R}^n \,,$$

where the superscript $\top$ means standard transposition, $\mathbf{I}_n$ is the identity in the space of $n \times n$ matrices, and the inequality is intended in the standard sense of square matrices. Assume also that there exists a constant symmetric matrix $\hat{\mathbf{A}}$ and two constants $\hat{a}$ and $\hat{b}$ such that

$$\mathbf{w}^\top \hat{\mathbf{A}} \mathbf{f}(t, \mathbf{w}) \leq \hat{a} |\mathbf{w}|^2 + \hat{b} \,, \qquad \forall (t, \mathbf{w}) \in \mathbb{R} \times \mathbb{R}^n \,.$$

Assume also that there exists a bounded domain $\Omega \subset \mathbb{R}^n$ in which $\deg(\mathbf{I} - \mathcal{P}, \Omega, 0) = \mathsf{d} > 0$, where $\mathbf{I}$ is the identity in $\mathbb{R}^n$ and deg is the Brouwer degree. Under these assumptions, there exist at least d stable T-periodic solutions of the above dynamical system. The proof is in [44]; it exploits the Schmidt's notion of amenable solution, which could allow us to reduce the question of the stability of a periodic solution to the stability of fixed points of certain discrete dynamical systems which are one dimensional. The evaluation of asymptotic stability may require recourse to the Leray-Schauder degree theory.

4 When the nonlinearity is small: an asymptotic estimate for the free vibrations of System (6)

Only in this section we restrict the generality of the nonlinear coupling in the two oscillators under analysis;

we consider that it satisfies some smallness in a sense defined below and, instead being only represented by a general continuous function g that is differentiable at 0, as assumed in the previous section, we presume also that g is monotonically increasing.

Consider the system (6). Look for solutions of the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + o(\varepsilon) \ ,$$
$$y(t) = y_0(t) + \varepsilon y_1(t) + o(\varepsilon)$$

and assume that g is of order ε; it means that we substitute in system (6) $g(y-x)$ with $\varepsilon g(y-x)$. We also take initial conditions of the form

$$x_0(0) = x(0), \quad \dot{x}_0(0) = \dot{x}(0),$$
$$y_0(0) = y(0), \quad \dot{y}_0(0) = \dot{y}(0)$$

and

$$x_1(0) = \dot{x}_1(0) = y_1(0) = \dot{y}_1(0) = 0 \ .$$

Set also $m_1 = m_2 = 1$ for the sake of simplicity. In what follows nothing will change for different values of the masses.
Set

$$\mathbf{w}_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ y_0 \\ \dot{y}_0 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ y_1 \\ \dot{y}_1 \end{bmatrix} \quad \text{and}$$
$$\mathbf{w} = \mathbf{w}_0 + \varepsilon \mathbf{w}_1 + o(\varepsilon).$$

Proposition 4.1 *When in* (6) *the function* g *is integrable and monotonously non-decreasing, the following estimate holds as* $t \longrightarrow 0$*:*

$$\|\dot{\mathbf{w}}(t)\| \le \varepsilon \frac{2C}{r^*} g(2\|\mathbf{w}_0(0)\|) + o(\varepsilon) \ , \tag{11}$$

where C *is a constant and* r^* *is the highest real part of eigenvalues of the matrix*

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2) & -(c_1+c_2) & k_2 & c_2 \\ 0 & 0 & 0 & 1 \\ k_2 & c_2 & -k_2 & -c_2 \end{bmatrix}.$$

Proof To within $o(\varepsilon^2)$ terms, system (6), with $f = 0$, reduces to

$$\begin{cases} \ddot{x}_0 = -(c_1+c_2)\dot{x}_0 + c_2\dot{y}_0 - (k_1+k_2)x_0 + k_2 y_0 \\ \ddot{x}_1 = -(c_1+c_2)\dot{x}_1 + c_2\dot{y}_1 - (k_1 \\ \quad +k_2)x_1 + k_2 y_1 + g(y_0 - x_0) \\ \ddot{y}_0 = -c_2\dot{y}_0 + c_2\dot{x}_0 - k_2 y_0 + k_2 x_0 \\ \ddot{y}_1 = -c_2\dot{y}_1 + c_2\dot{x}_1 - k_2 y_1 + k_2 x_1 - g(y_0 - x_0) \end{cases} . \tag{12}$$

Consider the first and the third equations. They constitute a linear system that we rewrite as

$$\begin{cases} \dot{x}_0 = x_{01} \\ \dot{x}_{01} = -(c_1+c_2)x_{01} + c_2 y_{01} - (k_1+k_2)x_0 + k_2 y_0 \\ \dot{y}_0 = y_{01} \\ \dot{y}_{01} = -c_2 y_{01} + c_2 x_{01} - k_2 y_0 + k_2 x_0 \end{cases} . \tag{13}$$

In terms of $\mathbf{w}_0$, system (13) becomes $\dot{\mathbf{w}}_0 = \mathbf{A}\mathbf{w}_0$, with $\mathbf{A}$ given in the proposition above, so that

$$\mathbf{w}_0(t) = \exp(t\mathbf{A})\mathbf{w}_0(0) \ . \tag{14}$$

Consider second and fourth equations in system (12). They can be rewritten as

$$\begin{cases} \dot{x}_1 = x_{11} \\ \dot{x}_{11} = -(c_1+c_2)x_{11} + c_2 y_{11} \\ \qquad -(k_1+k_2)x_1 + k_2 y_1 + g(y_0 - x_0) \\ \dot{y}_1 = y_{11} \\ \dot{y}_{11} = -c_2 y_{11} + c_2 x_{11} - k_2 y_1 + k_2 x_1 - g(y_0 - x_0) \end{cases} . \tag{15}$$

By setting

$$\mathbf{w}_1 = \begin{bmatrix} x_1 \\ x_{11} \\ y_1 \\ y_{11} \end{bmatrix} \quad \text{and} \quad \mathbf{G}(\mathbf{w}_0) = \begin{bmatrix} 0 \\ g(y_0 - x_0) \\ 0 \\ -g(y_0 - x_0) \end{bmatrix},$$

system (15) becomes $\dot{\mathbf{w}}_1 = \mathbf{A}\mathbf{w}_1 + \mathbf{G}(\mathbf{w}_0)$ with initial condition $\mathbf{w}_1(0) = 0$. Thus, we get

$$\mathbf{w}_1(t) = \exp(t\mathbf{A}) \int_0^t \exp(-\hat{s}\mathbf{A})\mathbf{G}(y_0(\hat{s}) - x_0(\hat{s}))\, \mathrm{d}\hat{s} \ . \tag{16}$$

Routh-Hurwiz's criterion [45] reveals that $\mathbf{A}$ is a stable matrix in the sense that all its eigenvalues have negative real part. We indicate by α^* the eigenvalue of $\mathbf{A}$ with the highest nonzero real part $r^* := \mathrm{Re}(\alpha^*) < 0$.

Consider a norm $\|\bullet\|$ that extracts the highest component of its real vector argument. Since g has been taken to be monotonously non-decreasing, we have

$$\|\mathbf{G}(y_0(\hat{s}) - x_0(\hat{s}))\| \leq 2|g(y_0(\hat{s}) - x_0(\hat{s}))| \\ \leq 2\, g(|y_0(\hat{s}) - x_0(\hat{s})|) \ .$$

Thus, by exploiting a classical result (see, e.g., [46, Th. 2.10]), we can say that there exists a constant C such that,

$$\begin{aligned} \|\mathbf{w}_1(t)\| &\leq \left\| \int_0^t \exp((t-\hat{s})\mathbf{A})\mathbf{G}(y_0(\hat{s}) - x_0(\hat{s}))\,\mathrm{d}\hat{s} \right\| \\ &\leq 2\int_0^t \|\exp((t-\hat{s})\mathbf{A})\| g(|y_0(\hat{s}) - x_0(\hat{s})|)\,\mathrm{d}\hat{s} \\ &\leq 2C\int_0^t \exp((t-\hat{s})r^*) g(|y_0(\hat{s}) - x_0(\hat{s})|)\,\mathrm{d}\hat{s} \\ &\leq 2C\int_0^t \exp((t-\hat{s})r^*) g(\max_{\hat{\tau}\in[0,\hat{s}]} |y_0(\hat{\tau})| \\ &\quad + \max_{\hat{\tau}\in[0,\hat{s}]} |x_0(\hat{\tau})|)\,\mathrm{d}\hat{s} \\ &\leq 2C\exp(tr^*)\int_0^t \exp((-\hat{s}r^*) g(2\|\mathbf{w}_0(0)\|)\,\mathrm{d}\hat{s} \\ &\leq 2C\exp(tr^*) g(2\|\mathbf{w}_0(0)\|) \left[-\frac{\exp((-\hat{s}r^*)}{r^*} \right]_0^t \\ &= \frac{2C}{r^*}(1-\exp(tr^*)) g(2\|\mathbf{w}_0(0)\|) \ . \end{aligned} \tag{17}$$

Thus, since $\mathbf{w} = \mathbf{w}_0 + \varepsilon\mathbf{w}_1 + o(\varepsilon)$, we have

$$\|\mathbf{w}(t)\| \leq \exp(tr^*)\|\mathbf{w}_0(0)\| \\ +\varepsilon\frac{2C}{r^*}(1-\exp(tr^*)) g(2\|\mathbf{w}_0(0)\|) + o(\varepsilon). \tag{18}$$

In full evidence, in a neighborhood of 0, $g(2\|\mathbf{w}_0(0)\|)$ becomes a bound for the velocity. □

5 Influence of non-linearity on the modulation of energy in Systems (5) and (6)

We thus return to full nonlinearity. First notice that the result in Theorem 3.3 does not depend on specific choices of the nonlinear coupling, which have, however, prominent influence on other aspects of the motion as shown below by evaluating numerically the spectra of systems (5) and (6). Before going into details, in summary we can conclude that the synchronization between periods of solution and forcing is stable, at least in the class of dynamical systems considered, with respect to specific choices of the nonlinearity, their prominent influence on the overall dynamics notwithstanding. The numerical examples below evidence this influence.

We adopt the following three specific choices for the nonlinear term g.

- The first option is a nonlinear elastic force (NEL). With $s := y - x$, we take

  $$g(s) = k_g s + a_1 s^2 + a_2 s^3 \ ,$$

  which is one of the nonlinear structures foreseen in the Fermi-Pasta-Ulam-Tsingou lattices. Such a form of g includes a nonzero value $\bar{g} = g(\bar{s})$ such that $g'(\bar{s}) = 0$, and $g''(\bar{s}) = 0$, where the prime indicates derivative with respect to s. Consequently, we have

  $$\bar{s} = \frac{3\bar{g}}{k_g} \ , \quad a_1 = \frac{3\bar{g} - 2k_g\bar{s}}{\bar{s}^2} \ , \quad a_2 = \frac{-2\bar{g} + k_g\bar{s}}{\bar{s}^3} \ .$$

  Analyses will be parameterized by $\bar{g}$.
- The second option deals with a simplified symmetric shape-memory behavior (SMA), also called pseudoelastic. With f_y the first threshold for the phase transition, the characteristic values f_{y1} and f_{y2} of the hysteresis cycles have been set equal to $f_{y1} = 0.8 f_y$ and $f_{y2} = 0.6 f_y$. The elastic coefficient after the first phase transition is $k_{gp} = k_g/20$.
- The third option is a piecewise linear elastic–plastic behavior with hardening (EPL). The first threshold for the phase transition has been indicated once again by f_y. The hardening stiffness is $k_{gp} = k_g/20$.

A schematic picture of the three choices is in Fig. 6.

Table 1 lists the values of characteristic parameters in the three previous cases used in the simulations.

Values of the linear constitutive parameters used in the computations are in Tables 2 and 3.

With these values, the natural angular frequencies are $\omega_1 = 7.82$rad/s and $\omega_2 = 15.37$rad/s for both systems. The principal damping ratios are $\xi_1 = 5.3\%$ and $\xi_2 = 5.4\%$. Initial conditions are null.

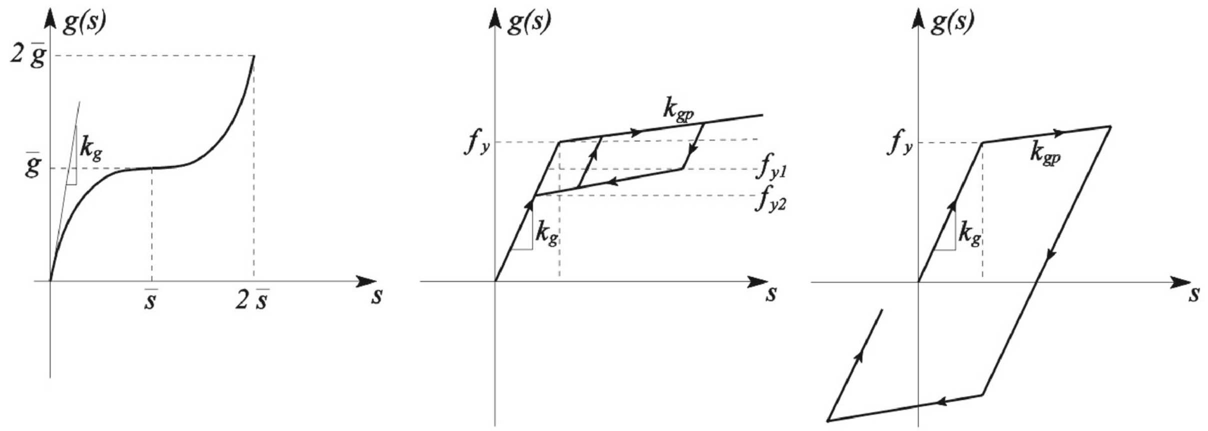

Fig. 6 Schematic representation of the three choices adopted for g

Table 1 Characteristic parameters of g in the three cases considered

NEL: $\bar{g}$(kN)	SMA: f_y(kN)	EPL: f_y(kN)
20	20	20
15	15	15
10	10	10
5	5	5

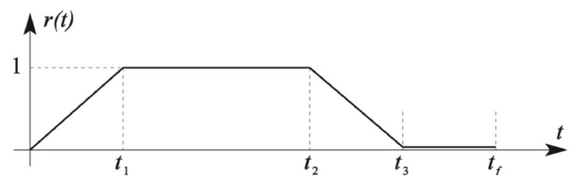

Fig. 7 The ramp function: $t_1 = 5s, t_2 = 15s, t_3 = 20s, t_f = 25s$

Table 2 Masses and damping coefficients

–	m_1(kNs^2/m)	m_2(kNs^2/m)	c_1(kNs/m)	c_2(kNs/m)
System 1	20	5	25	5
System 2	20	5	25	5

Table 3 Elastic coefficients

–	k_1(kN/m)	k_2(kN/m)	k_g(kN/m)
System 1	1700	0	850
System 2	1700	425	425

The external force is

$$\lambda f(t) = r(t) F_0 \sin(\omega_f t) \ , \tag{19}$$

where $F_0 = 25$kN in all simulations; ω_f has been considered in the interval $[1, 25]$rad/s with a step of 1rad/s, and $r(t)$ is the value of a ramp function depicted in Fig. 7, together with its characteristic instants.

For simulations we used an in-house code based on a unconditionally stable Newmark scheme with parameters $\bar{\alpha} = 0.5$ and $\bar{\delta} = 0.25$, and time step $\delta t = 0.002$s, for the time integration. Newton method has been used step-wise for calculating the tangential stiffness.

Figure 8 shows the energy of the elastic link connecting the oscillator with mass m_1 in Systems 1 and 2 with the environment, normalized by the elastic energy of the same element, stored under static conditions, versus the angular frequency of the loading. As expected, when g has an elastic–plastic character we record the least energy transfer between the two oscillators.

Figure 9 shows energy maps of the same elastic link in Fig. 8, normalized with respect to the energy of the same link when the system is purely linear elastic. In the case of System 2, the maps evidence a role of the linear elastic link between the two oscillator as amplifier of the nonlinear link in the energy transmission. Specifically, as shown in the shape-memory (SMA) case, after completing the hysteresis cycle, there is an increment of the energy transfer emphasized by the linear elastic link; such an emphasis is more pronounced in the nonlinear elastic (NEL) case after the plateau associated

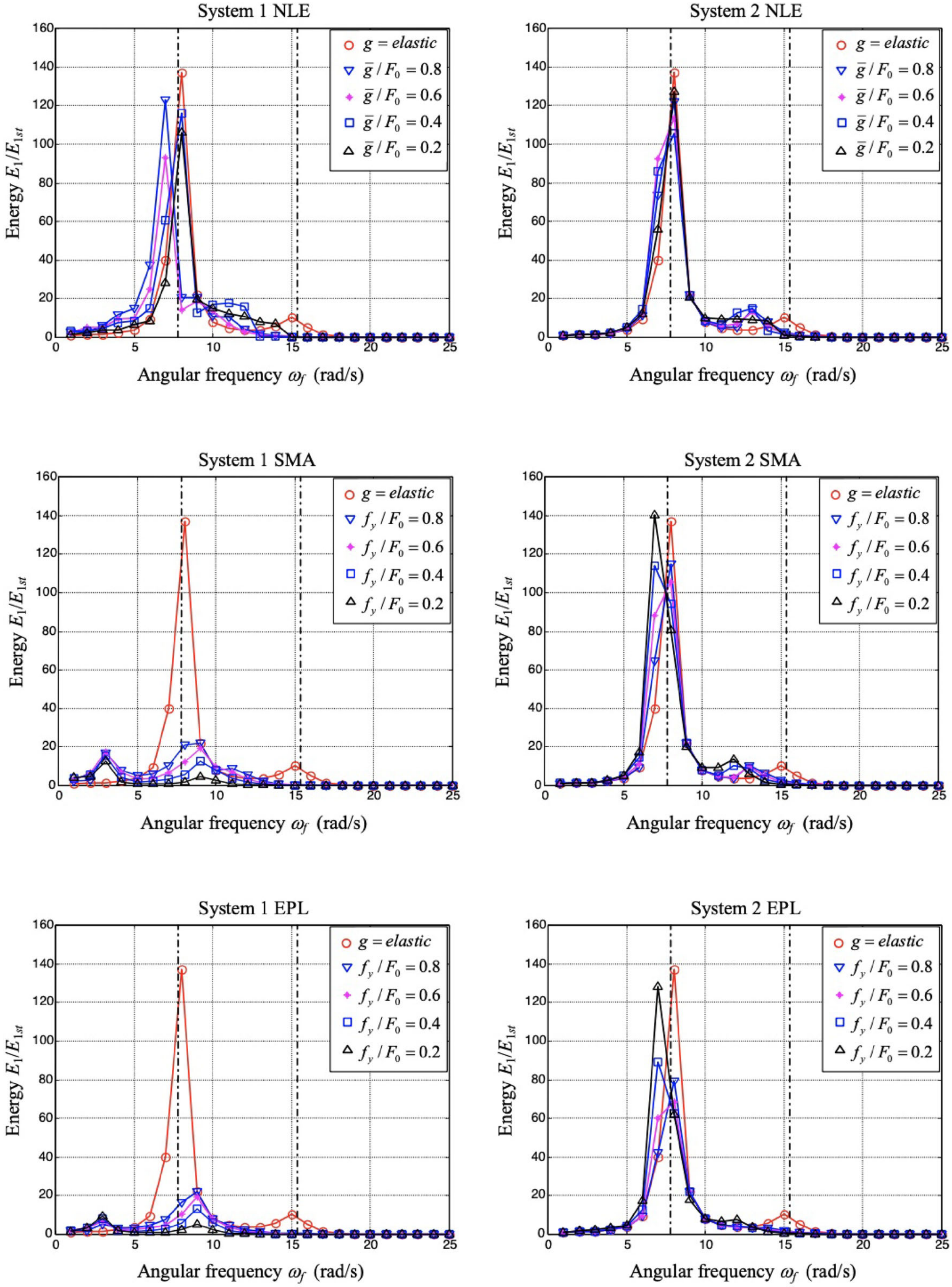

Fig. 8 Energy of the elastic link with stiffness equal to k_1, normalized with respect to the energy of the same link under static conditions

with $\bar{g}$; it is less evident in the elastic–plastic (EPL) case. Also, in both EPL and SMA cases, in System 2 the effect is more pronounced for small values of the phase transition first threshold, while in System 1 such an effect in the SMA case is for higher values of the threshold.

6 Final remark

With reference to prototypical coupled oscillators, we have only enlightened a family of peculiar behavior.

Analyses concerning the existence of branched periodic solutions of systems (5) and (6) can be carried out even when f is a live load depending on the history of motion, as in equation (1), provided that f is continuous. Pertinent analyses would require extensions of theorems in [43,47,48]. Here, we do not pursue further the topic.

A Appendix: Notes on the associated Lagrangian structures

The Lagrangian that correspond to the energy-type part of equation (1) is given by

$$L(t, \mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2}(\dot{\mathbf{q}} \cdot \mathbf{M}\dot{\mathbf{q}} - \mathbf{q} \cdot \mathbf{K}\mathbf{q} - 2V(t, \mathbf{q})) , \quad (A.1)$$

so that equation (1) is compatible with the d'Alembert-Lagrange principle

$$\delta L + (\mathbf{C}\dot{\mathbf{q}} + \mathbf{f}) \cdot \delta\mathbf{q} = 0 , \quad (A.2)$$

where δ represents the standard variation and the identity is assumed to hold for any choice of $\delta\mathbf{q}$, so that the Lagrange equations associated with the d'Alembert-Lagrange principle are obviously

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}; \mathbf{q}^{\varsigma}) = 0 . \quad (A.3)$$

Consider changes of observer governed by the action of generic diffeomorphisms on both the time scale and the ambient space $\mathbb{R}^n$. They are given by parameterized families of diffeomorphisms, namely $\chi_s \in \mathrm{Diff}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\mathbf{h}_s \in \mathrm{Diff}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, which are smooth with respect to s and such that χ_0 and $\mathbf{h}_0$ both coincide with the identity. For

$$\tau := \left.\frac{\mathrm{d}}{\mathrm{d}s}\chi_s(t, \mathbf{q})\right|_{s=0}, \qquad \text{and}$$

$$\mathbf{v} := \left.\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{h}_s(t, \mathbf{q})\right|_{s=0},$$

refer to the transformations

$$t \longmapsto t + s\tau + o(s), \qquad \mathbf{q} \longmapsto \mathbf{q} + s\mathbf{v} + o(s),$$
$$\dot{\mathbf{q}} \longmapsto \dot{\mathbf{q}} + s\dot{\mathbf{v}} + o(s), \quad (A.4)$$

and assume in addition that χ_s is chosen to be such that

$$\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} = 0 . \quad (A.5)$$

Define

$$\phi(t, \mathbf{q}, \dot{\mathbf{q}}) := \left(L - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}}\right)\tau + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{v}$$

Proposition A.1 *If L is invariant under the action of χ_s and $\mathbf{h}_s$, we have*

$$\frac{\mathrm{d}\phi(t, \mathbf{q}, \dot{\mathbf{q}})}{\mathrm{d}t} + (\mathbf{C}\dot{\mathbf{q}} + \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}; \mathbf{q}^{\varsigma})) \cdot (\mathbf{v} - \dot{\mathbf{q}}\tau) = 0 . \quad (A.6)$$

Proof Invariance of L means $L(\chi_s, \mathbf{h}_s, \dot{\mathbf{h}}_s) = L(t, \mathbf{q}, \dot{\mathbf{q}})$, thus $\frac{\mathrm{d}L}{\mathrm{d}s}|_{s=0} = 0$, which reads

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial \mathbf{q}} \cdot \mathbf{v} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{v}} = 0 . \quad (A.7)$$

By computing the time derivative of $\phi(t, \mathbf{q}, \dot{\mathbf{q}})$ and using equations (A.3), (A.5) and (A.7), we obtain the desired result. □

Corollary A.2 *If L is independent of time, $\mathbf{f}$ and $\mathbf{C}$ are nonzero, the energy decreases in time.*

Proof Set $\tau = const$ and $\mathbf{v} = 0$. The choice implies time independence of L. Thus, from (A.6) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}}\right) = (\mathbf{C}\dot{\mathbf{q}} + \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}; \mathbf{q}^{\varsigma})) \cdot \dot{\mathbf{q}} \leq 0 .$$

□

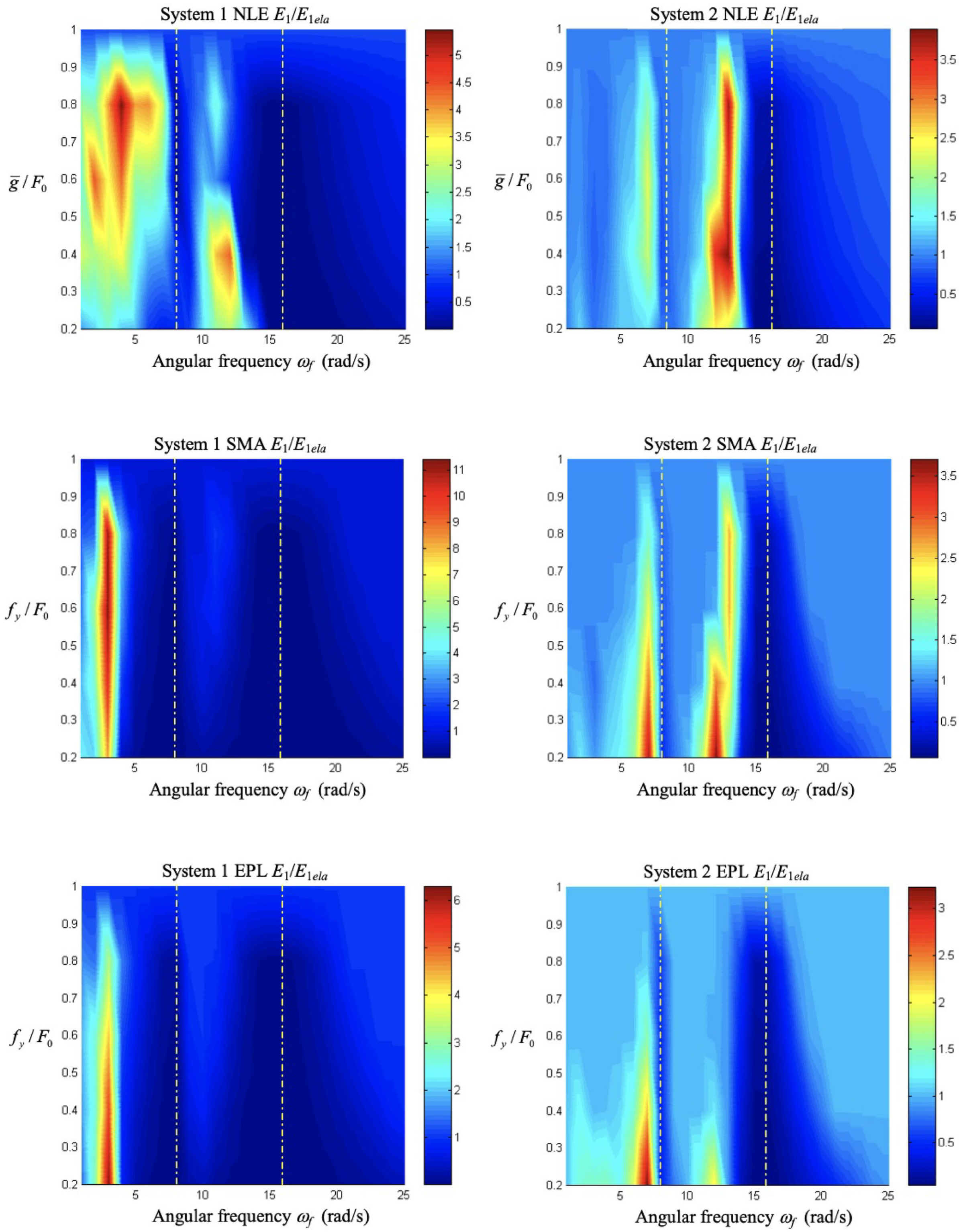

Fig. 9 Energy maps of the elastic link with stiffness equal to k_1, normalized with respect to the energy of the same link when the system is purely linear elastic

A.1 A special but not properly standard case

Consider the system

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \partial_{\mathbf{q}} V(t, \mathbf{q}) = 0 \tag{A.8}$$

and assume that all oscillators have the same mass and the same damping coefficient, namely $\mathbf{M} = m\mathbf{I}$ and $\mathbf{C} = c\mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ real identity matrix. Although dissipative, this system is fully Lagrangian. A Bateman-type Lagrangian [49] is, indeed, associated with equation (A.8) and is given by

$$L(t, \mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2} e^{t\frac{c}{m}} (m|\dot{\mathbf{q}}|^2 - \mathbf{q}\cdot\mathbf{K}\mathbf{q} - 2V(t, \mathbf{q})) , \tag{A.9}$$

so that, in components, the Euler-Lagrange equations are

$$(m\ddot{q}_i + c\dot{q}_i + \mathbf{K}_i^j q_j + \partial_{q_i} V(t, \mathbf{q}))e^{t\frac{c}{m}} = 0 ,$$

where summation over repeated indices is left understood. The previous equation implies (A.8) because $e^{t\frac{c}{m}} \neq 0$.

Consider transformations $\chi_s \in \mathrm{Diff}(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and $\mathbf{h}_s \in \mathrm{Diff}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ as in the previous section.

Proposition A.3 *If L is invariant under the action of χ_s and $\mathbf{h}_s$, the scalar $\phi(t, \mathbf{q}, \dot{\mathbf{q}})$ already defined is preserved along the motion.*

The proof follows strictly the one of Proposition A.1. Along the same path we see that the total time derivative of $\phi(t, \mathbf{q}, \dot{\mathbf{q}})$ vanishes.

Corollary A.4 *If L is independent of time, the energy is preserved along the motion.*

Proof Set $\mathbf{v} = 0$ and $\tau = const$. From Proposition A.3 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}}\right) = 0 ,$$

and the difference in the right-hand-side term is the total energy. However, Proposition A.3 requires that L be invariant under the assumed transformation, Thus, we need $L(\chi_s, \mathbf{q}, \dot{\mathbf{q}}) = L(t, \mathbf{q}, \dot{\mathbf{q}})$, with τ a constant; so, equation (A.7) implies

$$\frac{\partial L}{\partial t} = 0 .$$

□

Corollary A.5 *If L is invariant under the action of $SO(n)$, $m\dot{\mathbf{q}} \times \mathbf{q}$, that is the angular momentum, is preserved.*

Proof Set $\tau = 0$ and $\mathbf{v} = \mathbf{W}\mathbf{q}$, with $\mathbf{W}$ a skew-symmetric constant tensor. Proposition A.3 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \otimes \mathbf{q}\right) \cdot \mathbf{W} = \mathbf{e}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \times \mathbf{q}\right) \cdot \mathbf{W} = 0 ,$$

where $\mathbf{e}$ is third-rank Ricci's alternating index. Thus, due to the arbitrariness of $\mathbf{W}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \times \mathbf{q}\right) = 0 .$$

□

Corollary A.6 *If not only $\mathbf{M} = m\mathbf{I}$, but also $\mathbf{K} = k\mathbf{I}$, with $\mathbf{I}$ the unit matrix in $\mathbb{R}^{n\times n}$, with k a positive scalar, and L is rotationally invariant, $V(t, \cdot)$ must be a function of $|\mathbf{q}|$, defined to be $|\mathbf{q}|^2 = \mathbf{q} \cdot \mathbf{q}$.*

Proof Set $\tau = 0$ and $\mathbf{v} = \mathbf{W}\mathbf{q}$, with $\mathbf{W}$ a skew-symmetric constant tensor. Proposition A.3 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \mathbf{v}\right) = 0 .$$

By computing the time derivative, we thus get

$$(\mathbf{M}\dot{\mathbf{q}} \otimes \dot{\mathbf{q}} + \mathbf{K}\mathbf{q} \otimes \mathbf{q} + \partial_{\mathbf{q}} V \otimes \mathbf{q}) \cdot \mathbf{W} = 0 .$$

Due to the assumptions on $\mathbf{M}$ and $\mathbf{K}$, the second-rank tensor $\mathbf{M}\dot{\mathbf{q}} \otimes \dot{\mathbf{q}} + \mathbf{K}\mathbf{q} \otimes \mathbf{q} = m\dot{\mathbf{q}} \otimes \dot{\mathbf{q}} + k\mathbf{q} \otimes \mathbf{q}$ is symmetric. So, we have

$$\partial_{\mathbf{q}} V \otimes \mathbf{q} \in \mathrm{Sym}(\mathbb{R}^n, \mathbb{R}^n) .$$

A necessary and sufficient condition is that $\partial_{\mathbf{q}} V = a(t, \mathbf{q})\mathbf{q}$, with $a(t, \mathbf{q})$ a scalar-valued map. This is granted if and only if V is a function of $|\mathbf{q}|$, if $V(t, \mathbf{q}) = V(t, \mathbf{Q}\mathbf{q})$ for any choice of $\mathbf{Q} \in SO(n)$. □

An analogous analysis can be carried out if we relax the invariance requirement and impose that

$$\int_a^b L(\chi_s, \mathbf{h}_s, \dot{\mathbf{h}}_s)\, \mathrm{d}\chi_s = \int_a^b L(t, \mathbf{q}, \dot{\mathbf{q}})\, \mathrm{d}t + \frac{\mathrm{d}\omega(t, \mathbf{q})}{\mathrm{d}t} ,$$

with ω a scalar-valued function, as shown in one-dimensional setting in [50] (see also [51, Sec. 12.4.2]). In this case, $\phi(t, \mathbf{q}, \dot{\mathbf{q}})$ would become

$$\bar{\phi}(t, \mathbf{q}, \dot{\mathbf{q}}) := \phi(t, \mathbf{q}, \dot{\mathbf{q}}) - \omega(t, \mathbf{q}) \ .$$

Other symmetry transformations are also discussed for a Bateman's Lagrangian in [52].

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Declarations

Conflict of interest The authors declare no conflict of interest.

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References

1. Bountis, T., Skokos, H.: Complex Hamiltonian Dynamics. Springer, New York (2012)
2. Cveticanin, L.J.: Strong Nonlinear Oscillators. Analytical Solutions, 2nd edn. Springer, Cham (2018)
3. Guckenheimer, J., Holmes, P.J.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer Verlag, New York (2013)
4. Kovacic, I.: Nonlinear Oscillators. Exact Solutions and their Approximations, 2nd edn. Springer, Cham (2020)
5. Marsden, J.E., Ratiu, T.S.: Introduction to Mechanics and Symmetry. Springer, New York (1999)
6. Bera, B.K., Majhi, S., Ghosh, D.: Resurgence of oscillation in coupled oscillators under delayed cyclic interaction. Eur. Phys. J. B **90**, 132 (2017)
7. Saxenaa, G., Prasada, A., Ramaswamy, R.: Amplitude death: the emergence of stationarity in coupled nonlinear systems. Phys. Rep. **521**, 205–228 (2012)
8. Stefanovska, A., McClintock, P.V.E.: Physics of Biological Oscillators: New Insights into Non-Equilibrium and Non-Autonomous Systems. Springer, Cham, Switzerland (2021)
9. Ermentrout, B., Park, Y., Wilson, D.: Recent advances in coupled oscillator theory. Philos. Trans. R. Soc. A **377**, 20190092 (2019)
10. Hodgkin, A.L., Huxley, A.F.: A quantitative description of membrane current and its application to conduction and excitation in nerve, the. J. Physiol. **117**, 500–544 (1952)
11. Holden, A.V.: Nonlinear science - the impact of biology. J. Franklin Inst. **334B**, 971–114 (1997)
12. Wu, M., Liu, J.: Mechanobiology in cortical waves and oscillations. Curr. Opin. Cell Biol. **68**, 45–54 (2021)
13. Akhmet, M.U.: Self-synchronization of the integrate-and-fire pacemaker model with continuous couplings. Nonlinear Anal. Hybrid Syst **6**, 730–740 (2012)
14. Collins, J.J., Stewart, I.: A group-theoretic approach to rings of coupled biological oscillators. Biol. Cybern. **71**, 95–103 (1994)
15. Peskin, C.S.: Mathematical Aspects of Heart Physiology: Courant Institute of Mathematical Sciences. New York University, New York (1975)
16. Li, C., Chen, L., Aihara, K.: Synchronization of coupled nonidentical genetic oscillators. Phys. Biol. **3**, 37–44 (2006)
17. Russo, G., Bernardo, M., Sontag, E.D.: Global entrainment of transcriptional systems to periodic inputs. PLoS Comp. Biol. **6**, e1000739 (2010)
18. Zyglin, S.L.: Branching of solutions and nonexistence of first integrals in hamiltonian mechanics. I. Funct. Anal. Appl. **16**, 181–189 (1982)
19. Hildebrandt, T.H., Graves, L.M.: Implicit functions and their differentials in general analysis. Trans. Am. Math. Soc. **29**, 127–153 (1927)
20. Krasnosel'skii, V.M.: Topological Methods in the Theory of Nonlinear Integral Equations. GITTL, Moscow (1956)
21. Veinberg, M.: Variational Methods for the Study of Nonlinear Operators. Holden-Day, San Francisco (1964)
22. Arnold, V.I., Afrajmovich, V.S., Il'yashenko, Yu. S., Shil'nikov, L.P.: Bifurcation theory, in Dynamical Systems V, V. I. Arnol'd Ed., Encyclopedia of Mathematical Sciences, vol. 5, Springer Verlag, Berlin (1994)
23. Lloyd, N.: Degree Theory. Cambridge University Press, Cambridge, UK (1978)
24. Furi, M., Pera, M.P.: Co-bifurcating branches of solutions for nonlinear eigenvalue problems in banach spaces. Ann. Mat. Pura Appl. **135**, 119–131 (1983)
25. Gaines, R.E., Mawhin, J.: Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Math., **568**, Springer, Berlin (1977)
26. Prodi, G., Ambrosetti, A.: Analisi Non Lineare, Quad. 1, Scuola Norm. Sup. Pisa, (1973)

27. Furi, M., Spadini, M.: Branches of forced oscillations for periodically perturbed second order autonomous odes on manifolds. J. Differ. Equ. **154**, 96–106 (1999)
28. Gallavotti, G.: The Fermi-Pasta-Ulam problem. A Status Report. Springer, New York (2008)
29. Henderson, J.A., Dhamalac, M., Robinson, P.A.: Brain dynamics and structure-function relationships via spectral factorization and the transfer function. Neuroimage **235**, 117989 (2021)
30. Mariano, P.M., Spadini, M.: Evolution of neuron firing and connectivity in neuronal plasticity with application to parkinson's disease. Physica D **458**, 133993 (2024)
31. Dinca, G., Mawhin, J.: Brouwer Degree: The Core of Nonlinear Analysis. Birkhäuser, Cham, Switzerland (2021)
32. Milnor, J.W.: Topology for the Differentiable Viewpoint. Univ, Press Virginia, Charlottsville (1965)
33. Furi, M., Pera, M.P., Spadini, M.: A set of axioms for the degree of a tangent vector field on differentiable manifolds. Fixed Point Theory Appl. **2010**(1), 845631 (2010)
34. Mawhin, J.: The implicit function theorem and its substitutes in poincaré's qualitative theory of differential equations, stud. hist. philos. Sci. B Stud. Hist. Philos. Modern Phys. **47**, 124–130 (2014)
35. Cronin, J.: Differential Equations. Introduction and Qualitative Theory. Marcel Dekker Inc., New York (1994)
36. Bisconti, L., Spadini, M.: About the notion of non-t-resonance and applications to topological multiplicity results for odes on differentiable manifolds. Math. Met. Appl. Sci. **38**, 4760–4773 (2015)
37. Lewicka, M., Spadini, M.: On the genericity of the multiplicity results for forced oscillations on compact manifolds. Nonlinear Diff. Eq. Appl. - NODEA **6**, 357–369 (1999)
38. Yorke, J.A.: Periods of periodic solutions and the lipschitz constant. Proc. Am. Math. Soc. **22**, 509–512 (1969)
39. Galano, L., Mariano, P.M., Spadini, M.: Finite length fermi-pasta-ulam-tsingou-type chains: periodic forced dynamics and interaction inhomogeneity. Commun. Nonlinear Sci. Num. Sim. **151**, 109051 (2025)
40. Greuel, G.-M., Lossen, C., Shustin, E.: Introduction to Singularities and Deformations. Springer, Berlin (2007)
41. Arnold, V.I., Goryunov, V.V., Lyashko, O.V., Vasil'ev, V.A.:, Singularity Theory I, in Dynamical Systems VI, V. I. Arnol'd Ed., Encyclopedia of Mathematical Sciences, vol. 6, Springer Verlag, Berlin (1991)
42. Arnold, V.I., Goryunov, V.V., Lyashko, O.V., Vasil'ev, V.A.: Singularity Theory II: Classificatioon and Applications, in Dynamical Systems VIII, V. I. Arnol'd Ed., Encyclopedia of Mathematical Sciences, vol. 39, Springer Verlag, Berlin (1991)
43. Mariano, P.M., Spadini, M.: Periodic solutions to perturbed nonlinear oscillators with memory. Physica D **445**, 133635 (2023)
44. Ortega, R.: Topological degree and stability of periodic solutions for certain differential equations. J. Lond. Math. Soc. **42**, 505–516 (1990)
45. Gantmacher, F.R.: Applications of the Theory of Matrices. Dover, New York (2005)
46. Brauer, F., Nohel, J.A.: The Qualitative Theory of Ordinary Differential Equations: An introduction. Dover, New York (2012)
47. Furi, M., Spadini, M.: Periodic perturbations with delay of autonomous differential equations on manifolds. Adv. Nonlinear Stud. **9**, 263–276 (2009)
48. Furi, M., Pera, M.P., Spadini, M.: Periodic solutions of retarded functional perturbations of autonomous differential equations on manifolds. Commun. Appl. Anal. **15**, 381–394 (2011)
49. Bateman, H.: On dissipative systems and related variational principles. Phys. Rev. **38**, 815–819 (1931)
50. Sinclair, A.J., Hurtado, J.E., Bertinato, C., Betsch, P.: On noether's theorem and the various integrals of the damped linear oscillator. J. Astronaut. Sci. **60**, 396–407 (2013)
51. Novikov, S.P., Taimanov, I.A.: Modern Geometric Structures and Fields. AMS, Providence, Rhode Island (2006)
52. Campoamor-Stursberg, R.: Symmetry-preserving perturbations of the bateman lagrangian and dissipative systems. Phys. Atomic Nuclei **80**, 321–328 (2017)

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