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**The analysis of the
transplantability as a tool for the
development of mathematical
discourse**

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Abstract

The topic of this dissertation is a study conducted in mathematics education. The aim is to investigate the learning opportunities offered by dynamic explorative objects (DEO) to deal with algebraic notions concerning equations for high schools' students with a history of severe and persistent difficulties in mathematics. The investigation leans on previous findings on students' difficulties and misconceptions in tackling the notions of unknown and variable, equal sign and algebraic expressions, that together are basic in the processes of equation solving. I frame the investigation within the Theory of Commognition (Sfard, 2008). Mathematics, in this perspective, is a special discourse about mathematical objects and learning mathematics is seen as becoming participant to such discourse. Thus, investigating teaching and learning processes in this framework means analysing students' discourse according to four features that are words, visual mediators, routines and endorsed narratives. Two are the main research objectives. According to the will of looking at learning processes within digital environments, the first research objective is leaded by the aim of elaborating tools to guide the design of technology-enhanced activities. Indeed, the commognitive perspective offers refined tools to analyse students' discourse but does not explicitly deal with the involvement of digital technologies within learning processes. Thus, starting from the instruments offered by the Commognition, the aim is to develop a theoretical integration to investigate learning processes in digital environments. The second research objective involves the analysis of learning processes with respect to algebra and, especially, equations solving.

The study is conducted within three cycle of design-implementation-a posteriori analysis phases, according to the design base research (Cobb et al., 2003). It is developed through the design of a teaching sequence within DEOs to introduce and deal with the notions overmentioned and its implementation to look at students' learning processes according to a discursive perspective. The data collection, carried out in two different implementation's sessions involves 23 students of 10th grade classrooms from different high schools who recognize themselves as having a history of severe and persistent difficulties in mathematics. They have come, in the afternoon, to an out-of-school learning center once a week, for 5 or 6 weeks. Data available are audio-video recordings of the sessions, together with screen recordings of the tablet on which students interact with DEOs and accomplish the proposed task situation, and students written productions.

The study leads to the integration in the Theory of Commognition of some theoretical constructs. Specifically, I expand upon the notions of metaphor and transplanting proposed by Sfard (2008) to develop the construct of analysis of the transplantability. Moreover, I elaborate the notion of bridging discourse and blended discourse. These theoretical tools guide the design of new DEOs and task situations, as well as the elaboration of teacher's intervention. In addition, they offer new insight into the processes of learning algebra through digital environments, focusing on the treatment of indeterminate quantities and the relation of equality between algebraic expressions.

This dissertation is structured as follows.

In the first chapter I position our study within the literature concerning algebra and its teaching and learning processes, delving into the issues that such multifaceted topic carries along and pointing out the main students' difficulties with respect to this mathematical content. Relying on the previous research on mathematics education and digital environments, I stress the potentialities of technology-enhanced activities I want to take advantage of.

In chapter 2 I present the theoretical framework within which has been carried out this study, that is the Theory of Commognition. I discuss the position taken on by the commognitive perspective with respect to mathematics and its teaching and learning. In describing the theoretical tools offered by this theory to investigate the processes of teaching and learning, I put the light on some theoretical constructs I have needed to elaborate and refine in order to conduct the investigation. Thus, in conclusion of this chapter, I present the two research objectives that have guided this study. One is referred to the theoretical elaborations I aimed at developing, the other to the analysis of learning processes with respect to the algebraic discourse.

In chapter 3 I introduce the methodology used to deal with my research objectives. Together with this, I present the timeline of the study and the evolution of our research objectives according to the progression of the investigation.

In chapter 4 I propose a proposal to fulfil the first research objective. I present the notion of *analysis of transplantability*, as a theoretical construct to guide the design of digital environments and related task situations in terms of discursive aims. I propose a classification of discourses that can allow to clarify the aims of the elaborated design as well as to analyse some processes emerged during the experimentations. Specifically, I propose the construct of *bridging discourse* as the discourse about the relation between two different discourses, that in our case have been the discourse related to the DEO and the mathematical discourse. This

specific construct has guided the introduction of new task situations in our teaching sequence, involving the notion of *silent video*.

In chapter 5 I present and analyse the technology-enhanced activities I have developed within this study. Specifically, I use the theoretical developments introduced in Chapter 4 to provide the a priori analysis of each DEO involved in our teaching sequence, together with the task situations designed and the tutor's intervention elaborated.

In chapter 6 I provide information about the data collection, focusing on how data have been collected and analysed. The data analysis is proposed in chapter 7 and in chapter 8. In the first one I have collected the analysis of some episodes taken by the first cycle of implementation session, focusing on the features of emerged discourse and their relationship with the algebraic one. Chapter 8 is devoted to the analysis of episodes from the third cycle of implementation and is directed to the processes related to equations solving involving both DEOs and equations.

The concluding chapter (9) is dedicated to the research objectives and the reached proposals to fulfil them, highlighting the theoretical contributions offered by this study and contextualizing our findings within the commognitive framework. Moreover, I discuss possible didactical implications of our findings, focusing on the reinvestment of our results in a classroom context. In conclusion, I describe possible directions for further research, that include the refinement of the theoretical elaborations according to their use for the design and elaboration of other DEOs, as well as the investigation about the role of the teacher in promoting the transition from discourses related to DEO to the mathematical discourse. In addition, considering the context in which our study has been conducted, and especially the students involved, I could use our findings as a starting point to characterize *remedial interventions*.

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1. Introduction

In this chapter we aim at setting the research within the broad literature about the processes of teaching and learning algebra. Indeed, our study is focused on the domain of algebra, that has an essential role in the mathematical practice, and, at the same time, it is a delicate issue when we think about the processes of teaching and learning mathematics. In particular, with this study, we want to highlight the role of digital environments in promoting and supporting the growth of *algebraic thinking*, although we are aware of the ambiguity generated by this term (Kieran, 2022) that we try to discuss shortly.

As we explain in the next sections, the perceptions about and directions to introduce school algebra are various and sometimes intertwined (Bednarz et al., 1996); sure is that methodological choices and didactical decisions influence the perceptions and conceptions students' have concerning the theme. With respect to this issue, in this chapter we present students' common difficulties and misconceptions highlighted in the last four decades by research in teaching and learning algebra, relating them to common didactical approaches adopted in traditional school education.

More recently, technologies have gotten in the play when we talk about teaching and learning mathematics. Clearly, the study of digital artifacts has concerned the theme of algebra and has described potentials that digital technologies have in promoting the learning of this topic. Here we introduce and discuss some results taken from the literature about digital environments that put a stepping stone for our studies.

In the end, we describe the context of our research studies and, considering that, we propose a general formulation of our research question, that are elaborated and precisely framed in the next chapter.

1.1. Learning Algebra in the Italian context

This dissertation is focused on studying teaching and learning processes of algebra within digital environments. To frame this research, we need to discuss several aspects related to the theme. First of all, it is necessary to take into account the regulatory aspects that govern the Italian school context, the one in which our work moves.

Italian high schools are guided, in their educational aims, by Indicazioni Nazionali (MIUR, 2010), a document in which are collected recommendation and guidelines

for schools' teachers, with reference to contents, competencies and abilities that should be developed by students in their school years. In the section "General guidelines and skills (in Italian, Linee generali e competenze)", we read

"L'ampio spettro dei contenuti che saranno affrontati dallo studente richiederà che l'insegnante sia consapevole della necessità di un buon impiego del tempo disponibile. Ferma restando l'importanza dell'acquisizione delle tecniche, verranno evitate dispersioni in tecnicismi ripetitivi o casistiche sterili che non contribuiscono in modo significativo alla comprensione dei problemi."¹

Moreover, in the section "Specific learning objectives (in Italian, Obiettivi specifici di apprendimento)" for the first two years of high school, with reference to the nucleus "Arithmetic and Algebra (in Italian, Aritmetica e Algebra)" we find

"Il primo biennio sarà dedicato al passaggio dal calcolo aritmetico a quello algebrico. [...] Lo studente apprenderà gli elementi di base del calcolo letterale, le proprietà dei polinomi e le operazioni tra di essi. Saprà fattorizzare semplici polinomi, saprà eseguire semplici casi di divisione con resto fra due polinomi, e ne approfondirà l'analogia con la divisione fra numeri interi. Anche in questo l'acquisizione della capacità calcolistica non comporterà tecnicismi eccessivi. Lo studente acquisirà la capacità di eseguire calcoli con le espressioni letterali sia per rappresentare un problema (mediante un'equazione, disequazioni o sistemi) e risolverlo, sia per dimostrare risultati generali, in particolare in aritmetica."²

Finally, in the nucleus "Relations and functions (in Italian, Relazioni e funzioni)" we see

¹ "The broad range of content that the student will cover will require the teacher to be aware of the need for a good use of the available time. While the importance of mastering techniques remains, the focus will be on avoiding distractions from repetitive technicalities or sterile cases that do not contribute significantly to the understanding of the problems" (translated by the authors).

² "The first two years will be dedicated to the transition from arithmetic to algebraic calculation. [...] The student will learn the basic elements of literal calculus, the properties of polynomials, and operations between them. They will be able to factor simple polynomials, perform basic division with remainder between two polynomials, and explore the analogy with division between integers. Even in this case, the acquisition of calculation skills will not involve excessive technicalities. The student will gain the ability to perform calculations with literal expressions both to represent a problem (through an equation, inequalities, or systems) and solve it, as well as to prove general results, particularly in arithmetic" (translated by the authors).

“Lo studente apprenderà a descrivere un problema con un’equazione, una disequazione o un sistema di equazioni o disequazioni; a ottenere informazioni e ricavare le soluzioni di un modello matematico di fenomeni, anche in contesti di ricerca operativa o di teoria delle decisioni.”³

Taking into consideration what indicated here, high schools’ teachers, should avoid “excessive technicalities” in order to foster meaningful learning of algebra. Algebra and “its” calculations are described as tools for representing, describing and solving a problem, for studying and proving numerical general relationships, for modelling phenomena, while the corresponding manipulations are not the focus but just instruments to do so. The guidelines for Italian high schools are clear about the educational objectives of teaching algebra. Despite this, if we look at the main school texts adopted in the Italian classrooms, we can see a strong and spread attention to manipulations and calculations, while the use of algebra as an instrument to solve problems, to create models and relations is not emphasized as well. The teachers’ practice sometimes reflects this position. Consequent to this, Italian students are mainly exposed to “excessive technicalities” that often lead students to a sense of failure and lack of meaning with respect to algebraic symbols.

1.2. Contextualization of the research problem within the literature: teaching and learning algebra

Research in mathematics education has dealt with the issue of teaching and learning algebra since decades. Thematic conferences, special issues, volumes and so on make the interest in the theme well evident. Algebra and the processes of its teaching and learning are tackled assuming different standpoints: the dynamicity and the incessant concern that characterizes its study show the complexity as well as its relevance with respect to mathematics education. In the *Encyclopedia of Mathematics Education*, Kieran (2014) has had the task of defining school algebra and summing up the spread work of research in characterizing not only what is meant by the term but also how this conception has changed over years influencing both teaching practices and learning processes. The reflections that emerge from her work well represents the complexity that encompasses algebra

³ “The student will learn to describe a problem using an equation, an inequality, or a system of equations or inequalities; to obtain information and derive the solutions of a mathematical model of phenomena, including in the context of operational research or decision theory” (translated by the authors).

and its teaching. Indeed, the picture depicted by the researcher shows a many-sided topic, that has been faced and studied in different and evolving ways. Specifically, diverse perspective with respect to algebra refer both to the contents and the idea of the theme as well as to the didactical approaches. Thus, the interest in algebra and specifically in the processes of teaching and learning algebra is spread and active since years, and it has brought to light different perspectives that have highlighted the elaborate and multifaceted nature of algebraic thinking and the resulting complexity in addressing the topic from a didactic point of view.

1.2.1. What Algebra is about? A journey through the study of algebraic thinking

The necessity to define algebra intertwines with the will to characterize school algebra: different perspectives on the theme naturally reflect on different didactical approaches and choices. For this reason, it is unavoidable to weave together the nature of algebra and its activities, as has happened in the research about the topic. Within the years, this study has evolved. One of the main points is that while in the beginning school algebra was just perceived as a tool for manipulating symbols, within years the interest has moved and expanded to the aim of understanding the construction of meaning for algebra. The need to characterize the *algebraic thinking* has grown through years, but it has not reached a final point yet. Indeed, while in the first half of the twentieth century the focus of studying algebra and learning was on equation solving and the relative algorithms, in the second half the interest moved towards other aspects. For example, Freudenthal (1977) in his work *What is Algebra and What has it been in History?* reflect on the fact that algebra at school is solving linear and quadratic equations, while, from his point of view

“the ability to describe relations and solving procedures, and the techniques involved in a general way” (p. 193)

should be included in the name “algebra”. In her reflections on the changes in mathematics education research with respect to algebra, in the *Encyclopedia of Mathematics Education* (2014) Kieran emphasizes:

“In the late 1970s, when algebra education researchers began to increase in number and to coalesce as a community (Wagner and Kieran 1989), research began to center on the ways in which students construct meaning for algebra, on the nature of the algebraic concepts and procedures used by students during their initial attempts at algebra, and on various novel approaches for teaching algebra (e.g., Bednarz et al. 1996). While the study of students’ learning of algebra favored a cognitive orientation for some time, sociocultural

considerations have added another dimension to the research on school algebra since the end of the 1990s (Lerman 2000).” (Kieran, 2014, p.27)

Lesley Lee in her doctoral dissertation investigated algebraic understanding; a vast group of mathematicians, mathematics education researchers, teachers and students were asked by Lee the question “What is algebra?”. The answers depict a variety of ideas: algebra is *a school subject*, is *generalized arithmetic*, is *a tool*, is *a language*, is *a culture*, is *a way of thinking* and is *an activity*. As Kieran underlines, Lee wrote, “Algebra emerges as an activity, something you do, an area of action, in almost all of the interviews” (Lee, 1997, p. 187). This common idea goes with the perspective about algebra until the 1970s, as the science of equation solving and transformations and manipulations of algebraic expressions. On the other side, from the interviews, Lee brought to light also the rising interest in reflecting on the construction of algebraic objects and in the meaning making in algebra.

With respect to this, we have to consider that the growing concern in studying the meaning making in school algebra (e.g., Kaput 1989; Kirshner 2001) and in delving into the cognitive processes behind algebraic activities reflects in an emerging multitude of points of view about the theme. Bednarz and colleagues, in the introductory chapter of the volume “Approaches to algebra: perspective for research and teaching” (1996) well lead into the variety of theoretical perspective and consequent didactical approaches for algebra teaching and learning that are addressed in the book. Questions that contributors were asked to answer were “*What are the essential characteristics of algebraic thinking that are developed in each of these approaches (or what would we like students of algebra know)? On what thought processes can this learning rely (or what can algebra learning build on)? What thought processes must this learning work against (or what are the obstacles in algebra learning)? What situations might promote the emergence and development of algebraic thinking in students?*”. This work made emerge five main perspectives:

- Historical perspectives in the development of algebra
- A generalization perspective on the introduction of algebra
- A problem-solving perspective on the introduction of algebra
- A modelling perspective on the introduction of algebra
- A functional perspective on the introduction of algebra

These five perspectives, that do not complete the possible visions about algebra, give a fair idea of the lens through which we can look at the topic. For example, we

can think to algebra as a tool for problem-solving, taking into account the students' experience with problem-solving in arithmetic (Bednarz & Janvier, 1996). At the same time, we can look at algebra as the language for expression and manipulation of generalities. Mason affirms

“My long-standing conjecture, has been, and still is, that what awareness of generality permeates the classroom, algebra will cease to be a watershed for most people; that algebra as it is usually interpreted at school is a dead subject, akin to conjugating Latin verbs and memorizing part of flowers- that is, unless and until, expressing generality becomes natural and spontaneous in the conduct of mathematics, imbuing every mathematical encounter and topic. Generalization is the heartbeat of mathematics, and appears in many forms.” (Mason, 1996, p. 65)

The author uses the term “watershed” to describe algebra for most people. This expression particularly fit with this topic: how often people, in describing their experience with mathematics distinguish the period ‘before-symbols’ and ‘after-symbols’? The perception of a distinction between these two periods is widespread and quite familiar to researchers and teachers in mathematics. Moreover, Mason discusses the need of being sensitized to the distinction of *seeing a generality through the particular* and *seeing the particular in the general* as essential to develop awareness of generality. Another relevant perspective on algebra is the one that consider and use letters as variables and not as unknown, and thus, for example, considering the expression $3x + 5$, it is a function of x and x can take on a range of values (Kieran et al., 1996). This functional approach to algebra is documented since the last decades of the twentieth century by Kieran (2014)

“The years since the late 1980s have also witnessed a broadening of the content of school algebra. While functions had been considered a separate domain of mathematical study during the decades prior, the two began to merge at this time in school algebra curricula and research. Functions, with their graphical, tabular, and symbolic representations, gradually came to be seen as legitimate algebraic objects.” (Kieran, 2014, p.28)

In the years, the debate on algebra and on school algebra has been enriched by the overmentioned 12th ICMI Study, from 2000 to 2004, culminated in the book *The future of teaching and learning of Algebra*. Stacey and Chick (2004) introduce the composition, focusing on the reasons of this study. On one hand, the need of this study growth from the will to promote the “massification” of education, involving in this way issues about equity and relevance. The desire to promote equity rests on the idea of algebra as a “gateway to higher mathematics” and as an essential factor in mathematical activity; this requires that students should have the opportunity to

learn algebra in the best way. As Stacey and Chick (2004) stresses, it is not easy to achieve this outcome, that needs teaching approaches that are founded and take advantage of different students' intelligences. Another matter concerns the relevance algebra has for students: if it is perceived as symbolic manipulation and nothing more, its importance is lost among routines and procedures.

On the other hand, the progress of technologies has put in a new light competencies that were previously evaluated as fundamental in the mathematical practice. Anew, manipulations and procedures, that start to be carried out by a machine, should be left behind to leave space to other aspects. At the same time, technologies can offer new tools for the teaching practices that need to be investigated to improve the quality of algebra teaching and learning. The need is to shift from didactical approaches focused on manipulations and calculations to the promotion of meaning for objects of algebra, taking advantage of new technologies.

The will to characterize and describe algebraic activities has been kept alive during the years by researchers. Indeed, scholars in mathematics education have led several studies focused on the processes involved in the growth of *algebraic thinking*, even if this term has not found a shared definition yet (Kieran, 2022). For example, in the volume result of the 12th ICMI Study, Kieran (2004) describes a model for conceptualising algebraic activities. She identifies:

- *Generational activities*, as the formation of expressions and equations to represent situations, properties, patterns and relations
- *Transformational activities*, as the manipulations performed on expressions
- *Global/meta-level activities*, as the activities in which algebra is a tool like problem solving, modelling, generalizing, proving and so on

Kieran stresses the relevance of these three kinds of activities, also highlighting the widespread shift by research in mathematics education from attention to transformational activities to global/meta-level activities also aimed to meaning make of the objects of algebra. This shift, also due to the advent of technologies, took researchers to think that “the techniques would take care of themselves”. Unfortunately, Kieran point out that this has not happened: the necessity is to take care both of conceptual understanding and techniques as two sides of the same coin. In conclusion,

“Only an equilibrium among the different components and the various situations that make them meaningful can enable students to deeply understand the pertinence of algebra, its structure, the significance fundamental algebraic concepts such as variable, and the use of algebraic

reasoning (knowing to apply it in contexts where algebraic manipulations can be fruitful).” (Bednarz et al., 1996, p. 4)

We thus need to deal with the different aspects that characterize algebra to promote a meaningful learning and a perception of relevance of the theme in students, to avoid the common idea of algebra as a collection of mechanical procedures. At the same time, the techniques and manipulations should be considered by researchers and teachers and should have a role in the process of learning algebra.

In their work *The learning and teaching of Algebra: Ideas, Insights and Activities* (2017) Arcavi and colleagues reflect on aims, actions and entities of algebra. They sum up the aims of algebra as expressing generalizations, establishing relationships, solving problems exploring properties, proving theorems, calculating. The actions they pinpoint as “doing algebra” are noticing, describing, denoting and representing, handling symbolic expressions, symbolizing and reading, connecting representations, creating algebraic expressions. The entities identified as objects of the algebraic activity are signs, variables, algebraic expressions, equations, relations and functions. Arcavi and colleagues gather together the pieces of the puzzle that previous years of research have investigated and elaborated, sometimes separately. Their considerations show a complex system of aims, actions and entities that all together are algebra. This articulated perspective cannot be left behind in dealing with teaching and learning algebra, from a research point of view as well as from a teaching point of view.

In the journey to the characterization of algebraic thinking, research has started taking into account the role of primary education and in particular of arithmetic ways of thinking in the transition to formal algebra (Kieran, 2022). In fact, researchers have started discussing about how *early algebraic thinking* can be characterized and promoted within elementary education, to make it smoother the transition to the secondary school algebra. Kieran proposed this definition of early algebraic thinking:

“Algebraic thinking in the early grades involves the development of ways of thinking within activities for which the letter-symbolic could be used as a tool, or alternatively within activities that could be engaged in without using the letter-symbolic at all, for example, analyzing relationships among quantities, noticing structure, studying change, generalizing, problem solving, justifying, proving, and predicting.” (Kieran, 2004, p.149)

Researchers agree on the fact that it is possible to foster early algebraic thinking since primary school, not necessarily involving the algebraic symbols, and it seems necessary in order to promote a smooth transition to the formal algebra. For example, Radford (2014) presents an investigation involving young students, to study the genesis of embodied non-symbolic algebraic thinking. In presenting the research, he reflects on the relation between arithmetic and algebra, considering the fact that traditionally algebra is taught to students in high school, after many years of arithmetic practice but, if we choose to promote early algebraic thinking this traditional path is not completely followed anymore. In dealing with differences and rupture of arithmetic and algebra, Radford take into consideration students facing equation such as $Ax + B = Cx + D$ and affirms

“The students have to resort to a truly algebraic idea: to operate on the unknown. In order to operate on the unknown, or on indeterminate quantities in general (e.g., variables, parameters), one has to think analytically. That is, one has to consider the indeterminate quantities as if they were something known, as if they were specific numbers. [...] From a genetic viewpoint, this way of thinking analytically—where unknown numbers are treated on a par with known numbers—distinguishes arithmetic from algebra.” (Radford, 2014, p. 259)

Thus, Radford characterizes algebraic thinking as the activity of dealing with indeterminates quantities and considering them as if they were known. Moreover, he adds

“[...] the use of letters in algebra is neither a necessary nor a sufficient condition for thinking algebraically. [...] the rejection of the idea that notations are a manifestation of algebraic thinking opens up new avenues to the investigation of elementary forms of algebraic thinking in young students.” (Radford, 2014, p.260)

Thus, algebraic thinking is not characterized by the use of symbolism. He concludes his reflections describing three features to depicts algebraic thinking are indeterminacy (there are not-known numbers in the problem), denotation (not-known numbers are named or symbolised not necessary through algebraic symbolism), analyticity (indeterminate quantities are handled as known numbers). In this light, the study presented by Radford (2014) shows new possibilities to think about, involve and act with indeterminate quantities that can be developed also by younger students.

The interest in early algebraic thinking has grown and the results of research about the topic seem promising in showing that the introduction of informal algebra at

early stages seems an efficient strategy to avoid the gap from arithmetic to algebra that often generates difficulties and misconceptions (Kieran, 2018). More recently, Kieran, in her survey paper to introduce the Special Issue *Early Algebraic Thinking: A Multi-Dimensional and Ever-Expanding Global Vision* of ZDM (2022) identifies three dimensions for early algebraic thinking: analytic, structural and functional, while generalizing is considered the scarlet thread common to all of them. The analytic dimension put together the analyticity described by Radford (2014), that Kieran recalls

“Radford characterizes thinking as algebraic when it deals with indeterminate quantities in an analytic way, that is, the indeterminate quantities (unknowns or variables) are considered as if they were known and calculations are carried out with them in the same way as would be done with known numbers, with emphasis on the deductive nature of these calculations.” (Kieran, 2022, p. 1134)

with the notion of equality and equivalence, that for Kieran are considered key components of this dimension. She affirms:

“As Freudenthal (1983) reminds us, it is not the numerical value of left and right sides that remains the same throughout the equation-solving process, but rather the truth-value. Additionally, other properties are also involved in maintaining the balance of both sides within equation solving—the basic field properties applied to the individual expressions of the equations when they too are transformed.” (Kieran, 2022, p.1134)

Thus, the analytic thinking involves managing indeterminates quantities as if they were known as well as dealing with equality transformations and balancing. In this sense, the equal sign plays a central role, as a relational symbol and not as an operational one. I discuss these two perspectives in the next section. To conclude her thought, Kieran sums up:

“handling unknowns as if they were known, along with the thinking related to equivalence transformations and the notions of balance and truth-value, serve to characterize the overarching dimension of analytic thinking within early algebraic thinking.” (Kieran, 2022, p. 1135)

The aspects related to manipulations, transformations and treatment of expressions’ structures can interweave with the structural thinking that is described by Kieran as:

“The overarching dimension of structural thinking is characterized as encompassing the broader, Freudenthal-related perspective on structure and

properties within activity aimed at seeing relations, properties, and structure within numbers, operations, and expressions. Such early algebraic thinking is considered to underpin the kind of thinking involved in transforming and generating equivalent algebraic expressions.” (Kieran, 2022, p. 1138)

The last component that for Kieran characterizes early algebraic thinking is the functional one. The researcher, recalling decades of studies about the relationship between algebra and functional thinking, sums up

“the overarching dimension of functional thinking within early algebraic thinking is characterized as seeking, expressing, and reasoning about relationships between co-varying quantities (whose initial presentation might be in any of several possible formats) in a manner that makes explicit in a general way the underlying relation of the given function, by means of some type of conventional or unconventional representation.” (Kieran, 2022, p. 1141)

The three dimensions presented by Kieran gain together decades of research about the nature of algebra and the characterization of its activity. What is clear, from her considerations as well as from all the previous studies, is that algebra is a many-sided world. It can be a tool for problem-solving, for modeling, for generalize, and at the same time an instrument for functional thinking.

The theoretical and empirical research in the area gives us insight and instrument to deal with the topic, and at the same time furnishes us basis to delve into teaching sessions, research with special educations students and in digital environments. Specifically, in our study, we assume the position of Kieran (2022) with respect to analytic thinking, and in our design we developed activities to promote it while in the posteriori analysis look for learning processes related to this component of the algebraic thinking.

1.2.2. Students’ difficulties in learning algebra

If we think to algebra as something essential and fundamental for mathematical practice, in order to maximise learning opportunities, we need to investigate the process of learning algebra, focusing on difficulties and misconceptions. Considering the multifaceted nature of the topic, it does not surprise that since decades researchers in mathematics education have identified algebra as a challenging issue at all school levels. In particular, the transition from arithmetic to algebra, and so the development of algebraic thinking, plays a central role and can be complex and delicate. With respect to this, Lee (1996) introduces the term “cultural shock” to describe the students’ entry into the algebraic culture. The teaching of algebra is by her depicted as initiating students to a new culture, with its

own beliefs and custom, social forms and ways of communicating. The imagine offered by Lee particularly fits with what the research tells us about the process of learning algebra and about the students' difficulties in dealing with the topic. The "old" customs characterizing manipulating numbers are left behind to acquire new ways of thinking and acting that could disorient students: the computational nature of arithmetic activity is replaced by the relational thinking that characterize algebraic practice. In the work mentioned before, Mason (1996) talks about a "watershed" when people come to algebra. These and other ways of depicting the introduction of algebra do not surprise us: difficulties and misconceptions are spread and large, especially when the new rule and custom are not presented and perceived as an instrument to generalize, but just as rules and procedures. Considering this, as expert we cannot see the students' shock and the consequent difficulties as something unexpected.

One of the main protagonists of research studies about learning algebra is the equal sign. In 1981 Kieran talked about equal sign perceived by students as a "do something signal"; the equal sign is seen as something that "gives", a left-to-right directional signal, and this idea accompanies students along school years (Kieran, 1992). Other studies have shown that students' perception about this symbol has not changed yet (Carpenter et al., 2005). The operational idea of equal sign can find its roots in the approach to arithmetic during first schools' years. Indeed, within arithmetic, it is quite natural to execute calculations (addition, subtraction, multiplication, ...) to find a single number, "the solution" of the problem or exercise, the answer to the calculation. Such perception is reported in several studies. For instance, in front of the open number sentence $8 + 4 = \_\_\_ + 5$, elementary school's children respond that 12 (the sum of the numbers at the left of the equal sign) is the number to be put in the empty space, or that 17 (the sum of all the numbers in the sentence) should be put there. Only few pupils recognize that the equal sign indicates a relation of equality between the sums on the two sides of the equal sign (Carpenter et al., 2005). This operational view of the equal sign persists over years from elementary school to high school, overshadowing the relationship of equivalence the symbol expresses. When they deal with equations, pupils keep seeing the equal sign as a separator symbol and not as an indicator of the equivalence between the left and the right sides. Symmetry and transitivity of equality indicated by the equal sign are not perceived and in front of something like $2x + 3 = 5 + x$ students perform action as (Kieran, 1992)

$$2x + 3 = 5 + x \rightarrow 2x + 3 - 3 = 5 + x \rightarrow 2x = 5 + x - x - 3 \rightarrow 2x - x = 5 - 3 \rightarrow x = 2$$

In this light, after several years of high schools, students are able to carry out actions in order to solve successfully equation, but they lack in giving meaning to the equal sign.

This condition affects not only equal sign, but more in general letteral symbols. Indeed, Arcavi (1994) stresses

Even those students who manage to handle the algebraic techniques successfully, often fail to see algebra as a tool for understanding, expressing, and communicating generalizations, for revealing structure, and for establishing connections and formulating mathematical arguments (proof). Instruction does not always provide opportunities not only *not* to memorize, but also to "forget" rules and details and to be able to see through them in order to think, abstract, generalize, and plan solution strategies. (Arcavi, 1994, p. 24, italics in original)

In this paper, Arcavi questions himself about what symbol sense is. His characterization echoes with perspectives on algebra introduced before. In his discussion he identifies what an "acquired" symbol sense should include, shifting the perspective from what algebra is to what a person "doing" algebra should perceive. Indeed, among all, he talks about feeling, awareness and sensing, and says "symbol sense would be a quick or accurate appreciation, understanding, or instinct regarding symbols". Arcavi highlights, in this way, a personal dimension in doing algebra and dealing with symbols that is delicate and must be considered in the process of teaching. In particular, taking his perspective, it seems evident that a strong focus on manipulations lead naturally to a failure in constructing this instinct regarding symbols. Manipulations and procedures, if performed without awareness and meaning, are not instinctive but just something imposed by the teacher or the institution. Another problematic matter in this sense is the meaning attributed by students to the solution of an equation. Indeed, if in front of an equation a student feels the need just to complete a procedure, in the end, the inscription $x = 2$ will be for her the "solution of the equation". From a mathematical point of view, this makes no sense since $x = 2$ is still an equation, with the number 2 as solution. The operational meaning of the equal sign together with the lack of symbol sense take students to accomplish calculations not even thinking about the "problem" they are trying to solve and what a possible solution should be.

Recent studies (Xin et al., 2022) have pointed out that for students algebra continues to be just a manipulations of symbolic expressions without meaning. This is critical in particular for students with difficulties, who often refuses to learn by heart and repeat procedures to which they assign no meaning (Baccaglini-Frank, 2021). In this frame, literature shows us how the use and interaction with digital environments can foster the construction of algebraic objects, combining

operational aspects with relational ones. In the next section we give an overview of the main results in this field.

1.3. Contextualization of the research problem within the literature: digital tools for teaching and learning processes

In this study, our aim is to investigate the processes of teaching and learning algebra within well-designed digital environments for students with a history of difficulties in mathematics. If, on one hand, we rely on a large body of studies and perspectives about algebra and its teaching and learning, on the other hand during our investigation we have been guided by several results on the topic of learning through digital means. Indeed, since the last decades of the twentieth century, a large body of research has dealt with digital technologies, investigating potentialities and possible uses in teaching and learning mathematics. The digital technologies are perceived not only as an instrument to accomplish a task, but as a tool that transform the activity itself, as well as the relationship between the student and the “standard knowledge” (Mariotti, 2002). Behind a well-designed digital technology there is a mathematical knowledge of reference that guide the ways in which the digital objects and the software act and react according to the interaction of the user.

A wide strand of research about the topic has embraced the “instrumental” approach (Rabardel, 1995), that distinguishes between *artifact* and *instrument*. The artifact is the material object per se, while the instrument, according to Rabardel, is the dyad composed by the artifact (considering all its components) and the schematic components that are called *utilization schemes*. Thus, while the artifact is considered according to the intention of the designer, the instrument has a subjective component, because we look at it together with the use that a person makes of it. Rabardel calls *instrumental genesis* the process of elaboration and evolution of the instruments, looking at the instrument as a construction of an individual. Specifically, he makes a distinction between the processes of instrumentalization and instrumentation. The first one concerns the artefacts and its components, and in particular the recognition of its potentialities and constraints. On the other hand, instrumentation is referred to the emergence and development of the utilization schemes. Within this perspective, several studies have focused on the triad “student-task-digital tool” investigating the process of mediation triggered and encouraged by an instrument. Results show how students’ processes of learning are shaped by the tools and at the same time the latter are shaped by the students. For instance, various studies have faced the processes within dynamic geometric software strongly focusing on different modalities of

dragging and their relations with diverse phases in dealing with a geometric problem (Arzarello et al., 2002, Baccaglini-Frank & Mariotti, 2010). In the investigation of the triad overmentioned, the Theory of Semiotic Mediation, adopting the instrumental approach has faced the question of developing classroom interventions within digital (and not digital) artifacts, stressing the essential role played by teacher in exploiting the potential of these environments to promote meaningful learning processes (Bartolini Bussi & Mariotti, 2008). A strand of more recent research has dealt with the study of role of bodily experiences in the processes of learning with digital resources. Recent advancements in cognitive and educational sciences emphasize the importance of the body in the learning process. Considering the fact that emerging digital technologies increasingly support physical interaction, it has been (and it is) necessary to explore how the body contributes to learning mathematics through technology. Indeed, unlike traditional keyboard-based devices, tools such as multi-touch screens, augmented and virtual reality platforms, motion sensors, and gesture-recognition systems offer students enriched opportunities for embodied interaction. In mathematics education, these innovations have demonstrated their potential to improve students' number sense and enhance graph comprehension (Baccaglini-Frank & Maracci, 2015, Sinclair & Heyd-Metzuyanim, 2014, Nemirovsky et al., 2013). More recently, some researchers have introduced the notion of *body-artifact functional system* re-examining the instrumental approach through the lens of radical embodied cognitive science (Shvarts et al., 2021).

The investigation of the role of digital technologies in teaching and learning processes has been thus faced through different theoretical perspectives, all highlighting the role played by digital means in supporting and promoting learning outcomes (see for example Leung & Baccaglini-Frank, 2017, Hanna et al., 2019). Findings stress that if appropriately designed and used in the teaching process, these environments result helpful and efficient (Baccaglini-Frank, 2021, Falcade et al. 2007, Healy & Sinclair 2007). Indeed, digital environments, unlike paper and pencil, grant students the possibility to be engaged with diverse dynamic representations of the same mathematical object. Specifically, the dynamic nature of the available representations, that can be not just visualized but also manipulated, promotes meaningful participation to the mathematical activity (Sinclair et al., 2009) and foster the employment of different modes of communication (Ng, 2016). In this way, the interaction with digital environments can become multimodal, involving not only written signs, but gestures, bodily motion, speech and so on and can promote the interaction among peers.

Recently, within some studies, digital objects have been defined as DIMs, Dynamic Interactive Mediators: these objects are not only dynamic but also interactive, in the sense that when they are manipulated, they offer an immediate feedback (Antonini

et al., 2020). The potentialities of these kind of digital environments have shown to be efficient in several fields of mathematics, such as geometry (e.g. Leung et al., 2013), functional thinking (Ng, 2016; Antonini et al., 2020) and also algebra (Baccaglini-Frank, 2021). For example, an Italian group of research has developed AlNuSet (Algebra for Numerical Set) an environment in which we find three different components. The Algebraic Line component aims at developing the idea of independent and dependent variable dynamically while the Symbolic Manipulator was designed to promote a meaningful implementation of the algebraic manipulations in solving equations (Robotti, 2017). The Algebraic Line indeed represents the independent variable as a mobile point that can be dragged on the line. When the point is dragged, all algebraic expressions containing the variable move accordingly. Thus, the number line is a line on which the user can operate with algebraic expressions in a dynamic way. Another widespread model to specifically introduce and deal with equations and the equal sign is the balance model (Otten et al., 2019). The use of a digital balance model favours the meaningful construction of the notion of equality, as well as the interaction with objects on the pans that can be moved and thus promote considerations on conditions to keep the equality between the two sides. The feedback offered by a dynamic balance make possible to observe its behaviour according to addition or removal of objects from the pans. The large body of research about technologies and processes of teaching and learning mathematics offers us important results about potentialities of digital environments, and important insights about the process of mediation promoted by the use of an artifact within a teaching activity.

1.4. The research context: the DynaMat project

Our research project is embedded in the larger Italian research project Dynamic Math for Inclusive Education (DynaMat, <https://www.carme.center/progetto-dynamat/>) funded by MIUR (PRIN 2020BKWEXR). The principal investigator of the project is professor Anna Baccaglini-Frank, who leads the research group of the University of Pisa. The associated investigator is professor Samuele Antonini who leads the research group of the University of Florence.

DynaMat project is focused on students of the second year of Italian high schools (10th grade) that recognize themselves, or are recognized by their teachers, as having a history of failure and low achievement in mathematics. The choice of investigating cognitive processes of students with difficulties relies on the belief that a deep and detailed analysis of students' educational needs is essential to promote an efficient inclusive education. In particular, focusing on students with difficulties we can understand which are the activities and materials that meet their

needs in order to overcome the obstacles to an inclusive high-quality education in the field of mathematics.

With respect to this, the project has two main objectives. The first one consists in developing mathematics learning profiles of students with difficulties; the second one aims at investigating the role of activities developed within digital technologies on mathematics learning of such students. More specifically, the main theoretical objective is to provide insights into the impact of well-designed activities within digital environments on low achieving students both from a cognitive and an affective dimension. From a more practical point of view, the project aims at developing adaptable material for mathematical curricula, together with implementation strategies of digital activities for inclusive mathematics high school classrooms. The mathematical activities designed and experimented include mathematical notions mainly introduced in algebra during the first three years of high school.

In the beginning, the study had to be conducted within classroom settings, focusing on some low achieving students from each class as case studies. Due to the covid-pandemic we did not have the opportunity to enter classrooms, thus we decided to change the setting of our data collection. Specifically, we conducted our study in the Center for Advanced Research in Mathematics Education (CARME, <https://www.carme.center>) in Pistoia (Italy) where a ‘special’ room entitled to Maryam Mirzakhani equipped with cutting-edge technology allowed us to have at disposal all the audio-video recordings of the meetings with students. We have carried out two cycles of experimentation involving a total of 23 students of second year from different high schools. Students volunteer to participate to the project and came to the center during the afternoon once a week. The experimentation consists in a first individual interview, to point out the mathematics learning profile of the students and identify their strengths and weaknesses. After the interview there were five sessions of two hours each during which students were involved in activities with digital environments, guided by a researcher of the group, and a final individual interview. All the sessions were conducted using tablets to propose students digital environments and related tasks.

Within the DynaMat project, we specifically devoted to the study of digital activities’ impact on students learning processes. Thus, our role within the project has been to design digital environments and tasks and implement them with the students. In this context, our research problem has born and has evolved along the path through several steps, leading to focus on a model to describe both design phase and learning processes, as we briefly discuss.

1.5. Research objectives: a first formulation

The research here presented is focused on the development of well-designed activities in digital environments and on students' learning processes with respect to algebra when they deal with such activities. We have to consider that students attending the second year of high school, participating to DynaMat project, have already met some of the contents related to algebra that are the focus of our intervention. In fact, algebra is a topic that students have been grappling with since middle school and continue to encounter throughout high school. Moreover, participants to DynaMat experimentations are students with severe and persistent difficulties in mathematics, and, as we can infer from the initial interviews, perceive algebra as a matter of manipulations and rules, without meaning. The decision of the project was to frame the research in the Theory of the Commognition (Sfard, 2008) that sees mathematics as a discourse and learning as becoming more and more a participant of such a discourse. This theoretical framework allows us to study students' discourse focusing on what they are actually "saying" and not on what they do not "say". This for us has been essential in dealing with students with difficulties and a history of low achieving. Within this framework, we have not only conducted an a posteriori analysis of data from student experimentations, but we also aim to establish our a priori activity in designing and developing digital environments. .

Taking into account the research's context, the objectives of the DynaMat project, and the theoretical choices to conduct the study, we now introduce a first formulation of our research objectives.

1. Objective 1. Development of theoretical elaborations to:
 - a. design digital environments, related task situations and teaching intervention.
 - b. Investigate learning processes, considering the previous school experience of students and the presence of a digital environment in the teaching activity.
2. Objective 2. Identify processes within digital environments in which algebraic thinking is the core, taking into account the multi-dimensional nature of the topic. Specifically
 - a. Investigate the role of digital environments in promoting the analytic thinking (Kieran, 2022).
 - b. Investigate the role of digital environments in supporting the development of algebraic thinking, especially focusing on the processes of solving linear equations.

In chapter 2 I introduce and describe the theoretical framework in which this study is settled. Once I have discussed the theoretical principles that guide our investigation, I will clarify my research objectives according to the theoretical framework.

2.Theoretical framework

In this chapter I introduce and describe the theory of Commognition (Sfard, 2008), the theoretical framework within which we have conducted this study. Firstly, I describe the point of view taken on by this theory and the main theoretical tools it has offered us in our investigation. In particular, it is essential to focus on the perspective that Commognition has on mathematics, on mathematical activity and on the nature and the construction of mathematical objects. Once depicted this, we can grasp the consequent lens through which commognitive approach looks at mathematics teaching and learning, that is the lens we embrace to study students' learning processes. Specifically, I focus on the commognitive perspective about teaching and learning algebra. In addition, I discuss the role of artifacts in the theory of Commognition, converging to digital artifact, that are the core of our investigation. In this sense, I clarify our position about the didactical use and the students' interaction with dynamic and interactive environments and I highlight some aspects in the frame we want to expand with respect to the design and elaboration of activities within digital artifacts. Subsequently, I illustrate my research questions, framed in the theory of Commognition.

2.1. The theory of Commognition

The use of words *communication*, *thinking* and *discourse*, need to be clarified to grasp the essence of the commognitive perspective on mathematics and on teaching and learning mathematics.

First, to contextualize the theory, it is worth noting that it finds its roots in two main frames of reference. On one hand, it leans on the lens of Wittgenstein (1953/2003), who claims “the meaning of a word is its use in the language”. In this view, the meaning of a word is described in term of external features of the discourse: it turns into a public affair (Sfard & Kieran, 2001). On the other hand, also the learning process is perceived as a social issue. Such as Vygotsky discussed (1987) the social interaction of a person influences and shapes her learning process, such as the sociocultural context of learning. According to Vygotsky, the social dimension plays a crucial and indispensable role in the learning process studied by the commognitive perspective. In particular, Sfard quotes

“Any function in the child’s cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category.” (Vygotsky, 1987, p.11)

In this light, the ‘social plane’ is a forerunner for what is called, in the vygotskian perspective, *internalization*, or, in other words, “the internal reconstruction of an external operation” (Vygotsky, 1978, p. 56). This participationist view considers learning as something strongly social and highly situated. According to this, we look at mathematical learning processes that occur in social contexts within specific didactical activities aimed at promoting mathematical communication to grasp potential changes in the communication activities of students.

In this way, we arrive to the strong focus on communication that is essential in the theory. Indeed, communication is defined as follows:

“Communication is a collectively performed patterned activity in which action A of an individual is followed by action B of another individual so that

1. A belongs to a certain well-defined repertoire of actions known as communicational
2. Action B belongs to a repertoire of re-actions that fit A, that is, actions recurrently observed in conjunction with A. This latter repertoire is not exclusively a function of A, and it depends, among others, on factors such as the history of A (what happened prior to A), the situation in which A and B are performed, and the identities of the actor and re-actor.” (Sfard, 2008, p.86)

We have to notice that when it uses the word ‘patterned’ it refers to the fact that in communication, actions and reactions tend to align with each other in a repetitive and non-random way, and so to a certain communication action just certain re-actions can be performed. Taking this into consideration, we can think about a system as a collection of communicational actions that can be combined to create new communicational actions. The possibility to communicate is granted by the fact that the community got into a habit, in the sense that there are certain types of re-acting to certain actions; this collection of patterns is identified as historically established customs. In this sense, quoting Wittgenstein, the communication becomes a kind of game, because an action becomes a communicational one only when it is associated with specific situations and re-actions.

Considering the definition above, it is possible that the actor (A) and the re-actor (B) of a communication coincide ($A=B$). Indeed, thinking, at times characterized as ‘talking to oneself’, in this perspective, is considered as a variant of the activity of communication. Sfard (2008) writes

“Thinking is an individualized version of (interpersonal) communicating.”
(Sfard, 2008, p.81)

In this way, the theory leaves apart all the visions of thinking as something distinct, primary and independent from the acts of communication. Instead, it sees it as a form of communication.

The term *commognition* grows from this position: it is obtained by the mixture of the words communication and cognition, that are considered as two sides of the same coin.

We have now to take into account the notion of discourse, that Sfard (2008) defines as

“special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions; every discourse defines its own community of discourse; discourses in language are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives.” (Sfard, 2008, p. 297)

In this definition, the researcher introduces the notion of community of discourse, as something that is defined by a specific discourse. Indeed, each discourse is characterized by four features and the way in which they are implemented and used identify if a person belongs to the respective community of discourse. These communities of discourses are overlapping because it is not possible to enclose in a fence with clear borders a specific discourse, considering that it evolves and changes over time. To clarify the above definition, the four features depicted by Sfard as characterizing the discourse are:

- *Word use*, and thus specific keywords of a discourse and their use. In particular, “it is responsible for what the user is able to say about (and thus to see in) the world” (Sfard, 2008, p. 133). It is not just the word, but its specific use in the discourse that have to be considered.
- *Visual mediators* “are visible objects that are operated upon as a part of the process of communication” (Sfard, 2008, p.133).
- *Narrative* “is any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to endorsement or rejection with the help of discourse-specific substantiation procedures” (Sfard, 2008, p.134).
- *Routines* “are repetitive patterns characteristic of the given discourse” (Sfard, 2008, p.134) or, in other words, in order to attain communicational purpose routines are repetitive ordered actions that a person carry out.

In these features, we notice the narratives that are subjected to endorsement, and thus become *endorsed narratives*, or rejection. This process happens thanks to

procedures of substantiation guided by different criteria that vary from discourse to discourse.

2.1.1. Mathematics as discourse

In this framework, mathematics is a special type of discourse. Specifically, mathematics is defined as a discourse in which one builds theory about mathematical objects. These objects differ from objects of other discourses such as chemistry, history, zoology, that are concrete objects, which can be seen, touched, smelled, heard. Mathematical objects are abstract objects: in this sense, mathematics is an autopoietic system,

“a system that contains the objects of talk along with the talk itself and that grows incessantly ‘from inside’ when new objects are added one after another” (Sfard, 2008, p. 129).

The objects of mathematical discourse are discursive constructs build in the process of communication. To clarify this perspective, we could try to answer to the question ‘What are mathematical objects?’. On one hand, taking on the dualist perspective, we would say that they are independently existing objects, of which signifier are only representers of it. On the other hand, as we do in this study, we could embrace the commognitive perspective according to which mathematical objects are discursive devices that exist only in the discourse and are introduced to tell stories about phenomena (Figure 2.1). In this sense, discourse does not help in learning mathematics, but it is exactly what is learnt in mathematical classroom. Mathematics, being a discourse, has the four features described above:

- *Word use*, in mathematics there are words to talk about quantity and shapes. Even if these words can appear and be used in colloquial and everyday discourses, the way in which they are used in the mathematical discourse is specific and disciplined.
- *Visual mediators*, that in mathematics are mainly symbolic artifacts, such as algebraic notation or a function’s cartesian graph. The visual mediators of the mathematical discourse are created exactly for this kind of communication.
- *Narrative*, that in mathematics are mathematical theory, and more specifically definitions, theorems, proofs. The rules of endorsement of these narratives rely on “purely deductive relations between narratives” (Sfard, 2008, p.134).
- *Routine*, that in mathematics “can be seen in almost any aspect of mathematical discourses: in mathematical forms of categorizing, in

mathematical modes of attending to the environment, in ways of viewing situations as ‘the same’ or different” (Sfard, 2008, pp. 134-135).

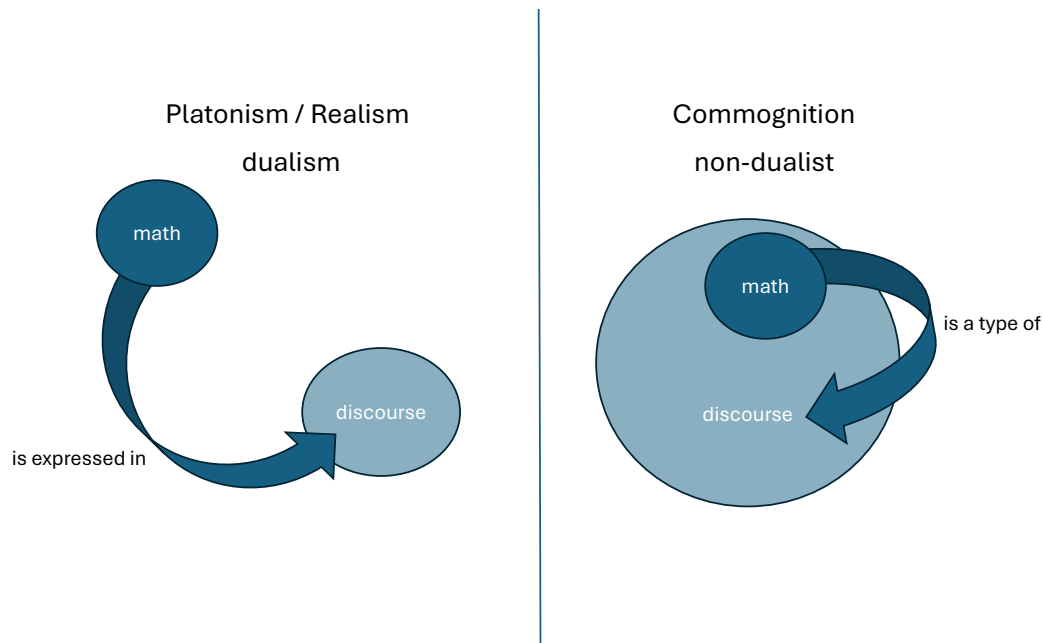


Figure 2.1: Distinction between the dualist and the commognitive perspective with respect to discourse and mathematics

Turning back to the idea of mathematics as an autopoietic system we now focus on the notion of discursive object. Its definition offered by Commognition is “The (discursive) object signified by S (or simply object S) in a given discourse on S is the realization tree of S within this discourse” (Sfard, 2008, p. 166). To understand this, we consider that “Signifiers are words or symbols that function as nouns in utterances of discourse participants, whereas the term realization of a signifier S refers to a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about S” (Sfard, 2008, p. 154). More specifically, “Realization of the signifier S is a perceptually accessible thing S' so that every *endorsed narrative* about S can be translated according to well defined rules into an endorsed narrative about S'” (Sfard, 2008, p. 154).

In this light, a mathematical signifier has several different possible realizations and each of the endorsed narrative that we can tell about it can be translated into an endorsed narrative about one of its realizations. Commognition stresses that realizations can be visual, such as verbal (written words or algebraic symbols), iconic, concrete, gestural, or vocal, as spoken words. In addition, the relation we have between a signifier and one of its realizations is symmetrical, in the sense that each realization can be itself a signifier and have its realizations. Indeed, Sfard (2008) stresses “almost any mathematical realization may be used as a signifier and then realized even further.” (Sfard, 2008, p. 165). Moreover, the notion of realization

tree is introduced: “hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations, as well as the realizations of these latter realizations, and so forth” (Sfard, 2008, p. 301). As a matter of fact, any signifier can be perceived as the origin of a tree of realizations. In this possible tree, each node is both a realization of the node above and together a signifier of the nodes just beneath it. For example, Sfard proposes the realization tree in Figure 2.2. At the top we see the signifier “Solution of the equation $7x + 4 = 5x + 8$ ” and some possible realizations such as the cartesian plane in which the two straight lines $y = 7x + 4$ and $y = 5x + 8$ intersect in the point with the coordinates $x=2$ and $y=18$, where 2 is the solution of the equation.

We can now appreciate the definition of discursive object mentioned above. Commognition specifies that realizations trees are personal construct even if they grow in social contexts. The same signifier could be realized in different ways by interlocutors: this could take to unrecognized ineffective communication. Moreover, observing them can grant us information about a person’s discourse. Specifically, it can offer us information about students’ discourse on a specific mathematical object. Finally, a realization tree could be strongly situated, influenced by situation and interlocutors, and so a certain realization could be taken into account by a student in a specific situation, but it could not be considered in another.

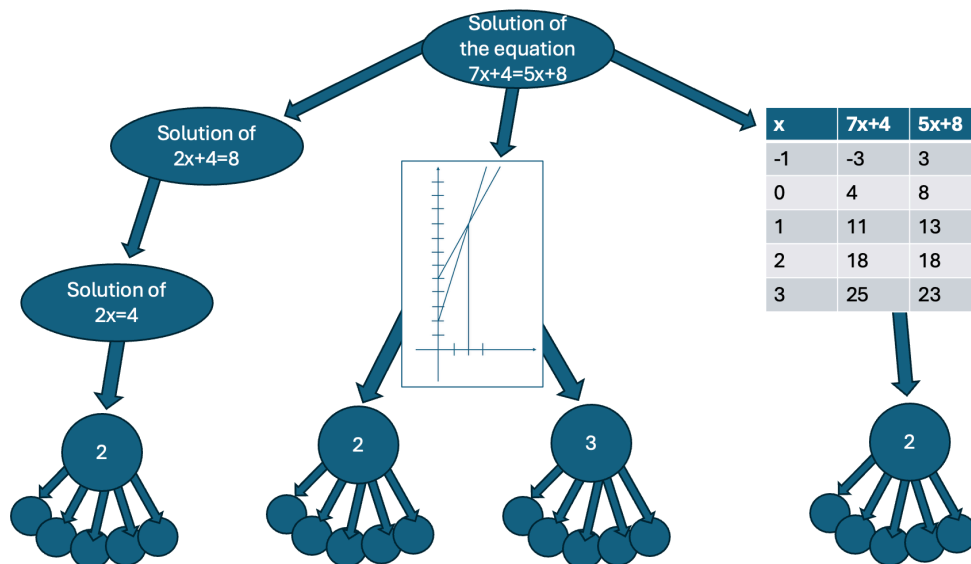


Figure 2.2. A realization tree of the signifier “solution of the equation $7x + 4 = 5x + 8$.” (Sfard, 2008, p. 165)

2.1.2. The construction of mathematical objects

The process of construction of mathematical objects, that make us use mathematical words and signifiers as if they signified discourse-independent objects is called objectification. This process is constituted by two discursive transformations that are *nominalization* and *alienation*.

Alienation consists in

“using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings” (Sfard, 2008, p. 295).

In this process, the subject and her actions disappear; implying an independent status for the object; for instance, there is an alienation when a subject goes from an utterance of the form “I sum 2 and 3 and I obtain 5” with the utterance “The sum of 2 and 3 is 5”. Hence, “this effect of the ‘objective existence’ is attained by eradicating the human subject and using the object-signifying noun in impersonal narrative” (Sfard, 2021).

The former, nominalization, is described as “replacing specific portions of text with a special noun” and can happen through three different processes that are now described.

- *Reification* consists in replacing stories about processes with stories of objects, or in other word, verbs with nouns. This requires involving new nouns or written symbols in the discourse. For example, the action of counting can be replaced with the mathematical object called number: the utterance ‘when I count the vertexes of a triangle, I arrive at the word three’ can be replaced with the word ‘three’.
- *Saming* consists in calling with the same name a number of things that previously were not considered to be the same. For example, considering Figure 2.2, the value of coordinate x of the intersection’s point of the two lines or the ‘solution of $2x = 4$ ’ could be considered as two different things that, after a process of saming, become ‘solution of the equation $7x + 4 = 5x + 8$ ’.
- *Encapsulation* consists in replacing plural with singular. In other words, a specific set of objects, about which stories have been told in plural, after the encapsulation process, are told in singular. For instance, if we consider a system of linear equations in two variables, such as

$$\begin{cases} x - y = -1 \\ x + y = 7 \end{cases}$$

its solution is composed by two numbers, 3 for the unknown x and 4 for unknown y , that as a pair (3,4) has to be considered an unique object. A mathematical object is an abstract object, because it is a discursive object, and so it exists only in the discourse, and its construction involves a reification process. On the opposite side, there are concrete primary objects, that are accessible objects on which a person can act. The transition from a primary concrete object to an abstract object occurs through an intertwining of processes thanks to which, in the end, the abstract object has many realizations and properties. In particular, we have to underline that all the objects constructed without reification are concrete. For example, the equation " $7x + 4 = 5x + 8$ " can be perceived, by a person, as a string of symbol on which perform a patterned scheme of actions, and in this it can be considered a primary concrete object. On the other hand, it could be seen as an abstract object, which realizations could be the representation on the Cartesian plane of two intersecting straight lines, or a table in which according to the value of the variable x , are evaluated the value of the two expressions ($7x + 4$ and $5x + 8$), or the equation " $2x = 4$ ". When a person starts to talk in term of realizations and properties of an objects, we can affirm that an objectification process has happened and so, a mathematical object has been constructed.

As a result of how mathematical objects are constructed, mathematical discourse develops recursively by the successive addition of ever-new discursive layers, recursive layers of discourse about discourse. Specifically, each discursive layer involves the construction of new objects which make it possible to express everything that could already be said, but, at the same time, also to produce a discourse on the previous discourses. In this sense, Sfard says "objectification of existing discourses is part and parcel of the construction process" (Sfard, 2008, p. 118). The growth in complexity of mathematical discourse is described by the processes of *expansion* and *compression* of the discourse. These processes are defined as follows:

- *discursive expansion* is a "growth in the volume and complexity of a discourse" (Sfard, 2008, p. 298);
 - *endogenous discursive expansion* is "an increase in the amount and complexity of discursive routines and endorsed narratives within the given discourse" (Sfard, 2008, p. 298); this happens when the volume of a discourse growth thanks to its use. For instance, the combination of different arithmetic procedures that leads to longer and more complex computational processes;
 - *exogenous discursive expansion* is "a discursive growth that expresses itself in proliferation of new discourses" (Sfard, 2008, p.

298). For instance, historically, two separated discourses, one about the now call whole numbers, and the other about ratios as pairs of numbers became subsets of the new discourse on rational numbers;

- *discursive compression* is the “developmental process complementary to discursive expansion that occurs through *intradiscursive saming* (endogenous compression) or by conflating of several different discourses into one *subsuming discourse* (exogenous compression)” (Sfard, 2008, p. 297). The former one consists in rising to the metalevel identifying commonalities between different processes in the same discourse; the latter, exogenous compression, consists in conflating several discourses into one, rising to the metalevel and at the same time objectifying the lower-level processes. “The result is an immediate drop in the perceived complexity of the discourse” (Sfard, 2008, p. 123).

Related to these processes of mathematical discourse’s development through the addition of new discursive lawyers, Sfard individuates two kind of discourses that are *object-level discourse* as a “discourse about objects” (Sfard, 2008, p. 300) while metalevel discourse as a “discourse about another (object-level) discourse” (Sfard, 2008, p. 299). Thus, while the first one is a discourse in which the discursants talk about objects, in the second one they talk about another discourse.

In conclusion, “mathematics is an autopoietic system that grows by annexing its own metadiscourses, and this means, among others, that what counts as a metarule in one mathematical discourse will give rise to an object-level rule as soon as the present metadiscourse turns into a full-fledged part of the mathematics itself” (Sfard, 2008, p. 201).

2.1.3.A way to classify the mathematical discourse: the four properties

Accordingly to the discussion of the previous sections, in Commognition mathematics is seen as a discourse based on a collection of historically established ways of communication. This system of habits has been developed through ages by mathematicians that agree about a shared way of talking about mathematics. Now, it seems natural to ask ourselves how a discourse can be identified as mathematical. We have to take into consideration the continuous evolution and transformation that, in general, discourses are subjected to, due to their specificity as a collective activity. At the same time, “they preserve their identities through the continuous transformations” (Sfard, 2008, p. 131). Commognition considers the four properties (words use, visual mediators, narratives and routines) as critical in deciding whether an instance is or not a mathematical one. I refer to this

characterization both in the elaboration of some theoretical tools and in the following analysis of students' discourses.

We observe that this characterization offered by commognitive perspective is a tool to affirm if a certain discourse is or not a mathematical one. A strand of research has recently addressed to investigating mathematics teaching and learning through the commognitive lens and thus using these operative tools. In particular, several studies deal with examining students' discourse, in particular the one emerging thanks to the interaction with digital environments (Sinclair & Moss, 2012; Ng, 2016; Lisarelli, 2019; Antonini et al., 2020). These investigations show that the analysis of word use, visual mediators, routine and narrative used by students give a way to look at their learning, as their changing in communication. In this study I follow the current overmentioned; in particular I focus on emerging discourses about indeterminate quantities and relation among them in well-designed digital environments and I study how these discourses shape the formal mathematical discourse. I give a deeper explanation of my intention in formulating my research questions later in this chapter.

Routine

The latter feature of the discourse described by Sfard, routine, worths an in-depth discussion, because after its definition above, this construct has changed and evolved within the theory. To accomplish this aim, it is necessary to introduce some preliminary definitions. According to Lavie and colleagues (2019) "learning is due to our ability to react to new situations by utilizing the memories they elicit. This tendency for modeling our present actions on what was done in the past results in patterns of actions that we term routines. Routinization of our actions is what learning seems to be all about" (Lavie et al., 2019, p. 156). They introduce the term individualizing as indicating the phenomena consisting in developing personal routine by acquiring those of experienced actors, in the transition from being a spectator to become an implementer of those routine. This is in line with the idea of individualization introduced by Sfard in talking about learning (2008) and discussed in the next section. Moreover, they define a task situation as "any setting in which a person considers herself bound to act—to do something" (Lavie et al., 2019, p. 159). To this set belongs all the situation in which a teacher asks something to students during a classroom activity. When she finds in a task situation, a person acts, even if it is new, thanks to her previous experience. Specifically, there are past situations to which the person can refer to because she perceives them as sufficiently similar to the task she is accomplishing. In this, she tries to perform actions that has been done by her or by another person in that situation. The notion of *precedent-search-space* (PSS) stands for the collection of situations and experiences, preselected by

restricting the set of possible choices, to which a person refers to perform an action previously accomplished. The replication involves a selection, because some aspects of the precedent situation could be the same while other could differ from the actual situations.

According to these, we can now introduce a new definition of routine thanks to task situation, as the composition of the task interpreted by the person together with the procedure she executed to perform the task. It must be underlined that within this definition, a routine is something extremely personal. Therefore, now on, we consider routine as the pair composed by a task and a procedure, within a task situation. Lavie and colleagues distinguish among several types of routine. They characterize:

- Practical routine: “a person interprets the task situation as requiring a change, re-organization or re-positioning of objects” (p.163); are part of this category biking, swimming, cleaning, and so on. These routines can be divided into two categories:
 - ritual is a process-oriented routine, strongly situated, that happens when we feel expected by others to do something, in particular by someone we perceived as superior; thus, ritual has a strong social component;
 - deed is an outcome-oriented routine, where the outcome is expected to be new, transformed or rearranged objects.
- Discursive routine: “a person interprets the task situation as requiring a communicational action” (p. 163). Discursive routines are pattern followed communicating with others or with ourselves. These routines can be divided into two categories:
 - ritual is a process-oriented routine, strongly situated, that happens when we feel expected by others to do something, in particular by someone we perceived as superior; thus, ritual has a strong social component;
 - exploration is a process-oriented routine, with the aim to have as outcome a new endorsable narrative.

Considering this distinction, we have to stress the fact that ritual on one hand and deed-exploration on the other are just the extreme cases in a continuum of forms. There is so a full range of possibilities that depends on the performer's ability to divide the task and the procedure (Lavie et al., 2019). Until when a procedure is considered as a part of the task, it cannot be taken as a full-fledged exploration: the

user feels that this exact procedure must be performed but does not focus on the specific outcome. Taking into account classrooms, we easily imagine the wide collection of procedures performed just in the will to please the teacher. For instance, if we think about solving the equation ' $7 + 3x = 10 + 2x$ ', the manipulations performed in order to obtain the expression ' $x = 3$ ' can often be described as ritual when students do not relate the above-mentioned expression with the fact that 3 is the number we have to substitute to x in order to obtain an equality between the two expressions given ($7 + 3x$ and $10 + 2x$). On the other hand, if we think about expert mathematicians, we imagine that their performed routines are strongly explorative.

Lavie and colleagues, in discussing about routine, highlight that it is not possible, as a learner, to deal with germinal routines (those routines from which new discourses are likely to develop (p. 167)) in an explorative way, but it is necessary to adopt them as rituals. These routines, and all the others developed as rituals, should transform into explorations, thanks to the process of de-ritualization. This consists in a shift of the performer's attention from the process to the outcome. This change is gradual and slow (Sfard & Prusak, 2005; Lavie & Sfard, 2016) and to see it is necessary to observe many students' performance in a specific task situation. These researchers pinpoint some characteristics that are indicators of an explorative routine that are flexibility, bondedness, applicability, performer's agentivity, objectification of the discourse, and substantiability (Lavie et al., 2019).

Endorsed narratives

Another feature that characterizes the discourse are the narratives, and, more specifically, the endorsed narratives, that for mathematics are theorem, definitions, proofs, and so on. The endorsed narratives are tied together with routine. Indeed, narratives can accompany the mathematical activity, such as accomplishing a specific routine, as well as can guide it. From expert point of view, in the specific case of algebra and equations, the typical change sides – change sign rule is guided by endorsed narratives about the ways in which, manipulating an equality between two algebraic expressions, the condition of equality is maintained. These narratives, endorsed in an expert's mathematical discourse, could not be endorsed in a student's discourse. Indeed, if a person looks at a task situation in which there is an equation to be solved as a request of performing actions with respect to the change sides – change sign rule, in order to please someone else, she is not guided by endorsed narratives, but only by procedures learnt by heart. This specific aspect is essential while facing, as teachers, procedure for solving equations. From our perspective, that is clarified later in the chapter, the role of endorsable narratives in the process of learning equations and respective procedures is crucial. Specifically, in our study we focus on the

assumptions that guide students' production of narratives and on the role played by the digital environments in the formulation of such assumptions. Our attention to the production of narratives is clarified in the formulation of the research objectives.

2.1.4. Learning Mathematics in the commognitive perspective

The vision of mathematics offered by Commognition influences the view about how learning mathematics is characterized. In the introduction of the chapter, it has been stressed the strong participationist perspective on learning the commognitive framework has, taking also into account its roots in the vygotskian way of thinking. Commognition characterizes learning as a process in which we change our patterned ways of doing things. In particular, Sfard stresses that these ways are changed in order to resemble ways of acting that have matured and established during years. In this sense, she calls activities all these instituted ways of acting, such as cooking, dressing, talking, and so on and she describes learning as something through which a learner becomes a participant of a historically developed activity, and thus becomes a participant in the community of that specific discourse. With respect to mathematics learning, Sfard and Cobb highlight "For Vygotsky, to learn meant to become a competent participant in activities that characterize the times and the culture into which people are born. From this theoretical perspective, learning mathematics is reconceptualized as becoming a competent participant in mathematical activity" (2022, p. 474). Taking this perspective, we naturally ask ourselves which are the activities, historically developed, that we try to be engaged in learning mathematics.

In answering this question, we have to take the commognitive lens about mathematics as a special type of discourse. In this light, "to learn mathematics means to individualize the historically established discourse known as mathematical" (Chan & Sfard, 2020). The term "individualize" again echoing Vygotsky with respect to *internalization*, here refers to the process of *individualization* that is described as "process that results in the ability to enact individually an activity that previously could only be performed with others" (Sfard, 2008, p. 299). Therefore, the process of changing ways of doing things for mathematics learning consists in changing the ways in which a person communicates about mathematical objects, in order to move closer to the potential experts' mathematical discourse. In conclusion, to learn mathematics means to change the mathematical discourse.

To describe processes of learning, Commognition distinguishes between *object-level learning* and *meta-level learning*. To understand this distinction, it is necessary to introduce two categories of rules governing the discourse. In the book Sfard writes,

“*Object-level rules* are narratives about regularities in the behavior of objects of the discourse, whereas *metarules* define patterns in the activity of the discursants trying to produce and substantiate object-level narratives” (p. 201). For instance, to the former category belongs the narrative “The sum of the angles in a polygon with n sides equals $(n - 2) \cdot 180^\circ$ ” (p. 201). To the latter belong to the rules thanks to which we formulate and substantiate the object-level rules. According to this, follow the definition of the two kinds of learning:

- *object-level learning* “resulting in endogenous discursive expansion” (Sfard, 2008, p. 300); in other word, there is an object-level learning when there is an expansion of the discourse thanks to the increase in the numbers of words and the production of new routines and endorsed narratives;
- *metalevel learning* that “expresses itself in the change in the metarules of the discourse” (Sfard, 2008, p. 300); this learning is usually related to an exogenous change in the discourse, in the sense that tasks that previously were done in a familiar way, now are no longer done as such.

While the object-level learning seems quite easy to promote, about metalevel learning we have to make some points. Taking the word of Sfard, “considering the contingency of metadiscursive rules – the fact that these rules are a matter of a useful custom rather than of necessity – it is rather implausible that learners would initiate a metalevel change by themselves” (p. 256). This perspective leads to see metalevel learning as a process that happens thanks to the direct encounter of the learner with the new discourse. In particular, to Sfard, this learning occurs within a *commognitive conflict*, “a situation in which different discursants are acting according to different metarules” (p. 256). Thus, learning as a process of changing one ways of communicating, is the result of an interaction with others, because it arose from differences in interlocutors’ communicational ways that generate a conflict. This change in what we say is necessary in order to change what we see: accordingly to the commognitive perspective, what we say about the world reflects what we are able to perceive and endorse. Our experience of reality determines what we are able to say. In this sense it is necessary to face with a different (with respect to metarules) discourse, to obtain a metalevel learning. Moreover, commognitive conflict is, in this perspective, an indispensable cause of metalevel learning. If a learner does not encounter other people’s example, he has no need and incentive to change his communicational ways. Finally, it is worth to notice that a commognitive conflict is originated by *incommensurable* discourses, or, in other words, discourses which use words in different ways, whose routine are different, which follow different rules of narratives endorsability. They have no common criteria to decide whether one should be accepted and the other refused. The

conflict is resolved only in accepting and choosing one of the discourses and abandoning the other.

To sum up, to observe learning through the lens of Commognition, means to look for changes in the students' discourses. The analysis of words' use, visual mediators involved, routine performed gains insight into their mathematical discourse and so their learning. To conduct our analysis, it is now worth to discuss which modification we look for in order to "observe" an algebraic learning.

Learning algebra in the commognitive perspective

I want to focus on the theme that is the core of my investigation in order to understand what algebra is and what means learning algebra within this theory, now that is clear what is meant by learning in the commognitive framework.

Algebra is presented as an example of compression for endogenous expansion. In this compression, through an intradiscursive saming, there is a shift to the metalevel identifying commonalities between different processes within the same discourse. That is, noticing the fact that the sum of any two numbers is equivalent to the sum of the same two numbers in reverse order, we declare that $p + q = q + p$. This "is a shortcut for the sentence *For every two numbers p and q , the numerical expression $p + q$ is equivalent to the numerical expression $q + p$* " (Sfard, 2008, p.120). Another example could be the expression $[(3 \cdot x - 4) \cdot 2 + 20]/6 - x$, corresponding to the discursive process "Choose a number; multiply by three; subtract four; multiply by two; add twenty; divide by six; subtract your starting number". The process could be repeated indefinitely for every number or can be compressed into the expression written above.

In this way "Algebra may thus be described as metaarithmetic or, more precisely, as the unification of arithmetic with its own metadiscourse" (Sfard, 2008, p. 120). In this light, algebraic expressions embed the duality of object and process: they stand for a compression of calculations on a set of numbers and at the same time they tell the story about this computational process. Taking this into account, school algebra is the mathematical discourse on arithmetical relations and processes (Caspi & Sfard, 2012).

This vision helps us in perceiving the complexity in the transition from arithmetic to algebra. The compression needed to shift from different processes to a unique discourse that encapsulates all of them is a delicate passage to which in school class is not devoted a lot of care. Instead, the focus is often on manipulations and calculations that stands for nothing but rituals to please the teacher. This could lead to a ritual participation to the mathematical discourse, guided by procedures unrelated from algebraic 'facts' such as the example above. One of our hypotheses,

that is clarified in the next chapter, is that the participation in a discourse ‘far’ from the mathematical one, but that still involves, from our perspective as experts, indeterminate quantities, equal sign, the relation between quantities and functions, could help in promoting the needed compression to pass from arithmetic to algebra fostering an explorative participation. In particular, we think that taking advantage of features of digital environments could even more foster this transition.

2.1.5. Digital environments through the commognitive lens

As discussed in the first chapter, the power of digital environments in promoting students’ mathematical achievement is by now largely recognized: their educational potential has been highlighted by research since decades. A recent strand of studies has dealt with the investigation of digital artifact and their impact on learning processes through the lens of the commognitive perspective. In particular, some research have focused on the learning of geometry describing how the interaction with dynamic geometry environments (DGE) allowed the shift from a tool-based discourse in which new terms were coined, such as *paralysed triangle*, to a generalized discourse in which the terms scalene, isosceles and equilateral triangles are used accordingly to the mathematical discourse (Kaur, 2015). At the same time, Commognition seems to offer powerful tools to describe learning about calculus in dynamic environments. Ng (2014, 2016) has dedicated to the interplay between gestures and dragging in the interaction with touchscreen DGE. She brings into play the commognitive definition of gestures as visual mediators and focuses on the differences between static and dynamic mediators. Indeed, when dealing with static environment such as paper and pencil, students mainly use utterances accompanied by deictic gestures and sometimes gestures to mediate temporal relationship. While using dynamic geometry environments, the students’ communication evolves in a new form of gesture that Ng has called *dragsturing* (2014). This term indicates an action that subsumes the dragging of an object on the screen and the consequent indirect motion of other objects that stands for communicational gestures. Dragsturing accomplish at two main aims: to move something on the screen and to communicate. Ng stresses that dragsturing promoted a new students’ communication, taking into account all the four characteristics of discourse, in which motion and temporality were protagonists in the description, changing the way of talking about calculus from something static and locked on the paper, to a dynamic process (Ng, 2016). In tune with the idea of dragsturing, other studies have focused on the learning of calculus and in particular on functions in digital environments. Antonini and colleagues (2020) propose a new characterization for a specific kind of digital visual mediators, considering not only their dynamic feature but also the interactive one. They define *DIMs*, *Dynamic*

Interactive Mediators, as digital objects that “are both dynamic – they change over time – and interactive – they respond to a person’s manipulations” (Antonini et al., 2020, p. 5). In this light, they describe objects like dynagraph, a dynamic representation of functions, as DIM, and they show how these DIMs can be not only mediators, but also interlocutors in the students’ discourse, which become discourse *with* a DIM. According to this characterization, Baccaglini-Frank (2021) conducted an investigation focusing on two low achieving students involved in a task about inequalities and having at disposal a DGE. In the study, she shows how the digital objects have been first of all dynamic interactive mediators of the students’ discourse but also have become protagonists of students’ stories. Indeed, the pupils were engaged in a discourse *about* the DIMs producing meaningful narratives. Moreover, their discourse was strongly explorative, leaving behind all the manipulations and procedures often associated to the resolution of algebraic inequalities.

In this study I talk about *dynamic explorative objects* (DEO) leaning on the notion of explorative object introduced by Muller and colleagues (2009) as “an interactive and dynamic computer-based model or tool that capitalizes on visualization and is developed to explore a mathematical concept or conjecture, or real-world situation” (p. 64). The notion of DEO allows me, in particular in describing the design phase, to talk about both the digital environments and all the objects it embeds as “just” objects that potentially can become something else for students. Indeed, they can become mediators in a students’ communication, but also interlocutors within a discourse, or protagonists for students’ stories, and this will be point out in the analysis session.

To conclude the section about the theoretical framework, we want to clarify why we have decided to use this lens for our study.

This perspective, for which learning mathematics involves building mathematical communicative competence, is suitable for the current study because it establishes a strong link between mathematics learning and communication. On one hand, according to Commognition, students could change their ways of communicating thanks to participation in mathematical activities. In this study we look at mathematical learning within social contexts, even if not in the classroom setting, through well-designed specific activities. On the other hand, the theory of Commognition furnishes not only operational tools to characterize what mathematical objects are and how they live in the communication but also instruments to identify a mathematical discourse. In particular, it provides us lenses through which look at the students’ communication and objectively observe and describe its features. Moreover, it puts the light on which changes in the discourse we should look for in order to identify learning processes about algebra,

and specifically the transition from an arithmetic discourse to an algebraic one. Finally, the vision on thinking as an individualized version of communicating allows us not to make hypotheses about students' thinking as something detached from what students communicate, and this is a relevant aspect in trying to avoid subjective researcher's choices in studying learning processes.

Thanks to this perspective, we thus have a collection of tools to talk about students' discourse and gain insight into their learning. In conducting the design and re-design of the activities within digital environments, we perceive the need to elaborate tools to talk about our activities as researchers. In other words, we found ourselves thinking about our process of elaboration and design but missing not only the words, but also a model to describe it within the commognitive framework. This leads us to interrogate ourselves in order to develop instruments to describe our design activities.

2.2. Research objectives: a second formulation

In the first chapter we have discussed multifaceted nature of algebra and the consequent possible didactical approaches to the theme, taking into account the long debate about what algebra is. Moreover, we have highlighted the central nature of algebraic thinking in mathematical activity at all levels that makes it essential to promote a meaningful learning of mathematical contents. Despite its essential role, algebra is often a source of difficulties that last from middle schools to university courses. Research reports the main problem of giving sense to symbols and to deal with them, and with the equal sign, in a relational way. In particular, several studies (Bednarz et al., 1996; Kieran, 2022) have emphasized that the intricate nature of algebra must be taken into account in didactical interventions to promote the construction of meaningful algebraic objects. A step towards this aim has been accomplished thanks to the advent of technologies in teaching and learning processes.

Thanks to the theoretical framework we have presented and discussed, we can now propose a second formulation of the research questions that lead this study.

Within the first design phase, we have realized the need to have theoretical tools to describe our considerations and choices about the DEOs; specifically, when we were elaborating the DEOs, we reflected upon their relations with the mathematical contents of our interest. Moreover, we aimed at reinvesting such tools to investigate learning processes taking into account the presence of pre-existing students' discourse and of DEOs. This is the focus of the first research objective discussed and elaborated in chapter 4.

On the other hand, we have previously discussed that our objective is to study the impact of teaching interventions within digital environments in promoting algebraic thinking. Framing our study in the commognitive perspective, we have to clarify our position with respect to the notion of analytic thinking offered by Kieran (2022). Within the commognitive framework, we embrace the idea of mathematics as a discourse about mathematical objects, characterized by four main features. In addressing mathematics learning, we focus on students' construction of mathematical discourse, examining the emergence and development of the four features with respect to a specific mathematical content. In this light, we frame analytic thinking as a specific discourse. We consider *analytic discourse*, in tune with Kieran (2022), a discourse in which the objects of the discourse are indeterminate quantities that are treated as if they were known and relations among them. Thus, recalling Wittgenstein perspective, we especially look at words and at their use in the discourse to identify instances of analytic discourse. In addition, according to Radford (2014), algebraic thinking involves indeterminate numbers that “have to be named or symbolised. Now this symbolisation may be accomplished in various ways. One can use alphanumeric signs—but not necessarily. The denotation of indeterminate quantities can also be symbolised through natural language, gestures, unconventional signs, or even a mixture of these” (Radford, 2014, p.260). In tune with these considerations, we have hypothesized the emergence of analytic discourse within task situations involving DEOs, in particular the use of “unconventional” words and visual mediators, but also the production of narratives and the development of routine that as experts we can recognize as analytic discourse even if not involving the alphanumeric symbolism. A specific aspect we want to focus on is how the interaction with specifically designed DEOs shapes the students' discourse the construction of endorsable narratives. A main feature of this study is the fact that the subjects are students with a history of difficulties in mathematics who have already encountered, in their past schooling, the themes we propose in the activities. In fact, algebra is a topic that students have been grappling with since middle school and continue to encounter throughout high school. This frame our intervention as a recover intervention: students, as we understand from the initial interview, already participate to algebraic discourse, mainly characterized by the enactment of meaningless procedures and manipulations. Our aim is to take advantage of the digital environments' potentialities and tools to promote the growth of a new discourse about these environments. This discourse should foster the re-construction of the previous students' algebraic discourse. We would like, within the analysis, to highlight some processes related to this re-construction and to characterize them. This is the core of the second research objective.

In summary, the two main objectives are:

- Objective 1: integration of theoretical tools, in the theory of Commognition (Sfard, 2008) to:
 - a. design DEOs, related task situations and teaching interventions in terms of promotion of specific discourses.
 - b. Investigate learning processes considering
 - i. the role of students' existing discourses;
 - ii. the role of a DEO and of discourses related to it.
- Objective 2: identify processes in the interaction with the DEOs that, as expert, we consider significant for the growth and the development of the algebraic discourse. Specifically:
 - a. investigate the role of DEOs in promoting analytic discourse;
 - b. Investigate the role of DEOs in supporting the theoretical organization of algebraic discourse. We want to focus on the formulation and organization of endorsed narratives with respect to the DEO that can be reinvested in the formulation and organization of endorsed narratives with respect to algebraic discourse, to guide the process of equations solving.

The two aspects concerning objective 2 are closely linked to each other. Analytic discourse involves dealing with indeterminate quantities as if they were known, with a specific focus on equality and properties that maintain equality between quantities depending on such indeterminate values. This aspect is related to what we identify as the theoretical organization of algebraic discourse. Indeed, with this expression we refer to the elaboration and organization of endorsed narratives that guide the process of equations solving, and thus the manipulations that maintain the relation of equality between quantities. With respect to objective 2 we thus aim at looking generally at students' discourse within DEO's task situations, analyzing its development along the activities with respect to our aim of promoting analytic discourse. We especially look at the formulation of narratives supporting processes that have the aim of finding values of indeterminate quantities. Such narratives could be reinvested in the process of solving linear equations: we are strongly interested in such transition, that is the key point in our perspective and choice of working with DEOs to promote the growth of algebraic discourse. To sum up, the a posteriori analysis is focused on episodes that shed a light on potentialities as well as weakness of the designed DEOs in promoting analytic discourse. In this, we specifically look at the construction and the use of narratives supporting processes we see related to equations solving.

3. The research process: an overview

In this chapter I aim at providing an overview of the research process that has developed from January 2022 to September 2024. Firstly, I offer a description of the research context in which I have been moving, the DynaMat project (PRIN 2020BKWEXR), describing its main objectives and the methodological approach it has chosen to accomplish them, and how it has evolved along the last three years. Secondly, recalling my research objectives, I describe how my research project has developed simultaneously with the DynaMat project. Specifically, I describe the methodology according to which my research has been conducted, and I especially focus on the first research objective and how it has evolved along the process.

In chapter 1 I have presented the research context in which my project has emerged and developed. The DynaMat project has started officially in May 2022 and concludes in May 2025. DynaMat aims at promoting an efficient inclusive education, and it has the firm conviction that to do so it is necessary to examine students' educational needs. The project focalizes on students with difficulties in mathematics, more specifically, students who recognize themselves and are recognized by their teachers as having a long history of failure and low achievement in mathematics. Resting on results of previous research (see for instance Leung & Baccaglini – Frank, 2017), DynaMat wants to take advantage of the potentialities offered by the digital technologies to investigate processes of teaching and learning of these students and to delve into their educational needs in order to develop theoretical and practical tools to promote a more inclusive education. The investigation of the project is conducted with the theory of Commognition that allows us to combine together individual and societal learning, avoiding the discussion about students' deficit. Indeed, persistent low achievement and failure in mathematics, through the lens of the commognitive framework are seen as an exclusion from a fully developed mathematical discourse (Heyd-Metzuyanim et al., 2016). This exclusion can find reasons in students' discourse on mathematical objects as well as in their way of identifying themselves while they are doing mathematics. Thus, Commognition permits to consider both cognitive and affective aspects in the processes of learning.

Actually, DynaMat project has these main objectives:

- At a theoretical level, the project aims at providing insights into:
 - impact of technologies-enhanced activities on low achieving students' mathematical discourse (cognitive aspects);

- impact of digital technologies-enhanced activities on these students' identifying discourse in mathematics (affective aspects);
- students' mathematical learning profile and their persistence over time.
- At a practical level, the results should involve:
 - the declination of tested principles for designing inclusive mathematical technologies-enhanced activities;
 - the development of adaptable mathematics curriculum material for high school in the form of technologies-enhanced activities sets for inclusive teaching and learning of algebra and geometry;
 - the identification of implementation strategies of technologies-enhanced activities for inclusive mathematics high school classrooms (this has important implications for teacher education);
 - a website for teachers and other educators organizing all the tested materials, with teacher guides providing the necessary support for their implementation also after the end of the project.

To reach these aims, DynaMat project embraces the Design Based Research (DBR) approach. Barab and Squire (2004) define Design Based Research as

“a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings”. (Barab and Squire, 2004)

The aims of DBR are mainly two: to elaborate and study a teaching and learning trajectory and to grasp new information about learning. The nature of Design Based Research is exploratory: it questions methods and activities that do not yet exist, but that are elaborated within their own study. Indeed, in describing design experiments with respect to DBR, Cobb and colleagues affirm

“Prototypically, design experiments entail both, ‘engineering’ particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting this. This design context is subject to test and revision and successive iterations that result play a role similar to that of systematic variation in experiment.” (Cobb et al., 2003, p. 9)

In doing so, DBR is conducted along iterative cycles. Specifically, these cycles consist of

- design

- implementation
- a posteriori analysis of data collected and re-design

Research conducted within this approach are strongly subjected to various factors according to the context in which they are conducted and depend on aspects that are not always available and characterizable a priori. In the setting of DBR, it is essential to clarify the context in which the research is conducted and the design principles according to which the activities are elaborated.

Taking this into consideration, DynaMat project develops a Design Based Research study aimed at investigating an inclusive educational approach to support mathematics learning of high school students with a history of low achievement or even diagnosed as having learning difficulty in mathematics. Specifically, the project wants to test some design principles to promote such an inclusive educational approach.

3.1. The development of the DynaMat project

According to DBR, we have developed our research in three cycles of design-implementation-a posteriori analysis and re-design. In Table 3-1 I report the timeline of the DynaMat project, starting from May 2022 to September 2024. As depicted in Table 3-1, the a posteriori analysis phase is, for the first and the second cycles, coincident with an a priori analysis and design phase. This happens because, in tune with DBR, during the a posteriori analysis we identified weakness in the design of DEOs and look for solutions to them through new DEOs or task situations, conducting a priori analysis and design. Therefore, there are periods in which one cycle and the successive one overlaps, because the a posteriori analysis of the first one intertwines with the re-design phase of the other.

Table 3-1: Timeline of DynaMat project

	May 2022 – Sep 2022	Oct 2022 – Dec 2022	Jan 2023 – Mar 2023	Apr 2023 – May 2023	Jun 2023 – Sep 2023	Oct 2023 – Dec 2023	Jan 2024 – Sep 2024
A priori analysis – design							
Session of experimentation							
A posteriori analysis							

In the beginning, the implementation's sessions were foreseen to be conducted in Italian high schools' classrooms of 10th grade, to involve in the study students at the beginning of their high school path, but far enough from their previous school cycle (middle school) that could develop a potentially advanced discourse with respect to the content of our interest. The focus, in the setting, would have been on some low achieving students of each class. Since the project started when some restrictions due to covid-pandemic were still active, we have had to reconsider our plan of go and conduct the data collection in schools. Indeed, as I have disclosed in the first chapter, the main implementations have been carried out at CARME (<https://www.carme.center>), a Center for Advanced Research on Mathematics Education settled in Pistoia. The 'Maryam Mirzakhani room' at CARME, where we have managed our sessions, is equipped with cutting-edge technology that yielded various types of data collected, soon to be described. Specifically, we have had the first data collection at the CARME center, the second one in a high school of Pistoia and the third one again at the CARME center. We have focused on the two collections at CARME center, considering that they have been conducted in the same context, while the one in the school has different characteristics. To clarify this point, I now give some details about each cycle focusing especially on the data collections.

- The first cycle has occurred from May 2022 and March 2023 (Table 3-1). The implementation, that has been conducted in the period October 2022 – December 2022, has involved 12 students from different high schools of Pistoia and surroundings. Students volunteer to participate to the study recognizing themselves as low achieving in mathematics (or being recognized by their teachers as so). They came to CARME center 5 or 6 times, once a week, in the afternoon. Students attended a first initial interview individually, during which they have been proposed some questions to investigate their identifying discourse and some task situations typical of classrooms' activities to investigate their mathematizing discourse. After that, they attended four or five sessions of two hour each, in pair or individually. During these sessions, they have dealt with our technology-enhanced activities guided by a tutor that is one of the researchers of the DynaMat team. We have decided, when possible, to keep the same pair students-tutor, in order to provide continuity and a serene and usual atmosphere. After the sessions with activities, students attended an informal interview to have some hints about their perceptions and feelings after the sessions.
- The second cycle has occurred from March 2023 to September 2023 (Table 3-1). Conducted between April 2023 and May 2023, the implementation has

been carried out in a 10th grade classroom of a vocational school in Pistoia. It has involved all the students of the classroom and lasted 6 sessions of one hour each. 3 of the 16 students in the classroom have attended the first session of implementation at CARME center. During these sessions, students have dealt with our technology-enhanced activities guided by some (2 or 3) researchers of the DynaMat team. They have worked in group of three or four people composed by their teacher.

- The third cycle has occurred from June 2023 to September 2024 (Table 3-1). The implementation, that has been conducted in the period October 2023 – December 2023, has involved 11 students from different high schools of Pistoia and surroundings that had not participated to the previous cycles. As the previous implementation at CARME, students volunteer to participate to the study recognizing themselves as low achieving in mathematics (or being recognized by their teachers as so). They came to CARME 6 times, once a week, in the afternoon. Students attended a first initial interview individually, during which they have been proposed some questions to investigate their identifying discourse and some task situations typical of classrooms' activities to investigate their mathematizing discourse. After that, they attended five sessions of two hour each, in pair or individually. During these sessions, they have dealt with our technology-enhanced activities guided by a tutor that is one of the researchers of the DynaMat team. We have decided, when possible, to keep the same pair students-tutor, in order to provide continuity and a serene and usual atmosphere. In the last session they have been dealt also with some task situations from the initial interview. Moreover, in the end of the last session, they have answered individually to some questions about their feelings and feedback about their experience.

In all the implementations, students interact with technology-enhanced activities thanks to tablets we gave them. Specifically, both at the CARME center and at schools, each pair or group has had at disposal a tablet to interact with the DEOs and a tablet with a blank page where they could write any possible annotation.

3.1.1.Types of data collected within DynaMat project

For the first and the third cycles, thanks to Maryam Mirzakhani room, we have collected different types of data. Indeed, we have:

- Audio-video recordings of every meeting (initial interview and sessions with technology)
- Screen recordings of each tablet students have used

- Pdf with students' annotations on one of the two tablets
- Notes of one of the researchers of the team that was observing the session (not in the same room of the session, but in another one thanks to the cameras present in the room)
- Notes written by the tutor that was conducting the session immediately after it

For the second cycle, we have collected:

- Screen recordings of each tablet students have used, equipped with the audio recording to have information about what students were saying while they were interacting with DEOs
- Pdf with students' annotations on one of the two tablets
- Notes written by the tutors that were conducting the session immediately after it

In both cases, for CARME center or for school, all students' parents (they were minors) signed an informed consent form that was demanded by the fact that the data collection was accomplished within a PRIN project and that has been approved by the Committee on Bioethics of the University of Pisa (Review No. 12/2022, Prot. 0049430/2022). In order to respect students' privacy, in accordance with the relevant regulations, all videos in which students appear have been edited to blur their faces. Moreover, their names have been changed, and in this thesis, as well as in all the articles reporting episodes from the data collections, pseudonyms are used. To give an example, in Figure 3.1 we can see Aldo (on the left) and Giulio (on the right) working with one of the team's members. Their faces are blurred. In front of them there are two tablets, the one on the left to interact with the DEO while the other present a blank page.



Figure 3.1: A snapshot of blurred audio-video recording from a session of the third cycle

3.2. My research project's development

Now I want to focus on my research project, describing its development along with the DynaMat project. It began in November 2021 with an in-depth literature review. Given that the work was to be carried out within the Dynamat project, the initial focus was placed on studies that constitute its theoretical foundations. In particular, I dedicated myself to exploring key studies related to digital environments and their potential from a didactic perspective, especially those involving students with learning difficulties. Moreover, taking into account the mathematical topics at the core of the interventions to be developed, I focused on the literature concerning the teaching and learning processes of algebra, with particular attention to studies addressing difficulties and misconceptions related to this topic. Together with this, particular attention was devoted to the study of the theory of Commognition as the reference theoretical framework of the project. Together with this, I deepen the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) as well as the construct of *semiotic bundle* (Arzarello, 2006) as possible lens through which conduct my research. Indeed, in the beginning, my study mainly concerned the understanding of learning processes by providing insights into the impact of activities with digital environments on low achieving students' algebraic learning. I was looking for a theoretical framework that could help me delving into the theme and considering the several aspects composing the study.

In chapter 2 I have introduced and discussed the theoretical framework within which I set my study, that is the Theory of Commognition. In the end, even if the

Theory of Semiotic Mediation (TSM) could have offered me tools for the design of DEOs, in the light of the construct of *semiotic potential of an artifact*, Commognition has seemed to us more adapt to the context in which the study was moving. Indeed, TSM is devoted to classrooms' activities and offers instruments to design and analyze processes involving material or dynamic artifacts where the teacher covers a key role. In the DynaMat project the context was different from the one of classroom, indeed, teacher and the presence of many students were missing. In the previous chapter, I have stressed how the commognitive lens allows me to investigate learning processes in a social dimension focusing on the communication of the participants. Moreover, adopting a commognitive perspective, I do not have to make hypothesis about what students are thinking as something internal and hided, because thinking is seen as intrapersonal communication. In this frame, our investigation aims at putting a light on the role of well-designed digital activities within DEOs in the development of students' algebraic discourse. To conduct the study, I adopted the DBR methodology, following DynaMat project. The timeline of the study is reported in Table 3-2. It mainly coincides with the timeline of DynaMat project in Table 3-1.

Table 3-2: Timeline of my research project

	Nov 2021 – May 2022	Jun 2022 – Sep 2022	Oct 2022 – Dec 2022	Jan 2023 – Mar 2023	Apr 2023 – May 2023	Jun 2023 – Sep 2023	Oct 2023 – Dec 2023	Jan 2024 – Sep 2024
Literature review								
A priori analysis – design								
Session of experimentation								
A posteriori analysis								

As I already discussed in the chapter 2, Commognition grants us sophisticated tools to analyze students' discourse, but during the first design phase and the first a posteriori analysis session we felt something was slipping away. In this sense, we have opened a discussion to clarify our design process and our didactical aims, in order to frame them in the commognitive perspective. Considering this, my research objectives (presented in chapter 2) have become:

- Objective 1: integration of theoretical tools, in the theory of Commognition (Sfard, 2008) to:

- c. design DEOs, related task situations and teaching interventions in terms of promotion of specific discourses.
 - d. Investigate learning processes considering
 - i. the role of students' existing discourses;
 - ii. the role of a DEO and of discourses related to it.
- Objective 2: identify processes in the interaction with the DEOs that, as expert, we consider significant for the growth and the development of the algebraic discourse. Specifically:
 - a. investigate the role of DEOs in promoting analytic discourse;
 - b. Investigate the role of DEOs in supporting the theoretical organization of algebraic discourse. We want to focus on the formulation and organization of endorsed narratives with respect to the DEO that can be reinvested in the formulation and organization of endorsed narratives with respect to algebraic discourse, to guide the process of equations solving.

I want now to focus on the first one and on the way in which we have acted to reach it within the study. It is a theoretical objective that has emerged during the first steps of the study, when we were conducting the first cycle of design-implementation-a posteriori analysis. I propose in Figure 3.2, Figure 3.3 and Figure 3.4 the three cycle that have composed our study, in tune with DBR. In the figures are reported some main questions emerged during the phases of each cycle and some 'answers' that we have found during the study.

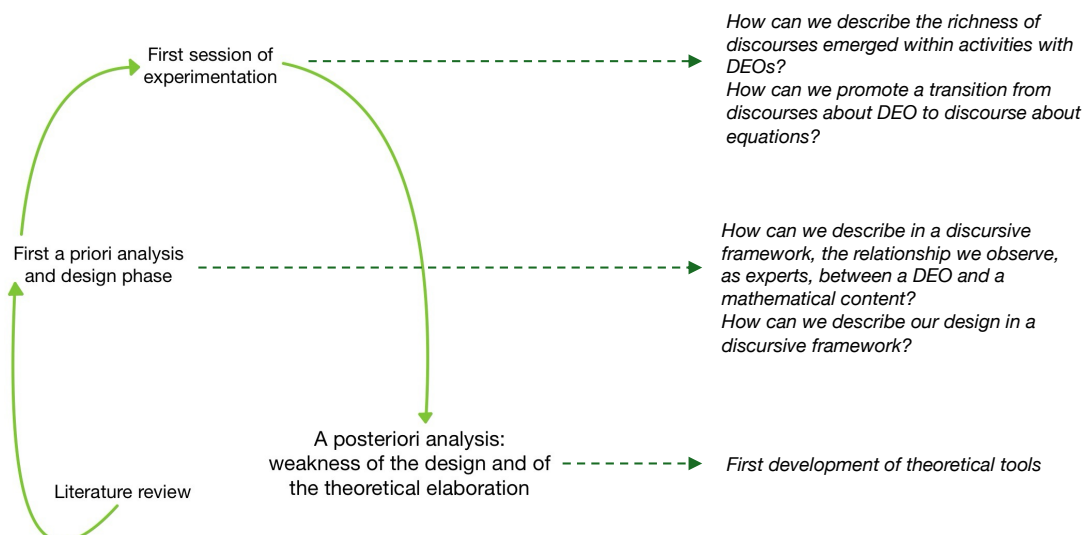


Figure 3.2: Description of the first cycle of design-implementation-a posteriori analysis

In Figure 3.2 we can find the description of the phases of the first cycle (May 2022 – March 2023) and some of the questions emerged within it. In particular, the main point of the first a priori analysis and design phase has been the description of the relationship as expert we perceive between a DEO and a mathematical content. This is related to the choices we made during the design phase to develop DEOs and related task situations. After the first session of experimentation, came to light the need of distinguish discourses produced by students during the activities with the DEOs. Thus, within the first a posteriori analysis we started highlighting in a systematic way the theoretical tools we need to answer the questions in Figure 3.3.

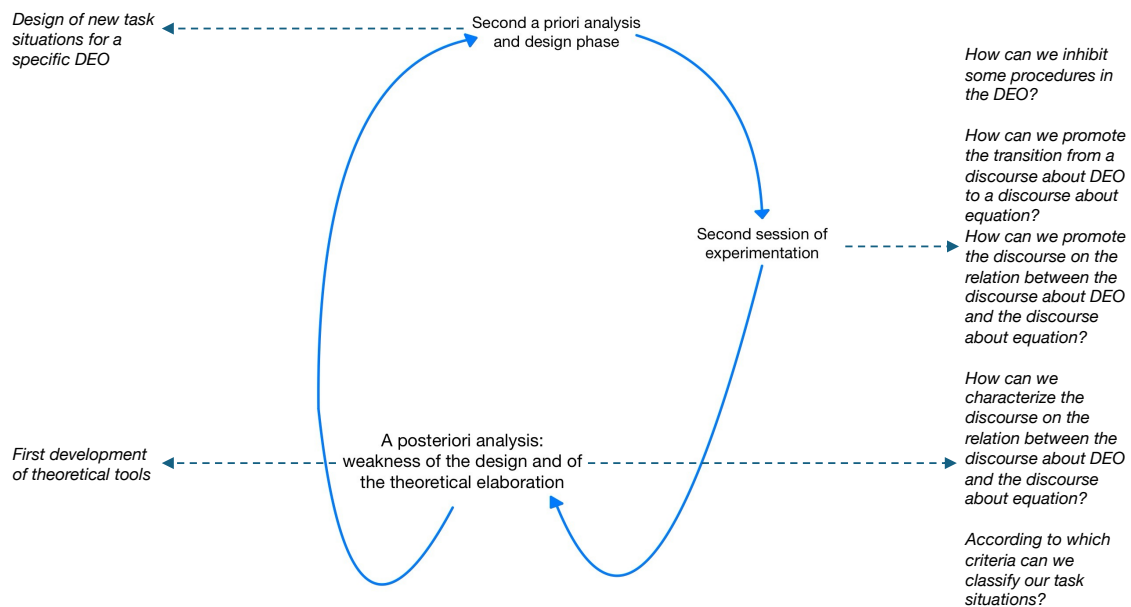


Figure 3.3: Description of the second cycle of design-implementation-a posteriori analysis

Moreover, in the re-design phase we started considering the introduction of new tasks situations to promote the transition from discourses about DEO to discourses about equation and, especially, to promote the construction of narratives with respect to the DEO that could be re-invested in the mathematical activity. During the second cycle of design-implementation-a posteriori analysis (March 2023 - September 2023) we focused on this new task situations. As shown in Figure 3.3, we deepen our a priori analysis on a specific DEO, for which we have developed two kinds of new task situations. After the second session of experimentation, we opened to a reflection on the task situations we have elaborated, looking for criteria to characterize them, according to a discursive perspective. Moreover, increasing attention has been devoted to the reflection on possible characterizations of the discourse on the relation between the discourse about DEO and the discourse about equation, investigating ways to promote such discourse.

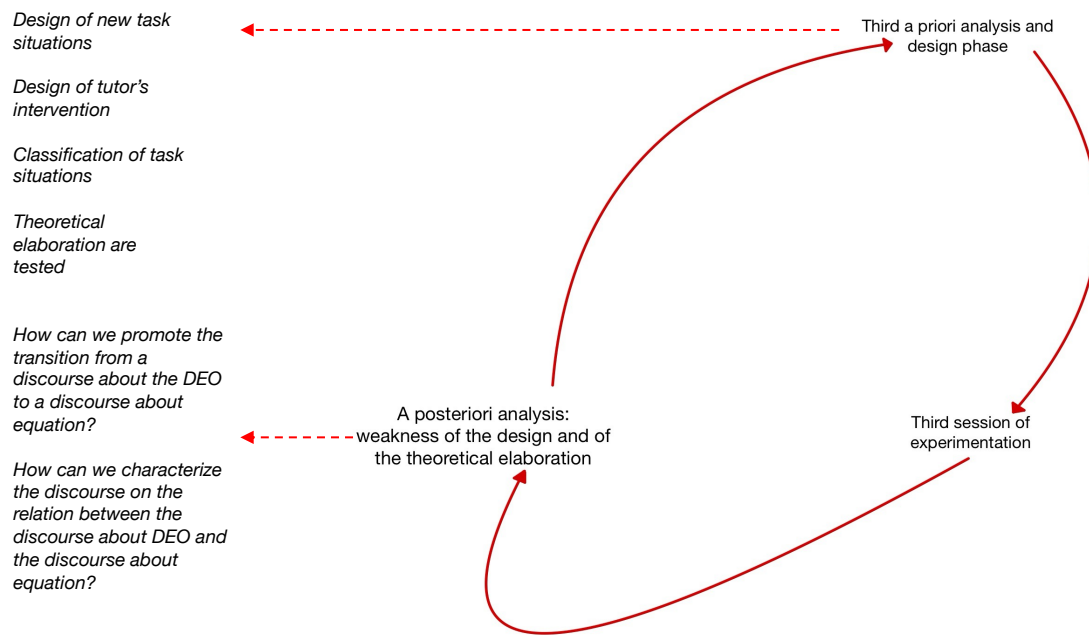


Figure 3.4: Description of the third cycle of design-implementation-a posteriori analysis

The third cycle of design-implementation-a posteriori analysis (June 2023 – September 2024), depicted in Figure 3.4 has grown from the need of the a posteriori analysis of the second cycle to answer the questions:

- *How can we promote the transition from a discourse about the DEO and a discourse about equation?*
- *How can we promote a reflection on the relation between the discourse about DEO and the discourse about equation?*

In this light, we developed new task situations and revised the ones introduced within the second cycle. Moreover, we have elaborated, for each DEO and related task situations, a collection of tutor's intervention to reach the objectives in the questions above. According to DBR, during the third a priori analysis and design phase we tested our theoretical elaborations developed in between the first and the second cycle to identify strengths and weakness of them in describing our design activity. After the third session of experimentation (October 2023 – December 2023), we look back at our last design and analyse if and to what extent task situations together with tutor's intervention have prompted the transition from DEO's discourse to mathematical discourse. Moreover, we developed a new classification of task situation.

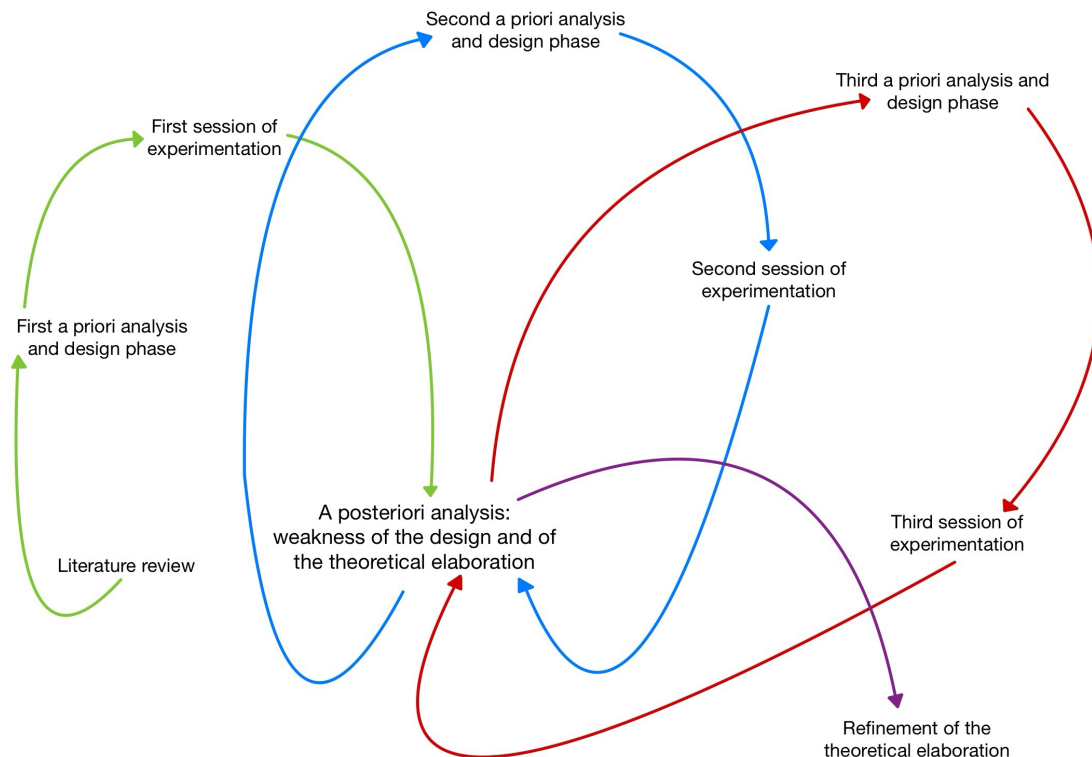


Figure 3.5: The complete development of the study according to the cycles conducted

To conclude the description of the path through which my study has moved, there has been a final session of refinement of the theoretical elaboration (September 2024 – April 2025). This has helped to look back at our research activity focusing on the theoretical and practical tools we have elaborated. This last refinement (Figure 3.5) has led us to the elaborations proposed in chapter 4. On one hand we have found theoretical tools to guide the design of DEOs in a discursive framework, providing criteria to elaborate task situations and tutor's intervention according to specific aim. On the other hand, we have developed theoretical tools that can help in the a posteriori analysis phase to stress the emerging complexity in the processes of teaching and learning within digital environments.

4. Our theoretical elaborations

In this study, in tune with recent research (Antonini et al., 2020), I want to investigate and deepen, within the commognitive perspective, the role of well-designed digital environments and related activities in promoting and fostering the construction of a mathematical discourse that is commensurable with the experts' mathematical discourse.

The main didactical aim of DynaMat activities is to foster and support the production of a discourse with respect to a DEO that could allow the entry and the participation to the algebraic discourse. Our hypothesis is that these environments grant support to the growth of the students' discursive production offering them concrete objects which they can operate upon and transform, but that can also be the protagonists for the construction of meaningful stories (Baccaglini-Frank, 2021). Moreover, we think that the activities within DEO promote the production of a discourse referred to the DEO that lays the foundations for the construction of a mathematical discourse commensurable with the canonical one. This hypothesis is in line with the *principle of the continuity of discourse* introduced by Sfard who stresses the relevance of getting involved in new discourses through other discourses in which the discursant is already fluent.

To accomplish this aim, we realize the need to outline a theoretical scaffold to support our study. Indeed, in the beginning, we felt the need to explicit the relation that as experts we perceive between a specific DEO we have chosen to be used in our study and the mathematical activity. Within the commognitive framework, this necessity shifted towards the study of the link between discourses related to a DEO and the mathematical discourse of our interest, and at the same time towards the development of theoretical tools to describe and talk about it. The need to elaborate this model has grown within the first design phase (June – September 2022); these instruments have been developed and refined simultaneously with the design phases that occurred during the project. Indeed, it was possible to clarify and make more accurate these theoretical tools only by looking a posteriori at what had been done previously. This scaffold allows us to describe the design phase a posteriori, but also to have a model to be followed in future activities of design of digital environments within commognitive framework.

Here I now introduce and discuss our theoretical constructs, as a first step towards a possible answer to the first research question. Some of these theoretical elaborations have been discussed in a recent work (Antonini et al., accepted). In chapter 5 I discuss the specific design of each digital environments together with the relative tasks proposed, according to the tools introduced in this chapter.

4.1. Some theoretical tools

We now give some definitions necessary to build our theoretical constructs. First of all, I want to stress again how the term discourse plays a central role in Sfard's theory. She defines

"The different types of communication, and thus of commognition, that draw some individuals together while excluding some others will be called *discourses*" (Sfard, 2008, p.91, italic in original).

According to this definition, she introduces the idea of communities of discourse, discussed in the previous chapter, that in our theoretical elaboration plays a central role. Indeed, within our study, the emerging need to investigate the relation between discourses related to DEO and mathematical discourse lead us to think about what we were considering as "discourse" and, in addition, what we were intending when talking of discourses related to DEO. We realized that also the idea of 'community of discourse', as the community of experts of a discourse, must play a role in our theoretical development.

In a more recent work, Sfard affirms

"Discourses are means, or sets of tools, for telling useful stories, or narratives, about specified types of objects. [...] The term discourse of X refers to a type of communication that is used in telling stories about X" (Sfard, 2021, p.4).

In our study, we decide to consider X as mathematics, DEO, algebra, arithmetic, and so on, but also as a collection of terms, as "relation between DEO and algebra", according to which a certain discourse is developed. Considering this, we define as

X-discourse the collection of specific words, visual mediators, routine and endorsed narratives referred to X and identified a priori by an expert or the community of discourse of X.

To give an example, let's choose as X volleyball. The experts are coaches, referees and experienced players. The *volleyball-discourse*, identified by them, could be composed by the following characteristics:

- Words: setter, attacker, libero, block, dig, and so on.
- Visual mediators: ball, playground, net, patterns of play written on a blackboard, scoreboard, hand gestures to indicate strokes, hand gestures to indicate choice of riser, and so on.
- Routine: patterns of play, movement patterns in the playing field, and so on.

- Endorsed narrative: rules of the game and empirically approved facts, such as, “If you throw the ball in this way in that direction, it will go out”.

I want now to focus on the expression “specific words”. To give an example, if we consider C as cinema, a specific word belonging to the C-discourse is *long take*, that is a cinematic technique consisting of a long shot that is intended to eliminate, or at least limit, the use of film editing.

With respect to this, we have to take into account that there are some words which could have various ‘natures’. For example, the verb “to shoot”, can be used as belonging to C-discourse to indicate the action of filming something. At the same time, it could refer to the action of using a gun to fire. In these cases, an expert of the community of C-discourse would recognize if a word or expression belongs to a specific discourse or another considering the way it is used, considering the historically established customs of the community of discourse in using such words. So, in using specific words we intend the words that are typical of a certain discourse; to identify if other words (such as ‘to shoot’) are used within a discourse or another one, we look at their use.

Sfard, in describing what she means by story about X, adds

“The term story about X, where X is a noun, is used as referring to a coherent sequence of utterances that, when taken together, can be said to be ‘about X’ (or ‘on X’ or ‘of X’). The ‘aboutness’ means that X is the grammatical object of some of the utterances in the sequence, and the sequence in its entirety is consistent and cohesive” (Sfard, 2021, p.3).

Considering the research aims, that consist in studying the emerging discourse of students working with digital activities and how this discourse allows the construction of the mathematical discourse, I now want to focus on the term ‘about’. The ‘aboutness’ for Sfard identify the object of the discourse, what is that discursants are talking about. From our perspective, the object of the discourse is an essential element in studying students’ discursive production and investigating learning processes within digital environments. Indeed, the object of the discourse in the digital task situations could be various (such as a digital object of the DEO or a mathematical symbol) and identify it could offer us essential information of the process of learning. In this sense, it is necessary to stress that the a posteriori (during the data analysis) ‘aboutness’ is inferred by the researcher who is looking at the students’ discourse: her interpretation could not correspond to the one could give students identifying the aboutness of their own discourse. This inference is thus strongly subjective and could open to interest discussion among researchers that could perceive different ‘aboutnessess’ in the same students’ discourses.

Thus, this classification cannot be considered as unique and exhaustive, but strongly personal and with a deep interpretational character. The identification of the aboutness depends on the choices made a priori for the analysis. In particular, the aboutness is subjected to the extent of the unit of analysis and during this time the aboutness can change. To locate the aboutness I am guided by the following questions.

1. What is the subject talking about?
2. Which is (are) the object(s) of her discourse?

The notion of ‘aboutness’ appears to be essential not only, as we discussed, as a posteriori analysis’ tool, but also as an instrument during the task situations’ design phase. As a matter of fact, the development of the digital activities and the script of tutor’s intervention has been guided by the notion of aboutness, that has become a design tool, thanks to which we are able to describe some of our aims.

We then affirm that a discourse is *about X* if the object of the subject’s discourse is an object of X-discourse. At the same time, we affirm that a task situation aims at promoting a discourse about X when, as designers of an activity, we foresee as possible answer a discourse in which the object is an object of X-discourse.

In this study, our aims are

- to describe our task design activity within DEO in term of promotion of discourses
- to look and analyze students’ discourse within specific moments, such as in the interaction with a DEO or the accomplishment of a specific task

For this reason, we decide to distinguish among the X-discourse, defined before as a collection of features, and a certain subject’s discourse that could be produced (a priori design) or produced (a posteriori analysis) in certain circumstances. Thus, in our framework we consider as *discourse about X* a specific communication involving one or more discursants in a specific moment and context about X, where the aboutness is inferred by an expert of X (a priori or a posteriori). We denote it as d_x .

Now that we have introduced and discussed our perspective, defining the notion of X-discourse and aboutness, we have the necessary tools to introduce other theoretical constructs.

Consider X and Y as two domains such volleyball, mathematics, DEO, cinema, and so on. We can pinpoint the following types of discourses, with reference to X and Y.

In our notation the subscript denotes the aboutness while terms in the brackets denotes the belonging of the features used to a specific N-discourse, where N can be mathematics, DEO, algebra and so on, as in the definition provided above:

- $d_x(X)$: discourse recognized by an expert of X as about X in which the discursant uses features that an expert of X recognizes as belonging to the X-discourse.
- $d_x(X,Y)$: discourse recognized by an expert of X as about X in which the discursant uses features that an expert of X and Y recognizes as belonging to X-discourse and to Y-discourse.
- $d_x(Y)$: discourse recognized by an expert of X as about X in which the discursant uses features that an expert of Y recognizes as belonging to Y-discourse.
- $d_{R(X,Y)}(X,Y)$: discourse recognized by an expert of X and of Y as about the relation between X and Y, in which the discursant uses features that an expert of X and Y recognizes as belonging to X-discourse and to Y-discourse.

We define $d_{R(X,Y)}(X,Y)$ as *bridging discourse*. An example of bridging discourse could be “the situation in which a scale is balanced is the same as the equality between these two numerical expressions”, where X is balance, Y is mathematics.

In the bridging discourse the discourse is about the relation between X and Y; it can be considered a meta-discourse because it is a discourse about discourses: the discourse about X, and the discourse about Y.

I recall that in this study we use the notion of DEO as dynamic explorative object. Moreover, now on, in describing the theoretical elaborations, I use the letter M that stands generically for mathematics. In the next chapter I use another notation for the specific content of our interest (linear equations).

We have the following constructs. At first, we consider DEO-discourse and M-discourse.

- *DEO-discourse*, as the collection of specific words, visual mediators, routine and endorsed narratives referred to the DEO and identified a priori by the designer of the DEO, as belonging to the community of discourse of DEO.
- *M-discourse*, as the collection of specific words, visual mediators, routine and endorsed narratives referred to mathematics and identified a priori by the community of experts of mathematics.

Moreover, I refer to d_{DEO} and d_{M} as specific discourses recognized by an expert to be respectively about DEO and about M, according to the definition previously introduced.

Consequently, we have these possible discourses.

- **$d_{\text{DEO}}(\text{DEO})$:** discourse recognized by an expert of DEO as about DEO in which the discursant uses features that an expert of DEO recognizes as belonging to DEO-discourse.
- **$d_{\text{M}}(\text{M})$:** discourse recognized by an expert of M as about M in which the discursant uses features that an expert of M recognizes as belonging to M-discourse.
- **$d_{\text{DEO}}(\text{DEO}, \text{M})$:** discourse recognized by an expert of DEO as about DEO in which the discursant uses features that an expert of DEO and M recognizes as belonging DEO-discourse and to M-discourse.
- **$d_{\text{M}}(\text{DEO}, \text{M})$:** discourse recognized by an expert of M as about M in which the discursant uses features that an expert of DEO and M recognizes as belonging to DEO-discourse and to M-discourse.
- **$d_{\text{R}(\text{DEO}, \text{M})}(\text{DEO}, \text{M})$:** discourse recognized by an expert of DEO and of M as about the relation between DEO and M, in which the discursant uses characteristics that an expert of DEO and M recognizes as belonging to DEO-discourse and to M-discourse.

To conclude this section, it is worth to recall that this classification of discourses can take part both in an a priori design phase of a didactical activity and during the analysis of episodes from the implementations of activities with students. As we describe in the following section, the distinction among aboutness and features of the discourse offers us, as designer of the activities, the possibility to clarify, in commognitive terms, what we mean with “relation between DEO and mathematics” and to formulate our didactical aims through a discursive perspective. On the other hand, within the analysis, the variety of discourses, pointed out by this classification, emerging from the activities with students can furnish us insights to investigate the role played by DEOs in the construction of meaningful narratives. In the end, it is worth to notice that it could be not always possible to identify a distinction among what the subject is talking about (the aboutness) and to which discourse the features belong to and, again, that the identification is strongly subjected to the researcher. With respect to the first cycle

of design-implementation-a posteriori analysis, this classification of discourses comes to light as an answer to the need of having instruments to describe the richness of discourses emerged within activities with DEOs during the first experimentation (Figure 4.1).

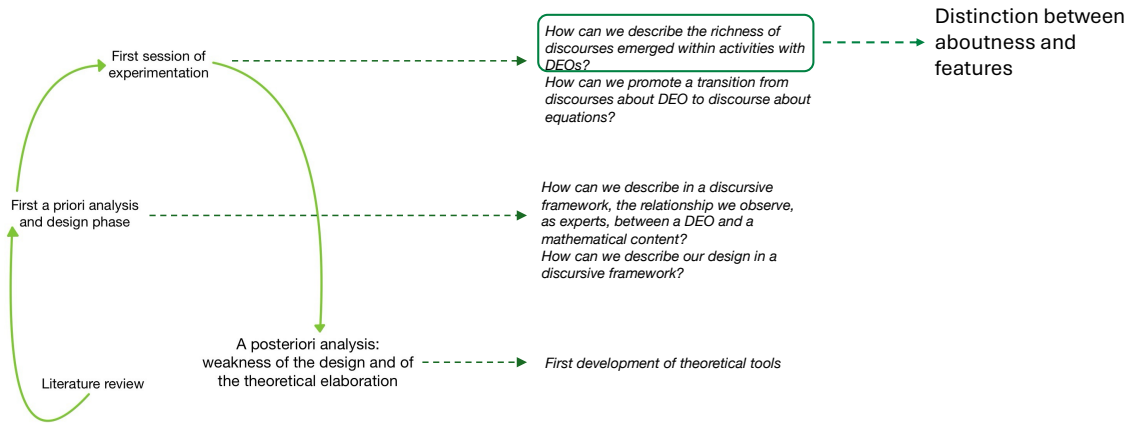


Figure 4.1: First cycle of design-implementation-a posteriori analysis and questions emerged within it. In green highlighted a specific question and the reference to our theoretical elaboration as a possible answer to it (see the classification of discourses proposed in this chapter)

4.1.1. The study of the relation between DEO-discourse and mathematics-discourse

With these instruments we are able to talk about the relation, seen by an expert, between the dynamic explorative objects we have designed and a mathematical content. According to the commognitive perspective, we are interested in promoting the development and the evolution of a discourse. In particular, we want to foster the growth of a DEO-discourse and exploit it to support a consequent shift towards the M-discourse. Thus, we aim at promoting two “directions” of growth in tune with two types of relation we perceive. The first one, among the components of a DEO, as an internal relation between objects and thus a consequent relation among the elements of DEO-discourse. The second one, between the DEO and mathematics, as an external relation in which the two poles are composed by DEO-discourse and M-discourse.

First of all, one of the main points concerns the construction of a DEO and its recursive structure. The notion of DEO as dynamic explorative object has already been presented in chapter 2. When we talk about DEO as a dynamic explorative object, we mean a digital environment in which different objects are embedded. These objects are dynamic and interactive: they can be defined as DEO too. For instance, considering a dynamic scale, as in our activities, each weight can be moved on and off a pan. In this DEO we have each object with known or not known

value, but we consider as object also the collection of objects on a pan, or the complete scale. This recursive structure offers the possibility to talk about many objects (in the example, known and not known values, a collection of objects, the whole scale) and, most of all, about relations among them. Moreover, the subject can construct narratives about his actions on the dynamic scale and the consequence of them according to the feedback given, relating the objects.

In the commognitive perspective, the evolution of the subject's discourse is identified in the use of words, in particular of specific words, accordingly to the established use, to construct endorsable narratives. In this light, our hypothesis is that the interaction with these DEOs, which embed other DEOs, offer the possibility not only to introduce new words, but also to use them in a discourse that becomes more and more articulated. In other words, within a DEO, the DEOs that compose it are in relation with each other. Some of these relations are realized by a DEO too (for instance, the position of the scale stands for the relation among the collection of objects on each pan). This recursive construction allows the user to produce narratives in which the words are used accordingly to the relation among the objects they refer to.

In the design phase, our aim was to exploit each DEO's recursive structure for the birth and the growth of students' DEO-discourse. According to the definition given by Sfard (2008), we consider the notion of discursive expansion as 'an increase in the amount and complexity of discursive routines and endorsed narratives within the given discourse' (p.298). To prompt this expansion, we thus developed a sequence of task situations for each DEO aiming at gradually promote the construction and the development of a specific DEO-discourse.

On one hand we have a DEO that, in our hypothesis, offers students objects to interact with and possibilities to participate to a discourse about them. On the other hand, our main didactical aim was to promote the growth of students' mathematical discourse. In the beginning of the first design phase, we decided to use specific digital environments to promote the construction and the development of algebraic discourse because, from our perspective, we saw a relation between the activity with these environments and this discourse. For example, in the balanced scale overmentioned, an expert can see the realization of an equation, where, to recall, "Realization of the signifier S is a perceptually accessible thing S' so that every *endorsed narrative* about S can be translated according to well defined rules into an endorsed narrative about S'" (Sfard, 2008, p. 154); the collection of objects on a pan is a realization of a quantity as a collection of known and not known value; each of these objects with known and not known value is a realization of a known or not known value in an algebraic expression. In designing a DEO, we thus

construct a collection of DEOs, each of which can be perceived, by an expert, as realization of a mathematical object

Nevertheless, the relation as experts we see, is not reduced to the presence of possible realizations of mathematical objects. Some of the features composing the DEO-discourse, such as routines and endorsed narratives, can be related with the features of the M-discourse. Thus, we can describe the relation between a DEO and a mathematical content in term of correspondence of elements in the respective discourses (DEO-discourse and M-discourse). An idea that can clarify and theoretically contextualize this consideration is the notion of *metaphor*. Sfard (2008) describes metaphor as “the action of ‘transplanting’ words from one discourse to another” (p. 39). The researcher discusses this notion in dealing with the fact that creating new discourses – or creating new knowledge – is inherently impossible. The key points she stresses is the fact that discourses exist only in action, and so to bring a new discourse into being is possible only by practicing it, and, on the other hand, if the rules have not yet been established, it is not possible to practice a discourse. The metaphor offers a way out of this cycle:

“familiar words, even if transplanted into a new discursive context, can still be used according to those of the old rules that seem consonant with the new context” (Sfard, 2008, p. 41).

Sfard considers the metaphor a way to introduce a new discourse by using words familiar to the subject in another context. Recalling Wittgenstein’s perspective, in this transplanting, familiar words will be used in the same way even if in a different context, and so, their meaning is kept. In this light, we interrogated ourselves about the idea of *transplanting*, not only words, but also visual mediators, routines and endorsed narratives, from one discourse to another. In our study, the idea of transplanting is used as a design tool. We are specifically interested in studying a priori pairs of discourses and, during the experimentation with students, in using their participation in one discourse to promote their participation in the other one. With respect to this, Sfard discusses the idea of *isomorphic discourses* as follows.

“[...] two discourses can be called isomorphic if there is an isomorphism that maps the set A of all the utterances of one of the discourses onto the set B of utterances of the other discourse. Isomorphism i between A and B is a one-to-one mapping that assigns to every utterance u from A an utterance $i(u)$ from B and preserves the truth-value of utterances and the relations between them, that is, fulfills the following conditions: (a) Utterance u of A is endorsed (true) if and only if $i(u)$ of B is endorsed, too; (b) if the utterance w in A is of the form “if u then v ” (or “ u and v ” or “ u or v ,” and so forth), then the utterance $i(w)$ in B is of the form “if $i(u)$ then $i(v)$ ” (or “ $i(u)$ and $v(u)$ ” or “ $i(u)$ or $i(v)$,”

and so forth, respectively. Of course, because of the blurriness of discourse boundaries and because of the context-dependence of its utterances, the relation of isomorphism is never as clearcut as required by this formal definition. Still, it is both useful and justified to think about some pairs of discourses as nearly isomorphic.” (Sfard, 2008, p. 122)

In this light, in our design we consider DEO-discourse and M-discourse and analyse a possible isomorphism among them. We define the *analysis of the transplantability* as the study of the relation between a DEO and a specific mathematical content. More specifically:

- Epistemological analysis of mathematical discourse of interest (arithmetic, algebra, functions, geometry, and so on)
- Study of DEO-discourse and M-discourse, and thus of features (words, visual mediators, routines and endorsed narratives) constituting the two collections.
- Study similar use of words and visual mediators, correspondence in routines and endorsed narratives, between DEO-discourse and M-discourse, and thus study of features of a discourse that can be transplanted in the other one (and vice versa) (the study of the isomorphism between DEO-discourse and M-discourse).
- Identify subsets of DEO-discourse which elements are transplantable in M-discourse and vice versa and subsets that are not (see Figure 4.2). This is the study of the limitations of the relation between the two discourses that, in our perspective, can have a role in the didactical process.



Figure 4.2: The analysis of the transplantability. The subset of a DEO-discourse corresponding to a subset of the M-discourse

Thus, with the analysis of the transplantability, we want to highlight the isomorphism between the two discourses. As depicted in Figure 4.2, we can imagine the analysis of the transplantability as the study of the two collections, DEO-discourse and M-discourse, and the identification of features in both collection that can be transplantable in the other. With this investigation, we identify two subsets of features that are transplantable: the one in the DEO-discourse is of our interest to develop the one in the M-discourse. Thus, DEO-discourse and M-discourse can be said *nearly-isomorphic*, considering the definition of isomorphic discourses provided above: not all the features, and thus

the utterances in DEO-discourse can be assigned to utterances in M-discourse and vice versa. Recalling the discussion in chapter 3, with respect to the first cycle and our need of finding a way to describe the relationship between a DEO and a mathematical content and our design, in a discursive framework, we have found an answer in the construct of analysis of the transplantability (Figure 4.3).

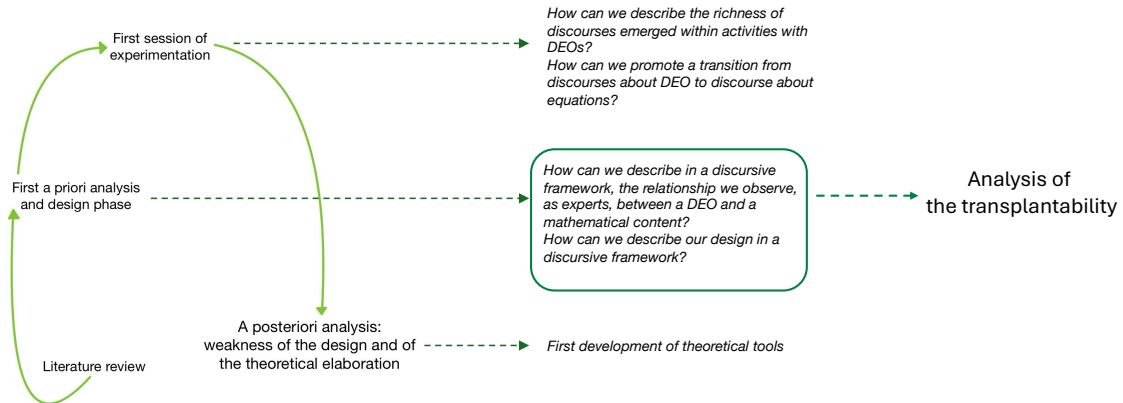


Figure 4.3: First cycle of design-implementation-a posteriori analysis and questions emerged within it. In green highlighted two specific questions and the reference to the theoretical elaboration we have developed as a possible answer to them

Thanks to this consideration, this study offers the opportunity to look at what is not transplantable from one discourse to the other, and thus at aspects that could be delicate but also crucial within the didactical activity. Indeed, in our design activity, we took into consideration the features in a discourse that could not be transplanted in the other, as possible emerging aspects in the experimentation's sessions both spontaneously in the students' discourse as well as prompted by our intervention.

To provide an example, in Table 4-1 I collect some of the features of the DEO-discourse identifying the corresponding feature of the M-discourse if they are transplantable, or writing *not-transplantable* if it is not, and vice versa.

Table 4-1: Examples of the analysis of the transplantability of DEO-discourse (two-pans-scale) and M-discourse

Feature	DEO-discourse	M-discourse
word	weight	number
words	object with not known weight	unknown
visual mediator	collection of weights on a pan	algebraic expression on one side of the equation
visual mediator	cancellation of weights on a pan	cancellation of terms in an algebraic expression

visual mediator	not-transplantable	negative numbers
routine	task: find the value of the weight of an object with not known weight routine: put the not known weight on a pan and known weights on the other until the scale is balanced	task: find the value of the unknown routine: not-transplantable
endorsed narrative	If in a balanced scale equal weights are put on or taken off the two pans, then the scale is balanced afterwards	if $A = B$ then $A \pm C = B \pm C$

The bridging discourse, that we have defined as

‘Discourse recognized by an expert of DEO and of M as about the relation between DEO and M, in which the discursant uses characteristics that an expert of DEO and M recognizes as belonging to DEO-discourse and to M-discourse’

is related to the analysis of transplantability. In our study, the promotion of a bridging discourse is a didactical strategy that can be implemented to highlight relations between DEO and M, and thus features that are transplantable in one or in the other discourse, as well as differences between the two discourses, or in other words, not-transplantable elements. For instance, with respect to the case of the two-pan scale, a bridging discourse could involve negative numbers as possible values for the objects’ weight and as element in algebraic expressions describing the equation. Our hypothesis with respect to this is that the development of a bridging discourse that highlights some critical aspects of the relation between DEO and M or some features, especially routines, of the DEO-discourse that are not transplantable in M-discourse and vice versa, can help the construction of a M-discourse coherent with the canonical one.

This analysis allows us to clarify, in a discursive framework, the reasons for the choice and the use of specific DEOs to achieve the didactical aim of promoting the construction and development of an algebraic discourse. Indeed, in designing task situations about DEOs, we expected the expansion of DEO-discourses. According to the analysis of the transplantability, to this growth corresponds, from our point of view, a M-discourse’s expansion. While a student’s DEO-discourse develops, we,

as experts, see a correspondent development of the M-discourse that is not actually perceivable through student's discourse, but that we can identify thanks to the a priori study we have done. Therefore, this analysis offers us tools not only to design DEOs and develop related task situations, but also to identify features in the students' DEO-discourses that are of our interest to develop an algebraic discourse and thus to guide expert's intervention to promote a shift from DEO-discourse to M-discourse. In this sense, with respect to the diagram in Figure 4.2, we can depict this potentiality as in Figure 4.4. Thus, during activities in which the subject's discourse is about DEO and characterized by DEO-discourse's features ($d_{DEO}(DEO)$) there is a 'behind' discourse that as experts we see expanding. In this sense, in Figure 4.4, the DEO-discourse and its subset of elements transplantable in M-discourse are coloured while the corresponding M-discourse and its subset are faded. We imagine it as something that develops and grows not effectively but through the development of DEO-discourse. For example, with respect to Table 4-1, when a student formulates and uses the endorsed narrative 'If in a balanced scale equal weights are put on or taken off the two pans, then the scale is balanced afterwards' as expert we see the possibility to transplant it in the M-discourse, and thus a possible expansion of this.

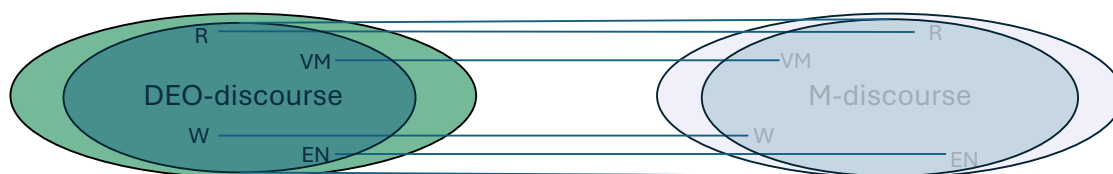


Figure 4.4: The representation of the development of DEO-discourse (coloured) and the corresponding M-discourse (faded)

In tune with these thoughts, we have to consider not only that the activities with DEOs offer the possibility to interact with objects and participate to a discourse that as expert we see resonant with the mathematical one, but also that the entering in the mathematical discourse could be facilitated by this discourse with respect to DEO. Indeed, “introducing a new discourse by transforming an existing one is certainly more effective than trying to build the new discourse from scratch” (Sfard, 2008, p. 254). We follow the *principle of continuity of discourse* as “pedagogical principle according to which new discourses should be developed by transforming discourses in which the learner is already fluent” (Sfard, 2008, p. 301). Accordingly to this, we think that the production of $d_{DEO}(DEO)$, thanks to the feature of dynamicity and interactivity, could be easier for students and could foster the development of explorative narratives. Thanks to the elaboration of the DEO-discourse, the successive development of M-discourse should be facilitated by the transformation of the previous one.

The analysis of transplantability is thus tied together with the principle of continuity of discourse: the first one allows us to describe the relation between DEOs and mathematics while the second one gives us a principle for the development of our activities. According to these considerations, we a priori imagine a transition from $d_{\text{DEO}}(\text{DEO})$ to $d_{\text{M}}(\text{M})$ (Figure 4.5).

$$d_{\text{DEO}}(\text{DEO}) \longrightarrow d_{\text{M}}(\text{M})$$

Figure 4.5: The a priori transition from DEO-discourse to M-discourse

According to the principle of continuity of discourse, the access to a new discourse is favored by the transformation of previous discourses in which the discursant is already fluent. In our design activity we took into account that students' discourse could not evolve from $d_{\text{DEO}}(\text{DEO})$ 'directly' to $d_{\text{M}}(\text{M})$ but, in tune with the continuity of discourse, there could have been the need of intertwining discourses, in which features from both DEO-discourse and M-discourse appear. These are what we denoted as $d_{\text{DEO}}(\text{DEO}, \text{M})$ and $d_{\text{M}}(\text{DEO}, \text{M})$.

These considerations lead us to develop a script for tutor's interventions that can become, in a classroom, teacher's interventions. We have identified, in the sequence of activities elaborated, task situations that require interventions to promote the expansion of the DEO-discourse, other in which the need is to require a compression of the DEO-discourse, or to move from $d_{\text{DEO}}(\text{DEO})$ to $d_{\text{DEO}}(\text{DEO}, \text{M})$, or to promote a bridging discourse. In the next chapter I discuss the possible intervention with respect to the task situations according to objectives in terms of promotion of discourses.

A specific characteristic of our study, that we have considered during the design phases, is the already developed M-discourse of students participating to the experimentation. As a matter of fact, we have grounded some activities and specific task situations on the expectation about M-discourse we hypothesized students had already built, to make it emerge and develop. We made these considerations according to the students' school age and the Italian school system and program, and we check them during the initial interviews with participants. Taking this into account, in our design we also considered, relying on the large body of research about the topic, that the participation of students could be strongly ritual, especially with respect to equation. Thus, in tune with the principle of continuity of discourse, we do not only count on DEO-discourse to promote the construction of M-discourse. We consider the interaction with the DEO essential with respect to the

participation it can prompt. Indeed, technology-enhanced task situations support an explorative participation that can be reinvested in the mathematical activity, developing a new way of constructing M-discourse and participating to $d_M(M)$. The activities with DEOs offer the opportunity to develop DEO-discourses in which words, visual mediators and routine are built thanks to the interaction with the environments which promote an explorative participation directed to the construction of endorsable narratives. Moreover, this students' explorative participation that have been developed thanks to the previous activity with DEOs can be transplanted in the mathematical activities, in which the old M-discourse changes according to new rules and in which the participation should be finalized to the construction of endorsable narratives. To conclude, we rely on the principle of continuity of discourse not only with respect to the use of the features (W, VM, R, EN) and their transplantability in the two discourses, but also referring to the new meta rules of participation to the discourse, developed thanks to a process of de-ritualization in which routines are not anymore a procedure accomplished to satisfy somebody else but a process thanks to which I obtain a product that I am able to interpret and use. In this sense, we can refer to the notions of *flexibility*, *applicability* and the *substantiability* offered by Lavie and colleagues (2019), to characterize the process of de-ritualization. The first one, flexibility, indicates that students have more than one way to perform a task, they have different procedures for the same task (Lavie, 2019, p. 167). For a routine, applicability (Lavie, 2019, p. 169) is the possibility for it to be used and performed in future tasks and not only in the one that is in this moment accomplished, but it also thus becomes a precedent for the student. While a ritual is self-sustained event, the exploration is a process that can be re-invested in future performances. The last one, substantiability, is related to the analysis of the obtained product of the process with respect to the task accomplished, specifically, is referred to the student's assessment of her own procedure (Lavie, 2019, p. 171). These three aspects offer instrument in order to look at students' discourse within activities to observe de-ritualization processes in technology-enhanced activities, as well as to underline changes in the M-discourse from the initial interview with respect to de-ritualization.

To conclude this section, it is worth to underline the relevance of the theoretical considerations here presented. The need to have tools to talk, in a discursive framework, about the relation an expert can see between a DEO and a mathematical content find a solution in the analysis of the transplantability as the investigation of DEO-discourse and of M-discourse of interest. This a priori analysis offers instruments to build, modify or guide the choice of specific DEOs taking into account a mathematical content, in order to promote the construction of a DEO-discourse in which as expert we see the correspondent features of the M-discourse.

Moreover, following the principle of continuity of discourse, not only DEO's design, but also related task situations and teacher's interventions can be developed, expressing their didactical aims in term of discourses to be promoted. Such a scaffolding wants to highlight the potential of a specific DEO to foster the production of discourses as well as the essential role played by task situations and teacher in shifting from a discourse strongly focused on the DEO ($d_{DEO}(DEO)$) to a discourse in which mathematical features are used ($d_{DEO}(DEO, M)$), finally reaching a discourse in which DEO disappears and students construct meaningful mathematical narratives ($d_M(M)$).

4.2. Educational implications of the theoretical framework

According to all these preliminaries, we can now elaborate the main objectives of our designed activities. I want to stress that these didactical aims can refer to other teaching experiments elaborated in the frame of Commognition within digital environments. Thus, they have the will to be general tools to guide didactical activities.

Within the first a priori design phase we adopted some design principles to follow both a theoretical-methodological direction and a didactical direction.

1) Foster students' discursive production.

This principle is coherent with the commognitive framework in which learning is perceived as becoming a participant of a specific discourse, historically established and organized. Thus, our main principle has been to give students tools to be involved in discourse, thanks to which they could develop words, visual mediators, routines and endorsed narratives. This discourse should evolve towards the mathematical discourse. For this reason, we ask students for descriptions, explanations, summaries about the objects involved in their discourses and their action on them.

2) Promote conjectures and argumentations.⁴

Our hypothesis is that dealing with concrete digital objects could help students in constructing endorsable narratives about them, that can be transplanted in the mathematical discourse. In this sense we ask students

⁴ This principle could have been considered part of the previous principle, in point 1. We have decided to distinguish it from the first one considering its epistemological and didactical relevance. Indeed, in our design, we have made a specific focus on the aspects involved in this principle.

for conjectures and argumentations, to promote an explorative participation that can be reinvested in the mathematical discourse.

- 3) Develop well-designed DEOs that give feedback coherent with the canonical mathematical discourse.

DEOs provide students with the possibility to interact with them and their responses follow precise rules based on how they are programmed, coherently with the mathematical discourse of an expert. This is in tune with the analysis of the transplantability and the principle of continuity of discourse. The interaction with the DEO (including its feedback) on one hand offers the possibility to access a discourse related to the canonical mathematical discourse, on the other hand, promote the entry in the mathematical discourse, considering the continuity of discourse according to which the growth of a new discourse is favoured by the transformation of a discourse in which the speaker is already fluent.

Our objectives can be described, thanks to our theoretical considerations in terms of ‘promotion of discourses’. The ‘path’ to follow, a priori, from our perspective was to go from the production of $d_{DEO}(DEO)$ to the production of $d_M(M)$ (Figure 4.5). In between we expected students to produce $d_{DEO}(DEO, M)$ and $d_M(DEO, M)$ both because we foresee that they could spontaneously introduce features from the M-discourse that they have already encountered and at the same time because we have proposed them tasks in which M-discourse’s features were involved, following the principle of continuity of discourse. Thus, we foresee an evolution as in Figure 4.6. As can be seen, the transition from a $d_{DEO}(DEO)$ to a $d_M(M)$ happens through $d_{DEO}(DEO, M)$ and $d_M(DEO, M)$, but we do not suppose a certain path of ‘discourse’s evolution’. In this progression we include the bridging discourse as a discourse on a meta-level, that can be developed anytime students intertwine but mostly relate DEO-discourse and M-discourse. In our hypothesis, the bridging discourse covers a key role in the processes of students’ learning, especially because we suppose it is essential in the development of the M-discourse. Moreover, considering our specific context of research, students have already participated to a M-discourse concerning equations. In this situation, we again underline that bridging discourse can prompt a new participation to M-discourse, giving up on a ritual participation in favor of an explorative one. The building of bridges, in the sense of being involved in discourses about the relation between DEO-discourse and M-discourse can help the development of the last one, prompting a new way of routine’s application supported by narrative endorsed according to well established rules.

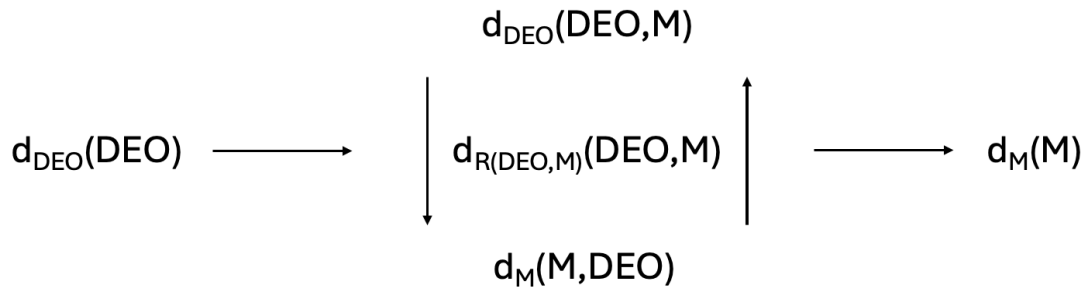


Figure 4.6: The scheme of specific discourses according to aboutness and features used

In summary, the five didactical objectives identified are:

- 1) O1: promote the production of $d_{DEO}(DEO)$ that, as expert, we see related to $d_M(M)$;
- 2) O2: promote the production of $d_{DEO}(DEO, M)$
- 3) O3: promote the production of $d_M(DEO, M)$
- 4) O4: promote the reflection on the relation among DEO and M, or, considering the tools introduced, promote the production of bridging discourses, discourses about the relation between DEO and M ($d_{R(DEO, M)}(DEO, M)$).
- 5) O5: promote the production of $d_M(M)$

These objectives regard both the design of the digital environments and the elaboration of the task situations within and without these DEOs.

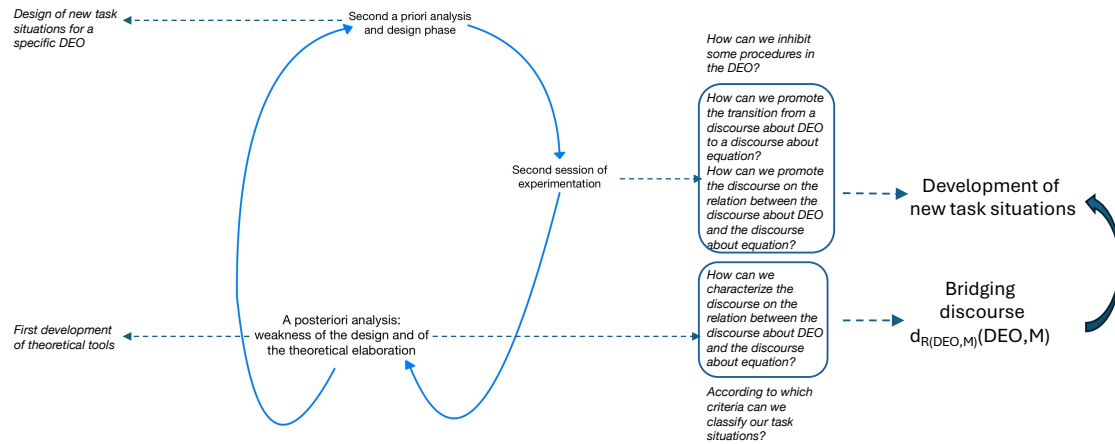


Figure 4.7: Second cycle of design-implementation-a posteriori analysis and questions emerged within it. In blue highlighted two specific questions and the theoretical elaboration developed by us to face these issues.

To conclude this section, I recall some of the questions emerged during and after the second session of experimentation and our answer (Figure 4.7). In particular, I highlight the construct of bridging discourse as a way to prompt the reflection on

the relation between DEO-discourse and M-discourse, that becomes one of the main objectives of our design.

I now want to concentrate on the classification of task situations we have developed according to the theoretical tools previously introduced and the didactical aims highlighted.

4.2.1. The classification of task situations

To present the considerations we have elaborated during the task design phase, I want to recall what is considered to be a task situation in the commognitive framework. Task situation is defined as “any setting in which a person considers herself bound to act—to do something” (Lavie et al., 2019, p. 159). This notion is related to the one of routine, that is the pair of a task interpreted by a subject and the procedure she implements to accomplish it (Lavie et al., 2019).

In the description of our didactical sequence in chapter 5, we will use the term task situation to indicate the initial task situation proposed to students and all the consequent sub-task situations growth within the initial task. We propose here a classification, that concerns the involvement of a DEO, of the task situations we have elaborated, according to specific criteria that have been developed through the three cycles of design. More precisely, this engagement can be *actual* when the subject directly interacts with the DEO, *partial* when the subject has at disposal a non-interactive representation (visual mediator) of the DEO or *artificial* when the DEO is mentioned but it has to be imagined. We are aware that this classification is not unique and exhaustive.

The main characteristic according to which we have decided to develop one of the classifications and distinguish the task situations is the presence of a request of action on the DEO involved. Thus, the two main categories are:

- *Task situations of effective action*: the subject is asked to
 - accomplish an action on the DEO to which correspond a feedback of the DEO, thus a reaction to the effective accomplished action;
- *Task situations of imagined action*: the subject is asked to
 - imagine her accomplishment of an action on the DEO, or
 - imagine accomplishing an action having at disposal specific visual mediators, involving the DEO,and to describe which is for her the expected re-action of the DEO consequent to that action.

The task situations of effective action require a real interaction of the user with the DEO, and thus an actual engagement. In this context, the user participates to a discourse in which the narratives she produces, together with possible visual mediators, are about what she sees happening in the DEO, or, in other words, the feedback offered according to specific actions. In this case, the narratives are for all intents and purposes stories about what the user observes on the screen. These narratives can involve cause-effect relationships but can also be simply description of actions and reactions. On the other hand, the task situations of imagined action are characterized, from a design point of view, by a discourse in which the description's narratives of what the user do and see cannot be endorsed because actions, and thus consequent feedback, are not possible. The user's engagement in this case is partial when she has at disposal a non-interactive visual mediator or artificial when she has nothing at all. We suppose the user to imagine the actions and the DEO's feedback according to the previous effective experience. In this case, the user is forced to develop narratives in which actions and feedback are linked by a causal relation. Students are thus prompted to make explicit the assumptions that guide their narratives in order to make them endorsable. This could be for them a new way to participate to the discourse, in which a process of de-ritualization starts up. Indeed, while within the task situations of effective action, narratives as "a thing is in this state because I see it" could be endorsed, in the task situations of imagined action this could no longer happen. An example of narrative similar to the one referred to scale, in the M-discourse could be "when I solve an equation, I have to take all the terms with x on the left side of the equal sign and all the numbers on the right side of it because you have to do so" that can be found in discourse of students with a strong ritualistic participation. The need to imagine both the action and the feedback could forces students to a more and more explorative participation, aimed at the development of endorsable narratives in DEO-discourse. At the same time, as already discussed, the process of de-ritualization with respect to M-discourse could be favoured by this halfway step in which students do not have action on and interaction with DEO and have to build narratives according to previous experience and established rules. The narratives as well as this new way of participating could be transplanted in the M-discourse, promoting the development of discourses in which the mathematical objects are abstract objects and not just concrete symbol that have to be manipulated.

Together with the request of action (effective or imagined), we have identified different types of requested actions. We have classified them as follows:

- *Free action*: the subject is asked to explore or imagine exploring the DEO, acting freely (for instance, “Drag the red tick mark and observe what happens”).
- *Bounded action*: the subject is asked to accomplish a specific action or to imagine accomplishing a specific action (for instance, “Drag the red tick mark on the value 8”).
- *Goal-oriented action*: the subject is asked to act or to imagine acting with a certain aim (for instance, “If possible, remove 3 objects from the scale so that it is still balanced afterwards”).

The different typologies of action (free, bounded and goal-oriented), in the effective case, are directed to the expansion of the DEO-discourse. Specifically, the free action can mostly prompt the involvement of new words and routines in the discourse, while bounded action and goal-oriented action can offer students the possibility to develop narratives to describe the DEO behaviour, both in a ritualistic or an explorative way.

4.2.2. The second classification of task situations: the model of use of a DEO

Our classification consists in identifying a task situation with respect to the kind of action requested as well as the objective it aims at reaching, from the students' perspective. To discuss the second aspect, we need to re-focus on the definition of DEO offered by Muller and colleagues (2009): “an interactive and dynamic computer-based model or tool that capitalizes on visualization and is developed to explore a mathematical concept or conjecture, or real-world situation” (p. 64). The dynamicity is a feature widely recognized as essential in the digital environments to promote the processes of teaching and learning mathematical contents. The interactivity allows the user to act on the digital environments and to observe feedback due to the action she has accomplished.

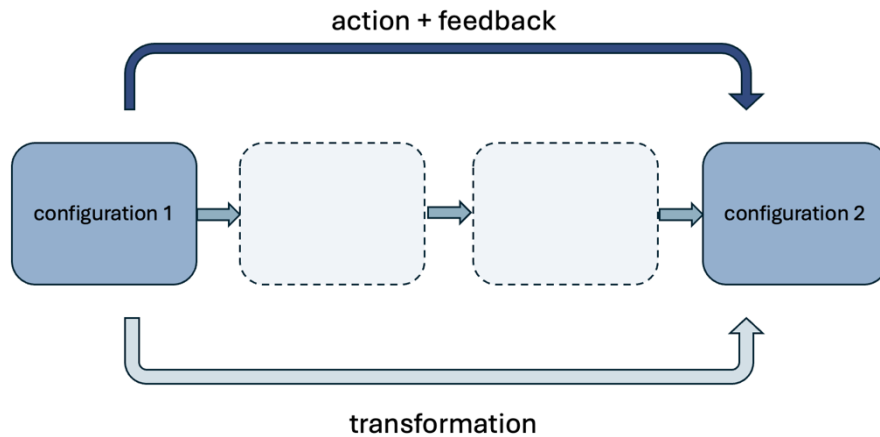


Figure 4.8: Model of use of a discrete DEO

With respect to this, we consider that the action of the user changes the configuration of the DEO, which goes from a first static situation to a second static situation. We call these two situations *configuration 1* (C1) and *configuration 2* (C2). The *transformation* is what happens to the DEO while the user is carrying out an action that take it from configuration 1 to configuration 2 (see Figure 4.8). We can imagine the transformation as a sort of ‘movie’ that tells the story of the DEO from C1 to C2: it is the representation of the action accomplished by the user and the feedback offered by the DEO together. What we are interested in, from a didactical point of view, are the user’s action and the DEO’s feedback, while with transformation we indicate that something has happened to the DEO, that it has changed. The fundamental educational aspects are what has been done to the DEO and how it has reacted to that action. With respect to the design of our DEOs, we have identified two possible categories of dynamic transformations: discrete and continue. On one hand, we can have a sequence of snapshots of the transformation and in this what for us matter are the intermediate static states, each characterized by an action and consequent feedback (Figure 4.8). On the other hand, within continue transformations, what is relevant from our didactical point of view is the continuous evolution from C1 to C2 and the focus for us is on the complete transformation, caused by an action or a continuous action and related feedback (Figure 4.9).

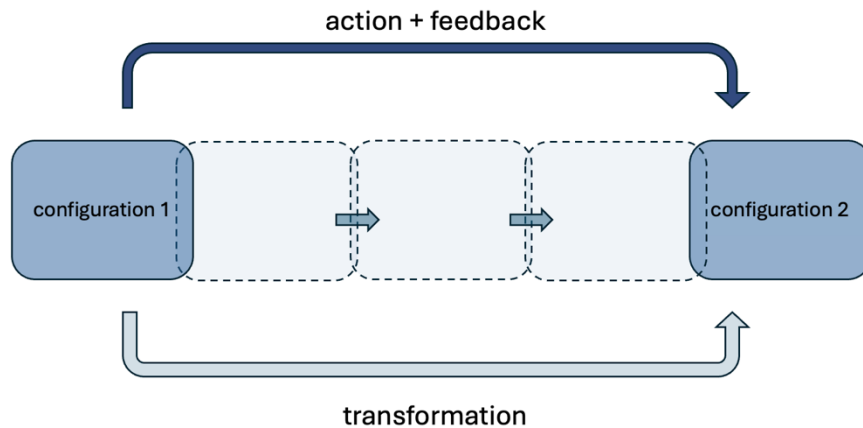


Figure 4.9: Model of use of a continuous DEO

An example of the first category is the dynamic two pan scale overmentioned. C1 could correspond to the balanced scale as in Figure 4.10a while C2 could be the scale's states in Figure 4.10b. The action accomplished is the lifting of the object of weight 7 from the right pan that together with the feedback of the scale that moves from a balanced position to hanging to the left, generate the transformation described in Figure 4.10. This example could also be a snapshot of a greater transformation related, for instance, to the request of lifting 3 objects from the pans of the scale to keep it balanced, and thus one of the steps of the discrete transformation.

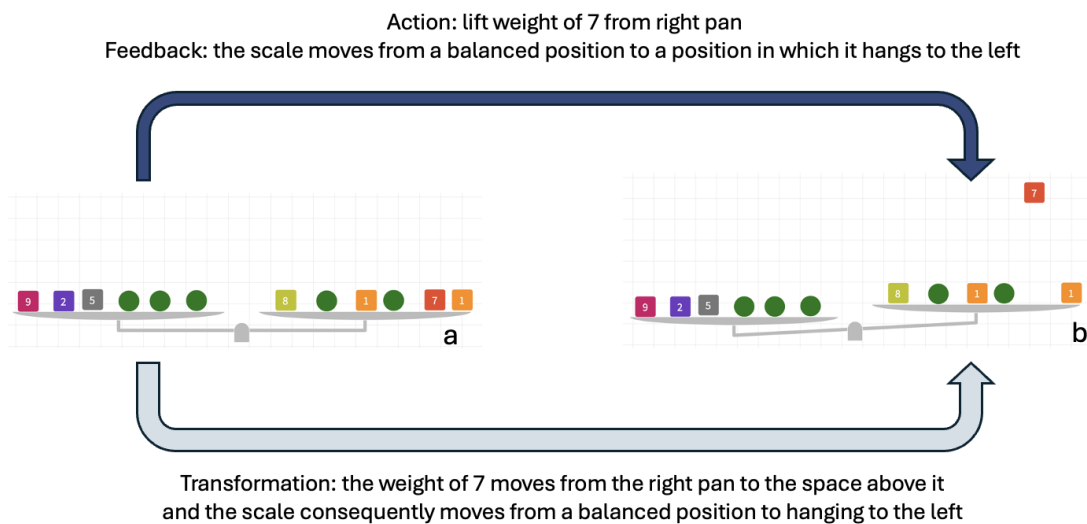


Figure 4.10: An example of C1 (a) and C2 (b) and the correspondent transformation

We want to underline that within a task situation, as well as during an activity, the successive configurations of a DEO could be more than two, generating a chain such as the one in Figure 4.11.

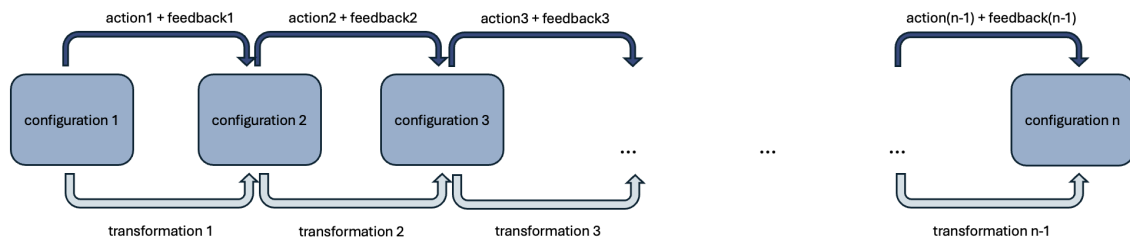


Figure 4.11: Chain of evolution of configurations and correspondent transformations of a DEO

This model of a DEO's use involving C1, C2, the action and the feedback and the correspondent transformation guided us in classifying from a different point of view the type of task situations developed. Indeed, considering the model, we developed two different typologies:

- a) *Descriptive task situation*: the subject describes an experience she has lived as participant in the accomplishment of the action on the DEO; she knows C1, C2 and the transformation (action and feedback), and thus the model is complete in all its parts. The request is to describe the action and what is the feedback offered by the DEO. The discourse here expected is characterized by descriptive narratives in which the objects of the DEO are the main protagonists.
- b) *Target task situation*: the subject has to reach an objective. This objective consists in discovering one or more parts of the model in Figure 4.8 or in Figure 4.9.
 - the subject knows some of the four elements (for example C1 and C2, but not the action and the feedback) and has to complete the model with the missing elements;
 - the subject knows partially all the four elements (C1, C2, action and feedback) and has to complete them with the missing aspects.

We expect here a discourse in which causal relation play an important role. Indeed, the subject knows some parts of the model, and she has to use them to develop the other parts, considering the relation among them. If for example she knows the initial configuration and the action, she has to consider them in order to develop narratives about the feedback and the final configuration of the DEO.

Some of the action to be accomplished could be described in the task situation, while other should be chosen by the subject. With respect to *target task situation*, we refer to the model (Figure 4.8 or in Figure 4.9) and identify missing aspects that have to be identified by the student (Figure 4.12, the component of the model

highlighted with green circles has to be discovered). Indeed, in the category of target task situations we mainly have

- 1) task situations in which the student has to anticipate the feedback of the DEO and C2 (Figure 4.12a);
- 2) task situations in which the student has to identify the action and thus the feedback that transform a DEO from C1 to C2 (both given) (Figure 4.12b);
- 3) task situations in which the student has to identify C1 when she is given the action accomplished and C2 (Figure 4.12c);
- 4) task situations in which the student know partially all the components, the initial configuration, the action and the final configuration, and has to identify the complete model (Figure 4.12d).

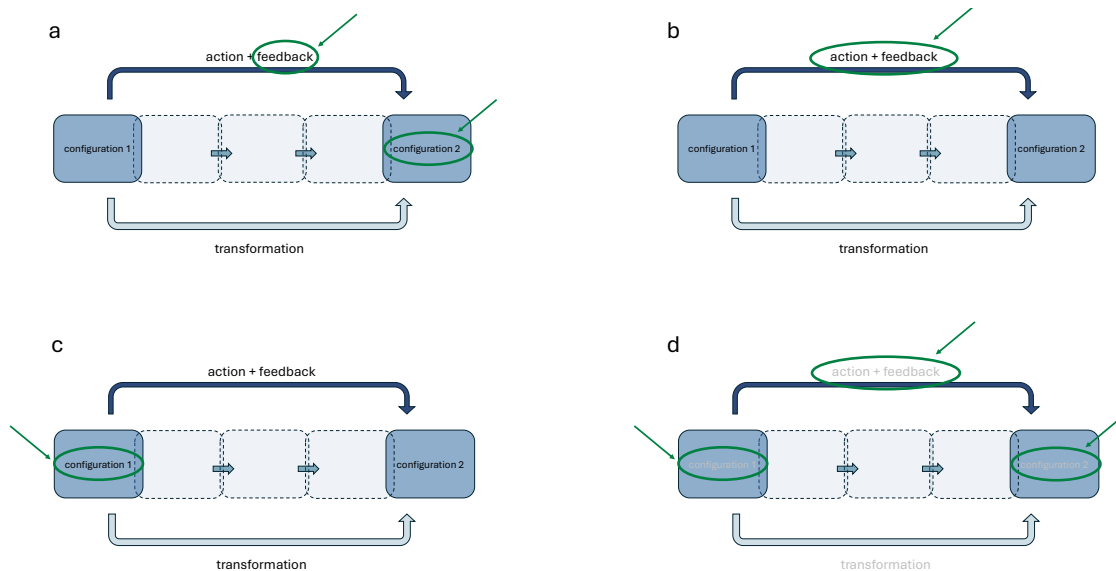


Figure 4.12: Examples of model of use of a DEO with highlighted components that have to be discovered in task situations

A dimension that is common to both kind of task situations, with respect to the narratives, is the request of producing argumentation about the object of the discourse. Indeed, with the accomplishment of the task situation, the tutor involved with the students asks them to justify and explain their considerations and actions. The classification discussed in this section can be summed up as in Table 4-2. In the first line we see the two possibilities of interaction or no-interaction with the DEO, considering effective and imagined actions, that can be both free, bounded or goal-oriented. At the same time, in the second line, we take into account which we expect is the aim of students' discourse in accomplishing the task situation. Thus, we can see descriptive and target task situations that can involve both effective or imagined action of all kinds.

In chapter 5, when the DEOs are presented and analysed, I also discuss the related task situations according to the two classifications proposed.

Table 4-2: Classifications of task situation with respect to the action (first line) and to the students' discourse objective (second line)

Classification with respect to the action	Effective action	Imagined action	Free action	Kind of action
			Bounded action	
			Goal-oriented action	
Classification with respect to the students' discourse objective	Descriptive task situation			
	Target task situation			

Taking into account the questions emerged within the second cycle (Figure 4.13), these two classifications can offer principles to generate other task situations. In particular, the classification with respect to actions allows to develop task situations in which the action on the DEO is not allowed and to describe it in term of free, bounded or goal-oriented imagined action, relating it with previous task situations in which the actions were effective.

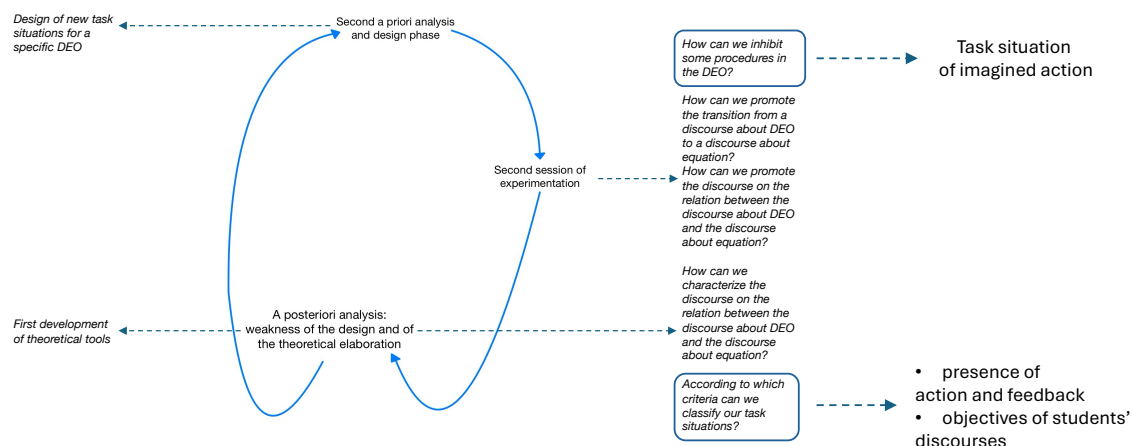


Figure 4.13: Second cycle of design-implementation-a posteriori analysis and questions emerged within it. In blue, highlighted two questions and our developed answers according to the theoretical elaborations

In the next subsection I present a task situation that moves away from the classification here discussed. It involves a video that is used, from a design point of view, as a possible visual mediator for students' discourse.

4.2.3.A special kind of task situation: silent video

A recent strand of research has opened to the use of video in teaching and learning processes. In particular, I want to focus on the work of Bjarnheiður Kristinsdóttir (2021) that has discussed *silent video tasks*. Silent videos are defined as “animated, short films without any text or sound that show mathematics dynamically, each video focusing on one mathematical concept” (Kristinsdóttir et al., 2018). In the silent video tasks students are required to watch a silent video as many times as they want, to discuss in group of two what they have seen and to record an audio that they believe suitable for the video. The researcher identifies 9 main characteristics of silent video (Kristinsdóttir, 2021):

- C1. Keep it short (less than 2 minutes long).
- C2. No words or sound, the silence is intended to start an inner discourse (thought) and thus give a reason for a voice-over to be made (words, outer situated discourse).
- C3. Include some change or movement (otherwise there would be no need for it to be a video).
- C4. Keep it simple (not too overwhelming) and thus accessible for learners to narrate.
- C5. Pay attention to the pacing of movement (timing of events). It should allow for learners to describe, explain or narrate what is happening in the video.

- C6. Focus on one mathematical concept, its definition and/or properties thereof.
- C7. Do *not* show a one-way-to-describe process (e.g. do not display one certain problem solution procedure) but rather make the video offer possibilities for various (also creative) responses—in other words, make the video possess some sense of openness such that there is no single correct way of describing, explaining or narrating the silent video.
- C8. Make it offer learners an opportunity to relate mathematical ideas, possibly deepening their mathematical understanding.
- C9. Make it offer an opportunity for learners to express a generality.

Kristinsdóttir stresses that

“silent video tasks are designed to encourage students to communicate verbally about mathematics, to inform teachers about their students’ conceptual understanding, and to make a ground for group discussion, e.g. about precision in language use. Not only do silent video tasks encourage classroom discussions about mathematics, but they also seem to open teachers’ eyes to students’ ability to use new technologies” (Kristinsdóttir, 2020, pp. 189 - 190).

Kristinsdóttir, in her doctoral dissertation, investigate teachers’ expectations and experiences in using silent video to identify opportunities and challenges in using these tasks, from a teachers’ perspective. In our study, we want to focus on the other side of the coin, investigating the students’ processes in the accomplishment of a silent video task. In the commognitive framework, the task situation of elaborating an audio track for a video seems particularly fitting. Kristinsdóttir (2021) opens to this strand of research:

“Another interesting path would be to define in what ways silent video tasks can become a mediator between different types of discourses, and if they might shorten the so-called discourse-gap described by Leung and Bolite-Frant (2015). This gap between the phenomenal world and the conceptual world exists because students rather use their own presumed mathematical ideas and discourses in learning activities that promote discussion instead of the more formal mathematical concepts defined by teachers or textbooks. In that work, focusing on the learner, Sfard’s commognitive (Sfard, 2008) view might be helpful in the process of analysing of data.” (Kristinsdóttir, 2021, p. 94)

Our research focus is on the processes that grow when the students are involved in the construction of an audio track for the video, especially looking at aspects that

involve the algebraic discourse. Specifically, we decided to use this kind of task situation to promote a bridge between a DEO and mathematics, considering the video as a mediator between different types of discourses. We aim at promoting a bridging discourse between DEO and mathematics, focusing on the respective discourses. From a commognitive perspective, the task situation of producing a voiceover for a video is complex. Indeed, with this request, we ask students

- to produce a discourse related to a visual mediator already provided, that is the video;
- to participate to a discourse in which the object of the discourse is the voiceover (and thus a discourse) for the video.

In developing this task situation, we are interested in the discourse developed within the process of construction and of transformation of the voiceover. Our focus is thus to investigate not the product, but the growth of the discourse for the video: students, producing the voiceover, are involved in a meta-discourse. Indeed, in accomplishing this request, students are asked to participate to a discourse about a discourse. We expect students, that work in pairs, to debate about what they see in the video, but most of all to find a voiceover that for them fits with the video. In doing so, they are prompted to discuss about the relation they see between DEO and mathematics. This task could gain insight into students' interpretation of relations between (concrete) objects and processes with objects that are dynamically shown in the video. We thus take into consideration to develop silent video task situations to promote students' investigation of the relation between two discourses, DEO-discourse and M-discourse, aiming at the production of bridging discourses.

With respect to our cycles present in chapter 3, the second one has been strongly focused on the development of new task situations. In particular, within its experimentation we tested a first silent video task situations, and after the a posteriori analysis we have developed two different silent video task situations to prompt the transition from DEO-discourse to M-discourse as well as the production of bridging discourses (Figure 4.14).

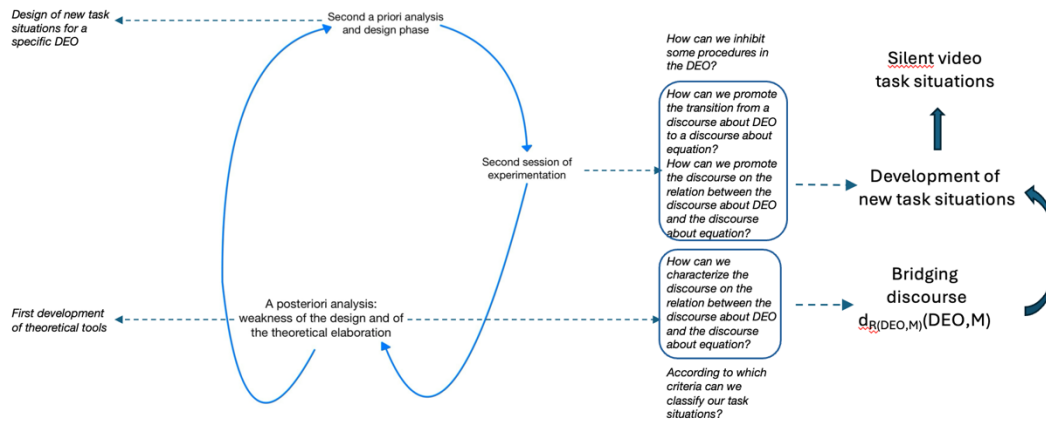


Figure 4.14: Second cycle of design-implementation-a posteriori analysis and questions emerged within it. Highlighted the silent video task situations as a possible answer to the need of relating discourse about DEO with discourse about equation.

4.3. Conclusions

To conclude this chapter, I want to sum up the main points presented.

In the first part, I have introduced and discussed our theoretical elaboration starting from the notion of discourse and its possible faces. Indeed, the distinction between the specific discourse and what we called X-discourse allows us to develop the construct of analysis of transplantability considering DEO-discourse and E-discourse. This analysis permits to describe the relation an expert can see between a specific DEO and a mathematical content, in discursive terms, stressing also the ‘weak’ points of the relation that can become, from our point of view, didactical strengths. On the other hand, the focus on specific discourses leads to the distinction of the object of a discourse and the features a discursant uses to talk about it. The consequent classification is useful in describing aims of task situations and of teacher intervention, as well as in the a posteriori analysis of the activity.

In the second part of the chapter, I have introduced two classifications for task situations. The first one puts the light on task situations that exploit the interaction or the missing interaction with the DEO, to prompt the expansion of the DEO-discourse and the correspondent, from our point of view, expansion of the M-discourse. Specifically, the main aims of these task situations are:

- to promote the use of new words, visual mediators and routine to enrich the DEO-discourse ($d_{DEO}(DEO)$)

- to promote the emergence of the relation between DEOs embedded in the DEO to prompt the production of narratives in the DEO-discourse ($d_{\text{DEO}}(\text{DEO})$)
- to promote the production of narratives in the DEO-discourse that once transplanted can be endorsable in the M-discourse ($d_{\text{DEO}}(\text{DEO})$)
- to promote the development of a bridging discourse ($d_{R(\text{DEO},M)}(\text{DEO},M)$)

The second one is based on the model of use of a DEO we have constructed and is centred on what we expect to be the objective of students while accomplishing that task situation.

The classifications, together with the theoretical elaborations introduced and described before, will be implemented in chapter 5 within the presentation and the analysis of the DEOs we have designed for our experimentation. We will present not only the task situations related to this classification, that involve a DEO (effectively or not) but also the “mathematical task situations” we have considered in our experimentation, some of which follow the typical task situations of schoolbooks and teacher practice, while other are related to the activity with the DEO.

The description and the analysis that will follow are related to the mathematical content of our interest, that is algebra and, in particular, equations. The theoretical elaborations and the task situations’ classification here offered aim to have a general nature. They are indeed tools that could be implemented in studies involving digital environments that assume a discursive perspective. Specifically, they can become useful instruments to develop didactical activities, from a research point of view as well as from a teacher perspective.

5. A priori analysis of DEOs and task situations

This chapter's main aim is to illustrate the different DEOs we have elaborated during our design phases. In designing the environments and the corresponding task situations, we realized the need to elaborate tools to describe our design and our related objectives. For this reason, we developed some theoretical tools, presented and described in the fourth chapter, that we now use to introduce, discuss and analyse our design activity.

We have chosen two models to deal with the mathematical contents of our interest, that is algebra and more specifically equations. From now on I want to focus on this mathematical content, thus I substitute the general notation M-discourse with E-discourse, where E stands for equations. The first model we use in our design is the balance model, a common metaphor in teaching linear equations, to introduce the equal sign and to gain familiarity with known and not known quantities. Otten and colleagues (2019) have discussed the use of this model within research projects about algebraic reasoning and introduction of the linear equation. The relation of equality between two expressions, that must be kept all along the solving process, is a crucial aspect in equations' solving, but it is often disregarded, in particular because of students' operational perception of equal sign. Thus, the balance model can help in supporting and developing strategies for linear equation solving, improving the relational role of the equal sign. In particular, together with dealing with not known quantities, the balance model promotes the "doing the same things on both sides" principle. In other words, in the process of equation solving, students often talk about "change sides, change sign", that in the Italian school is called "rule of transport" (in Italian, "Regola del trasporto"). As expert we know that this "rule" derives from the principle that, when two quantities (known or not known) are equal, the act of adding (algebraically) the same quantity to both keeps them equal. This principle seems ignored in students' practice, together with the relational idea of equal sign, in favour of the change sides, change sign rule. One of the key points in using the balance model is that it well deals with this principle of equivalence, taking into account the role of equilibrium as a good representative of the relational idea of equal sign.

At the same time, we are aware of the limitations of the balance model. Negative numbers cannot be represented in this model, together with the restriction to the representation of linear equations only. Moreover, in using the balance model the risk is to promote an additive perspective about not known values. In our design we have decided to use the balance model and to focus on its potentialities, but also to deal with its weaknesses. Specifically, we have decided to use two diverse balance models to stress different aspects we see related to equations and

equations solving and to face the limitations of the balance model. This will be clarified along this chapter, in the description of the dynamic explorative objects designed where I present the analysis of the transplantability of each balance model used, putting the light on transplantable features and not-transplantable features of the two discourses (DEO-discourse and E-discourse).

The second model is based on dynagraph, a representation of functions described in the literature (Goldenberg et al., 1992). Dynagraph is a dynamic artifact that visually realizes functions. It consists in representing two horizontal lines with one point on each line, one is the independent variable, which is dynamic, while the other is its image. Recent studies have investigated the role of dynagraphs in promoting the emergence of a discourse about a function and its features (Ng, 2016; Lisarelli, 2019; Antonini et al., 2020). The idea of the Algebraic Line in AlNuSet (Robotti, 2017) is related to the dynagraph's construction. In this environment, an algebraic variable is represented as a mobile point on a line that can be labelled with a letter. In the same line there can be algebraic expressions dependent on the variable, that moves accordingly to the movement of the point "algebraic variable". In the Algebraic Line students can explore equations, inequalities, systems of equations and inequalities.

Our second DEO is called moving-arrows-DEO and takes advantage of the dynamicity and interactivity of the software GeoGebra to deal with equations and inequalities, echoing the idea of dynagraphs. The aim is the production of a discourse about independent and dependent quantities and their relation.

The two models are embedded together in the mobile-arrows-DEO. We design this environment to put together the affordances offered by the two separated DEOs; moreover, this union can promote a saming process within the students' discourse thanks to which an algebraic discourse could grow.

In this chapter I present the analysis of the transplantability for each DEO (Figure 5.1). In this sense, I describe words and visual mediators, routines as the pair composed by task and procedure, narratives that can be endorsable, that belong to a DEO-discourse, stressing their relations with the E-discourse. Moreover, I examine the task situations proposed in the light of the didactical aims previously described and the classification introduced before.

Together with the DEO and the related task situations, students interact also with a tutor. The tutor's role is to accompany the students during the activities, making sure that the use of the software and the task situations are clear. Moreover, he intervenes to prompt students' discursive production by making requests such as "Can you explain to me again?", "Can you repeat it?", "Could you tell it in other

words?”. We have developed, in the light of the theoretical development described before, a ‘script’ for tutor’s intervention within a certain DEO. Some of these interventions are collected in this chapter for each DEO after the description of the task situations proposed.



Figure 5.1: Representation of the analysis of the transplantability

To sum up, I now describe DEOs we have used during tutoring sessions. With respect to the theoretical perspective and tools previously introduced, I decide to follow this path for each DEO:

- Discuss the analysis of the transplantability as the study of the relation between a DEO and a specific mathematical content. More specifically:
 - Epistemological analysis of mathematical discourse of interest (arithmetic, algebra, functions, geometry, and so on)
 - Study of DEO-discourse and M-discourse, and thus of features (words, visual mediators, routines and endorsed narratives) constituting the two collections
 - Study similar use of words and visual mediators, correspondence in routines and endorsed narratives, between DEO-discourse and M-discourse, and thus study of features of a discourse that can be transplanted in the other one (and vice versa)
 - Identify subsets of DEO-discourse and M-discourse that are interchangeable and subsets that are not (see Figure 5.1). This is the study of the limitations of the relation between the two discourses that, in our perspective, can have a role in the didactical process.
- Description of the main task situations within the DEO, according to the classifications we propose in the fourth chapter. We classify the task situations in two ways. The first one concerns the presence of a request of action that can be effective or imagined. In both cases, the action can be free, bounded or goal oriented. I classify each of the task situation presented here according to this classification. On the other hand, the second classification refers to the aim a subject has to reach within the accomplishment of the task; indeed, with respect to the model described

and reported in Figure 5.2 (that has also a “continuous version”), we expect students to discover one or more components of the scheme (that are configuration 1, configuration 2, action and feedback). I describe the task situations of the first two DEOs with respect to this model.

- Collection of some tutor’s intervention with respect to the specific DEO and certain aims.

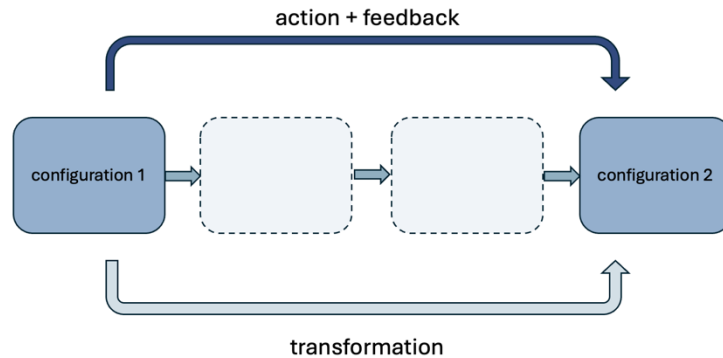


Figure 5.2: Model of use of a DEO

The sequence here presented is the last elaboration of our design activity. We have used it during the third experimentation with students in autumn 2023. It is the result of the design process depicted in chapter. At the end of the chapter, I briefly present the sequence of activities of the first experimentation and the consequent evolution of it to the actual product I here discuss, putting the light on the main differences. Some of the task situations described below can be found in the DynaMat project website (<https://www.carme.center/risorse-dynamat/>) which is under continuous development and will be updated with additional activities.

The task situations, that have been formulated in Italian, are translated in this chapter into English. I enumerate the task situations from 1 to 48, thus I do not restart the enumeration for each DEO. Only the final ‘mathematical’ task situations present a new enumeration (from 1M to 5M). I underline that I analyse here only examples of task situations. Of the same typology, in our sequence, there could be more than one. In the text, when this happens, I point it out in brackets near the number of the task situation or along the text. In Appendix A there are all the task situations in Italian and in their English translation.

5.1. The two-pans-scale-DEO

One of the DEOs chosen during the design phase of the activities is a dynamic two-pans-scale, now on TPS-DEO or TPS, available on the DESMOS platform. The scale moves assuming a balanced or unbalanced position depending on the sum of the weights’ values on the two pans. The user can move one object at a time, and when

an object is placed or removed from a pan, the DEO produces dynamic feedback in response to the action accomplished. Specifically, as in Figure 5.3, the objects that can be moved on the pans are:

- coloured shapes with a number inside which refers to the value of the weight corresponding to that object; objects with the same shape and colour realize the same numerical value; we call these objects *known weights*
- coloured empty shapes, whose weight's value is not known; objects with the same shape and colour realize the same not known value; we call these objects *unknown weights*.

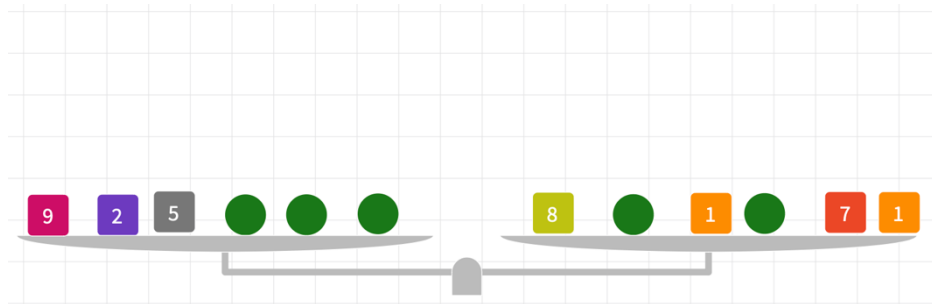


Figure 5.3: An example of two-pan-scale DEO

All the weights have been set up previously by the designer and the user cannot create new weights. From a mathematical point of view, the collection of all the objects on the left and on the right pans realize, respectively, two algebraic expressions. In particular, known weights realize numbers in algebraic expressions while unknown weights are realization of the mathematical object unknown in the algebraic expressions. The balanced scale is a realization of the condition of equality between the two algebraic expressions while the not balanced scale is a realization of an inequality between the two algebraic expressions. Let's consider the scale in Figure 5.3. The algebraic expression realized by the left pan is $9 + 2 + 5 + x + x + x$, while the algebraic expression realized by the right pan is $8 + x + 1 + x + 7 + 1$. The scale is balanced: it realizes the condition of equality between the two algebraic expressions, or, in other words, the equation $9 + 2 + 5 + x + x + x = 8 + x + 1 + x + 7 + 1$. It is worth to underline that the value of an unknown weight is previously settled by the designer of the DEO. With the use of this DEO, it is possible to study the relation between a variety of different algebraic expressions depending on the same unknown (the value of which is chosen by the designer of the DEO). The possible action a user can do are putting or removing objects on/from the two pans of the scale. For instance, in Figure 5.4 we see the scale of Figure 5.3 where have been removed the weight of value 9 from the left pan and the weight of value 7 from the right pan. The scale is no longer balanced after these actions.

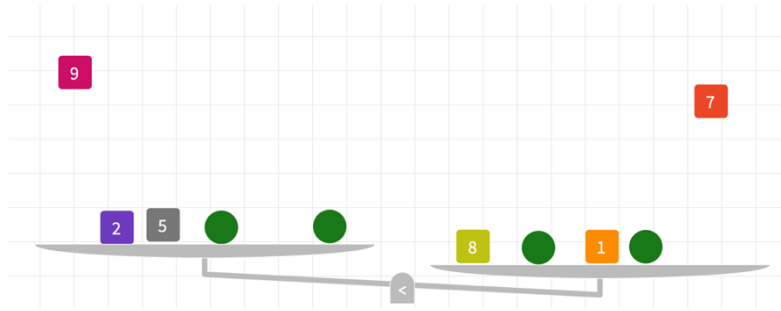


Figure 5.4: Scale of figure 3 with some objects lifted from the two pans. Since the value of the weight removed from the left pan is greater than the value of the weight removed from the right pan, the scale is hanging to the left

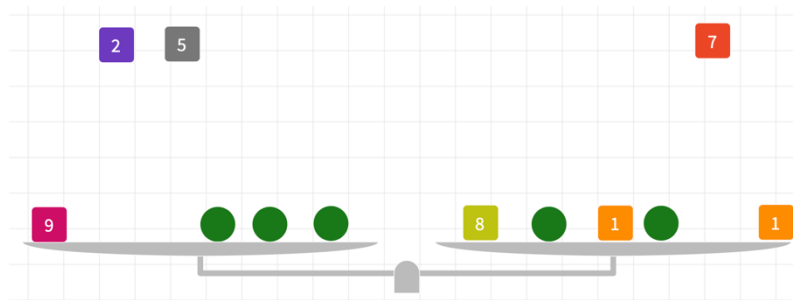


Figure 5.5: Scale of figure 3 with objects lifted from the two pans. Since the value of the weights removed from both pans is the same, the scale is balanced

As already discussed, the two-pans-scale DEO here described has been involved in our design because the balance model has been proved to be effective in dealing with the relation of equality among quantities and with known and not known quantities. Indeed, the possibility of adding or removing objects to or from both pans allow the user to discuss about the actions that keep a scale balanced or that change its condition (see for instance, the difference between Figure 5.4 and Figure 5.5). With the analysis of the transplantability conducted, we can highlight in which ways the activity with TPS and the consequent development of TPS-discourse can help the construction and the development of E-discourse, according to the principle of continuity of discourse. In Table 5-1 I present the analysis of the transplantability of the two-pans-scale DEO. In column 1 we find the feature of the discourse to which we refer, in column 2 we find the TPS-discourse feature while in column 3 there is the correspondent feature of the E-discourse. If, according to the analysis, the feature is not-transplantable I write not-transplantable, in bold.

Table 5-1: The analysis of the transplantability of TPS-DEO

Feature	TPS-discourse	E-discourse
word	Weight	Number

word	Object with not known weight/Little red (blue, green, ...) ball	Unknown
word	Equilibrium	Equality
word	Pan	Algebraic expression
visual mediators	Not-transplantable	Negative numbers
visual mediators	Weights without number	Unknowns
visual mediators	Weights with the same colour and shapes (with or without numbers)	Equal numbers
visual mediator	Collection of weights on a scale's pan	Algebraic expression on the left or right of the equal sign
visual mediator	Equilibrium of the scale	Equality between two algebraic expressions $f(x_0) = g(x_0)$ where x_0 is a specific, but not known, number
visual mediator	Scale that hangs to the right	Inequality $f(x_0) < g(x_0)$ where x_0 is a specific, but not known, number
visual mediator	Scale that hangs to the left	Inequality $f(x_0) > g(x_0)$ where x_0 is a specific, but not known, number
procedure	Putting in weights on one scale's pan	Performing an algebraic sum on an expression
procedure	Removing weights from one scale's pan	Performing an algebraic sum on an expression
procedure	Putting in the same weight or a collection of weights with the same sum to both scale's pan	Adding up algebraically the same numbers or algebraic expressions to two sides of an equation
procedure	Removing the same weight or a collection of weights with the same sum from both scale's pan	Adding up algebraically the same numbers or algebraic expressions to two sides of an equation
procedure	Replacing a weight or sum of weights in a pan with a sum of weights or a weight of the same value	Replacing a term or a collection of terms of an expression with a sum of

		terms or a term with the same value
routine	Task: find the value of the weight of an object with not known weight procedure: put the weight with not known value on a pan and weights with known values on the other until the scale is balanced	Task: find the value of the unknown procedure: not-transplantable
routine	Task: find the value of the weight of an object with not known weight procedure: removing weight with the same value or a collection of weights with the same sum value from both scale's pan	Task: find the value of the unknown procedure: adding up algebraically the same value to two sides of an equation
routine	Task: find the value of the weight of an object with not known weight procedure: not-transplantable	Task: find the value of the unknown procedure: change side change signs rule
endorsed narrative	P ₀ : A scale is balanced if and only if the total weights' value on the two pans is the same. It hangs on the side of the pan where the total weights' value is greater.	P ₀ : The equality " $f(x_0) = g(x_0)$ " (where x_0 is a specific, but not known, number) is true if and only if $f(x_0) = g(x_0)$
endorsed narrative	P ₁ : If objects that collectively have the same weights' value are added to or removed from both pans from a balanced scale (not balanced), then the scale keeps being balanced (not balanced).	P ₁ : if $A = B$ then $A \pm C = B \pm C$
endorsed narrative	P ₂ : If, in balanced scale (not balanced) I remove an	P ₂ :

	object from one of the two pans and replace it with one or more objects whose added weights' values are equal to the value of the initial weight, the scale will remain balanced (not balanced).	if $A = B \pm C$ and $C = D + E$ then $A = B \pm (D + E)$ And: if $A \neq B \pm C$ and $C = D + E$ then $A \neq B \pm (D + E)$

The relations we see between TPS-discourse and E-discourse are many and interest all the features of the discourse (word, visual mediators, routine, endorsed narratives). Words as weight, little red ball, pan, balanced scale, of TPS-discourse are transplantable in E-discourse in place of number, unknown, expression, equality, and vice versa. At the same time, all the DEOs composing TPS (weights, collection of weights, equilibrium of the scale and so on) are visual mediators of the DEO-discourse transplantable in E-discourse as described in Table 5-1. Moreover, different types of procedures can be implemented thanks to the action of adding or removing objects to/from the two pans, that correspond to different procedure in the E-discourse, as described in Table 5-1. What we are mainly interested in is the procedure of removing weights with the same value or a collection of weights with the same value from both scale's pan because it is transplantable in the E-discourse, where we have the procedure of adding up algebraically the same value to two sides of an equation. As highlighted in Table 5-1, these procedures can be paired with the tasks of finding the value of an object and of finding the value of the unknown respectively, composing possible students' discourse routines. As I describe in the next paragraph, we focus several of our task situations on this TPS procedure, also not related to the request of finding the not known weight's value. With respect to the tasks overmentioned, the analysis of the transplantability makes emerge that they can be paired with procedure in a discourse that do not find a correspondent in the other one. One is the case of removing all the objects from the scale's pans, put a weight with not known value on one pan and weights with known values on the other until the scale is balanced. This procedure that can be implemented in the TPS-discourse cannot be transplanted in E-discourse, in which 'removing' all the terms and posing an unknown on one side of the equal sign and known values is not coherent with the mathematical discourse. At the same time,

the procedure known as ‘change side, change signs’ rule, widespread in the Italian school curricula, do not find a correspondent in the TPS-discourse. Indeed, moving objects from one pan to the other of the scale would unbalance it and thus not keep the initial relation among quantities. Within the description of the TPS task situations I illustrate how we considered to use these not-transplantable procedure as didactical tools. Another essential role in our design is covered by the narratives that can be produced in TPS-discourse and are transplantable in E-discourse. Specifically, as reported in Table 5-1, we focus on three principles in TPS-discourse (P_0 , P_1 and P_2) that, once transplanted in E-discourse become assumptions to guide students’ narrative when they solve linear equations. In the task situations I describe, when of the main aims is to develop these principles in TPS-discourse and to transplant them in E-discourse. We foresee that this helps also in the development of a justification for the change side, change signs rule. As is described in the next paragraph, we have introduced some task situations in which the dynamic scale is substituted by a picture of the balanced scale. In this task situation, visual mediators that cover a relevant role are the cancellation marks that can be produced on the weights of a scale, when there is a picture of it and not a dynamic one, as well as on the terms of an equation. In this case the visual mediator and its use in the two discourses are the same, what differs are the object on which it is produced. On the dynamic scale the user can perform the action of adding or removing objects to/from the pans and obtaining feedback according to it.

5.1.1.TPS task situations

In this section we present some task situations related to this DEO and their analysis according to the classification we proposed in chapter 4. With respect to the model of use of a DEO, we refer to the discrete one, considering that in the TPS the user can perform a discrete action (add or remove one object) and the scale offers a feedback. For each task situation I write in brackets if students have at disposal a dynamic scale or a picture of a scale.

Task situation 1 *You have the objects in the blue box at your disposal. If possible, put some of these objects on the pans of the scale so that it is balanced. If it is not possible, explain why.* (dynamic scale, Figure 5.6)

This first task expects the solver to choose among the objects in the blue box and puts them on the pans of the scale to have it balanced. This is a task situation of effective goal-oriented action because the subject has to act on the DEO in order to obtain a certain situation. The task wants to promote a $d_{DEO}(DEO)$ in which the discursant starts talking about balanced scales, weights, pans (new words) together with gestures with the two hands that stand for balanced/not balanced

scale (new visual mediators). Moreover, we expect the discursant to produce endorsable narrative about the conditions to have a balanced situation (P_0), such as “To have a balanced scale, I have to put objects with the same value on the two pans. Otherwise, I have to put a collection of objects on a pan that has the same value of the collection on the other pan”. Thus, this task situation aims at an endogenous discursive expansion. In this task the subject only knows completely C1, that consists in the empty scale and the collection of weights and their values at disposal. She knows that she has to put some objects on the pans and obtain a balanced scale, and thus she is aware of a generic C2 to reach. The action to be made (in the sense of the choice of objects) and the feedback are not known. The request aims at the development of action that should be supported by narratives about the scale according to the pair action accomplished-feedback received.

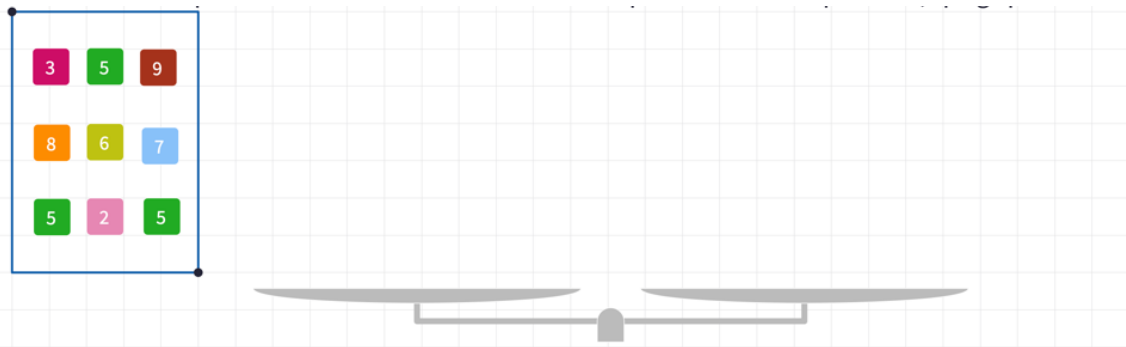


Figure 5.6: Two-pans-scale of task situation 1

Task situation 2 Here you see a balanced scale. If possible, add 2 objects from those available in the box so that it is still balanced afterwards. If it is not possible, explain why. (dynamic scale)

Task situation 3 Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why. (dynamic scale)

Task situation 4 Here you see a balanced scale. If possible, change only the (things on the) right pan so that it is still balanced afterwards. If it is not possible, explain why. (dynamic scale)

These are task situations of effective goal-oriented action: they ask to obtain a certain situation in a certain way (adding 2 or removing 3 objects, or changing the composition of one pan, to keep the scale balanced). The task situations want to promote, in line with the analysis of the transplantability, an expansion of TPS-discourse, promoting d_{DEO} (DEO) in which could emerge the narratives that in Table 5-1 I called P_0 , P_1 and P_2 (with respect to the DEO). Indeed, accomplishing the task

situations students should observe reaction of the DEOs according to their action (adding/removing objects) and, after several attempts, formulate general observations that can become P_0 , P_1 and P_2 . Moreover, in these scales there are only weights with known values. The subject, as the task situation before, knows C1 completely, C2 partially and the general action (adding or removing objects, suggested by the task).

These four task situations just discussed have the aims of

- expanding students' TPS-discourse: within these task situations students have to implement new actions, but also use new words to construct new narratives
- constructing ways of talking about different objects in the DEO according to its feedback
- elaborating and producing P_0 , P_1 and P_2 for scales with known weights only.

Task situation 5 (and similarly, task situation 6) *Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.* (dynamic scale, Figure 5.7)

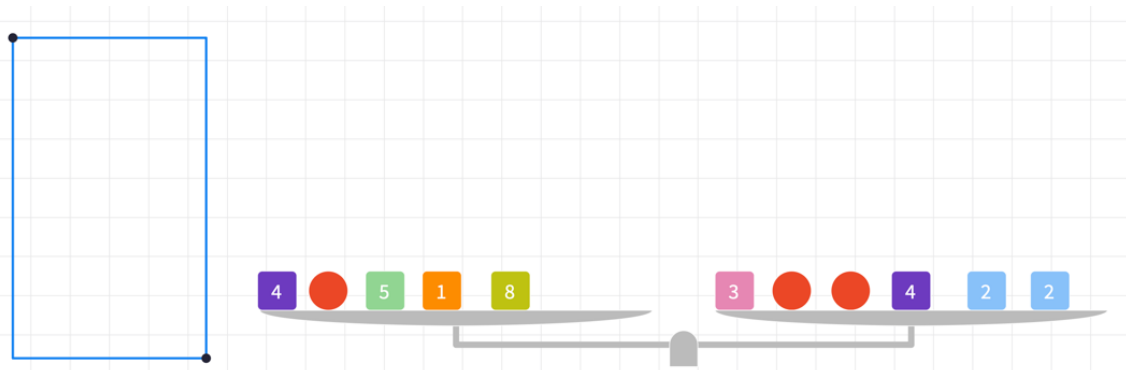


Figure 5.7: Two-pans-scale of task situation 5

This task situation is goal-oriented and is proposed together with the scale in Figure 5.7. This is the first task situation where objects with unknown weights are present. The task situation wants to promote d_{DEO} (DEO) in which are involved terms to refer to the weight with not known value, new narratives about it are developed, together with action where the red ball is moved. Moreover, the task situation aims at re-considering the narratives emerged within task situations 1, 2, 3 and 4 (P_0 , P_1 and P_2), taking into account the presence of objects with a not known value. In addition to this, following the principle of continuity of discourse, we expect students to deal with the little red balls as they have previously done with other objects which have known values. In the light of the analysis of transplantability, this treatment of the

little balls can become the way in which students deal with unknown in E-discourse, and thus develop discourses about indeterminate quantities as if they were known.

Task situation 7 (and similarly, task situation 8) *Here you see a balanced scale. Could you find out how much a little blue ball weighs?* (dynamic scale, Figure 5.8)

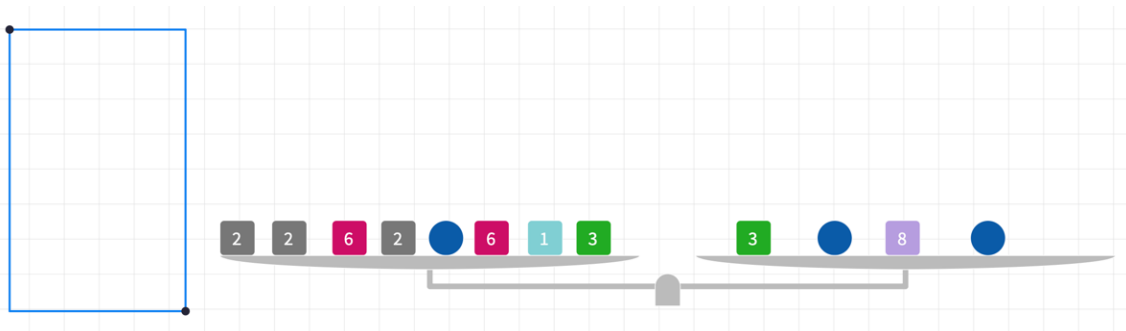


Figure 5.8: Two-pans-scale of task situation 7

This task situation, which balance is the one in Figure 5.8, asks explicitly to find out the value of a not known object. It is a goal-oriented task. This request aims at making emerge procedures to discover the unknown value, in particular the one as experts we see related to the procedure of solving linear equations, discussed in the previous analysis. Despite this, we are aware of other possible strategies to find the blue ball's value, such as leaving one ball on a pan and put different objects in the other till the scale is balanced. In this case the student is free to act as she prefers, we do not prompt for a specific procedure, aiming at developing the basis for a bridging discourse about the different procedures to accomplish this task situation.

Task situation 9 (and similarly 10, 11, 12) *Could you find out how much a purple square weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.* (picture of a scale, Figure 5.9)

The task is a goal-oriented task in which we furnish not a dynamic balance on which the solver can perform action, but just a picture of a balanced scale (see Figure 5.9). The request is to find out the value of the purple square. The aim is to promote a $d_{DEO}(DEO)$ in which the solver uses the principles described in Table 5-1, that she should have elaborated within previous task situations, to explain and argument her reflections to find the solution. In addition, she has the possibility to write on the picture: this feature is given to promote the production of signs on the scale, such as cancellation marks, that are transplantable in the process of resolution of linear equations. This task could open to $d_{DEO}(DEO, E)$. The student knows C1, but she does not know the action, feedback and C2. She is here required to develop a discourse

in which she makes hypothesis on the action and the consequent feedback, justifying them.

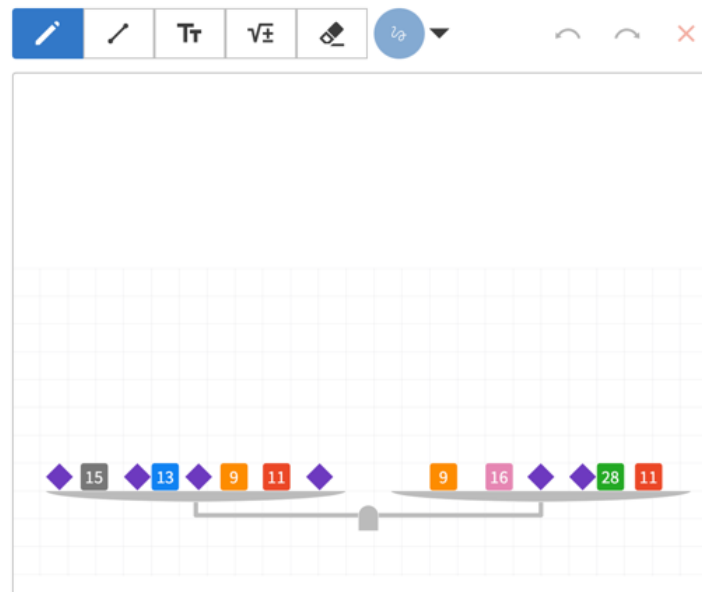


Figure 5.9: two-pans-scale of task situation 9

Task situation 13 Here you see a balance in equilibrium. What happens if we remove an object from both pans? Justify your answer. (picture of a scale)

Task situation 14 (and similarly, task situation 15) Imagine a scale that is inclined to the left. On the left pan I add a weight of 9. On the right pan I add a weight of 5 and two red balls. The scale is now balanced. What can you say? Justify your answer. (nothing)

These two are argumentation tasks. In the 13 solver has at disposal a picture of a balanced scale while in the 14 he has no scale, but he has the possibility to draw, because he has to imagine a scale hanging to the left. Students have to construct argumentation, using principles P_0 , P_1 and P_2 with respect to the scale that they have elaborated in the previous task situations to discuss their ideas and different situations according to what is added or removed from the pans. In task situation 13 students know $C1$ and partially the action to be accomplished. They have to imagine what could be the feedback to all the possible action and thus $C2$. Instead, in task situation 14 they partially know $C1$, but they completely know the action, the feedback and $C2$. They have to develop a discourse in which they make hypothesis that could refer to $C1$ as well as one or more elements of the scale, such as the composition of the pans as collections of weights, the value of the red balls and so on.

Task situations 9, 13 and 14, and all the other developed proposing only a picture of a scale are task situations of imagined action. The scale is not dynamic and interactive anymore: objects cannot be moved, and feedback is absent. Students are asked to satisfy a goal (task situation 9) or to imagine possible situations according to different actions (task situations 13, 14). We consider inhibiting actions, and thus the consequent re-action of DEOs, in order to promote the development of a discourse in which the feedback of the DEO is substituted by students' narrative. Indeed, the absence of DEO's answer requires students to make hypothesis on the consequence of their actions and to justify them, especially according to the principles they have developed. Thus, students are forced to enrich their discourse explicating the assumptions they make to develop narratives that can become endorsed narratives. These task situations have been designed to prompt the use of the principles previously elaborated (P_0 , P_1 , P_2) to support the narratives about the imagined action and, especially, the imagined reaction of the DEO. Indeed, within these task situations, narratives as 'after I remove these objects the scale is balanced because I see it' are not possible anymore. We consider introducing this typology of task situations to promote an explorative participation to the discourse that can be transplanted in the E-discourse, where a feedback is not present and the solver must guide his action with meaningful narratives.

To conclude this section, I present in Table 5-2 some of the tutor interventions we have developed for the task situations. Specifically, in Table 5-2 we find some of these interventions are collected, together with their objectives (column "Objectives") and the task situations in which these interventions were used (column "Task situations"). The objectives of the interventions are described according to the objectives of our sequence of activities. In particular, we focus on the expansion of TPS-discourse, and thus the use of words, visual mediators, routine and endorsed narratives. With respect to the last features, the main aim is to foster the development of the principles collected in Table 5-1 with respect to the scale. Moreover, these task situations can open to $d_{DEO}(DEO, E)$. Indeed, some students could introduce some features of the mathematical discourses to describe the scale (for instance, the word "unknown" to talk about objects with not known value or the inscriptions $x + 15 + x + 13 + 9 + 11 + x = 9 + 16 + x + x + 28 + 11$ to deal with the request in task situation 9). Thus, we have developed some intervention in order to prompt the processes of intertwining of discourses.

Table 5-2: Some of the tutor's intervention developed for TPS-task situations

Intervention	Objectives	Task situations
Can you find other ways?	Promote different ways to accomplish a specific request (discursive expansion)	1; 2; 3; 4; 5
Before you act on the balance, write on the notes what the balance looks like in the beginning, so we don't forget it	Promote a $d_{DEO}(DEO)$ and a $d_{DEO}(DEO,E)$	2; 3; 4; 5; 7
What if I now asked you to remove a different number of objects? For example, 4? (the number change according to the task we refer to)	Promote a $d_{DEO}(DEO)$, in which seeds for principles in Table 5-1 are developed (discursive expansion)	2; 3
Show me on what you wrote what you have just done on the balance.	Promote a $d_{DEO}(DEO,E)$	2; 3; 4; 5; 7
How would you say that in general? Can you write it?	Promote a compression of the discourse thanks to a general formulation of P_0 , P_1 and P_2	1; 2; 4; 5
Can you explain and justify each step?	Promote a $d_{DEO}(DEO)$ in which the principles in Table 5-1 are used to argument	9; 13; 14

In order to promote a bridge between TPS-discourse and E-discourse, after this session with two-pans-scale, we introduce a sequence of activities that do not strictly involve a dynamic environment.

Silent video 1

I have discussed in chapter 4 the notion of silent video task introduced by Kristinsdóttir (2021). I now settle silent video tasks within our sequence of activities,

underlying that some design choices are quite different from the characteristics identified by Kristinsdóttir (2021). Specifically, we diverge from Kristinsdóttir's definition since our design choice has been to develop some videos in which parts of the voiceover are present.

Task situation 16: The first silent video

We have developed a silent video task situations to put the light on the relation between TPS-discourse and E-discourse. Indeed, as discussed in the previous paragraph, task situations with respect to two-pans-scale DEO have the aim to prompt the production of $d_{DEO}(DEO)$ and $d_{DEO}(DEO,E)$. As a matter of fact, while dealing with TPS, the tutor is supposed to intervene to foster the possible use of visual mediators and words of the E-discourse, as well as some of the task situations could favor this use. In this frame, we consider appropriate to push for the development of a bridging discourse. Indeed, recalling our scheme of discourses (Figure 5.10) and the analysis of the transplantability of TPS previously discussed, we want to take advantage of TPS-discourse to promote the development of E-discourse.

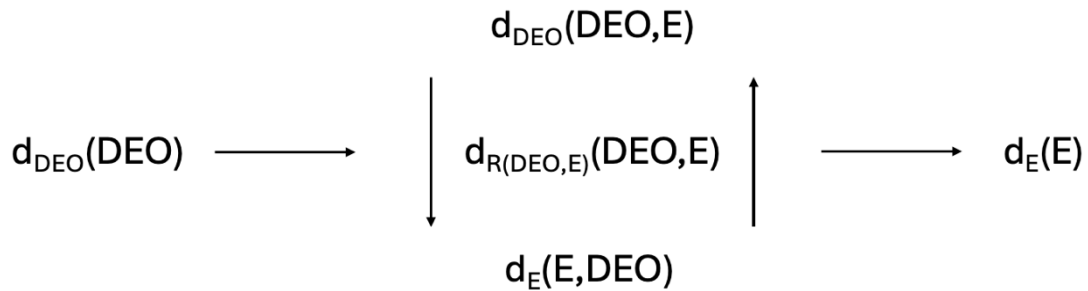


Figure 5.10: scheme of discourses within DEO-discourse and E-discourse according to our classification

Thus, we have designed this task situation that consists in a video vertically divided into two parts: on the left there are subsequent pictures of a two-pans-scale while on the right there is an equation, that correspond to the two-pans-scale on the left, on which are performed manipulations. The sequence of pictures of TPS show a scale on which are made cancellation marks or substitution of weights according to the principles previously formulated. On the equation on the right are performed the 'same' action. When the video starts, a voice recorded on the video says "Consider now a scale with these weights on the two pans. We note that the scale is balanced. We want to solve this problem: figure out how much a red ball weighs". Thus, the objective of the person making the video is to find out the value of the red ball. The voice of the narrator is present all along the video, but it disappears in

some specific moments. We left this silence-moments asking the students to fill them. Specifically, with respect to the left side, the silence-moments are anticipated by a ‘because’ of the voiceover and correspond to the justification of some statements about the maintained equilibrium. On the right side of the video, referred to the equation, a viewer does not effectively see the equation and the related manipulations. Indeed, what appears on the right side is blurred. It is accompanied by the voice over, that has some silence-moments here too. Specifically, these moments correspond to the ‘translation’ in the written form, as an equation, of the scale on the left.

The request for the students is to imagine what has been written in the blurred part and to think about how to complete the voiceover in the moment of silence. After that, they have to record their voiceover together with the inscription on the right side of the video. The structure of the video assumes that the two sides, one composed by a sequence of picture of the two-pans-balance, the other constituted by a sequence of blurred strings, go on together and related to each other. The relation between the two sides is stressed by the voiceover, which, for each picture of the scale, affirm “we can write what we did on the scale as...”.

In summary, the aims of this video are:

- promote a $d_{DEO}(DEO,E)$ and/or a $d_E(DEO,E)$: the request to imagine and write the right part of the video wants to foster the students’ use of visual mediators and words from the E-discourse. In other words, we expect students to write an equation with two expressions that correspond to the collection of objects of the two scale’s pans;
- promote a $d_{R(DEO,E)}(DEO,E)$: the request to complete the voiceover and to imagine the right part of the video wants to foster an emerging discourse about the relation between the objects of the two-pans-balance and the visual mediators from E-discourse.

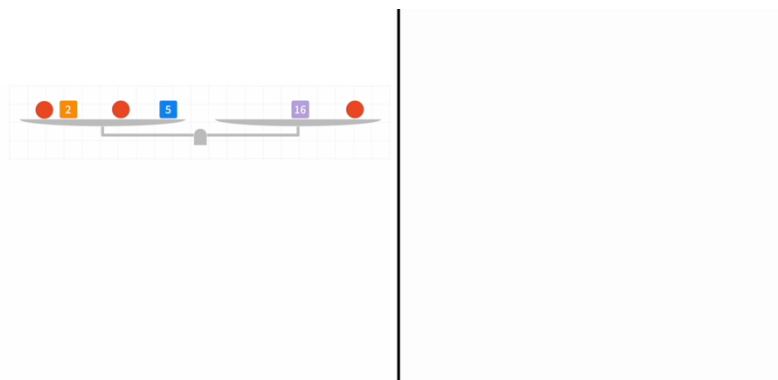


Figure 5.11: Snapshot of the silent video (00:05s)

In Figure 5.11 there is a snapshot of what is seen in the beginning of the video. The black line was added a posteriori to highlight the division of the screen into two parts. It was not present in the version of the video shown to the students. While the scale appears, the voice over says “Consider now a scale with these weights on the two pans. We note that the scale is balanced. We want to solve this problem: figure out how much a red ball weighs”. After that, the voice over says “We can write the situation like this. We write what is in the left pan as” and accordingly something blurred appears on the right of the screen (Figure 5.12a); the voice then says, “and what is in the right plate as” and accordingly something blurred appears on the right of the screen (Figure 5.12b). After that, the voice says, “Then we write the equal sign because” and something appears between the two blurred ‘things’ on the right.

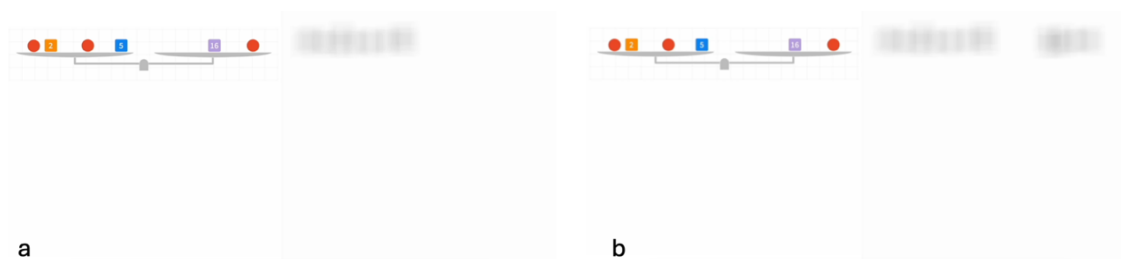


Figure 5.12: Snapshots of silent video at 00:28 (a) and at 00:35 (b)

The video continues in the same way. The final screen of the video is in Figure 5.13 while the complete voiceover pronounced is in Table 5-3, together with what happens in the video, described in italics in square brackets (refer to Figure 5.13).

Table 5-3: Voiceover of the silent video (column 2) and related

Time	What is said	What is shown
00:05	Let us now consider a scale with these weights on the two pans [1]. We notice that the scale is	[1] the first screen of the scale appears on the left

	balanced. We want to solve this problem: figure out how much a red ball weigh.	
	We can write the situation like this. We write what is in the left pan as [2] And what is in the right pan as [3]. Then we write the equal sign [4] because [...]	[2] something blurred appears on the right [3] something blurred appears on the right [4] something blurred appears on the right
00:41	Now we can remove a red ball from both pans [5]. We are sure that the scale is still balanced [6]. This is because [...].	[5] cancellation marks appear on two red balls on the left [6] the second screen of the balance appears on the left
	We can write what we just did on the scale as [7].	[7] something blurred appears on the right
01:14	Now on the right pan of the scale we can change a weight of 16 with a weight of 9 and a weight of 7 [8]. The scale remains balanced because [...].	[8] the third screen of the scale appears, on the right pan there are a weight of value 9 and a weight of value 7 instead of a weight of value 16
	We can write what we just did on the scale as [9].	[9] something blurred appears on the right
01:48	Now, on the scale, we can remove the weight of 2 and the weight of 5 from the left pan, and the weight of 7 from the right pan [10].	[10] cancellation marks appear on the weights of values 2 and 5 on the left pan and the weight of value 7 on the right pan
	We are sure that the scale is still balanced [11]. This is because [...]	[11] the last screen of the scale appears on the left
	At this point we can conclude that the red ball weighs 9, because [...]	
	We can write what we just did on the scale as [12].”	[12] something blurred appears on the right

Within the audio-video recording, some students could have introduced algebraic symbols, while others could not. Moreover, the justifications required involve only the two-pans-scale and its principles, while the blurred “things” do not require any, they are considered just as a sort of “translation” using different visual mediators. For this reason, we have decided to propose a sequence of task situations not

involving silent video that consists only in algebraic symbols, to construct and develop the principles P_0 , P_1 and P_2 with respect to the mathematical discourse.

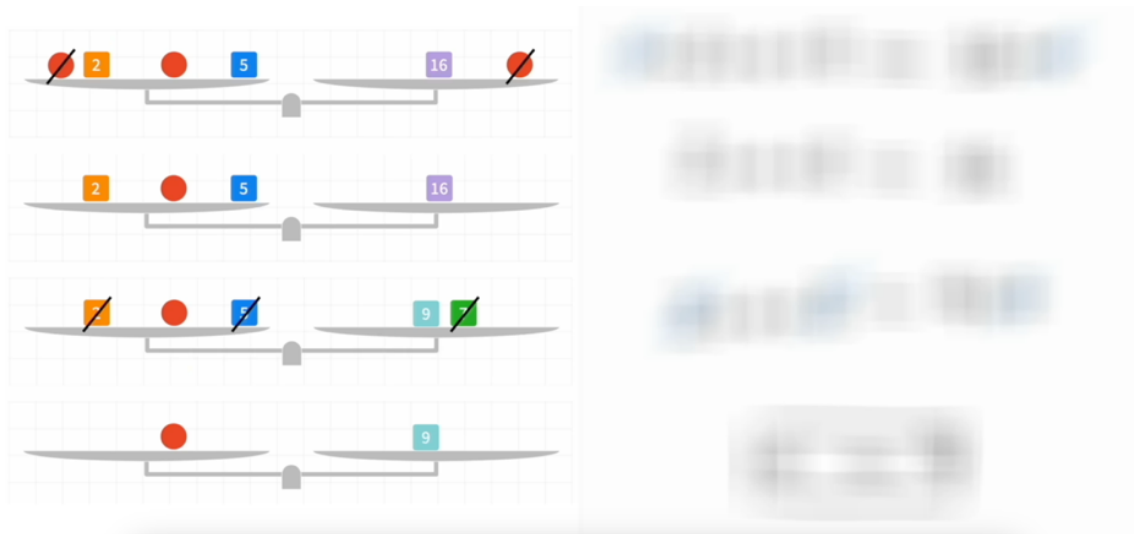


Figure 5.13: Final snapshot of the silent video

Before describing the sequence of task situations involving equations, I present other two task situations based on videos. Indeed, in addition to the previous silent video, to take advantage of the correspondences as well as the missing correspondences between DEO-discourse and E-discourse, we have developed two other videos. The aim of the task situations involving these two videos is to foster the development of a discourse about the relation between DEO-discourse and E-discourse focusing on the not-transplantable features. According to the analysis of the transplantability, the procedure of removing all the objects from the pans of a balanced scale, position a not known object on one of the two pans and add known objects to the other one, till the scale balances, is not transplantable in the E-discourse as a procedure to find out the value of the unknown. Indeed, differently from the dynamic scale that provides feedback, assuming a balanced or not balanced position, transplanting this procedure to the equations would not correspond to a meaningful procedure, but to a sequence of manipulations that do not consider the relation of equality between the two algebraic expressions involved.

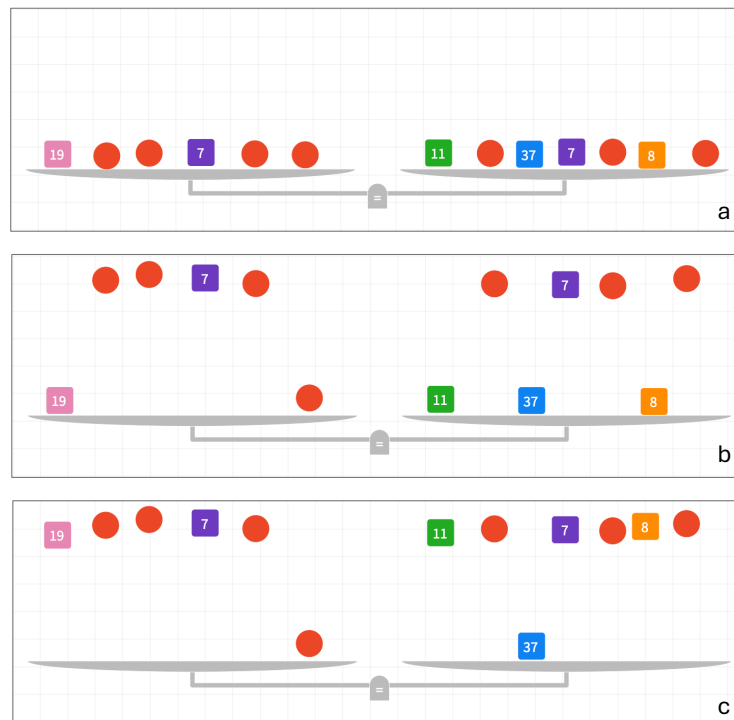


Figure 5.14: Snapshot from the first video (procedure of removing the same value from both sides to discover the value of the not known weight)

Thus, we have taken into account to introduce in our sequences a task situation with a video in which from a balanced scale (Figure 5.14a), an object or a collection of objects with the same value are removed from both pans (see for instance Figure 5.14b). These actions are repeated till the scale presents only a not known object on one pan and a known object (or a collection of them) on the other (see Figure 5.14c). The request is to translate into inscriptions the processes observed in the video. We expect students to write down the scale as an equation and to produce cancellation marks and equivalent equations to describe each step in the video. Together with this task situation, we have designed another one involving a video. In this video too, in the beginning, there is a balanced scale (Figure 5.15a). Differently from the other one, the user starts removing all the objects from the left pan and then from the right pan (see the objects above the scale in Figure 5.15b). Then, he positions a not known object (little red ball) on the right pan and starts adding weights with known values on the left pan (Figure 5.15b). By trials and errors, he obtains a balanced scale, finding out the value of the not known object (Figure 5.15c). In this case too we foresee to ask students to translate into inscriptions what they see happening in the video. We expect students to write down an equation to describe the scale and then to interrogate themselves about the way of representing the steps accomplished on the scale to the equation. The two task situations are supposed to be proposed together to foster a discussion about the

differences between the two videos (bridging discourse) and especially to promote the development of reflections on the not-transplantable procedure performed in the second video. We hypothesize that such discussion strengthens the development of an algebraic discourse coherent with the canonical one.

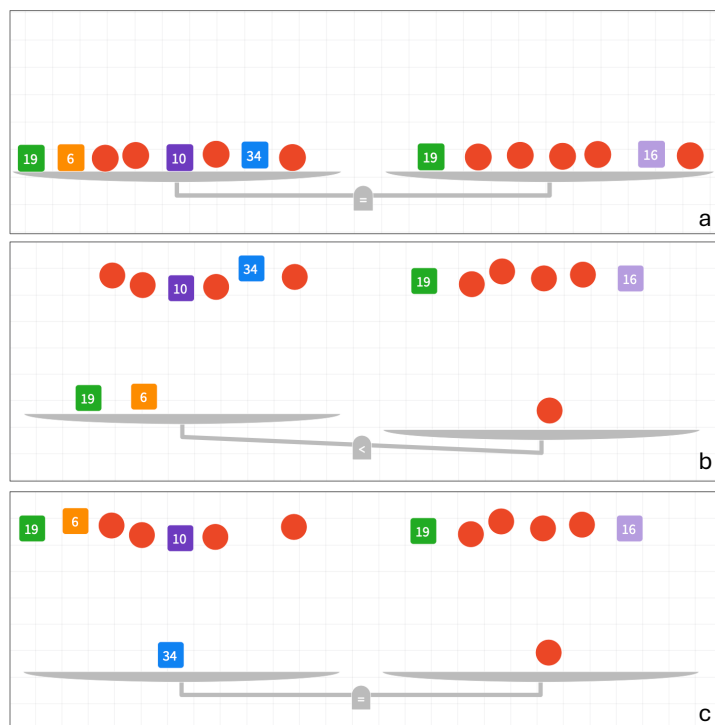


Figure 5.15: Snapshot from the second video (procedure of removing all the objects from both pans and weigh the little red ball to discover its value)

The sequence of equations

Task situation 17, Task situation 18, Task situation 19 and Task situation 20

I now discuss some task situations elaborated to be proposed after the first silent video. During the experimentations, we proposed these task situations on a tablet where students can write. Task situation 17 and task situation 18 ask to justify the steps of the inscriptions respectively in Figure 5.16a and Figure 5.16b, they are argumentation tasks. For instance, we decided to propose students a tablet with the inscriptions in Figure 5.16a, and ask them to justify the subsequent inscriptions. The steps in Figure 5.16a could be justified using the two-pans-balance's principles P_0 , P_1 and P_2 : there are only positive terms, and the solver can describe a complete correspondence between the steps accomplished and the possible actions on the balance. The steps in Figure 5.16b cannot be described in scale's terms because there are negative values, and thus in this task situation can emerge a feature identified in the analysis of the transplantability as not-transplantable. This is a

situation in which the tutor could prompt the production of a bridging discourse focusing on the presence of negative numbers in E-discourse that do not find a correspondence in the TPS-discourse. This should open to the need of elaborating equations' principle that I collected in Table 5-1.

~~$x+8+x$~~ = ~~$4+x$~~ + ~~13~~ + ~~5~~ $8+x$ = ~~4~~ + ~~13~~ + ~~5~~ ~~8~~ + ~~x~~ = ~~4~~ + ~~13~~ + ~~4~~ + ~~1~~ x = ~~14~~ a	~~$x-2$~~ + ~~x~~ + ~~6~~ = ~~-3~~ + ~~8~~ + ~~$x-2$~~ ~~x~~ + ~~x~~ + ~~6~~ = ~~-3~~ + ~~8~~ + ~~x~~ ~~x~~ + ~~6~~ = ~~-3~~ + ~~6~~ + ~~2~~ x = ~~-1~~ b

Figure 5.16: Visual mediators for task situation 17(a) and task situation 18(b)

In order to involve the 'change sides, change sign' rule in students' discourse, according to the fact that they have encountered it during school years, we propose the two argumentation task situations, task situation 19 and task situation 20 (respectively, Figure 5.17a and Figure 5.17b). The request of task situation 19 (Figure 5.17a) is to solve the equation proposed on the tablet, using the blank space under it. If students use the overmentioned rule, we ask to justify each step. We expect students to not have principles as the one they have elaborated for the scale to support their procedure. Otherwise, if they proceed according to the principle in Table 5-1, we pass to task situation 20 and ask to justify the steps in Figure 5.17b. Also this task situation is supposed to be proposed in a tablet. Within the resolution of this equation, the change sides, change sign rule has been applied. We imagine students recognize it but cannot justify it; indeed, according to the initial interview and previous pre-experimentation, we can infer that students apply the change sides change sign rule but are not able give an explanation about why it works. For this reason, we have designed a second silent video that I briefly introduce.

$x+5-x$ = $8-x+2$ a	$8-x+2$ = $-x+3-x$ $-x+x+x$ = $+3-8-2$ x = -7 b

Figure 5.17: Visual mediators for task situation 19(a) and task situation 20(b)

Silent video 2

Task situation 21: The second silent video

We designed a second silent video to foster the production of a discourse about the “change sides, change sign” rule. The video starts with the voiceover saying, “In this video, we want to explain all the steps in solving an equation”. The screen is vertically divided into two sides that at the beginning both shows the equation $2x - 3 = x - 1$ (Figure 5.18a). The video goes on with no audio. The end of the video is showed in Figure 5.18d.

The two sides of the video show two solving processes of the same equation. On the left side is applied the “change sides, change sign” rule while on the right side the P_1 principle is used. The two equations’ solving processes go on quite simultaneously. Indeed, each step in the left side is followed by two steps in the right side. For instance, in Figure 5.18b, we see the first step on the left side, and the consequent two steps on the right side are introduced in Figure 5.18c when the value 3 has been summed to both sides of the starting equation. We chose to emphasize this operation by giving another color to the terms added. Once reached on both sides the equation $2x - x = -1 + 3$, first on the left and then on the right appears $x = 2$.

As underlined by Otten and colleagues (2019) with respect to the two approaches in solving equations,

“despite the close relationship between these two approaches and the related underlying principles, in only a few articles of our review study when authors refer to the use of the balance model, they also refer to the inverse method [change sides, change sign rule]. This indicates that there has not been much research in which both approaches have been put in relation or contrasted”.

The aim of this silent video is exactly to promote a comparison between the two approaches and to underline their common nature.

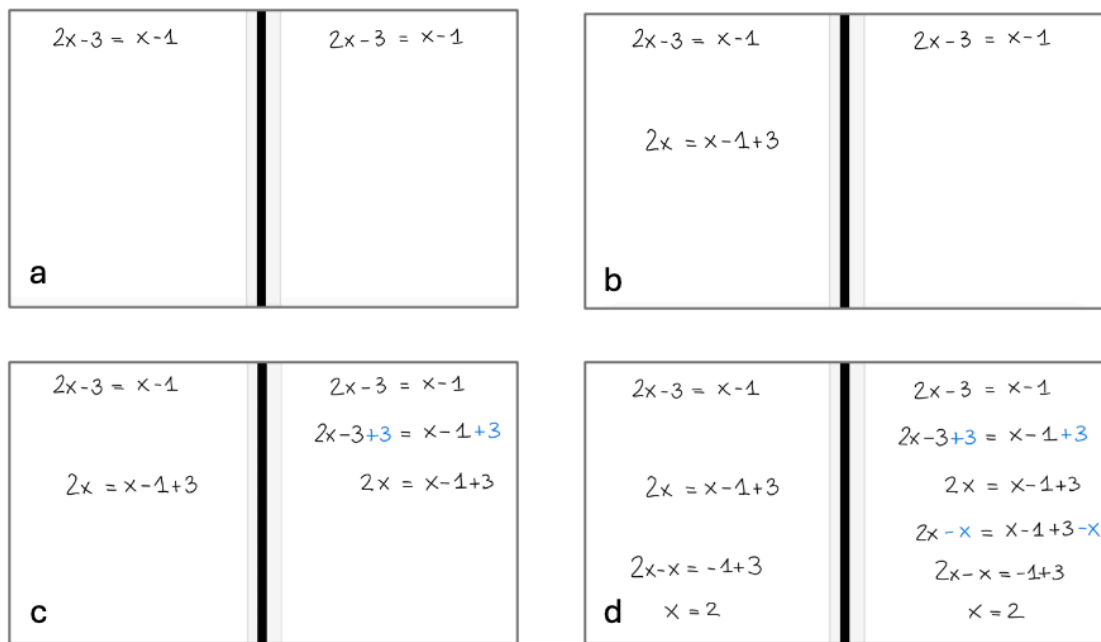


Figure 5.18: Some snapshots of the silent video (task situation 21)

The task for students is to reflect on what is shown and discuss it. The aim of this argumentation task is to make students produce narratives to justify the change sides change sign rule. It is a request that from our perspective involves the construction of a $d_E(E)$ as in the previous activities with equations; moreover, we expect students to produce narratives in which the use of equations' principle appears. Specifically, the discursive objects P_0 , P_1 , P_2 (with respect to equation) should be used to justify the change sides change sign rule.

5.2. The moving-arrows-DEO

The moving-arrows-DEO (Figure 5.19) is elaborated in GeoGebra software. This DEO echoes both the Dynagraph (Lisarelli, 2019) and the Algebraic Line in AlNuSet (Robotti, 2017).

It consists in two or more horizontal line(s) one of which is oriented and numbered (Figure 5.19). There is a vertical line, intersecting the oriented one in 0. On the oriented number line there is a red arrow, labelled " a "; the point 0 is the origin of the arrow while its endpoint is marked with a red tick mark and is directly draggable and bound to the line. Parallel and below the red arrow, two black arrows labelled " a " and " $-a$ " whose endpoints move indirectly according to the dragging of the red arrow's endpoint. There are two other black arrows labeled as 1 and -1 that are fixed.

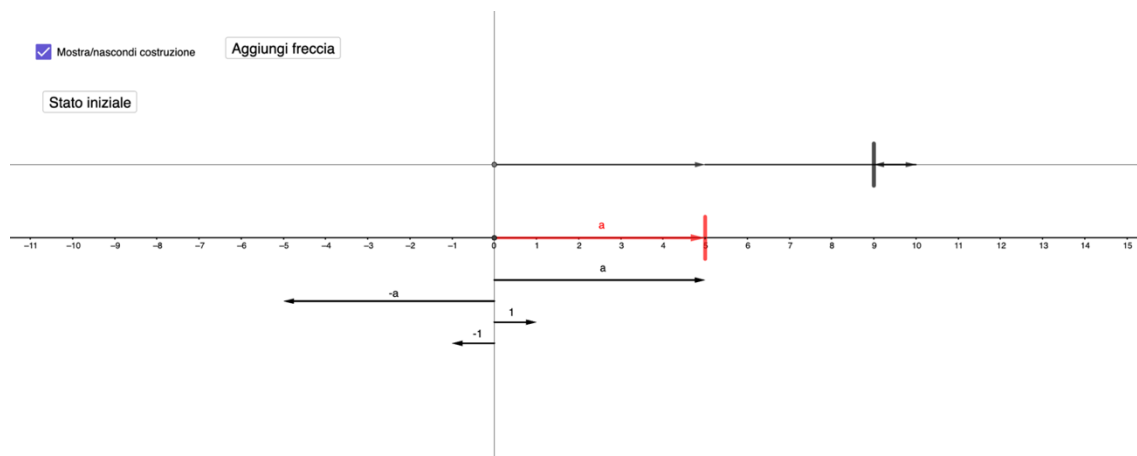


Figure 5.19: Moving-arrows-DEO

The empty non-numbered horizontal line(s) is used to construct other new arrows. This can happen with the ‘Add arrow’ button (in Italian ‘Aggiungi freccia’). It creates different consecutive copies (that is arrows with the same length and same orientation) of the black arrows a , $-a$, 1 and -1 . The origin of each other arrow is the endpoint of the previous one and the endpoint of the last copied arrow is marked with a black tick mark that cannot be directly dragged, but it moves in correspondence with the dragging of the red arrow. In the GeoGebra’s screen the user finds the checkbox ‘Show/hide construction’ (in Italian ‘Mostra/nascondi costruzione’). This command allows to visualize all the black arrows copied on the horizontal line, the entire construction, or only the endpoint of the last arrow, that is the black tick mark alone. Moreover, thanks to the ‘Reset’ button (in Italian ‘Stato iniziale’), the user is allowed to re-establish the initial configuration by deleting any construction made.

Figure 5.20 shows the result of the construction through the use of the ‘Add arrow’ button of two consecutive copies of a followed by a copy of -1 . Figure 5.20a displays the entire construction while Figure 5.20b shows only the endpoint of the last arrow. The red tick mark is displaced on the value 5 and the new built arrow is consequently positioned on 9. In Figure 5.20a the overlapping of arrows worths to be observed. The last arrow, a copy of -1 , overlaps the other black arrows; however, the black tick mark is positioned at the endpoint of the last arrow inserted.

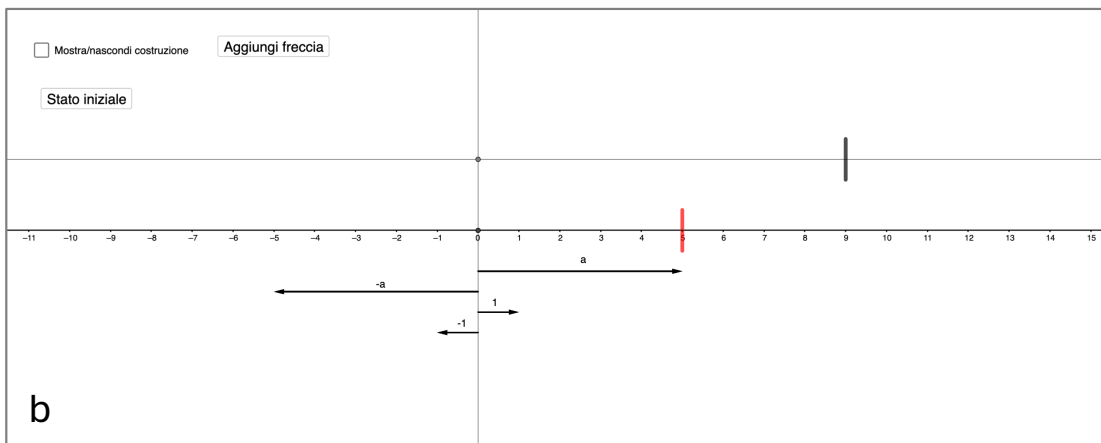
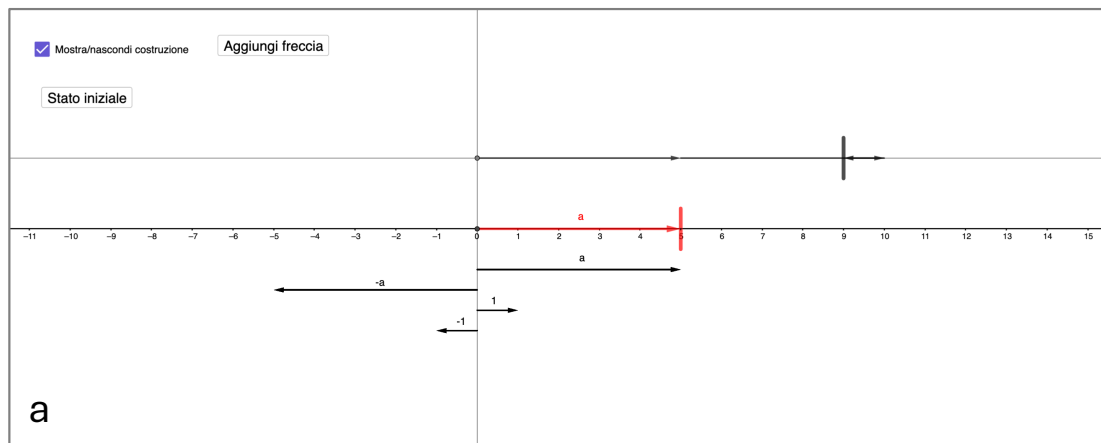


Figure 5.20: Construction of the arrow double of a minus 1 (a) and black tick mark corresponding to double a minus 1 (b)

When, in addition to the numbered one, two horizontal lines are present, it is possible to build two different arrows that depend by the dragging of the endpoint of the red arrow (Figure 5.21). In this case, two “Add arrow” buttons are present that allow the user to choose on which line she wants to add arrows. The endpoints of the last copied arrow on the two lines are marked with a black and a blue tick mark respectively.

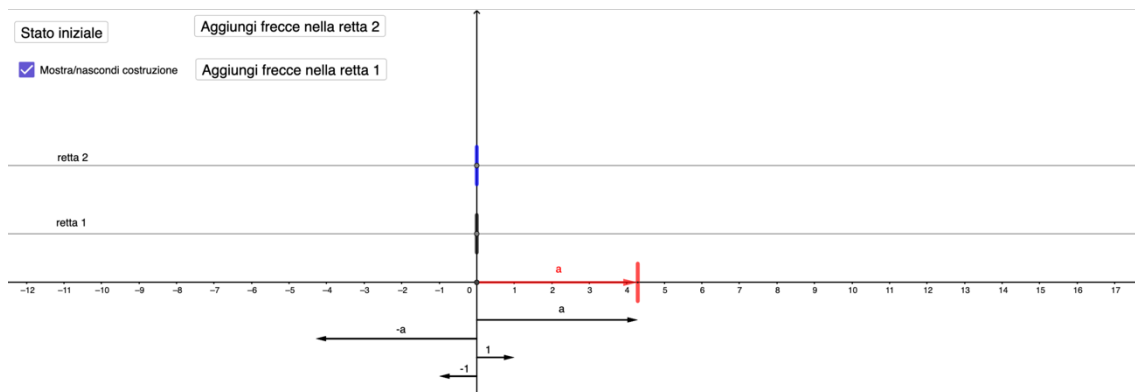


Figure 5.21: Moving-arrows-DEO with two horizontal lines and two ‘add arrows’ buttons

A second type of moving-arrows-DEO consists in the same disposition of horizontal lines (a numbered one and two or more not numbered, Figure 5.22). This new DEO presents one or more ‘Input field’ (in Italian ‘Campo di inserimento’) in place of the ‘Add arrow’ button, with the label ‘Grey tick mark’ (if one input field is present, if there are two there is also ‘Blue tick mark’). The user can insert in the input field an algebraic expression and a grey tick mark appears on the horizontal non-numbered line, correspondent to that expression. According to the dragging of the red tick mark, the grey tick mark will move. We have used this DEO to foster the involvement of an algebraic language in the student’s discourse, as is explained in the following description of the task situations.

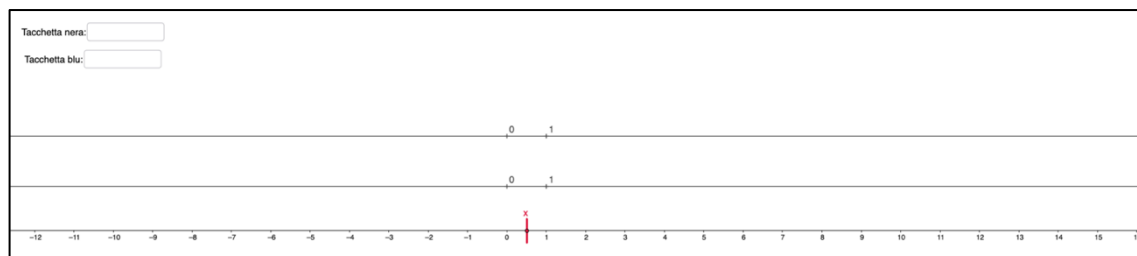


Figure 5.22: Moving arrows-DEO with two horizontal lines and two input fields

The first DEO has been described and analysed, according to the Theory of Semiotic Mediation, with respect to its semiotic potential (Lisarelli et al., 2024). Here, we want to focus on the description of the relation we see between this DEO and the mathematical contents of our interest, in term of words and their use, visual mediators, routines and possible emerging narratives, in tune with the commognitive framework and with the analysis of the transplantability.

This moving-arrows-DEO, in the first or in the second version, strives to foster a discourse in which movements and positions of tick marks play a central role. We believe that the objects’ dynamicity in the DEOs can offer access to a discourse in which the relation between quantity make sense for students. This discourse can

be a steppingstone towards a mathematical discourse about variables in which narratives related to the narratives produced within the DEO are endorsable. In particular, we have conducted an analysis of the transplantability to highlight the correspondence we see between features of the moving arrows-discourse and features of the E-discourse. In Table 5-4 I present this analysis, considering words, visual mediators, procedure and routine, and endorsed narratives. With respect to these latest features, I propose some examples referred to specific situations but that can be used in many others.

Table 5-4: The analysis of the transplantability for the moving arrows-DEO

Feature	TPS-discourse	E-discourse
words	Red arrow and red tick mark	Independent variable
words	Black arrow and black tick mark	Dependent variable
words	To move	To vary
visual mediator	Red arrow and red tick mark	Independent variable
visual mediator	Black arrows a and $-a$	Dependent variable
visual mediator	Black arrows 1 and -1	Constant numerical value
visual mediator	Sequence of arrows on the non-numbered line and its corresponding black tick marks	Variable $f(a)$ dependent on the variable a
visual mediator	Position of a sequence of arrows and of its black tick marks on a specific value	Evaluation of the quantity $f(a)$ for a given value of the independent variable a
visual mediator	Position of two sequences of arrows and their corresponding blue and black tick marks	Relationship (greater, less, equal) between the values of two functions for a given value of the independent variable a
procedure	Dragging the red tick mark on the oriented number line	Variation of the independent variable

procedure	Dragging the red tick mark to specific numerical value	Assignment of a specific value to a variable
procedure	Adding black arrows on the not-numbered line to construct a sequence of arrows	Construction of a linear algebraic expression in the variable a
routine	Task: find on which value the black tick mark goes when the red tick mark is on a specific value Procedure: dragging the red tick mark to the specific value and look at the value reached by the black tick mark	Task: find the value of $f(a)$ when a has a specific value Procedure: assignment of the specific value to the independent variable and evaluate the value of $f(a)$
routine	Task: find on which value you have to put the red tick mark to have the black tick mark on a specific value Procedure: not-transplantable	Task: find the value of a to have a specific value of $f(a)$ Procedure: write and solve an equation in which $f(a)$ is equal to the specific value
Routine	Task: find on which value you have to put the red tick mark to have the black tick mark and the blue tick mark on the same value Procedure: dragging the red tick mark until the two other tick marks are aligned	Task: find the value of a to have $f(a) = g(a)$ Procedure: assignment of a specific value to $f(a)$ and $g(a)$ and find out if they have an equal value. Repeat it until the value of a is found
routine	Task: find on which value you have to put the red tick mark to have the black tick mark and the blue tick mark on the same value	Task: find the value of a to have $f(a) = g(a)$ Procedure: solve the equation $f(a) = g(a)$ with the standard procedure

	Procedure: not-transplantable	
narrative	If I drag the red tick mark a , the black tick mark a follows it	If I assign a value to the variable a , the function $f(a) = a$ assumes the same value
narrative	If the red tick mark is at 2, the black tick mark is at 5	If $a = 2$ $f(2) = 5$
narrative	(After having built an arrows) If the red tick mark is on value greater than 3, the black tick mark is always on value smaller than 2	If $a > 3$ then $f(a) < 2$
narrative	(After having built the arrows corresponding to $2x - 1$ and $x + 3$) If the red tick mark is on values smaller than 4, the blue tick mark is always in a position on the left with respect to the grey tick mark	Consider $f(x) = 2x - 1$ and $g(x) = x + 3$ If $x < 4$, then $f(x) < g(x)$

With respect to the analysis of the transplantability (Table 5-4), the activities with the moving-arrows-DEO can promote the development of a discourse about the movement that, according to the principle of continuity of discourse can become a discourse about relations between variables. Indeed, the dragging of the red tick mark can foster the growth of a discourse about arrows, and in particular of the dependent movement of the black ones (a and $-a$) with respect to the red one. The main concrete object in the applet are the red arrow and its tick mark that is freely draggable along the oriented numbered line. With this dragging, the user moves indirectly the black arrows a and $-a$. In addition, the construction of a sequence of black arrows allows not only to experience the dependence between the movements of the red and the black tick marks, but also the contribution of each black arrow to this movement. Indeed, the user can perceive simultaneously the movement of the sequence of arrows and the movement of each arrow composing it. For instance, if we imagine building the arrow corresponding to $2a - 1$ (Figure 5.20), when we drag the red tick mark a the three arrows above move accordingly.

In this collective movement of the arrows, we can observe the movement of each arrow in our construction and their contribution to the arrow built. The focus can be both on the red tick mark and on the structure of the new arrow constructed. As experts, the possible emerging discourse about movements and their relation of dependence seems to us related to a discourse about independent and dependent variables (Baccaglini-Frank et al, 2023). Specifically, these black arrows are for us realizations of the mathematical object ‘dependent variable’. At the same time, the arrow 1 and -1 are potential realizations of the mathematical object ‘constant numerical value’. Moreover, the sequence of arrows constructed with the ‘Add arrows’ button realizes a function dependent on the variable a : the relationship that binds $f(a)$ to a is invariant with respect to the dragging. Together with this, the simultaneous perception of the collective movement of the sequence and the change (or constancy) of each arrow enables to grasp together the function and its variation. Not only a discourse about independent and dependent movements is possible, indeed we can promote a discourse about the relation among two arrows depending on the same red tick arrows. This opens to a discourse about the relation among two functions depending on the same variable.

5.2.1. Moving-arrows-DEO task situations

The task situations related to the moving-arrows-DEO aim at taking advantage of its dynamic features to promote the emergence of a discourse about the movements described above. In other words, the use of the DEO, fostered by these task situations, puts the light on the dependent nature of the black arrows and, consequently, the sequences of black arrows, from the red arrow. Moreover, the task situations allow students to discuss the relation between two sequences of black arrows and their tick marks, promoting a discourse that we can relate to a discourse about the relation between two functions depending on the same variable.

Both the task situations in the first and in the second version of the DEO do not present algebraic symbols. When is required to construct an arrow, this is described in natural language. We do not aim at introducing algebraic symbols ourselves and we want students to be free the use visual mediators they prefer to describe and discuss what happens in the DEO.

I now present the proposed task situations according to the theoretical tools introduced in the previous chapter. I use the term “activity” because within this DEO there are more task situations collected together. These task situations are proposed to students in one of the two tablets at their disposal, on which they can also write. The other table is dedicated to the Geogebra applet.

Activity 22 and Activity 23 are proposed within the first moving-arrows-DEO; in this case, the environment offers the “Add arrows” button. The other activities are proposed within the second moving-arrows-DEO in which the user finds the “Input field”.

Activity 22 *Explore the file. What happens if you try to drag the red mark?*

We want to construct the arrow representing the double of a , plus 1.

To do this, use the “Add arrow” button.

How does this new arrow move as a varies?

What happens if you tick the checkbox “Show/hide construction”?

Activity 23 *We want to construct the arrow representing the triple of a , minus 1.*

To do this, use the “Add arrow” button.

Can the tick mark at the end of the arrow you have built reach every position? Explain why.

Where do we need to move the red tick mark if we want the black one to reach the value 8?

And if we want it to reach the value -1?

Where do we need to move the red tick mark if we want the black one to be always on values that are greater than 5?

Activity 22 and Activity 23 are fulfilled in the DEO in Figure 5.19.

The first one is a task situation of free action because it wants to promote a preliminary exploration of the DEO. The main aim is to highlight the differences between the fixed (1 and -1) and the movable (a and $-a$) black arrows. Moreover, students are expected to observe and describe movements dependent from the dragging of the red tick mark of the black arrows a and $-a$. In this sense, the first part of the activity (the first task situation) aims at promoting a discursive expansion and the production of discourses of the type $d_{\text{DEO}}(\text{DEO})$. The second part of the activity is a task of bounded action. Students are supposed to gain familiarity with the ‘Add arrow’ button at first, and to discuss about the construction they have elaborated consequently. In particular, we want to foster the production of $d_{\text{DEO}}(\text{DEO})$ in which the ‘issue of movement’ is central. For this we ask students to describe how the constructed arrow moves as a varies. Within Activity 23, after the construction of the arrow corresponding to triple of a , minus 1 (task of bounded action) there is a task about the possibility for the arrow to reach every possible value. After that, there are three task situations of imagined action in which students are asked to foresee the position of the red tick mark to have a certain position of the black tick mark. These requests highlight the asymmetry between

the red and the black tick mark: we look for the value of the red one such as the black one has a certain value. Thus, the focus is more on the tick marks and less on the arrows, even if they are still there and move along the lines, because the task situations' focal point is the position of the tick marks. These tasks open to a possible $d_{DEO}(DEO, E)$: the request to foresee could push some students to involve in their discourses algebraic expressions (oral or written) that belong to E-discourse. The first task situation of the two activities, the request of construction, can be classified, according to the model of use of a continuous DEO (Figure 5.23) as situations in which the user know C1 and partially the action to be accomplished (add some arrows). He has to act in order to obtain C2. Once the construction is done, the user can drag the red tick mark and observe what happens. The request of describing it is thus a descriptive task situation (present in Activity 22 and Activity 23). In the three task situations of imagined action the user knows C1 and partially C2, because he knows the position of the black tick mark after a specific dragging. He does not know where the red tick mark has to be put to have so, and thus the action to be accomplished.

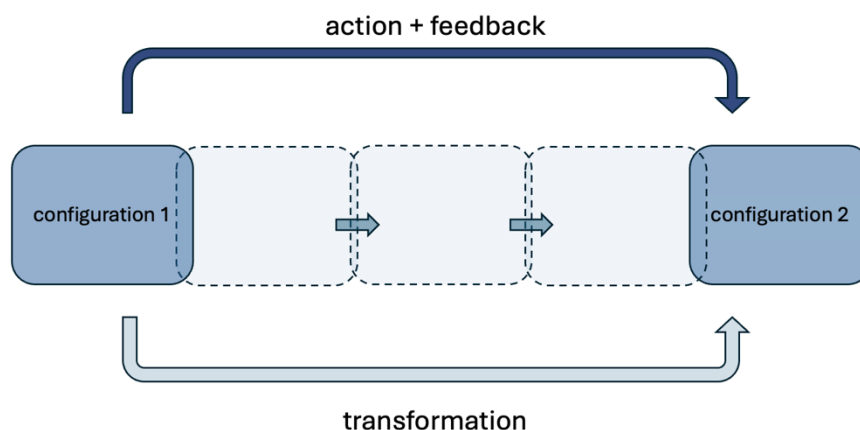


Figure 5.23: Model of use of a continuous DEO

Activity 24a (and 24b, 24c, 24d) The blue tick mark above has been constructed with the “Add arrow” button. Can you discover how it has been constructed? (Figure 5.24)

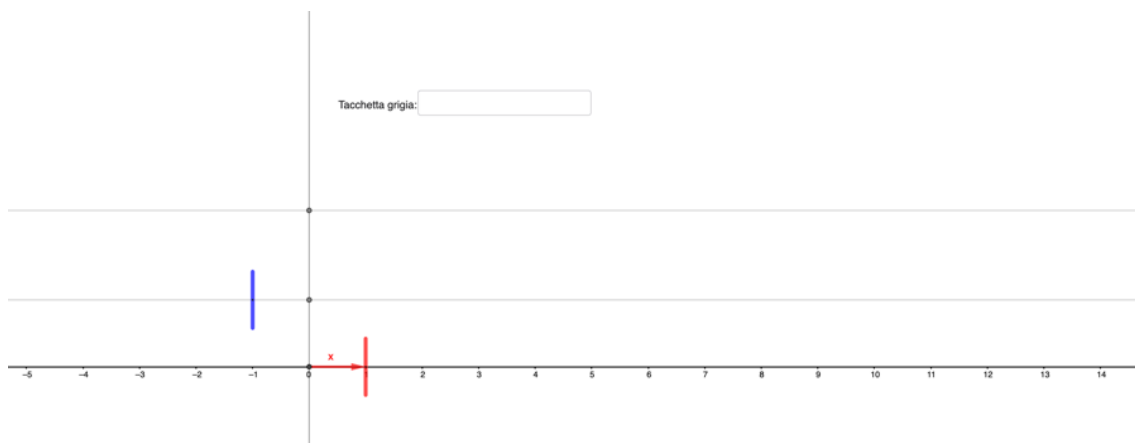


Figure 5.24: Moving arrows-DEO of activity 24a

The activity is settled in the second type of the moving-arrows-DEO. It presents, together with the red arrow, a blue tick mark on a not-numbered horizontal line, without the correspondent arrows (see Figure 5.24). The task is to discover, imagining having at disposal the “Add arrow” button, how the blue tick mark has been constructed. The students are now asked to interpret a done construction: in dragging the red tick mark, they should focus on pair of values (red tick mark – blue tick mark) and on the mutual movements. Specifically, it is necessary to find an arrow and a correspondent tick mark that, during the dragging of the arrow x , always stays aligned with the blue tick mark. This task promotes a discourse in which could be developed new ways of representing information such as a table with the pair value of the red tick mark-value of the blue tick mark or an expression (written or verbal) describing the blue tick mark depending on the red one. In so, one of the aims is to foster the production of $d_{DEO}(DEO, E)$. In the input field the student has to insert an algebraic expression to receive feedback by the software. After inserting the expression “ $x - 2$ ”, the software responds representing a grey arrow labeled “ $x - 2$ ” that, dragging the red tick mark x , moves assuming the value of the correspondent function.

Activities 24b, 24c, 24d present the same request with different algebraic expressions realized by the blue tick mark.

Activity 25 Enter expressions in the input fields so that:

one tick mark corresponds to double of a , minus 1.

another tick mark corresponds to a , plus 3.

Drag the red tick mark and describe the positions of the blue and the black tick marks.

The activity is proposed within the moving-arrows-DEO in Figure 5.25. The user has to fill the two input fields with expressions to obtain the two tick marks requested.

This is a task of bounded action that aims at fostering a discourse about the comparison of two tick marks. We imagine the students' discourse focused on the movements of the two tick marks and on their mutual positions. Moreover, the request of describing the tick marks' positions could promote the use of features from the E-discourse, such as symbols $<$, $>$, $=$ and written literal expressions. Thus, the production of a $d_{DEO}(DEO, E)$ is encouraged. The reference to movements and mutual positions can be seen as a realization of the relation between two functions depending on the same variable and, as experts, we can see many of these narratives correspondent to narratives about relation among functions, in a mathematics discourse. This is a descriptive task situation because student has to observe the movement of the tick marks and describe it.

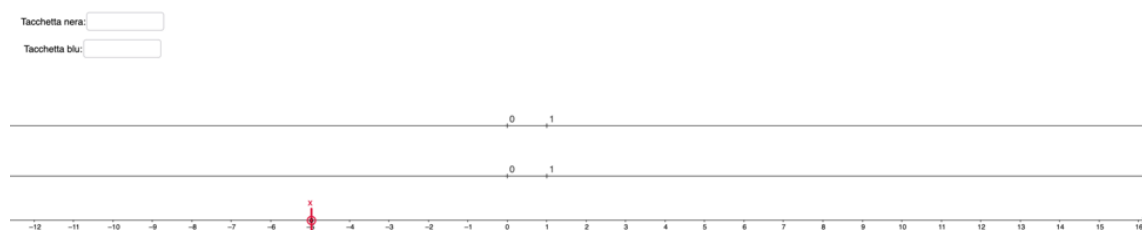


Figure 5.25: Moving arrows-DEO of activity 25

Activity 26. *Imagine that in the input fields there is written*

$$2x$$

$$2x + 2$$

How would the tick marks move?

Activity 27. *Enter two expressions into the input fields so that the blue tick mark and the black tick mark meet only when the red tick mark is on 5.*

Activity 26 consists in a task of imagined action. The solver is asked to imagine the movement of the tick marks corresponding to $2x$ and $2x + 2$. This task aims at promoting a compression of the previous discourses to give a description of the expected DEO's feedback. Moreover, it could promote the description of the imagined tick marks and their supposed movements in term of algebraic symbolism ($d_{DEO}(DEO, E)$). After the description of what, from their perspective, should be, students are allowed to test their hypothesis using the input fields and dragging the red tick marks. If they have imagined movement that do not correspond to what happens in the DEO, they are asked to discuss it and justify why their hypothesis do not correspond to what effectively happens in the DEO. In this case, students know partially C1, that consists in the static situation of the two expressions inserted in the input field and the correspondent tick marks present on the lines even if they do

not know their related position, and the action of dragging the red tick mark, but do not know how the two built tick marks would move according to the dragging.

Activity 27 is a task of goal-oriented action. Here students have, from a mathematical point of view, to construct two expressions, depending on the same variable, that are equal only when the variable's value is 5. This task too could promote the emergence of a $d_{DEO}(DEO, E)$ and of a $d_E(DEO, E)$. In this task situation students know partially C2, because they are asked to build two tick marks that meet only when the red tick mark is on 5. They do not know C1 and thus which expressions are inserted in the input fields: they have to insert them considering the request of the task situation.

In Table 5-5, I collect some tutor's interventions during the activities with moving-arrows-DEO; the columns indicate the intervention, the respective objective, and the number of activities in which it is proposed.

Table 5-5: Some of the tutor's intervention developed for moving arrows-task situations

Intervention	Objectives	Activity
How does the arrow $-a$ move as a varies?	Promote a $d_{DEO}(DEO)$	22
Why in some cases are all the arrows on the same side of the tick mark, and in other cases some are to the right and some to the left?	Promote a $d_{DEO}(DEO)$, in particular a reflection about the construction of the sequence of arrows	22
Where do you expect the black tick mark goes if I put the red tick mark on 4? (the number changes with respect to the construction)	Promote a $d_{DEO}(DEO)$ and a $d_{DEO}(DEO, E)$	22; 23

Could you write the relation between the two expressions as a varies?	Promote a $d_{DEO}(DEO,E)$ and a $d_E(DEO,E)$	25
What is the difference between the symbol x in this file and the symbol x within scale?	Promote a $d_{DEO}(DEO,E)$ and a $d_{R(DEO,E)}(DEO,E)$	25

5.3. The balancing-mobile-DEO

We designed another dynamic-scale-model to deal with linear equations. This DEO was elaborated within the free online software DESMOS. The balancing-mobile-DEO has been described and analysed in the light of Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) according to the construct of semiotic potential (Bonadiman, 2024). Here we want to describe this environment and its features according to the theoretical tools elaborated in the previous chapter.

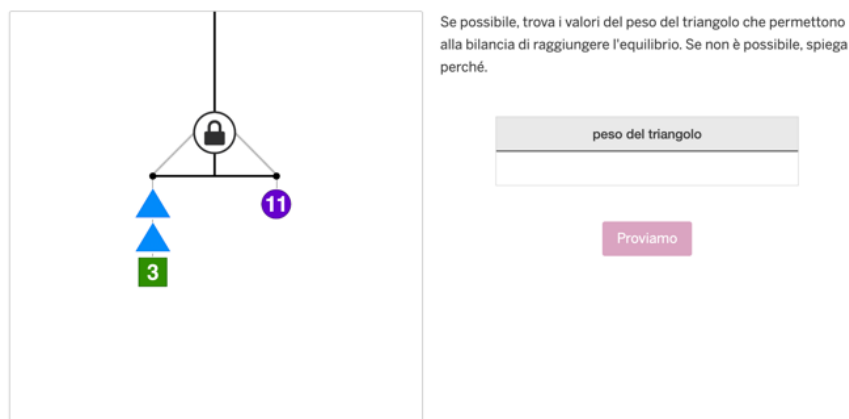


Figure 5.26: Test balancing mobile

In the beginning, this DEO, that we call “Test balancing mobile” consisted in a locked dynamic mobile, as in Figure 5.26. The two pans of the mobile are filled with different coloured shapes; some of these shapes have a number inside, that states the weight’s value, other do not have a number inside because their value is not defined. Indeed, the user can choose the value to be assigned to these shapes inserting it in the “input field” and pressing the key “Let’s try!” to activate the mobile. The mobile reacts to the user’s choice moving accordingly to the sum of the weights’ value in each pan. If the values’ sum of the left pan is greater (less) than the

values' sum of the right pan, the balance dangles to the left (right). If the sum of the values on the two pans is the same, the balance assumes the equilibrium position. For example, in Figure 5.26, the mobile has on the left pan two shapes with undefined value and a shape which value is 3, while on the right pan it has a shape of value 11. If the user inserts the number 4 in the input field, after pressing the key, the mobile assumes a balanced position (Figure 5.27a). Instead, if the user chooses the value 6 (Figure 5.27b), the mobile dangles to the left because the sum of the values on the left ($6 + 6 + 3 = 15$) is greater than the sum of the values on the right (11). Together with the position assumed by the mobile, the user can also observe another feedback of the DEO. Once she presses the key and the mobile moves, the lock is substituted by a green tick when the mobile is balanced (Figure 5.27a) while, when the two pans are not equal, a red X appears (Figure 5.27b). From a mathematical point of view, in using this DEO, we can see the realization of the mathematical object "variable" while the mathematical object "unknown" remains hidden. Indeed, the DEO in Figure 5.26 allows us to study the relation between the two functions, $f(x) = x + x + 3$ and $g(x) = 11$ with respect to the value assigned to x . The balance can be in equilibrium, realizing the equality between $x + x + 3$ and 11, it can dangle to the left, realizing the relation $x + x + 3 < 11$, it can dangle to the right, realizing the relation $x + x + 3 > 11$. More precisely, the mobile position realizes a relationship between the two expressions corresponding to a specific value of the variable x , the one entered by the user. In Figure 5.27a the user has inserted the number 4: the mobile realizes the relation $4 + 4 + 3 = 11$, while in Figure 5.27b the mobile realizes the relation $6 + 6 + 3 > 11$, because of the user entering the value 6. Moreover, the feedback given by replacing the lock with a green tick or a red X realizes the function that associates the chosen value of the variable x with the output "true" (green tick) or "false" (red X) with reference to the equality between the value of the two functions.

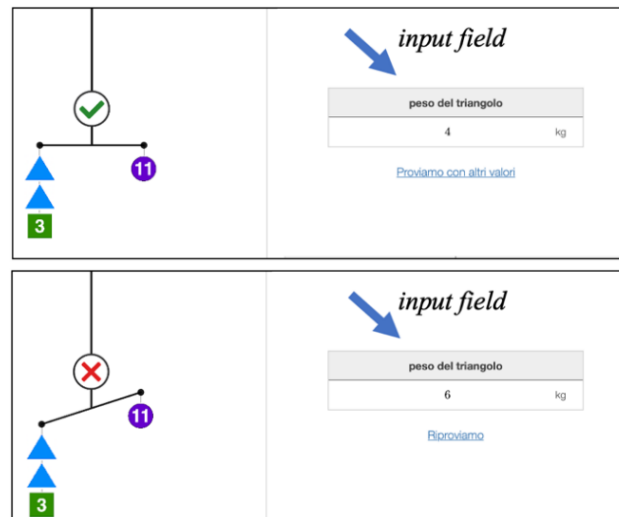


Figure 5.27: Balancing mobile when inserted the value 4 (a) and the value 6 (b)

As the description of the DEO stresses, the use of this artifact fosters the study of the relation between two functions, realized by the collection of shapes on each pan, depending on the same variable. The mathematical object “variable”, as an indeterminate quantity that can vary and to which I can assign a value, which we, as experts, see involved here, is essential in the construction of the mathematical object “equation”. However, it is equally important to treat the mathematical object “unknown”, as a specific but not known value that makes the two algebraic expressions involved equal, for the construction of the mathematical object “equation”. In this light, since in the DEO thus proposed the unknown comes into play only when the user enters the input field the value that allows the mobile to be balanced, we decided to modify the DEO as follows.



Figure 5.28: Visual mediator introduced in the test balancing mobile

We consider a visual mediator as in Figure 5.28, that we call “Fixed mobile”. It consists in a fixed balanced mobile with a green tick. The two pans have the same value, even if some shapes have a not known value. From a mathematical point of view, the mobile realizes the condition of equality between the two expressions composed by numbers, known and not known. In this case, shapes with a not known value realize the mathematical object “unknown” since we do not know their value, but we know that it is such that the mobile is balanced.

The DEO used within the teaching experiments is in Figure 5.29. We call it balancing-mobile-DEO. It involves both the test mobile on the right and the fixed mobile on the left. The need to combine them grows from the didactical aim of introducing the object variable and unknown, in dealing with equations, and thus to promote a DEO-discourse that could open to it. The fixed mobile, with the green tick, realizes the equality between the two expressions involved. The test mobile, for which it is necessary to choose a value to be assigned to the undefined shape, realizes the functional relation between the two functions realized by the two collections of shapes, one on the left pan and the other on the right pan. In particular, considering Figure 5.29, in the fixed mobile the shapes realizing unknown (triangles) are coloured while in the test mobile they do not have a colour. This choice is related to the consideration about unknown and variable. Dealing with $x + x + 3 = 11$, x realizes both the unknown and the variable. In the fixed mobile the not known value of the shape realizes the not known value of an unknown, while, on the other hand, the undefined value realizes a variable, because it can be chosen by the user; it realizes a varying quantity (only on the positive numbers) and it determines the position of the mobile.

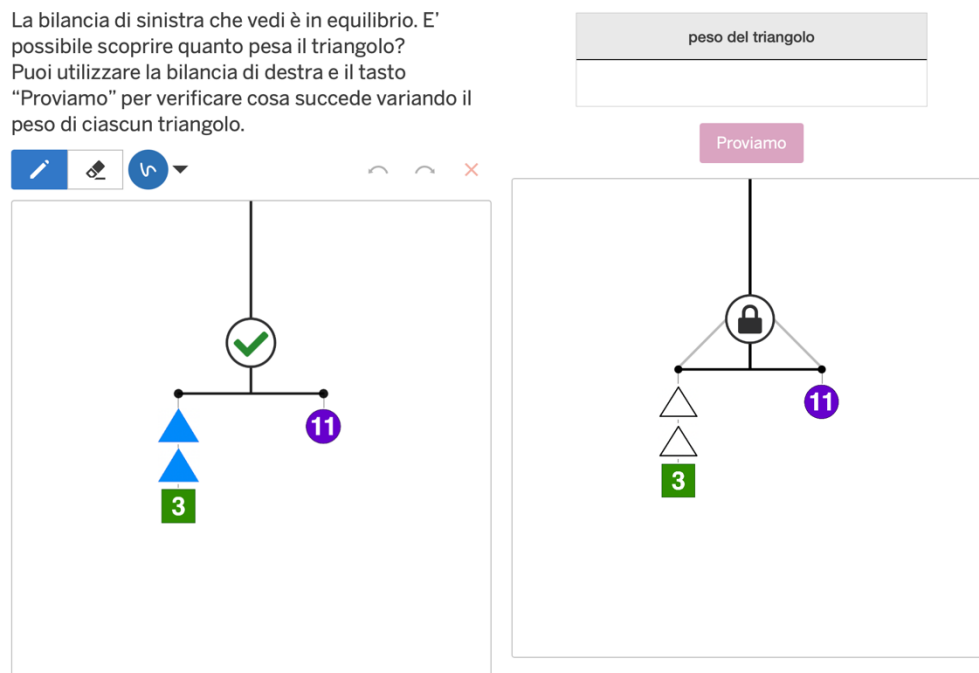


Figure 5.29: The balancing mobile-DEO

In Table 5-6 I present the analysis of the transplantability of balancing mobile-DEO. I focus on visual mediators, procedures and endorsed narratives. The words we see transplantable from DEO-discourse to E-discourse and viceversa are the same as in the TPS-DEO, so we can refer to Table 5-1. Specifically, within this DEO the user has the possibility to perform different actions, that an expert can see related to

action on the equations. On one hand students can insert the value in the input field and press the button “Let’s try!” to see the related movement of the test mobile. We expect them to discuss the relation between the two sides of the test-mobile according to the choice of the not known weight’ value. On the other hand, they can draw on the picture of the fixed mobile: we decided to offer this possibility to promote cancellation marks that can also be done during the solution of an equation. The narratives we expect could emerge in activities with this DEO regard the relation between the fixed mobile and the test mobile. In particular, we expect students highlight the different features of objects with not known values in the two mobiles. To sum up, within this DEO, the analysis of the transplantability is focused on narratives that stresses characteristics of the fixed mobile and the balancing mobile, related on one hand to narratives about equation while on the other on the study of the relations among two functions. We designed the DEO in this way to promote a way of talking about the two mobiles that can be transplanted in the DEO-discourse. In Table 5-6 I collect the analysis of the transplantability of this DEO.

Table 5-6: The analysis of the transplantability of balancing mobile-DEO

Feature	TPS-discourse	E-discourse
visual mediator	Weight	Number
visual mediators	Weights without number	Unknowns
visual mediators	Weights with the same colour and shapes (with or without numbers)	Equal numbers
visual mediator	Weights with the same colour and shape without number	Equal unknowns
visual mediator	Collection of weights on a balance’s pan	$f(x)$ and $g(x)$ on the left or right of the equal sign
visual mediator	Green tick	Equality of the two functions for a specific value of the independent variable
visual mediator	Red X	Inequality of the two functions for a specific value of the independent variable
visual mediator	Fixed balance	Relation of equality $f(x_0) = g(x_0)$ where x_0

		is a specific, but not known, number
visual mediator	Test balance	Relation (greater, less, equal) between $f(x)$ and $g(x)$ for a given value of the independent variable x
visual mediator	Coloured and not numbered weight in fixed balance	Unknown
visual mediator	White (not numbered) weight in test-balance	Variable
visual mediator	Fixed balance – Test balance	Different nature of x and of $f(x)$ and $g(x)$
procedure	Inserting a value into the input field	Assignment of a value to the variable x
procedure	Press the “Let’s try!” button	Study of the relationship between $f(x)$ and $g(x)$ once a value is assigned to the variable x
visual mediators	Cancellation marks on the fixed balance	Cancellation marks on the equation
endorsed narrative	If I put in the input field the weight of value 3, the mobile is balanced	$f(3) = g(3)$
endorsed narrative	If I put in the input field a weight of value greater than 3, the balance hangs to the left	If $x > 3$, then $f(x) > g(x)$
endorsed narrative	For any value of the weight of the square, the mobile is always balanced	For any value of the independent variable x , $f(x) = g(x)$
endorsed narrative	The value of the weight of the triangle is always the value of the weight of the square plus 3	$y = x + 3$

5.3.1. Balancing mobile-DEO task situations

As discussed, within this DEO we want to promote students' discourse about the relation among the two sides of the mobile with respect to the varying value of the not known weight. We propose two main kinds of task situations.

Task situation 28 *On the left you have a balanced mobile. Can you discover the weight of the triangle? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each triangle.*

This task situation refers to the DEO in Figure 5.29. It is a task situation of goal-oriented action: the solver is asked to find the value of the triangle's weight of the left mobile. The task situation also suggests using the test-mobile to observe what happens varying the value of each triangle.

This kind of task situation is proposed with different mobiles: in the session we have task situation 29, task situation 30, task situation 31 with the same request.

Task situation 32 *For what values of the weight of the circle does this mobile hang to the right? And to the left? And for what values is it balanced?*

This task situation is a free action task: the solver is not asked to accomplish a specific action or to find a certain solution. An example of the DEO proposed can be found in Figure 5.30a.

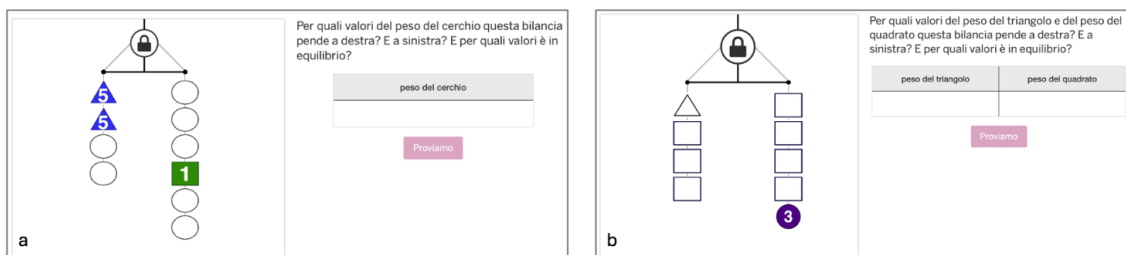


Figure 5.30: Balancing mobiles of task situation 32 (a) and of task situation 36 (b)

The environment presents only the test-mobile and, according to this, the request is to study the behaviour of the mobile, or, more specifically, to find out for which values of the circles' weight it assumes the positions of equilibrium and/or disequilibrium. This kind of task situation aim at promoting a d_{DEO} (DEO) in which the relation among values assigned to the not determined weight and the correspondent mobile's position is described (for example, in the case of mobile in Figure 5.30a, narratives as “If I give circles a weight's value greater than 3, the mobile hangs to the left”).

Task situation 33, task situation 34, task situation 35 present the same request. In particular, the mobile in task situation 34 is always balanced, realizing an equation with infinite solutions while the one in task situation 35 is never balanced, realizing an equation with no solutions. These two task situations are proposed to promote narratives such as the one in Table 5-6: the relation among the two quantities on the arms keeps being the same despite the value of the not determined weight. Considerations about the mobile's positions promote narratives that could be starting points for narratives about relations among functions.

Task situation 36 *For what values of the weight of the triangle and of the weight of the square does this mobile hang to the right? And to the left? And for what values is it balanced?*

The mobile for task situation 36 is in Figure 5.30b. It is a task of free action. It presents a hanging mobile in which there are two shapes with no determined values. This task aims at promoting a d_{DEO} (DEO) in which some narratives describe the mutual relation of weights' value to have a balanced mobile. From a mathematical point of view, the mobile realizes the relation among two quantities depending on two different variables. The situation of equilibrium realizes the infinite pairs of values for the two variables that make the two quantities equal.

Task situation 37 and task situation 38 are similar to task situation 36 and refers to the relation between two functions depending on two variables.

Task situation 39 (and similarly, 40 and 41) *On the left you have two balanced mobiles. Can you discover the weights of the triangle and the circle? You can use the mobile on the right and the button "Let's try!" to verify what happens when you vary the weight of each triangle and of each circle.*

This last type of task situation proposes to students an environment as in Figure 5.31 and it is a task of goal-oriented action. The setting is the same of the first group of task situations, but the mobiles are two. On the left there is a picture in which they are balanced while on the right there are two test-mobiles. The solver has at disposal two input fields to assign value to both the triangle and the circle's weights. As experts, we see this DEO as a realization of a system of equations. During the accomplishment of the task situation, could emerge narratives related to the routine of solving a system of equations. For instance, with respect to Figure 5.31, students could notice that a circle with a specific value (the purple one) is balanced with two triangles with a certain value (the blue ones) and an object of value 3. This could be used in the second mobile: substituting the purple circle with two triangles and an object of value 3, the second mobile presents only weights with known

values and triangles. In this way, the not known value of triangles' weight can be discovered. This corresponds to the strategy of substituting, in an equation, a variable with a correspondent collection of known terms and variables, to solve the equation.

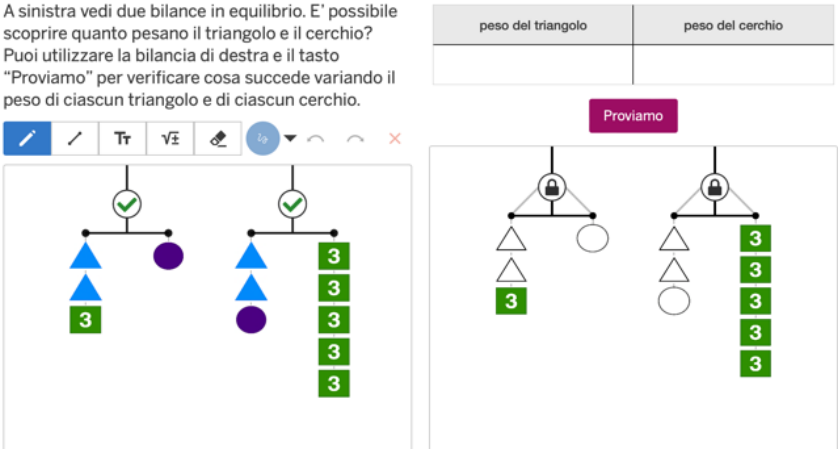


Figure 5.31: Balancing mobiles of task situation 39

In Table 5-7 I summarize some of the tutor’s interventions within these task situations to promote a $d_{DEO}(DEO,E)$.

Table 5-7: Some of the tutor's interventions developed for balancing-mobile-DEO

Intervention	Objectives	Task situation
Can you write what you have discovered?	Promote a $d_{DEO}(DEO,E)$	32
How could you synthetize what you have just discovered?	Promote a $d_{DEO}(DEO,E)$; in particular a discursive compression	32
How could you write it?	Promote a $d_{DEO}(DEO,E)$	36 and 39

5.4. The mobile-arrows-DEO

The last DEO proposed embeds both the test-mobile-DEO and the moving-arrows-DEO. We designed two different types of this DEO. The first one is presented in Figure 5.32. On the left parts it shows a balancing mobile, on the right part it presents moving arrows. The mobile’s arms in Figure 5.32 consist of coloured and

not-coloured shapes. The former stand for objects with a known weight, that is indicated by the number inside the shape. The latter are objects with a not defined value that is determined by the right part of the screen. Indeed, dragging the red tick mark in the moving-arrows-DEO, the user varies the weight's value of the not-coloured objects. Moreover, the two arrows, respectively with a blue and a grey tick mark as endpoints, move depending on the dragging.

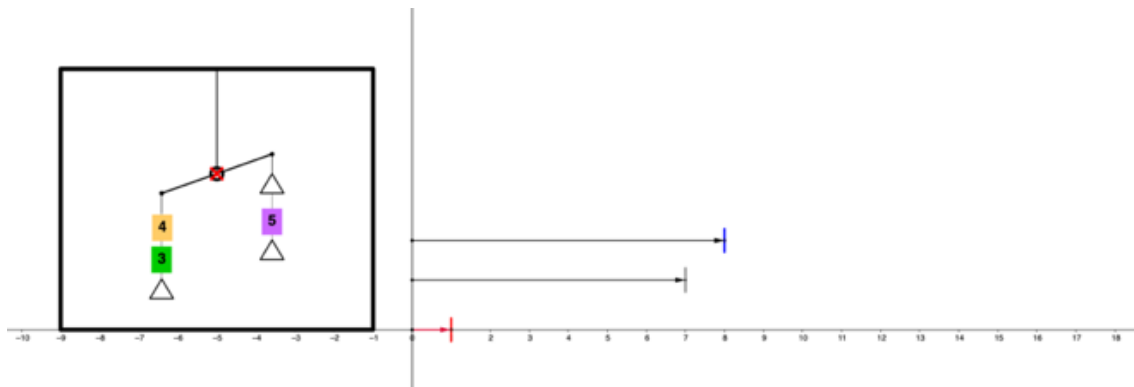


Figure 5.32: The first type of mobile-arrows-DEO

The left pan and the blue tick mark correspond to the same value, whether is the value of the red tick mark, as the right pan and the grey tick mark. Therefore, the two DEOs are bonded together because they describe the same relationship between two functions depending on the same variable realized by the red tick mark: when the red tick mark is on the value that makes the two functions equal (in the case of Figure 5.32 it is 2), the mobile is balanced, presenting a green dot, where in the fixed mobile-test mobile-DEO there was a green tick, while the two tick marks are aligned. Otherwise, the mobile is not balanced and the tick marks are not aligned: the one corresponding to the heavier pan is on the right of the one corresponding to the lighter pan. This DEO presents a mobile that echoes the test mobile in the previous DEO. In this case, the user assigns values to the not determined weight thanks to the dragging of the red tick mark: there is no more a discrete evaluation of the mobile position according to the insertion of a value in the input field, but a continuous movement of the artefact.

The second mobile-arrows-DEO aims at developing a bridging discourse that can favour the development of the E-discourse. Indeed, it is composed by a balancing mobile on the left and three moving-arrows on the right, but it presents two input fields through which the user writes expressions corresponding to pans and arrows. With respect to the first DEO, the difference is that here the user can choose which expressions he wants to see represented. On the mobile appear two labels with the expressions inserted, one blue-coloured, corresponding to the blue tick mark, and one grey-coloured, corresponding to the grey tick mark. Moreover, above the mobile

there is a red label indicating the value that the variable is assuming according to the position of the red tick mark (for instance $x = 2$ in Figure 5.33a), and under the mobile there is a label in which appear the numerical values of the two pans and their mutual relationship (greater, less, equal in symbol). Both these labels change simultaneously with the dragging of the red tick mark. Till the user has inserted in the input fields the two expressions, the DEO presents only the red tick mark and the two input fields.

For instance, in Figure 5.33 the DEO presents in the input fields the expressions $2x + 3$ and $x + 5$. After these inputs, in the balance-DEO appear two labels as pans, in which are written $2x + 3$ and $x + 5$ while in the arrows-DEO appear two arrows with a blue and a grey tick mark respectively labelled $2x + 3$ and $x + 5$ too. When the user drags the endpoint of the arrow x , both the mobile and the arrows move accordingly to the value on which the red tick mark is positioned. In Figure 5.33a the red tick mark is on 2, the value that assigned to the variable x make the two expressions equal. The mobile is balanced: there is the green dot as before and the label, coloured in green, tells " $7 = 7$ ". At the same time, on the right, the two tick marks are aligned on the value 7. The situation corresponding to the red tick mark on 3 is shot in Figure 5.33b. The blue pan is heavier such as the blue tick mark is ahead of the grey one and the yellow label indicates the value of the two pans and the relation among them, the one on the left is heavier than the right one, thus $9 > 8$.

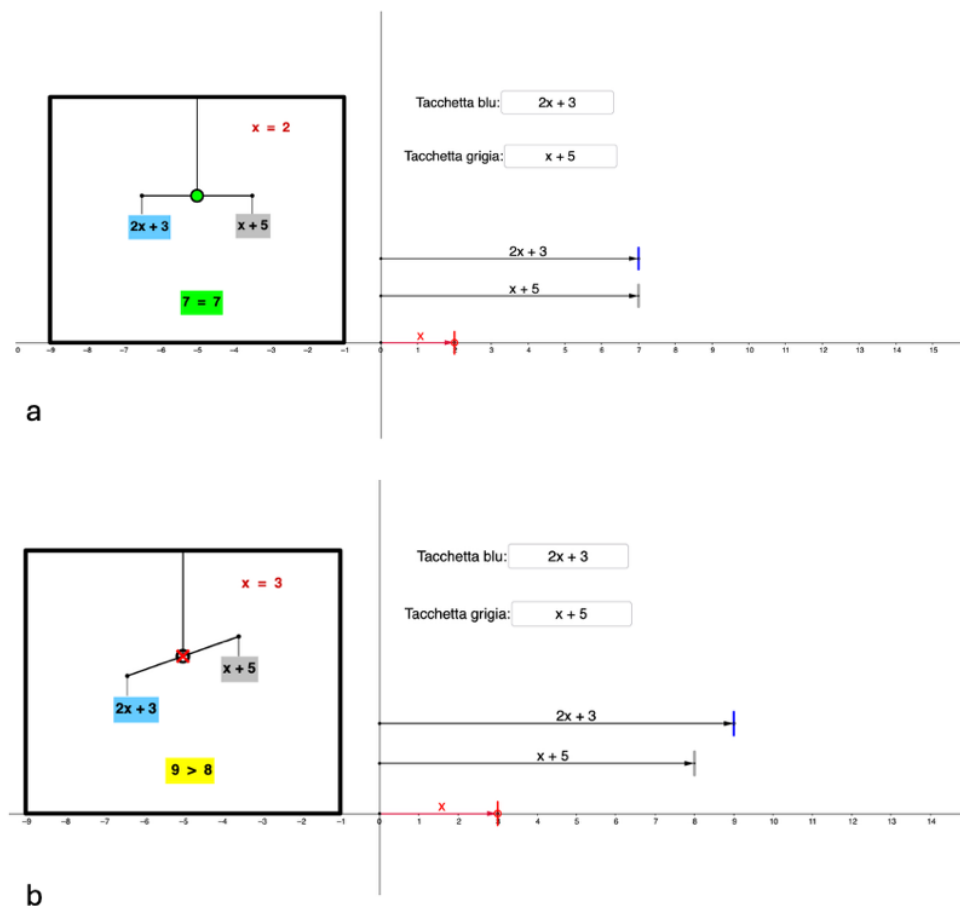


Figure 5.33: The second type of mobile-arrows-DEO

The visual mediators and the actions regarding this DEO are described in the previous sections within the description of the FB-TB-DEO and the moving-arrows-DEO (see Table 5-4 and Table 5-6 for the visual mediators, Table 5-6 for the actions, considering that the user can insert the algebraic expressions in the input field and drag the red tick mark along the axis). With respect to the narratives, the analysis of the transplantability makes emerge the potential this DEO has in order to promote the discourse in two directions. The former concerns the relations about the two DEOs, and thus a sort of bridging discourse among mobile-discourse and arrows-discourse. On the other hand, we expect the development of a discourse that can refer both to the mobile as well as the moving arrows. In this sense, we foresee students to feel the need to have tools to talk about both the DEOs and thus to be forced towards an E-discourse which can offer instruments, such as an algebraic expression, to refer both to a pan as well as to a tick mark (the grey or the blue one), or to the word 'x' to talk about the variable weigh and the red tick mark at the same time. We have designed the DEO in this way especially to prompt the development of narratives in which the subjects are both the mobile and the arrows, and the way to talk about them is an unique one.

5.4.1.Mobile-arrows-DEO task situations

With this DEO we want to accomplish the following main aims:

- Promoting the process of saming between mobile-DEO and arrows-DEO, thus in recognizing the mobile movement and the arrows movement as “the same thing”.
 - Within this saming process, we conjure mobile and arrows become realizations of the same relation between two algebraic expressions depending on the same variable. In this sense, the construction of abstract objects could be promoted.
- We suppose that the use and interaction with this DEO encourage the emergence of new discourses and in particular the growth of a subsuming discourse to talk about the two sides of the DEO simultaneously. This discourse could involve algebraic discourse: for this reason, we have proposed a DEO in which the mobile has expressions as pans and several visual mediators are algebraic symbols.

Task situation 42. *What do you observe in this file? Drag the red tick mark and describe what happens.*

The DEO proposed within this activity is the one in Figure 5.32. This is a task of free action to promote a discourse about the new DEO, in which previous DEO are embedded. In particular, we want to encourage a discourse about the relation between equilibrium/disequilibrium of the mobile and alignment/misalignment of the tick marks.

Task situation 43. *Explore this file and try to reconstruct this mobile in the other file.*

Within this task of goal-oriented action, we give students two DEOs: one of them is similar to the one in Figure 5.32, where, from a mathematical point of view, the expressions are now $7 + 2 + x$ and $x + x + x$, and the other is the one in Figure 5.33, with no expressions in the input field. Students are asked to explore the first DEO in which there are a mobile and correspondent arrows and use the second one to reconstruct the mobile. This task should foster the use of algebraic symbols and the production of a subsuming discourse to talk about the two DEOs simultaneously.

Task situation 44. *Enter expressions into the input fields so that:
a tick mark indicates 12.
A tick mark indicates $2x$.*

What do you expect will happen by dragging the red tick mark?

Task situation 45. *Enter expressions into the input fields so that:*

a tick mark indicates $x + 5$.

A tick mark indicates $2x + 1$.

What do you expect to happen by dragging the red tick mark?

These two task situations, proposed in the DEO in Figure 5.33, differently from the task situations in the moving-arrows-DEO with the input fields, gives expressions in symbols and not in natural language. This because, again, we want to foster the use of algebraic symbols as a subsuming discourse. These are imagined action task situations because, after having inserted the expressions in the input field, students are asked to describe what they think should happen according to the dragging of the red tick mark. More specifically, after a “general” description, students are also asked, by the tutor, to foresee the position of mobile and tick marks for specific value of the red tick mark.

All these task situations can promote a $d_{DEO}(DEO)$, a $d_E(DEO, E)$ and a $d_E(DEO, E)$. At the same time, they could open to a discourse about the relation between balancing-mobile-DEO-discourse and moving-arrows-DEO-discourse, thus a bridging discourse on the relation between the discourses of the two different DEOs (DEO_1 and DEO_2), that we can indicate as $d_{R(DEO_1, DEO_2)}(DEO_1, DEO_2)$. In addition, students could be involved in a discourse on the relation between the discourses of the two DEOs using features of the E-discourse, thus producing $d_{R(DEO_1, DEO_2)}(DEO_1, DEO_2, E)$. There are two DEOs together, about which students have already produced discourses. Moreover, the DEO proposes visual mediators from DEO-discourse together with visual mediators from E-discourse. These factors together could open to different students’ discursive production and could make emerge differences among students with respect to their interaction and use of DEO in order to produce a mathematical discourse. In Table 5-8 I collected some of the tutor’s intervention we have identified to promote the development of E-discourse.

Table 5-8: Some of the tutor's interventions developed for mobile-arrows-DEO

Intervention	Objectives	Activity
For instance, what happens if you drag the	Promote $d_{DEO}(DEO)$ and $d_{R(DEO_1, DEO_2)}(DEO_1, DEO_2)$	42;44

red tick mark to the right?		
For which value of the red tick mark is the mobile balanced? And for which does it dangle to the right?	Promote $d_{DEO}(DEO, E)$ and $d_{R(DEO1, DEO2)}(DEO_1, DEO_2)$	44;45
What do you expect if you drag the red tick mark on the value 4? What will change?	Promote $d_{DEO}(DEO)$, $d_{DEO}(DEO, E)$ and $d_{R(DEO1, DEO2)}(DEO_1, DEO_2)$	45

5.5. The mobile-DEO

Related to the previous DEO, we have designed also this mobile-arrows-DEO, in which only the red arrow and the related tick mark appear. Indeed, as it can be seen in Figure 5.34, the applet is the previous one, but the dependent arrows are missing.

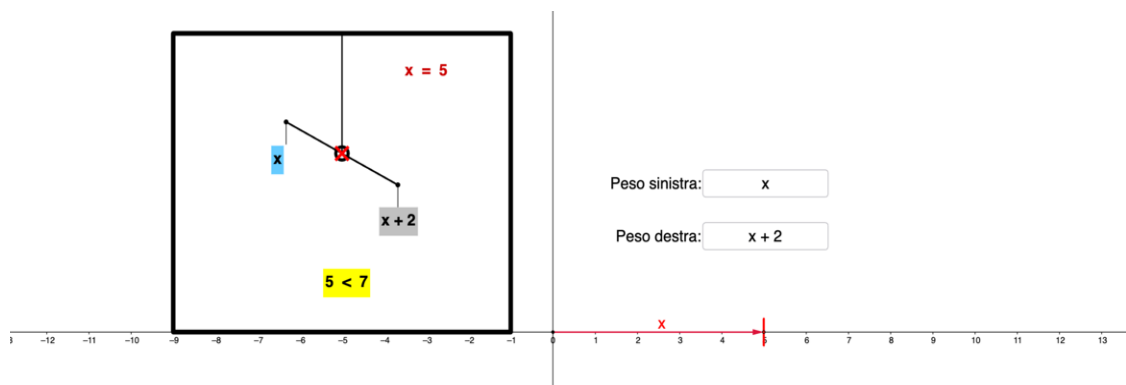


Figure 5.34: The mobile-DEO

The user can insert in the input fields linear expressions that correspond to the pans of the mobile on the left. With the dragging of the red tick mark, the mobile moves accordingly.

5.5.1. Mobile-DEO task situations

Task situation 46. *Construct a mobile that is always balanced.*

Task situation 47. *Construct a mobile that always hangs to the right.*

These task situations of goal-oriented action and the corresponding DEO are proposed to give the possibility to students to “construct” a balance and to reflect on two “special” mobiles that are the one always balanced and the one hanging always to one side. From a mathematical point of view, these mobiles are realizations respectively of an equation with infinite solutions and of an equation with no solution. They have previously encountered similar requests with the moving-arrows-DEO (Activity 26 and Activity 27): for this reason, we decide to propose such questions in a DEO just with mobile.

5.6. The ‘mathematical’ task situations

We propose five task situations that are common within school mathematical activities. These requests are the last part of our tutoring intervention and, while solving, students have at their disposal some of the DEO we have described before: the moving-arrows-DEO, the mobile-arrows-DEO and the mobile-DEO.

These task situations have the aim to study how, at the end of our tutoring intervention, students’ deal with common mathematical task situations. In particular, we want to focus on mathematical discourse of students investigating the possible role of previous discourses in their processes of solution of the task situations. Moreover, considering their initial interviews, we want to examine if their discourses have changed and eventually how have changed. Indeed, except for task situations 1 and task situations 5, the others are equal or similar to task situations proposed during the initial interview. Our didactical aim is to foster the production of $d_E(E)$. This discourse allows us to investigate if, when students face traditional scholastic mathematical requests, after having encountered and dealt with our DEOs, something has changed. In other words, we are aware, thanks to the initial interviews, that students attending our experimentations, participate to the mathematical discourse in a strongly ritualistic way. Facing equations and inequalities, they implement procedures and manipulations focusing only on the process. Within accomplishing these task situations, their participation to the mathematical discourse could show a change, shifting from a ritualistic to a more explorative nature. Furthermore, students have at disposal some DEOs that they can use to answer our tasks. The eventual use of these environments could gain us insight about their relations with the growth and development of students’ mathematical discourse. To conclude, these tasks offer us the opportunity to investigate if and how a remedial intervention has taken place with our tutoring sessions.

Task situation 1M. *For which values of x , $3x + 1$ is equal to $x + 2$?*

For which values of x , $3x + 1$ is greater than $x + 2$?

For which values of x , $3x + 1$ is less than $x + 2$?

Task situation 2M. $3 + x = x + 3$

Does it have any solution?

Can you find them or possible explain why there cannot be any?

Task situation 3M. $3x + 7 = 13$

Does it have any solution?

Can you find them or possible explain why there cannot be any?

Task situation 4M. $13 - x = 13 + 11$

Does it have any solution?

Can you find them or possible explain why there cannot be any?

Task situation 5M. $4x + 5 < x + 12 + x$

Does it have any solution?

Can you find them or possible explain why there cannot be any?

To conclude the chapter, I describe our transition from the first to the second sequence of activities elaborated, according to the theoretical tools.

5.7. The development of our teaching sequence

The sequence of task situations I have presented is the result of a long process in which have taken turns design phase, implementation with students and a posteriori analysis. In this section I want to sum up what has been discussed in this chapter, looking back at what in the beginning was our sequence. I think this clarifies the reasons for the theoretical effort we made to construct tools to describe the design phase and our didactical aims.

During the first cycle of experimentation with students, occurred in Autumn 2022, we proposed the sequence of activities within the DEOs described as in Figure 5.35. The sequence is represented in the blue shapes. The arrows head towards the didactical aims of these DEOs. In particular, the objectives referring to the scales, moving arrows and mobiles had not been formulated in this way within the first design phase. Indeed, this is the a posteriori analysis of the design in the light of our theoretical elaboration presented in chapter 4.

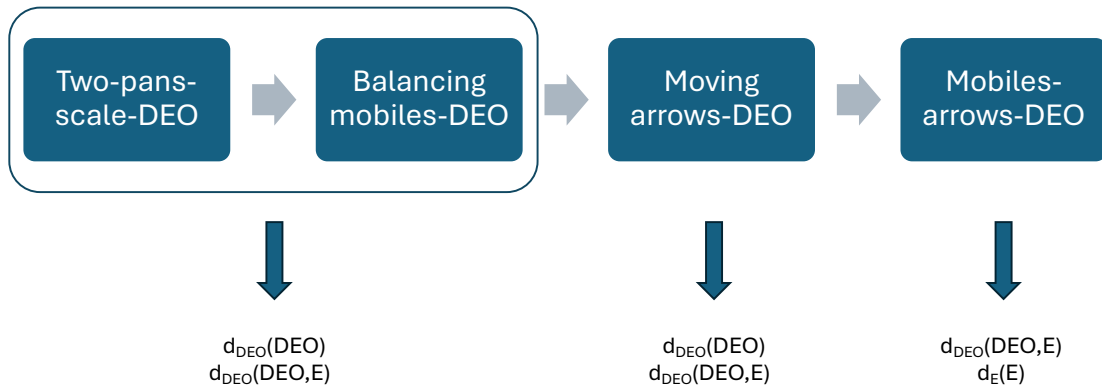


Figure 5.35: The sequence of DEOs of the first experimentation

The research aim of the first experimentation was to investigate the role of our digital environments in the development of an algebraic discourse. In particular, we wanted to test our design hypothesis. What we supposed a priori was that these DEOs could offer tools and ways of acting that echoes the algebraic discourse, that is, in the terms of the analysis of the transplantability, that they allows the construction of the subset we have identified for each DEO-discourse. Moreover, taking into account that students participating to our study have already met algebraic symbolism and procedures related to equations, we expected them to naturally be involved in what we later called “bridging discourse”. Our expectations were that they took part to our activities and, within accomplishing our task situations, their discourse became of the type $d_{DEO}(DEO,E)$ and a $d_E(DEO,E)$. The analysis in the sixth chapter shows that some students have involved in their discourse tools that as experts we recognized belonging to the E-discourse. Nevertheless, several students, that we identify as the ones having more difficulties in mathematics, remained strongly tied to the DEOs and produced discourses which did not involve E-discourse. Roughly speaking, they did not “make the bridge”.

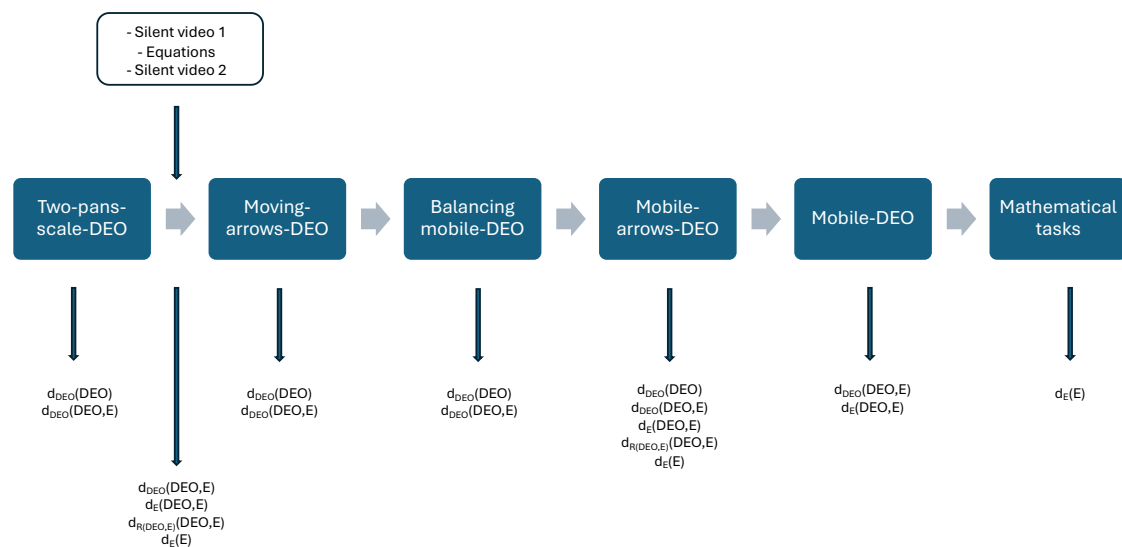


Figure 5.36: the sequence of DEOs of the third experimentation

For this reason, after the first experimentation, we turn back to our design and we try to find different ways to make emerge a $d_{DEO}(DEO,E)$ and a $d_E(DEO,E)$. Thus, we choose to involve in our sequence the two silent videos presented and to propose new task situations to force a new ‘use’ of DEOs (activities where actions were not possible and feedback has to be imagined). We tested some of these new tools within the second experimentation of the DynaMat project in a high school of Pistoia. Thanks to this experimentation, we have been able to refine our sequence of task situations.

Moreover, we develop a script for the tutor’s interventions, to promote the shift from a discourse about DEO and with features of the DEO-discourse to discourses in which features of the E-discourse are used and the aboutness could change from DEO to mathematics. In Figure 5.36 the final didactical sequence is sum up together with our didactical aims with respect to the possible production of discourses. As it jumps out, the choice within the design phase for the third experimentation, has been to introduce new environments and task situations that directly ask for the involvement of features from E-discourse. This happens especially with the introduction of the two silent videos in which the visual mediators of the E-discourse are explicitly involved, and the task situations about equation that are presented and formulated using tools of the E-discourse. Moreover, we decided to force the translation from DEOs-discourse to E-discourse with the transition from the first balance-arrows-DEO to the second one with the input fields. We have also considered to change the order of the DEOs, proposing the moving-arrows-DEO before the balancing mobiles to prompt for the development of a discourse about quantities that vary that can later be reinvested with the balancing mobile-DEO, together with the discourse about not known quantities that has been developed with the TPS-DEO.

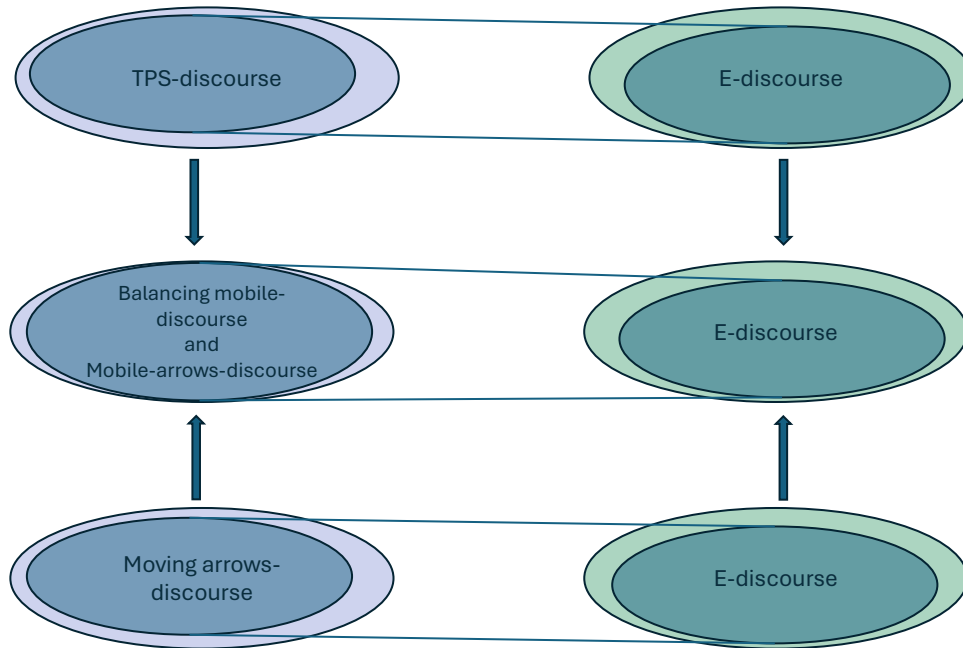


Figure 5.37: Representations of the analysis of the transplantability and relations between discourses of different DEOs

We thus identify two discourses with respect to the DEOs. The former, from an expert point of view, is characterized by features transplantable in a discourse about equation (equal sign, equality among algebraic expressions, unknown, principles). The latter is depicted as a discourse in which features would be transplantable in a discourse functions and relations among them. The two of them together are the discourse that characterizes the activity within balancing-mobile-DEO and mobile-arrows-DEO and that for us have elements transplantable in E-discourse that are essential to develop it coherent with the one of an expert. In Figure 5.37 we see the TPS-discourse and the moving arrows-discourse that, according to the analysis of the transplantability introduced in this chapter, are constituted by subsets of features transplantable in E-discourse. These subsets partially compose the balancing mobile-discourse and the mobile-arrows-discourse and together form the subset transplantable in E-discourse.

6. Data analysis

In this chapter I want to introduce the a posteriori analysis I present and discuss in the following chapters. In wider terms, the analysis is devoted to the examination and the study of processes of teaching and learning algebra within the DEOs we have elaborated. According to the multifaceted nature of algebra and the theoretical framework in which we have conducted our study, I formulated in the fourth chapter the second research objective, to which is committed the analysis, as follow:

- Objective 2: identify processes in the interaction with the DEOs that, as expert, we consider significant for the growth and the development of the algebraic discourse. Specifically:
 - a. investigate the role of DEOs in promoting analytic discourse;
 - b. Investigate the role of DEOs in supporting the theoretical organization of algebraic discourse. We want to focus on the formulation and organization of endorsed narratives with respect to the DEO that can be reinvested in the formulation and organization of endorsed narratives with respect to algebraic discourse, to guide the process of equations solving.

This second objective can be reformulated according to the theoretical tools developed in chapter 4. In general, in the a posteriori analysis, we are interested in investigating development of DEO-discourses and especially of elements that we have identified as transplantable in E-discourse, as well as elements that are not transplantable but could be relevant didactical tools. Specifically, we look for instances of analytic discourse in the form of $d_{DEO}(DEO)$ that can be transplanted into $d_E(E)$. Moreover, we want to focus on the process of construction of P_0 , P_1 , P_2 and their use in solving task situations about not known objects in the DEO-discourses. According to the analysis of transplantability, such principles can be transplanted in E-discourse, together with their use. From our perspective, the elaboration and use of such principles in DEO-discourse and the related transplanting in E-discourse could offer evidence of de-ritualization processes in students' discourses.

Before presenting the course of data analysis, I recall the main characteristics of the two data collections that are of our interest.

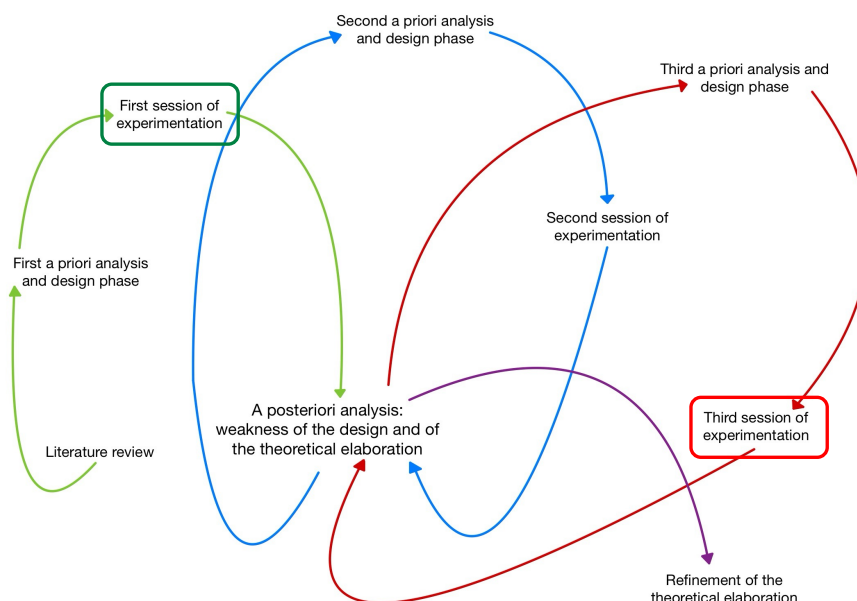


Figure 6.1: The complete development of the study with the two experimentations analysed highlighted

6.1. Data collections: an overview

For our research purpose, we have chosen to focus on the first and the third session of implementations (Figure 6.1). They were conducted at the CARME center in Pistoia (<https://www.carme.center>) while the second one has been led in a 10th grade class of a vocational high school in Pistoia. Students participating to the first and the third cycles have attended an initial individual interview led by a member of the research team. This interview was directed to the accomplishment of one of the main purposes of the project, that is to characterize mathematical learning profile of students both from an identifying and a mathematizing point of view. To fulfil this aim, the interview has been composed by a sequence of task situations in which students were asked about their relationship with mathematics, in the past and nowadays, to promote a discourse in which they could tell stories about them and mathematics. After that, they have faced some mathematical task situations concerning various mathematical content that we expect students to know, at least partially, considering their school age and Indicazioni Nazionali's recommendations (MIUR, 2010). All the task situations were proposed to students on a computer in front of them. They had at disposal a tablet with blank pages in the case they need to write. For each task situation students were asked if they want to answer or if they want to skip it, explaining why.

In tune with the interest of the study here presented, I briefly discuss in the analysis some task situations submitted to students during the initial interview. Some of them indeed, regard equations. These task situations were all proposed by the

interviewer in the same way. She showed on her personal computer a slide as in Figure 6.2.

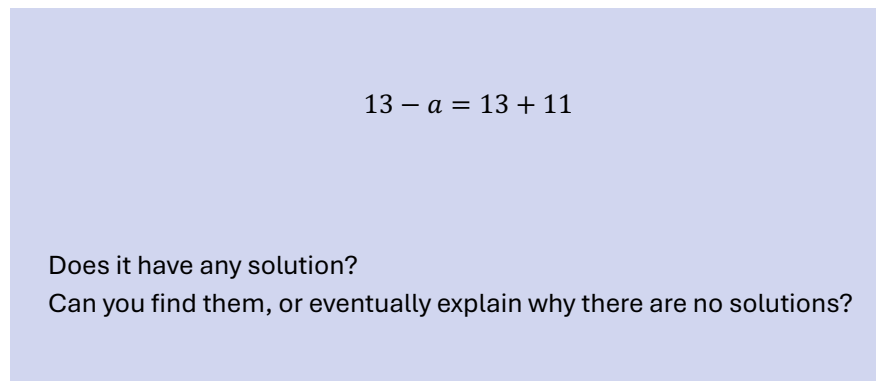


Figure 6.2: Example of task situation from the initial interview

In the interview there were two groups of task situations concerning equations, as the one in Figure 6.2. In both group the same equations were displayed with the only difference in the visual mediator that realizes the unknown. For instance, the equation above was presented also in the form $13 - x = 13 + 11$. Moreover, a third group involved the same equations, with a square instead of a letter for the unknown, as in Figure 6.3. In this case the request was about a possible value that once inserted in the square makes true the inscription. The decision of submitting the same task situations, from an expert point of view, in three different forms aimed at making emerge eventual differences perceived by students and investigating the implementation of, for them, familiar rituals (Macchioni, 2025).

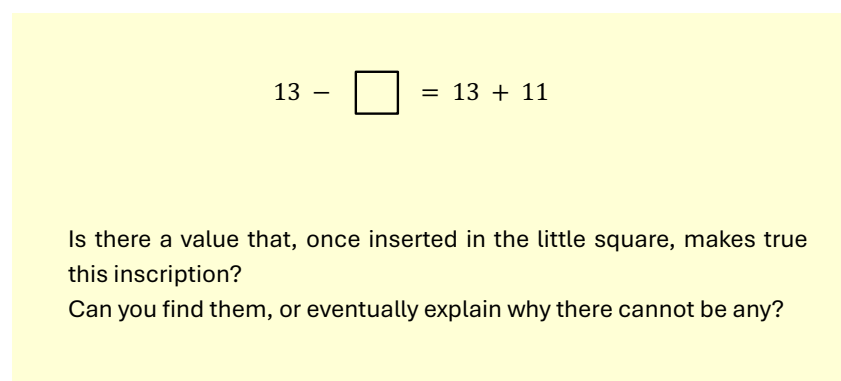


Figure 6.3: Example of task situation from the initial interview

In general, we inserted these kinds of requests to deepen our knowledge about students' incoming algebraic discourse. We aimed at observing eventual procedures and manipulations implemented and the narratives accompanying them. I discuss, in the following chapters, some episodes taken from task situations of the initial interviews to present initial students' discourse.

The complete version of the initial interview with a detailed description and analysis of each task situation can be found in Macchioni (2025). In the analysis that follows, I focus on a restricted number of task situations from the interview, that are collected in Table 6-1.

Table 6-1: Task situations of the initial interview that are discussed in the following analysis

$13 - a = 13 + 11$ Does it have any solution? Can you find them, or eventually explain why there cannot be any?
$3 + x = x + 3$ Does it have any solution? Can you find them, or eventually explain why there cannot be any?
$3 + a = a + 3$ Does it have any solution? Can you find them, or eventually explain why there cannot be any?
$13 - x = 13 + 11$ Does it have any solution? Can you find them, or eventually explain why there cannot be any?

6.1.1.Data collections: the students involved

The first data collection has taken place in the autumn of 2022 (Oct – Dec 2022) and involved 12 students from different high schools of Pistoia. All the students attended the initial interview and after it they were assigned to the ‘algebra’ or ‘functions’ course. These assignments have been made principally due to organizational reasons. Indeed, we have elaborated a schedule of the sessions according to students’ availability and to their requests of working with a mate from the same class. In Table 6-2 are collected the information about the 7 students who have participated to the algebra course. In the first column there is the pseudonym we assigned them, in the second the school they were attending during the experimentation. In the third column we find if they participate to the sessions alone or in couple, and if so with whom, while in the last column there is the number of sessions they attended (in the count the initial interview is excluded).

Table 6-2: Information of students who attended the first data collection

Name	School	Alone/pair	Sessions attended
Andrea	vocational school	with Hugo	4
Dario	vocational school	alone	5
Gioele	vocational school	alone	4

Grazia	classical high school	with Lucia	5
Hugo	vocational school	with Andrea	4
Ilenia	vocational school	alone	5
Lucia	classical high school	with Grazia	5

The third data collection has taken place in the autumn of 2023 (October 2023 – December 2023). It has involved 11 students from different schools of Pistoia and its neighborhoods that have worked in pairs for 5 sessions. As the students of the first cycle, after the initial interview they have been assigned to the ‘algebra’ or ‘functions’ course, according to organizational issues. Differently from the first experimentation, in the third one, while attending the last sessions, students have faced some mathematical task situations recalling the initial interview (for a detailed description see chapter 5). The information about students attending the algebra course are collected in Table 6-3.

Table 6-3: Information of students who attended the third data collection

Name	School	Alone/pair	Sessions attended
Alba	arts high school	with Elsa	5
Aldo	technical institute	with Giulio	5
Alice	arts high school	with Rina	5
Elsa	arts high school	with Alba	5
Giulio	technical institute	with Aldo	5
Rina	arts high school	with Alice	5

6.1.2. Data collections: the generation of data and the tools for the analysis

In tune with our theoretical framework and with our research objectives, we decided to look at students working in pairs guided by a tutor (in the first implementation three students have worked alone only due to organizational matters). During the sessions, they have at disposal two tablets, one to work with the DEOs, the other to write down notes. We have chosen this setting to prompt students talking, according to the discursive analysis, as well as producing visual mediators to support their narratives. Moreover, they can develop routine both in the DEO and on the possible visual mediators they produce. The task situations proposed are unfamiliar, also considering that they are settled in new environments (the DEOs),

to look at discourses evoked in such contexts and to compare them with the expected DEO-discourses.

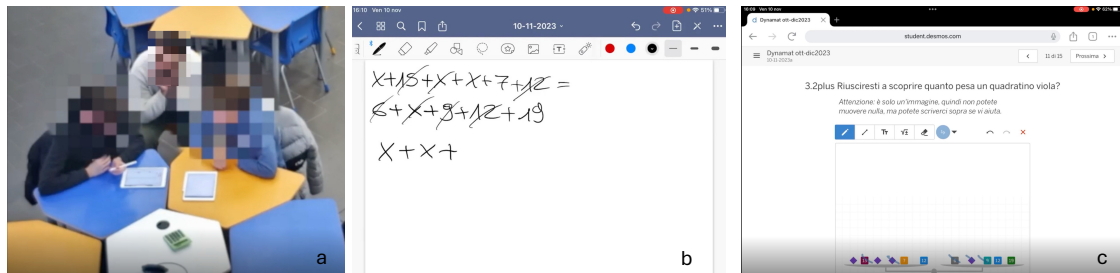


Figure 6.4: Examples of typologies of data collected. Audio - video registration of the sessions (a); tablets screen recordings with students' inscriptions (b) and with the DEO (c)

- Data generated during the sessions:
 - a. Audio-video recordings of every meeting (initial interview and sessions with DEOs) (Figure 6.4a)
 - b. Screen recordings of each tablet students have used (Figure 6.4b and Figure 6.4c)
 - c. Pdf with students' annotations on one of the two tablets (Figure 6.4b)
 - d. Notes of one of the researchers of the team that was observing the session (not in the same room of the session, but in another one thanks to the cameras present in the room)
 - e. Notes written by the tutor that was conducting the session immediately after it.
- Data generated after the sessions:
 - a. Edited video in which we find the audio-video recording of the session and the screen recording of each tablet together (Figure 6.5).
- Data generated after a process of sift through the videos:
 - a. Written transcripts of selected parts of the edited video
 - b. Coding with MAXQDA software

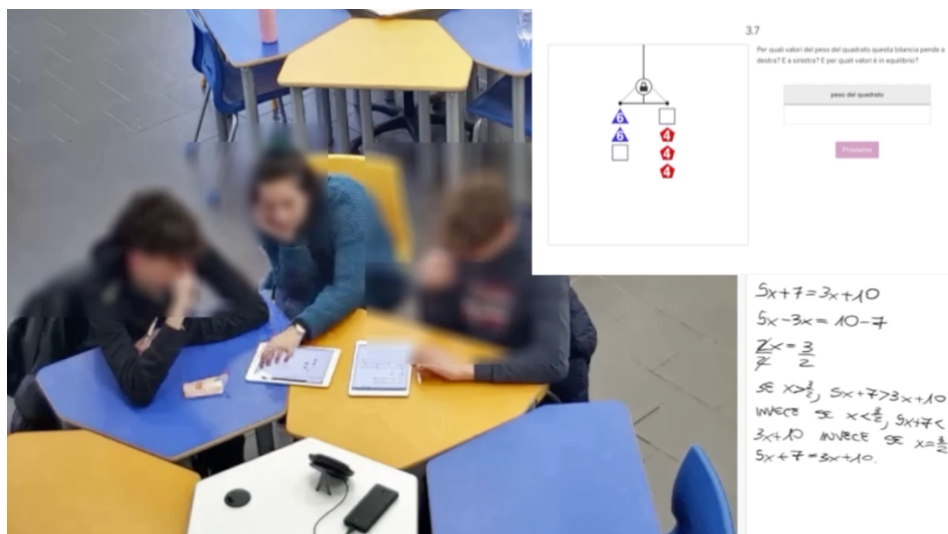


Figure 6.5: Example of edited video with the montage of all the available recordings (of participants and of tablets used)

In between the session and the processes of editing, we take into account notes of the researcher observing the sessions and of the tutor that has conducted them to identify specific episodes that could be of our interest. Once accomplished the process of video editing, thanks to the notes, we proceeded with the process of videos' sifting selecting the episodes more relevant for our research purposes in terms of discursive opportunities. The analysis has been conducted taking as unit of analysis the process of resolution of an entire task situation. After a first analysis of the episode, we have focused on the parts of the episodes for us more significant. In this dissertation I report only some of such excerpts.

Now I distinguish the analysis after the first and the third sessions of implementation.

With respect to the first session of implementation, we transcribed (in Italian) the selected episodes and conduct a first analysis. With this analysis we aim at identifying strengths and weakness of the design and the implementation of our task situations. In more detail, we looked for episodes in which students interact and talk with and about the DEO and its components as we aimed during the a priori analysis and design phase. Within and after the first implementation sessions, we identified crucial episodes that put the light on opportunities offered by the DEOs designed to talk about DEOs' objects in a way that 'resonate' with the mathematical discourse about equations. At the same time, some episodes drew our attention towards some processes that we did not expect or that we expect to happen differently, or to processes that from our point of view were missing. As I have previously discussed, this led us to consider the re-design of some DEOs features, to elaborate new task situations, to develop a script for tutor's intervention. In the

same breath, the processes emerged within the first cycle made grow the need of theoretical tools to describe them as well as the a priori design.

For the analysis conducted after the theoretical elaborations, I created in MAXQDA the labels presented in chapter 4 ($d_{DEO}(DEO)$, $d_{DEO}(DEO,E)$, $d_E(DEO,E)$, $d_E(E)$ and $d_{R(DEO,E)}(DEO,E)$) to label students' and tutor's discourses. Specifically, the identification of discourses is guided by two questions: *What is the object of the specific discourse?* and *To which discourse (DEO-discourse or E-discourse, or both) belong the features used in this specific discourse?*. Moreover, I created some labels to pinpoint, in the students' discourses, words, visual mediators, routine and endorsed narratives from the DEO-discourse and from the E-discourse that we have identified, in the analysis of the transplantability, as essential features for our didactical aims. Specifically, I chose the label *unknown*, *variable*, *equality*, *change sides change sign rule*, P_0 , P_1 and P_2 to investigate the possible development of analytic discourse and specifically the construction and use of narratives to support processes involving the discovery of indeterminate values. To provide an example, in Figure 6.6 we can see the MAXQDA project referred to Grazia and Lucia, two students of the first cycle about which I talk in the next chapter. On the left, below, we find the labels I have used for the analysis. On the right, above there is the video edited with the screen recordings of the two tablets used, while below we find the transcript labelled. In the transcript we can see the line starting with T., L., or G., that stands for the initial of the tutor and of the pseudonyms of the students (in this case Lucia and Grazia).

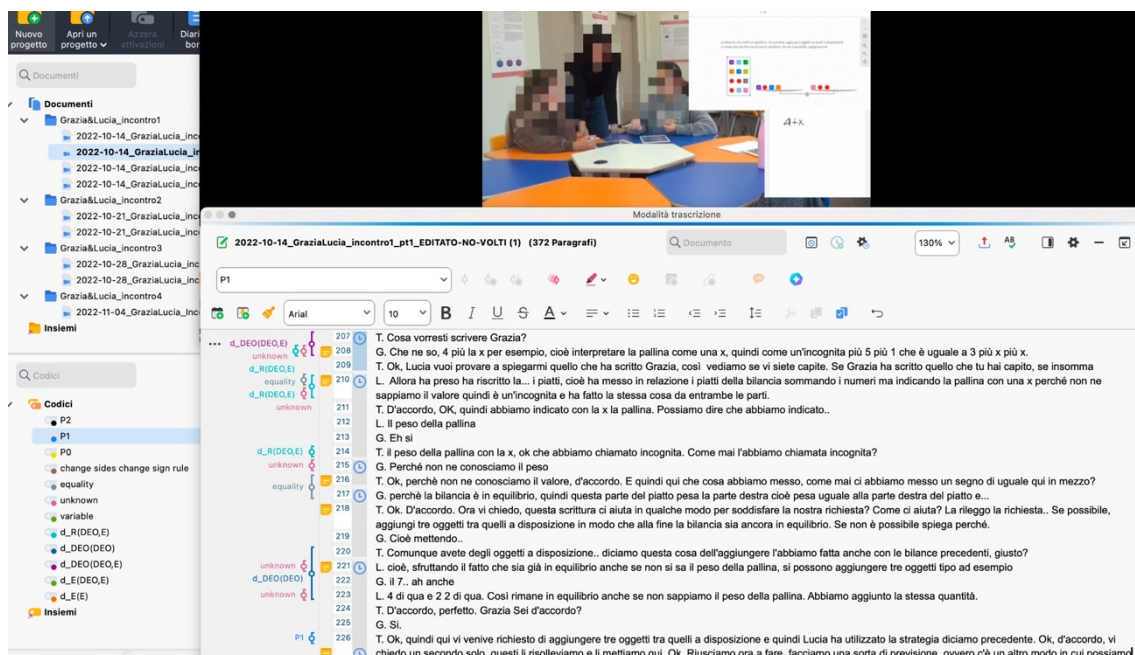


Figure 6.6: Snapshot of a MAXQDA project to analyse the case of Grazia and Lucia

To clarify the use of the label, it is worthwhile to point out that, even if they all refer to E-discourse, I use them both in d_{DEO} and in d_E . For instance, the label *unknown* has been used, in Figure 6.6 line 223, for the sentence “This way, it remains balanced even if we don’t know the weight of the little ball”, because Lucia’s narrative about the balanced scale and the little red balls (of which they do not know the value) is transplantable in E-discourse. Indeed, the way in which Lucia talks about the little red ball is exactly the way in which an expert would talk unknown in the procedure for solving linear equations. Such labelling has helped identifying in the huge number of episodes, excerpts that could be interesting according to our didactical aim of promoting analytic discourse and specifically of supporting the construction and the development of P_0 , P_1 and P_2 . On one hand, with respect to the first cycle of experimentation, labelling has put the light on excerpts in which emerge strengths and weakness of the design, according to our theoretical tools. On the other hand, considering the third cycle of experimentation, labelling has helped in pinpointing crucial moments in students’ learning processes, such as the first introduction of elements from E-discourse in their discourses, as well as the explicit development of principles in DEO-discourse and their consequent use. Moreover, it allows to identify moments in which such principles were used with respect to E-discourse. The discourses’ coding with respect to the aboutness and the features highlights wide differences among students’ learning processes especially with respect to the introduction and use of features from E-discourse. After the coding in MAXQDA I thus identified episodes that were more significant with respect to the objective of the analysis. In addition, in the analysis of episodes from the third session of implementation, we concentrated on the tutor’s intervention investigating if and how they have prompted the production of specific discourse in tune with what we expected.

6.2. Data analysis: guidelines

The episodes discussed in chapter 7 and in chapter 8 have been selected according to specific aims that are presented in the beginning of each chapter. Chapter 7 is dedicated to the analysis of episodes from the first cycle of experimentation while chapter 8 is devoted to the analysis of episodes from the third cycle of experimentation. The analysis that follows will not be in fine-grained as in the typical commognitive framework. It is based on the types of discourses identified a priori and aims at describing holistically the discourses emerged within the activity with DEOs.

The excerpts are presented through the transcriptions, considering for each line its number (in order of appearance in this text), the speaker and what is said. The

enumeration of the line restarts from 1 when a new session is faced. In the spoken parts we find square brackets ([]) and inside them the description of what is done within that line. I underline the line according to aboutness and features of the specific discourse, referring to the classification of discourses presented in chapter 4 and used in the analysis in MAXQDA. The discourses are underlined as follow:

- $d_{DEO}(DEO, E)$ and $d_E(DEO, E)$
- $d_E(E)$
- $d_{R(DEO, E)}(DEO, E)$

I decided to underline and thus put the light on aboutness and features only in specific parts that were crucial for the analysis. Thus, not all the lines are underlined.

All the excerpts have been analyzed in MAXQDA where they had previously been transcribed in Italian. For this dissertation they have been translated into English by the author. In translating the excerpts, I chose to translate literally what was said by the students and the tutor. In particular, when the term ‘peso’ (in Italian) was used, I translated it literally as ‘weight’. In some cases, in the analysis, I refer to the value of the weight when the term ‘weight’ is used, in order to use a more appropriate term according to the context.

7. The development of new discourses

In this chapter I present the a posteriori analysis we have conducted after the first data collections at the CARME center (Autumn 2022), keeping in mind the aims of it. Indeed, conducting the first cycle of experimentation we wanted to have an insight of the potential of our DEOs and related task situations in order to promote the development of E-discourse. To conduct the analysis, I present some excerpts from the sessions with students. To make the analysis clearer, with respect to some students I discuss short episodes happened during the initial interviews, to put the light on students' algebraic discourse at the beginning of the experimentation.

This experimentation has been conducted when the a priori design had strongly relied on the Theory of the Semiotic Mediation. The theoretical tools presented in the fourth chapter had not yet been developed when we have carried out the sessions with students and our design was based on the study of the literature about the main results on the processes of teaching and learning algebra as well as on the potentiality of digital environments in promoting meaningful learning. The sequence of activities proposed to students was composed as in Figure 7.1, described in chapter 5.



Figure 7.1: The sequence of activities of the first experimentation

In this section I want to show and discuss some specific excerpts of the activities with students that are crucial for three specific aspects:

1. put the light on the essential role played by DEO with respect to the didactical aim of promoting analytic discourse; thus, I have selected some of the following episodes because they offer insights into the potentiality of the designed DEOs to promote:
 - a. dealing with indeterminate quantities as if they were known
 - b. talking about the relation of equality between collections of weights and the condition to maintain such equality;
2. make emerge the need of a re-design of some task situations and of an introduction of new ones to promote more explicitly bridges with E-discourse, and thus in particular, to promote the production of $d_{\text{DEO}}(\text{DEO}, E)$, $d_E(\text{DEO}, E)$ and $d_E(E)$;

3. make emerge the need of developing theoretical tools to describe the new design aims and the discourses produced by students within the task situations.

Specifically, considering the focus of the study on the learning processes involving unknowns and variables and equations solving, with a specific attention to the narratives produced within the resolution of equations, I have decided to focus on episodes involving the two pans scale-DEO and the balancing mobile-DEO. Indeed, these two DEOs are focused on these aspects and, especially the first one, are designed to promote the elaboration and the use of the principles P_0 , P_1 and P_2 . Moreover, thanks to the MAXQDA analysis, I have selected episodes that could offer an overview on the variety of processes emerged within the interaction with these DEOs and highlight differences between students' discourses within such interaction.

I underline that, within the analysis, I use the theoretical tools developed after the data collection here presented.

The analysis

To conduct the analysis, as I have anticipated in chapter 6, I report some excerpts from the sessions with DEOs and mathematical task situations. I underline the line according to aboutness and features of the specific discourse. I recall that the discourses are underlined as follow:

- $d_{DEO}(DEO, E)$ and $d_E(DEO, E)$
- $d_E(E)$
- $d_R(DEO, E)(DEO, E)$

I decided to underline and thus put the light on aboutness and features only in specific parts that were crucial for the analysis. Thus, not all the lines are underlined according to the previous criteria.

7.1. The case of Dario

Dario, 15 years old student, was attending a vocational school of Pistoia. He came to our experimentation at the CARME center 5 times, and he participated to all the sessions alone. The case of Dario has been investigated in the project comparing his discourse during the initial interview with his discourse within the sessions of technology-enhanced activities. To have a deeper insight with respect to his initial mathematical learning profile, and how we have related it with the consequent activities with DEO, refer to Bonadiman and Macchioni (2024).

7.1.1. The case of Dario: the initial interview

During the initial interview, facing a task situation in which he is asked to find the solution of an equation, Dario acts as in the following excerpt.

- 1 I: Have you ever seen it written like this? [the interviewer shows the equation $13 - a = 13 + 11$]
- 2 D: Mmm, well, a sum and a subtraction, so mmm, I think a is the result, I mean... I don't know. I think it's the result, I think. So, boh... I mean, do I have to calculate it, or...? [...] Looking at it this way, it would come to mind to do $13 a$ equal 13 plus 11 , but... [laughs embarrassed]. 13 plus 11 seems like it gives the same 24 , 13 minus a mmm... I don't know, maybe I move the result to the other side with the a [...] So, boh, I think, I mean... eh...eeeh... I'm not reasoning, but I'm trying to think of... a method like, boh, if I move the a over there, 13 plus 11 is 24 , minus 13 ... no, it wouldn't work. Mmm, boh, I don't know.
- 3 I: The question is whether 13 minus a equal 13 plus 11 has any solutions. If there are, if you know how to find them, or eventually explain why you think there can't be any.
- 4 D: [laughs] Mmm... A calculation like this, boh, I mean... I don't really think anything [laughs]. That's the problem... uh... for sure 13 minus a , I mean, I wouldn't know how to explain it, I mean... or it's a monomial, so you do $13a$, but... 13 plus 11 mmm... I mean, 13 minus a is equal to 13 plus 11 . We need to discover what a is. [...] Uh, I think a is the result of 13 plus 11 , in my opinion, it's... so... I think, then...
- 5 I: Ok, so would you say that this has solutions?
- 6 T: I mean... mmm... uh, yes, I think it is... well, aside from 13 plus 11 being 24 ... I think it should be done, I mean... [...] I'm thinking of a calculation [...] Uh, I can't think of any calculation... uh... mm... that I do $24, 24$ uh... in my opinion, 24 plus 13 ... but, I mean, 24 plus 13 is 37 , so if I do 13 minus 37 it gives 24 . I think it is... [...] I think it should be done this way, but I don't know. Can I know if I did it right or not?

In this task situation, as well as in the other task situations of the initial interview, the student's discourse is strongly driven by the need to accomplish a procedure, not well-defined, that he seems to recall from previous experiences. The will to think of a method or a calculation recurs in the lines ("I'm not reasoning, but I'm trying to think of... a method like" (line 2), "I'm thinking of a calculation" (line 6)). He mentions the possibility of moving terms from one side to the other (line 2), probably referring to the procedure for solving linear equations. His focus is on the unknown of the equation a , "I think a is the result" (line 2), "We need to discover what a is" (line 4), "I think a is the result of 13 plus 11 " (line 4), but he is not able to follow a

way to find the solution that satisfies him. Indeed, having accomplished some calculations, in the end he asks, “Can I know if I did it right or not?”. Dario’s discourse is centered on looking for a method to find the ‘result’, the value of a . Once he has reached a number from his calculations, he needs to refer to someone else in order to know if his procedure has taken him to the right solution. The discourse of Dario, in this task situations but also in other, during the initial interview, is widely ritual and concentrated upon the process.

7.1.2.The case of Dario: the first session

During the first session, I was the tutor for Dario. We have dealt with the task situations with two-pans-scale-DEO and with a pair of task situations with balancing mobiles-DEO.

Excerpt 1

Dario has dealt with two-pans-scale with weights of known values only, in which the main didactical objectives were to gain familiarity with the DEO and to build the routine composed by the task situation ‘Add objects from those available in the box/remove objects from the scale so that it is still balanced afterwards’ and the procedure of adding or removing weights with the same values from both pans, even considering collection of weights for one pan. The student satisfies the task situations proposed, accompanying his actions with endorsable narratives. The next excerpt concerns Dario’s discourse during the accomplishment of the first task situation in which objects have known and not known values (Figure 7.2).

Task situation 5 *Here you see a balanced scale. If possible, add 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.*

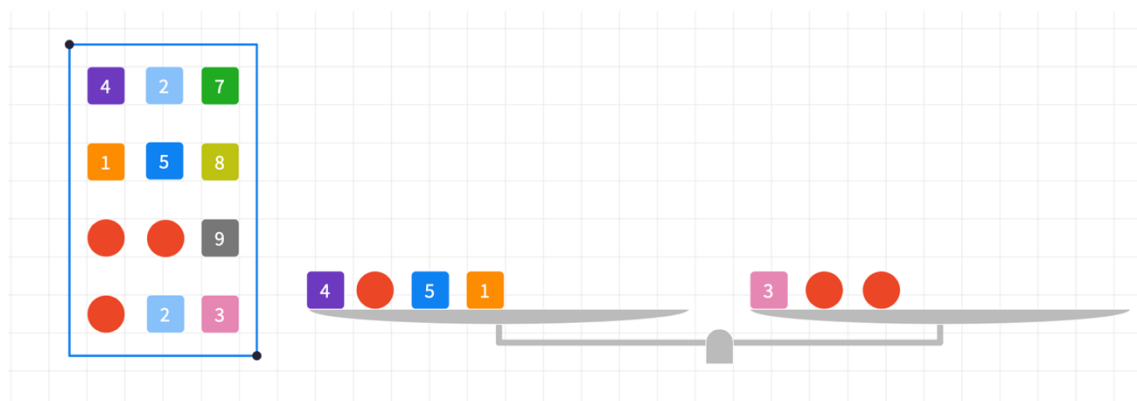


Figure 7.2: Scale of task situation 5

- 1 D: So... um... add three objects. Except that it should be understood the little balls, what weight they have.

- 2 T: Okay. So, you would like to know what weight these little red balls have?
- 3 D: Yes then... [...] They all have the same weight. Mmm... surely here, trying to sum it up again, 5 and 1 make 6, and 4 make 10. So, this is 10, and here there's a 3, which, however, is equivalent—that is, the two sides are balanced.
- [...]
- 4 D: That is, I don't know, try to understand why here there is a 3 and two little balls, that is, especially understand what these two little balls make.
- [...]
- 5 D: [after having added a little red ball to the right pan] Because, in fact... the right side of the scale would have gone down, just like the left side, because going back to the previous exercise, with one, in my opinion, it cannot be measured, that is, it cannot be balanced.
- [...]
- 6 D: Well, I have tried to do, I do not know, maybe it is 10, 10, I do not know, a little red ball is worth 10... [...] 10 and 10, 23... 10, 15, 20...
- 7 T: Ok, thus, giving value 20, 10 sorry, to the weight of the little red ball, what would happen?
- 8 D: That is, it would not be balanced
- 9 T: Ok
- 10 D: Because on one side it would make 20 and on the other 23
- [...]
- 11 D: Otherwise, trying to make 6 and 6, 12 plus 3, 15 but however, it still wouldn't be balanced, because here it would make 16, that is...

The first thing Dario says, after reading the request, is “it should be understood the little balls, what weight they have” (line 1). He does not move objects on the scale, but he reflects on the composition of the two pans, adding up the known values (line 3). The tutor asks him about his thinking, and he explicates that his focus is on “understand what these two little balls make” (line 4). He decides to add a little red ball to the right pan, and he discusses the obtained situation in line 5. The feedback of the scale is in line with what he expected also with respect to a previous task situation to which he refers, where he was asked to add only one object to keep the scale balanced (line 5). The tutor takes back the attention on the written request, that Dario accomplishes in different ways, adding three objects to the scale. Despite this, the attention keeps being grasped by the little red balls. He tries to assign a specific value to their weight, counting the sum of the values of the objects

once he has chosen 10 or 6. He considers that in the first situation the left pan weighs 20 and the right 23, while in the second one the left weighs 16 while the right is 15, but he does not take into account the two results together, trying to assign other values among 6 and 10. In this case, he does not take advantage of the feedback that could be offered by the scale, and he makes hypothesis resting only on his calculations.

Excerpt 2

Dario accomplishes the task situation 6 (Figure 7.3). Once he has removed the requested number of objects (5), he focusses again on the objects with not known value, and he affirms “I would try to do this thing” (line 10) while he starts acting on a scale as in Figure 7.3. The task interpreted by Dario is once again finding the value of the not known weight.

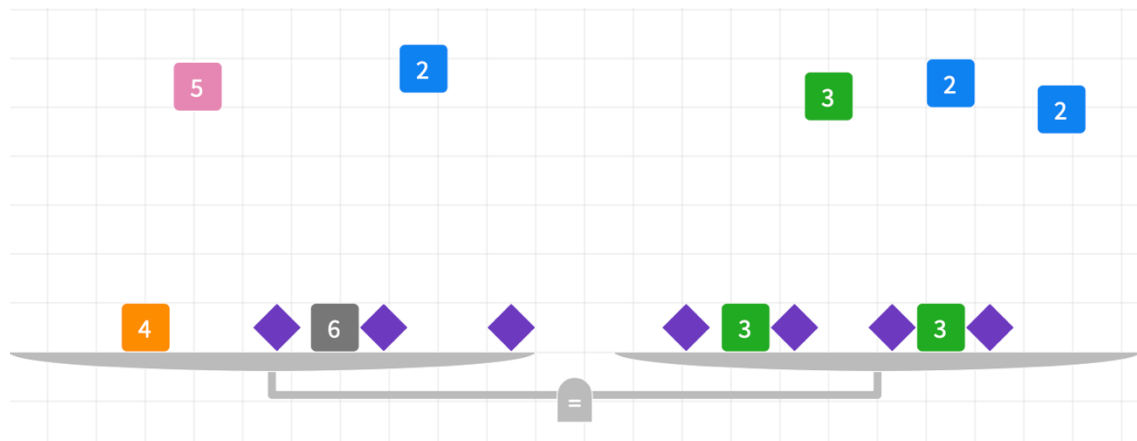


Figure 7.3: Scale of task situation 6 once Dario has removed 5 objects

- 12 D: Well, I would try to do... this thing [he removes all the weights from both pans]
- 13 T: So you are removing everything...
- 14 D: That is, I would try, to understand [he sets the not known weight on the left pan], well ok, this is natural, that the left side is lower, I try to put the 3..." [he sets a weight of value 3 on the right pan]
- 15 T: And what happens?
- 16 D: That it is greater than 3 [...] With respect to the right side. Let's try to put another 3 [he adds a weight of value 3 on the right pan]... and it is also greater than 6 [...] With 2 [he adds a weight of value 2 on the right pan]... actually no, wait, let's try with 3 [he removes the weight of value 2 from the right pan and adds a weight of value 3 on the right pan]... it is also greater than 9 [...] [he adds a weight of value 2 on the right pan] Eh!
- 17 T: What has happened?

- 18 D: That... That the sum of 3, 6, 9... that is, 3, 3 and 3 and 2 makes 11 is, that is 11 equals a rhombus, that is...
- 19 T: Ok, and thus, why are you saying that it is equal?
- 20 D: Because, indeed, it is balanced, the scale, only with in the left side the purple shape and in the right side instead eleven weights.
- 21 T: Ok
- 22 D: That is, 11... four objects that form 11

He lifts all the weights from the two pans (line 12), sets the purple square on the left pan (line 14) and affirms “this is natural, that the left side is lower”; then he starts adding objects on the right pan. Dario’s discourse is characterized by the verb ‘to try’ (lines 12, 14, 16). This verb could be associated to his participation during the initial interview, while he was trying to find ‘a calculation’. In this case instead, his attempts are accompanied by narratives in which he relates the action accomplished with the feedback of the scale. Moreover, in line 16, he does not need to ask if his actions have taken him to the right solution, because, thanks to the feedback of the scale, he can (with joy) evaluate it alone. The narratives he produces to justify his sentence “that is 11 equals a rhombus” (line 18) are endorsable in the DEO-discourse. Dario has found a way to answer the task situation he himself has created.

Excerpt 3

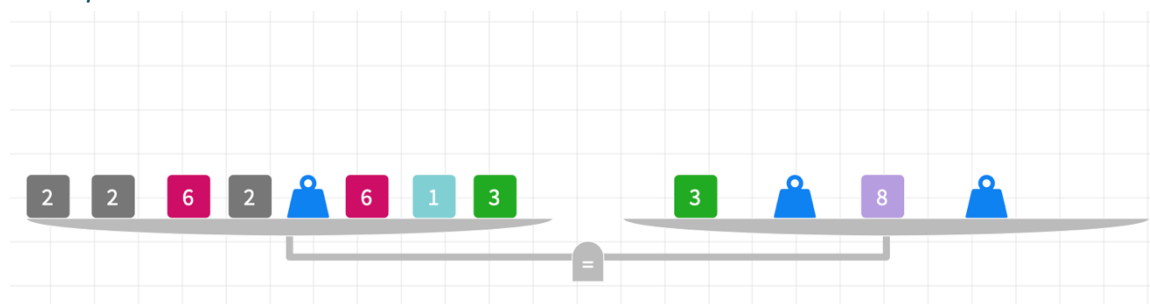


Figure 7.4: Scale of task situation 7

The following task situation offers a scale as in Figure 7.4 and asks to remove 4 objects. To satisfy the task situation, students have to lift also objects with not known value.

- 23 D: Otherwise, yes wait... I could try to do a thing
- 24 T: What would you do?
- 25 D: Remove this... that is, the blue shape from here and here [he lifts a weight with not known value from both pans], it would be equal, then I would remove the 3 both from here and here [he lifts a weight of value 3 from both pans], I removed four objects, taking off the same, the same shapes, the same...

- 26 T: Ok, because the request was to remove four objects... thus you have lifted exactly four, and why does it work?
- 27 D: Because anyway, the blue shape is equal to the other blue shape and the 3 is equal, that is, the weight of 3 is equal to the weight of 3
- 28 T: Do we know how much the blue weight is worth?
- 29 D: No but it could be understood now considering that here there is an 8 and here there is a blue weight, thus here there is 2, 4, 6, actually, no, let's do 6 and 2 and thus these two [...] And thus it makes 8, it is like it makes this [he points to the object of value 8 on the right pan]... instead 6 and 1, 7, 9 and 11 and the weight would be 11

Dario at first is hesitant and seems willing to discover the value of the blue object. But then he proceeds as in the excerpt (line 23). He manipulates the blue objects developing a meaningful narrative in the DEO-discourse, "Because anyway, the blue shape is equal to the other blue shape and the 3 is equal, that is, the weight of 3 is equal to the weight of 3" (line 27). Specifically, he acts with the not known weights in the same way as he acts with the known weights: he has introduced a new routine in his discourse, in which also not known objects can be moved. In the excerpt 1 his need of finding out the value of the little red balls stopped him for minutes and he ceased to focus on them only due to tutor's intervention. In this task situation objects with not known values have become a way to solve the request (line 25), they can be moved even if Dario does not know their value. When the tutor asks him about the value of the blue objects, Dario starts counting with respect to the objects on the left pan. At first, he sums up the two object of value 2 ("2,4" line 29) and then he wants to add the object of value 6 (line 29). But then he changes the objects, focusing on a weight of value 2 and a weight of value 6. He affirms "let's do 6 and 2 and thus these two [...] And thus it makes 8, it is like it makes this [he points to the object of value 8 on the right pan]" (line 29). In these utterances, as expert, we see the use of P_1 for the scale. Dario chooses two objects on the left pan that "make" the object of value 8 on the other. In this way, such objects cannot be considered, in order to discover the value of the not known object. Indeed, Dario goes on "instead 6 and 1, 7, 9 and 11 and the weight would be 11" (line 29) reaching a conclusion for the value of the blue object. Only at this point the tutor asks him if he wants to try to do the action he has described. In this excerpt we can see that Dario has developed a new routine: his discourse manifests a spontaneous change in the way of finding the value of the not known weight that bring him closer to the routine accomplished by expert in solving linear equations.

Excerpt 4

In this excerpt, Dario deals for the first time with the balancing mobiles-DEO (task situation 28). The mobile presents on the left arm two triangles with not known value and a square of value 3, while on the right arm there is an object of value 11.

- 30 D: Thus, I already have an idea
- 31 T: Ok, what is your idea?
- 32 D: So, I can tell that I try to do $11-3$, the two numbers we can see. And, indeed, it makes 8. That are two triangles. Eh... divided, 4 and 4. 4 and 4, 8, plus 3, 11 and 11 on the other side thus it is balanced, it is balanced.
- 33 T: Ok, perfect. So, you have made 11 minus 3
- 34 D: Yes.
- 35 T: How come?
- 36 D: Eh, because, indeed, for me, considering that they are the only two numbers, it is like doing, I do not know, just to say, x plus x plus 3 equal 11.
- [...]
- 37 D: Well, I was thinking, I do not know... The triangles, so, I do not know... square x plus 3 is equal [he writes, Figure 7.5a], that is, x plus x , I mean... oh god, wait because I do not remember if square x is x times x or x plus x ... Well, whatever
- 38 T: Let's try to understand, what do you want to write here?
- 39 D: That is... It is easier to do so... let's make triangle plus triangle plus 3 [Figure 7.5b]
- 40 T: Ok and here you are writing what, with respect to your mobile?
- 41 D: That triangle plus triangle plus 3 is balanced with 11, so it is equal to 11
- 42 T: Ok
- 43 D: So, I think to calculate 11, that are the two numbers we can see, minus 3, equal... and you discover what are these two triangles...
- [...]
- 44 D: It's just that those are the only two numbers that... are visible.
- 45 T: Ok, and if... let's say, if you had, now I'll draw too... So, let's always make the little triangle, the little triangle... and 3 [writes as in Figure 7.6]. Ok? Here too, you would... you told me 'these are the only two numbers that appear on this scale, and I do 11 minus 3'. In this case too, you have—this scale is a bit different from this one, would you have done 11 minus 3?
- 46 D: No, here I wouldn't... I mean, I wouldn't have subtracted, but instead, I would have added.

- 47 T: OK, why?
- 48 D: Because, by adding 11 plus 3, which makes 14, then I would have divided by 2, these two [he points to the two triangles on the left arm], and it would be 7 and 7.
- 49 T: Ok. Alright. And in this case [she points to the mobile on the DEO], you do subtract, right?
- 50 D: It made sense.

Figure 7.5 consists of two rectangular boxes, labeled (a) and (b) at the bottom right. Box (a) contains the handwritten mathematical expression $x^2 + 3 =$. Box (b) contains the handwritten mathematical expression $\Delta + \Delta + 3 = 11$.

Figure 7.5: Visual mediators produced by Dario in task situation 28. On the left (a) the inscription made in line 37. On the right (b) the inscription made in line 39

As soon as Dario reads the task, he affirms “I already have an idea” (line 30). He proposes to make the difference between the two numbers he can see, and divide the obtained number by 2, to find out 4 (line 32). He also verifies the obtained value “4 and 4, 8, plus 3, 11 and 11 on the other side thus it is balanced, it is balanced”, producing a meaningful narrative. When the tutor asks him why he has proceeded in this way, Dario introduces words that belong to E-discourse (line 36). He relates the position of the balanced mobile with a condition of equality between algebraic expression, “ $x + x$ plus 3 equal 11”. Prompted by the tutor, he writes $x^2 + 3$, typical visual mediators of the E-discourse (Figure 7.5a). Once he has written them, he gets stuck: he does not remember “if square x is x times x or x plus x ” (line 37). Despite this, he finds a way out (line 39). The visual mediator produced by Dario is an intertwining of DEO-discourse and E-discourse, that create a $d_{DEO}(DEO, E)$; indeed, triangles stand for the triangles on the left arm of the mobile, while the plus and equal signs are used to mediate the relation between the object on the scale (Figure 7.5b). He goes on repeating the procedure previously introduce, that is to make the difference between 11 and 3, finding “what are these two triangles” (line 43). After having tested the value of 4 finding out that is the one that makes the mobile balanced, the tutor asks Dario why he has made a difference between the two numbers. Having the student answered that “It's just that those are the only two numbers that... are visible” (line 44), the tutor proposes him a drawing of a new mobile Figure 7.6, with weights of value 11 and 3 both on the right arm (line 45). The objective of tutor's intervention was to make emerge a bridge between the calculations Dario has decided to perform with respect to the mobile and the action accomplished on the two-pan scale within previous task situations. In this sense, the tutor's aim was to promote the development of narratives such as “I subtract

from 11 the number 3 that is as removing from both arms a weight of value 3, and the mobile keeps being balanced”. In response to the tutor’s mediator, Dario affirms that he would sum up the two numbers (11 and 3) and then divide by 2. The reason why Dario proposes these two procedures his “it made sense”. He accomplishes the right procedure, coherent with the mathematical one, to find out the not known values, but if he is asked to explain why, he does not produce meaningful narratives about it. In particular, he does not relate the previous routine of removing the same value from both pans to the action of subtracting 3 to 11.

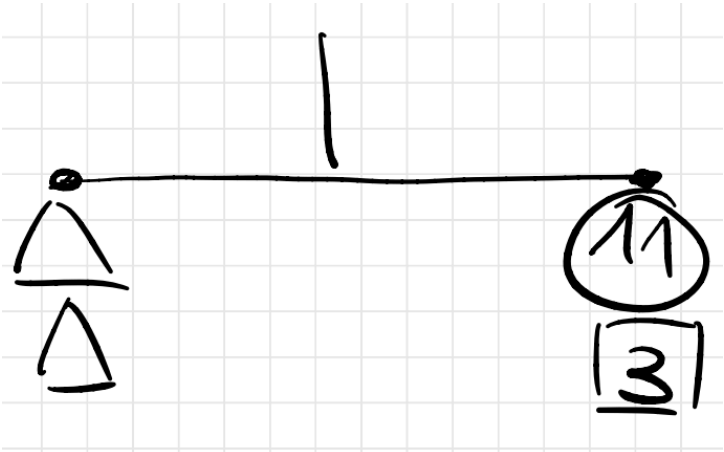


Figure 7.6: Visual mediators proposed by the tutor

From the reported excerpts, we can highlight several aspects of Dario’s discourse. We can underline a huge difference in his participation between the initial interview and the session with DEO. While in the first one he kept looking for calculations and methods and trying to remember a procedure to satisfy the request that has been proposed to him, in the activity with DEOs he himself develops tasks to be accomplished.

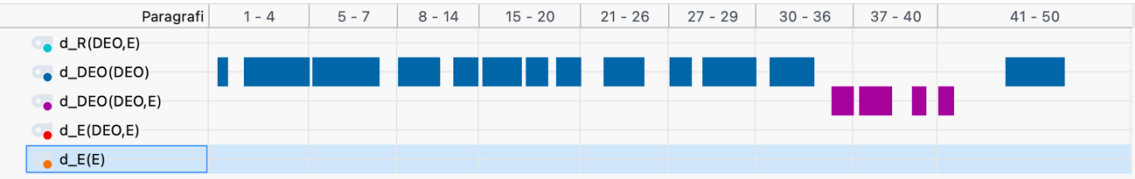


Figure 7.7: MAXQDA codeline of the episodes analysed from Dario's first session

In Figure 7.7 we find the MAXQDA codeline of the excerpts reported in the analysis of Dario’s first session. In the column there are the ranges of paragraphs of the transcript while in the row there are the labels referring to the coding of discourses according to aboutness and features used. As discussed in chapter 6, in MAXQDA I have coded also the lines involving $d_{DEO}(DEO)$. In this case, only the lines of the student have been coded. The codeline developed in MAXQDA shows that his discourse is strongly attached to the DEOs proposed to him. Only within task situation 28 Dario involves in his discourse features from the E-discourse that he

quickly abandons affirming “It is easier to do so” and changing his mediators from Figure 7.5a to Figure 7.5b. The opportunity to interact with the DEOs seems for Dario effective in promoting analytic discourse. In particular, the presence of not known objects has been for him stimulating. Indeed, his strategies to find out the value of these objects evolve along the excerpts. In the first activity he tries to assign values to the little red balls and makes related calculations, while in the second one he ‘weighs’ the not known object. Finally, in the third activity, he reinvests the procedure of removing the same value from both pans to find out the value of the blue object, showing the applicability of the procedure. All his actions are supported by meaningful narratives, such as “That... That the sum of 3, 6, 9... that is, 3, 3 and 3 and 2 makes 11 is, that is 11 equals a rhombus, that is...” (line 18) and “Because, indeed, it is balanced, the scale, only with in the left side the purple shape and in the right side instead eleven weights” (line 20). The relation of equilibrium among the two pans is used to understand the value of not known objects, that are considered and treated, in the third excerpt, as known objects (“Because anyway, the blue shape is equal to the other blue shape and the 3 is equal, that is, the weight of 3 is equal to the weight of 3”, line 27). The blue shape, that for us realizes the mathematical object unknown, can be removed by both pans of the scale, as all the other known objects. Recalling the idea of analytic discourse, we can underline that Dario has started dealing with indeterminate quantities as if they were known and he uses the balanced scale as a relation among quantities. Considering such narratives, with an appropriate intervention of the tutor, Dario could have developed P_1 for the scale. He uses it in his discourse, even if it has not been formalized yet. Moreover, in his discourse Dario uses on several occasion P_0 . For instance, when he tries to assign values to the not known weight in task situation 5 (lines 10 and 11), as well as when he starts to ‘weigh’ the not known object in task situation 6 in lines 14, 18 and 20 (“Because, indeed, it is balanced, the scale, only with in the left side the purple shape and in the right side instead eleven weights”). Such utterances could have evolved towards a formalization of the principle as we have prompted within the third cycle of experimentation. Despite this, we can affirm that the interaction with the DEO and its feedback allow Dario to develop a discourse in which he looks no more for rituals, favouring actions that are always accompanied by meaningful narratives. Considering Dario’s participation to the discourse, it is worth to highlight that he makes decisions autonomously and he is always focused on the product of his actions that he is able to ‘evaluate’ alone according to his own aim. The last excerpt makes emerge a difference in the metarules that guide Dario’s discourse when he is working with the DEOs and when he is involved in mathematical activity. Indeed, when he tries to write through algebraic symbols the mobile he is interacting with, he gets stuck, and he affirms that he does not remember if “x square x is is x times x or x plus x”. Thus, he goes

back to the mediators offered by the DEO. Dario shows us that to produce a mathematical discourse he needs to remember facts and procedures, and if he does not recall something, he cannot go on. On the other hand, when he is involved in a discourse with respect to DEOs, he has tools to proceed autonomously. This is one of the main opportunities offered by the DEO: it opens to a new participation that for Dario means making decisions alone and develop a discourse about them.

7.2. The case of Hugo and Andrea

Andrea and Hugo are two classmates from a vocational high school of Pistoia. They came for the initial interview and 4 times for the sessions with DEOs-activities. They attended the session together. During the initial interviews, that were conducted separately, they are asked to face some task situations about equations. I report here short excerpts to give some hints about their initial discourse about equations.

7.2.1. The case of Hugo and Andrea: the interview of Hugo

The interviewer asks to Hugo if he has ever seen written like $13 - a = 13 + 11$, and he answers that this inscription is familiar for him. In particular, he focuses on the letter a , saying “It remembers me that the letter a is a tacit letter, we can say, because with the brain you have to reason because thirteen minus a equal thirteen plus eleven”. Taking into account that it is familiar for him, the interviewer asks Hugo if this inscription has solutions and if he is able to find them. Hugo affirms that he wants to try to find them and says, “For me, considering that thirteen minus a equal thirteen plus eleven, the result should be thirteen plus eleven, that would be twenty-four, so the result, so thirteen minus a would be equal to twenty-three minus thirteen, that is twenty-four minus thirteen”. Hugo makes all these considerations only by words, without writing. The interviewer suggests, if it can help him, to write, and Hugo produces the visual mediators in Figure 7.8. The interviewer affirms “So at the request, does it have any solutions you have said yes. You can find it. So, which are the solutions in this case?”. Hugo answers “twenty-four minus thirteen equals thirteen plus eleven”. In accomplishing this task situation, Hugo seems well focused on the unknown, and on the fact the two expressions are equal. Once he is asked to find the solutions, his discourse becomes characterized by calculations and manipulations and in the end the solution for him is an equality (incorrect).

$$\begin{array}{l}
 13 - a = 13 + 11 \\
 a = ? \\
 a = 13 + 11 = 24 \\
 24 - 13 = 13 + 11
 \end{array}$$

Figure 7.8: Visual mediators proposed by Hugo during the task situation

7.2.2. The case of Hugo and Andrea: the interview of Andrea

When he faces task situations involving equation, Andrea affirms that he has already seen inscriptions like the ones proposed. When he sees the equation $13 - a = 13 + 11$, he says “I have already seen it, but I do not remember how to do it”. Then he is asked if the equation has any solution, Andrea says “For me no, because the minus has become plus and the a has become eleven, from nothing, we can say. In another task situation of the interview, with the equation $3 + x = x + 3$, Andrea makes considerations about the fact that the expressions on the two sides are equal, he affirms “They would be the same thing”. When the tutor asks him if it has any solutions, Andrea answers “In which sense? I cannot understand in which sense it has solutions”. The interviewer then tries to change the question, “If I ask you, the number one is a solution of this thing?” and Andrea says, “I do not know”. From the two examples we can affirm that Andrea’s discourse is focused on the relation of equality of the two sides of the equation, with respect to $3 + x = x + 3$. Instead, when he deals with $13 - a = 13 + 11$, he seems puzzled by the fact that some terms of the first expression are changed, out of the blue. He does not involve in his discourse any procedure or manipulations, affirming that he does not remember how to do it. All along the interview, Andrea is quiet and do not speak a lot, he does not write anything in trying to answer the questions. The interviewer has to prompt him several times to have answers, that are often “I do not know”.

7.2.3. The case of Hugo and Andrea: the first and the second sessions

I now discuss three excerpts from different activities conducted with Andrea and Hugo during the experimentation. Some aspects of these excerpts have been discussed in a paper already published (Antonini et al., 2023).

Excerpt 1

Andrea and Hugo deal with task situation 28, that involves a balancing-mobile-DEO. They have to find out the value of the white triangle to keep the mobile balanced. In this short excerpt emerge some characteristics of Andrea and Hugo's discourse that are relevant for our analysis.

- 1 A: 4.
- 2 H: I already figured it out... 4. Because here, on one side, I have three. Like, if I put a weight here [he points to one of the white triangles] worth 4, this is worth 4. 4 plus 4, 8 + 3, 11.
- 3 T: Perfect. Let's try—this is what you were saying, right? Let's take a look for a moment. Ok, fine, let's say—let's try with other values. Can we, for example, make the test scale dangle to the left?
- 4 H: Well, instead of 4, if you put 5, the scale tends to dangle that way, because here, while there are 13, there are only 11 on the other side.
- 5 T: Ok, perfect. And what happens if I put—actually, sorry—what if we wanted to make it dangle to the right?
- 6 H: To make it dangle to the right, you need to put a number smaller than...
- 7 A: 1.
- 8 H: ...smaller than 4.
- 9 T: Andrea suggested 1. But let's see what happens with 1.
- 10 H: Or we could say we need to put a number smaller than 4, so that—so that here, the 11 is still larger. If you put 3 here, this side becomes 9, while the other side remains 11.
- 11 T: If we wanted to describe all of this, in general, for this mobile, how could we say it? In a more summarized way, how would you put it?
- 12 H: To keep the scale balanced, the number here—a triangle—is worth 4. To make it dangle to the left, we need to find a number greater than 4, while to make it dangle to the right, we need a number smaller than 4.

Andrea and Hugo's discourse is focused on numbers and on the act of 'putting' them into the field. Indeed, the attention is on the interaction of the two students with the DEO and especially on the action of inserting a value for the triangle, that they indicate with the verb 'to put'. For instance, Hugo points at the white triangles of the mobile on the left when he affirms "If I put a weight here [1] worth four, this is worth 4. 4 plus 4, 8 + 3, 11". Actually, they cannot insert value in the white triangles but only in the input field offered by the DEO. Once they have tested their hypotheses to have the mobile balanced, the tutor asks for other configurations.

Their discourse keeps being strongly related to the action on the mobile, offering examples to have the mobile dangling to the right or to the left. In line 11 the tutor asks for a generalization of the mobile's behaviour according to the value of the white triangle. Hugo sums up the possible situations, always referring exclusively at the mobile and its positions. In this task situations students' discourse is about the mobile and uses features only from the DEO-discourse. Even if it is not formalized, they implicitly use P_0 to study possible configurations of the mobile. Their discourse is guided by such principle that support the construction of meaningful narratives with respect to the mobile.

Excerpt 2

The two students are dealing with task situation 29, where they have the DEO in Figure 7.9 and they have to discover the value of the circle to have the mobile balanced.

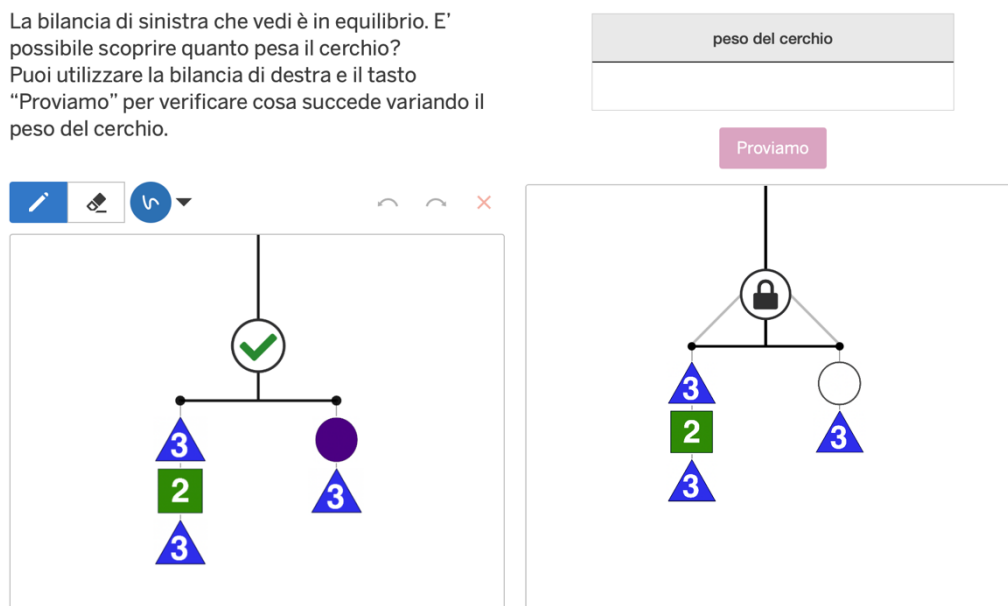


Figure 7.9: DEO of task situation 29

As soon as Andrea and Hugo see the mobile, they affirm “Five!”. The tutor asks for clarification of their reasoning. Andrea and Hugo seem to offer two different justifications for their answer, and the tutor decides to focus on Hugo's one (line 19, line 21). Indeed, Hugo is focused on the possibility of ‘ignoring’ some of the weights on the arms (one triangle of value 3 for each arm) to discover the not known value (lines 15 and 20). Thus, the tutor with his intervention aims at discussing this strategy, that, according to our analysis of the transplantability, is a transplantable procedure in the E-discourse to solve linear equation.

- 13 H: Because here I know that one is worth 3 [he points to the left arm of the mobile], so you just add up, this square with the 3 to find out how much a circle weighs [he points to the left arm of the mobile making circles around the above triangle and the square].
- 14 T: Wait, I didn't really get this.
- 15 H: So, because... here on one side, there's both 3 kilos [he points to the left arm of the mobile] and 3 kilos [he points to the right arm of the mobile].
- 16 A: Wait, do the calculation here on the left side, which is 8.
- 17 T: Ok...
- 18 A: Do you have the weight written... 3, 5+3 is 8.
- 19 T: But maybe Hugo was saying something slightly different.
- 20 H: For example, here, if you take the same thing from both sides, it's like removing these 3 kilos [he points to the left and to the right arms of the mobile], and what remains is summed up to find how much a circle is worth.
- 21 T: Ok, maybe I'm starting to understand. Andrea, did you follow what Hugo said?
- 22 A: Yes.
- 23 T: Try to explain it to me as well.
- 24 A: You can remove one from here [he points to the left arm of the mobile], a triangle from here [he points to the right arm of the mobile], and then add these two together [he points to the left arm of the mobile] to find...
- 25 H: ...you find how much a circle is worth.
- 26 T: Ok, Ok, wait, because here it looks like you can write over this, it's a drawing, right?
- 27 A: Yes.
- 28 T: Try to show it to me.
- 29 H: Like, if you remove both this [he produces the visual mediator in Figure 7.10a]...
- 30 T: Yes.
- 31 H: ...and you remove this [he produces the visual mediator in Figure 7.10b], then you're left with just a circle [he produces the visual mediator in Figure 7.10c].
- 32 A: Only a circle.
- 33 H: So, you know that this whole part [he produces the visual mediator in Figure 7.10d] is equal to a circle.
- 34 A: Makes the circle.

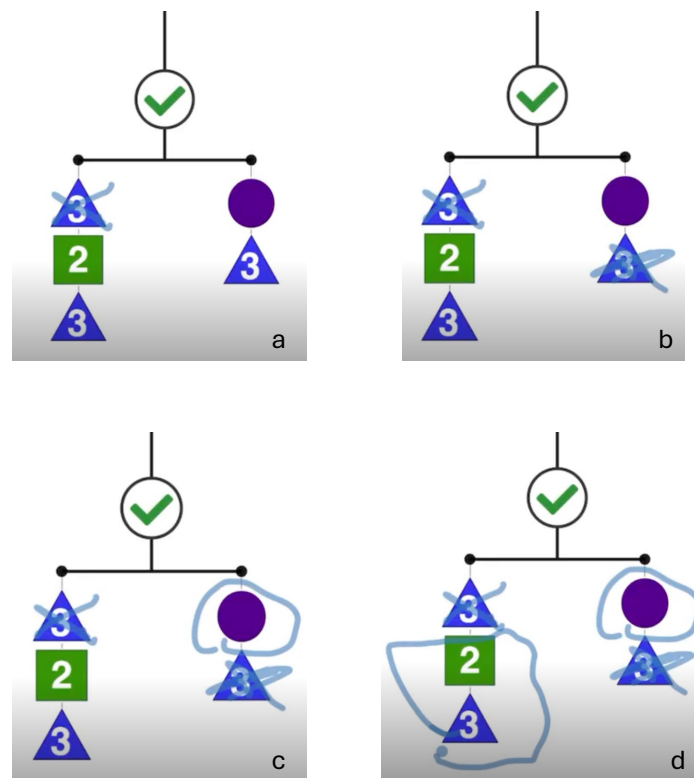
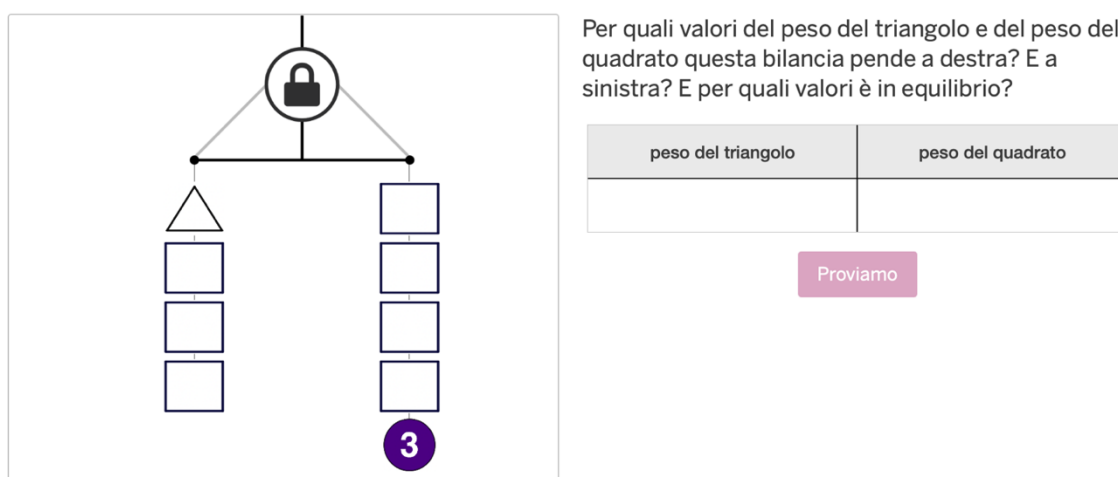


Figure 7.10: Visual mediators produced by Hugo within task situation 29

What it is worth to notice in this excerpt, is Hugo's discourse. For one thing, he quickly proposes his considerations about the two objects of value 3 to discover the value of the circle (line 20), producing endorsed narratives for the DEO-discourse. Moreover, once the tutor proposes him to write, he produces several visual mediators to support his discourse. Differently from the visual mediators produced within the initial interview, each of these visual mediators (Figure 7.10) is produced to support Hugo's explanations of his strategy, developing meaningful narratives. These visual mediators can be transplanted in E-discourse in the procedure of resolution of linear equations. Andrea follows his mate. Indeed, at first, he develops narratives (line 24) to explain to the tutor Hugo's considerations. Then, he 'completes' his mate's words (lines 32 and 34) according to his visual mediators. In line 32 he affirms "only a circle" referring to what is 'left' on the right arm, while in line 34 he says, "makes the circle", underling that the value of the two objects left on the left arm are equal to the value of the circle. As in the previous task situation, Andrea and Hugo's discourse keeps being strongly related to the DEO. They use the terms triangle/s and circle to refers to quantities (known and not known) and Hugo talks about 'kilos' to refer to some objects. Finally, the verb to remove (lines 20, 24, 29 and 31) echoes previous task situations with the two-pans-scale and the action of lifting some objects from the pans of the scale. In describing the action of

removing, they use P_1 , even if only Hugo in line 20 mentions the fact that they “take the same thing from both sides”. In the other lines, they refer to the action of removing but not to the choice of removing things with the same value. Moreover, in their narrative is only implicit that ‘what is left’ on the two sides of the mobile must be equal because from a situation of balanced mobile they have removed the same quantity from both sides. Despite this, their discourse presents several features that are transplantable in E-discourse and that re-invested in the resolution of equation could prompt for an explorative participation to the mathematical discourse.

Excerpt 3



Per quali valori del peso del triangolo e del peso del quadrato questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

peso del triangolo	peso del quadrato

Proviamo

Figure 7.11: DEO of task situation 36

In this excerpt, Andrea and Hugo face one of the mobiles with two not known values (Figure 7.11).

- 35 A: But also, with the 10 and the triangle 13.
- 36 T: Let's take a look.
- 37 H: That's what I was doing too, just now...
- 38 A: Because the triangle must have, let's say, a weight...
- 39 H: Greater [he inserts the value 13 for the triangle and the value 10 for the square in the two input fields].
- 40 A: Greater, meaning, like, the square is 5, it [the triangle] must be 8, plus 3 kilos, let's say.
- 41 T: Ok, so we're saying the triangle...
- 42 H: The triangle should be a square plus 3
- 43 T: The triangle should be a square plus 3
- 44 A: Plus 3.

45 T: Wait, let me say it again: the triangle must be a square plus 3. Ok.

46 H: Like, if I put 11 here [he inserts the value 11 for the square in the input field]

47 A: Then here it must be 14.

48 H: Here it must be 14 because it has to be a triangle [he inserts the value 14 for the triangle in the input field], it has to be a square plus three.

49 T: Ok, why do you think we could see this from this mobile? That, in fact, a square—sorry, a triangle—could be a square plus three?

50 H: Because it's as if this circle [he points to the circle on the right arm] were worth as much as a triangle.

51 T: Which one, sorry?

52 H: If the circle [he points to the circle on the right arm], which would be 3, the triangle was worth as much as the same value...

53 A: No, otherwise it wouldn't work.

54 T: Why wouldn't it work? Andrea...

55 A: It wouldn't work because here there are four squares [he points to the squares on the right arm]. And if it were three, like I did earlier with 5, here it would be 15, 18, and here 23 [Hugo inserts the value 104 for the square in the input field].

56 T: Hmm... that makes sense. Do you get it, Hugo? [Hugo inserts the value 107 for the square in the input field]

57 H: Yes.

58 T: What were you trying now?

59 H: 104, 107.

60 T: Ok, so this confirms what we were saying before, right? That the triangle is a square plus 3. Perfect. Now, going back to the question I asked earlier, without trying too many different values again, how can we see that the triangle can be a square plus 3?

61 H: Because it's as if...

62 A: Like maybe by removing...

63 H: If you remove three triangles [he points to the right arm of the scale]...

64 A: All the squares [he points to the right arm of the scale].

65 H: Three squares here [he points to the right arm of the scale], you can see that a triangle is a square plus 3.

66 T: Aldo, what were you saying?

- 67 A: Like, by removing three squares from here [he points to the right arm of the scale] and three from there [he points to the left arm of the scale]... [Hugo starts writing on the tablet]
- 68 T: What's left?
- 69 A: What's left is this plus three [he points to the right arm of the scale].
- 70 T: Which one, sorry? I didn't see.
- 71 A: Like, by removing three squares from here [he points to the scale], what's left here is just the triangle [he points to the left arm of the scale], and on the other side remains... [he points to the right arm of the scale]
- [from line 67 to line 71 Hugo has produced the visual mediators in Figure 7.12a]
- 72 H: That's what I was saying—if you remove 3 squares from both sides [he produces the visual mediator in Figure 7.12b], you're left with a triangle [he produces the visual mediator in Figure 7.12c], and you know that the triangle is a square plus three [he produces the visual mediator in Figure 7.12d].

They propose some combinations of values to have the mobile balanced. In proposing one of this combinations, Andrea and Hugo start making some considerations about the relation between the value of the triangle and the value of the square (lines 38 and 40). In line 42 Hugo compresses his discourse, encapsulating in the words 'triangle' and 'square' all the possible pairs of numbers that would keep the mobile balanced, according to the relation they have identified. The tutor asks for a justification of this observation (lines 49 and 60). The two students refer again to the action of removing (lines 62, 63, 67, 71 and 72) objects from the two arms to highlight the relation between the triangle and the square. Andrea refers to the mobile (lines 67, 69, 71) and explicates that considering the action of removing three squares from each arm, are left a square and a circle of value 3 on one side and the triangle on the other side. In the meantime, Hugo produces some visual mediators (Figure 7.12) and uses them to support his discourse (line 72), accompanying each part of his narrative with an enrichment of the reproduction of the mobile. What is worth noting is that when the tutor first asks about the motivation for the relationship between the triangle and the square, Hugo provides a narrative that is not coherent with what they were saying or with the configuration of the mobile (lines 50 and 52). Andrea then intervenes, emphasizing that if things were as Hugo described, it wouldn't work. Andrea's participation, that I stress in this excerpt but that characterizes several task situations, is different from the one we have seen during the initial interview. Indeed, he participates to the discourse, offering his proposals and reflections, and correcting his mate when he says something incorrect, offering examples to support his intervention. I stress again the strong attachment of the two students to the DEO-discourse: the use of

the verb to remove, the reference to the objects with the term ‘triangle’, ‘square’, ‘circle’, the production of mediators only from the DEO-discourse, show us that Andrea and Hugo keep being anchored to the world of the DEO. Their DEO-discourse expands and is enriched by new words and mediators, and at the same time finds way to be compressed (line 42). We can see that in the DEO-discourse Andrea and Hugo have developed some seeds that can be useful in the process of transplanting. Indeed, the way in which they use the words ‘triangle’ and ‘square’ is the same use that an expert would do of the words ‘x’ and ‘y’, as well as the visual mediators produced referring to the scale could be used to develop a discourse about algebraic relations and quantities. Related to this, we can notice instances of analytic discourse. Andrea and Hugo combine visual mediators (Figure 7.12) with narratives involving indeterminate quantities. For example, in lines 64 and 65 they propose to remove three squares from both arms of the mobile, and, later, Hugo produces visual mediators with respect to this imagined action (Figure 7.12). These narratives and mediators have been involved in their discourse about known quantities within the previous task situations. In this excerpt such features are used referred to indeterminate quantities, that they treat as if they were known. In addition, in Andrea and Hugo’s discourse the words ‘triangle’ and ‘square’ are used as two specific numbers, and at the same time are involved to realize all the possible numbers for the values of the two objects. Moreover, they implicitly use P_1 . In this case to conclude his narrative, Hugo affirms “you know that the triangle is a square plus three” (line 72). The expression ‘you known’ could have been a starting point towards the generalization of P_1 .

In Andrea and Hugo’s sessions, the process of transplanting has not been activated yet, indeed they do not involve any feature of the E-discourse. We consider that with more time at disposal and a longer teaching intervention, they could shift towards the E-discourse, being involved in a transplanting process. Considering their DEO-discourse, along their session involving mobiles, we have highlighted instances of analytic discourse. Moreover, within task situation 28 they implicitly use P_0 (lines 10 and 12): such principle supports their narratives in explicating the behaviour of the mobile according to the value inserted in the input field. Their discourses during the initial interviews were characterized by ritualistic manipulations (Figure 7.8) and by several “I do not know” narratives. Within the excerpts here discussed, Andrea and Hugo’s discourse is rich of explorations that aim at the production of endorsable narratives. From the first excerpt in which they use P_0 , they arrive to the third excerpt in which their discourse is enriched using P_1 and developing narratives that, once transplanted in E-discourse, can be endorsed.

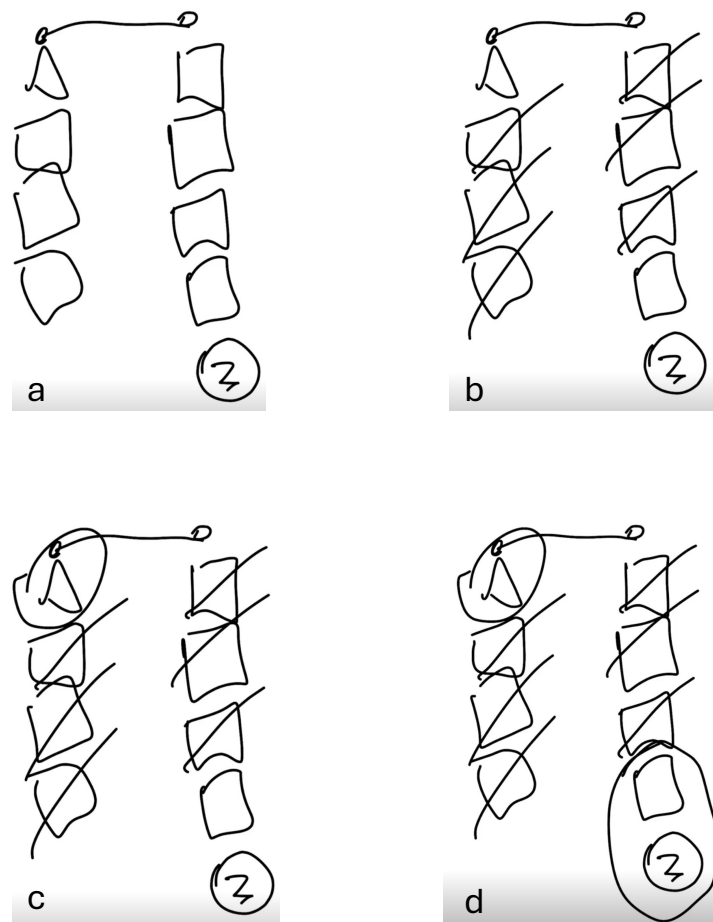


Figure 7.12: Visual mediators produced by Hugo in task situation 36

7.3. The case of Grazia and Lucia

Grazia and Lucia are two students from a classics high school of Pistoia. They have attended five sessions of the experimentation together.

7.3.1. The case of Grazia and Lucia: the interview of Lucia

Lucia is facing the task situation involving the equation $13 - a = 13 + 11$.

- 1 I: So, I ask you if you have ever seen written like this, what it is for you, what comes to your mind, thirteen minus a equals thirteen plus eleven?
- 2 L: So, since there is an equal sign, surely it is an equality, so this means that thirteen minus a has the same value, we can say, as thirteen plus eleven, in some ways.
- 3 I: Ok. Is it something that you have seen written like this?
- 4 L: I don't think so... probably yes, but I do not think so.

5 I: What I ask you, is if this thing has any solutions and I ask you if you want to try to look for them or if you prefer skip this and go to the next one.

[13 seconds of silence]

6 L: I do not know, it does not come to my mind how to solve it

7 I: Ok, so you are thinking if it comes to your mind how to solve it?

8 L: I am thinking if there is a solution but...

[7 seconds of silence]

9 L: I do not know.

[10 seconds of silence]

10 L: I do not know.

In this short excerpt we can observe that Lucia recognizes the relation of equality among the two expressions proposed (line 2) but she affirms that she does not think that she has seen something like that before. Moreover, once she is asked if the equation has any solutions and if she can find them, she gets stuck. There are several moments of silence in which she seems thinking about the request, but she does not reach a proposal. She says, “I do not know, it does not come to my mind how to solve it”. Lucia decides to skip the request and goes to the next one.

When Lucia faces the equation $3 + a = a + 3$, she affirms that it is an equality, and that the two sides are ‘the same thing’. But again, when she is asked about the solution, she gets stuck in a long silence. When the tutor asks her about her thinking, she says, “I am thinking about possible solutions but...” and again a long silence comes. Prompted by the interviewer, she affirms “But maybe there are no solutions, it is simply an equality, there are no calculations or operations to do, so I do not think there is a solution” and goes to the next task situation. Lucia thus affirms that since there are no “calculations or operations to do” there is no solution. In her discourse, the manipulations performed on an equation seem to correspond to the solution of such equation.

Lucia, also in other task situations, seems hesitant. In front of equations, she puts the light on the relation of equality and gets stuck when she is asked about the solution of the equation. Considering her narrative when she is facing the equation $3 + a = a + 3$, we can infer that for Lucia the solution of an equation coincides with calculations or operations to be performed on the expression she has in front of her. This aspect relates to the role of memory, that is central for Lucia: it does not come to her mind how to solve the equation, she decides to skip it.

7.3.2. The case of Grazia and Lucia: the interview of Grazia

As her mate Lucia, Grazia is in front of the equation $13 - a = 13 + 11$.

- 1 I: I ask you if you have ever seen written like this, what it is for you, what comes to your mind, thirteen minus a equals thirteen plus eleven?
- 2 G: So, it is a proportion, and I try to solve it.
- 3 I: Ok, if you need you can write...
- [15 seconds of silence]
- 4 I: How do you do to solve it? While you are thinking...
- 5 G: Ok maybe I would try to do, eleven multiplied by thirteen, divided by thirteen... [she takes the calculator and makes some calculations] but it is not a proportion, because there is an addition, that is, there are an addition and a subtraction...
- 6 I: Ok, and this comes to your mind according to what you have seen or...?
- 7 G: That... If a is, according to my calculations, eleven, the equality is not true. Because thirteen minus eleven makes two and thirteen plus eleven makes twenty-four... so it cannot add up.
- 8 I: And what else comes to your mind?
- 9 G: So... [she makes some calculations on the calculator] I have to find a number that subtracted from thirteen gives me as a result twenty-four.
- 10 I: In the meantime, I show you the request written... You have already gone on... Does it have any solutions? Can you find them or explain why there are no solutions?
- 11 G: Maybe there is no solution... Because if I subtract whatever number to thirteen, I cannot arrive to twenty-four, if anything at minus twenty-four.

Differently from Lucia, she seems confident and affirms “So, it is a proportion, and I try to solve it” (line 2). Then she starts thinking and after seconds of silence the tutor asks her about process of resolution. Grazia’s discourse is mainly ritualistic: she finds herself in front of an inscription and performs a ritual. She proposes the typical way taught in school for solving proportion and, once she has done some calculations on the calculator, she affirms that it is not a proportion because there are an addition and a subtraction (line 5) and, moreover, substituting the term a with the value she has found (eleven) the equality would not be true (line 7). In considering what she has found (line 9), Grazia’s discourse is coherent with the one of an expert, but then she is blocked by the fact that, for her, whatever number subtracted to thirteen would never give twenty-four. Thus, she thinks that there is no solution.

Both Grazia and Lucia's discourse involve the relation of equality between the algebraic expressions are shown to them. Nevertheless, when they are asked to find the solution, their discourses are suspended and become hesitant. On one hand, Lucia affirms many times "I do not know" (line 9). On the other hand, Grazia at first associates the visual mediators proposed to proportion, then affirms that since there is that minus sign, the equation cannot have any solution.

7.3.3. The case of Grazia and Lucia: the first session

Excerpt 1

After the sequence of task situations involving only objects with known values, in this excerpt Grazia and Lucia are dealing with the first task situation with TPS-DEO in which appear weight with not known value (task situation 5, Figure 7.2).

- 1 L: So, the scale is still in balanced, just like before. So, we are given a balanced scale to which we have to add three objects in a way that it remains balanced afterwards. And in addition, there are not only numbers anymore, but there are also little balls.
- 2 T: Ok, so there are little balls. There aren't just little squares with numbers inside anymore. Ok. How can we interpret these little balls? What meaning do you give them, if you assign them any meaning?
- 3 G: Like, they're like little weights, but without a number on them...
- 4 L: They all have the same weight.
- 5 T: They all have the same weight. I confirm. They are weights without the weight written on them.
- 6 G: There's no number.
- 7 T: OK. Alright, so we can say that they are objects that all have the same weight, but we don't know...
- 8 G: ...their weight.
- 9 T: Alright, OK. So... three objects. What do we do? Lucia, I see you're thinking.
- 10 L: So, um, the sum of... of these [she points to the left pan] is 10, without counting the ball. But here there's a 3 [she points to the right pan]. And the weight has to be the same, so surely the little balls are not worth 1. I mean, they must have a different weight, because the scale is balanced
- 11 T: Ok, the scale is balanced, because we are in a situation where the two sides are at the same, let's say, level. OK. If you want to help yourselves with this [she points to the other tablet, with the blank page]... I don't know.
- 12 G: Uh, yes.

- 13 T: What would you like to write, Grazia?
- 14 G: I don't know, like, 4 plus the x, for example [she writes the visual mediators in Figure 7.13, left side of the equation]. Like, interpreting the little ball as x, so as an unknown [she points to the two pans of the scale], plus 5 plus 1, which is equal to 3 plus x plus x [she writes the visual mediators in Figure 7.13].
- 15 T: Ok, Lucia, do you want to try explaining to me what Grazia wrote? [she moves the tablet with the visual mediator nearer Lucia] That way, we can check if you both understood each other—if what Grazia wrote is what you understood.
- 16 L: So, she rewrote the... the scales, I mean she related the two sides of the scale by adding up the numbers but indicating the little ball with an x because we don't know its value, so it's an unknown. And she did the same on both sides.
- 17 T: Alright, OK. So, we have indicated the little ball with x. Can we say that we have indicated...
- 18 L: The weight of the little ball.
- 19 G: Uh, yes.
- 20 T: The weight of the little ball with x, OK, which we called an unknown. Why did we call it an unknown?
- 21 G: Because we don't know its weight.
- 22 T: Ok, because we don't know its value. Alright. And so, what have we put here? Why did we put an equal sign here in the middle?
- 23 G: Because the scale is balanced, so this side of the scale [she points to the left pan of the scale] weighs the same as the right side of the scale [she points to the right pan of the scale], and...
- 24 T: Ok. Alright. Now I ask you: does this equation help us in some way to satisfy our request? How does it help us? Let me read the request again... If possible, add three objects from those available so that the scale is balanced afterwards. If it is not possible, explain why.
- 25 G: By placing...
- 26 T: Anyway, you have objects available... We've done this adding process with previous scales too, right?
- 27 L: Like, taking advantage of the fact that it's already balanced, even though we don't know the weight of the little ball, we can add three objects, like, for example...
- 28 G: The 7... oh, also...
- 29 L: 4 on this side [she takes the object of value 4 on the left pan] and 2 and 2 on the other side [she takes the objects of value 2 on the right pan]. This way, it remains balanced even if we don't know the weight of the little ball. We've added the same quantity.

$$4 + x + 5 + 1 = 3 + x + x$$

Figure 7.13: Visual mediators produced by Grazia in task situation 5

Lucia describes the situation stressing the presence of the little balls differently from the previous situations (line 1). The first part of the excerpt is devoted to the description of the red balls, focusing on the fact that they have all the same value, but the students do not know what it is. After that, Lucia's discourse shifts to the need of finding the value of the red balls (line 10), once the tutor asks them to satisfy the request of adding three objects (line 9). She considers the known numbers and, according to them she makes reflections about the possible value of the red balls. Adding up the known values of each pan, she affirms "so surely the little balls are not worth 1. I mean, they must have a different weight, because the scale is balanced" (line 10). She thus considers the balanced situation to exclude value for the unknown. In this line Lucia uses the relation of equality between the collections of objects on the two pans: this can be a seed towards the analytic discourse. Since the two students seem hesitant, the tutor intervenes suggesting that they can write on the blank page of the tablet, if they need it (line 11). Grazia immediately says "Uh, yes" and produces the visual mediators in Figure 7.13. The discourse of Grazia and Lucia in lines 14, 16, 20 and 21 is an example of bridging discourse. Indeed, Grazia in line 14 says "interpreting the little ball as x, so an unknown" explicates a first bridge between DEO-discourse and E-discourse. She transplants the words 'little ball' in the E-discourse and "interprets" them with the word 'unknown' and the visual mediator 'x'. Lucia describes what her mate has done as "rewriting the scale". These words Lucia uses are an instance of bridging discourse: the visual mediators Grazia has used are a way to rewrite the scale, they are related to some of the visual mediators of the DEO (weights, red balls, pans, balanced scale). Moreover, she adds that Grazia has "related the two sides of the scale" (line 16). The tutor focuses on the use of the equal sign in the middle of the inscriptions, and Grazia again produces a bridging discourse relating the situation of balanced scale with her use of the equal sign, underlying that the two sides must have the same value (line 23). Thus, the intervention of the tutor about the use of the equal sign in an inscription typical of the E-discourse that relates two algebraic expressions, moves to the reflection on the relation of the use of it with the position of the scale, thus a relation between the elements of two discourses. In the end of the excerpt the tutor goes back to the initial request and Lucia (lines 27 and 29) proposes a combination of weights to be removed, stressing that even if they do not know the value of the little balls, they can act anyway. In the discourse of Lucia, the little red balls are treated as if they were known, "taking advantage of the fact that it's already

balanced, even though we don't know the weight of the little ball, we can add three objects" (line 27), producing a narrative that is an instance of analytic discourse. Indeed, she affirms that they can act to satisfy the task situation, even if they do not know the value of the red balls but considering them as known values and exploiting the balanced position of the scale.

Excerpt 2

Grazia and Lucia are dealing with task situation 30 in Figure 7.14. As the task situation 28 described in chapter 5, the two students have to find out the value of the square that makes the mobile balanced.

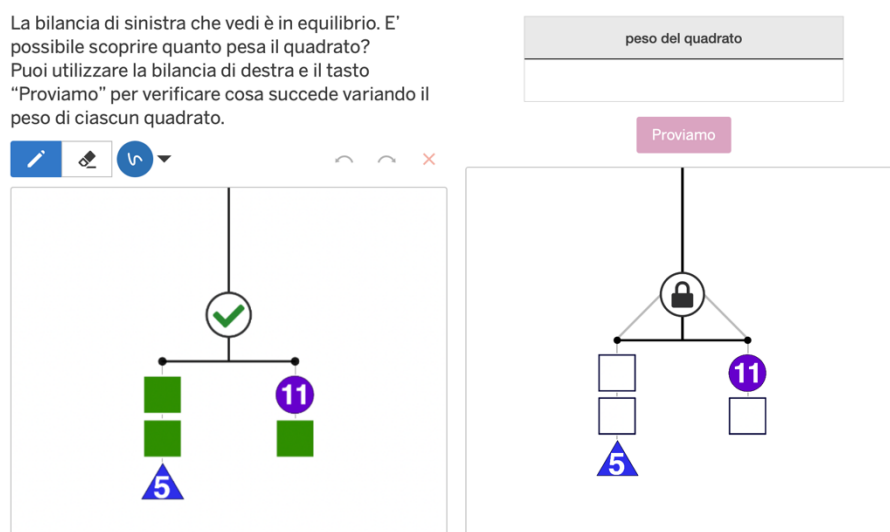


Figure 7.14: Task situation 30

- 30 L: Certainly, the sum of 11 plus the square and the sum of 5 plus the two squares has to be equal... I mean, they must have the same value.
- 31 T: Ok. Let's say, because here it tells us that...
- 32 L: ...the scale is balanced.
- 33 T: It's balanced, ok.
- 34 G: We can do what those students did before that you told us about, like simplify [she takes the pen and makes a cancellation mark on the first square on the left arm and a cancellation mark on the square on the right arm]. See the visual mediators in Figure 7.15a], and in this case, I know that a square plus 5—that is, the square plus the triangle, which has a value of 5—results in 11. So, I subtract 5 from 11 and I can find the value of... I mean, I can find how much the square weighs.
- 35 T: Ok, so you said, "I'm going to simplify," right? Referring to the action we did earlier? On the scales, what did you do here?
- 36 G: I removed the same weight from both sides of the scale.

- 37 T: Ok, great. And would you be able to write that down for me—could we, could you... How would you describe the initial situation, the one before...
- 38 G: This one—the one before simplifying? [she points to the fixed mobile]
- 39 T: Yes...
- 40 G: Like this [she takes the pen and writes the visual mediators in Figure 7.15b]
- [...]
- 41 T: Perfect, very good. So, when we then go and do this kind of... we go and remove the same weight from both sides of the scale—the same weight that we don't know, but we know it's the same—then we get to a scale that, how would you write it in symbolic form? You can turn the page if you prefer.
- 42 G: In this way... x plus 5 equals 11 [she writes the visual mediators in Figure 7.15c]
- 43 T: Does that make sense to you, Lucia? Do you want to try saying it again? I don't know if you want to write it in a different way...
- 44 L: She indicated with two x —well, with x —the weight of the square, and since the square appears twice, she wrote x twice, plus 5, and that represented the first pan of the scale. Then she put the equals sign because the scale is balanced and added 11 on the other side, which is the same weight x plus 5. Then when she simplified it, she removed one x from each side, and what she had left was that something—some weight—plus 5 equals 11.
- 45 T: Ok, when she simplified it, how would we say that in terms of the scale?
- 46 L: She removed... she removed weights.
- 47 T: Ok, did she remove random weights?
- 48 L: No, the weights of the square.
- 49 T: Ok, she removed the same weight...
- 50 L: From both sides...

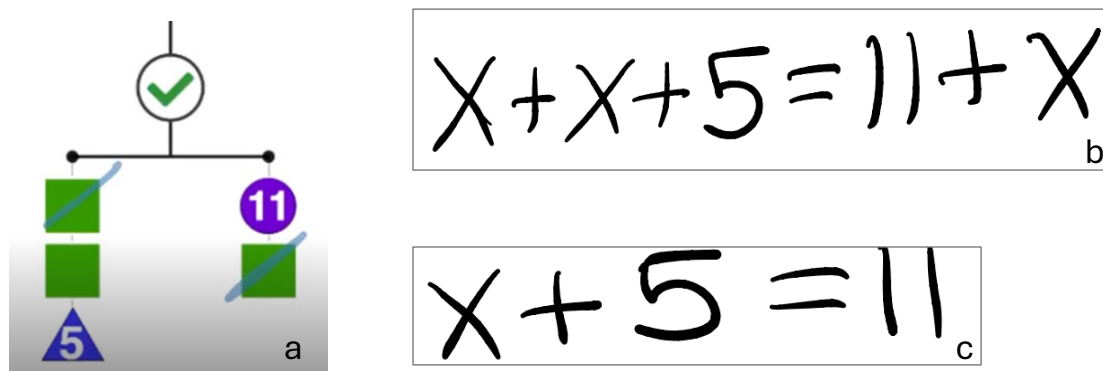


Figure 7.15: Visual mediators produced by Grazia in task situation 30

They observe that, differently from the previous mobiles, in this case the ‘unknown’ is present on both arms. Lucia observes that considering the balanced mobile “the sum of 11 plus the square and the sum of 5 plus the two squares has to be equal” (line 30). Grazia recalls a tutor’s previous suggestion about the possibility of cancelling some objects on the fixed mobile (line 34) and talks about ‘simplifying’ when she produces the visual mediators in Figure 7.15a. She uses a word together with visual mediators that we see belonging to the E-discourse in describing action on the mobile, producing a $d_{DEO}(DEO, E)$. The tutor grasps this to promote a bridging discourse. Indeed, she asks how this ‘simplifying’ is referred to the previous scale and related actions (line 35). Grazia relates this simplification with the action of removing the same values from both sides of the scale (line 36). In line 37 the tutor again intervenes to promote the intertwining of discourse, taking into account the previous visual mediators produced by the students. Grazia writes as in Figure 7.15b and as in Figure 7.15c, producing visual mediators of the E-discourse to describe the mobiles before and after the ‘simplification’. What is here missing, is the request to show the simplification also on the visual mediators Grazia has produced. The tutor involves Lucia (line 43), asking her if what Grazia has written makes sense for her. Lucia, as in the previous excerpt, produces a bridging discourse describing the relation between each mediator in Grazia’s inscription and the component of the mobile. She uses the verb ‘to simplify’ too but she explains it affirming “she removed one x from each side”: she is describing the transition from Figure 7.15b to Figure 7.15c, a description that in Grazia’s discourse was missing. In the end of the excerpt the tutor stresses the fact that the same value has been removed from both sides, since from the discourse of the two students this aspect has not emerged. In the end, it is worth noting that, even if the two students find the value for the not known object to keep the mobile balanced, they do not relate it with the equation Grazia has written and with the value of x to have an equality.

7.4. Dario, Hugo and Andrea, Grazia and Lucia: an overview

The excerpts I reported in this section are good examples to discuss what, after the first cycle of experimentation have been our considerations. Indeed, according to the Design Based Research, with the a posteriori analysis conducted after this data collection (September 2022 – December 2022) we looked for strength points as well as weak points of our design and at the same time we made emerge theoretical needs that I have discussed in chapter 4. In the following points I put the light on what these excerpts show us and how they help us in developing a new design and new theoretical tools.

- Dario, Hugo, Andrea, Grazia and Lucia, within the task situations here described, has expanded their DEO-discourse. They use new words and visual mediators (for instance, Figure 7.5a, Figure 7.10, Figure 7.12, Figure 7.13, Figure 7.15), as well as they have developed different procedure to accomplish the same task (for instance Dario in the will of finding the not known values). Specifically, they have developed parts of the subset of DEO-discourse that with the analysis of the transplantability we have identified as transplantable in E-discourse. Moreover, an aspect that emerge from all the excerpts is that students deal with indeterminate quantities as if they were known and often refer to the situation of balanced scale to underline the relation of equality between the objects on the two pans, showing instances of analytic discourse.
- The participation of Dario, Andrea and Hugo, Grazia and Lucia has slightly changed. While in the initial interviews we have identified a strong ritual participation, focused on manipulations of symbols, the need to find a calculation or a method, or the necessity to remember, within the activities with the DEO students' discourse has shown instances of de-ritualization.
 - The students have shown substantiability: differently from the task situations of the initial interview they have been able to evaluate their procedure with respect to the task situation they were accomplishing.
 - In the episode we have highlighted students' agentivity. Especially Dario, Andrea and Lucia, that during the initial interview were blocked in a collection of 'I do not know' and 'I do not remember how to do it' instances, within DEO's activities have shown a new way to participate to the discourse. For instance, Dario proposes by his own different ways to proceed in order to find the value of the not known

object. When he finds himself stuck (in front of the E-discourse visual mediators), he proposes a way out. The initial interview of Andrea has been characterized by long silence and ‘I do not know’ utterances. While interacting with the DEO, he has proposed strategies to answer task situations and has corrected his mate when needed.

- The procedure discovered within one task situation has been reinvested in different task situations, showing applicability.
- Dario, Grazia and Lucia in some episodes use features from the E-discourse to talk about the scales. These and other examples opened to two directions:
 - The activities with the DEOs prompted sometimes the use of E-discourse and to relate DEOs and equations. This can be explicit or not, it can work or not (see the case of Dario), it can be complete or not. The examples from the experimentation made us think that bridges are possible but there was the need to prompt them more, and more explicitly. Thus, we designed new task situations and new tutor interventions to foster the promotion of intertwining discourses. For example, in the third cycle, an explicit established intervention of the tutor is to write down the scale in the beginning of the task situation (for the two-pans-scale DEO). Thus, could open to the use of equations (as Grazia and Lucia did) to write the scale, as well as open to other ways of writing that could be essential in order to promote the transition from DEO-discourse to E-discourse. Moreover, we have added the task situations involving explicitly equation, in order to shift from $d_{DEO}(DEO, E)$ to $d_E(DEO, E)$. In addition, we have introduced task situations to foster the production of discourses about the relations between DEO-discourses and E-discourse to promote the overmentioned shift towards d_E .
 - The episodes such as the one with Dario and the triangles, or the others with Grazia and Lucia, put the light on the fact that specific discourses cannot always be classified as ‘mathematical discourse’ or ‘DEO discourse’, because, as stressed, they present a mixed nature. This has led to the distinction of aboutness (what is the object of the discourse) and features used in the discourse and to the classification I discussed in chapter 4. Thanks to the episodes of the first cycle indeed, we observe that students sometimes talk about the scales using features from the E-discourse and viceversa. This phenomenon seemed to us relevant from a didactical point of view as well from a designed point of view.

With respect to this, thanks to some episodes such as the one of Grazia and Lucia, we feel the need to characterize and promote what we have called bridging discourse, to discuss the relations between DEO-discourse and E-discourse but also all the limitation, highlighted with the analysis of the transplantability.

- An aspect that for us was essential during the first design phase was the interaction with the two-pans-scale DEO and the possibility to remove the same values from both pans as an action transplantable in the E-discourse and the procedure for solving linear equations. With the first cycle of experimentation, we have observed that these actions have been accomplished and developed several times: we can see examples of such interaction or imagined interaction in the third excerpt of Dario, or in the cancellation marks produced by Hugo, in the utterances and visual mediators of Grazia and Lucia. What is missing, is the explication of the principle (P_1) 'If objects that collectively have the same values are added to or removed from both pans from a balanced scale (not balanced), then the scale keeps being balanced (not balanced)'. Indeed, if we observe the excerpts of Hugo and Andrea or Grazia and Lucia, they 'remove' the same values, they produce cancellation marks to indicate that action, but they never refer to the fact that the scale is balanced after their actions. In order to transplant this and the other two principles (P_0 and P_2) in E-discourse, we decided to focus more on them within the third experimentation, prompting a formulation and a formalization of them within the first group of task situations, and involving task situations in which we have required to use them to produce justification.

To sum up, on one hand, the activity with the DEOs has shown to be effective, according to our analysis of the transplantability, to expand some parts of the subset of DEO-discourse of our interest. This has led to a new participation of students to the activities. Moreover, some uses of features of E-discourse or relations with it emerge. They need to be characterized. On the other hand, these relations do not emerge in the discourse of all the students (see for example Hugo and Andrea) and are not always explicit and used in an effective way. Thus, there is the need to develop instruments to promote a bridging discourse. Moreover, the organization of the knowledge with respect to the DEO-discourse, in particular the formulation of principles, seems necessary to promote the organization of knowledge with respect to the E-discourse (principles for equations).

8. The theoretical organization of discourses

As I have previously anticipated, in this chapter I aim at discussing some episodes based on the third experimentation, conducted at CARME center in Pistoia. In this study we do not present and analyse episodes from the second cycle of experimentation, according to research needs and objectives.

This implementation session has been conducted in Autumn 2023 and has involved 11 students from different high schools. Six of these students have attended, in pairs, the algebra course participating to 5 sessions, after the initial individual interview.

The last cycle of design-implementation-a posteriori analysis (in red in Figure 8.1) has been devoted:

- in the design session to:
 - the refinement of new task situations developed and implemented in the second session of experimentation
 - the elaboration and refinement of theoretical tools as the classification of discourses ($d_{DEO}(DEO)$, $d_{DEO}(DEO,E)$, $d_E(DEO,E)$, $d_E(E)$) and the construct of bridging discourse ($d_{R(DEO,E)}(DEO,E)$)
 - the development of the classifications for task situations
 - the development of a script for tutor's intervention
- in the experimentation session to:
 - the implementation of task situations' sequence revised in the design session
 - the implementation of tutor's intervention developed
- in the a posteriori analysis session to:
 - the examination of the impact of new task situations in students' discourse especially focusing on:
 1. production of $d_{DEO}(DEO,E)$ and $d_E(DEO,E)$, in particular the use of E-discourse's features to talk about DEOs
 2. production of $d_{R(DEO,E)}(DEO,E)$
 3. theoretical organization of DEO-discourse as well as E-discourse, with respect to scale's principles and equations' principles to guide the production of endorsable narratives
 - the implementation of theoretical tools to conduct parts of the analysis

Thus, all along this analysis, I wish to shed lights onto the three overmentioned points, observing students' discourse within the task situations and relating it to our a priori design objectives. I focus on the task situations that were most revised

between the first and third cycles of experimentation, particularly the ones with the aim of promoting the theoretical organization of discourses—namely, those involving two-pan scale DEO, silent video tasks, and tasks explicitly involving equations. Moreover, I stress the relevance of tutor’s intervention in order to reach some of our a priori didactical objectives.

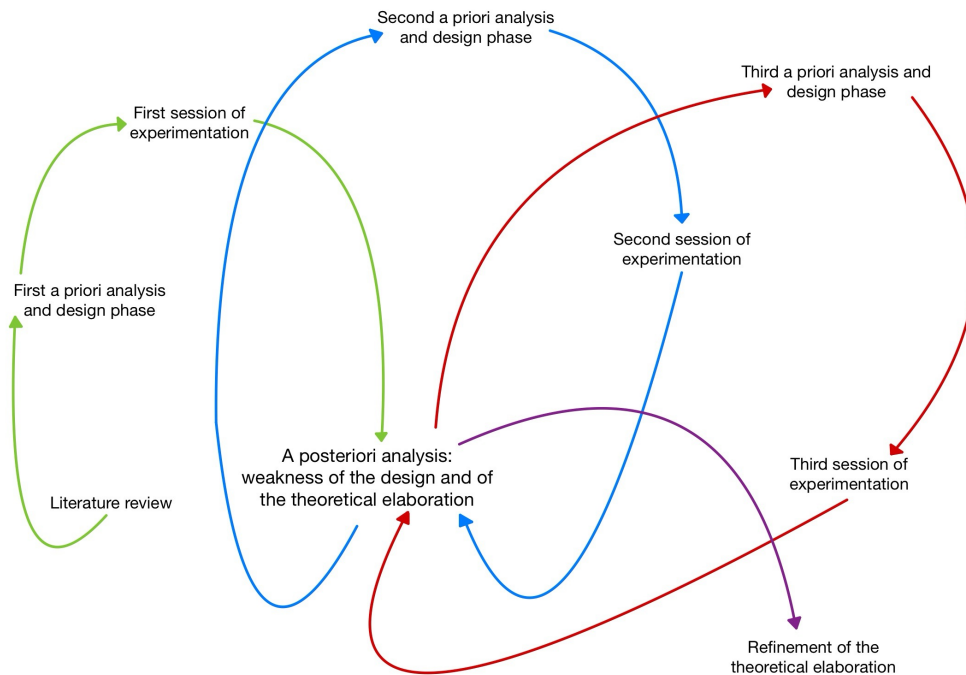


Figure 8.1: The complete development of the study according to the cycles conducted

The analysis

To conduct the analysis, as I have anticipated in chapter 6, I report some excerpts from the sessions with DEOs and mathematical task situations. The excerpts are presented considering for each line its number (in order of appearance in this text), the speaker and what is said. The enumeration of the line restart from 1 when a new session is faced. In the spoken parts we find square brackets ([]) and inside them the description of what is done within that line. I underline the line according to aboutness and features of the specific discourse. Specifically, the identification of discourses is guided by two questions: *What is the object of the specific discourse?* and *To which discourse (DEO-discourse or E-discourse, or both) belong the features used in this specific discourse?*. The discourses are underlined as follow:

- $d_{DEO}(DEO, E)$ and $d_E(DEO, E)$
- $d_E(E)$
- $d_{R(DEO, E)}(DEO, E)$

I decided to underline and thus put the light on aboutness and features only in specific parts that were crucial for the analysis. Thus, not all the lines are underlined according to the previous criteria. Moreover, I highlight in bold tutor's intervention that have been elaborated within the a priori design phase and that play a role in the development of task situations.

In this experimentation students were asked explicitly to write a general formulation of scale's and equations' principle. They have written them in Italian. In the analysis, I report their formulation in Italian (in figures) and my translation in English in the text.

8.1. The case of Aldo and Giulio

Aldo and Giulio are two classmates that have participated to the third cycle of DynaMat project's experimentation, during which they had an initial individual interview (1 hour long) and 5 sessions of 2 hours each. In the last session they had an individual interview lasted about 20 minutes.

8.1.1. Aldo's initial interview

Aldo's interview was conducted by one of the members of the research team and lasted about an hour. In the beginning he has answered the questions about his relationship with mathematics. He describes himself as a student that in the first years of elementary school was good, but, going on in the school path, he has faced his first difficulties, in particular with geometry. He affirms "there [at the end of middle school] I realized that math was not really my subject, that is, my favorite subject". Moreover, he tells that in the current school year he has done the first test about inequalities and he has got a bad mark.

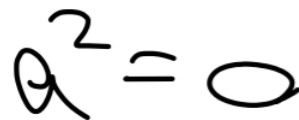
Task situation 1

$$3 + a = a + 3$$

Have you ever seen something like this? Does it have any solutions?

In front of this task situation, Aldo affirms "3 plus a equals a plus 3, that is, here, it states that 3 plus a is equal to a plus 3, thus, that is, they are equal [...] The two, the two sums". His discourse seems coherent with what an expert would say in front of this equation, but, when the interviewer asks what this says to him about the solutions, he states "I do not think it has solutions". Prompted by the tutor, he justifies his statement "Maybe because if a goes to the left. It becomes, it becomes squared a , and then the 3 goes to the right, it becomes -3 and then the one to the right gets simplified so there's no number left, so squared a equals zero wouldn't make...sense, let's say". Aldo writes, to explain better his reasoning, as in Figure

8.2. He develops manipulations that recall the usual procedure to solve linear equation, making some mistakes that lead him to Figure 8.2. For him that inscription does not make sense, so he stops. His discourse seems guided by memorized rituals that he implements without relating the obtained inscription to his initial considerations.



A handwritten mathematical expression showing the number 0 with a superscript 2, followed by an equals sign and another 0. The handwriting is somewhat informal and sketchy.

Figure 8.2: Visual mediators produced by Aldo during the initial interview

Task situation 2

$$3 + x = x + 3$$

Have you ever seen something like this? Does it have any solutions?

Aldo, as first statement, focuses again on the ‘structure’ of the two sides of the equation and he seems aware that the two sides are always equal, affirming “We have the 3 that is part of both sides, then we have x on both sides, that is unknown, that is positive too”. When asked about the possible solutions, he states “As before, thus, since they are equal, if [he writes $3 + x = x + 3$ on the notes, Figure 8.3a], since if it is moved to the left, they will have the same place, it can be simplified both of them [he makes cancellation marks Figure 8.3a] and it remains 0 [he writes $=0$ after the right side of the equation]” (see Figure 8.3a). The interviewer asks him if it thus has any solutions, and Aldo answers “No, that is, it is true...”. Prompted by the tutor to explain what he means in saying that it has no solution and that it is true the student says, “That is, this I mean you could simplify it and when on the same side such as in different parts they always have the same numbers the solution is always 0”. Thus, the tutor asks him if he can verify if he has done well and Aldo implements the change sides change sign rule obtaining the visual mediator in Figure 8.3b. Then he simplifies on each side, as in Figure 8.3c (second line) saying “and thus, it equals 0 [writing the last line in Figure 8.3c]”. In the end, when the tutor asks him again about the solution, he affirms “Thus the solution is 0”.

Figure 8.3: Visual mediators produced by Aldo during the initial interview

Task situation 3

$$13 - x = 13 + 11$$

Have you ever seen something like this? Does it have any solutions?

Aldo writes on the notes the equation $13 - x = 13 + 11$ and affirms “This does not have any solution [...] because the x is negative and thus there is not a number, if... if, even if we put a number greater than 11, or lesser than 11, the result will never be equal to the one on the right”. Within a previous task situation, proposing the same equation with a square instead of the symbol ‘ x ’, Aldo has produced quite the same narrative relating the minus sign of the x to the fact that the left side ($13 - x$), since there is a subtraction, cannot be equal to $13 + 11$.

Within these three episodes we can notice that Aldo’s discourse about the given relation (such as $13 - x = 13 + 11$) is coherent with the canonical mathematical discourse. Moreover, we observe that the task situation that requires to find the solution of an equation is familiar for him. To accomplish such tasks he implements the change sides change sign rule (even if he does not explicitly mention it) both in the first and in the second request. Nevertheless, the procedure seems to lead him distant from his initial statements about the equations $3 + a = a + 3$ and $3 + x = x + 3$. The relation of ‘always true’ equality between the two sides of the equation is described by Aldo who does not relate his statements with the output of the implemented manipulations and, in particular, with the mathematical object ‘solution of an equation’. Aldo’s discourse is mainly characterized by concrete objects and is guided by procedures which products he does not relate to his initial narratives.

8.1.2. Giulio’s initial interview

Giulio describes himself as a student that during the elementary and the middle school was good at math, he says “I was doing well. Now I have found some more difficulties because the topics are more, there is more to study and the topics are

more difficult. [...] I am more inclined to some topics and not to others. So, when I succeed, I have a good relationship with math, when I don't succeed, I try to get them back anyway and understand them". Giulio correlates his "good relationship with math" to his success with respect to some topics. When he talks about succeeding, he relates it to the correct resolution of task situations, such as equations and inequalities, and thus about procedures to be implemented in the right way to reach the right solutions.

Task situation 1

$$13 - a = 13 + 11$$

Have you ever seen something like this? Does it have any solutions?

In front of this inscription, Giulio immediately recognizes the task situation as familiar and says "It is an equation, I have seen it before. We did it in classroom" and at the question "Does it have any solution?" his discourse involves the change sides change sign rule. Indeed, he affirms "Then to find them you have to carry all the integers on one side and the unknown on the other, so it would come 13 plus 13 minus 11 equals minus a ... and... it would come 15, but not, that is, in this case there are no solutions, that is, it is impossible in my opinion". Thus, the interviewer asks Giulio what he has thought, and he replays "Because, 13 plus 13 minus 11 equals 15 and therefore it cannot be, that is, it cannot return minus a [...] Because 13 minus, minus 15, is not equal to 13 plus 11". Giulio's discourse is strongly guided by procedures that he implements only by voice, without producing any visual mediators. The output of the implemented procedure (for him a worths 15) makes him say that the equation is impossible. His justification of this statements is coherent with the canonical mathematical discourse about the solution of an equation; indeed he considers the value of a as 15 and conclude that there cannot be a relation of equality between "13 minus, minus 15 and 13 plus 11".

Aldo and Giulio's discourses in the accomplishment of task situations are mainly guided by procedures and manipulations for them familiar because learned at school. We can notice, from the excerpts discussed, that both students deal with the equal sign coherently with the canonical mathematical discourse. Indeed, for example, in task situation 3, Aldo affirms "[...] even if we put a number greater than 11, or lesser than 11, the result will never be equal to the one on the right" showing that for him there has to be a relation of equality between the two sides of the equation. At the same time, also Giulio treats the equal sign as a relational object when he reaches the number 15 as solution of the equation. Despite this, the manipulations learned at school are not put in relation with the relation of equality that must be maintained among the two algebraic expressions involved in the

equation. This led to a ritualistic participation in which the procedures cover a main role and the meaningful narratives developed by the two students about the relation of equality are left behind to give space to manipulations. Within all the interviews, Aldo and Giulio seem confident and often propose to accomplish actions such as the ones we have seen in these episodes. Their discourse is guided by learn by heart manipulations and procedure that make them feel often sure about what they are doing and saying, differently from other participants (see for example Dario or Andrea of the first cycle). This participation influences the sessions with the DEO, as I discuss in the next paragraphs.

8.1.3. Aldo and Giulio's first session

During the first session, lasted about an hour and a half, Aldo and Giulio have accomplished the activity concerning the first DEO, the two-pans-scale, both dynamic and static.

The first group of task situations aims at promoting the expansion of DEO-discourse and a possible involvement of features from the E-discourse in the specific discourses. It specifically concerns the elaboration of the “scale's principles”, I have introduced in chapter 5.

Within the first four task situations (1, 2, 3, 4) Aldo and Giulio elaborate and write on the tablet the three principles with respect to a scale with objects with known values. I now present some excerpts from the session that are particularly significant.

Excerpt 1

Task situation 1: *You have the objects in the blue box at your disposal. If possible, put some of these objects on the pans of the scale so that it is balanced. If it is not possible, explain why.* (the scale is empty, and the students have at disposal some objects with different values in a blue box)

Aldo and Giulio propose different ways to accomplish the given task situation. After some attempts, the tutor changes the task situation, asking them which is the strategy they are using to answer the request (line 1) to make explicit the principle⁵ according to which the scale is balanced. The two students formulate it (lines 2 and 4) and when the tutor asks them to write it down, Aldo starts proposing the same sentence offered by Giulio in line 2 but then interrupts himself and changes the structure of his proposal. Indeed, he formulates the sentence in the form ‘if...then’ (line 6), aligning himself to the prompt the tutor (line 3, “[...] if it's equal to the sum

⁵ Now on I use the term “principle” according to what I have introduced in the previous chapters. When this term is used explicitly, by tutor or students, during the sessions, I will underline it.

of the numbers on the other pan, what happens?"). After that, Giulio writes "If the sum of the numbers on one pan is equal to the sum of the numbers on the other pan, the scale will be balanced" (Figure 8.4).

- 1 T: OK, for example, 5 and 5 is fine. And **what strategy are you using, then, to respond to this**, let's say, to this...
- 2 G: So, the sum of the numbers on one pan must be equal to the sum of the numbers on the other pan.
- 3 T: OK, so you were saying that the sum of the numbers on one pan, if it's equal to the sum of the numbers on the other pan, right, what happens?
- 4 A: The scale is balanced.
- 5 T: I'm going to ask you to do a bit of a strange exercise, I realize this, **to try to write down exactly what you've just said**, OK, on that other tablet. So, take a moment to think about the words you'd like to use.
- 6 A: So, the sum of the addends on the two pans... If the sum of the two, of two addends, that is, two or more addends on both pans are equal, the scale will be balanced. [Giulio writes down on the blank page as in Figure 8.4]

SE LA SOMMA DEI
NUMERI PRESENTI
SU UN PIATTO E'
UGUALE ALLA
SOMMA DEI NUMERI
DELL'ALTRO PIATTO
LA BILANCIA STA
IN EQUILIBRIO.

Figure 8.4: Written formulation by Giulio of the principle to have a balanced scale (P_0)

- 7 T: OK, let's read that back for a moment. If the sum of the numbers on one pan is equal to the sum of the numbers on the other pan, the scale will be balanced. OK, that's good. It makes sense to me. **Just one more question for you, well, another question, and what happens in the case where it's just one and one?**
- 8 G: If the number on one pan is equal to the number on the other pan, the scale will be balanced.

- 9 T: Yeah, OK, and can we manage to say, whether it's one or a sum, in a single sentence to cover all these possibilities?
- [...]
- 10 G: If the sum of one or more numbers...
- 11 T: Yeah, we could say the sum of one or more numbers, it works, like, do we understand what the sum of a single number means? I mean, I'm asking if you think someone, like not one of us three because I understand what you mean, but someone reading it [points to the inscriptions produced by Giulio, Figure 8.4] from outside, do you think they would understand it, or would we need to say...
- 12 G: I think if they read it like this, they'd only understand if it's two or more numbers, not just one.

The tutor chooses to focus on the fact that there could be just one object on each pan (line 7), prompting a discussion about a possible formulation of the principle to include also this case. In line 9 the tutor tries to compress the students' discourse asking for a formulation that includes both the case of one object for each pan and the case of more than one object for each pan ("Can we manage to say, whether it's one or a sum, in a single sentence to cover all these possibilities?"). Giulio agrees with the tutor about the need to make clear both cases (line 12), because, he says, from his inscriptions an external person would think about two or more numbers and not the case with just one and arrives to the sentence "If the sum of one or more numbers on one pan is equal to the sum of one or more numbers on the other pan, the scale will be balanced", employing his previous proposal (Figure 8.5). This formulation corresponds to only one of the two implications in P_0 we have proposed in chapter 5, that is "A scale is balanced if and only if the total value of the objects on the two pans is the same". The choice of the tutor here has been to accept this proposal and, actually, the two students do never reach the missing implication. Despite this, as I point out in the following excerpts, in their discourse appears and is used the implication they do not have formulated. Thus, according to cognitive perspective (Ferrari, 2021) we could hypothesize that, with their proposal (Figure 8.5), they aim at including both the implications. Thus, Aldo and Giulio have accomplished the first step towards the development of DEO-discourse's subset that is transplantable in E-discourse. Within this task situations, Aldo and Giulio's develop $d_{DEO}(DEO)$, using words from the DEO-discourse such as pan, scale, to be balanced. In their discourse appears some words that are typical of the mathematical discourse (add up, sum, number, to be equal) but we do not identify these as $d_{DEO}(DEO, E)$ because they are mainly features of the arithmetic discourse that is not our research focus.

SE LA SOMMA DI UNO O PIU' NUMERI PRESENTI SU UN PIATTO E' UGUALE ALLA SOMMA DI UNO O PIU' NUMERI DELL' ALTRO PIATTO LA BILANCIA STA IN EQUILIBRIO.

Figure 8.5: Revised written formulation of the principle to have a balanced scale (P_0)

Excerpt 2

Task situation 2: Here you see a balanced scale. If possible, add 2 objects from those available in the box so that it is still balanced afterwards. If it is not possible, explain why. (Figure 8.6)

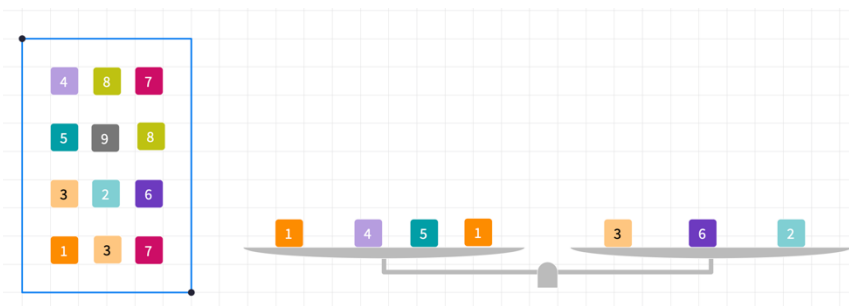


Figure 8.6: Initial scale of task situation 2

Aldo and Giulio easily propose different combinations to answer the request. The tutor, as foreseen by the script we have developed during the task situations design, says “OK, since I’ll make you return to the initial balance many times, write down the numbers that are on the pans now. This way, you can modify them as you wish and return to the initial setup” with the aim of prompting an inscription as $1 + 4 + 5 + 1 = 3 + 6 + 2$. Giulio effectively writes the numbers on the two pans listing them as in Figure 8.7. Giulio’s inscription is ‘similar’ to the form of the scale on which pans there are two collections of numbers with no symbols relating them, despite their use of the words ‘sum’ and ‘equal’ that could have led them to produce a different visual mediator to relate the objects on the pans, as the one expected by the tutor.

The image shows handwritten visual mediators. On the left, there is a sequence of characters: 'l', a semicolon, 'h', a semicolon, '5', a semicolon, and '1'. To the right of this sequence is a lowercase letter 'e'. Further to the right is another sequence: '3', a semicolon, '6', a semicolon, and '2'.

Figure 8.7: Visual mediators produced by Giulio in task situation 2 to write down the scale

After this inscription, the tutor changes the task situation, asking to add 3 objects so that the scale is balanced afterwards. After Aldo interventions on the DEO, who adds two objects weighing 8 on the left pan and an object of value 9, one of value 2 and one of value 5 on the right pan, the tutor asks him to explain his choices. Aldo, according to tutor's request, produces a first seeds for the formulation of P_1 , providing a generalization of his action: "it's like adding the same number to both pans" (line 48).

- 13 A: I made the sum of 8 and 8, and it gave me 16. Then I added up these three numbers too, and they gave me the same number. [...] So, it's like adding the same number to both pans.
- 14 T: OK. So, if we add the same number to both pans, OK, great. So, it's a bit like what I made you do before. So, **what's the general strategy you're using in this case?**
- 15 A: You keep in mind, I mean, you first add two or more numbers and keep in mind the total, the sum, and then you find numbers that, when added, give the same number of the first one, that is of the first sum.
- 16 T: OK, perfect, that's how you thought about it to solve it. OK, so **if we wanted to generalize?**
- 17 G: In general, keeping in mind the sum of the first, the sum of the numbers on the first and on the second pan, you add one or more numbers to both sides to keep the scale balanced.
- 18 T: OK, but did you keep track of all the numbers? [he points towards the two pans] The ones that were already on the first and second pan?
- 19 A: No, I mean, you don't need to know. If it says it's balanced, we just need to find numbers that have the same sum, that is the same result.
- [...]
- 20 T: Aldo was saying it doesn't matter, you just need to know it's balanced.
- 21 A: If it's balanced
- 22 T: So, we can say, if the scale is balanced
- 23 A: If the scale is balanced

- 24 G: That is, you do not need to consider the numbers that were already there.
- 25 T: Yes, exactly. So **how do you now see the rule**, or maybe it doesn't matter if we call it a rule, but **what's the general property?** If the scale is balanced...
- 26 A: If the scale is balanced.
- 27 T: So, starting from a balanced scale, what did you do?
[...]
- 28 T. So, if we want to add objects in a way that the scale is balanced afterwards
- 29 G: If the scale is balanced, you just need to add the same total to both sides of the scale.
- 30 T: OK, great. So, if the scale is balanced, you just need to add the same total to both sides.
- 31 G: To have that it remains balanced.
[...]
- 32 T: OK, I would write that down.
- 33 A: If a scale is balanced and you add two equal sums to both sides, the scale stays balanced.

In the first part of the excerpt (lines 13 – 19), when the tutor asks for the general strategy, Aldo is focused on his procedure of finding two groups of numbers that together give the same sum (line 15). On the other hand, in line 17, Giulio talks about “keeping in mind” the sum of the values of the initial weights on each pan and adding one or more numbers to both sides. In the short exchange that follows (lines 18 - 24), an essential aspect about the scale. Indeed, prompted by the tutor (line 18), Aldo and Giulio affirm that, to have a balanced scale adding some objects on the pan, it is not necessary to know the values of the initial weight of each pan, if the scale is balanced (lines 19 and 24). Then, Giulio verbally formulates the principle (lines 29 and 31). After that, solicited by the tutor (line 32, students arrive to a formulation of the principle that Aldo verbally formulates as “If a scale is balanced and you add two equal sums to both sides, the scale stays balanced” (line 33) while Giulio writes down as ‘if the scale is balanced and you add the same sum of numbers to both sides of the scale, it will stay balanced’ (Figure 8.8a). After this exchange, the tutor proposes a reflection on the situations of unbalanced scale. He asks them what happens if, starting from an unbalanced position, they add the same amounts of numbers on both sides of the scale. Giulio affirms “So, if the sum of the initial numbers on a scale’s pan is greater than on the other, if you add a sum of numbers equal to both sides of the scale, it will stay unbalanced afterwards”, while Aldo proposes to write down “So, that is, it would be like this [he points to the previous inscriptions of Giulio (Figure 8.8a)], but we can modify it, as if the scale is

not balanced and we add the same sum of numbers on both sides of the scale, it will always stay, that is, it will not be balanced.” They write down the revised principle, copying and adjusting P_1 , as ‘If the scale is not balanced and you add the same sum of numbers to both sides of the scale, it will stay not balanced’ (Figure 8.8b).

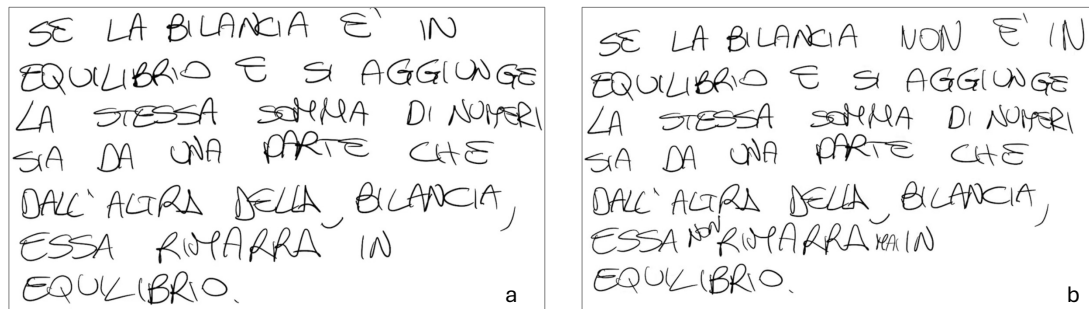


Figure 8.8: Written formulation of the principle to have a balanced scale adding objects (P_1)

In this excerpt, have emerged some aspects that from an expert point of view are crucial to construct a meaningful procedure for solving linear equations. Indeed, Aldo for instance, but we can see this also in Giulio’s discourse, affirms “If it says it's balanced, we just need to find numbers that have the same sum, that is the same result” (line 54). This relates to the endorsed narrative of the E-discourse ‘if $A=B$ and $C=D$ then $A+C=B+D$ ’ that is used in solving linear equations (where A , B , C and D are algebraic expressions). It is not necessary to know the value of the algebraic expressions, and this is crucial when we deal with indeterminate quantities, to have the above-mentioned implication. We can thus perceive an expansion of DEO-discourse to which correspond, from our point of view, an expansion of E-discourse. Indeed, these considerations offer a basis on which develop the principles and the procedures to deal with linear equations.

Excerpt 3

Task situation 3: *Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.*

After Aldo and Giulio have removed some objects from the two pans, making different attempts, the tutor asks them to bring the scale back to the initial configuration and make a note of it (line 83). Giulio writes on the blank page as in Figure 8.9.

h; 5; 7; 6; 3 e 3; 5; 8; 2; 2; 5

Figure 8.9: Visual mediators produced by Giulio to write down the scale of task situation 3

- 34 T: OK perfect, you write it down this way OK. I might, I mean, we could have noted the weights like this. Tell me if this convinces you. **I, for example, could have noted them down like this 4+5+7+6+3 and here, I would put 3+5+8+2+2+5 and I would put an equal sign** [he writes under Giulio's visual mediator as in Figure 8.10]
- 35 G: Yes
- 36 T: Does this notation make sense to you, or could it be strange?
- 37 G: Yes, because if it is balanced
- 38 A: If it is balanced, yes
- 39 G: We're okay with the equal sign in the middle
- 40 T: We're okay with the equal sign in the middle because it is balanced, and the plus signs, **what do you think they could represent in relation to the scale?**
- 41 G: The sum
- 42 A: The sum of all the weights

$$4 + 5 + 7 + 6 + 3 = 3 + 5 + 8 + 2 + 2 + 5$$

Figure 8.10: Visual mediators produced by the tutor to write down the scale in task situation 3

The tutor chooses to introduce himself new visual mediators realizing the balanced scale (Figure 8.10). This choice was planned according to the a priori design. The aim of the request to make a note of the scale was to involve visual mediators from the E-discourse in the students' discourse about scales. Thus, in the case students did not use them, we decided to propose ourselves the plus and the equal sign symbols. Aldo and Giulio relate the balanced scale with the equal sign (lines 37-39). Giulio affirms "because if it is balanced" (line 37) "We're okay with the equal sign in the middle" (line 39). Here, as I anticipated in excerpt 1, they are using the converse implication with respect to their formulation of P_0 (Figure 8.5). They accept the plus sign standing as the sum of all the values on the pans (lines 41-42). After this exchange, the tutor recalls the sentence they have written before and ask them to adjust it according to what they have done within this scale. Indeed, the request of

removing objects from the scale to keep it balanced afterwards has been designed to reconsider P_1 and reformulate it. The sentence is changed in 'if the scale is balanced and you add or subtract the same sum of numbers to both sides of the scale, it will stay balanced' (Figure 8.11).

SE LA BILANCIA E' IN
EQUILIBRIO E SI AGGIUNGE O SOTTRA
LA STESSA SOMMA DI NUMERI
SIA DA UNA PARTE CHE
DALL'ALTRA DELLA BILANCIA,
ESSA RIMARRA' IN
EQUILIBRIO.

Figure 8.11: Written formulation of the principle to have a balanced scale adding or removing objects (P_1)

Excerpt 4

Task situation 4: *Here you see a balanced scale. If possible, change only the (things on the) right pan so that it is still balanced afterwards. If it is not possible, explain why.*

$$2 + 6 + 4 + 5 = 8 + 3 + 6$$

Figure 8.12: Visual mediators produced by Giulio to write down the scale of task situation 4

Within this activity, Aldo and Giulio again propose several possibilities to accomplish the task situation. For example, Aldo says "Yes, you can remove the 8 and add 4, 3, 1" (while saying it he takes away 8 from the right pan and put on it 4, 3 and 1) while Giulio affirms "Or the 6 and 3 instead of the 9". Moreover, while Aldo proposes the overmentioned solution, Giulio writes down the scale, using the visual mediators previously introduced by the tutor (Figure 8.12). After that, the tutor asks again to write down the generalization of what they have done (line 43).

- 43 T: So, I'll have you write this one too, how could we say this?
- 44 A: So, if the sum of... If the sum is equal to the number we removed, the scale will always remain balanced.
- 45 T: So, by sum you mean that what you put equals the number you removed or earlier we also had a case where you removed two

- and put one, **so maybe we need to find a more general way to say it**. Anyway, the scale is initially balanced, right? So, I'd start by saying, if a scale is balanced, or in a balanced scale. What did you do? You modified, or you changed
- 46 G: So, in a balanced scale, if you add or subtract a quantity that is... [silence]
- 47 T: It's normal that it's hard to say, I'm purposely making you do it as an exercise.
- 48 G: I mean, I can explain it for the sum.
- 49 T: Well, let's start with the sum then.
- 50 G: So, in a balanced scale... I mean, in a balanced scale, if you take away numbers, I mean, if the sum of the numbers you take away is equal to the sum of the numbers you add, the scale will stay balanced.
- 51 T: OK, but on one side of the scale, though.
- 52 G: Yes, they're on the same side.
- 53 T: Fine, it seems like this is a very general concept. OK, let's try to state it. So, it seems I've started again, if on a balanced scale...
- 54 G: On a... [he starts writing 'On a balanced scale']
- 55 T: On a balanced scale... [Giulio writes 'If you take away a sum of numbers']
- 56 G: If you take away a sum of numbers from the same side... [he writes 'from the same side']
- 57 T: OK, so if you take away a sum of numbers, which could also mean just a single number, that's fine. So, if you take away some numbers from one side, and then?
- 58 G: On the same side, you add the same amount [writes 'and you add']
- 59 T: By "amount," do you mean sum again?
- 60 G: Yes.
- 61 T: Alright, then let's use "sum" again to make it clear [Giulio writes 'the same sum']
- 62 A: The scale will remain balanced [Giulio writes 'the scale will remain balanced']

The process of construction of what we call P_2 reveals for Aldo and Giulio more challenging than the others. In line 44, Aldo proposes "If the sum is equal to the number we removed, the scale will always remain balanced" referring to a specific case of accomplishing the task, that is substituting an object on one pan with more objects whose values added up equals the removed value. The tutor proposes to find "a more general way to say it" (line 45). Moreover, he stresses that in the beginning the scale is balanced and this could be underlined in the initial part of the

sentence. In response to this, Giulio seems stuck in finding a way to elaborate the principle (line 46). In particular, he talks about “adding or subtracting a quantity”. In line 48 he affirms “I can explain it for the sum” and in the sentence he proposes (line 50) the verb ‘to subtract’ is replaced by the verb ‘take away’. Thus, the actions on the scale are indicated as ‘taking away’ and ‘adding’ and the term ‘sum’ stands both for the possibility of considering just one object or more than one. Tutor’s interventions (lines 53, 55, 57, 59 and 61) are several but all with the same aim of clarifying aspects of the principle, such as the fact they are acting on one side of the scale, or that the term ‘sum’ could stand for more numbers but also for just one.

Within task situations from 1 to 4, Aldo and Giulio accomplish smoothly the requests offered by the tutor involving actions on the scale. Their discourse is mainly characterized by words that belong to the DEO-discourse such as pan, scale, balanced, weights, and we could label their utterances as $d_{DEO}(DEO)$. When, during the first activity the tutor asks which is their strategy, Giulio affirms “the sum of the numbers on one pan must be equal to the sum of the numbers on the other pan” (line 2). Within this sentence, Giulio uses words (‘sum’ and ‘equal’) that are typical of the mathematical discourse. Despite this, when they are asked to write down the scale to keep track of it, Giulio produces the inscription in Figure 8.7, in which no visual mediators related to the term ‘sum’ and ‘equal’ appear. When the tutor proposes his way of writing down the scale (line 34) they accept the visual mediators introduced (the symbol plus and the equal sign) and they well justify their uses relating them to the DEO, and, in the subsequent activity, Giulio uses them to support his discourse (Figure 8.12).

Moreover, we see a change in the two students’ discourse, prompted by tutor’s interventions, with respect to the way of formulating the principles. In the beginning (lines 2), Giulio explicates the need to have the same sum for both pans, but is the tutor who asks explicitly “what happens?” (line 3). Aldo then tries to formulate the principle in the form “if...then” considering the scale’s starting position. In the successive requests of elaborating verbally the general strategy, Giulio seems still more focused on the action to do, while Aldo again refers to the scale’s initial and final positions.

I recall that the main a priori objective of this first group of activities was an expansion of the DEO-discourse. According to the a priori analysis, Aldo and Giulio have expanded it, with words, visual mediators and routine, but most of all with new endorsed narratives (P_0 , P_1 , P_2). Thus, the DEO-discourse of the two students at the end of this first group is characterized by the features described in Figure 8.13, where the correspondent features of the E-discourse are presented.

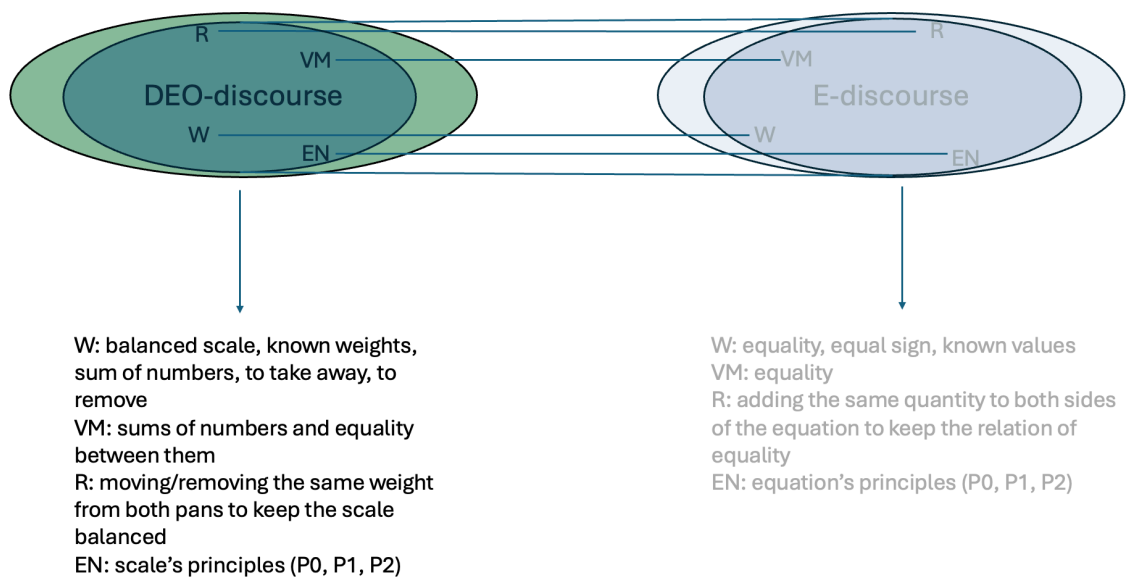


Figure 8.13: Aldo and Giulio's DEO-discourse after the first group of activity (on the left) and correspondent E-discourse identified by us according to the analysis of the transplantability (on the right)

Excerpt 5

I split the following excerpt in three parts in order to make it easier the reading. This is the first task situation in which appear objects with not known value.

Task situation 5: *Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.* (Figure 8.14)

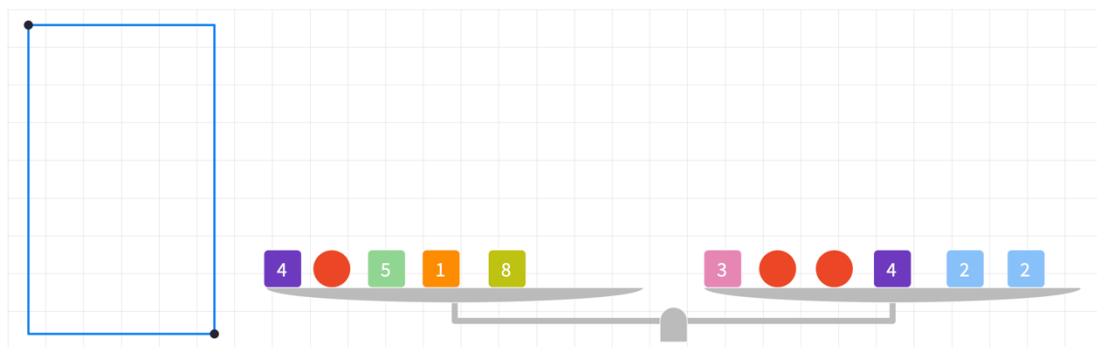


Figure 8.14: Initial scale of task situation 5

- 63 T: First of all, **how could we write down this scale?** Using the notation from before...
- 64 G: Four, plus x, plus 5... [he writes the equation $4 + x + 5 + 1 + 8 = 3 + x + x + 4 + 2 + 2$]
- 65 T: OK. What do you mean by x?

- 66 G: That is, the little ball, because it doesn't have a value.
- 67 A: It doesn't have a value.
- 68 T: OK. What do you mean by it doesn't have a value?
- 69 G: It doesn't have a value [points to the scale]... expressed as a number in this case.
- 70 T: OK, so, more than...
- 71 G: I mean, we don't know the value of the little ball.
- [...]
- 72 T: We don't know the value of the little ball, OK, that makes sense to me. But someone might say, no, it does have a value!
- [...]
- 73 G: In this case, we don't know.
- [...]
- 74 T: Why did it remind you of x?
- 75 G: Because x is an unknown, and in this case, the little ball...

After reading the task situation, the tutor asks the students to write down the scale involving the notation used before (line 63). Giulio introduces a new visual mediator, the symbol 'x', developing a first shift towards the E-discourse. The utterances from line 63 to line 75 are characterized by a discourse about the little red ball and the symbol 'x'. In answering to tutor question (line 65, "What do you mean by x?"), Giulio's narratives about the little red ball change. Indeed, in describing it, Giulio goes from the sentence "It doesn't have a value" (line 66), to "It doesn't have a value... expressed as a number in this case" and in the end to "we don't know the value of the little red ball" (lines 66, 69, 71). This last narrative is coherent with a possible expert's narrative about the mathematical object unknown. Thus, Giulio involves in his discourse a new word, 'unknown' (line 75, 'incognita' in Italian), probably borrowing it from his previous existing discourse, affirming that "x is an unknown, and in this case, the little ball...". The use that Giulio makes of the visual mediator 'x' and of the word 'unknown' is coherent with the mathematical discourse and is aligned with what we expected by asking to write down the scale and to talk about the little red ball. The tutor's intervention (lines 68, 70, 72) prompt for the explication of the fact that the little red ball has a value, but it is not actually known. With the introduction of the visual mediator 'x' and the words 'x' and 'unknown', Giulio produces $d_{DEO}(DEO, E)$ in which the object of the discourse is the little red ball, but the features used to talk about it are taken by the E-discourse (see lines 64, 66, 69, 71 double underlined).

- 76 T: For example... If we wanted to do it with two objects, how could we do it?

- 77 A: 4 and 4? [he removes a 4 from the left pan and a 4 from the right pan]
- 78 T: 4 and 4... Or? [Aldo puts the weights back on the scale]
- 79 G: Or two little balls, I mean, two unknowns [Aldo removes one red ball from the left and one from the right]
- 80 T: OK, two unknowns. Why are we sure that two unknowns would stay balanced?
- 81 G: Because they are equal [points to the tablet with the scale]
- 82 T: But someone might say, yeah, but we don't know the value!
- 83 A: Right, we don't know the value...
- 84 T: We don't know the value...
- 85 G: But visually, they are equal.
- [...]
- 86 T: I can also tell you that. The little red balls all have the same value. We don't know how much, but they have the same value. So, if I remove them from both sides?
- 87 G: It stays balanced.
- 88 T: It stays balanced [nods], does that make sense to you, Aldo?
- 89 A: Yes.
- 90 T: So, it doesn't matter how much I remove... [...] I just need to know that I removed the same thing from both sides.

Giulio, in order to accomplish the request of removing two objects from the pans to have the scale balanced afterwards, proposes to remove “two little balls, I mean, two unknowns” (line 79) (one from each pan), “because they are equal” (line 81). In the first utterance we can see that in the discourse of Giulio, the words ‘two little balls’ and ‘two unknowns’ are exchangeable, they are two different ways to talk about the same objects. I underline them as $d_{DEO}(DEO, E)$ considering that the object of the discourse are the possible combinations of weights that can be removed to have the scale balanced. The tutor too is involved in this discourse using the terms unknown while referring to the position of the scale (line 80). Moreover, Giulio’s proposal of taking away the little red balls show an instance of analytic discourse, dealing with indeterminate quantities as if they were known. On the other hand, in the beginning Aldo is not convinced by the fact that they can be removed, considering that their value is not known (line 83), while, after tutor’s intervention, he seems to agree. The moment in which the tutor asks to find out how much a little red ball weighs changes the task situation for the two students, who now have to face another request (line 91).

- 91 T: Can we find out how much the little ball weighs?

- 92 A: Mhm...
- 93 G: Yes
- [10 seconds of silence]
- 94 G: You take... I mean, you take the... [he points to the scale]
- 95 T: If you want to do it, you can also use your finger, eh, if you want, on the tablet. If you wanted to write something or move something.
- 96 G: You take... I mean, you move all of these to one side [he points to the scale].
- 97 T: The little balls?
- 98 G: Yes. So, it becomes... it becomes three x [he writes on the tablet as in Figure 8.15a] and, on this side, you sum all these numbers [points to the scale]
- 99 T: Okay. Does this make sense on the scale? What happens if you move all these to one side?
- 100 G: 4, 9, 10, 18... 18... 20 [Counting quietly]. 29 [he writes as in Figure 8.15b]. Then you do...
- 101 A&G: Divided by 3 [in chorus]
- 102 G: Divided by 3 [he writes as in Figure 8.15c]. And x equals...
- 103 A: It can't be simplified.
- 104 G: Uh, it becomes 29/3.
- 105 T: Alright, if we want to do it with the calculator... [he takes out the calculator and computes 29/3, getting 9.66667] 29 divided by 3... 9.6 recurring.

a

b

c

Figure 8.15: Visual mediators produced by Giulio in task situation 5

The tutor's request to find out how much the little ball weighs prompt the entrance of the 'change sides, change sign' rule in Aldo and Giulio's discourse. It is Giulio again who engages this new element in the session, affirming "I mean, you move all of these to one side" (line 96), pointing to the scale. He is proposing to implement the change sides, change sign rule on the scale, to find out the value of the red ball (line 96). He confirms this in line 98, saying "it becomes three x" and working on the equation he has written before (Figure 8.15). These lines show some procedures in Giulio's discourse that are not coherent with the canonical discourse. In particular he performs the change sides change sign rule without changing the signs. Aldo

follows his mate and does not underline such issues. Giulio, even if he does not accomplish the procedure on the scale, he ‘transplants’ it on the visual mediator produced in line 64 (that is, $4 + x + 5 + 1 + 8 = 3 + x + x + 4 + 2 + 2$). Giulio says “So, it becomes... it becomes three x (Figure 8.15a) and, on this side, you sum all these numbers” (line 98). On one hand, Giulio’s discourse in line 96 is about the balls (d_{DEO}), that must be moved on one side, and, immediately after, it shifts towards the inscription (d_E). The terms ‘it becomes’ (line 98) put in relation the $d_{DEO}(DEO)$ of line 96 with the $d_E(DEO,E)$ of line 98 stressing how for Giulio talking about the red balls and the x is the same, they are both the same ‘unknown’. Even if the tutor asks about the scale (line 99), trying to focus the two students’ attention on what happens if all the red balls are moved on one side of it, Aldo and Giulio’s discourse is completely devoted to the inscription $4 + x + 5 + 1 + 8 = 3 + x + x + 4 + 2 + 2$ and its evolutions (lines from 100 to 103), showing, as in the initial interview, a strong attachment to manipulations and procedures. In this short part we can say that their discourse is about the equation and is developed through E-discourse’s features, thus can be labelled as $d_E(E)$. We can observe that once they have entered in the procedure (change sides, change sign rule) they are completely involved in it, producing words and visual mediators that they are familiar with (see for instance line 101 in which in chorus they say “divided by three” and Figure 8.15) and accomplish the expected steps not relating them to the situation (the scale and its balanced position), developing a strongly ritual participation.

- 106 T: Okay, explain to me what you did.
- 107 G: I mean, we brought all the unknowns to one side and summed them up, which were three, so three x.
- 108 A: Yes
- 109 G: And all the known numbers to the other side, which were... so the sum was 29. And then, since to find x the 3 needs to go away [he points to 3x]...
- 110 A: x needs to be left alone, the unknown.
- 111 T: Okay. So, you were reasoning about this writing [points to the tablet with the equation]
- [...]
- 112 T: Okay, on that writing. Because do you remember doing it this way in school? Why did you do it like that...?
- 113 G: Yes.
- 114 A: Yes, it’s an equation.
- 115 T: Okay, do you usually solve equations this way?
- 116 A: Yes

117 G: Yes

Once the tutor asks for an explanation, changing the task situation (line 106), Giulio affirms “I mean, we brought all the unknowns to one side and summed them up, which were three, so three x ” (line 107) and “And all the known numbers to the other side, which were... so the sum was 29. And then, since to find x the 3 needs to go away...” (line 109), giving voice to the implemented procedure. Aldo echoes him, saying “ x needs to be left alone, the unknown” (line 110). The tutor’s intervention aims at understanding about what the two students are referring to (line 111). Since they are focusing on the inscriptions, the tutor explicitly asks if they remember this way of doing from school and the two students confirm it. Thus, in this excerpt, Aldo and Giulio recall a known routine from school, composed by the task of finding out the value of the unknown in an equation (line 114) and the procedure of taking all the unknowns on one side and all the numbers on the other side (line 109), to leave the unknown alone (line 110). We can label all their utterances, in this short excerpt, as $d_E(E)$: they are talking about the inscriptions, and they use features that belong to E -discourse. In this discourse, as in the one before (lines from 96 to 105), that aims at explaining what they have done, the relation between the output of the implemented procedure with their initial aim is not considered. Thus, once involved in $d_E(E)$, or better, once they start producing $d_E(E)$, they seem less devoted to the construction of endorsable narratives (as they have instead done in the previous task situations), favouring familiar rituals, learnt by heart (for instance, line 110).

118 T: Okay, fine. But what if we wanted to do it on the scale?

119 G: On the scale...

120 T: Let’s see... really reasoning it on the scale, how could I discover the weight

121 A: Mhm...

122 [8 seconds of silence]

123 G: I mean, still like this [points to the equation]. I mean, you would move [points to the scale].

124 T: Try, try [brings the tablet with the scale closer to Giulio] let’s see what happens if you try.

125 G: I mean, you would move all the numbers to one side and the balls to the other.

126 T: What happens if you move all the numbers to one side and all the balls to the other? Let’s see...

127 A: [moves some weights with known values from the left pan to the right pan, then the red ball from the right pan to the left and finally the remaining weights with known values from the left pan to the right pan, reaching the scale in Figure 8.16]

- 128 G: Then you would sum these here [points to the right pan of the scale].
- 129 T: But the scale is not balanced
- 130 A: Uh-huh.
- 131 G: Uh-huh.
- 132 [8 seconds of silence]
[...]
- 133 A: The sum is 29.
- 134 T: This makes 29 [he points to the right pan of the scale], you summed it up [he points to the equation].
- 135 G: Yes
- 136 T: Now we can recheck with the calculator if it makes 29. But it's not... I mean, if it were balanced... then I could do 29 divided by 3 and get the weight of the balls, right? [gestures with flat hands, one higher and the other lower]
[...]
- 137 T: But it's not balanced, so moving left and right on the scale doesn't keep it balanced, right?
- 138 A: Yes.
- 139 T: Think about it, like, if you have a scale like before with the numbers. If I move from right to left [gesture of moving an object from left to right], why should it stay balanced?
- 140 G: Mhm.
[...]
- 141 T: So, moving this way [points to the equation], the scale doesn't... doesn't give us the same feedback. Okay?
[...]
- 142 T: So maybe this isn't the strategy. **Try to go back to the starting point.**

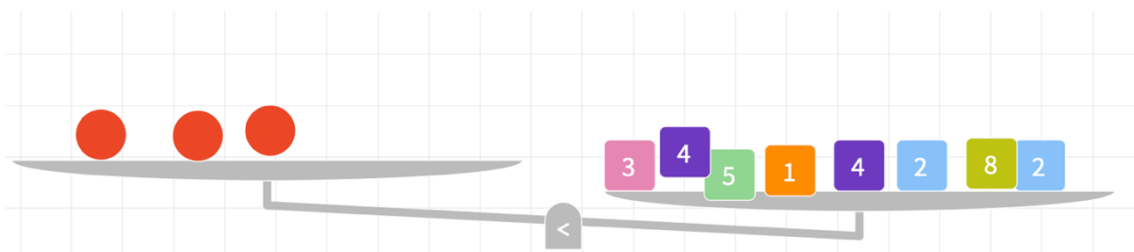


Figure 8.16: The scale obtained once Aldo have implemented the change sides change sign rule on it

Since they have applied wrongly the procedure and, mostly, it does not work on the scale (see the analysis of the transplantability of this DEO in chapter 5), the tutor insists to focus on the DEO and to find out the value through the scale (line 120), trying to take advantage of the not transplantable procedure to promote a discourse on what they have done and its justification. This new task situation causes the implementation of the previous procedure on the scale (lines 123-128). In particular, Aldo moves all the objects (line 127, Figure 8.16), and Giulio wants to sum the known values (line 128), according to what has been done on the equation. Neither of them realizes that the scale is not balanced anymore: it is the tutor who highlights it (lines 129, 136, 137), while Aldo keeps being focused on adding up the known values (line 133). The exchange from line 129 to line 142 is essentially characterized by the tutor's interventions to describe the behaviour of the scale after students' action and on the reason of this behaviour. Aldo and Giulio's answers are mainly 'yes' and 'mhm', and, after the suggestion of the tutor (line 142), they restore the objects on the two pans as in the beginning. In this excerpt we can see the application of the change sides, change sign rule to the scale, causing the development of $d_{DEO}(DEO, E)$. Also in this exchange, even if they are not dealing with algebraic symbols, once they have started replicating the procedure for them familiar, they develop a ritualistic participation not considering the consequences of their action (position of the scale) with respect to their task. Despite the tutor's intervention they do not seem convinced that what they were doing wasn't working.

After students have restored the scale to the initial configuration, the tutor recalls previous actions to keep the scale balanced, going back to the initial task situation of the activity, and they remove several objects from the two pans. Then, in line 247, the task situation consists again in finding out how much one ball weighs. Aldo and Giulio have in front of them the scale as in Figure 8.17.

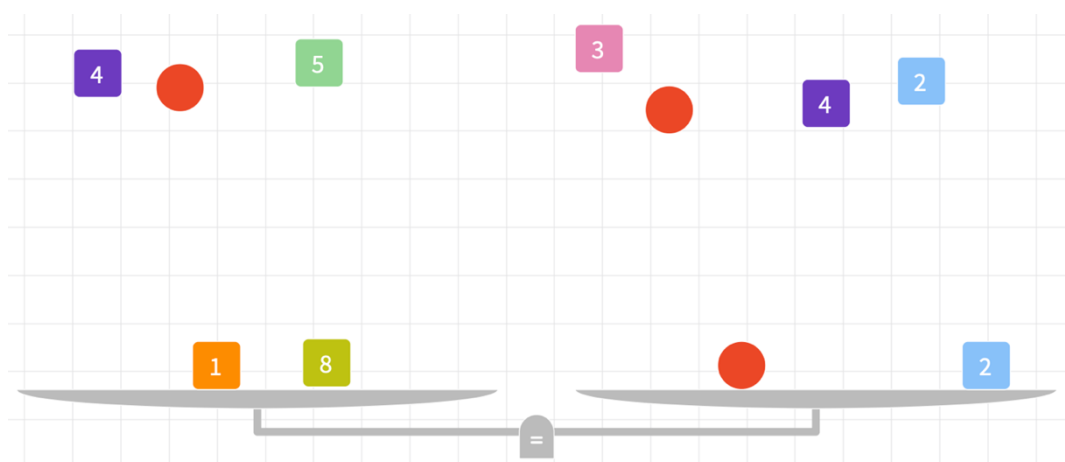


Figure 8.17: Scale of task situation 5 after the actions of Aldo and Giulio

- 143 T: Okay. Can we find out how much one ball weighs?
- 144 G: Um...
- 145 [7 seconds of silence]
- 146 A: Now... it's balanced, so 9 equals $x + 2$.
- 147 G: Yes. So, we move the 2 to the other side and it becomes minus 2.
- 148 T: It becomes minus 2?
- 149 A: No. It will... the little ball will be 7.
- 150 T: The little ball will be 7... Okay, try to explain to me.
- 151 A: Because if here it's 9 [points to the left pan of the scale] then 9 equals $x + 2$, so...
- 152 G: Uh, you move the 2 to the other side and it becomes 7 equal x
- 153 A: So, $7 + 2$ equals 9. And they must have the same number [points the two pans of the scale] because they are balanced.
- [...]
- 154 G: That is... on one side the sum of 8 and 1 is 9.
- 155 T: Okay... on the other side?
- 156 G: On the other side it's x plus 2.
- 157 T: Okay.
- 158 G: So, it should be... that is, the 2 should be moved over there [gestures with hand moving from right to left] and it changes sign, and it becomes...
- 159 A: Not necessarily. I mean, you can also do this in your head.
- 160 T: Okay, we can also do it in our heads, yes. How would you do it in your head?
- 161 A: So, looking at the left [points the left pan], the sum is 9 and as we said the rule is that a scale is balanced if the sum of the first pan equals the second, but we have an unknown on the right and a 2. So, to find the unknown, we can do the sum of this pan [points the left pan] minus 2
- 162 T: Okay
- 163 A: At least we can find the unknown, that is, the little ball.

At this point, the two students' discourses diverge. Aldo's discourse is about the scale (the "it's balanced" refers to the scale) and intertwines words from the DEO-discourse and from the E-discourse. Indeed, in line 146, Aldo affirms "it's balanced, so 9 equals $x+2$ ", using the verb 'to be balanced' (DEO-discourse) together with the terms 'equals' and ' x ' (E-discourse). We find in this utterance a $d_{DEO}(DEO,E)$ that shows us for Aldo talking about the little red ball is the same as talking about the term ' x '. He says that "the little ball will be 7" (line 149) and justify it (lines 151 and

153) considering the relation of equality between the two sides of the scale (“they must have the same number because they are balanced”). When the tutor asks again to explain their reasoning, Giulio takes the floor and again is more taken with the change sides, change sign rule (lines 147, 152, 158). The last time Giulio proposes to move the 2 on the other side and change its sign (line 158), Aldo intervenes “you can also do this in your head” (line 159). He recalls the principle (P_0) they have written within the first activity, even if, again, he uses the converse implication (line 161) with respect to the one they have elaborated (Figure 8.5). Aldo produces a narrative that is a merging of features from DEO-discourse and E-discourse (line 161). Specifically, even if in line 153 his discourse justifies the result of 7 according to what seems a ‘completing strategy’, in these utterances it is not clear if he is still using this way of acting or the change sides change sign rule, since he affirms “to find the unknown, we can do the sum of this pan minus 2”. In line 163, differently from before, he relates his procedure to the aim he has: “At least we can find the unknown, that is, the little ball”.

- 164 T: What about your written reasoning? How did you want to do that?
[turning to Giulio, points at the notes on the tablet]
- 165 G: So...
- 166 T: Try rewriting from here [he points to the scale]. So, it's $1 + 8$.
- 167 G: It's 9.
- 168 T: Yes...
- 169 G: That is, it comes out [he takes the pen from the tablet].
- 170 T: Yes, yes, you can write directly.
- 171 G: 1 plus 8 equals x plus 2 [writes as in Figure 8.18a].
[...]
- 172 G: Comes out 9 equals x plus 2 [writes as in Figure 8.18b], which becomes 9 minus 2 equals x [writes as in Figure 8.18c], because I moved the 2 to the other side and it becomes minus 2.
[...]
- 173 G: And it comes out 7 equals x [writes as in Figure 8.18d]
- 174 T: Okay. Why is it that when you moved the 2 to the other side, you did minus 2? [points to the equation]
- 175 G: Because if you move it to the other side of the equal sign, the sign changes.
- 176 T: Okay. This is how equations work?
- 177 G: Yes.
- 178 T: Do you remember why? Or had you ever thought about it? Has anyone ever told you why we do something like that?

[...]

179 A: No, no, I mean... I had never thought about it.

180 G: We have never thought about it.

Figure 8.18 displays four handwritten equations, labeled a, b, c, and d, which represent the steps of solving for the value of the unknown (x) in a task situation. The equations are:

- a: $1 + 8 = x + 2$
- b: $9 = x + 2$
- c: $9 - 2 = x$
- d: $7 = x$

Figure 8.18: Visual mediators produced by Giulio in task situation 5 to find out the value of the little red ball

Giulio, prompted by the tutor, offers his reasoning writing the sequence of visual mediators as in Figure 8.18. Differently from Aldo, he is strongly devoted to visual mediators and procedure of E-discourse: he writes the scale as $1 + 8 = x + 2$, and he starts implementing the change sides change sign rule again. Since he has affirmed “I moved the 2 to the other side and it becomes minus 2” (line 172), the tutor asks him the reason to do so (lines 174 and 178). The two students say that they have never thought about it (lines 179 and 180).

Their discourse, particularly Giulio’s one, is thus strongly guided by a ritual they have learned at school. They do not provide a meaningful narrative to explain why they can accomplish this procedure to find out the value of the unknown. This excerpt confirms what has partially emerged during their initial interviews: Aldo and Giulio’s discourse is characterized by procedures and manipulations that they do not completely control and that they accomplish without accompany them by meaningful narratives. According to what has happened in this episode, the change sides change sign rule is not related by them to the equality relation that must be maintained among the two expressions involved in the initial equation. In particular, this emerge when they accomplish the same procedure on the scale (see lines 127-141).

To conclude, this excerpt is crucial in the story of Aldo and Giulio. Within activity 5 their discourse shift from $d_{DEO}(DEO)$ to $d_{DEO}(DEO,E)$ and $d_E(E)$. They involve in their

discourse words, visual mediators, routines and endorsed narratives from the E-discourse. Specifically, they use in their discourse the little red ball and the symbol 'x' both as realization of the mathematical object unknown. The task situation proposed by the tutor of finding the value of the little red ball is interpreted by the two students as the request of finding the value of the unknown in a given equation. On the other hand, the routine guided by the change sides change sign rule they are used to perform in this situation is not related to meaningful narratives to justify it, explicating a ritualistic participation of Aldo and Giulio to solve the task situation when it is interpreted by them in the E-discourse. Such participation interferes with their previous discourses. Indeed, while they were developing $d_{DEO}(DEO)$ they show an explorative participation finalized to the construction of endorsable narratives. Once they introduce, in their specific discourses, E-discourse's features, they return to the participation we have identified also in the initial interview and start implementing procedures on the scale, or referring to it, that cannot be accompanied by endorsable narratives, but this seems not to matter to them.

Excerpt 6

Task situation 9: *Could you find out how much a purple square weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.*

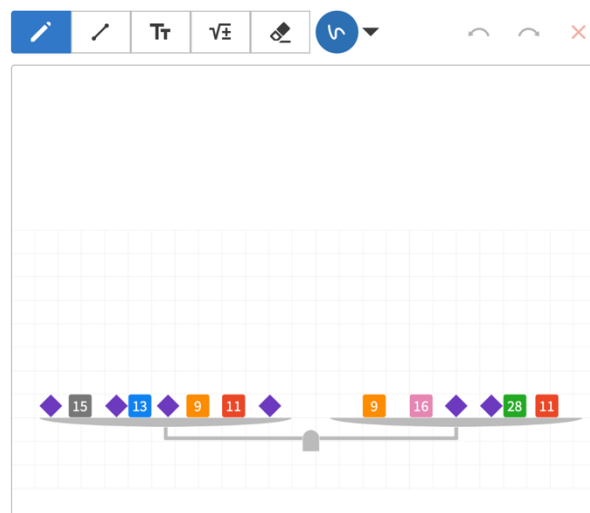


Figure 8.19: Picture of the scale at disposal in task situation 9

In this excerpt, Aldo and Giulio deal with the first activity in which the dynamic two pan scale is substituted by a picture of a balanced scale on which students can write (see for instance Figure 8.19, picture of task situation 9). As starting point, Giulio writes down the scale with an equation ($x + 15 + x + 13 + x + 9 + 11 + x = 9 + 16 + x + x + 28 + 11$, Figure 8.20 before cancellation marks). After that, Aldo

affirms “Let’s start simplifying, that is...” and produces cancellation marks on the picture accompanied by utterances such as “OK. So, this and this go away with this and this [cancels two little squares on the left and two little squares on the right] 15 and 13 go away with 28 [cancels 15, 13, 28]”. In the end Aldo reaches the inscriptions in Figure 8.21. Simultaneously to this, Giulio produces the same cancellation marks on the equation he has written before, obtaining the visual mediator in Figure 8.21. Then the exchange goes on with the two students trying to find out the value of the purple square.

$$\begin{array}{r} X + 15 + X + 13 + X + 9 + 11 + X = \\ 9 + 16 + X + X + 28 + 14 \end{array}$$

Figure 8.20: Visual mediators produced by Giulio in task situation 9

- 181 G: So, it would come x...
- 182 A: x plus x and that's it, here nothing is left [writes over the left pan as in Figure 8.21a]
- 183 G: Yes
- 184 A: So, x plus x equals 16 [he writes as in Figure 8.21b] [Giulio writes as in Figure 8.22a] so it would be
- 185 G: Yes, it makes two x [he writes as in Figure 8.22b]
- 186 A: Two x [he writes as in Figure 8.21c]
- 187 G: Equals 16 [he writes as in Figure 8.22b]
- 188 A: Equals 16 [he writes as in Figure 8.21c]
- 189 G: Divided by two [he writes as in Figure 8.22c]
- 190 A: And so, two of these unknowns make 16 [Giulio writes as in Figure 8.22d]. And since they are two, we can divide 16 by two and find the number of just one little square 8 [he writes as in Figure 8.21d]

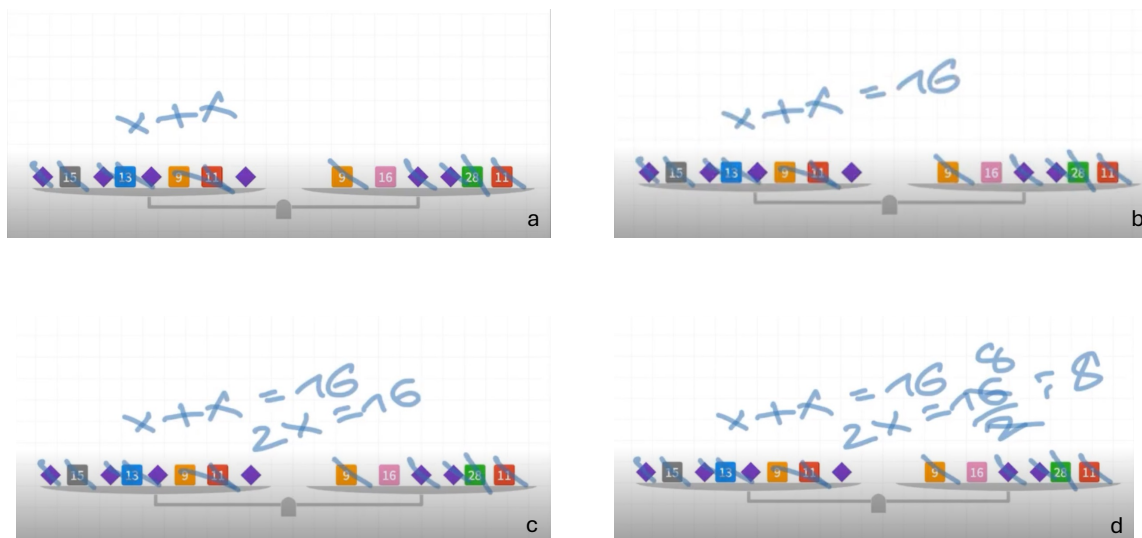


Figure 8.21: Visual mediators produced on the picture of the scale by Aldo in task situation 9

Aldo and Giulio proceed simultaneously narrating the procedure they are implementing on each tablet (lines 181 – 190, Figure 8.21 and Figure 8.22). In particular, once Aldo has reached the inscription ‘ $2x=16$ ’ (Figure 8.21c) he affirms “two of these unknowns make 16. And since they are two, we can divide 16 by two and find the number of just one little square 8” (line 332, Figure 8.21d), producing a $d_{DEO}(DEO, E)$. In his discourse he highlights the equivalence relation among the two quantities (see also line 184) and offers a meaningful narrative to accompany the procedure to find out the value of the unknown, that is the value of one little square, as he says. Indeed, the action of dividing 16 by two, that could be just another manipulation to be implemented within the process, is justified by Aldo with the fact that there are two purple squares. At the same time, Giulio develop a $d_E(E)$ writing on the blank page the equation and the subsequent steps, narrating his procedures (Figure 8.22).

$$x + x = 16$$

a

$$2x = 16$$

b

$$\frac{2x}{2} = \frac{16}{2}$$

c

$$x = 8$$

d

Figure 8.22: Visual mediators produced by Giulio in task situation 9

- 191 T: Which comes to 8. That's what you wrote in this way. And if I didn't want to do divided by two but wanted to reason let's say out loud, [Aldo writes a fraction line and 2 under 16 and then '=8', then he simplifies] I could also do right from x plus x equals 16? What could I say about 16?
- 192 G: Yes, because 16, that is here, this part is 16 [he points to the right side of the equation], and thus also here should be 16 [he points to the left side of the equation]
- 193 T: Yes
- 194 G: Thus 8 plus 8 makes 16
- 195 T: Sure, exactly. Strange question, couldn't I say, ok well 8 plus 8 makes 16, but someone could say, 9 plus 7 makes 16
- 196 A: No, because the two unknowns are the same, the two x.
[...]
- 197 T: And all those cancellations you made are right because now the scale remains—it's an image. So, we're not sure if it stays balanced. **Who guarantees you that it stays balanced?**
- 198 G: Because we removed the same things at the same time, from both sides.
- 199 T: Exactly.
- 200 A: We removed the same amounts.
- 201 T: You removed the same amounts, so at one point—were you using something? Let's say, the things we wrote earlier, for example.

202 A: Yes, when the scale is balanced, if you subtract or add the same amount from both sides, it always remains balanced.

When the tutor asks about reasoning on the inscription ' $x+x=16$ ' to find out the value of the purple square, Giulio participates to the discourse too. The student offers a narrative in which he involves the relation of equivalence between the two sides of the equation (line 192) to justify the value of 8 for the square's value. Moreover, in answering the question of the tutor (line 195), Aldo in lines 196 constructs a narrative in which the two unknowns are abstract objects which have specific properties, they "are the same", they have to be "two equal numbers that add up to 16". The last tutor question is about the scale and (line 197) aims at promoting the use of principles to construct meaningful narratives about the behaviour of the scale imagined by the two students according to their cancellations. The students answer coherently with has been said and, prompted by the tutor, Aldo recalls the principle they have written before (line 202). Thus, in this occasion they have the discursive tools (the principles) to justify the maintained balanced scale with respect to what they have done. The aim of this task situation, and the other with pictures of the scale, is mainly to promote a discourse in which the students explicate the assumptions on which their actions and considerations are made. In particular, we want to promote the use of the principles previously written to describe the imagined behaviour of the scale. In this sense, the task situation, together with the tutor's intervention, have reached their aims.

Excerpt 7

Task situation 11: *Could you find out how much a purple square weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.*

As within task situation discussed in the previous excerpt, after having read the task situation, Giulio writes on the notes an equation (Figure 8.23).

$$\begin{array}{l} X+15+X+X+7+12 = \\ 6+X+9+12+19 \end{array}$$

Figure 8.23: Visual mediators produced by Giulio in task situation 11

After that, Aldo starts talking about action referring to the picture of the scale. He reaches the situation in Figure 8.24.

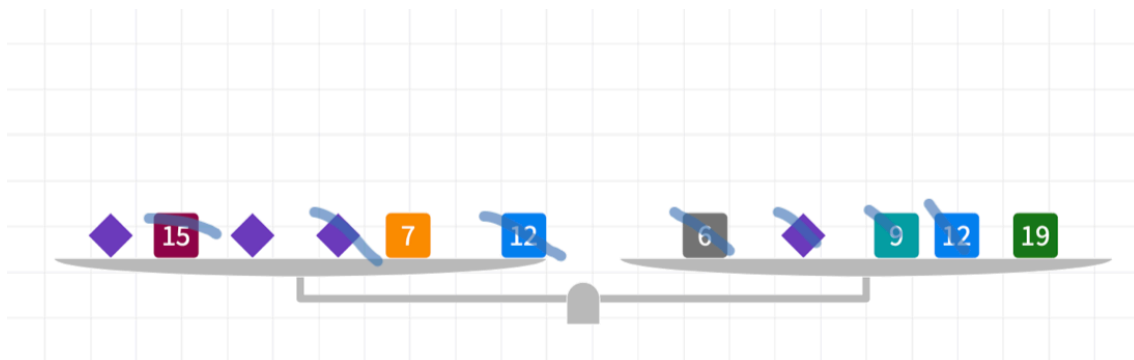


Figure 8.24: Visual mediators produced by Aldo on the picture of the scale in task situation

11

- 203 A: So, these two unknowns plus 7 make 19 [Aldo writes as in Figure 8.25a, Giulio writes on the notes ' $2x=19-7$ ', so we could subtract 7 from 19 [Aldo writes as in Figure 8.25b]. That gives us 12 [Figure 8.25b][Giulio writes underneath ' $2x=12$ ' then he writes a fraction line and a 2 under both sides of the equation, then he writes underneath ' $x=6$ '], and 12 indicates x plus x , and since the two x are equal, we can divide by 2 and it gives 6 [Aldo writes as in Figure 8.25c], so one x equals 6 [Aldo writes as in Figure 8.25d].

[...]

- 204 T: **On the scale, what would mean subtracting 7?**

- 205 A: To take away, I mean taking away 7, so it was no longer balanced.

- 206 T: Okay, so it was no longer balanced, but you changed the 19, right? I mean, if you took away 7 from here, then you wrote $19-7$ there... how could we...

- 207 A: Ehm... [silence]

- 208 T: No, no, it's fine. You said, I take away the 7, so the scale is no longer balanced. In fact, **on the balance, we can't just take the 7 and put it over there, right?** Because we saw earlier that if I put it over there, the scale is no longer balanced, **so perhaps with this -7, we took this 7 away and wrote -7, we're doing something else. It's not like we're moving the 7 over there.**

- 209 G: To simplify it on the scale, to make it a bit simpler, we should have subtracted 12 and 7 with 19, and it would have remained here, $x+x$, and here 12.

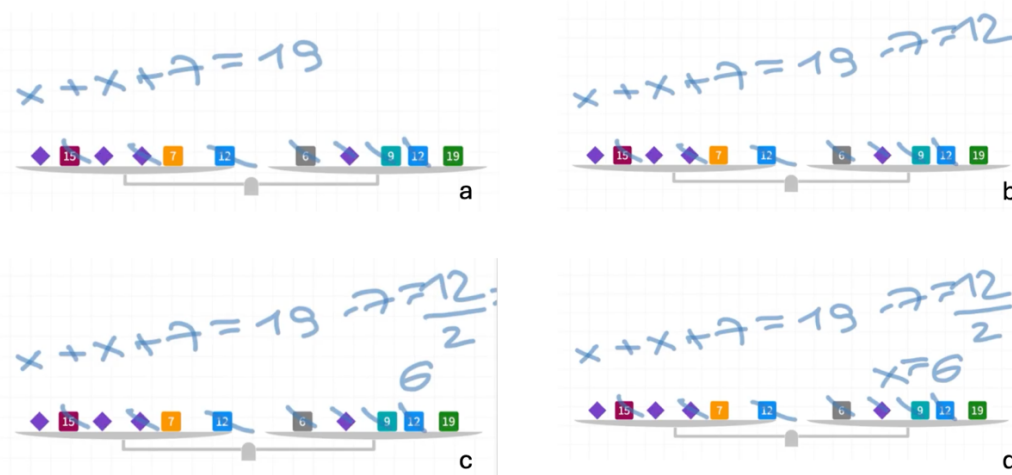


Figure 8.25: Visual mediators produced by Aldo on the picture of the scale in task situation 11

The two students fulfil quite simultaneously the same procedure to accomplish the task of finding the value of the purple square (line 203). They produce several visual mediators (see Figure 8.25 for Aldo's mediators) to accompany their discourses. They both arrive to the correct solution of the equation and the tutor asks Aldo to explain again the chain of equal signs he has written (Figure 8.25d). The tutor, with his interventions, aims at promoting a bridging discourse between students' deeds on the equations and the possible (imagined) action on the scale. The tutor asks explicitly what it means to subtract 7 on the scale. Aldo's answer "To take away, I mean taking away 7, so it was no longer balanced" (line 205), is not coherent with the action on the scale that for an expert corresponds to the one of subtracting 7 on the equation. In particular, for Aldo the 'taking away' seems to be performed only on one pan of the scale and it thus leaves the scale unbalanced. With the intervention in lines 204 and 208 the tutor tries to prompt the development of a bridging discourse, about the scale and about the equation, and specifically about the action performed on the equation and the correspondent on the scale. Giulio's intervention (line 209) shifts the attention towards another way of acting on the scale. Thus, the tutor's attempt does not reach its aim: his prompt does not solicitate a bridging discourse by the students who are focused on reaching the solution of the equation and that do not relate the action of removing the same value from both pans to the change sides change sign rule.

Excerpt 8

Task situation 14: *Imagine a scale that is inclined to the left. On the left pan I add a weight of 9. On the right pan I add a weight of 5 and two red balls. The scale is now balanced. What can you say? Justify your answer.* (blank page were students can write)

The task situation proposed in this excerpt is an argumentation task that has the aim of promoting a discourse about an imagined scale using the principles written and discussed with the students before. Despite an initial confusion due to the task that can be tricky, the two students well justify the fact that the sum of 5 with the two little red balls cannot be 9. Indeed Aldo (line 210), affirms that “Since if a scale hangs one way and if you add the same amount, like now let’s say 9, if you add 9, the scale remains—it doesn’t remain balanced but still hangs to the left. Whereas if you add a larger number, for example, a number larger than 9, there’s that number, ah right, but a number that can bring it into equilibrium”. This utterance recalls what the students have discussed during previous activities. From line 215 Aldo, Giulio and the tutor’s discourse is about the value put on each pan. Aldo concludes that 5 and the two little red balls are greater than 9 (line 224). After that, the tutor asks what they can say about the little red balls (line 227). Giulio (lines 228, 231 and 233) offers meaningful narratives to explain why the red ball must have a value greater than 2. Along the episode we observe several moments in which the previous written principles are used, not explicitly recalling them, to construct narratives about the scale and the red balls.

- 210 A: Since if a scale hangs one way and if you add the same amount, like now let’s say 9, if you add 9, the scale remains—it doesn’t remain balanced but still hangs to the left. Whereas if you add a larger number, for example, a number larger than 9, there’s that number, ah right, but a number that can bring it into equilibrium. Yes, it’s difficult to explain.
- 211 T: I think I’m getting it, so you were saying... If I put 9 on both sides, it wouldn’t be...
- 212 A&G: It wouldn’t be balanced.
- 213 T: I mean, it wouldn’t be balanced, but we know that it is balanced, so what can I conclude?
- 214 G: So, I mean 9...
- 215 T: So, I didn’t put 9 on both sides; you convinced me, but you told me, **so I put more on which two sides?** I don’t even remember anymore.
- 216 G: From here, I mean, in theory, if it was hanging to the left, it means there was less weight here; 9 is greater than 5 plus two little balls.
- 217 T: Wait, if it was hanging to the left...
- 218 A: No, if it was hanging to the left, it was heavier.
- 219 T: On 9 there’s more weight.
- 220 A: And indeed, so I was right. Here it was heavier because it went down.

- 221 T: Okay, so it's down to the left. I'll help with my hands, yes, it's down to the left.
- 222 A: You add 9, and so...
- 223 G: So that...
- 224 A: It's true that I told you it was greater than 5. In fact, it puts it on the right, so it can balance. So, 5 and the two balls are greater than 9.
- 225 T: So, 5 and two balls are greater than 9.
- 226 G: Yes.
- 227 T: Okay, so how much will the balls weigh? I mean, obviously, we can't say the exact value, but if it has to be greater than 9, how much?
- 228 G: I mean, each ball should have a value greater than 2.
- 229 A: Yes.
- 230 T: How come, what reasoning?
- 231 G: Because $5+2+2$ always equals 9, so it couldn't be that.
- 232 T: If it were, instead, greater than 2, the ball...
- 233 G: And it would be 5 plus a value greater than 2 for each ball, so it would be greater than 9.

At the end of the session, Aldo and Giulio have accomplished the second group of activities. Their DEO-discourse has expanded, due to the new task situations, involving words and visual mediators to talk about objects with not known values, and new routine, for them, to find their value. Specifically, this last aspect has been introduced after the involvement in their discourse of the change sides change sign rule. Indeed, the two students, during activity 5, introduce an equation to write down the scale and recognize as familiar the task situation of finding the not known value, involving the procedure 'from school'. This led them to a product that is not correct and, once the tutor asks to act on the scale, they proceed in the same way. Thus, the involvement of a procedure for them familiar take them away from the endorsed narratives developed within the previous group of activities and put the light on an aspect already perceived during their interviews. The tutor's intervention takes them back to the use of principles that they use in their discourse to justify their action. In the successive task situations, they recall the principle to guide their actions and narratives, but the spectre of the change sides change sign rule is still there. Despite the tutor's attempts to make a bridge between this procedure and their action on the scale, Aldo and Giulio's discourse do not relate the two aspects, that at the end of the session seem still detached.

To conclude, we can affirm that within the first session the two students' DEO-discourse has expanded according to our a priori analysis of the transplantability. Moreover, as we aimed, they have involved several features from the E-discourse in their discourses, highlighting some relations between them and some features of the DEO-discourse. It has covered a crucial role the presence of the tutor and his intervention. Indeed, his request of writing the scale as well as asking to find the value of not known weights have prompted the use of E-discourse's features that are of our interest. To conclude the analysis, with respect to Figure 8.13, we can notice that at the end of the session the E-discourse is not 'faded' anymore (Figure 8.26). Some features have emerged and have been used, but some aspects remain without justification (change sides change sign rule).

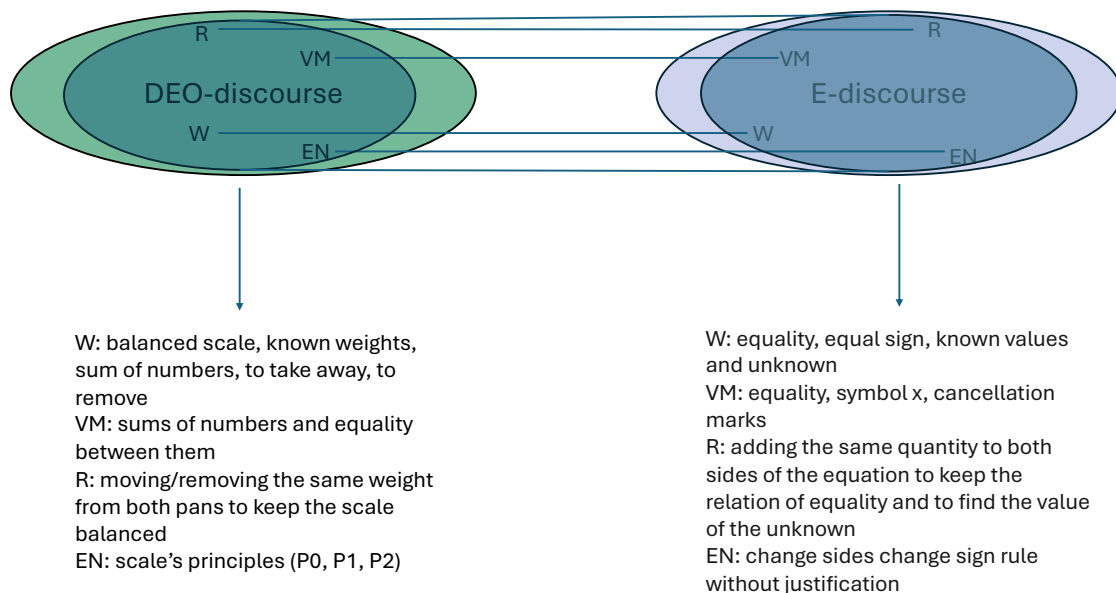


Figure 8.26: Aldo and Giulio's DEO-discourse and E-discourse at the end of the first session

8.1.4. Aldo and Giulio's second session

During the second session, Aldo and Giulio deal with the task situations with explicitly reference to equation. We decided to print the pages with the principles they have written during the previous session and leave them on the table at disposal of students. They face the first silent video task situations using the principles they have written to justify each step, and to complete the blurred part on the right, they use a sequence of equations, as we aimed. I now discuss some excerpts from activities with equations.

Excerpt 1

Aldo and Giulio are facing task situation 17, in which they are asked to explain each step of the writings they see (Figure 8.27). This task aims at prompting the formulation of equations' principles related to the principles elaborated within scales.

$$\begin{aligned} \cancel{x} + 8 + x &= 4 + \cancel{x} + 13 + 5 \\ 8 + x &= 4 + 13 + 5 \\ \cancel{8} + x &= \cancel{4} + 13 + \cancel{4} + 1 \\ x &= 14 \end{aligned}$$

Figure 8.27: Visual mediator of task situation 17

- 1 T: Let's continue. Okay, so actually, this part is a bit long. So, now you can see this series of writings. I'm asking you to explain the steps in between these writings.
- 2 A: So, x and x cancel out [points to the equations], because if we move the x to the left, it becomes negative [points first to the right side then to the left side of the equation], and so a negative and a positive cancel each other out.
- 3 T: Okay, repeat that part for me because I'm not sure I understood. There's a blue mark there.
- 4 G: So, when two numbers, two equal values are on opposite sides of the equals sign, that is, if they're the same, they...
- 5 A: You can simplify them.
- 6 G: They cancel out because they have the same sign, and they're on opposite sides of the equal sign. So, they can be simplified.
- 7 T: Okay, **but why?** Maybe you already told me, and I didn't get it. That's probably the case.
- 8 G: Because...
- 9 A: When it's moved to the left, the number becomes negative, so it cancels out with the positive number.
- 10 T: Okay, I think you're telling me two different things. Or maybe I misunderstood.
- 11 G: Like...
- 12 T: Because here [points to the tablet], **it is not moved to the left, right?**

- 13 G: Yes.
- 14 T: So here, there are these x with blue marks on them [points to the x with cancellation marks on the tablet], and then these x aren't there anymore. So, what's happening?
- 15 G: He's simplifying two equal values on opposite sides of the equal sign.
- 16 A: He's removing two equal values.
- 17 T: Removing two equal values...
- 18 A: From two different sides.
- [...]
- 19 T: Yeah, on this second line? [points to the second line of the equation]
- 20 A: Yes, since we removed the two x, the following numbers remain, and there's still an unknown on the left. Then, we do the same thing with the 8, removing the two 4's on the right because the two 4's add up to 8, and we can remove them. And finally, we can figure out that the unknown equals 14, which is 13 plus 1, which together makes 14.
- 21 G: Yes
- 22 T: Okay, so going back to the beginning, you're telling me we can make the jump from here to here because?
- 23 A: Because we have two equal values, so if we remove them, it doesn't change anything.
- 24 T: Okay, what do you mean by "doesn't change anything"? That's a tough question, huh?
- 25 G: Because if one of these values that both have the same sign were to move to the other side, it would change sign, while the other would remain with the same sign, so they would cancel each other out. Therefore, without moving them to the other side, they could already be eliminated like this.
- 26 T: OK, they can already be eliminated like this because you were telling me they are the same, they are x and x, so they have the same value.
- 27 A: It's like on the scale.

In the beginning Aldo's discourse is characterized by the justification of some steps relating it to the change sides change sign rule. For instance, he affirms "x and x cancel out, because if we move the x to the left, it becomes negative" (line 2) and it repeats the same considerations in line 9. In the meantime, Giulio's discourse (lines 4 and 6) seems more focused on the possibility of cancelling the same value on both sides of the equation. Tutor's interventions (lines 7, 12 and 14) aim at stressing the difference in the two students' discourse. Specifically, when she underlines that

nothing is moved to the left (line 12), Aldo's discourse change and aligns with the one of Giulio. Indeed, he says "He's removing two equal values" (line 16) and completes, "From two different sides" (line 18). The tutor asks again to explain why we can move from the first and the second line of the inscription (line 22) and Aldo answers "Because we have two equal values, so if we remove them, it doesn't change anything" (line 23), and he produces an instance of bridging discourse: "It's like on the scale" (line 27). Giulio in line 25 makes some considerations that could offer a way to discuss the relation between the change sides change sign rule and the procedure of removing the same value from both sides, but it is not exploited by the tutor that prefers to grasp the link with the scale offered by Aldo (line 28), who has explicated a bridge with the two-pans-scale DEO.

- 28 T: It's like on the scale. OK, so, here you see that we are no longer using scales, and you are telling me things that may resemble what you also wrote about the scales, but we are in a slightly different situation. So, one effort I am asking you to make, which you have already told me, I hope it won't be too much of a request, is to try to write down what you just told me. **OK, so you did it with scale? Now let's try to move into this world here, OK? So, regarding what you told me, what do you connect it to, among these things you wrote?** [points to the sheets of paper with the principles written by Aldo and Giulio]
- 29 A: I think this [points to the sheet of paper with P_1], if it is balanced and you subtract the same sum of numbers, then the scale will remain balanced
- [...]
- 30 T: What do we have here?
- 31 G: Eh... An equation.
- 32 T: What do you mean by equation?
- 33 G: When there is an equal sign in the middle, and on both sides, there is a series of numbers and unknowns.
- 34 T: Ok.
- 35 G: But since there is an equal sign in the middle, the numbers on one side are equal to the numbers on the other side, their sum.
- [...]
- 36 T: Alright? OK, **so let's try to formulate. Do you want to try writing already?** Do you have something? Let's think, go ahead, I don't know which of you will do it.
- 37 A: If in an equation... [Giulio writes on the notes 'If in an equation you subtract'] we add or subtract because even if you add two x from two different sides, the equation will remain equal...

- 38 G: If in an equation... But should we only write in this case or also in the case of adding a sum?
- 39 T: Well, let's say we are trying to say it a bit more generally.
[...]
- 40 G: An equation... subtract or add the same value on both sides of the scale...
- 41 A: Well, it's...
- 42 G: Yes, of the equation... on both sides of the equation. [writes 'on both sides of']
- 43 T: Can we say on both sides of the equal sign, maybe?
- 44 A&G: Yes... [Giulio writes 'the equal sign']
- 45 A: The equation will remain... I mean...
- 46 G: The equation will remain an equality.
[...]

The episode goes on with the prompted by the intervention of the tutor to bridge DEO-discourse and E-discourse. Specifically, she asks to relate the narratives on equations with the principles they have written about the scale (line 28) and Aldo identifies P_1 . The exchange goes on focusing on the fact that there are no scales anymore (line 30) but equation (line 31). The tutor asks them to clarify what they mean (line 32) and Giulio (lines 33 and 35) describes the two sides of an equation as a collection of numbers and unknowns as well as the relation of equality that is between them. Since Giulio is writing the generalization considering only the subtraction, Aldo proposes to consider in the principle also the 'sum', because, he affirms "even if you add two x from two different sides, the equation will remain equal..." (line 37). In conclusion of the episode, they arrive to formulate the principle as 'If in an equation we subtract or add the same value to both sides of the equal sign, the equation will remain an equality' (Figure 8.28).

SE IN UN'EQUAZIONE
SI SOTTRA E O AGGIUNGE
UNO STESSO VALORE SIA
DA UNA PARTE CHE
DALL'ALTRA DELL'UGUALE
L'EQUAZIONE RIMARRA
UN'UGUAGLIANZA

Figure 8.28: Written formulation by Aldo and Giulio of the principle to keep the relation of equality adding or removing known or not known values (P_1)

Excerpt 2

Task situation 19

In task situation 19 Aldo and Giulio have transplanted also P_2 in the E-discourse, thus they have at disposal the tools to construct endorsable narratives about equation. In this task situation they are asked by the tutor to 'make some steps' as the one discussed in the previous activity, and justify them, with respect to the equation in Figure 8.29.

$$x + 5 - x = 8 - x + 2$$

Figure 8.29: Visual mediator of task situation 19

- 47 T: Now I ask you to try to make some steps on this writing and then justify them to me.
- 48 G: So...
- 49 A: You can simplify $-x$ with $-x$ [he points to the left and right of the equal sign]
- 50 G: Yes [he produces the visual mediators in Figure 8.30a]
- 51 A: You can do something like eight equals five plus three [points to the equation]

- 52 G: he produces the visual mediators in Figure 8.30b
 53 A: It cancels...
 54 G: The 5 cancels [he produces the visual mediators in Figure 8.30b]
 55 A: So, x equals 5.
 56 G: writes below as in Figure 8.30c
 57 T: What can you tell me?
 58 G: At the beginning, we removed minus x with minus x.
 59 A: We used the first... I mean, the rule we had written.
 [...]
 60 A: While in the second... I mean, [points to the tablet with the equation] we used the second... I mean, the second law, that is, we replaced this value with a sum of two numbers...

Figure 8.30 displays three visual mediators (equations) produced by Giulio during task situation 19. The equations are handwritten and show the process of simplifying a linear equation.

Equation (a) shows the initial simplification: $x + 5 - x = 8 - x + 2$. The x terms on both sides are crossed out, indicating the cancellation of $-x$ from both sides.

Equation (b) shows the next step: $x + 8 = 5 + 3 + 2$. The x and 8 on the left side are crossed out, and the right side is simplified to $5 + 3 + 2$.

Equation (c) shows the final result: $x = 5$.

Figure 8.30: Visual mediators produced by Giulio in task situation 19

From a design point of view, we expected that with task situation 19 would have emerged, in students' discourse, the change sides change sign rule, as a procedure for them familiar in the process of solving linear equations. In this short excerpt, Aldo and Giulio's discourse is characterized by a different (and probably new for them) procedure to solve equation. Indeed, they proceed at first 'simplifying' the ' $-x$ ' from both sides of the equation, producing visual mediators that previously were produced on the scale, and thus that have been transplanted (lines 49 – 50, Figure 8.30a) and then they substitute the number 8 in the right side of the equation with the sum $5+3$ (lines 51 – 52, Figure 8.30b). Prompted by the tutor, they justify their action referring to the 'rules' they have written (lines 59 and 60). This task situation is the first one, since the first session, in which they find themselves in front of an equation and they are asked to perform action on it. Their discourse is characterized

by different features from the ones in the initial interviews. The E-discourse of Aldo and Giulio has expanded with new principles thanks to which they can act on the equation and construct endorsable narratives about their action. They have re-invested the routine, composed by the task situation of acting and justifying and the procedure of using the principles, previously developed within the two-pans-scale, in their discourse about equation.

Excerpt 3

Task situation 20

In this task situation, Aldo and Giulio are required to justify each step in the procedure they see (Figure 8.31). This request aims at, again, making emerge the change sides change sign rule explicitly ‘using’ it in the process of solving the equation proposed.

$$\begin{array}{l}
 8 - x + 2 = -x + 3 - x \\
 -x + x + x = +3 - 8 - 2 \\
 x = -7
 \end{array}$$

Figure 8.31: Visual mediator of task situation 20

- 61 T: Go ahead, tell me a bit.
- 62 G: So... at the beginning, he moved all the unknowns...
- 63 A: He moved two x's to the left.
- 64 G: ...to the left.
- 65 A&G: And the numbers to the right.
- 66 A: Changing signs!
- 67 G: Yes.
- 68 T: **Why?**
- 69 G: Because when you move from one side to the other of the equal sign, you change signs...
- 70 A: You always change signs.
- 71 G: ...to the value.
- 72 T: And why?
- 73 A: Uh

- 74 [15 seconds of silence]
- 75 T: You told me. What did he do here? [points to the tablet] Repeat it for me so we can go back.
- 76 A: We moved the unknown to the left, changing signs, and the numbers to the right, changing signs.
- 77 T: OK. And so, my question was
- 78 A: Because... it moves to the left because the unknowns are always on the left, so we put the x's on the left.
- 79 T: Ok.
- 80 A: While changing signs because...
- 81 G: Because when you move from one side to the other of the equal sign, you change the sign of the number.
- 82 T: OK, and why?
- 83 A: Uh, I don't know. [laughs]
- [...]
- 84 T: OK, fine, let's move on now. Then tell me, you told me that he moved to the left and to the right.
- 85 G: Yes.
- 86 A: And we summed the x... and we did the sum of all the x, so minus x and plus x cancelled out, so only one x was left, and plus three minus eight, minus two equals minus seven. So, the unknown was minus seven.

The object of this excerpt's discourse is the procedure the two students' see implemented and that they recognize as familiar. Indeed, from line 61 to line 71 Aldo and Giulio describe the steps quite naturally with no signs of doubt. What it is worth to notice is that, asked by the tutor about the 'why' of the steps performed, Aldo and Giulio's discourse presents a ritual participation strongly guided by actions that 'you' have to do. For instance, Aldo affirms "Because... it moves to the left because the unknowns are always on the left, so we put the x's on the left" (line 78), while Giulio says, "Because when you move from one side to the other of the equal sign, you change the sign of the number" (line 81). The unknowns always on the left side of the equation, the changing sign when you move from one side to the other, the sum of the x, are all parts of the procedure the two students identify and to which they seem to be accustomed. The two students' discourse is strongly concentrated upon the actions and when they are asked to explain why, when moving terms from one side to the other, it is necessary to change their sign, they do not know the reason. Thus, we go back to what we notice in the first session: if they are prompted to use the procedure to which they are used to, Aldo and Giulio implement a

ritualistic participation in which the ‘because’ are only related to actions that must be done.

Excerpt 4

Task situation 21: the second silent video (Figure 8.32)

$2x - 3 = x - 1$ a	$2x - 3 = x - 1$ $2x = x - 1 + 3$ b
$2x - 3 = x - 1$ $2x = x - 1 + 3$ c	$2x - 3 = x - 1$ $2x - 3 + 3 = x - 1 + 3$ $2x = x - 1 + 3$ $2x - x = -1 + 3$ $x = 2$ d

Figure 8.32: Some snapshots from the silent video of task situation 21

Within this entire task situation, the two students’ discourse is about the two equations and involves only features of the E-discourse, thus it can be labeled as $d_E(E)$. Aldo and Giulio are asked to look at the video and discuss it. The tutor leaves them alone, and after having watched the video, the students have this short exchange from line 151 to line 161 in which the change sides change sign rule covers a main role in their discourse. Indeed, since the first utterances of Aldo, the procedure of ‘moving a number from one side to the other’ emerges. The student interprets the action of adding a number (3 in this case, Figure 8.32c) to the two sides of the equation as a way to move the number “to the right”. Also from Giulio’s discourse, when he tries to generalize probably recalling the previous activities, the aim of the solver (in this case the person who is writing in the video), is to “move a value to the other side of the equal sign” (line 156). They are focused on the procedure of changing sides, that for them is the goal of the procedure, and they try to describe the other actions as a function of it.

- 87 A: So... when you want to move the number to the right, you add or subtract the opposite number from both sides...
- 88 G: Yes.
- 89 A: So, if, for example, here it was 2 and you wanted to move it to the right, to the left you had to put minus 2, and so minus 2 on the

- right as well. So, on the left plus 2 and minus 2 cancelled out, and then minus 2 remained on the right.
- 90 G: Um...
- 91 A: But, like, how to explain it...
- 92 G: But... if... in an equality, you want to move a value to the other side of the equal sign...
- 93 A: It should be subtracted or added with the number with the opposite sign.
- 94 G: Yes... with the number that we wanted to move to the other side of the equal sign.
- 95 A: Mhm [nods] Like with the same value but with the sign...
- 96 G: Opposite
- 97 A: Opposite
- [...]
- 98 T: **OK. What happens on this side instead?** [points to the right part of the video]
- 99 A: The same thing happens, but when you wanted to move a number to the right and the unknown to the left... that is, you added the number from both sides... that is, the same number but with the opposite sign. So...
- 100 T: OK. Give me an example so maybe we can understand each other...
- 101 A: For example, if we have a 2 on the left and we want to move it to the right, we add it to both sides with minus 2. And since on the left 2 and minus 2 cancel each other out, minus 2 remains on the right.
- 102 T: OK, I think I understand. **Do you want to try writing this example down?** Then we can think about it a little more... or even referencing to what's over there, I mean. [places the tablet with the blank sheet in front of Aldo]
- 103 G: I mean...
- 104 T: Also, with an example like you said, where there's a 2 on one side and a 2 on the other... I mean, I don't know.
- 105 A: Like... $x + 2 = 3 + x$ [writes $x + 2 = 3 + x$]. Like if you want to move the 2 to the right [writes as in Figure 8.33a], you add this number to both sides but with the opposite sign, so if it was plus two you add minus 2 [writes -2 on the left side of the equation] and here you add minus 2 [writes -2 on the right side of the equation]. And so, the plus 2 and minus 2 cancel out [writes as in Figure 8.33b], and then it remains...
- 106 T: OK, OK. Let's stop for a moment at what you did here. [points to the tablet where Aldo was writing] Here you told me... so what did you do?

- 107 A: We added the same number but with the opposite sign.
- 108 G: Yes.
- 109 T: OK, and you added it...
- 110 A: To both sides of the equal sign.
- 111 T: OK. And... why do you do it?
- 112 G: So... if you put the number with the opposite sign on both sides, the side with the number you wanted to move to the other side cancels out. So only this negative one remains [points to the '-2' not cancelled in Aldo's writing].
- 113 T: OK, alright. **And can you justify this in some way?** I mean, this thing that Aldo proposed to put the -2 here and here [points to Aldo's writing, then to the left and right of the equal sign].
- [...]
- 114 G: Because this one cancels out what's on this side, since we only wanted to move it to the other side [points first to '+2' on the left and then to the right]. So, there's no need for it to be here too.

The tutor comes in, and Aldo and Giulio describe to her the left part of the video recognizing the change sides change sign rule and the steps performed. Then, the tutor asks about the right part of the video (line 98). Aldo's discourse is still focused on describing the action performed according to the will of moving number to the right side and unknowns to the left side of the equation (lines 99 and 101). He says "the same thing happens" referring to what he has just described for the left part of the video (line 99). Aldo offers a visual mediator to accompany his sentence "like if you want to move the 2 to the right" (line 105), drawing an arrow that goes from the number 2 on the left side of the equation he has written to the right side of it (Figure 8.33a). His narratives are centered on the action of moving the number 2: from his discourse we can infer that, when he accomplishes the change sides change sign rule, for him it is that specific number 2 that moves. Giulio, when asked why the same number has been added on both sides of the equation, justifies it referring to the number 'to be moved' (line 112). The tutor asks for a justification of this action (line 113) and Giulio again focuses on the aim of moving the number to the other side (line 114). Thus, it seems that the tutor's questions are interpreted as a request of describing the aim of the choice of adding '-2' to both sides (line 114) and not, as it is in the will of the tutor, to explain why Aldo and Giulio can perform the action of adding the same number on both sides of the equation.

Figure 8.33 consists of two panels, (a) and (b), showing handwritten mathematical equations. Panel (a) displays the equation $x + 2 = 3 + x$. A curved arrow originates from the '2' on the left side of the equation and points towards the 'x' on the right side. Panel (b) displays the equation $x + 2 - 2 = 3 + x - 2$. In this equation, the '+2' and '-2' on the left side are crossed out with diagonal lines, indicating a simplification step.

Figure 8.33: Visual mediators produced by Aldo in task situation 21

115 T: Okay, got it. And here, instead, we're adding... maybe you had written it down [changes page on the tablet to P₁ for equations].

Let's try to reread what you wrote earlier. "If from an equality between sums of values the same value is subtracted or added from both sides of the equal sign, it remains an equality."

[...]

116 T: Okay, and what was done?

117 A: It was...

118 G: I mean... we added the same number on both sides.

[...]

119 T: And by doing that? What do we know?

120 G: That the equation stayed an equality, it didn't change.

[...]

121 T: Only the plus three remains on the other side. Okay. Got it. And compared to what happens here? [points to the left side of the video]

122 A: Well here, instead...

123 G: Here, instead, it's done directly, that step over there isn't shown

124 A: Yes, they move the sign directly.

125 T: Okay, so you're saying they do it directly, that's fine.

[...]

126 T: Okay. And so, then we get to two x minus x equals minus one plus three. And what happened here instead? [points to the left side of the video]

127 G: The same thing, but without the intermediate step, and they immediately moved the x to the other side.

[...]

- 128 T: Okay. Got it. So you're telling me that here [points to the left side of the video] there are fewer steps than here [points to the right side of the video]?
- 129 A&G: Yes.
- 130 T: Okay. And what's happening here [points to the right side of the video]?
- 131 A: I mean, they also moved the three to the right. I mean the numbers.
- 132 T: And they changed its sign.
- 133 A&G: Yes
- 134 T: Okay. So, if I now ask you, why do we change the sign? Can you explain it a bit?
- 135 G: You change the sign to... I mean, initially to add the same number with the opposite sign on both sides, which helps cancel out the number that was originally there and add it to the other side.
- 136 T: Okay. Great. Aldo, do you want to try repeating it? Does what Giulio said make sense to you?
- 137 A: Yes
- 138 T: Okay. So, for example, here they do it with this minus three [points to the second line on the right side of the video], and here below [points to the fourth line on the right side of the video]?
- 139 A&G: With the x.
- 140 T: For example, what do they do with the x, Aldo?
- 141 A: I mean here they move it to the left, so they change sign
- 142 T: And why do they change the sign?
- 143 A: Because it's added—I mean, x is added on both sides with minus x.

After some exchanges in which the two students are still focused on the fact that the reason of adding the same value is that we want to take some terms from one side to the other of the equation, the tutor intervenes explicitly referring to what they have written before and going back on the tablet to their utterances (line 115). This seems a turning point in Aldo and Giulio's discourse. Indeed, Giulio affirms "I mean... we added the same number on both sides" (line 118) and "the equation stayed an equality, it didn't change" (line 120) showing that he has related the action performed in the video with the utterances recalled by the tutor. When the tutor asks to relate this action with the other side of the video, Giulio says "Here, instead, it's done directly, that step over there isn't shown" (line 123) while Aldo affirms "Yes, they move the sign directly" (line 124). Moreover, in line 127, Giulio, at the question of the tutor to confront the two sides of the video, restates "The same thing, but without the intermediate step, and they immediately moved the x to the

other side". The two sides for the students are related, and specifically on the left part are just missing some "intermediate steps". When again the tutor asks to clarify why when a term is moved from one side to the other (line 134), the two students seem to have changed their narratives.

This entire episode shows us the strong attachment Aldo and Giulio have to the familiar, for them, procedure for solving linear equations. When the tutor tries to ask the reason, and more explicitly the justification to the actions performed on the right side of the video (adding the same value to both sides), Aldo and Giulio repeat several times that this is done in because the solver in the video wants to take some terms from one side to the other of the equal sign. The aim is to accomplish the procedure for them familiar (change sides, change sign rule), and the action on the right side are just done to favour it. It's only after an explicit intervention of the tutor (line 115) who refers to their principles, that their discourse changes. It shifts towards the explication and the justification of the procedure accomplished on the right side, guided by the questions of the tutor. In the end of the excerpt, the narrative of Giulio seems nearer an expert one with respect to the actions accomplished: "You change the sign to... I mean, initially to add the same number with the opposite sign on both sides, which helps cancel out the number that was originally there and add it to the other side" (line 135). In his sentences, the action of adding the same number with opposite sign on both sides is done to help cancelling out a number on one side and add it to the other side. It is not clear if for him the procedure of adding the same value is justified by the fact that we want to take a term on the other side of the equal sign or that we can accomplish the procedure of changing sides changing sign because we implicitly add the same value on both sides of the equation. The answer of Aldo too is not clearly explicating the relation of the two things. Despite this, we can highlight that at least some aspects of the right side have been clarified and justified according to the principle, that are reinvested in the following task situations till the last session.

8.1.5. Aldo and Giulio's fifth session: the mathematical task situations

Aldo and Giulio, during the last session, deal with some of the mathematical task situations having at disposal the DEOs they have used during the experimentations. I present here task situation 1M and task situation 2M that involve equations and inequality and that provide us insight into students' discourse after the activities with our DEOs.

Task situation 1M:

For which values of x , $3x + 1$ is equal to $x + 2$?

For which values of x , $3x + 1$ is greater than $x + 2$?

For which values of x , $3x + 1$ is less than $x + 2$?

After having read the task situation, the tutor leaves the two students alone. Giulio grasps the tablet with the moving-arrows-DEO and insert in the input fields the expressions $3x + 1$ and $x + 2$, generating the two tick marks corresponding to the expressions. In the meantime, Aldo takes the tablet with the blank page. While Giulio drags the red tick mark along the enumerated axes, Aldo writes, as in Figure 8.34, an equation and performs some manipulations on it.

$$\begin{array}{l} 3x + 1 = x + 2 \\ \hline 2x = 1 \\ \hline x = \frac{1}{2} \end{array}$$

Figure 8.34: Visual mediators produced by Aldo in task situation 1

- 1 T: What did you do?
- 2 A: I did that three x plus one equals x plus two, so I considered that three x plus one would be balanced with x plus two.
- 3 G: Yeah.
- 4 A: If I were using a scale, I would have simplified one x with one x, and what was left was two x plus one and x plus two. So, then I could simplify the two as one plus one, I removed the one from the other side, and I had two x equals one. Then, to find x, I had to divide both sides by two.
- 5 T: So here you wrote an equation. And the solution?
- 6 G: x equals one-half.
- 7 T: And what does this x equals one-half tell me?
- 8 G: That for x equals one-half, the equation is an equality.
- 9 T: And what does that mean?
- 10 G: That for x equals one-half, what's on one side is equal to what's on the other side... so there are the same quantities on both sides of the equal sign.
- 11 T: Aldo, what do you say?
- 12 A: I mean... they're equal... they form an equality if x is equal to one-half.

The tutor intervenes referring to Aldo and asking him what he has done. In line 2 he uses the instance “would be balanced” to refer to the relation he expressed with his inscriptions among the two given expressions. Then he goes on (line 4), explaining his manipulations according to what he would have done on a scale. In these utterances Aldo makes a mistake, probably due to his difficulties with the language, affirming that on the right, after the first action, would remain x . Indeed, in going on, he does not consider this ‘ x ’ anymore and just refers to known values. His inscriptions are compressed: he has not developed in the writings all the steps he instead describes to the tutor. While Aldo explains his writings, Giulio pays attention to his words and, when the tutor asks about the solution of the equation he takes part in the discourse. The two students’ answer (lines 10 and 12) about the ‘meaning’ (line 9) of the statement “for x equals one-half, the equation is an equality” (line 8) constitute endorsable narratives for the E-discourse and are instances of the occurred construction of the mathematical object ‘solution of an equation’. In this episode, we see an example of merging TPS-DEO-discourse and E-discourse. In lines 2 and 4 Aldo’s discourse is about the equation he has written and, in describing it, he uses terms that belong to TPS-DEO-discourse. Aldo chooses to refer to the scale to explain his procedure, developing a $d_E(\text{DEO}, E)$. In his discourse we see instances of a happened transplanting process of the procedure of removing the same value from both pans to find out the value of the not known objects in a discourse about equations. Actually, in this excerpt emerges the ambiguity of discourses in which we see features from DEO-discourse and E-discourse. Indeed, Aldo is talking about his actions on the equation but then affirms “If I were using a scale”. The object of his discourse could have changed towards the scale. Or, the object of his discourse could be both the scale and the written inscriptions, that together realize an equation. This episode makes emerge that some discourses cannot be codified as d_{DEO} and d_E because in them appear mixed features that could refer to both the DEO and an equation.

Task situation 2M:

$$3 + x = x + 3$$

Does it have any solution?

Can you find them or possible explain why there cannot be any?

As soon as the tutor leaves them working alone, Aldo writes on the notes the equation above. Then, he writes under it $0 = 0$ and he cancels the terms in the equation (see Figure 8.35).

$$\cancel{3} + \cancel{x} = \cancel{x} + \cancel{3}$$

$$0 = 0$$

Figure 8.35: Visual mediators produced by Aldo in task situation 2

Once Aldo has finished writing, Giulio inserts in the input fields of the moving-arrows-DEO.

- 13 T: What do we expect could happen?
- 14 A: That for every value of x we give, the arrows will always be in the same place.
- 15 T: What does that mean? Always still?
- 16 A: I mean, no...
- 17 G: They'll always be at the same level. [dragging the red tick mark]
- 18 A: Same value.
- 19 T: Okay, same level, same value. And this [points to Aldo's writing, Figure 8.35], what does it tell us about our equation? So, the answer to this question—does it have solutions?
- 20 G: Yes... for every x belonging to R .
- 21 T: What does "for every x belonging to R " mean?
- 22 G: It means that for every value of x ... [...] for every value of x , the equation will always be an equality.
- 23 T: If we look at this one [points to Aldo's equation] $3 + x$ equals $x + 3$, compared to that, if I ask you [she refers to Aldo] whether it has any solutions and you tell me "for every x in R ," what do you mean in relation to this equation here?
- 24 A: I'd say that 0 equals 0.
- 25 G: For every value that x takes—like, for example, two—so if x were two, three plus two makes five on both sides. And that works for every value x takes.
- 26 A: Yes.
- 27 T: So, what is this "for every x in R " telling us?
- 28 G: For each value we give to x , that is, the two, the equation will always be equal ('l'equazione sarà sempre uguale', in Italian)
- [...]
- 30 T: For each value we give to x
- 31 G: The equation will always be an equality

- 32 T: So, we say three plus x
- 33 A: Will always be equal to x plus three
- 34 G: To x plus three
- 35 T: Thus, if we try to remove as you have done, we reach something like zero equals zero. What is this zero equals zero telling us?
- 36 G: That for each value it happens that we arrive always to zero equal zero, thus... Every value is fine... That x takes on.
- 37 T: Are we convinced, Aldo?
- 38 A: That is, yes, every equation that is equal will have as result zero equals zero
- 39 T: Every equal equation, what do you mean?
- 40 A: Such as, four plus x equals four plus x, it will be zero equals zero too.

In line 13 the tutor intervenes asking what they expect to happen once they have inserted the two expressions in the input field of moving-arrows-DEO. Before Giulio starts dragging the red tick mark, Aldo gives an answer that probably sounds incorrect for his choice of the words, but that is coherent with what later happens (line 14). Indeed, he completes his mate words with “same value” (line 18) referring to the position of the two tick marks. The tutor’s request (line 19) to relate the tick marks’ position with the question ‘Does it have any solution?’ open to a new discourse. Indeed Giulio, probably recalling a formulation common at school, that he has used also in other task situations, answers “for every x belonging to R” (line 20). The tutor, recognizing this as a typical narrative associated to equations, that is often use in a ritualistic way not relating the output of the manipulation with the idea of infinite solutions, asks to clarify the expression. While Giulio seems to use the narrative coherently with the canonical mathematical discourse (lines 22 and 25), Aldo’s discourse seems attracted by the equation $0 = 0$ (lines 24). The tutor goes back to his inscriptions (line 35), and Giulio answers producing a narrative endorsable in E-discourse (line 36). In conclusion, also Aldo seems to relate the initial equation (and other possible equations of the same ‘form’) with the output he reached in Figure 8.35, showing a change with respect to the initial interview about the same task situation.

8.1.6. The case of Aldo and Giulio: some considerations

To conclude, I want to sum up some points emerged within the analysis of Aldo and Giulio’s experimentation. I focused on the activities with two-pans-scale DEO and the subsequent ones with silent videos and equations. I have made this choice because I wanted to deepen our understanding about the potentiality and the discursive opportunities of some of these activities with respect to our design

objectives. Indeed, as I have discussed in the previous chapters, after the first cycle of experimentation we felt the need to make more explicit the ‘bridge’ between DEO-discourse and E-discourse and to involve more explicitly features of the last one in the activities with DEOs. Moreover, between the first and third cycles, one of our main objectives became to foster the development of the principles of equations (referred to as P_0 , P_1 , and P_2 in chapter 5), to sustain the theoretical organization of the E-discourse and to support the construction of narratives to accompany procedures for solving linear equations. To do so, we have:

1. Introduced new task situations:
 - a. task situations of imagined action
 - b. silent video task situations to prompt the development of bridging discourse
 - c. task situations explicitly involving equation to foster the transplanting of routine developed in DEO-discourse into the E-discourse
2. Developed a script for tutor’s intervention with respect to each task situation

In the case of Aldo and Giulio we can observe that:

- Aldo and Giulio, in the episodes discussed, as well as in many of the others that I have not presented here, expand their DEO-discourses in tune with our analysis of transplantability (see for instance Figure 8.13).
- specific tutor’s interventions combined with certain task situations have facilitated first instances of bridging discourse. Indeed, the request of write down the scale as well as the tutor’s proposal with respect to this request, has prompted, in Aldo and Giulio’s discourse:
 - the production of visual mediators as the one in Figure 8.12, in which the equal sign and the plus symbol are used.
 - In addition, this has led to the written formulation of the scale with known and not known values as an equation (see the first part of excerpt 5). Moreover, the discourse developed around the little red ball and the symbol ‘x’ can be a seed of bridging discourse.
 - In the same excerpt, the equation written by Giulio allows to talk about the mathematical object ‘unknown’ as well as to deal with the change sides, change sign rule. Indeed, once clarified what the two students mean by unknown, they are focused only on the equation to find out the value of the little red ball, and they perform, wrongly, the overmentioned procedure. Thus, students develop $d_{DEO}(DEO, E)$ and $d_E(E)$. The intertwining of discourses, especially with respect to the change sides, change sign rule, makes emerge critical points that we

wanted to deepen and clarify with our activities. Indeed, Aldo and Giulio have wrongly transplanted the change sides change sign rule on the scale and this become an opportunity to develop a bridging discourse about the actions performed on the scale/on the equation and the consequent positions of the scale/relation between the two algebraic expressions.

- The task situations of imagined action in which students have at disposal a picture of the scale and are asked to find out the value of the not known object, together with tutor's prompt of justifying the imagined feedback to specific imagined action, has changed the way in which Aldo and Giulio participate to the discourse. Indeed, both in the task situations 9, 11 and 14 that involve scales and in the task situation 17 and 19 with equations, they refer to the principles they have elaborated to justify their considerations. There is a new way of producing narratives, indeed, they are always supported by specific assumptions, that are P_0 , P_1 and P_2 (with respect to the scale and to the equations).
- The formulation of principles has offered them a new way to solve equation. Indeed, when in task situation 19 they are asked to solve an equation, Aldo and Giulio do not implement the change sides, change sign rule but use the principles they have previously elaborated.
- According to the two previous points, the routine developed in the DEO-discourse has been transplanted in Aldo and Giulio's E-discourse, as well as the way in which they justify it, and thus the way in which they participate to the discourse. They have developed a new routine, that is not been imposed by an external individual, as the teacher, but that has been transplanted by themselves from other discourses.
- The second silent video task situation has made emerge a difference in the tutor's and in the students' perception of what a justification can be. Indeed, while the tutor asked to justify the inscriptions appeared in the right part of the video (for instance the 'plus 3' on both sides of the equation) aiming at promoting a bridge with P_1 , the students were focused on the will of the person writing in the video of taking the 'minus 3' from the left to the right part of the equation. Despite this, at the end the two students agree that on the left part there are some steps missing, or, to say with the words of Giulio, on the right part there are "intermediate steps".
- In the last session, accomplishing the mathematical task situations, Aldo and Giulio's E-discourse shows new features. They refer to the principles (and also to the scale) to solve the equation. Moreover, the mathematical object 'solution of an equation' is described through narratives coherent

with the mathematical ones. Their E-discourse has expanded. They relate the procedures accomplished and the obtained product with the initial task situation, instances of explorative participation. This happens also through the merging of different discourses and the multivalence, offered by the intertwining of discourses, thanks to which Aldo and Giulio, in this case Aldo (see task situation 1M), produces narratives that could be about both the scale and the equation. Finally, some procedures and narratives that they show within the initial task situations have changed. For instance, Aldo in the last session relates the output of the implemented procedure, that is $0 = 0$ with the fact that the equation considered has infinite solutions.

8.2. The case of Alba and Elsa

Alba and Elsa are two students from the arts high school. They came from the same classroom and attended 5 sessions after the initial interview. During their initial interview have emerged several aspects of their mathematical discourse and their participation to it. Here we focus on short excerpts in which they face task situations with linear equations in order to understand the main features of their initial algebraic discourse.

8.2.1. Elsa's initial interview

Elsa's interview was conducted by one of the DynaMat team's researchers and lasted about an hour and a half. When, in the beginning, she is asked if there is something she really does not like about mathematics, she says "Well, many many, because let's say I don't have a good relationship with mathematics. Since elementary school, I have always been dyscalculic". Then she adds "Like very long expressions, that there it confuses me; I can't do them...". Elsa presents quite soon herself as a student with deep difficulties in mathematics. She describes herself with respect to mathematics as in the following sentences, "I mean, I don't consider myself very good at mathematics also in everything" and "Well, even in elementary school, I could see that I wasn't... how do you say it? I wasn't cut out for mathematics".

Task situation 1

To introduce Elsa mathematizing with respect to algebraic discourse, I discuss here a brief excerpt, when she is asked to deal with the equation $13 - a = 13 + 11$.

- 1 I: What comes to mind to you? What does this thing here remind you of? What comes to mind when you see this thing here? [showing the overmentioned equation]

[...]

- 2 E: That ... [Elsa remains silent for 14 seconds]
- 3 I: What are you looking at of this thing here? Even parts of this thing here, what do you see in it?
- 4 E: Actually, I was thinking about how, how, how to solve it, first of all.
- 5 I: Ah, you're thinking about solving, so look, let me ask you the next question right away. It doesn't matter what it is, does it have solutions? Because you mentioned solving it in some way.
- 6 E: Yes
- 7 I: We also ask ourselves: can you find them or explain why there might not be any? So you thought about solving it. What does solving it mean to you, and how would you do it?
- 8 E: So, how I would do it is a bit... well, seeing the equals sign, I'm used to seeing an x there; so, I would move everything with the x to this side [gestures with both hands as if she is moving something] and those without the x over here, but there isn't one, so...
- 9 I: There isn't [an x]?
- 10 E: No
- 11 I: There isn't an x , OK. Is there something that is, let's say, similar to the x ?
- 12 E: Which is the a .
- 13 I: OK. So, if the a were the x , let's say, how would you do it?
- 14 E: I would move it [gestures as she is moving something] over here.
- 15 I: OK, could you write for me how you would do it?
- 16 E: Like here... I would take the a , and then the other num[bers]... and then I would do equals three... [she writes a , Figure 8.36a] minus, minus, wait, it's minus a [she adds the symbol minus in front of a , Figure 8.36a], thirteen, mmm, wait [she writes $13 + 13 + 11$, Figure 8.36a].
- [...]
- 17 E: And then to finish calculating everything
- 18 I: OK, so how would it come out for you?
- 19 E: So a
- 20 I: Ah you can use the calculator if you want [Elsa nods and the interviewer opens an app of the tablet with the calculator] here it is.
- [...]
- 21 E: 13 plus 13 plus 11... 37 [Elsa types $13 + 13 + 11$ into the tablet's calculator, getting 37]

- 22 I: [Elsa closes the tablet calculator page and reopens the page where she was writing] you got 37
- 23 E: So, I would do... [Elsa first adds another equal sign to what she had written in Figure 8.36a] minus a [she writes as in Figure 8.36b]
- 24 I: So, OK, can you read to me quickly what you meant by this thing here?
- 25 E: That for me, solving means first doing [something], first thinking about all the steps, then writing them down, and then solving them.
- [...]
- 26 E: a I move it.. it stays here [the letter a on the left side], 13 I move it [the number on the left side of the equation, she makes a gesture as to take it to the other side of the equal sign] to the right, that becomes 13 plus 13 plus 11; that gives minus a minus 37

A handwritten equation on a tablet screen. It reads: $-a = 13 + 13 + 11 =$. The numbers are written in a cursive, slightly slanted style. The equals sign is in the middle of the expression.

a

A handwritten equation on a tablet screen. It reads: $-a - 37$. The letter 'a' is written in a cursive style, and the number '37' is also written in a cursive style. The minus signs are clearly visible.

b

Figure 8.36: Visual mediators produced by Elsa within task situation $13 - a = 13 + 11$.

In the beginning, after the initial questions of the interviewer, Elsa remains silent. Prompted by the tutor, she affirms that she is thinking about how solve (what she is looking at). In line 8 Elsa refers to the fact that she is used to see the symbol x when there is an equal sign, and in that case, she would move things from one side to the other, distinguishing the terms with x and the ones without it (line 8). Since there are no x there, Elsa seems stuck. The interviewer's question about the possibility that there could be something similar to the x (line 11). According to the interviewer's suggestion, she imagines the a to be an x , and proposes her procedure to solve the inscriptions proposed to her (she does not use the term equation). She seems quite confident with the procedure she is implementing (lines 16 to 23), and, despite the need of the calculator that is offered by the interviewer, she proceeds describing her steps. The procedure she executes remind the change sides change sign rule. She seems attached to it and it seems to offer her a distinct way to proceed. Solving for her is "first thinking about all the steps, then writing them down, and then solving them" (line 25). She does not mention the solution of the equation and just stops

when she reaches $-a - 37$, probably because she has obtained a familiar inscription composed by just one letter and one number.

8.2.2. Alba's initial interview

Alba's initial interview was conducted by one of the researchers of the DynaMat team and lasted about one hour and fifteen minutes. In the beginning, when she is asked if there is something that she does not like of mathematics, she affirms "No, it's really just something... because it really tires my mind out. Like, most of the time when I do math—whether it's in class or at home—I get a headache because it involves so much thinking, and for me that's hard... thinking about all these things, all these calculations, and all the different methods for doing math. So that's mostly what bothers me about math. It's how exhausting it is". From these first utterances, Alba expresses her difficulties in doing mathematics, especially because for her is strenuous, involving a lot of thinking, calculations and methods to do maths. These considerations return in the following excerpts.

Task situation 1

In this first excerpt, Alba is dealing with the task situation involving the equation $13 - a = 13 + 11$.

- 1 I: So the first thing I'm asking you is if you've seen something written like this: *13 minus a equals 13 plus 11*.
- 2 A: Yeah, I'd say I've seen it.
- 3 I: What comes to your mind?
- 4 A: Well, since there's an equals sign, what comes to mind is an equation [speaks more softly]. But... um, I don't know if it's possible to do the equation because, like I said, it's hard to apply and understand what I'm supposed to do, because there are so many ways to do math, so the right method is hard to find. Just like that, with no context. Ugh. But I have to solve it because... um, what I'm thinking is that I can't, because I don't have any reference for what I'm supposed to... like, I don't have a guide, and like this, I just... I feel kind of stuck.
- 5 I: What... help me understand better what you mean by "guide."
- 6 A: Because if I'm, if I'm um... given something to calculate, to solve like this out of nowhere, without saying what it is, without any reference, I feel kind of stuck, because I don't—don't have a reference, and I don't know what to look for in my brain to apply it.
- 7 I: Do you usually get a reference in class?
- 8 A: Yeah, like for example we'll do "Factor decomposition." I know we're doing that. But if no one tells me, I feel stuck.
- 9 I: OK.

- 10 A: Even if I might actually know that thing. It's still hard to find it without.
- [...]
- 11 I: Here the question would be this: whether this thing here has any solutions—okay—if there are any, if you know how to find them; and if there aren't, whether you can explain why you think there can't be any.
- 12 A: Oh gosh... [whispered]
- 13 I: Again, I'm asking you first if you'd like to try to tackle it, or if you'd rather move on to the next question.
- 14 A: I'd rather move on, because like I said, I just can't figure out what I'm supposed to do—what I'm supposed to do right now.

Within this task situation, Alba's discourse is mainly characterized by narratives on her way of doing mathematics. Indeed, in line 4, she affirms "it's hard to apply and understand what I'm supposed to do, because there are so many ways to do math, so the right method is hard to find". In front of a task situation in which she is not given 'a reference' of the topic she is dealing with (line 6), she does not know how to act (line 10). Thus, despite she mentions herself the word 'equation' (line 4) that could become a reference for her actions, once she is asked to find the solutions, she decides to move on because "I just can't figure out what I'm supposed to do—what I'm supposed to do right now".

Task situation 2

Immediately after the discussed episode, Alba faces the task situation involving the equation $3 + a = a + 3$.

A handwritten equation in black ink on a white background. The equation is $-a + a = 3 - 3$. The minus sign at the beginning is written as a short horizontal line. The letters 'a' are written in a cursive, handwritten style. The numbers '3' are also handwritten. The equals sign is a simple horizontal line.

Figure 8.37: Visual mediators produced by Alba within task situation $3 + a = a + 3$

- 15 I: Ok. So now this changes a bit, and the first thing I'm asking you, again, is what comes to mind when you see this—if you've ever seen it before.
- 16 A: Okay, what comes to mind here—
- 17 I: —written like this—
- 18 A: An equation. Where on one side, you have to put the numbers, and on the other side, the letters, right?

- 19 I: Okay, yes.
- 20 A: So it would be... um, I don't remember which side the letters go, which side the numbers... but I do remember that the signs change—I remember that the signs change. You could do... oh, excuse me.
- 21 I: There's also an eraser, if you prefer, you know.
- 22 A: So, if this bit changes, it would be... [writes $-a + a =$, see Figure 8.42] um wait, I think like this [Figure 8.42] something like that—maybe I got the signs wrong because I don't remember which part moves. But I do remember that they move, and the signs can change. I mean... and that it's an equation.
- 23 I: Do you remember seeing this at school?
- 24 A: Yes, yes, I saw it at school—we did equations. So, it might be wrong, but that's what I think.
- 25 I: Okay, and so to this question—does this have any solutions?
- 26 A: Yes, in this case it's an equation, um I think it has a solution. The terms are pretty simple. Twos, threes, and two *as*. So, if you move them the right way, I think the solution comes out.
- 27 I: Okay, and would you know how to find that solution?
- 28 A: Yeah, I think I can find it. Like I said, I don't remember everything right now, but I do remember—I mean, I had understood how to solve equations—I'd just need to review it a bit [laughs], but I know how to solve them.
- 29 I: Ok, so in this case?
- 30 A: Well, right now I've forgotten a few small details.
- 31 I: But if you had those small details—
- 32 A: I think I could do it. I could solve the equation.

Differently from the previous task situation, in this case Alba decides to act. This could happen because, as she says, “the terms are pretty simple”. She recognizes the visual mediators proposed as an equation and proposes a procedure related to it (line 18). She mentions the fact that she does not remember on which side she has to put the letters and on which the numbers, but she remembers the need to change the signs (line 20). This is confirmed when she takes the tablet and writes as in Figure 8.37: “I do remember that they move, and the signs can change. I mean... and that it's an equation” (line 22). Once she has implemented the procedure she remembers, she is stuck again. Indeed, she does not remember everything (line 28) and, in this case, she cannot find the solution. Despite this, she affirms that if she had the small details she has forgotten, she could solve the equation.

In both task situations, we can see that Alba's discourse when she finds herself in front of equations, is characterized by words and verbs that refer to memory and

procedures. Indeed, in the first one she affirms that she needs a reference to guide her, thus a word or similar, that give hints in order to proceed. In the second task situation, her procedural participation to the mathematical discourse emerges even more. She finds herself in front of an equation, she starts acting as she remembers must be done, but once asked to find the solution, she needs more details to proceed. Alba produces rituals accompanied only by narratives that refers to how she is supposed to act, according to her previous experience and especially about what she remembers. She does not focus on the relation of equality between the algebraic expressions proposed, she does not mention what for her is the solution of an equation, she does not try to find other ways to accomplish the request. Once the hints and the references miss, she skips to the next task situation.

8.2.3. Alba and Elsa's first session

Alba and Elsa within the first session deal with task situation 1, 2, 3, 4 and 5.

Within task situations 1, 2, 3 and 4 they have elaborated the principles P_0 and P_1 . Respectively, they formulated them as:

- P_0 : to balance a scale, you need to have equal weight on both platforms (Figure 8.38a)
- P_1 : If we have in the beginning a balanced scale, we can add or remove to/from the two pans 2 or more numbers to have it balanced afterwards. The 2 or more numbers can be sum or single numbers but they must make the same weight to both the pans (Figure 8.38b)

We can notice, as in the case of Aldo and Giulio, that in the formulation of P_0 proposed, Alba and Elsa include only one of the two implications that we included in P_0 proposed in chapter 5. Indeed, from their inscriptions, we see the implication 'If a scale is balanced, the total value on the two pans is the same'. Thus, they are not considering the converse implication, 'If the total value on the two pans is the same, the scale is balanced'. In this case too, from a cognitive perspective (Ferrari, 2021), we could suppose that they are including both the implications.

PER METTERE IN EQUILIBRIO
UNA BILANCIATA BISOGNA AVERE
UN PESO UGUALE SU TUTTI
E DUE I PIANI

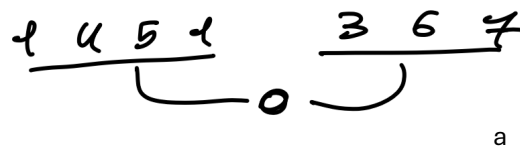
a

Se abbiamo all'inizio una
bilancia già equilibrata
possiamo aggiungere
o togliere sui/dei 2
piatti 2 o
più numeri per farla stare
in equilibrio. I 2 o più
numeri possono essere somme
o numeri soli ma devono
fare lo stesso peso da tutti e
due piatti.

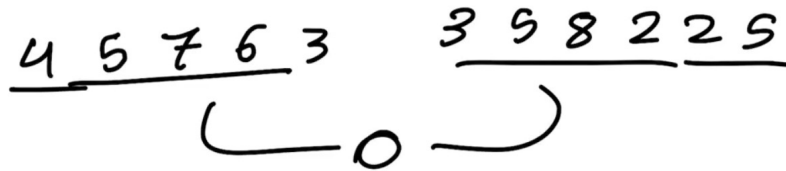
b

Figure 8.38: Written formulation of $P_0(a)$ and $P_1(b)$ by Alba and Elsa

Moreover, within these task situations, they perform several actions, engaging in the procedure of adding or removing the same value from both pans from a balanced scale to have it balanced afterwards. As emerged during the initial interviews, Elsa does not participate actively to the task situations, while Alba seems more engaged, proposing ways to accomplish the request and often writing on the blank page of the tablet. Indeed, when they are asked to write down the scales, she proposes the visual mediators as in Figure 8.39. In task situation 3 (Figure 8.39b), the tutor, aiming at prompting the transplanting the visual mediator 'equal sign' in the DEO-discourse to realize the balanced position of the scale, proposes to write down the pans as two sums of numbers and to put an equal sign in the middle. This proposal is accepted by the two students, and thus the possibility of transplanting E-discourse's features in DEO-discourse is open. Despite this, in the following excerpts we can see that they do not use such visual mediators when they are asked to write down the scales.



a



b

Figure 8.39: Visual mediators produced by Alba to write down the scale. Task situation 2 (a, wrongly written 7 instead of 2 on the right pan) and task situation 3 (b)

Excerpt 1

Task situation 5: *Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.* (Figure 8.14)

In this excerpt, Alba and Elsa deal with the first scale with objects with not known values. When the tutor, at first, ask them to write it down, Alba produces the visual mediators in Figure 8.40, despite the previous attempt of the tutor to transplant the equal sign in their DEO-discourse. The first part of the episode is devoted to the little red balls that attract since their appearance the two students. Indeed, even if the tutor leaves them alone to accomplish the task (removing 3 objects), they start moving the little red balls, outside and inside the pans, observing the feedback of the scale.



Figure 8.40: Visual mediators produced by Alba to write down the scale in task situation 5

After some attempts in order to ‘understand’ these little balls, the tutor leaves Alba and Elsa working alone, when they have reached the situation in Figure 8.41, in order to reflect about the position of the scale after they have removed one red ball from each pan. Alba and Elsa’s discourse take another direction. More specifically,

Alba starts to focus on the red ball left on the right pan, while Elsa looks at her quite silently.

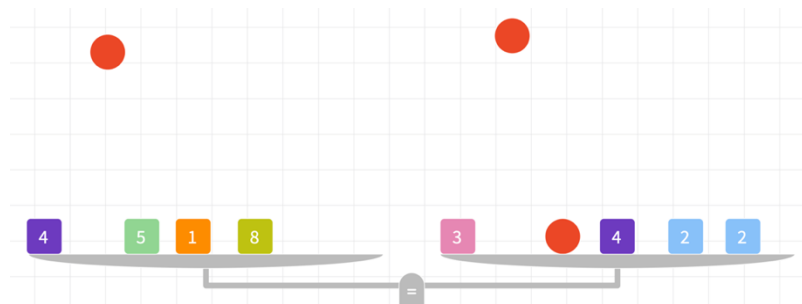


Figure 8.41: Situation on the DEO reached by Alba and Elsa after several actions on the red balls

- 1 A: Wait, wait. I think there's something about this little ball [she points to the red ball on the right pan]... do they have the s... the sum? Yeah, the sum of this [she points to the right pan] and the sum of that [she points to the left pan] give the same product—oh god, my Italian. So, I'd like to try [she takes the tablet with her inscriptions, Figure 8.40] If we do 4 plus 5 plus 1 plus 8 [she writes the left part of equality in Figure 8.42], and here there's 3 plus little ball plus... [she writes the right part of equality in Figure 8.42]
- 2 E: But how much is the little ball worth?
- 3 A: We're about to find out. Because, um, if we add this—4 plus 5 is 9, plus 1 is 10, plus 8 is 18 [she makes the line under the sum on the left side of the equality, Figure 8.42]
- 4 E: It makes 18?
- 5 A: This one too will be... you'll see... 18 [she makes the line under the sum on the right side of the equality, Figure 8.42]
- 6 E: 3 plus 1 plus 4 plus 22...
- 7 A: So, if we do... if we... oh man... Okay
- 8 E: Okay
- 9 A: Let's subtract, I need to subtract. From 18 we take away 3, that makes 15 [she writes 18, then she makes a cancellation mark on it and write 15, Figure 8.43a].
- 10 E: Mmm
- 11 A: Take away 4, that's 11 [she makes a cancellation mark on 15 and write 11, Figure 8.43b], take away another 4 and it's—ss—wait, 11 minus 4 is seven [she makes a cancellation mark on 11 and write 7, Figure 8.43c]. So, with that logic, this would be 7 [she makes an arrow above the black ball in Figure 8.42 and writes above it 7]. 3 plus... where's the calculator? The calculator, where is it? [she goes out the blank page on the tablet]
- 12 T: You need the calculator? Here it is [open the calculator on the tablet]

- 13 A: Wait, what did I just do? [she goes back to her inscriptions] Um, 3 plus 7 plus 4 plus 2... um... [opens again the calculator] 3 plus 7 plus 4 plus 2...

[...]

- 14 A: Yes. It makes 18. So, in this way I have found the value of the little ball.

$$\begin{array}{c} 4 + 5 + 1 + 8 = 3 + \text{●} + 4 + 22 \\ \hline \quad \quad \quad | \quad \quad \quad | \\ \quad \quad \quad 18 \quad \quad \quad 18 \end{array}$$

Figure 8.42: Visual mediators produced by Alba in task situation 5 (she later adds the plus symbol between the numbers 2 and 2 in the right pan)

Within this short excerpt, Alba's discourse is focused on the little red ball. She aims at finding its value even if the tutor has left them with another task. Starting from considering that the two sums of the pans of the scale should be equal (line 1), she writes as in Figure 8.42 an equation, where she uses a little black ball as a realization of the red ball. In proceeding, she is strongly involved in what she is doing, and she is confident in her procedure. Considering the fact that both pans must be worth 18 (lines 3 and 5), she 'removes' the weights with known values from the value 18, finding 7 (line 11). Then she checks her procedure, adding up the values of the objects on the right pan, considering the ball as 7. She concludes: "Yes. It makes 18. So, in this way I have found the value of the little ball" (line 14). Alba has just implemented a new procedure accompanied by meaningful narratives, and once she has reached an output, she relates it to the initial problem (proposed by herself, line 1).

Figure 8.43: Visual mediators produced by Alba when she does calculations to find out the value of the little red ball

After that, Elsa asks for clarification that Alba offers her retracing her reasoning. Since Elsa seems still confused, the tutor decides to intervene and asks herself Alba to repeat what she has just explained. After some considerations about the little balls, while they are still dealing with the proposal of Alba to find the value of the

little red ball in front of the scale in Figure 8.44, Alba introduces a new way to proceed.

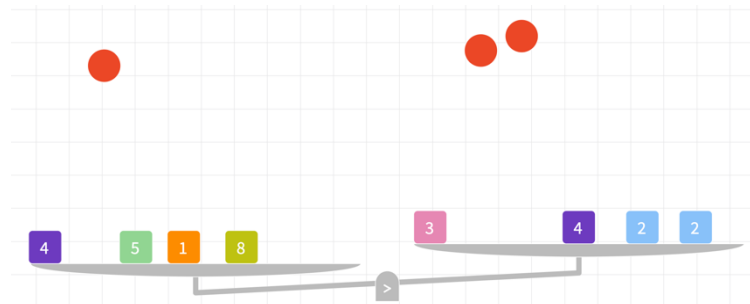


Figure 8.44: Configuration of the scale in line 15

- 15 A: Here it was eighte... here it was 18 [she points to the left pan], I mean without the little balls, and here it makes 11 [she points to the right pan]. So, if I add a little ball, it balances out [she adds a little red ball to the right pan].
- 16 T: If you add a little ball, it balances out. Okay.
- 17 A: So, before it was 11 [she writes 11 on the other tablet], and the other side was 18 [she writes 18 above the just written 11], and it was unbalanced. We need to add a weight to this [she makes a circle around 11] to, to, to make it stay—to bring it back into balance with 18.
- 18 T: Okay.
- 19 A: So, with this, we can just do 18 minus 11, which is 7, so you can figure out that the little balls are actually worth 7. This just came to me now, but before I had done it in a more interesting way

Alba, prompted by the position of the scale when all the red balls are removed, produces narratives and mediators about the relation of the two pans, when a red ball is added to the right pan (lines 15 and 17). Considering the difference among the two pans when they are not balanced, she finds out again that the little red ball worth 7.

The episode goes on with Alba that explains his reasoning to Elsa and the tutor. After that, the tutor goes back to the initial task situation of removing 3 objects from the pans of the scale to have it balanced. Alba and Elsa propose various ways of acting, according to the request of the tutor that from 3 objects goes to 2 and then to 4. The tutor asks Alba and Elsa to show on the inscriptions made by Alba (Figure 8.40) some of the actions they have made on the scale to satisfy the request. Is Alba again who takes the tablet and open a new blank page to make a new picture of the scale. Despite her previous use of the mediators in Figure 8.43, she remains attached to the draw of the scale as in Figure 8.45. They remember some attempts considering two objects to be removed. In the first one (Figure 8.45a) they have removed the objects of value 4, while in the second one, that has been proposed by Elsa, they have removed two little red balls (Figure 8.45b). Alba affirms “Even if we do not know

the value, if we know that they all have the same value, if we cancel one from here [points to the left pan of her inscription and produce the cancellation mark on the little ball, Figure 8.45b] and one from here [points to the right pan of her inscription and produce the cancellation mark on the little ball, Figure 8.45b] it remains the same” and in the meantime she produces cancellations marks on the little balls. The tutor grasps this narrative to go back to the formulation of the principle they have written before (Figure 8.38b). In considering the action of removing the two red balls from the pans, Alba affirms “It applies here too”, referring to their inscriptions.

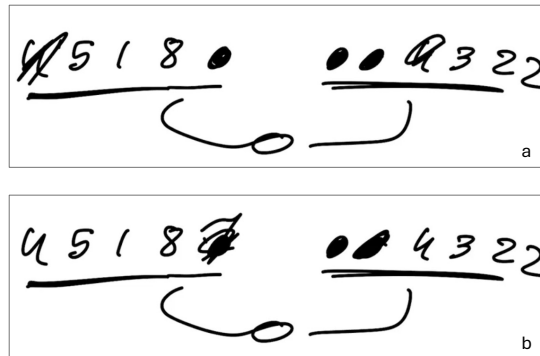


Figure 8.45: Visual mediators produced by Alba when the tutor asks Elsa and her to show on their inscriptions the action accomplished on the scale

The first session of Alba and Elsa concludes in this way. Their discourse, within this session, has expanded. In particular, their DEO-discourse has grown, thanks to the task situations proposed, together with the tutor’s intervention. They have formulated P_0 and P_1 , developing new narratives, they have used new words and visual mediators (for instance, Figure 8.45), they have introduced new routine. In particular, in task situation 5 Alba herself formulates the task of finding the value of the little red balls and proposes different way to accomplish it. Moreover, she shows instances of analytic discourse, explicating that the red balls can be removed even if they do not know their value.

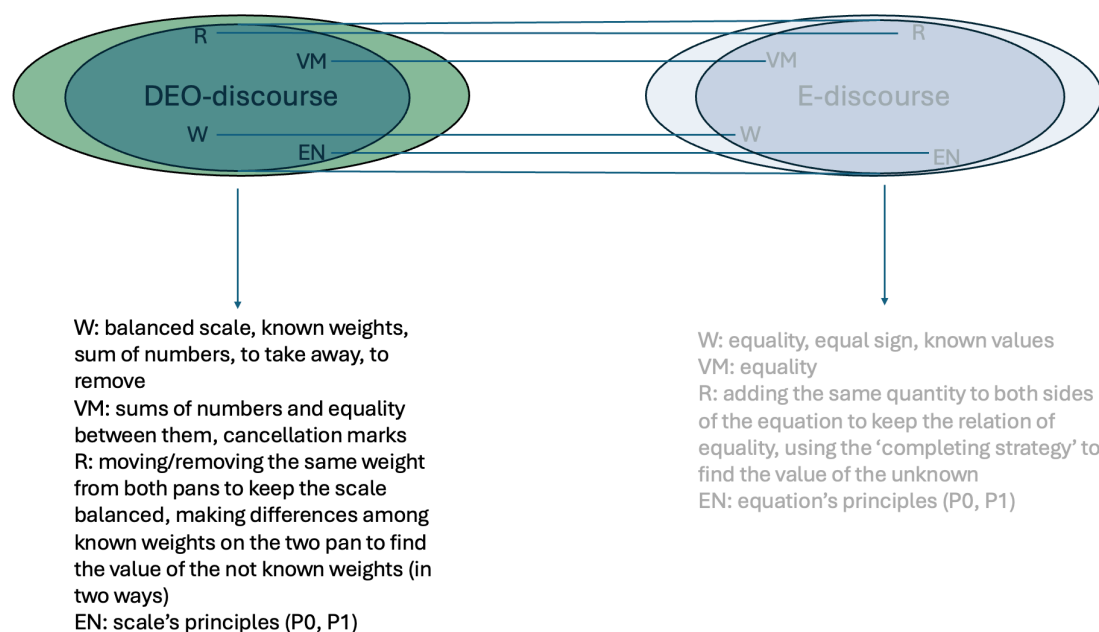


Figure 8.46: Alba and Elsa's DEO-discourse at the end of the first session

In Figure 8.46 I sum up the main features of DEO-discourse developed by Alba and Elsa. In this analysis I have reported the episode referring to the first scale with not known values, in which their discourse involves all the features reported. It is worth to notice that, differently from Aldo and Giulio, Alba and Elsa's discourse do not include features from the E-discourse. All their utterances can be labelled as $d_{DEO}(DEO)$, and even if for some procedures they use the equality form for the scale, they remain attached to the draw of the scale till the end of the session. This attachment offers them the opportunity to deal with task proposed by the tutor and task proposed by themselves in an autonomous way, without the need to remember procedures and manipulations. In conclusion, it is appropriate to notice that is mainly Alba who participate to the session. Elsa often seems a spectator while her mate, together with the tutor, are actress. Their interaction in this last task situation has been investigated in a paper, according to the method of re-enactment to address some emotional and affective aspects (Baccaglini-Frank et al., submitted).

8.2.4. Alba and Elsa's second session

During the second session, Alba and Elsa deal with some task situations involving dynamic two-pans-scale and accomplish the request of finding the value of not known objects using the principles they have written and that we have printed for them and given at their disposal. In the following short excerpt, we see Alba and Elsa facing the first task situation with a picture of a balanced scale.

Excerpt 1

Task situation 10: *Could you find out how much a purple square weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.*

The two students are left alone to face the request by the tutor, and after some seconds, Alba proposes “Ok, wait, I have an idea. We can start with the same technique as before and remove the same things”. They start, together, to find group of numbers on both side of the equation that could be removed because with the same value, obtaining the picture as in Figure 8.47.

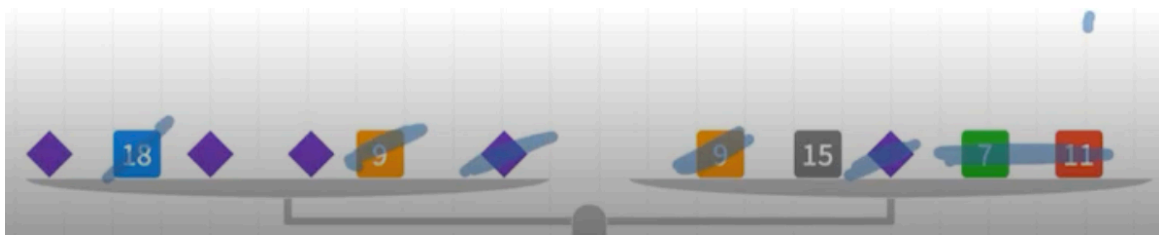


Figure 8.47: Visual mediators produced by Alba and Elsa in task situation 10

- 1 A: In theory, it should stay balanced...
- 2 E: Yes
- 3 A: So, we're left with this 15 here [she makes a circle around the object of value 15 on the right pan, Figure 8.48]
- 4 E: Which has to be...
- 5 A: Which for sure is these—oh my... which for sure is these three [she makes a circle around the two near square and a circle around the square on the left, both on the left pan, Figure 8.48].
- 6 E: Yes.
- 7 A: So, since there are three, we do 15 divided by 3, which is 5... So in theory—
- 8 E: Each one is worth 5, in theory [Alba writes a square, the equal sign and 5, Figure 8.48]

Alba, after their cancellations, affirms as in line 1, referring to the imagined position of the scale after having removed all the objects they cancelled. Then she proceeds considering the objects on the pans finding the value of the not known objects. Differently from the previous session, Elsa seems more involved in the discourse and follows Alba's discourse (for instance, line 8). Moreover, after this exchange, they check their solutions adding up all the values, considering the purple square as 5. It is Elsa who does the calculations on the calculator offered to her by the tutor finding out that the sum is 47 for both pans. In addition, when the tutor asks them to explain their reasoning, Elsa develops narratives in order to retrace their

procedure. In this excerpt we thus see both Alba and Elsa implementing a procedure in the DEO-discourse that is accompanied by endorsable narratives. Such procedure, together with the narratives, are transplantable in the E-discourse as a procedure to solve equations. I want to underline, as I also discuss in the following lines, that Alba and Elsa keep being anchored to the DEO. They do not introduce visual mediators to realize the not known object, they do not produce bridging discourses relating DEO-discourse and E-discourse. This absence of E-discourse seems for them effective: differently from the initial interview, they make decisions and actions producing meaningful narratives.

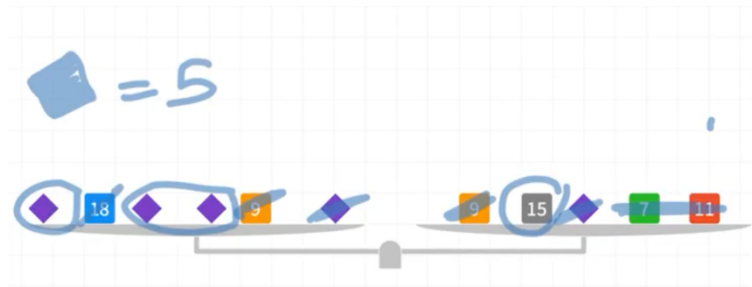


Figure 8.48: Visual mediators produced by Alba to discover the value of the purple square in task situation 10

In this second session, Alba and Elsa face other task situations of imagined action. After them, in the end second part of the meeting, they deal with the first silent video task situation. To recall, this task situation consists in a video divided in two parts, on the left one we see some pictures of a scale on which some objects are removed or changed. On the right part, that is blurred, the viewer can perceive that something is being written, but he cannot see it. There is an audio track that accompany the video, to which miss some parts. The students are required to imagine what is going on in the right part of the video and write it, and complete the audio track in its missing part. A snapshot of the video can be seen in Figure 8.49.

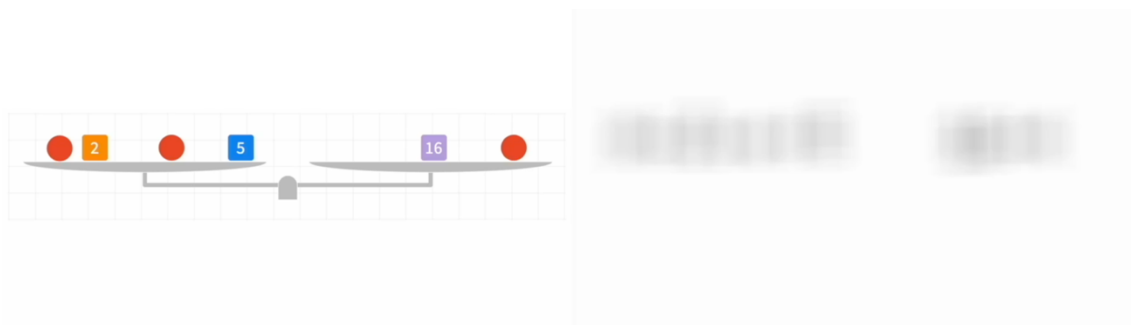


Figure 8.49: Snapshot of the first silent video

I do not discuss here the entire episode, but I report the way in which Alba and Elsa imagine the right part of the video (Figure 8.50). Once again, to deal with and to mediate not known objects, they use black little balls. Thus, also at the end of the

second session, Alba and Elsa's utterances can be labelled all as $d_{DEO}(DEO)$. They do not use words or visual mediators taken by the E-discourse to refer to the not known objects. Instead, they have expanded their DEO-discourse, formulating P_2 , using the principles to justify some of their narratives, developed routines to find the not known value (such as the one presented before), according to our a priori analysis of the transplantability.

$$\begin{array}{c}
 \cancel{x} + 2 + \cancel{x} + 5 = 16 + \cancel{x} \\
 2 + \cancel{x} + 5 = 16 \\
 \cancel{2} + \cancel{x} + \cancel{5} = 9 + \cancel{7} \\
 \textcircled{x} = 9
 \end{array}$$

Figure 8.50: Visual mediators produced by Alba and Elsa to complete the right part of the silent video

8.2.5. Alba and Elsa's fourth session

Since during the third session only Elsa was present, I now focus on the fourth session in which the two students have dealt with the task situations involving explicitly equation and the second silent video.

Excerpt 1

Alba and Elsa are in front of the equation in Figure 8.27 (task situation 17) and are asked to justify the steps among the inscriptions.

- 1 E: Last time I thought that they removed these two x's, then...
- 2 A: From here—
- 3 E: From here, they rewrote $8 + x$ and $4 + 13 + 5$ without the x
- 4 A: Here you decomposed down 5—you took 5 and turned it into 4 plus 1.
- 5 E: Yes, I decomposed it down into 4 plus 1...
- 6 A: That way you can simplify 8 with two 4s.
- 7 E: Yes, 4 plus 4 makes 8.

- 8 A: So, you simplify.
 9 E: I remove the 8, the 4, and the 4, and I'm left with 13 plus 1, which is the value of x, so it's 14.

In this short excerpt, we see Alba and Elsa talking about the steps accomplished on the equation, using mainly words from the E-discourse. In lines 4 and 5 instead, they use the verb 'decompose', that they have previously used with respect to the scale, transplanting it into E-discourse. For this reason, I labelled the two lines as $d_E(\text{DEO}, E)$. Then, the tutor asks them to explain their reflections.

- 10 A: So, here I see simplification...
 11 T: Ok, what do you mean by simplification?
 12 A: Like removing the same value from both sides, like always.
 13 T: **What do you mean by "like always"?**
 14 A: I mean, every time—uh, when there's the same weight on both sides, you can take it away because it's the same, so it doesn't change anything in the end. [...]
 15 T: **We don't have a scale here, right?**
 16 A: No... it's a...
 17 T: What do we have?
 18 A: It's an equation.
 19 T: **What do you mean by equation?**
 20 A: I mean, the value is equal on both sides—in theory.
 [...]
 21 A: Like this one here at the end... like this—I mean, this could also be shown as a balance scale [points to the written expressions].
 22 T: It can be shown as a balance scale, ok. I think you and Elsa did that last time. **Do you want to do it too?**
 23 A: Like, I take the x as a little ball, like those little balls from earlier when we had to figure out the value, right?



Figure 8.51: Visual mediators produced by Alba to write the equation as a scale

- 24 T: Ok, but are you representing this situation here? [points to the first line of the expressions]

- 25 A: Yes
- 26 T: Ok, so you're saying, I can see this as a balanced scale—Elsa did it too. **Did you draw the scale balanced?**
- 27 A: Huh?
- 28 T: Is it in balanced or not?
- 29 A: It is balanced
- 30 T: It's balanced. Ok, so actually, if we think about the fact that here we have one expression, and here we have another, and there's an equal sign in the middle...
- 31 A: It means that there is an equality.
- 32 T: There's an equality.
- 33 A: So, the weight will be the same, so the scale is balanced.

In this second parts, the exchange involves Alba and the tutor. Alba refers to the action of simplification as removing the same value from both sides, “like always” (line 12). To explain these expressions, she refers to the action of removing the same values from both sides (line 14) and the tutor grasps this narrative to promote a bridge between DEO-discourse and E-discourse. In line 21 is Alba that mention the possibility to show the equation as a scale, producing the drawings in Figure 8.51. She explicitly relates the little balls of before with the symbol x (line 23), producing a bridging discourse. The tutor aims at clarifying also the relation between the position of the scale and the equal sign (lines from 24 to 33), developing herself a bridging discourse. Once Alba has proposed her realization of the actions that were shown on the equation on her scale, the tutor aims at developing P_1 for E-discourse. Thus, she fosters the production of a discourse on the equation and the following steps (Figure 8.27), relating them to the scale and its principles. Alba and Elsa arrive to the principle ‘In the beginning we have an equality. If we remove or add the same value to both sides, it always turns out to be an equality’ (Figure 8.52).

In partenza abbiamo un'uguaglianza. Se togliamo o aggiungiamo lo stesso valore da entrambi i lati, risulta sempre un'uguaglianza.

Figure 8.52: Written formulation of Alba and Elsa of P_1 for E-discourse

After that, the students, solicited by the tutor, the two students transplant P_2 that they have written for the scale to P_2 for equation. Indeed, Alba writes what Elsa dictates to her. Elsa is looking at the principle in Figure 8.53a, thus the principle for the scale ('If we have a balanced scale, we can substitute an element with another element or a sum of elements with the same value, to have the scale balanced afterwards'). They adjust it to obtain the principle in Figure 8.53b: 'Starting from an equality, we can substitute an element with another element or a sum of elements with the same value, to have an equality afterwards'. In this case we thus see an instance of transplanting from DEO-discourse to E-discourse. Indeed, they use the same sentence's structures as well as the same words, except for the ones that explicitly refer to the scale or to the equation.

Se si parte da una bilancia
equilibrata possiamo
sostituire un elemento con
un altro elemento o una
somma di elementi con lo
stesso peso per mantenere
l'equilibrio.

a

Partendo da un'uguaglianza, si
può sostituire un elemento
con un altro elemento o una
somma di elementi con lo
stesso valore per mantenere
l'uguaglianza.

b

Figure 8.53: Written formulation of P_2 : (a) for the scale and (b) for equations

Excerpt 2

Alba and Elsa are in front of the equation in Figure 8.54. They are asked to accomplish some steps with respect to the initial inscriptions.

$$x + 5 - x = 8 - x + 2$$

Figure 8.54: Task situation 19

The episode is short. The two students work alone for the first part, producing narratives according to the principles they have written, while Elsa writes the visual mediators in Figure 8.55.

$$\begin{aligned}
 &x + 5 - x = 8 - x + 2 \\
 &x + 5 - \cancel{x} = 8 - \cancel{x} + 2 \\
 &x + 3 + \cancel{2} = 8 + \cancel{2} \\
 &x + 3 = 8 \\
 &\cancel{x} + \cancel{3} = 5 + \cancel{3} \\
 &x = 5
 \end{aligned}$$

Figure 8.55: Visual mediators produced by Elsa in task situation 19

When the tutor goes back to them and asks for explanation, Alba takes the floor (line 34). Her explanation implicitly refers to the principles they have written (for instance, “we started by removing, well, what can be removed—so the same value”, “I decomposed this five and made it into three plus two because there’s a two on the other side, and I wanted to remove it”). In front of a task situation involving an equation to solve, Alba and Elsa’s discourse is changed with respect to their initial interviews. It involves new procedures and narratives, but most of all, it does not require to remember steps to be accomplished, because they proceed autonomously. Their participation to the algebraic discourse is quite different from the beginning. Indeed, thanks to the new E-discourse’s features they have developed, they can act and relate their action to meaningful narratives. In addition, we can notice the use of verbs as ‘to decompose’ and ‘to remove’ that refers to actions previously accomplished on the scale. These verbs have been largely used

in the task situations involving scales, and have here been transplanted into E-discourse, allowing Alba and Elsa to produce endorsed narratives.

- 34 A: I'll do it [laughs]... Okay, we started by removing, well, what can be removed—so the same value. Here there's a minus x , and there's another one here, so we removed them, and we're good up to that point. Then ummm... wait, I lost track [laughs]. Okay! So, then I decomposed this five and made it into three plus two because there's a two on the other side, and I wanted to remove it. So, I decomposed five into three plus two and removed the +2 from both sides. That leaves x plus three equals eight. But I also wanted to remove this three, so I decomposed the eight again into five plus three, removed the three from here and the three from there, and I was left with x equals five...

Excerpt 3

After task situations 19, Alba and Elsa deal with task situation 20 in which they are asked to justify the steps they see in the inscriptions (Figure 8.31). Task situation 20 aims at making emerge the change sides change sign rule that in the discourse of Alba and Elsa has not appeared yet. In justifying the steps, they recognize it as something familiar, but when they are asked why there is the need to change the sign when they move a term from one side to the other, they say that they do not know the reason.

Here I present some exchanges from task situation 21, when Alba and Elsa are dealing with the second silent video task situation.

When they look at the video for the first time, they recognize what happens in the left part of the video as the change sides change sign rule. They instead are quite confused by the appearance of '+3' and '- x ' on both sides of the equation on the right part. When the tutor asks them for a feedback, Alba poses a question: "But is this connected with this? [points at the tablet, indicating the two sides of the video]". After a short exchange, they agree with the fact that in the video are shown two different ways or methods to find the value of x . Then, they rewatch the video. The exchange goes on reflecting on the two sides. Here I report an excerpt when they have in front of them the video stopped as in Figure 8.56.

$2x - 3 = x - 1$ $2x = x - 1 + 3$		$2x - 3 = x - 1$ $2x - 3 + 3 = x - 1 + 3$ $2x = x - 1 + 3$

Figure 8.56: Snapshot of the silent video about which Alba, Elsa and the tutor are discussing

- 35 T: I mean, we're doing it, but I feel like the reason behind it doesn't totally make sense to us... **Does this part here help us understand why we change the sign?** [5 seconds of silence] Like, where does this plus three come from [she points to the +3 added in the equation on the left side, Figure 8.56]?
- 36 A: He adds it... We're the ones adding it.
- 37 T: We add it, yes.
- 38 A: He added it... Here, here—oh gosh—here he simplified it [she points to the left side of the equation on the right] but it stays here [she points to the right side of the equation on the right], it's like he moved it.
- 39 T: He simplified it here and it stays here.
- 40 A: It becomes zero, so basically it disappears from here, but you add a +3, so it's like moving it with one extra step.
- 41 T: Okay... let's try to keep going, let's see if we can explain ourselves...
- 42 E: Yes
- 43 A: I was thinking of it like a ghost step, because this, in my opinion, is extra—because you could just directly move from here to...
- 44 T: Maybe this helps us explain it a little better [she points to the left part of the video, with the change sides change sign rule]...
- 45 A: Honestly, it confused me even more [she points to the right part of the video], but I understood it, yeah [laughs].
- 46 T: Can you try repeating it for me, so Elsa can listen too and we can see if it makes sense?
- 47 A: Repeat it... So you add the same value, and so on one side, the opposite signs, it becomes... uh... they cancel out to zero, because it's the same value but with opposite signs, so you get zero, and on the other side where we added the same value, it stays. It's like you remove this minus three, and a plus three remains on the other side. I don't really know how to explain it, but I understood it. Really.

The tutor fosters for a bridge between the two sides of the video and especially for finding a justification of the change sides change sign rule in the right part of the video (lines 35 and 44). Alba seems confused by the right side of the video, but along the excerpts, she develops narratives that seem to explain the relation between the two sides. In line 38 she relates the fact that on the right side of the equation the '+3' remains with the action of moving it ("it's like he moved it"). Moreover, she adds "It becomes zero, so basically it disappears from here, but you add a +3, so it's like moving it with one extra step" (line 40), introducing in her discourse the idea of an extra step, that soon becomes a "ghost step" (line 43). To sum up and explain to Elsa, in line 47 Alba develops narratives that well relate the two sides of the video, and in particular the change sides change sign rule with the action of adding or removing the same value to both sides of an equation. Going on, despite her explanation, Alba repeats that for her on the right side are performed extra steps. For instance, she says: "I mean, I understand why he does it, but I don't see why he *should* do it... because I, I would just move it directly, that's all". In conclusion she says "This should be an explanation of the change signs... because we add the same value to both sides, from one it is removed, on the other it stays, but the one that stays has the opposite sign of the one we have removed, that is, more or less...", providing again the explanation for the change sides change sign rule.

8.2.6. Alba and Elsa's fifth session

The two students, during the course, worked through the task situations in the two-pans-scale DEO and the moving arrows DEO, and, due to time constraints, did not work with the balancing mobile-DEO. During the first part of the fifth session, they have faced task situations involving the mobile-arrows-DEO.

In the end of the last session (the fifth) the tutor proposes to Alba and Elsa two mathematical task situations. Specifically, they face:

Task situation 1M:

For which values of x , $3x + 1$ is equal to $x + 2$?

For which values of x , $3x + 1$ is greater than $x + 2$?

For which values of x , $3x + 1$ is less than $x + 2$?

Task situation 4M:

$$13 - x = 13 + 11$$

Does it have any solution?

Can you find them or possible explain why there cannot be any?

As in the case of Aldo and Giulio, Alba and Elsa had at disposal all the DEOs (in tablet) they have worked with. In both task situations, once they are left alone, Alba takes the tablet with the moving arrows-DEO and Elsa the one with the mobile arrows-DEO. They insert the algebraic expressions in the input fields and drag the red tick mark, observing what happens. In both task situations they bridge the alignment of the blue and the black tick marks for a certain position of the red tick mark with the equality of the two expressions. They do not write on the blank pages offered to them in order to answer the request solving equations. Thus, as already pointed out in the previous excerpt, Alba and Elsa, despite the fact that task situations about equations are being proposed to them, keep going back to DEO-discourses. They produce endorsable narratives in the DEO-discourse, and relate them with the two algebraic expressions, but never produce $d_E(E)$ working on the equations and inequalities proposed.

8.2.7. The case of Alba and Elsa: some considerations

As in the case of Aldo and Giulio, also for Alba and Elsa I focused on the task situations involving two pans scale-DEO, silent videos and equations to have some hints about the discursive opportunities offered by these activities with respect to the aim of developing an algebraic discourse. In the case of Alba and Elsa we can observe that:

- The two students, in the episodes here presented as well in the other not discussed in this chapter, have expanded their DEO-discourse in tune with our analysis of the transplantability (see Figure 8.46).
- The appearance of objects with not known values (task situation 5) has fostered for Alba the production of new narratives and procedures that we see transplantable in the E-discourse.
- The formulation of principles has been reinvested by the two students both in the dynamic two pans scale-DEO and in the task situations involving pictures of scales, showing a new way to participate to the discourse. Specifically, Alba and Elsa implement procedures such as removing the same value from both pans accompanying them with endorsable narratives based on the principles.
- It is the explicit appearance of equations that prompt bridging discourses relating features of DEO-discourse with features of E-discourse. Before it, nor Alba or Elsa have introduced features of E-discourse to talk about the scale.
- The process of transplanting the principles from DEO-discourse to E-discourse offers new tools to face equations: Alba and Elsa in task situation

19 do not need someone who remember them how to act. Indeed, they proceed using the principles and justifying coherently with an expert's mathematical discourse each step in their procedure.

- Alba and Elsa's discourse is strongly attached to DEO till the last session. Indeed, even when the tutor proposes them task situation 1M and 4M in which equations appear explicitly, they refer to the DEOs and answer the requests using their feedback.
- Alba and Elsa's participation has changed with respect to the initial interview. Thanks to the interaction with the DEOs, they make decisions independently and for each step of the procedures they accomplish, they produce meaningful narratives. Even if in the first session Elsa seems a spectator of Alba's performance, we can notice that in the other sessions she takes part to the discourse and offers her contribution. To conclude, there is an opening from a strongly ritual participation, to an explorative one. This last one is deeply related to the presence of DEOs, to which Alba and Elsa stay attached.

8.3. Concluding remarks

With respect to the aims of the analysis, we can underline that the re-design occurred between the first and the third cycle has reached several objectives. Indeed, both the pairs here presented and discussed, have developed a DEO-discourse in which the features, for us essential in the light of the analysis of transplantability and the principle of continuity of discourse, appear. Indeed, one of our design focuses was on the elaboration and the formal formulation of principles to guide the production of narratives. Specifically, we aim at fostering the use of these principles when students were interacting with the scales or with equations. The introduction of task situations of imagined action has prompted the use of principles with respect to the scale. We have seen examples of the students using them to justify their considerations and the procedure to find out the values of not known objects. Such new participation has then been reinvested once students have faced task situations explicitly involving equations. Indeed, both Aldo and Giulio and Alba and Elsa, in front of an equation to solve, transplant the procedures of removing the same values from both sides of the equation and of substituting a term with a pair of terms with the same value to reach the solution. They have accompanied these new procedures with narratives guided by the principles they have elaborated. Thus, the theoretical organization of E-discourse has occurred and has offered students tools for a new participation to the algebraic discourse. Moreover, the design of some of our task situations and of tutor's intervention, guided by the construct of bridging discourse, has shown to be effective to relate

DEO-discourse's features and E-discourse's features. To conclude, the re-design has reached some of its aims. Despite this, we can notice that Aldo and Giulio and Alba and Elsa have followed a different course, even if they have faced the same task situations.

In addition, we can stress that in their discourses there are instances of de-ritualization processes. Specifically, we can highlight a new performer's agentivity together with substantiability. Both pairs of students in task situations involving DEOs as well as equations have proposed new ways to proceed and have justified their acting relating the task situation with the outcome of the implemented procedure. Moreover, Aldo and Giulio and Alba and Elsa in their $d_{DEO}(DEO, E)$ and $d_E(DEO, E)$ have shown applicability: the procedure to find out the value of not known objects has been reinvested in task situations of equations solving. In their discourse they have highlighted similarity between these task situations and used the procedure of removing the same quantities from both sides to accomplish them. This aspect relates with substantiability: the formulation of principles P_0 , P_1 and P_2 in DEO-discourse have offered them new tools to justify their action, that they have applied also in task situations involving equation. This has successively evolved in the formulation of principles for equations and in their use to justify their procedures. At the end of the sessions students have developed a new participation to the mathematical discourse, characterized no more by application of rituals to satisfy someone else, but by relations between the obtained product and the task situation and the justification of each step of the implemented procedure according to new endorsed narratives.

In the analysis, I have focused on these two pairs of students because, since the initial interview, their participation has been quite different. On one hand we find Aldo and Giulio, two students attending a technical institute that show themselves quite confident in dealing with the mathematical task situations. Moreover, they describe themselves as quite good students with respect to mathematics in the past. Nowadays they are facing some difficulties, but they deal with task situations and implement procedures with self-confidence. On the other hand, we find Alba and Elsa, two students attending the arts high school that within the initial interview present themselves as two students with difficulties in mathematics. Elsa affirms that she isn't "cut off for mathematics" while Alba refers that she gets headache caused by the too much thinking required in the mathematical activity. During the sessions with the DEOs, their discourses evolve quite differently with respect to the same task situations. This has emerged within a first analysis through the labelling with MAXQDA. In Figure 8.57 is reported the codeline produced in MAXQDA for the first session of Aldo and Giulio. We can notice that while in the beginning the two

students developed only $d_{DEO}(DEO)$, as soon as appear weights with not known values (line 63), they involve features from E-discourse. At the same time, Alba and Elsa's discourse remains strongly attached to the DEO until it is not present anymore (task situation 17, Figure 8.59).

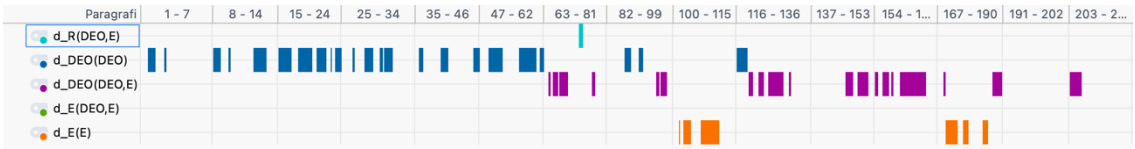


Figure 8.57: MAXQDA codeline of Aldo and Giulio's first session

In Table 8-1, I sum up the labels used to diversify the students' discourses occurred within the task situations presented in this chapter. In the first column I report the task situations grouped according to their typology. For instance, in the first row of the first column there are the 4 task situations involving TPS-DEO with only known values. In the third row, task situations involving a picture of a two pans scale. In the brackets there are the number of task situations to which refers the labelling of the same row. In addition, if the two pairs of students have faced different task situations of the same group, I have stressed it in the brackets. In the second column we find the labels referred to Aldo and Giulio's discourse along the task situations, while on the third there are Alba and Elsa's. The table highlights how Aldo and Giulio and Alba and Elsa have had quite a different course. On one hand, Aldo and Giulio have introduced features of the E-discourse during task situation 5, at the first appearance of weights with not known values. Since that moment, in all the task situations they have used features from the E-discourse to deal with our requests. It is worth to notice that when we have proposed them the task situations involving only equations (17, 19, 20, 21) Aldo and Giulio have mainly developed $d_E(E)$ and $d_{R(DEO,E)}(DEO,E)$, and only in the last mathematical task situations, Giulio has used the moving arrows environment to write the algebraic expressions in the input fields and drag the red tick mark to observe the movements of the dependent tick marks. Despite this, when they are asked to discuss and talk about the solutions of the equations proposed, he refers to written inscriptions and to narratives from the E-discourse (for instance, "for every x belonging to R ", line 20). On the other hand, Alba and Elsa have used and developed only features of the DEO-discourse till the moment in which we have forced them dealing with elements of the E-discourse (task situation 17). As it is highlighted in the third column, even if they are in front of task situations explicitly involving equations (17, 19, 21), they keep including features of the DEO-discourse in discourses about equations. For the two pairs, the moments of bridging the two-discourses and of transplanting processes of routines

and narratives from one discourse to another happen in different occasions with respect to our sequence of task situations.

Table 8-1: Sequence of labels in the task situations analysed for Aldo and Giulio (second column) and Alba and Elsa (third column)

	Aldo & Giulio	Alba & Elsa
Task situations with TPS-DEO with only known weights (1,2,3,4)	$d_{DEO}(DEO)$	$d_{DEO}(DEO)$
First task situation with known and not known weights (5)	$d_{DEO}(DEO,E)$ $d_{R(DEO,E)}(DEO,E)$ $d_E(DEO,E)$ $d_E(E)$	$d_{DEO}(DEO)$
Task situations with picture of balanced scale (9 and 11, A&G, 10 A&E)	$d_{DEO}(DEO,E)$ $d_E(E)$	$d_{DEO}(DEO)$
Task situation with an equation solved through removing the same value from both sides (17)	$d_E(E)$ $d_{R(DEO,E)}(DEO,E)$	$d_{DEO}(DEO,E)$ $d_{R(DEO,E)}(DEO,E)$ $d_{DEO}(DEO)$
Task situations involving equations and change sides change sign rule (19,20 and 21, A&G, 19 and 21 A&E)	$d_E(E)$	$d_{DEO}(DEO,E)$ $d_{R(DEO,E)}(DEO,E)$ $d_{DEO}(DEO)$
Task situations involving resolutions of equations (1M and 2M A&G, 1M and 4M, A&E)	$d_E(E)$ $d_E(DEO,E)$	$d_E(DEO,E)$ $d_{DEO}(DEO)$

For Aldo and Giulio, the crucial point is task situation 5, accomplished during the first session. We can see how the aboutness and the features used in their discourse change in Figure 8.58, where in the MAXQDA codeline of the first session I have highlighted in red the labels of task situation 5 (lines 63-180).

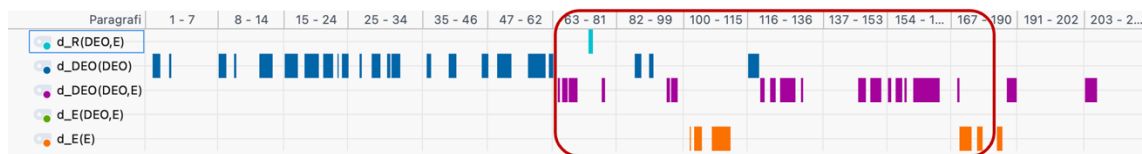


Figure 8.58: MAXQDA codeline of Aldo and Giulio's first session. In red are highlighted lines 63-180 (task situation 5)

Alba and Elsa, till task situation 17 have developed $d_{DEO}(DEO)$, with some seeds of E-discourse in inscriptions as in Figure 8.42 and Figure 8.50 to support their narratives about the scales. Once in front of such task situation that is about an equation (thus, d_E), asked to justify the steps in the procedure of equation solving, they immediately involve DEO-discourse's features (see Figure 8.59). They refer to the scale spontaneously and perform the actions they see implemented on the equation on the scale drawn by Alba (Figure 8.51).

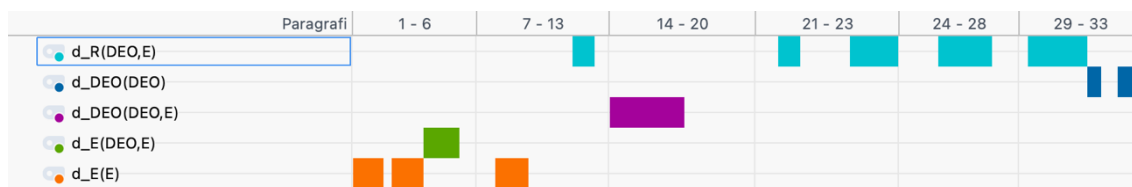


Figure 8.59: MAXQDA codeline of Alba and Elsa's task situation 17

Considering these two pairs of students, we could think about two different processes of transplanting. Indeed, Aldo and Giulio in task situation 5 transplant the features they have at disposal from the E-discourse in the DEO-discourse (visual mediators and words as 'x' and 'unknown' as well as procedure, change sides change sign rule). Since then, they keep using such features and other from the E-discourse to talk about equations. Instead, Alba and Elsa stay in the DEO-discourse for many task situations. When they face the first equation (task situation 17) they transplant in the E-discourse supposed to be produce according to the task situation the features of the DEO-discourse previously developed. In this sense we can see two different processes of transplanting, which reasons we could make some hypothesis. Aldo and Giulio in the initial interview have shown to be quite confident of and 'devoted' to the rituals proposed them at school. In task situation 5 we have highlighted it, when they transplant the change sides change sign rule on the scale. On the other hand, Alba and Elsa, who perform manipulations too in the initial interviews, seem less attached to the procedures. Alba, for instance, mentions the fact that she needs someone to remember her how to proceed. For them, the interaction with the DEOs have offered a 'way out' of the rituals they have difficulties to perform, and they use them to develop meaningful narratives about equations. Aldo and Giulio, from the excerpt analysed, seem to 'adapt' their E-discourse to the task situations involving DEOs.

These differences between students, of which Aldo and Giulio and Alba and Elsa are only an example, open to further research directions that I discuss in the following chapter.

9. Discussion and conclusions

In this concluding chapter I discuss how the study we have conducted has led to significant findings with respect to our research objectives. I recall that in the beginning, embedded in the DynaMat research project, our study aimed at providing insights into the learning processes involving algebraic discourse of students with a history of severe and persistent difficulties in mathematics, within technology-enhanced activities. According to the design based research (Cobb et al., 2003), we arranged the study in consecutive cycles of design-implementation-a posteriori analysis. During the first cycle, the three phases have made emerge the needs to build and develop some theoretical tools that could describe our design choices as well as some processes occurred during the implementations. Simultaneously to this, the a posteriori analysis of the first implementation's session has pointed out the need of a re-design of the teaching sequence in order to foster more explicitly the bridge between the activities with DEOs and the mathematical activity. Thus, the study has undertaken two research directions, one theoretical and the other practical, which might appear separate, but were in fact investigated simultaneously and in an interconnected manner.

I now discuss the research objectives emerged due to these two directions, highlighting the elaborations and findings that fulfil them. After that, I contextualize our findings within the literature, and I discuss some didactical implications. To conclude, I put a light on limitations of the study exploring possible directions for further research.

The analysis and the findings here discussed have no statistical intentions, due to the reduced numbers of cases studied. Our study has had an explorative nature towards two main points: the integration of theoretical tools to guide and support the design of digital environments in a discursive framework and the investigation of the role of such environments in the construction and expansion of algebraic discourse.

9.1. Research objectives: our findings

The research objectives we aim to fulfil were:

- Objective 1: integration of theoretical tools, in the theory of Commognition (Sfard, 2008) to:
 - c. design DEOs, related task situations and teaching interventions in terms of promotion of specific discourses.

- d. Investigate learning processes considering
 - i. the role of students' existing discourses;
 - ii. the role of a DEO and of discourses related to it.
- Objective 2: identify processes in the interaction with the DEOs that, as expert, we consider significant for the growth and the development of the algebraic discourse. Specifically:
 - a. investigate the role of DEOs in promoting analytic discourse;
 - b. Investigate the role of DEOs in supporting the theoretical organization of algebraic discourse. We want to focus on the formulation and organization of endorsed narratives with respect to the DEO that can be reinvested in the formulation and organization of endorsed narratives with respect to algebraic discourse, to guide the process of equations solving.

Now I focus on each research objective, highlighting the results emerged within the analysis that can provide meaningful insights with respect to our aims.

9.1.1. Research objective 1

In this study we focused on the elaboration of theoretical tools to integrate the Theory of Commognition (Sfard, 2008) with respect to the design of digital environments to promote the construction of specific discourses, in our case the algebraic one. Indeed, this theoretical framework assumes a specific position on mathematics and on its learning and offers us sophisticated instruments to investigate teaching and learning processes of construction of mathematical discourse, but we have not found detailed studies and theoretical elaborations that could guide teaching interventions with technology.

The need of such theoretical integrations has emerged because, framed in the commognitive perspective, our study foresees to involve discourses 'different' from the mathematical one. Indeed, our investigation has relied on the conviction that teaching intervention settled in digital environments can be effective and powerful for students with difficulties in mathematics, as many studies have highlighted. The possibility of interacting with the digital environments and of receiving feedback, related to the dynamic nature of the objects represented, and the possibility of having multiple representations of the same mathematical objects, are only some of the potentialities pointed out as crucial in the learning processes.

In our study, we decided to assume the position of Muller and colleagues (2009), framing the digital technology as dynamic explorative objects (DEO), as I have discussed in chapter 2. In this context, we selected DEOs that for us could be

incisive to promote the construction and the development of the algebraic discourse. Expanding upon the idea of metaphor offered by Sfard as “the action of ‘transplanting’ words from one discourse to another” (p.39) and on the principle of continuity of discourse, that is a “pedagogical principle according to which new discourses should be developed by transforming discourses in which the learner is already fluent” (Sfard, 2008, p. 301), we elaborated the construct of *analysis of the transplantability*, as the study of the relation between a DEO and a specific mathematical content. With this analysis, we have had the instruments to clarify, from a discursive perspective, the choices and the design of specific DEOs explicating in which sense we see them related to the content of our interest. To develop the construct of analysis of the transplantability, we had to take into account the notion of *discourse* proposed by Sfard and to elaborate on it. Indeed, as discussed in chapter 4, we propose the notion of *X-discourse* as the collection of specific words, visual mediators, routine and endorsed narratives referred to X and identified a priori by an expert or the community of discourse of X. This allows us to identify a priori the two collections of our interest, DEO-discourse and M-discourse, and to refer to them in order to characterize the analysis of the transplantability. Indeed, we depict it as:

- Epistemological analysis of mathematical discourse of interest (arithmetic, algebra, functions, geometry, and so on)
- Study of DEO-discourse and M-discourse, and thus of features (words, visual mediators, routines and endorsed narratives) constituting the two collections.
- Study similar use of words and visual mediators, correspondence in routines and endorsed narratives, between DEO-discourse and M-discourse, and thus study of features of a discourse that can be transplanted in the other one (and vice versa) (the study of the isomorphism between DEO-discourse and M-discourse).
- Identify subsets of DEO-discourse which elements are transplantable in M-discourse and vice versa and subsets that are not (see Figure 9.1). This is the study of the limitations of the relation between the two discourses that, in our perspective, can have a role in the didactical process.

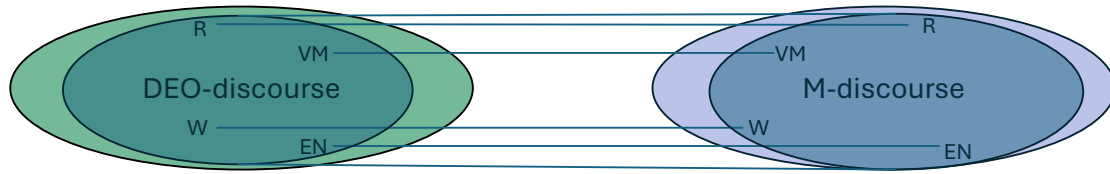


Figure 9.1: The analysis of the transplantability. The subset of a DEO-discourse corresponding to a subset of the M-discourse

According to the analysis of the transplantability, we can focus on a mathematical content and, as experts, identify a priori which are the features (W, VM, R, EN) we are interested in, in order to develop the mathematical discourse. At the same time, we can choose a DEO and study its discourse, highlighting the features we evaluate as related to the mathematical discourse we aim at promoting. In this light, we identify the subsets that from our point of view present correspondences, as well as subsets that do not correspond. In our perspective, these missing correspondences can be tools in the learning process. To explain in which sense we intend this, I recall that we propose a classification of specific discourses according to the criteria of *aboutness* and *features used*. In particular, we identify the following cases:

- **$d_x(X)$** : discourse recognized by an expert of X as about X in which the discursant uses features that an expert of X recognizes as belonging to the X-discourse.
- **$d_x(X,Y)$** : discourse recognized by an expert of X as about X in which the discursant uses features that an expert of X and Y recognizes as belonging to X-discourse and to Y-discourse.
- **$d_x(Y)$** : discourse recognized by an expert of X as about X in which the discursant uses features that an expert of Y recognizes as belonging to Y-discourse.
- **$d_{R(X,Y)}(X,Y)$** : discourse recognized by an expert of X and of Y as about the relation between X and Y, in which the discursant uses features that an expert of X and Y recognizes as belonging to X-discourse and to Y-discourse.

We have defined the last discourse *bridging discourse*. Considering that the object of this discourse is the relation between discourses, and thus other discourses, it can be considered a meta discourse. From a didactical point of view, within this discourse a subject could identify relation between the two discourses, as well as missing correspondences, that we have highlighted in the a priori analysis of the transplantability, that could promote the construction of the mathematical

discourse (since one of the two discourses considered, in our case, is always the mathematical one).

The analysis of the transplantability, together with the classification of discourses allow us to

- clarify the choice of specific DEOs and the design of new DEOs, describing the related DEO-discourses and the correspondence of their subsets with subsets of M-discourse
- design task situations according to the specific discourses we aim at fostering
- design teacher's intervention within the task situations according to the specific discourses we aim at fostering

In chapter 5 I used our theoretical developments to describe the design of our DEOs and to analyse them in a discursive perspective. Moreover, in discussing the task situations for each DEO, I used the classification of discourses to describe their aims, as well as I have done with respect to the tutor's interventions.

To better clarify how the classification of discourses can also be a tool to analyse discursive production during the sessions with DEOs, it is worth to underline that it has emerged within the a posteriori analysis of the first implementation session. Indeed, during the first data collection, we have observed and analysed episodes that in our perspective were crucial and we need instruments to mark them out. Some of these episodes are characterized by the students' spontaneous introduction of elements from the E-discourse. For example, in Excerpt 4 with respect to Dario, we have observed him introducing the word and the visual mediator 'x' (lines 36 and 37) to refer to the triangle (object with not known value).

36 D: Eh, because, indeed, for me, considering that they are the only two numbers, it is like doing, I do not know, just to say, x plus x plus 3 equal 11.

[...]

37 D: Well, I was thinking, I do not know... The triangles, so, I do not know... square x plus 3 is equal [he writes, Figure 9.2a], that is, x plus x, I mean... oh god, wait because I do not remember if square x is x times x or x plus x... Well, whatever

38 T: Let's try to understand, what do you want to write here?

39 D: That is... It is easier to do so... let's make triangle plus triangle plus 3 [Figure 9.2b]

- 40 T: Ok and here you are writing what, with respect to your mobile?
- 41 D: That triangle plus triangle plus 3 is balanced with 11, so it is equal to 11

$X^2 + 3 =$

a

$\Delta + \Delta + 3 = 11$

b

Figure 9.2: Visual mediators produced by Dario in task situation 28. On the left (a) the inscription made in line 37. On the right (b) the inscription made in line 39

We find another example in Grazia and Lucia's analysis, when, in excerpt 1, Grazia introduces the symbol 'x' and the inscriptions as in Figure 9.3, taken by the E-discourse, to write down the scale they are interacting with.

- 14 G: I don't know, like, 4 plus the x, for example [she writes the visual mediators in Figure 9.3, left side of the equation]. Like, interpreting the little ball as x, so as an unknown [she points to the two pans of the scale], plus 5 plus 1, which is equal to 3 plus x plus x [she writes the visual mediators in Figure 9.3].
- 15 T: Ok, Lucia, do you want to try explaining to me what Grazia wrote? [she moves the tablet with the visual mediator nearer Lucia] That way, we can check if you both understood each other—if what Grazia wrote is what you understood.
- 16 L: So, she rewrote the... the scales, I mean she related the two sides of the scale by adding up the numbers but indicating the little ball with an x because we don't know its value, so it's an unknown. And she did the same on both sides.
- [...]
- 22 T: Ok, because we don't know its value. Alright. And so, what have we put here? Why did we put an equal sign here in the middle?
- 23 G: Because the scale is balanced, so this side of the scale [she points to the left pan of the scale] weighs the same as the right side of the scale [she points to the right pan of the scale], and...

$$4 + x + 5 + 1 = 3 + x + x$$

Figure 9.3: Visual mediators produced by Grazia in task situation 5

The classification introduced allow us to identify as $d_{DEO}(DEO, E)$ the lines 36, 37 and 39 of Dario's discourse and the line 14 of Grazia and Lucia's discourse. At the same time, the excerpt taken from Grazia and Lucia's sessions made emerge a

phenomenon that we identify as essential in our study. Indeed, both relate features of the DEO-discourse to features of the E-discourse, developing bridging discourses ($d_{R(DEO,E)}(DEO,E)$).

These episodes, that are only explicative examples, have solicited us to distinguish the specific discourses that emerge within the activities with DEOs and to develop a classification that highlight the introduction of features from E-discourse caused by school's students experience. At the same time, the discourses emerged in these episodes have become for us examples of processes we aimed at prompt thanks to our design. In particular, I refer to the introduction of features from the E-discourse to talk about DEOs and to the production of bridging discourses. In this sense, the classification of discourses has helped us to clarify the aims of our design as well as to underline learning processes during the sessions.

The blended discourse

To conclude, in relation to the classification of discourses, I would like to make some considerations to distinguish the a priori analysis and design, and the a posteriori analysis. The distinction of aboutness and features of the specific discourse has been a key instrument in the design of task situations and tutor's intervention. Indeed, while constructing a task situation, we envisioned possible specific discourses that might emerge, and we reflected on which ones we were most interested in and which directions we would have liked to pursue in a teaching intervention. This analysis has been conducted according to the criteria of aboutness (What would we like the students' discourse to be about? What would we want them to talk about? What would we want as the object of their discourse?) and to the features we would have liked to be used while talking about that specific object. In this sense we made emerge our intentions to promote $d_{DEO}(DEO,E)$, $d_E(DEO,E)$ and $d_{R(DEO,E)}(DEO,E)$ and we elaborated examples of such specific discourses we wanted to foster. On the other hand, in the analysis of episodes with students, the complexity that always characterizes learning processes has emerged. Indeed, as for instance shows line 14 of Grazia and Lucia previously discussed, these specific discourses we have a priori identified intertwines together. In addition, it is not always possible to identify the object of the students' discourses. On one side, the use of expressions such as 'it', 'that', 'them', when employed in a context in which, for example, we have a scale, and an equation could in some cases refer to both. On the other side, the nature of these discourses is ambiguous. In this sense, elaborating on the classification proposed and taking into account the processes emerged within the sessions, we have started considering the idea of *blended discourse*, as a discourse in which blend together

features from DEO-discourse and from E-discourse, where could develop individually $d_{\text{DEO}}(\text{DEO}, E)$, $d_E(\text{DEO}, E)$ and $d_{R(\text{DEO}, E)}(\text{DEO}, E)$, as well as all of them together. With respect to our scheme of specific discourses according to aboutness and features used, we identify the blended discourse as the ‘space’ highlighted in blue in Figure 9.4 that collects different discourses. The multivalence of this discourse is something that, from our perspective, needs to be exploited. Indeed, it is such openness that allows students to talk, for instance, about both the little red ball and the unknown in the same way, or to deal with the equilibrium of the scale and with the equality between two algebraic expressions in the same moment. This blended space is where the transplanting processes can happen (and has happened in some of our examples), and thus its development must be supported and at the same time used in learning processes.

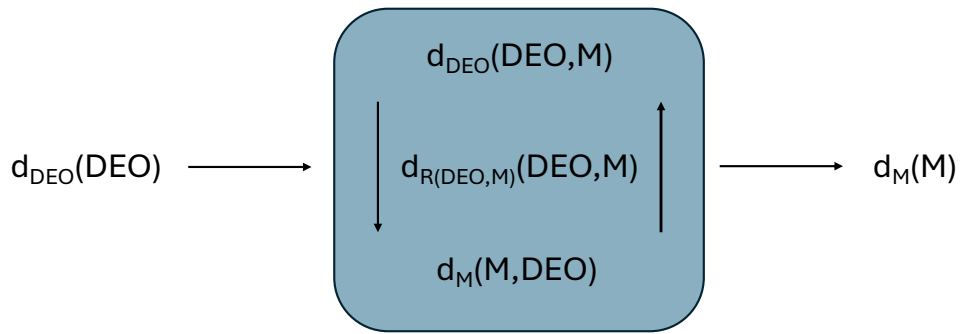


Figure 9.4: The scheme of specific discourses where is highlighted the blended space

The classification of task situations

In addition to the theoretical elaborations just discussed, we have proposed two classifications of task situations according to different criteria. The first one refers to the presence or absence of an action to be accomplished on the DEO. Indeed, we have identified as *task situations of effective action* the ones in which the student is required to perform an action on the DEO and thus receive a reaction, that is feedback, corresponding to it. On the other hand, the *task situations of imagined action* are characterized by the imagination of actions on the DEO together with the description of expected re-actions according to what has been done. This second type of task situation has revealed to be effective with respect to some of our aims, as I discuss in the following section.

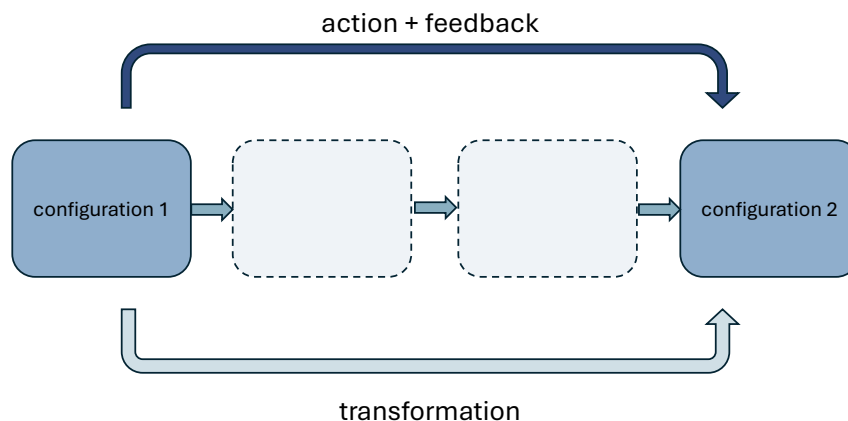


Figure 9.5: Model of use of a discrete DEO

Moreover, we have elaborated on the notion of DEO (Muller et al., 2009), developing a *model of use of a DEO* (discrete or continuous), as in Figure 9.5. Such model has guided the development of some task situations. Specifically, thinking about students' discourse, we interrogate ourselves about what could be the aims of such discourse, according to the model. Indeed, we have identified the components of the model as two static situations, that are configuration 1 (C1) and configuration 2 (C2), while what takes the DEO from C1 to C2 is an action, accomplished by the user, to which follows feedback. According to our task situations, students' discourse aims at discovering one or more parts of the model. This classification has helped us in clarifying our expectations with respect to students' discourse. Indeed, we have identified task situations in which students have to develop narratives about C2, knowing C1 and the action to be accomplished, or knowing C2 identifying C1 and the action accomplished to reach such final configuration.

The two classifications, developed according to different criteria, have helped us in expressing our designed choices and aims for each task situation.

To sum up, the theoretical scaffold we have elaborated offers us, in the commognitive framework, a guide to justify the use of DEOs in our teaching intervention and to contextualize our design with respect to discursive aims.

9.1.2. Research objective 2

In chapter 7 and 8, I put the light on some episodes occurred during the first and the third session of implementations with students that for us have been particularly significant with respect to the second research objective.

The first point of the research objective regards the role of DEOs and related task situations in the promotion of analytic discourse. I recall that in this study, we take on the perspective offered by Kieran, who affirms:

“In sum, handling unknowns as if they were known, along with the thinking related to equivalence transformations and the notions of balance and truth-value, serve to characterize the overarching dimension of analytic thinking within early algebraic thinking.” (Kieran, 2022, p. 1135)

In our framework, relying on Kieran’s constructs, we have looked for instances of analytic discourse, and thus words and their use, visual mediators, routine and endorsed narratives that involve handling of indeterminate quantities as if they were known and the notions of equality and equivalence, central in activity with equations and equations solving. According to this perspective, since the first cycle, we focused our design in order to promote such two components identified as characterizing the analytic discourse. In our analysis we focused on the students’ discourse about indeterminate quantities. In the episodes of both chapter 7 and chapter 8 we have observed that students involved in activities with DEOs are attracted by the objects for us realizing indeterminate quantities, and in some case, feel the need to discover their values before accomplishing other requests. This has happened especially during the task situation, involving two pans scale DEO, that for the first time present objects with not known value (little red balls, see Figure 9.6). In this task situation students were explicitly asked to add (in the first cycle) or to remove (in the third cycle) three objects from the pans of the scale to have it balanced afterwards.

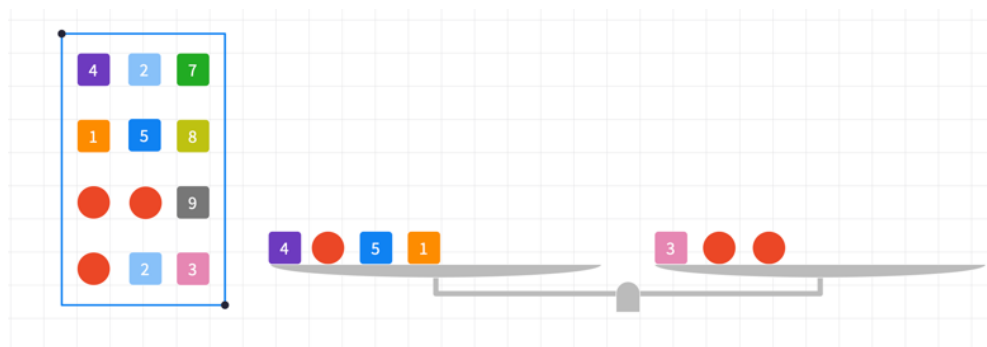


Figure 9.6: Scale of task situation 5

For instance, Dario’s discourse, as soon as he sees the scale in Figure 9.6, is about the little balls.

- 1 D: So... um... add three objects. Except that it should be understood the little balls, what weight they have.
- 2 T: Okay. So, you would like to know what weight these little red balls have?
- 3 D: Yes then... [...] They all have the same weight. Mmm... surely here, trying to sum it up again, five and one make six, and four make

ten. So, this is ten, and here there's a three, which, however, is equivalent—that is, the two sides are balanced.

[...]

- 4 D: That is, I don't know, try to understand why here there is a 3 and two little balls, that is, especially understand what these two little balls make.

He shows the will to understand “what these two little balls make” (line 4) and this need characterizes all along the episode involving this task situation. In the end he has accomplished the request, but he feels unsatisfied by the fact that he has not discovered the value of the little red balls. In proceeding with the task situations, the presence of not known objects stimulates him to find procedures that satisfy his will of finding their value (see in chapter 7, The case of Dario, excerpt 2 and excerpt 3) and shift is participation towards a more explorative nature. In particular, in excerpt 3, forced by the task situation, he removes also not known objects from the pans.

- 23 D: Otherwise, yes wait... I could try to do a thing
24 T: What would you do?
25 D: Remove this... that is, the blue shape from here and here [he lifts a weight with not known value from both pans], it would be equal, then I would remove the 3 both from here and here [he lifts a weight of value 3 from both pans], I removed four objects, taking off the same, the same shapes, the same...
26 T: Ok, because the request was to remove four objects... thus you have lifted exactly four, and why does it work?
27 D: Because anyway, the blue shape is equal to the other blue shape and the 3 is equal, that is, the weight of 3 is equal to the weight of 3

We can see in Dario's discourse a first instance of analytic discourse: in line 27, in justifying his actions, he underlines that “Because anyway, the blue shape is equal to the other blue shape”. Indeed, despite his discourse is a $d_{DEO}(DEO)$, the use he makes of words (blue shape and equal) is the same use an expert would make of the words ‘unknown’ and ‘same value’. Thus, recalling Wittgenstein position “the meaning of a word is its use in the language” (1953/2003), such words, and their use, are transplantable in the E-discourse. This, along with others in our excerpts, is an example of a specific discourse about DEO that is isomorphic to a specific discourse about equations.

Also Alba and Elsa, in their first interaction with the scale in Figure 9.6, have shown an intense attraction towards the red little balls. They spent minutes in removing and adding the three little objects on the pans of the scale, and their presence, as in the case of Dario, prompts Alba to uncover strategy to find out their value (see

excerpt 1). It is always Alba that, after having discovered the value of the little balls, and having proposed, together with Elsa, several combinations to satisfy the task, affirms “Even if we do not know the value, if we know that they all have the same value, if we cancel one from here [points to the left pan of her inscription and produce the cancellation mark on the little ball, Figure 9.7b] and one from here [points to the right pan of her inscription and produce the cancellation mark on the little ball, Figure 9.7b] it remains the same”.

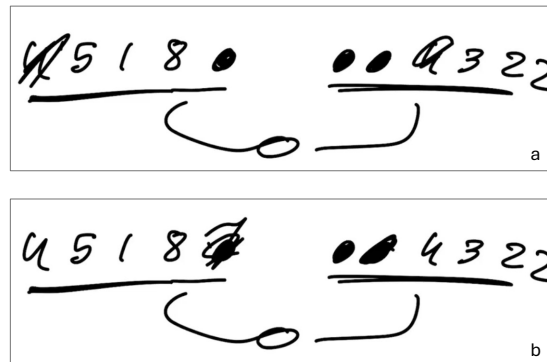


Figure 9.7: Visual mediators produced by Alba when the tutor asks Elsa and her to show on their inscriptions the action accomplished on the scale (task situation 5)

In both the case of Grazia and Lucia and Aldo and Giulio, in dealing with task situation 5, we see the students focused on the request. In accomplishing it, Lucia says:

- 27 L: Like, taking advantage of the fact that it's already balanced, even though we don't know the weight of the little ball, we can add three objects, like, for example...

[...]

- 29 L: 4 on this side [she takes the object of weight 4 on the left pan] and 2 and 2 on the other side [she takes the objects of value 2 on the right pan]. This way, it remains balanced even if we don't know the weight of the little ball. We've added the same quantity.

While Giulio affirms:

- 76 T: For example... If we wanted to do it with two objects, how could we do it?

[...]

- 79 G: Or two little balls, I mean, two unknowns [Aldo removes one red ball from the left and one from the right]

- 80 T: OK, two unknowns. Why are we sure that two unknowns would stay balanced?

- 81 G: Because they are equal [points to the tablet with the scale]

The lines reported here are only few examples of the discourse developed within our activities about indeterminate quantities. As I have highlighted, the students, even if in some cases aim as first thing to discover the value of the not known objects, deal with them as if they were known, and develop narratives that point out how for them indeterminate quantities can be treated as determinate ones. Thus, our design of DEOs and related task situations has fostered the growth of narratives and procedures that highlight the dealing with indeterminate quantities as if they were known. It is worth to notice that students use different mediators for indeterminate quantities. Indeed, in some cases they refer only to the mediators offered by the DEO, while in other they introduce the symbol ' x '. As underlined by Radford (2014), the denotation process, that is the naming or symbolizing of indeterminate quantities must happen, but this does not necessarily involve algebraic symbolism. Even if students do not involve in their discourse visual mediators taken by the E-discourse, according to our position, we can affirm that they are developing analytic discourses. Indeed, the use of words and visual mediators (with respect to DEO-discourse) is the same an expert would make of words and visual mediators referred to equation. Thus, such features and most of all their use, are transplantable in the E-discourse. Moreover, according to the principle of continuity of discourse, their narratives about indeterminates quantities in the DEO-discourse can be transplanted in the E-discourse and can become a tool to talk about unknown and variables in the symbolic form.

The second aspect characterizing analytic discourse, that are the notions of equality and equivalence, central to activity involving equations and equations solving, emerges as well within our sessions. Indeed, especially the two pans scale DEO together with the task situations designed have forced a multiplicity of discourses engaging balancing transformations. As we can notice, this aspect intertwines with the previous one about indeterminates. For instance, in Lucia's narratives (lines 27 and 29) the treatment of indeterminate quantities as if they were known is tied together with considerations about the conditions that keep the scale balanced. In Giulio's discourse too (lines 79 and 81), the action that allows to keep the scale balanced involves dealing with the little balls as if they had known values. These are only few examples offered by the analysis of instances of analytic discourse with respect to the DEO-discourse. At the same time, we find episodes in which the transplanting process has happened, and thus these features of DEO-discourse have been transplanted in the E-discourse. For instance, in the case of Aldo and Giulio we have highlighted the process in which they deal with the process of resolution of an equation and transplant in their discourse features from the E-discourse. They are dealing with task situation 17, and Aldo affirms:

- 20 A: Yes, since we removed the two x, the following numbers remain, and there's still an unknown on the left. Then, we do the same thing with the 8, removing the two 4's on the right because the two 4's add up to 8, and we can remove them. And finally, we can figure out that the unknown equals 14, which is 13 plus 1, which together makes 14.
- 21 G: Yes
- 22 T: Okay, so going back to the beginning, you're telling me we can make the jump from here to here because?
- 23 A: Because we have two equal values, so if we remove them, it doesn't change anything.
- 24 T: Okay, what do you mean by "doesn't change anything"? That's a tough question, huh?
- [...]
- 27 A: It's like on the scale.

Aldo's discourse is characterized by the verb 'to remove' that recalls previous actions on the scale, but he is justifying successive steps of the resolution of an equation. In line 23 his narrative echoes the narratives formulated about the DEO, and he is the one who make the bridge between scales and equations (line 27).

To sum up, the choices we made during the design phases have shown to be effective with respect to our aim to prompt the development of analytic discourse. Indeed, according to the commognitive framework and to the principle of continuity of discourse, we have words, visual mediators, routine and endorsed narratives of the DEO-discourse that characterize the interaction with DEOs. In tune with the analysis of the transplantability, these features find correspondences in the E-discourse we aimed at promoting. The analysis has highlighted in which ways our design has prompted for the development of that specific features of DEO-discourse we recognize as characterizing analytic discourse, as well as episodes of successful transplanting.

With respect to the second point of the second research objective, it is worth to notice that it intertwines with the investigation about the processes related to analytic discourse. As a matter of fact, the notion of equality and equivalence, together with the balancing transformations for equations, are what characterize the principles (P_0 , P_1 and P_2) in which we are interested in when we talk about the theoretical organization of E-discourse.

The analysis discussed in chapter 8 considers episodes from the third cycle of experimentation in which one of the explicit aims was the theoretical organization of DEO-discourse and, as a mirror, the E-discourse. Such analysis has put a light on

how our DEOs' design related to tutor's interventions have fostered the elaboration and the formulation of the principles with respect to the scale. Together with this, the integration in the sequence of task situations of imagined action has solicited the employment of such principles in the process of accomplishing the proposed requests. Thus, on one hand we find the elaboration of the principles that have been formulated and re-formulated according to the needs, for instance, the transition from considering only the action of adding objects to the scale to considering both adding and removing, as well as the transition from only known values to known and not known. On the other hand, the design has forced the use of such principles to support student's narratives when they have dealt with pictures of the scale. This new way of participating to the discourse, developed through task situations of effective and imagined actions has been transplanted into E-discourse, for which students have developed the principles as well as a new way of proceeding in solving equations. The theoretical scaffold offered by the principles has allowed them to relate each manipulation accomplished on an equation to a narrative that justifies the action and expresses the maintained relation of equality between the algebraic expressions involved.

9.2. Contextualization of findings within the Theory of Commognition

In this section I aim at position our findings within the commognitive perspective. I discuss how our study could offer a different position, in this framework, on how learning can occur. Moreover, I stress how our theoretical development can contribute to the studies conducted within this perspective involving digital environments.

According to the Theory of Commognition (Sfard, 2008), mathematics is a special discourse about mathematical objects, that are discursive objects. This perspective allows us to look at learning mathematics as becoming a participant of the mathematical discourse, that is becoming fluent in the communication about mathematical objects. The characterization offered by Sfard (2008) of the discourse according to the four features that are words, visual mediators, routine and endorsed narratives, provides tools to investigate students' discourse. Indeed, the use of such features give insight into students' learning processes. This framework seemed to us particularly fitting because it allows us to talk about what the students 'say' (in a wider conception of the term, thus including not only spoken production, but also written, gestural and so on), and not focusing on what they 'do not say'. Specifically, considering the way in which the features are used and the discourse is developed, Sfard provides instrument to distinguish between a ritual and an

explorative participation. With respect to this, according to the commognitive framework, the learner has access to a new discourse thanks to her exposure to it through the interaction with an expert. In this sense, the learner initially must go through a ritual participation in which the imitation of others plays an essential role in the development of new rituals. Sfard affirms,

“[...] on their way to new routines involving new metarules or new mathematical objects, learners must pass, if only very briefly, through the stage of ritualized performance” (Sfard, 2008, p. 249)

Our investigation, as other developed in the commognitive perspective (Lisarelli, 2019), has pointed out that the passage through ritualized performance is not always necessary. Indeed, in our teaching sequence, the students have had the chance to develop a discourse that echoes the mathematical one, in an explorative manner. Since the beginning our DEOs have offered students words and visual mediators, as well as a way of developing routines and endorsed narratives that foster an explorative participation for the purpose to construct endorsable narratives. This participation, as discussed, is transplanted in the mathematical discourse. Considering that our students already ‘speak’ an algebraic discourse, we want to stress that the innovation for them is exactly the way of participating. This ‘jump’ towards a new (in the sense of participation) mathematical discourse in which the rule of narratives’ endorsability are different, and are elaborated by students themselves, is not imposed by an authority (usually the teacher) but is developed from the reflections on DEO-discourse and E-discourse and their relations (bridging discourse).

Moreover, I have broadened and deepened the notion of metaphor proposed by Sfard, developing the construct of analysis of transplantability. Such elaborations, together with the classifications of discourses, provide theoretical tools, that were missing, to talk about artifacts (not only digital) in the commognitive perspective. Specifically, these theoretical integrations and elaborations can become tools to design activities with artifacts, guiding their design and the formulation of task situations.

In conclusion, our notions can become tools to develop, in a discursive perspective, teaching interventions with artifacts to promote the entrance in the mathematical discourse through an explorative participation, leaving behind the need of imitating and of referring to an authority to develop a canonical mathematical discourse.

9.3. Didactical implications

In terms of didactical implications, this study has dealt with different aspects that can be essential in the teaching activities.

First of all, we have developed tools for the teacher's design of interventions, starting from the choice and the elaboration of DEOs. Such tools can turn out to be essential in the development of teaching intervention. Indeed, the analysis of the transplantability can guide a teacher in clarifying why she chooses to use a specific digital environment in her teaching intervention, and which are the features of the discourse that she aims at promoting, relating them to the features of the mathematical discourse she is interested in. Another significant point with respect to the design of classrooms' activities is touched by the classifications of task situation. Indeed, for instance considering the TPS-DEO, the introduction in the sequence of task situations where the actions on the scale and the consequent feedback were inhibited has pushed students to imagine DEO's feedback according to their actions and to justify their hypotheses, according to the principles they have elaborated. This has favoured a smoother transition to the discourse on equation in which manipulations on the algebraic symbols are allowed, but the related feedback miss. Moreover, their new habit of justifying their hypotheses, developed in task situations of imagined action, has been transplanted in E-discourse. Thus, a relevant didactical implication can regard the need, when dealing with technology-enhanced activities, of task situations in which actions on environments and thus feedback are inhibited, to promote both hypotheses and justification, and in this sense a way to participate that can be reinvested in the mathematical discourse.

In addition, the theoretical elaborations have allowed us to develop instruments to guide the interventions in specific task situations. Indeed, having elaborated the analysis of transplantability for a teaching intervention provides tools to identify, during the implementation, within students' discourses, transplanting opportunities and to take advantage of them, together with occasions of prompting processes of compression or expansion of the discourse. At the same time, the construct of bridging discourse can be employed in the design of a teaching sequence in identifying task situations or specific moments in which the development of a discourse on the relation of the two (or more) discourses involved. Together with this, the construct of aboutness can conduct the teacher in the shift from a discourse strongly related to the DEO to a discourse about equations, using DEO-discourse's features, and vice versa.

Moreover, this study proposes an example of the implementation of such theoretical tools to develop a sequence of well-designed DEOs and related task

situations. The sequence proposes description of DEOs and their analysis of transplantability, together with the formulation of task situations and their discursive aims, and related teacher's intervention to promote specific discourses. The sequence described in chapter 5 can be used to support the development of students' discourse with respect to equations and equations solving. Specifically, the sequence can be employed to promote a discourse about indeterminate quantities as well as the theoretical organization of E-discourse, according to the principles used to solve equations. It offers DEOs to build and expand DEO-discourse according to features we are interested in. Moreover, it provides DEOs and task situations to foster the transition towards the development of the E-discourse aligned with the canonical one.

Finally, our study provided further evidence that well-designed activities can also be effective for students with severe difficulties in mathematics. Indeed, in the analysis, we have highlighted how the discourses developed by students within DEO-situated activities are rich of instances of $d_{\text{DEO}}(\text{DEO})$ in which the use of words, visual mediators, routines and endorsed narratives is transplantable in the E-discourse we have had as main didactical aim.

9.4. Limitations and further research

To conclude this chapter, I would like to discuss some limitations we recognize in this study, that open to new general questions, and consequently, to new research directions.

The study here discussed has been conducted in a peculiar context that presents several differences with the canonical classroom's environment. Indeed, the implementations analysed have been conducted at the CARME center (<https://www.carme.center/>), during the afternoon, with students who have volunteered to participate to the data collection. Thus, the participants, despite recognizing themselves as having persistent and severe difficulties in mathematics, were interested in participating to the experimentation. Moreover, they have worked in pair guided by a tutor that interact only with them. These characteristics outdistance the study from the classroom context in which the students are obliged to stay, and most of all, work under the guide of the teacher that has to follow the entire class. Our study has had an explorative nature and one of the aims was to identify discursive opportunities offered by our design. The implementations have put the light on several potentialities of our DEOs with respect to this in a context of a pair of students and a tutor. We expect that these potentialities would emerge also in a classroom context. What needs to be studied are the steps subsequent to the interaction with the DEOs, in particular the role of the teacher in guiding

classroom's discussion and in fostering the transition from DEO-discourse to E-discourse.

With respect to this, another point to consider is the typology of students involved in the study. Indeed, when we have gone to classrooms presenting the project and looking for participants, we explicitly asked for students that recognized themselves as having difficulties in mathematics. In this sense our analysis has involved only students of which mathematical discourse was not aligned with an expert one and who presented a strongly ritual participation. A study engaging students at various levels, with respect to the mathematical discourse, could offer new insights into the potentialities of our designed sequence and new hints about learning processes. Moreover, the same didactical sequence could be proposed to students of different scholar levels, such as the last year of the middle school or the first one of the high school, in which equations are introduced.

This last consideration leads to a crucial feature of our study. As I discuss in the following lines, our experimentation can be considered as a 'remedial intervention', because the students participating to the data collection have already encountered equation in their school course. Our task situations' design in some cases has strongly rested on this aspect, especially when we aim at promoting $d_{\text{DEO}}(\text{DEO}, E)$ and $d_E(\text{DEO}, E)$. Further investigations are needed to understand how such task situations and interventions intertwines with discourses of students that do not have at disposal algebraic symbolism or that have not been introduced to equations and equations solving.

The study has led to the development of theoretical tools to guide the design of technology-enhanced activities in a discursive framework. These tools have been elaborated according to our design needs and thus refer, even if implicitly, to our DEOs. We ask ourselves how the analysis of the transplantability applies to other DEOs. In this sense, the analysis of different DEOs and the elaboration of other task situations could offer insights into the theoretical tools elaborated and could help in clarifying some aspects of them. In relation to this, considering the classification of task situations we have developed, we believe that the analysis of interactions and feedback of other DEOs could lead to different models of use of the DEO and thus to different criteria for the classifications of task situations. Finally, considering the notion of blended discourse introduced in this chapter (Figure 9.4), further research is needed in order to deepen the understanding of learning processes involved in such 'space' as well as to develop criteria in order to foster the growth and the evolution of such blending of discourses.

One of the aims of the DynaMat project has been to characterize mathematical learning profiles. To do so, some researchers of the team have developed the sequence of task situations of the initial interview and analysed the data obtained through the commognitive lens to investigate the identifying discourse and the mathematizing discourse of students approaching our experimentations. This study has led to significant results in bridging cognitive and non-cognitive perspectives in the investigation about persistent difficulties in mathematics (Macchioni & Baccaglini-Frank, 2025). In relation to this, the characterization of Mathematics Discourse Participation Profiles could offer interesting instruments to look and deepen students' discourses in the activities with the DEOs. A first attempt to relate the initial student's profile and his discourse in the experimentation's sessions has been made in the analysis of the case of Dario (Bonadiman & Macchioni, 2024). The study has underlined a possible opening to change of Dario's discourse metarules prompted by our technology-enhanced activities. Indeed, differently from the initial interview where we have identified two main metarules that are the necessity of perform calculations to do math and the need of remember by heart rules and procedures to act, the analysis of some episodes has pointed out how Dario makes autonomous decisions and does not rely on memory to satisfy the proposed requests. Thus, the case of Dario could be only the first of several investigations in order to deepen the relation of the initial profile with the students' discourse during the activities with the DEOs. In particular, this combining could provide new insight with respect to some differences between students' processes emerged during the analysis.

To this last point relates another possible research directions. During our study, we have deepened the position of Lewis (2017) with respect to what she calls *remedial intervention*. In her studies she has focused on students with mathematical learning disability for whom standard education has shown to be ineffective. Framing her investigation in a vygotskian perspective, she looks at the human development as a phenomenon that happens through two different directions, the biological and the sociocultural one, that intersect and blend together. This occurs thanks to mediators, that can be tools and signs, which are fundamental in cognitive development and allow for the intertwining of the biological and sociocultural dimensions. This happens for individuals with typical development, whereas for individuals with disabilities, the process is usually more complex. Everything related to human culture and its development is rooted in a "traditional" developmental condition that assumes certain biological prerequisites in humans. When these are lacking, the tools and signs of mediation become ineffective. Lewis speaks of *re-mediation* in the context of students with disabilities, for whom standard educational approaches often prove ineffective precisely because they

rely on tools and signs designed for typical development. The researcher identifies as the main aim of re-mediation the creation of unconventional tools and signs that considers the qualitative differences of the student's development. In her investigation she ties together the vygotskian perspective and the commognitive framework to develop a re-medial intervention based on discursive patterns about fractions of a student with learning disabilities. This re-medial intervention is constituted by the development of a *bridging discourse* that “attempted to connect the student's discourse to the canonical discourse” (Lewis, 2017, p.326). This bridging discourse is a construct different from the one proposed in this dissertation, and it relies on the detailed analysis of initial student's discourse about fractions, identifying the ways in which it is persistently far from the canonical one. These ways have become a tool to guide the design of new and alternative mediators to promote the connection of the student's discourse with the canonical mathematical discourse. Our study distances from the one carried out by Lewis. Indeed, first of all, we have worked with students who recognize themselves or are recognized by their teachers as having a history of severe and persistent difficulties in mathematics, despite the eventual presence of learning disabilities. Moreover, while Lewis focuses on a single case study, deeply investigating student's discourse about fractions and identifying her discursive patterns (Lewis, 2014) to develop a re-medial intervention addressed to these discursive patterns, we have designed tutoring sessions about algebra that are the same for all the students participating. The design of these tutoring sessions has relied on the study of the literature about main students' difficulties and misconceptions in learning algebra and not on specific discourses of the participants. Despite this, I think that our intervention could be framed as a remedial one. From the initial interview, we have collected instances of students' algebraic discourse characterized by manipulations learnt by heart, and of their strongly ritual participation focused on processes to be accomplished. These discourses are far from the mathematical discourse we would like students participate to, in doing mathematics. Thus, considering the term *re-mediate*, from the Latin *remediare*, derived from *remedium*—meaning both “remedy” and “medicine”—we could refer to it with a dual meaning. On one hand, it refers to the idea of “providing a remedy,” that is, addressing the need to reconstruct mathematical discourse that the students in question appear not to have built. On the other hand, in line with Lewis's proposal, it expresses the intention to provide new tools and signs for students that can foster the construction of mathematical discourse aligned with an expert's one. In relation to this, we can consider our DEOs as the new mediators for students' discourse. As emerged from our implementation' sessions, our sequence prompts a multiplicity of learning processes, according to different students. Leaning on the elaboration of Lewis and expanding them, we could deepen the idea of re-medial intervention

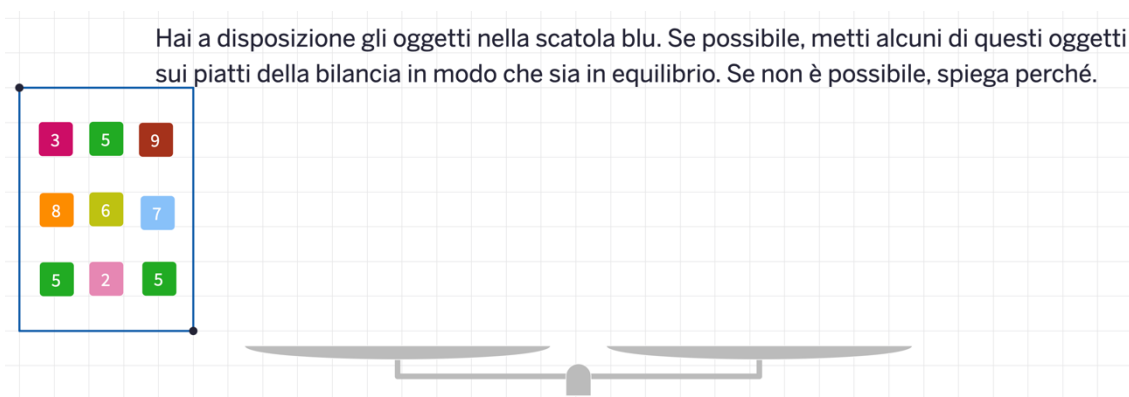
and join it with the construct of Mathematics Discourse Participation Profiles (Macchioni & Baccaglini-Frank, 2025). On one hand, the construct of re-medial intervention could be deeper characterized and described. Moreover, relating the initial student's profile with her learning processes emerged during the teaching sequence could offer insights into features and criteria to develop re-medial intervention. Indeed, we have seen how the same task situations have prompted different discourses for different students. Understanding, thanks to the identification of the initial profile, the reasons for these discursive differences could provide enhance our understanding of students' educational needs on which build criteria to develop different re-medial interventions for different students. This research direction could have the potential to yield meaningful results toward the development of an increasingly inclusive mathematics education.

Appendix A

In this appendix I report the task situations as they were originally formulated in Italian and that have been implemented during the data collection. I report also the translation in English. I collect here the sequence described in chapter 5 and used within the third cycle of implementation session (chapter 8). If the verbal formulation of the task situation is included in the DEO, I report them. Otherwise, I add it above the picture of the DEO.

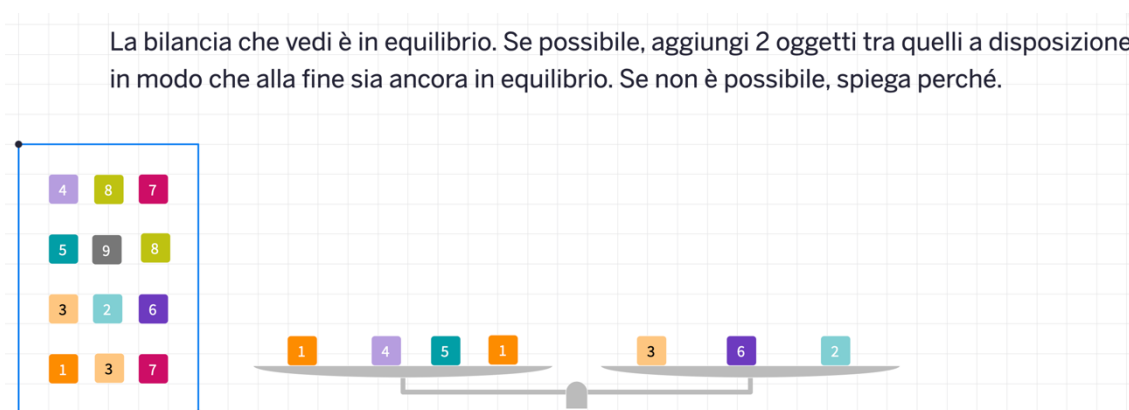
Task situation 1:

You have the objects in the blue box at your disposal. If possible, put some of these objects on the pans of the scale so that it is balanced. If it is not possible, explain why.



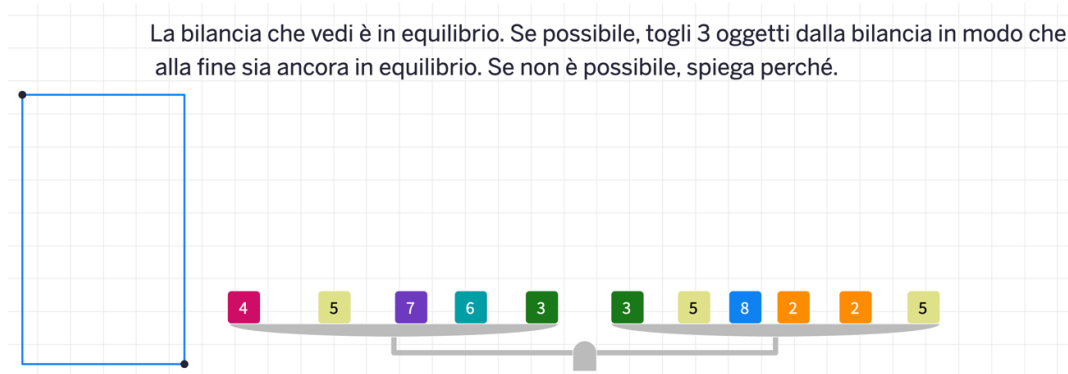
Task situation 2:

Here you see a balanced scale. If possible, add 2 objects from those available in the box so that it is still balanced afterwards. If it is not possible, explain why.



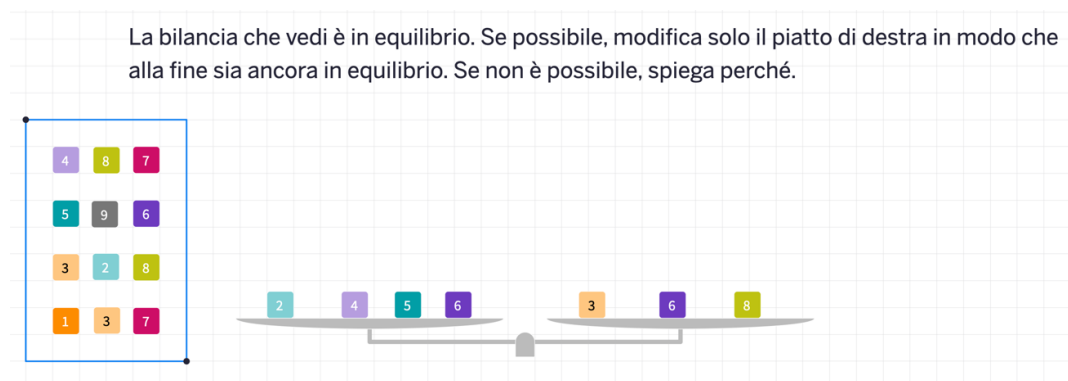
Task situation 3:

Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.



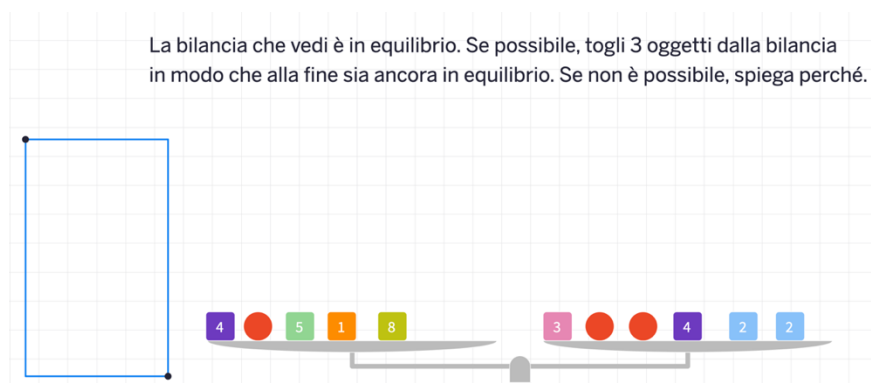
Task situation 4:

Here you see a balanced scale. If possible, change only the (things on the) right pan so that it is still balanced afterwards. If it is not possible, explain why.



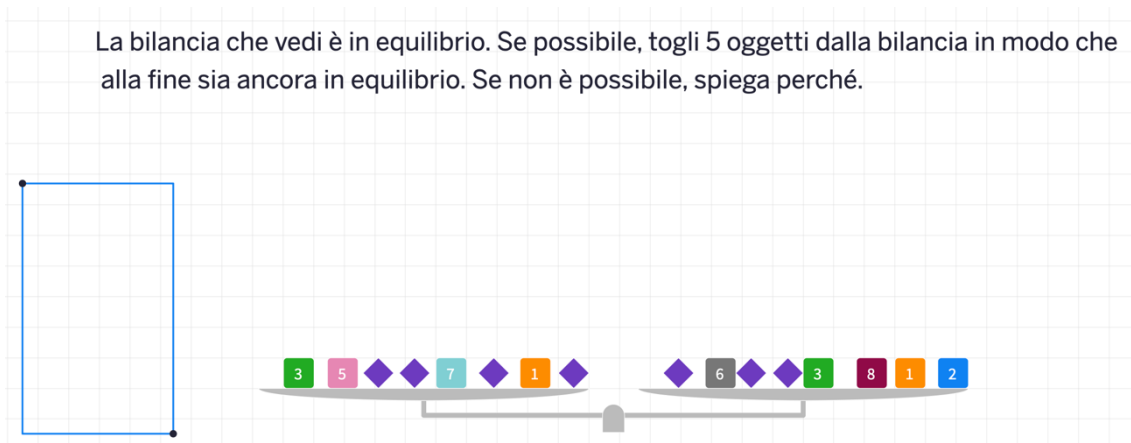
Task situation 5:

Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.



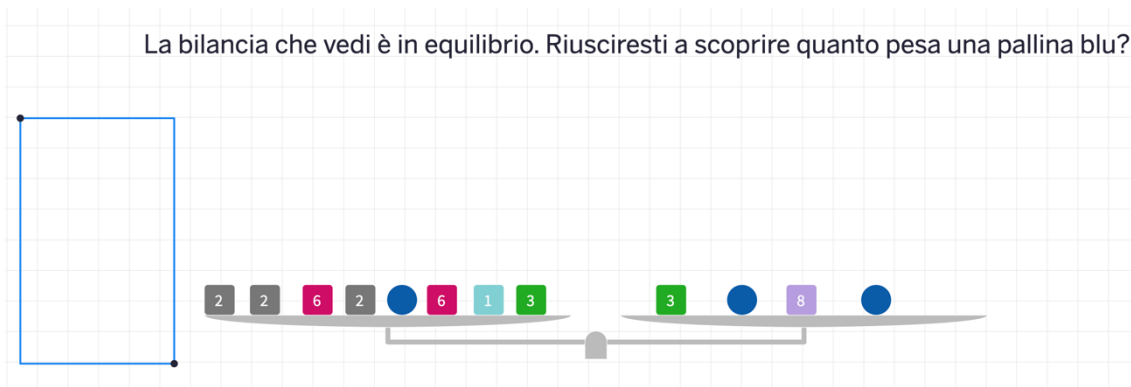
Task situation 6:

Here you see a balanced scale. If possible, remove 5 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why.



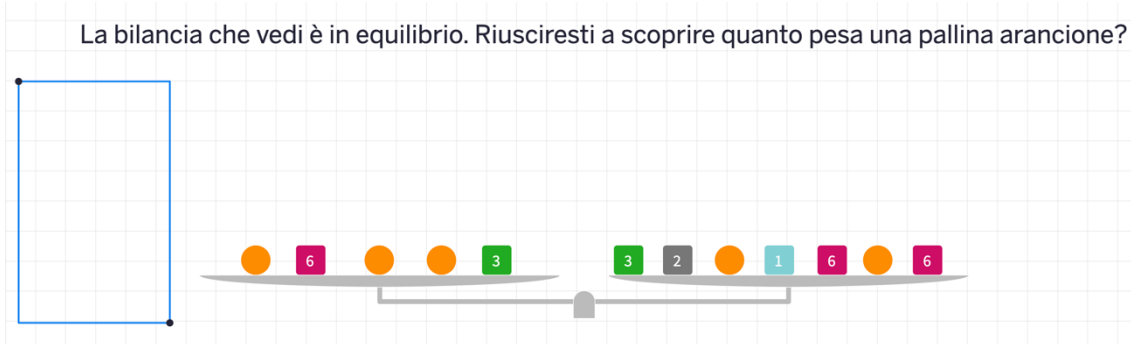
Task situation 7:

Here you see a balanced scale. Could you find out how much a little blue ball weighs?



Task situation 8:

Here you see a balanced scale. Could you find out how much a little orange ball weighs?

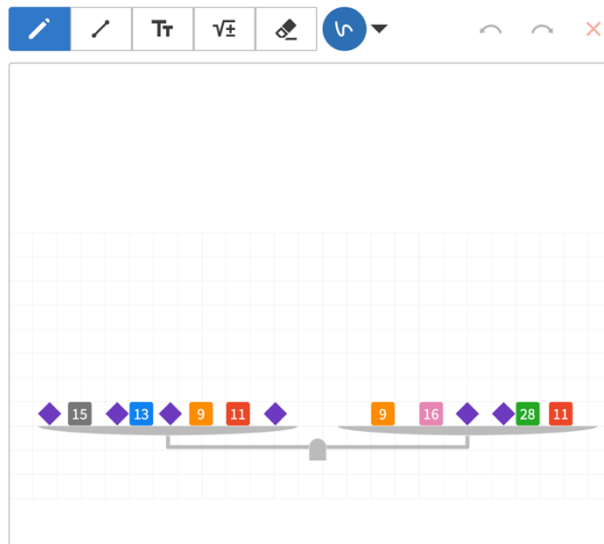


Task situation 9:

Could you find out how much a purple square weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.

Riusciresti a scoprire quanto pesa un quadratino viola?

Attenzione: è solo un'immagine, quindi non potete muovere nulla, ma potete scriverci sopra se vi aiuta.

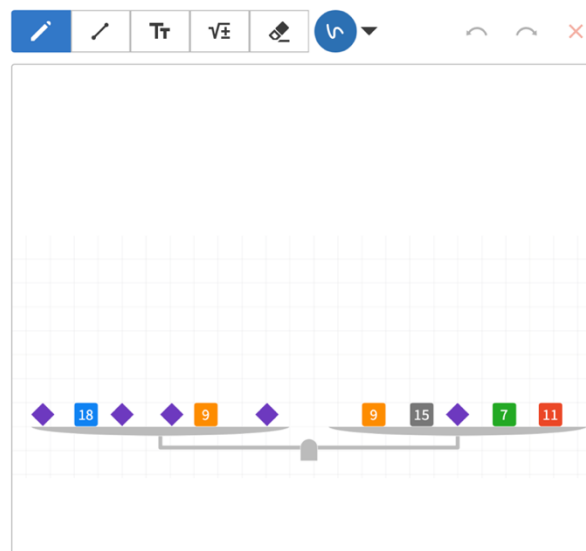


Task situation 10:

Could you find out how much a purple square weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.

Riusciresti a scoprire quanto pesa un quadratino viola?

Attenzione: è solo un'immagine, quindi non potete muovere nulla, ma potete scriverci sopra se vi aiuta.

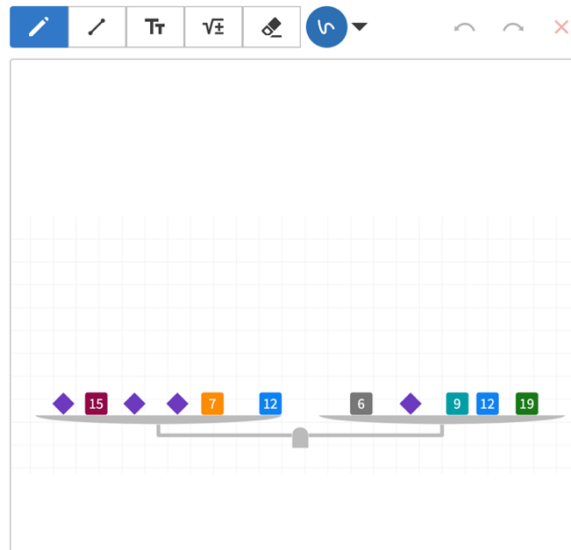


Task situation 11:

Could you find out how much a purple square weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.

Riusciresti a scoprire quanto pesa un quadratino viola?

Attenzione: è solo un'immagine, quindi non potete muovere nulla, ma potete scriverci sopra se vi aiuta.

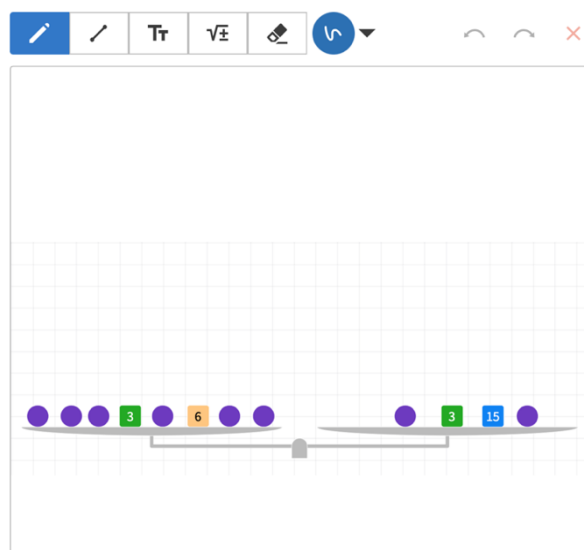


Task situation 12:

Could you find out how much a purple ball weighs? Please note it is only an image, so you cannot move anything, but you can write on it if it helps you.

Riusciresti a scoprire quanto pesa un cerchio viola?

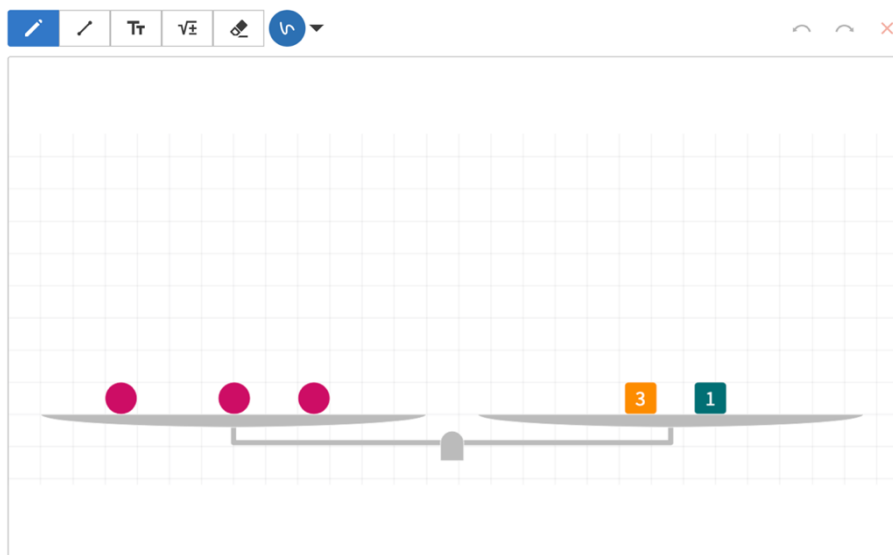
Attenzione: è solo un'immagine, quindi non potete muovere nulla, ma potete scriverci sopra se vi aiuta.



Task situation 13:

Here you see a balanced scale. What happens if we remove an object from both pans? Justify your answer.

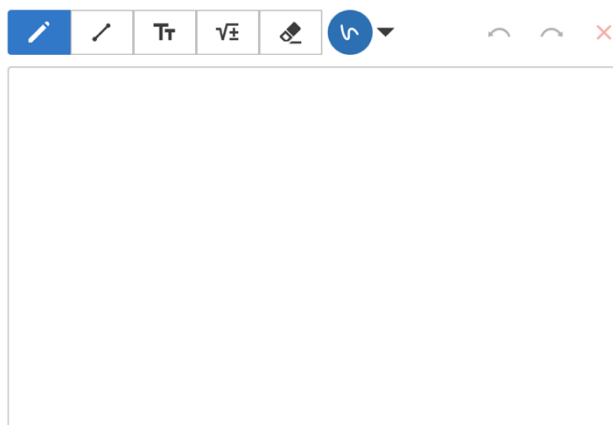
Questa bilancia è in equilibrio.
Cosa succede alla bilancia se togliamo un oggetto da
entrambi i piatti?
Giustifica la tua risposta.



Task situation 14:

Imagine a scale that is inclined to the left. On the left pan I add a weight of 9. On the right pan I add a weight of 5 and two red balls. The scale is now balanced. What can you say? Justify your answer.

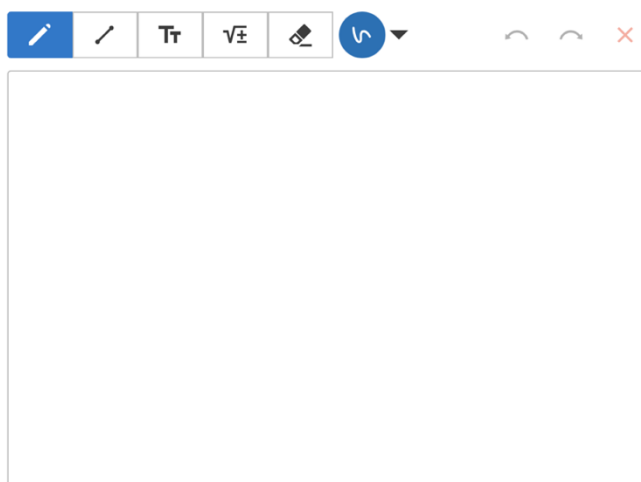
Immagina una bilancia che pende a sinistra.
A sinistra aggiungo un peso da 9.
A destra aggiungo un peso da 5 e due palline rosse.
La bilancia è in equilibrio.
Cosa puoi dire?
Giustifica la tua risposta.



Task situation 15:

Imagine a balanced scale. From the right pan I remove a weight of 5. From the left pan I remove a weight of 2 and a little red ball. The now hangs to the right. What can you say? Justify your answer.

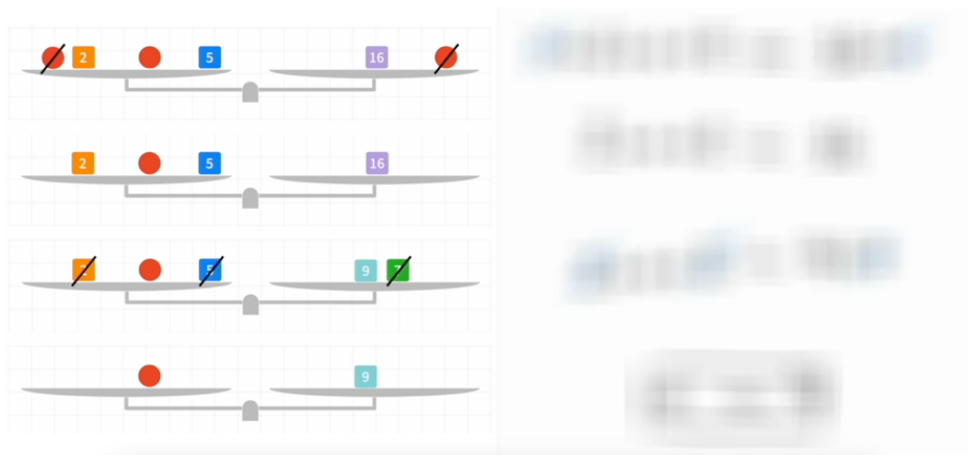
Immagina una bilancia in equilibrio.
Da destra tolgo una peso da 5.
Da sinistra tolgo un peso da 2 e un peso rosso.
La bilancia pende a destra.
Che cosa puoi dire? Giustifica la tua risposta.



Task situation 16:

Watch this video. Complete its audiobook in its missing parts. Imagine the right side of the video that is blurred.

Guarda questo video. Completa l'audio nelle sue parti mancanti. Immagina la parte di destra che nel video è oscurata.



Task situation 17:

Justify the steps in this inscription.

Giustifica i passaggi che vedi in questa scrittura.

$$\cancel{x} + 8 + x = 4 + \cancel{x} + 13 + 5$$

$$8 + x = 4 + 13 + 5$$

$$\cancel{8} + x = \cancel{4} + 13 + \cancel{4} + 1$$

$$x = 14$$

Task situation 18:

Justify the steps in this inscription.

Giustifica i passaggi che vedi in questa scrittura.

$$x - \cancel{2} + x + 6 = -3 + 8 + x - \cancel{2}$$

$$\cancel{x} + x + 6 = -3 + 8 + \cancel{x}$$

$$x + \cancel{6} = -3 + \cancel{6} + 2$$

$$x = -1$$

Task situation 19:

Solve the following equation.

Risolvi l'equazione che ti viene proposta.

$$x + 5 - x = 8 - x + 2$$

Task situation 20:

Justify the steps in this inscription.

Giustifica i passaggi che vedi in questa scrittura.

$$8 - x + 2 = -x + 3 - x$$

$$-x + x + x = +3 - 8 - 2$$

$$x = -7$$

Task situation 21:

Watch this video. What do you think it wants to show us?

Guarda questo video. Secondo te che cosa ci vuole mostrare?

$2x - 3 = x - 1$	$2x - 3 = x - 1$
a	

$2x - 3 = x - 1$	$2x - 3 = x - 1$
$2x = x - 1 + 3$	
b	

$2x - 3 = x - 1$	$2x - 3 = x - 1$
$2x = x - 1 + 3$	$2x - 3 + 3 = x - 1 + 3$
	$2x = x - 1 + 3$
c	

$2x - 3 = x - 1$	$2x - 3 = x - 1$
$2x = x - 1 + 3$	$2x - 3 + 3 = x - 1 + 3$
$2x - x = -1 + 3$	$2x - x = x - 1 + 3$
$x = 2$	$2x - x = -1 + 3$
d	$x = 2$

Activity 22:

Explore the file. What happens if you try to drag the red mark?

We want to construct the arrow representing the double of a, plus 1.

To do this, use the "Add arrow" button.

How does this new arrow move as a varies?

What happens if you tick the checkbox "Show/hide construction"?

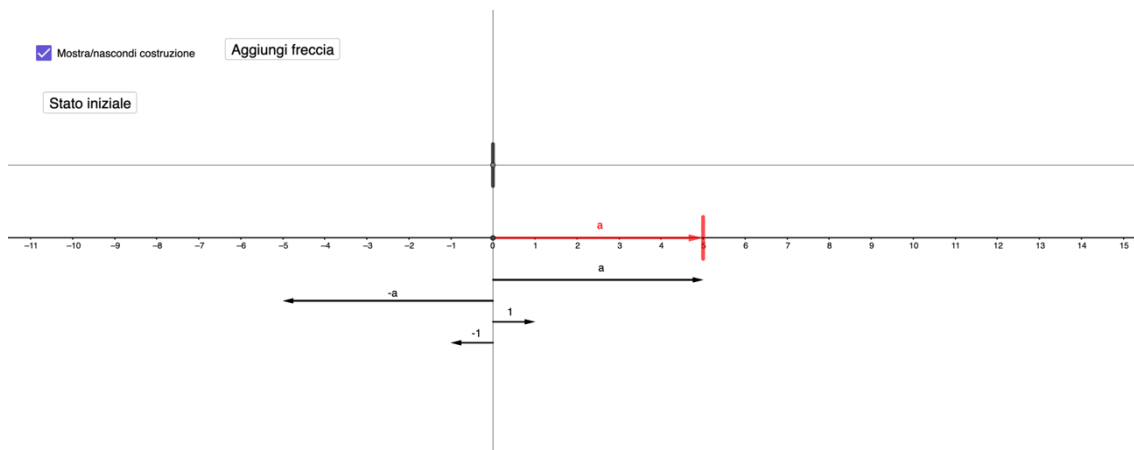
Esplora il file. Cosa succede se provi a trascinare la tacchetta rossa?

Vogliamo costruire la freccia corrispondente al doppio di a, più uno.

Per farlo, utilizza il pulsante “Aggiungi freccia”.

Come si muove questa nuova freccia al variare di a ?

Cosa succede se clicchi il pulsante “Mostra/nascondi costruzione”?



Activity 23:

We want to construct the arrow representing the triple of a , minus 1.

To do this, use the “Add arrow” button.

Can the tick mark at the end of the arrow you have built reach every position? Explain why.

Where do we need to move the red tick mark if we want the black one to reach the value 8?

And if we want it to reach the value -1?

Where do we need to move the red tick mark if we want the black one to be always on values that are greater than 5?

Vogliamo costruire la freccia corrispondente al triplo di a meno 1.

Per farlo, utilizza il pulsante “Aggiungi freccia”.

La tacchetta della freccia che hai costruito può raggiungere qualsiasi posizione?

Secondo te, perché?

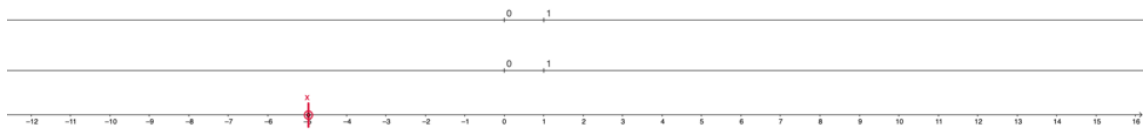
Dove dobbiamo spostare la tacchetta rossa perché quella nera raggiunga il valore 8?

E affinché raggiunga il valore -1?

Dove dobbiamo spostare la tacchetta rossa perché quella nera stia sempre su valori maggiori di 5?

Tacchetta nera:

Tacchetta blu:



Activity 26:

Imagine that in the input fields there is written

$2x$

$2x + 2$

How would the tick marks move?

Immagina che nei campi di inserimento ci sia scritto

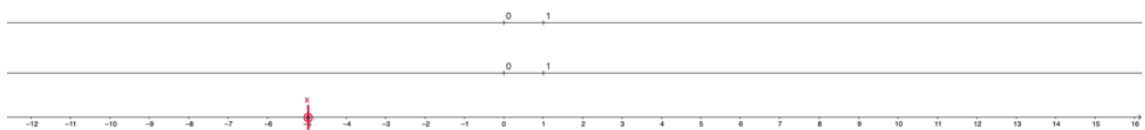
$2x$

$2x + 2$

Come si muoverebbero le due tacchette?

Tacchetta nera:

Tacchetta blu:



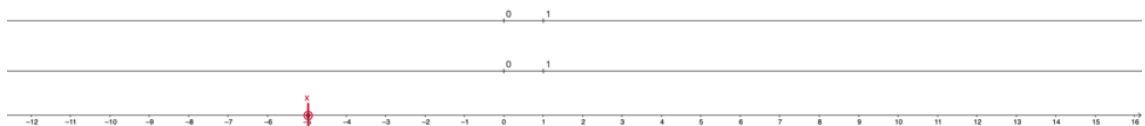
Activity 27:

Enter two expressions into the input fields so that the blue tick mark and the black tick mark meet only when the red tick mark is on 5.

Inserisci nei campi di inserimento due espressioni in modo che la tacchetta blu e quella nera si incontrino quando la tacchetta rossa è sul 5.

Tacchetta nera:

Tacchetta blu:



Task situation 28:

On the left you have a balanced mobile. Can you discover the weight of the triangle? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each triangle.

La bilancia di sinistra che vedi è in equilibrio. E' possibile scoprire quanto pesa il triangolo? Puoi utilizzare la bilancia di destra e il tasto “Proviamo” per verificare cosa succede variando il peso di ciascun triangolo.

peso del triangolo

Proviamo

Task situation 29:

On the left you have a balanced mobile. Can you discover the weight of the circle? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each triangle.

La bilancia di sinistra che vedi è in equilibrio. E' possibile scoprire quanto pesa il cerchio? Puoi utilizzare la bilancia di destra e il tasto “Proviamo” per verificare cosa succede variando il peso del cerchio.

peso del cerchio

Proviamo

Task situation 30:

On the left you have a balanced mobile. Can you discover the weight of the square? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each triangle.

La bilancia di sinistra che vedi è in equilibrio. E' possibile scoprire quanto pesa il quadrato? Puoi utilizzare la bilancia di destra e il tasto "Proviamo" per verificare cosa succede variando il peso di ciascun quadrato.

peso del quadrato

Proviamo

The left mobile balance is in equilibrium, indicated by a green checkmark. The left pan contains two green squares stacked vertically, with a blue triangle labeled '5' below them. The right pan contains one green square with a purple circle labeled '11' next to it. The right mobile balance is locked, indicated by a padlock icon. It has two pans: the left pan contains two white squares stacked vertically with a blue triangle labeled '5' below them, and the right pan contains one white square with a purple circle labeled '11' next to it. Above the right mobile is a text input field labeled 'peso del quadrato' and a pink button labeled 'Proviamo'.

Task situation 31:

On the left you have a balanced mobile. Can you discover the weight of the triangle? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each triangle.

La bilancia di sinistra che vedi è in equilibrio. E' possibile scoprire quanto pesa il triangolo? Puoi utilizzare la bilancia di destra e il tasto "Proviamo" per verificare cosa succede variando il peso di ciascun triangolo.

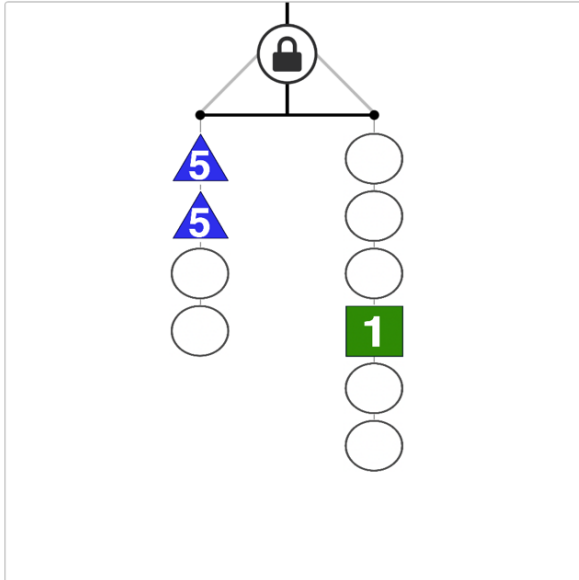
peso del triangolo

Proviamo

The left mobile balance is in equilibrium, indicated by a green checkmark. The left pan contains three blue triangles stacked vertically, with a purple circle labeled '8' next to them. The right pan contains three blue triangles stacked vertically, with a green square labeled '5' next to them. The right mobile balance is locked, indicated by a padlock icon. It has two pans: the left pan contains three white triangles stacked vertically with a purple circle labeled '8' next to them, and the right pan contains three white triangles stacked vertically with a green square labeled '5' next to them. Above the right mobile is a text input field labeled 'peso del triangolo' and a pink button labeled 'Proviamo'.

Task situation 32:

For what values of the weight of the circle does this mobile hang to the right? And to the left? And for what values is it balanced?



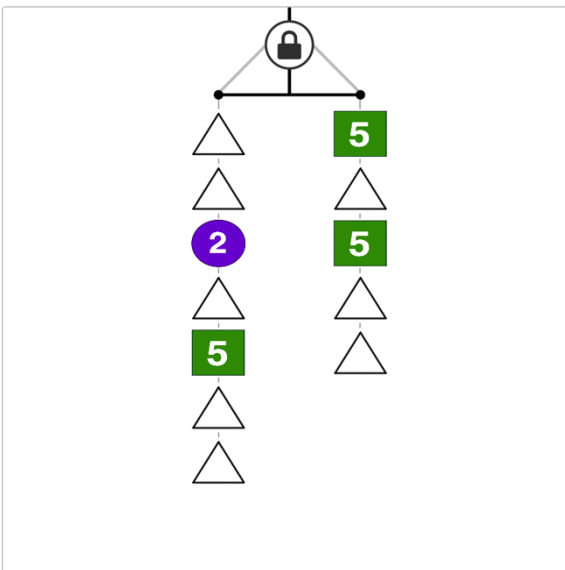
Per quali valori del peso del cerchio questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

peso del cerchio

Proviamo

Task situation 33:

For what values of the weight of the triangle does this mobile hang to the right? And to the left? And for what values is it balanced?



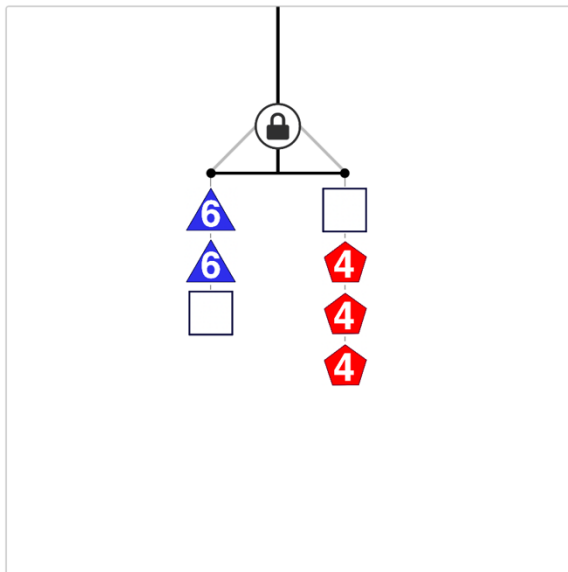
Per quali valori del peso del triangolo questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

peso del triangolo

Proviamo

Task situation 34:

For what values of the weight of the square does this mobile hang to the right? And to the left? And for what values is it balanced?



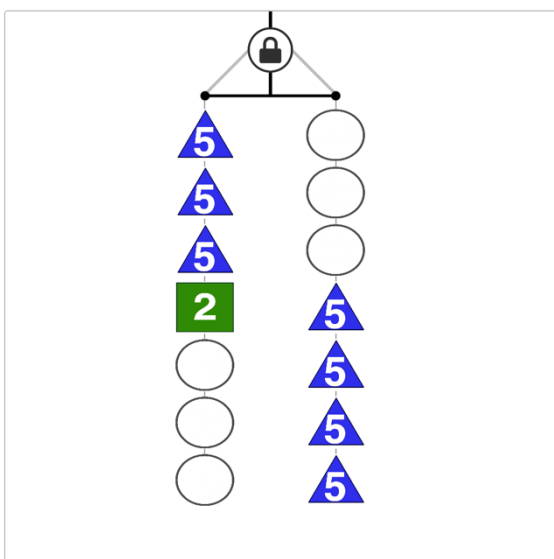
Per quali valori del peso del quadrato questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

peso del quadrato

Proviamo

Task situation 35:

For what values of the weight of the circle does this mobile hang to the right? And to the left? And for what values is it balanced?



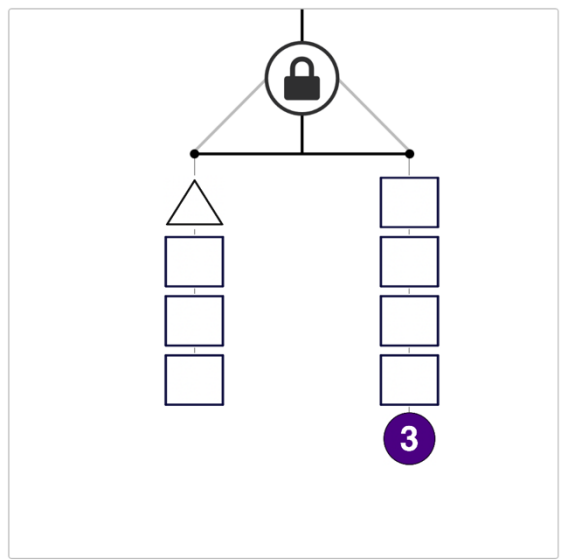
Per quali valori del peso del cerchio questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

peso del cerchio

Proviamo

Task situation 36:

For what values of the weight of the triangle and of the weight of the square does this mobile hang to the right? And to the left? And for what values is it balanced?

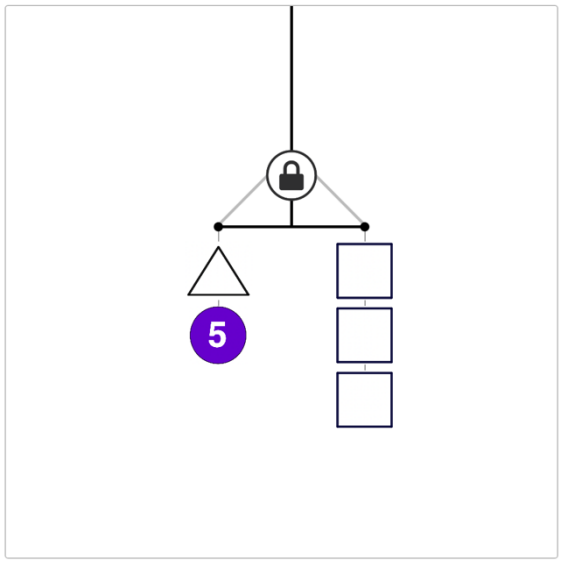


Per quali valori del peso del triangolo e del peso del quadrato questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

peso del triangolo	peso del quadrato

Proviamo

Task situation 37: For what values of the weight of the triangle and of the weight of the square does this mobile hang to the right? And to the left? And for what values is it balanced?



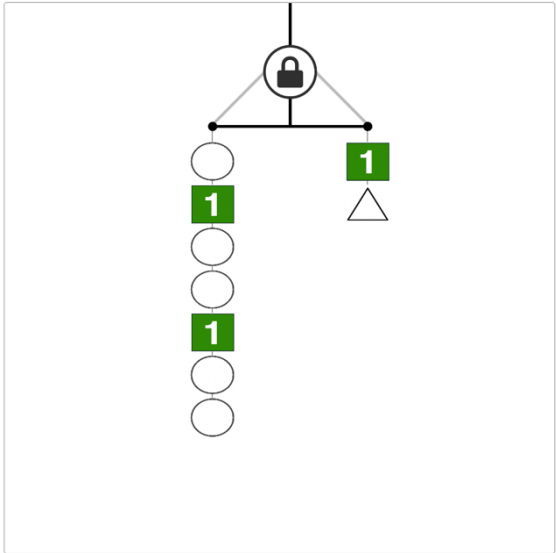
Per quali valori del peso del triangolo e del peso del quadrato questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

peso del triangolo	peso del quadrato

Proviamo

Task situation 38:

For what values of the weight of the circle and of the weight of the triangle does this mobile hang to the right? And to the left? And for what values is it balanced?



Per quali valori del peso del cerchio e del peso del triangolo questa bilancia pende a destra? E a sinistra? E per quali valori è in equilibrio?

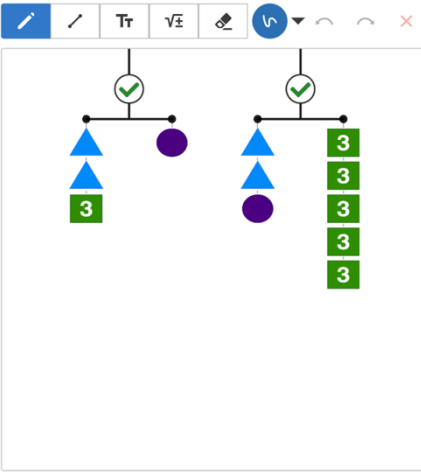
peso del cerchio	peso del triangolo

Proviamo

Task situation 39:

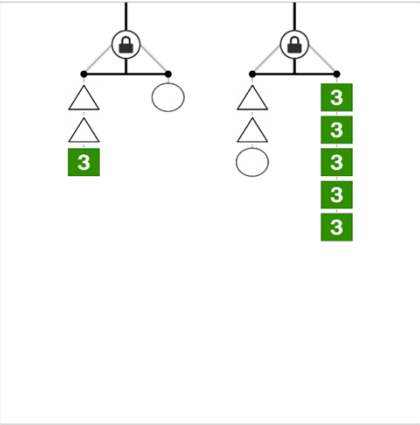
On the left you have two balanced mobiles. Can you discover the weights of the triangle and the circle? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each triangle and of each circle.

A sinistra vedi due bilance in equilibrio. E' possibile scoprire quanto pesano il triangolo e il cerchio? Puoi utilizzare la bilancia di destra e il tasto “Proviamo” per verificare cosa succede variando il peso di ciascun triangolo e di ciascun cerchio.



peso del triangolo	peso del cerchio

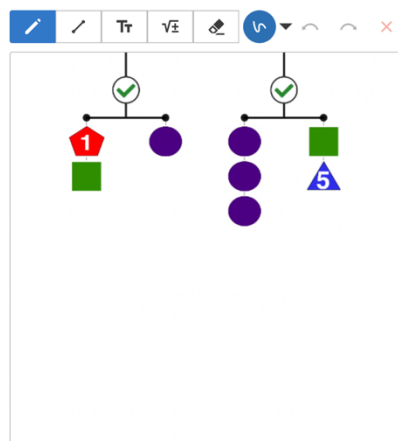
Proviamo



Task situation 40:

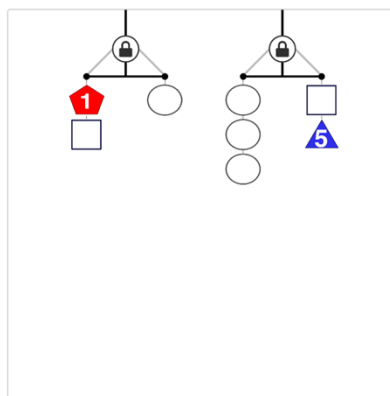
On the left you have two balanced mobiles. Can you discover the weights of the square and the circle? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each square and of each circle.

A sinistra vedi due bilance in equilibrio. E' possibile scoprire quanto pesano il quadrato e il cerchio? Puoi utilizzare la bilancia di destra e il tasto "Proviamo" per verificare cosa succede variando il peso di ciascun quadrato e di ciascun cerchio.



peso del quadrato	peso del cerchio

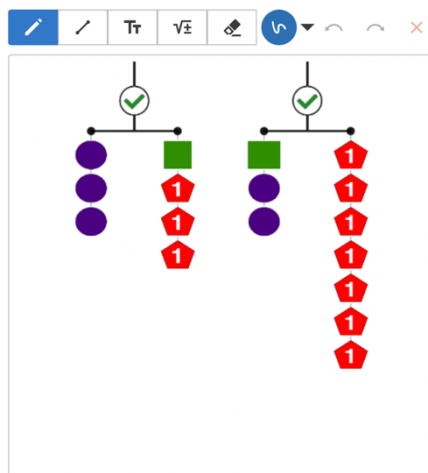
Proviamo



Task situation 41:

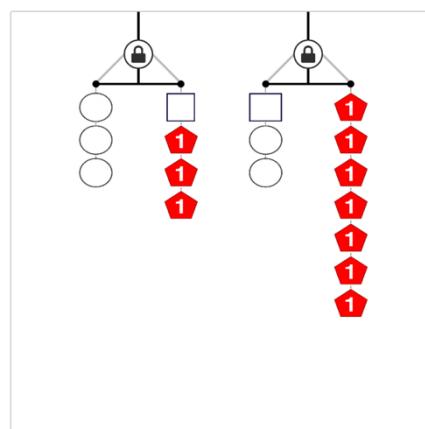
On the left you have two balanced mobiles. Can you discover the weights of the square and the circle? You can use the mobile on the right and the button “Let’s try!” to verify what happens when you vary the weight of each square and of each circle.

A sinistra vedi due bilance in equilibrio. E' possibile scoprire quanto pesano il cerchio e il quadrato? Puoi utilizzare la bilancia di destra e il tasto "Proviamo" per verificare cosa succede variando il peso di ciascun cerchio e di ciascun quadrato.



peso del cerchio	peso del quadrato

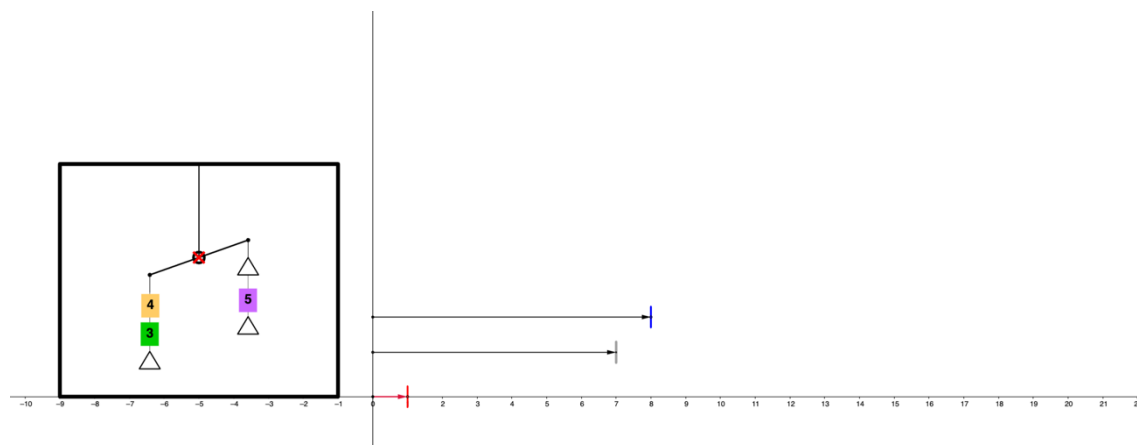
Proviamo



Task situation 42:

What do you observe in this file? Drag the red tick mark and describe what happens.

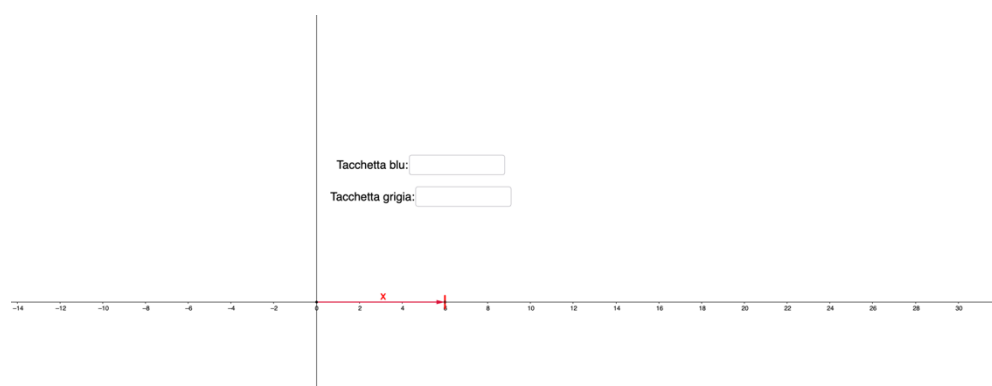
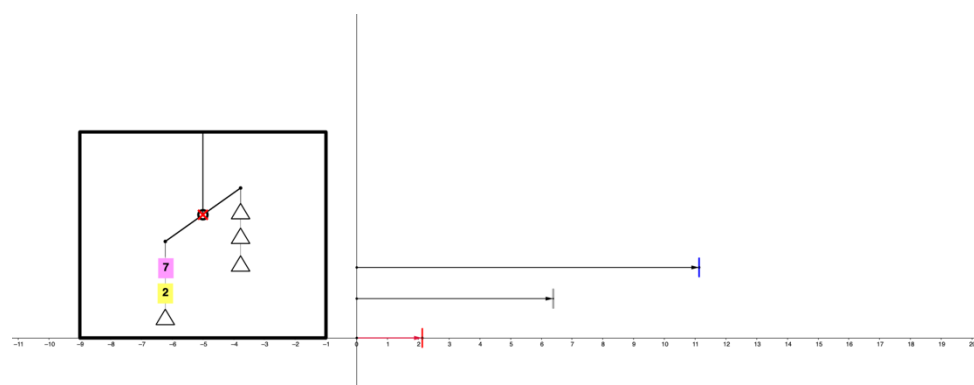
Che cosa osservi in questo file? Trascina la tacchetta rossa e descrivi che cosa succede.



Task situation 43:

Explore this file and try to reconstruct this mobile in the other file.

Esplora questo file e prova a ricostruire questa bilancia nell'altro file.



Task situation 44:

Enter expressions into the input fields so that:

a tick mark indicates 12.

A tick mark indicates $2x$.

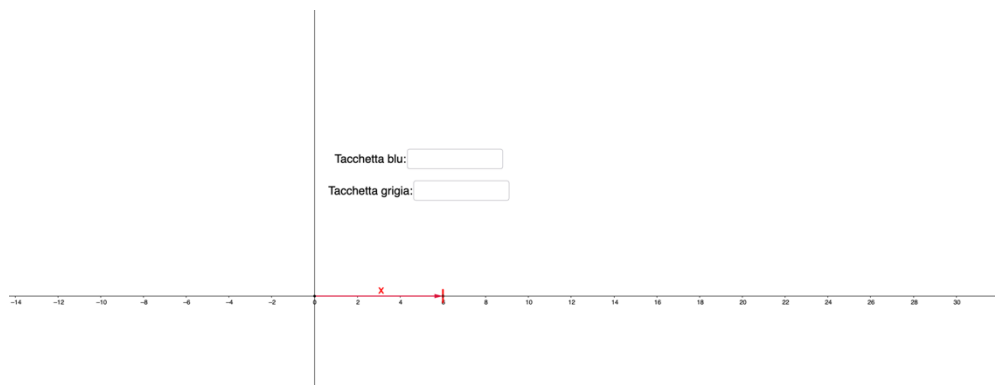
What do you expect will happen by dragging the red tick mark?

Inserisci nei campi di inserimento delle espressioni in modo che:

una tacchetta indichi 12.

l'altra tacchetta indichi $2x$.

Cosa ti aspetti che succeda trascinando la tacchetta rossa?



Task situation 45:

Enter expressions into the input fields so that:

a tick mark indicates $x + 5$.

A tick mark indicates $2x + 1$.

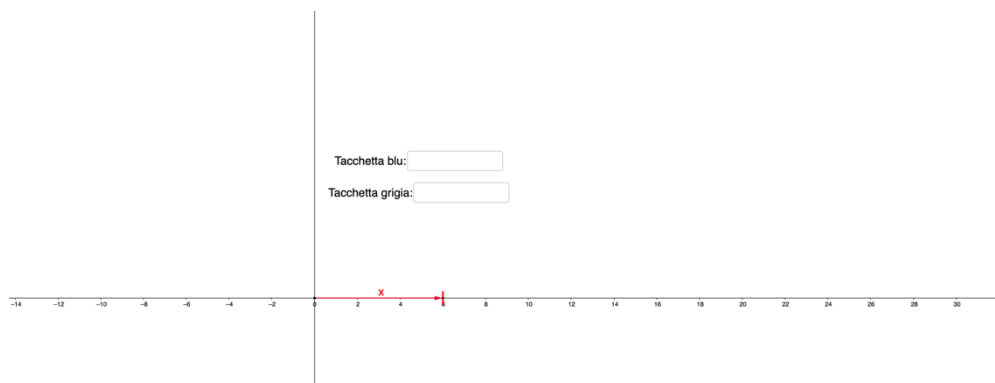
What do you expect will happen by dragging the red tick mark?

Inserisci nei campi di inserimento delle espressioni in modo che:

una tacchetta indichi $x + 5$

l'altra tacchetta indichi $2x + 1$.

Cosa ti aspetti che succeda trascinando la tacchetta rossa?



Task situation 46:

Enter expressions into the input fields so that:

a tick mark indicates $x + 2$.

A tick mark indicates $5x$.

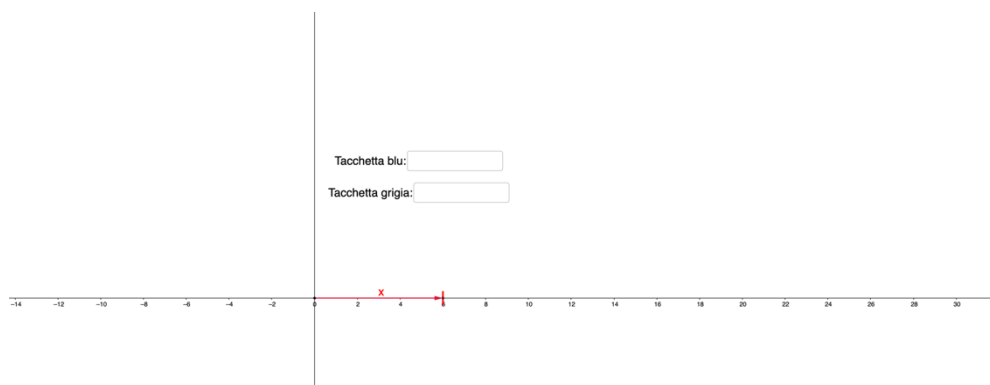
What do you expect will happen by dragging the red tick mark?

Inserisci nei campi di inserimento delle espressioni in modo che:

una tacchetta indichi $x + 2$

l'altra tacchetta indichi $5x$.

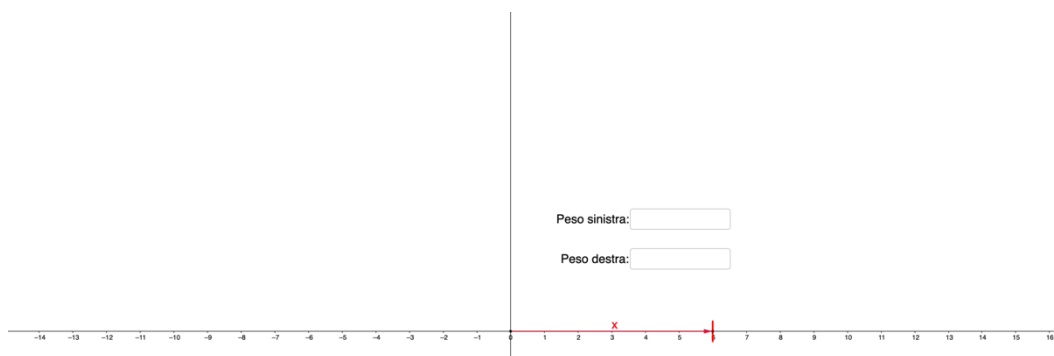
Cosa ti aspetti che succeda trascinando la tacchetta rossa?



Task situation 47:

Construct a mobile that is always balanced.

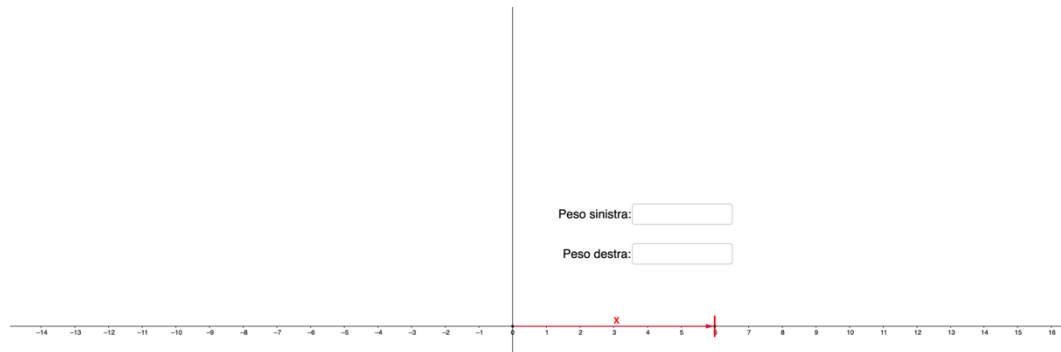
Costruisci una bilancia che sia sempre in equilibrio.



Task situation 48:

Construct a mobile that always hangs to the right.

Costruisci una bilancia che penda sempre a destra.



Task situation 1M:

For what values of x , is $3x + 1$ equal to $x + 2$?

For what values of x is, $3x + 1$ greater than $x + 2$?

For what values of x , is $3x + 1$ less than $x + 2$?

Per quali valori di x , $3x + 1$ è uguale a $x + 2$?

Per quali valori di x , $3x + 1$ è maggiore di $x + 2$?

Per quali valori di x , $3x + 1$ è minore di $x + 2$?

Task situation 2M:

$$3 + x = x + 3$$

Does it have solutions?

Can you find them or explain why there cannot be any?

Ha delle soluzioni?

Sai trovarle o eventualmente spiegare perché non ci possono essere?

Task situation 3M:

$$3x + 7 = 13$$

Does it have solutions?

Can you find them or explain why there cannot be any?

Ha delle soluzioni?

Sai trovarle o eventualmente spiegare perché non ci possono essere?

Task situation 4M:

$$13 - x = 13 + 11$$

Does it have solutions?

Can you find them or explain why there cannot be any?

Ha delle soluzioni?

Sai trovarle o eventualmente spiegare perché non ci possono essere?

Task situation 5M:

$$4x + 5 < x + 12 + x$$

Does it have solutions?

Can you find them or explain why there cannot be any?

Ha delle soluzioni?

Sai trovarle o eventualmente spiegare perché non ci possono essere?

Bibliography

- Antonini, S., Baccaglini-Frank, A., & Lisarelli, G. (2020). From experiences in a dynamic environment to written narratives on functions. *Digital Experiences in Mathematics Education*, 6(1), 1-29. <https://doi.org/10.1007/s40751-019-00054-3>
- Antonini, S., Bonadiman, C., & Nannini, B. (2023). Dynamic interactive mediators in discourse on indeterminate quantities: A case study. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel, & M. Tabach (Eds.), *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 43–50). University of Haifa, Israel: PME.
- Antonini, S., Bonadiman, C., & Nannini, B. (accepted). The analysis of transplantability between discourses: the case of scales and equations. In *Proceedings of the 48th Conference of the International Group for the Psychology of Mathematics Education*.
- Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics*, 14(3), 24–35.
- Arcavi, A., Drjvers, P., & Stacey, K. (2017). *The learning and teaching of algebra: ideas, insights, and activities*. Routledge.
- Arzarello, F. (2006). Semiosis as a multimodal process. *RELIME. Revista Latinoamericana de Investigación En Matemática Educativa*, 9(Extra 1), 267–300.
- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM Mathematics Education* 34 (3), 66–72. <https://doi.org/10.1007/BF02655708>
- Baccaglini-Frank, A. (2021). To tell a story, you need a protagonist: how dynamic interactive mediators can fulfill this role and foster explorative participation to mathematical discourse. *Educational Studies in Mathematics*, 106(2), 291-312. <https://doi.org/10.1007/s10649-020-10009-w>
- Baccaglini-Frank, A., & Mariotti, M.A. (2010). Generating Conjectures in Dynamic Geometry: the Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning*, 15(3), 225-253. <https://doi.org/10.1007/s10758-010-9169-3>
- Baccaglini-Frank, A. & Maracci, M. (2015). Multi-touch technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education*. 1, 7-27. <https://doi.org/10.1007/s40751-015-0002-4>
- Baccaglini-Frank, A., Antonini, S., Lisarelli, G. (2023). Exploiting the Potential of Dynamic Asymmetry in Dragging to Foster Students' Understanding of Functions and Their Cartesian Graphs. In: Pepin, B., Gueudet, G., Choppin, J. (eds)

Handbook of Digital Resources in Mathematics Education. Springer International Handbooks of Education. Springer, Cham. https://doi.org/10.1007/978-3-030-95060-6_14-1

- Baccaglini-Frank, A., Sinclair, N., Macchioni, E., Bonadiman, C., & Poli, C. Re-enacting a digital environment tutoring episode: New insights. Submitted.
- Barab, S., & Squire, K. (2004). Design-Based Research: Putting a Stake in the Ground. *Journal of the Learning Sciences*, 13(1), 1–14. https://doi.org/10.1207/s15327809jls1301_1
- Bartolini Bussi, M.G. & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In Lyn D. English & David Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (pp. 746–783). Routledge. <https://doi.org/10.4324/9780203930236>
- Bednarz, N., Kieran, C., & Lee, L. (Eds.). (1996). *Approaches to algebra: Perspectives for research and teaching*. Dordrecht, The Netherlands: Kluwer.
- Bednarz, N., Kieran, C., Lee, L. (1996). Approaches to Algebra: Perspectives for Research and Teaching. In: Bednarz, N., Kieran, C., Lee, L. (eds) *Approaches to Algebra*. Mathematics Education Library, vol 18. Springer, Dordrecht. https://doi.org/10.1007/978-94-009-1732-3_1
- Bednarz, N., Janvier, B. (1996). Emergence and Development of Algebra as a Problem-Solving Tool: Continuities and Discontinuities with Arithmetic. In: Bednarz, N., Kieran, C., Lee, L. (eds) *Approaches to Algebra*. Mathematics Education Library, vol 18. Springer, Dordrecht. https://doi.org/10.1007/978-94-009-1732-3_8
- Bonadiman, C. (2024). Unknown and variable: The semiotic potential of a digital balance- model. In P. Drijvers, C. Csapodi, H. Palmér, K. Gosztonyi, E. Herendiné-Kónya (Eds.), *Proceedings of the 13th Congress of the European Society for Research in Mathematics Education*, (pp. 4393-4400). <https://hal.science/hal-04407923v1>
- Bonadiman, C., & Macchioni, E. (2024). Aperture a cambiamenti nel discorso. *Didattica della Matematica. Dalla ricerca alle pratiche d'aula*, 15, 9 – 31. <https://doi.org/10.33683/ddm.24.15.1>
- Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in elementary school: Developing relational thinking. *ZDM Mathematics Education*, 37(1), 53–59. <https://doi.org/10.1007/BF02655897>
- Caspi, S., & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step toward school algebra. *International Journal of Educational Research*, 51-52, 45-65. <http://dx.doi.org/10.1016/j.ijer.2011.12.006>
- Chan, M. C. E., & Sfard, A. (2020). On learning that could have happened: The same tale in two cities. *The Journal of Mathematical Behavior*, 60. <https://doi.org/10.1016/j.jmathb.2020.100815>

- Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design Experiments in Educational Research. *Educational Researcher*, 32(1), 9-13. <https://doi.org/10.3102/0013189X032001009>
- Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, 66(3), 317–333. <https://doi.org/10.1007/s10649-006-9072-y>
- Ferrari, P. L. (2021). *Educazione matematica, lingua, linguaggi. Costruire, condividere e comunicare matematica in classe* (Vol. 1, pp. 1-192). De Agostini.
- Freudenthal, H. (1977). “What Is Algebra and What Has It Been in History?” *Archive for History of Exact Sciences*, vol. 16, no. 3, 1977, pp. 189–200. JSTOR, <http://www.jstor.org/stable/41133469>
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Reidel.
- Goldenberg, E. P., Lewis, P., & O’Keefe, J. (1992). Dynamic representation and the development of an understanding of functions. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, 25. MAA Notes.
- Hanna, G., Reid, D.A., & de Villiers, M., (2019). *Proof Technology in Mathematics Research and Teaching*. Mathematics Education in the Digital Era. Springer.
- Healy, L., & Sinclair, N. (2007). If this is your mathematics, what are your stories? *International Journal of Computers for Mathematics Learning*, 12(1), 3–21. <https://doi.org/10.1007/s10758-006-9109-4>
- Heyd-Metzuyanim, E., Tabach, M., & Nachlieli, T., (2016). Opportunities for learning given to prospective mathematics teachers. *Journal of Mathematics Teacher Education*, 19(6), 547– 574. <https://doi.org/10.1007/s10857-015-9311-1>
- Kaput, J.J. (1989) Linking representations in the symbol systems of algebra. In: Wagner S, Kieran C (eds) *Research issues in the learning and teaching of algebra*, vol 4, Research agenda for mathematics education. National Council of Teachers of Mathematics, Reston, pp 167–194.
- Kaput, J.J., Carraher, D.W., & Blanton, M.L. (2008). *Algebra in the early grades*. Routledge.
- Kaur, H. (2015). Two aspects of young children’s thinking about different types of dynamic triangles: prototypicality and inclusion. *ZDM Mathematics Education* 47(3), 407–420 <https://doi.org/10.1007/s11858-014-0658-z>
- Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12(3), 317–326. <https://doi.org/10.1007/BF00311062>
- Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 390–419). Macmillan Publishing Co, Inc.

- Kieran, C., Boileau, A., Garançon, M. (1996). Introducing Algebra by Means of a Technology-Supported, Functional Approach. In: Bednarz, N., Kieran, C., Lee, L. (eds) *Approaches to Algebra*. Mathematics Education Library, vol 18. Springer, Dordrecht. https://doi.org/10.1007/978-94-009-1732-3_19
- Kieran, C. (2004). The Core of Algebra: Reflections on its Main Activities. In: Stacey, K., Chick, H., Kendal, M. (eds) *The Future of the Teaching and Learning of Algebra The 12th ICMI Study*. New ICMI Study Series, vol 8. Springer, Dordrecht. https://doi.org/10.1007/1-4020-8131-6_2
- Kieran, C. (2014) Algebra Teaching and Learning. In S. Lerman, *Encyclopedia of Mathematics Education* (pp. 238-241). Springer Reference.
- Kieran, C. (Ed.). (2018). *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice*. Springer.
- Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions. *ZDM Mathematics Education* 54(6), 1131-1150. <https://doi.org/10.1007/s11858-022-01435-6>
- Kirshner, D. (2001). The structural algebra option revisited. In: Sutherland R, Rojano T, Bell A, Lins R (eds) *Perspectives on school algebra*. Kluwer, Dordrecht, pp 83–98
- Kristinsdóttir, B., Hreinsdóttir, F., & Lavicza, Z. (2018). Realizing students' ability to use technology with silent video tasks. In H.-G. Weigand, A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, N. Grønbaek, & J. Trgalova (Eds.), *Proceedings of the 5th ERME Topic Conference MEDA 2018* (pp. 163–170). University of Copenhagen. <http://www.math.ku.dk/english/research/conferences/2018/meda>
- Kristinsdóttir, B., Hreinsdóttir, F., & Lavicza, Z. (2020). Using silent video tasks for formative assessment. In B. Barzel, R. Bebernik, L. Göbel, M. Pohl, H. Ruchniewicz, F. Schacht, & D. Thurm (Eds.), *Proceedings of the 14th International Conference on Technology in Mathematics Teaching* (pp. 189–196). https://duepublico2.uni-due.de/receive/duepublico_mods_00070763
- Kristinsdóttir, B. (2021). *Silent video tasks-their definition, development, and implementation in upper secondary school mathematics classrooms*. PhD Thesis, University of Iceland, School of Education, Faculty of Subject Education
- Lavie, I., & Sfard, A. (2019). How children individualize numerical routines – Elements of a discursive theory in making. *Studies in Mathematics Education*, 28(4-5), 419-461. <https://doi.org/10.1080/10508406.2019.1646650>
- Lavie, I., Steiner, A. & Sfard, A. (2019). Routines we live by: from ritual to exploration. *Educational Studies in Mathematics* 101 (2), 153–176. <https://doi.org/10.1007/s10649-018-9817-4>
- Lee, L. (1996). An Initiation into Algebraic Culture through Generalization Activities. In: Bednarz, N., Kieran, C., Lee, L. (eds) *Approaches to Algebra*. Mathematics

- Education Library, vol 18. Springer, Dordrecht. https://doi.org/10.1007/978-94-009-1732-3_6
- Lee, L. (1997). *Algebraic understanding: The search for a model in the mathematics education community*. Unpublished PhD Thesis. Université du Québec à Montréal.
- Lerman, S. (2000) The social turn in mathematics education research. In: Boaler J (ed) *Multiple perspectives on mathematics teaching and learning*. Ablex, Westport, pp 19–44.
- Leung, A., Baccaglini-Frank, A. & Mariotti, M.A. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics* 84(3), 439–460. <https://doi.org/10.1007/s10649-013-9492-4>
- Leung, A., Bolite-Frant, J. (2015). Designing Mathematics Tasks: The Role of Tools. In: Watson, A., Ohtani, M. (eds) *Task Design In Mathematics Education*. New ICM Study Series. Springer, Cham. https://doi.org/10.1007/978-3-319-09629-2_6
- Leung, A., & Baccaglini-Frank, A. (2017). *Digital technology in designing mathematics education tasks potential and pitfalls*. Mathematics education in the digital era. Springer.
- Lewis, K. E. (2014). Difference not deficit: Reconceptualizing mathematical learning disabilities. *Journal for Research in Mathematics Education*, 45(3), 351–396. doi: 10.5951/jresmetheduc.45.3.0351
- Lewis, K. E. (2017). Designing a bridging discourse: Re-mediation of a mathematical learning disability. *Journal of the Learning Sciences*, 26(2), 320–365. <https://doi.org/10.1080/10508406.2016.1256810>
- Lisarelli, G. (2019). *A dynamic approach to functions and their graphs: a study of students' discourse from a commognitive perspective*. PhD Thesis, University of Florence, Florence, Italy.
- Lisarelli, G., Maracci, M., & Nannini, B. (2024). The moving arrows environment: a digital artifact for mediating the meanings of variable and unknown. In D. Diamantidis, M. Karavakou, M. Grizioti, & C. Kynigos (Eds.), *Proceedings of the 16th International Conference on Technology in Mathematics Teaching*, (pp. 69–76). Athens, Greece.
- Macchioni, E. (2025). *Capturing struggling students' mathematics difficulties holistically: towards a characterization of Mathematics Discourse Participation Profiles*. PhD Thesis, University of Pisa, Pisa, Italy.
- Macchioni, E., & Baccaglini-Frank, A. (2025). When the whole is greater than the sum of its parts: Towards a characterization of students' Mathematics Discourse Participation Profiles. *Educational Studies in Mathematics*. <https://doi.org/10.1007/s10649-025-10410-3>
- Mariotti, M. A. (2002). Influence of technologies advances on students' math learning. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.),

- Handbook of international research in mathematics education* (pp. 695–724). Mahwah, NJ: Erlbaum. <https://hdl.handle.net/11365/39308>
- Mason, J. (1996). Expressing Generality and Roots of Algebra. In: Bernarz, N., Kieran, C., Lee, L. (eds) *Approaches to Algebra*. Mathematics Education Library, vol 18. Springer, Dordrecht. https://doi.org/10.1007/978-94-009-1732-3_5
- MIUR (2010). Schema di regolamento recante “Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali”. http://www.indire.it/lucabas/lkmw_file/licei2010/indicazioni_nuovo_impaginatio_d_ecreto_indicazioni_nazionali.pdf
- Muller, E., Buteau, C., Ralph, B., & Mgombelo, J. (2009). Learning Mathematics through the Design and Implementation of Exploratory and Learning Objects. *International Journal for Technology in Mathematics Education*, 16(2), 63–73.
- Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom—The case of function. *International Journal of Educational Research*, 51-52, 10-27. <https://doi.org/10.1016/j.ijer.2011.12.007>
- Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education*, 44(2), 372–415. <https://doi.org/10.5951/jresmetheduc.44.2.0372>
- Ng, O. (2014). The interplay between language, gestures, and dragging and diagrams in bilingual learners’ mathematical communications. In P. Liljedahl, S. Oesterle, C. Nicol, D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36*, 4, (pp. 289–296). Vancouver: PME.
- Ng, O. (2016). Comparing calculus communication across static and dynamic environments using a multimodal approach. *Digital Experiences in Mathematics Education*, 2(2), 115–141. <https://doi.org/10.1007/s40751-016-0014-8>
- Otten, M., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (2019). The balance model for teaching linear equations: a systematic literature review. *International Journal of STEM Education*, 6(1), 1-21. <https://doi.org/10.1186/s40594-019-0183-2>
- Rabardel, P. (1995). *Les hommes et les technologies – Approche cognitive des instruments contemporains*. Paris: A. Colin.
- Radford, L. (2014). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal*, 26, 257–277. <https://doi.org/10.1007/s13394-013-0087-2>
- Robotti, E. (2017). Designing Innovative Learning Activities to Face Difficulties in Algebra of Dyscalculic Students: Exploiting the Functionalities of AlNuSet. In: Leung, A., Baccaglini-Frank, A. (eds) *Digital Technologies in Designing*

- Mathematics Education Tasks*. Mathematics Education in the Digital Era, vol 8. Springer, Cham. https://doi.org/10.1007/978-3-319-43423-0_10
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, UK: Cambridge University Press.
- Sfard, A. (2023). Taming Fantastic Beasts of Mathematics: Struggling with Incommensurability. *International Journal of Research in Undergraduate Mathematics Education*, 9, 572–604. <https://doi.org/10.1007/s40753-021-00156-7>
- Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification - The case of algebra. *Educational Studies in Mathematics*, 26 (2-3), 191–228. <https://doi.org/10.1007/BF01273663>
- Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, 8(1), 42–76. https://doi.org/10.1207/S15327884MCA0801_04
- Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34(4), 14–22. <https://doi.org/10.3102/0013189X034004014>
- Sfard, A., & Cobb, P. (2022). Research in Mathematics Education: What Can It Teach Us about Human Learning? In R. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (Cambridge Handbooks in Psychology, pp. 467–485). Cambridge University Press. <https://doi.org/10.1017/9781108888295.029>
- Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics*, 107 (3), 447–469. <https://doi.org/10.1007/s10649-021-10053-0>
- Sinclair, N., Healy, L., & Sales, C.O.R. (2009). Time for telling stories: narrative thinking with dynamic geometry. *ZDM Mathematics Education*, 41(4), 441–452. <https://doi.org/10.1007/s11858-009-0180-x>
- Sinclair, N. and Moss, J. (2012). The more it changes, the more it becomes the same: the development of the routine of shape identification in dynamic geometry environments. *International Journal of Education Research*, 51-52, 28–44. <https://doi.org/10.1016/j.ijer.2011.12.009>
- Sinclair, N., Heyd-Metzuyanim, E. (2014). Learning Number with TouchCounts: The Role of Emotions and the Body in Mathematical Communication. *Technology, Knowledge and Learning*, 19(1-2), 81–99. <https://doi.org/10.1007/s10758-014-9212-x>
- Stacey, K., Chick, H., Kendal, M. (eds) (2004). *The Future of the Teaching and Learning of Algebra The 12th ICMI Study*. New ICMI Study Series, vol 8. Springer, Dordrecht.

- Stacey, K., Chick, H. (2004). Solving the Problem with Algebra. In: Stacey, K., Chick, H., Kendal, M. (eds) *The Future of the Teaching and Learning of Algebra The 12th ICMI Study*. New ICMI Study Series, vol 8. Springer, Dordrecht. https://doi.org/10.1007/1-4020-8131-6_1
- Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
- Vygotsky, L. S. (1987). *Thinking and speech*. In R. W. Rieber, & A. C. Carton (Eds.), *The collected works of L. S. Vygotsky*. New York: Plenum Press.
- Xin, Y.P., Thur, R., & Thouless, H. (2022). *Enabling mathematics learning of struggling students*. Springer.
- Wagner, S., & Kieran, C. (Eds.). (1989). *Research Issues in the Learning and Teaching of Algebra: the Research Agenda for Mathematics Education*, Volume 4 (1st ed.). Routledge. <https://doi.org/10.4324/9781315044378>
- Wittgenstein, L. (1953/2003). *Philosophical investigations: The German text, with a revised English translation* (3rd ed., G. E. M. Anscombe, Trans.). Malden, MA: Blackwell.

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