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A new construction for Melnikov chaos in piecewise-smooth planar systems

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Abstract

In this paper we consider a piecewise-smooth **2**-dimensional system

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon)$$

where $\varepsilon > 0$ is a small parameter and $\vec{f}$ is discontinuous along a curve Ω^0 . We assume that $\vec{0}$ is a critical point for any $\varepsilon \geq 0$, and that for $\varepsilon = 0$ the system admits a trajectory $\vec{\gamma}(t)$ homoclinic to $\vec{0}$ and crossing transversely Ω^0 in $\vec{\gamma}(0)$. In a previous paper we have shown that, also in an n -dimensional setting, the classical Melnikov condition is enough to guarantee the persistence of the homoclinic to perturbations, but more recently we have found an open condition, a geometric obstruction which is not possible in the smooth case, which prevents chaos for **2**-dimensional systems when $\vec{g}$ is periodic in t .

In this paper we show that when this obstruction is removed we have chaos as in the smooth case. The proofs involve a new construction of the set Σ from which the chaotic pattern originates. The results are illustrated by examples.

Keywords: Melnikov theory, piecewise-smooth systems, chaos, homoclinic trajectories, non-autonomous dynamical systems, non-autonomous perturbation

MSC Classification: Primary 34A36 , 37G20 , 37G30; Secondary 34C37 , 34E10 , 34C25 , 37C60

1 Introduction

In this paper we deal with planar, piecewise-smooth systems and we prove, by Melnikov theory, the occurrence of a chaotic behavior when the fixed point of the unperturbed system lies on the discontinuity level.

More in detail, we consider the piecewise-smooth system

$$\dot{\vec{x}} = \vec{f}^\pm(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad \vec{x} \in \Omega^\pm, \quad (\text{PS})$$

eq-disc

where $\Omega^\pm = \{\vec{x} \in \Omega \mid \pm G(\vec{x}) > 0\}$, $\Omega^0 = \{\vec{x} \in \Omega \mid G(\vec{x}) = 0\}$, $\Omega \subset \mathbb{R}^2$ is an open set, G is a C^r -function on Ω with $r > 1$ such that 0 is a regular value of G . Next, $\varepsilon \in \mathbb{R}$ is a small parameter, and $\vec{f}^\pm \in C_b^r(\Omega^\pm \cup \Omega^0, \mathbb{R}^2)$, $\vec{g} \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R}, \mathbb{R}^2)$ and $G \in C_b^r(\Omega, \mathbb{R})$, i.e., the derivatives of $\vec{f}^\pm$, $\vec{g}$ and G are uniformly continuous and bounded up to the r -th order, respectively, if $r \in \mathbb{N}$ and up to r_0 if $r = r_0 + r_1$, with $r_0 \in \mathbb{N}$ and $0 < r_1 < 1$ and the r_0 -th derivatives are r_1 Hölder continuous.

Here and in the sequel, we use the shorthand notation $\pm$ to represent both the $+$ and $-$ equations and functions.

We assume that both the systems $\dot{\vec{x}} = \vec{f}^\pm(\vec{x})$ admit the origin $\vec{0} \in \mathbb{R}^2$ as a fixed point, and that $\vec{0}$ lies on the discontinuity level Ω^0 . Moreover, the 2×2 matrices $\mathbf{f}_x^\pm(\vec{0})$ have real eigenvalues of opposite sign, and the corresponding eigenvectors are transversal to Ω^0 at $\vec{0}$, see Section 2 for more details. Further, we assume that for $\varepsilon = 0$ system (PS) admits a piecewise-smooth homoclinic trajectory $\vec{\gamma}(t)$, crossing transversely the discontinuity level Ω^0 at some point $\vec{\gamma}(0) \neq \vec{0}$. That is, the homoclinic trajectory $\vec{\gamma}(t)$ satisfies:

$$\vec{\gamma}(t) \in \begin{cases} \Omega^-, & t < 0, \\ \Omega^0, & t = 0, \\ \Omega^+, & t > 0. \end{cases}$$

So $\Gamma = (\{\vec{\gamma}(t) \mid t \in \mathbb{R}\} \cup \{(0, 0)\})$ crosses Ω^0 transversely twice.

Our aim is to obtain a sufficient condition for the occurrence of a chaotic pattern, when we turn on a small perturbation $\varepsilon \neq 0$ in (PS).

Discontinuous problems are motivated by several physical applications, for instance mechanical systems with impacts, see e.g. [11], power electronics when we have state dependent switches [1], walking machines [20], relay feedback systems [10], biological systems [40]; see also [17, 31] and the references therein. Further they are also a good source of examples since it is somehow easy to produce piecewise linear systems exhibiting an explicitly known homoclinic trajectory, so giving rise to chaos if subject

to perturbations, whereas this is not an easy task in general for the smooth case (especially if the system is not Hamiltonian).

In the smooth case, the problem of detecting the occurrence of chaotic solutions for non-autonomous dynamical systems is now classical. In particular, it is known that perturbing periodically a (smooth) dynamical system which admits a transversal homoclinic point of the period map, an invariant Smale horseshoe set originating a chaotic pattern arises; we refer the interested reader, e.g., to [23, 36, 42, 44, 45]. More in detail, let us consider the smooth planar system

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad \vec{x} \in \Omega, \quad (\text{S})$$

eq-smooth

having a homoclinic trajectory, where both $\vec{f}$ and $\vec{g}$ are of class C^2 in the whole Ω . Melnikov theory gives an integral condition sufficient to ensure the persistence of the homoclinic trajectory for $\varepsilon \neq 0$, and, if the system is periodic in t , the insurgence of a chaotic pattern: the classical requirement is that a certain computable function $\tau \mapsto \mathcal{M}(\tau)$, which for system (PS) is given by formula (2) below, has a non-degenerate zero. In this framework, after the pioneering work of Melnikov [34], an important step forward was performed by Chow et al. in [16]. Another milestone in this context is [36] where Palmer addressed the n -dimensional case where $n \geq 2$. When $n > 2$ an additional non-degeneracy condition on the family of unperturbed homoclinic is needed, a condition which is automatically satisfied in the $n = 2$ case. Palmer's approach is analytic and combines Fredholm alternative with the use of exponential dichotomy and shadowing lemmas. These results have been improved from many points of view, in particular they have been extended to the almost periodic case, see e.g. [38, 41], and to the case where the zeros of $\mathcal{M}(\tau)$ are degenerate, see e.g. [2, 3]. Afterwards the so-called perturbation approach has been widely developed by many authors. Such a theory is by now well-established for smooth systems, and there are many works devoted to it. For example we refer to [9, 16, 21, 23, 35–38, 41, 43–45].

In the last 20 years many authors addressed the problem of generalizing Melnikov theory to a discontinuous piecewise-smooth setting: among the other papers, let us mention e.g., [4, 6, 12, 15, 28–30, 32]. In fact most of the cited results are set in the planar case $n = 2$. However we wish to quote the nice work performed by Battelli and Fečkan, who managed to extend the theory to the $n \geq 2$ case, but assuming that the critical point of the unperturbed system *does not* belong to the discontinuity hypersurface (in the case $n = 2$, the curve Ω^0), see e.g. [6]. For a generalization of Melnikov theory in other directions, see e.g. the recent [8, 22, 24, 33, 46].

Let us assume for illustrative purposes that $\vec{g}$ is 1-periodic in the t -variable (in fact the results of Battelli and Fečkan require very weak recurrence properties, see e.g. [6]). Roughly speaking, in order to prove the existence of the chaotic pattern in a discontinuous setting, Battelli and Fečkan glued together infinitely many solutions of the perturbed problem, which mimic different time-translation of the unperturbed homoclinic. For this purpose, they had to solve infinitely many equations via implicit function theorem, using a Lijapunov-Schmidt reduction (needed in the $n > 2$ case) together with the Melnikov condition. A key point of their contribution, and one of the main technical and theoretical issues, was to understand how the discontinuity of the

flow affects the jumps in the projections of the exponential dichotomy, and how the non-degeneracy condition required in the $n > 2$ case is translated in this discontinuous setting. This way they managed to prove the persistence of the homoclinic [4], the insurgence of chaos when the unperturbed homoclinic exhibits no sliding phenomena [6] and when it does [5].

If, instead, we assume that the critical point *lies on the discontinuity hypersurface* Ω^0 , the problem of detecting a chaotic behavior becomes even more challenging. Why shall one insist on studying this case? In many real applications this is what really happens, e.g. in systems with dry friction. In this setting, as a preliminary step, in [12] it was shown that the Melnikov condition found by Battelli and Fečkan together with a further (generic) transversality requirement (always satisfied in two dimensions) are enough to prove the persistence of the homoclinic trajectory.

Subsequently, in [15] we considered the case of a “one-sided” homoclinic. In other words, in the discontinuous setting like (PS) we assumed that $\vec{\gamma}(t) \in \Omega^+$ for any $t \in \mathbb{R}$. In this case, with suitable assumptions we proved that the chaotic trajectories are still located in Ω^+ . We point out that, even if the main motivation and applications are set in the discontinuous case, a crucial idea in both the papers [12] and [15] was that the problem addressed could be equivalently stated for continuous systems and reduced, in some sense, to a problem of location of trajectories. Moreover, we stress that in [12] and [15], like in the paper by Battelli and Fečkan [6], we studied systems in $\mathbb{R}^n$.

Hence, in the case in which the origin lies on the discontinuity hypersurface Ω^0 and Γ crosses Ω^0 transversely twice, so that it splits in two branches, the problem of the occurrence of chaotic patterns was still open. The purpose of the paper is to fill this gap, at least in the two-dimensional case.

We think it is worth mentioning the recent papers [24, 25], still concerning the n -dimensional case, $n \geq 2$, and in which the origin lies on the discontinuity hypersurface Ω^0 , proving persistence of the homoclinic trajectory in the $n > 2$ degenerate case, and persistence of the periodic solutions.

As pointed out, here we focus on the two-dimensional case. In this setting, in the recent paper [18], quite surprisingly, it was shown that the Melnikov condition, which ensures the persistence of the homoclinic [12] and the transversality of the crossing between stable and unstable leaves, *does not guarantee* the insurgence of chaos differently from the smooth setting, and also from the piecewise-smooth setting considered, e.g., in [4–6, 28, 29, 32]. In fact, anytime we have sliding phenomena close to the origin, a geometrical obstruction prevents the formation of chaotic patterns, and new bifurcation phenomena take place, scenarios which may exist just in a discontinuous context.

In this paper we complete the picture of the two-dimensional, discontinuous case showing that, when close to the origin no sliding phenomena occur, system (PS) exhibits a chaotic behavior as in the smooth setting; while according to [12], if there is sliding close to the origin, even if it does not involve the homoclinic trajectory, we have persistence of the homoclinic, but no chaos when the perturbation is turned on.

For the proofs we profit of the results obtained in our previous papers [13, 14]. Let us now summarize the main ideas. In [14] we have proved the existence of trajectories which are chaotic only in the future or chaotic only in the past. Assume again for

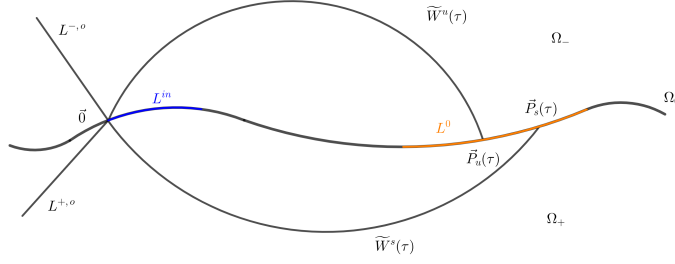


Fig. 1: Stable and unstable leaves.

LinL0

illustrative purposes that, $\vec{g}$ is 1-periodic in t and that the Melnikov function $\mathcal{M}(\tau)$ has a non-degenerate zero, i.e., $\mathcal{M}(\tau_0) = 0 \neq \mathcal{M}'(\tau_0)$ for some $\tau_0 \in [0, 1]$. Let $\tilde{W}^s(\tau)$ and $\tilde{W}^u(\tau)$ denote the branch of the stable/unstable manifolds between the origin and Ω^0 and denote by $\vec{P}_s(\tau)$ the (transversal) intersection between $\tilde{W}^s(\tau)$ and Ω^0 and by $\vec{P}_u(\tau)$ the (transversal) intersection between $\tilde{W}^u(\tau)$ and Ω^0 . Let $\vec{x}(t, \tau; \vec{\xi})$ denote the trajectory of (PS) leaving from $\vec{\xi}$ at $t = \tau$. Let $\mathbb{Z}^+ = \{j \in \mathbb{Z} \mid j \geq 1\}$, $\mathbb{Z}^- = \{j \in \mathbb{Z} \mid j \leq -1\}$, $\mathcal{E}^+ = \{0; 1\}^{\mathbb{Z}^+}$, $\mathcal{E}^- = \{0; 1\}^{\mathbb{Z}^-}$, $\mathcal{E} = \{0; 1\}^{\mathbb{Z}}$.

Firstly in [13] we have developed a careful analysis of the fly time needed for a trajectory starting in Ω^0 close to $\vec{P}_s(\tau)$ at $t = \tau$ to perform a loop close to $\tilde{W}(t) = \tilde{W}^s(t) \cup \tilde{W}^u(t)$, till it intersects again Ω^0 close to $\vec{P}_u(t)$, together with a careful evaluation of the space displacement with respect to $\tilde{W}(t)$.

Then in [14], using the analysis of [13] we have shown that there is ε_0 such that for any $0 < \varepsilon \leq \varepsilon_0$ there are compact connected sets $X^+(\tau, \mathcal{T}^+, e^+)$ and $X^-(\tau, \mathcal{T}^-, e^-)$ such that if $\vec{\xi}^+ \in X^+$ then $\vec{x}(t, \tau; \vec{\xi}^+)$ is “chaotic in the future”, i.e., it mimics a translate of $\vec{g}(t)$ if $e_j = 1$ and it is close to the origin if $e_j = 0$ in $t \in [T_{2j-1}, T_{2j+1}]$ and $j \in \mathbb{Z}^+$; while if $\vec{\xi}^- \in X^-$ then $\vec{x}(t, \tau; \vec{\xi}^-)$ is “chaotic in the past”, i.e., it has a similar property when $j \in \mathbb{Z}^-$.

In this article we show some sort of continuity of the sets $X^+(\tau, \mathcal{T}^+, e^+)$ and $X^-(\tau, \mathcal{T}^-, e^-)$ with respect to τ so that we can apply the intermediate value theorem to find a $\tau_\star \in [0, 1]$ such that the intersection $X^+(\tau_\star, \mathcal{T}^+, e^+) \cap X^-(\tau_\star, \mathcal{T}^-, e^-)$ is non-empty.

Then we construct the set $\Sigma(\tau_\star, \mathcal{T})$ made up by all the initial conditions giving rise to a chaotic pattern. We stress that an advantage of the application of the intermediate value theorem, instead of the implicit function theorem, is that we can allow $\mathcal{M}(\tau_\star)$ to be a degenerate zero (as far as we know, this fact is new in the piecewise-smooth but discontinuous context, while analogous results in the smooth case have been obtained via degree theory in [2, 3]). However, a drawback is that we just obtain a semi-conjugation with the Bernoulli shift.

Further, our approach, inspired by the work by Battelli and Fečkan, see [4–7], allows to weaken the recurrence properties of the Melnikov function $\mathcal{M}(\tau)$.

The paper is divided as follows: in §2 we list the main hypotheses and we state our main results, i.e. Theorems 2.6, 2.8 and 2.14; in §3 we introduce some preliminary constructions and some key results concerning time needed to perform a loop and space displacement with respect to $\tilde{W}(t)$, mainly borrowed from [13]; we close the section by illustrating how the results are applied to similar scenarios; in §4, roughly speaking, we show that the compact sets $X^+(\tau, \mathcal{T}^+, e^+)$ and $X^-(\tau, \mathcal{T}^-, e^-)$ of initial conditions giving rise to trajectories chaotic in the future and in the past respectively, depend continuously on τ , see Theorems 4.1 and 4.2: the argument relies heavily on [14] and it is detailed in §4.1 in the weaker setting of Theorem 2.14, and then quickly adapted to the setting of Theorems 2.8 and 2.6 in §4.2; in §5 we use the intermediate value theorem to prove that the Melnikov condition guarantees the existence of τ_* such that there is a non-empty intersection between the sets $X^+(\tau_*, \mathcal{T}^+, e^+)$ and $X^-(\tau_*, \mathcal{T}^-, e^-)$, thus proving Theorems 5.1 and 5.5, from which Theorems 2.6, 2.8 and 2.14 easily follow; in §6 we show that in the setting of Theorems 2.6 and 2.8 the action of the flow of (PS) on the set Σ giving rise to chaos is semi-conjugated to the Bernoulli shift in $\{0; 1\}^{\mathbb{Z}}$; finally in §7 we illustrate our hypotheses and our results by some examples.

S.setting

2 Statement of the main results

2.1 Basic assumptions, constants and notation

Following [13, 14] firstly we give a notion of solution and we collect the basic hypotheses which we assume through the whole paper together with the main constants. We emphasize that we keep the notation as in [13] and [14]. By a *solution* of (PS) we mean a continuous, piecewise C^r function $\vec{x}(t)$ that satisfies

$$\dot{\vec{x}}(t) = \vec{f}^+(\vec{x}(t)) + \varepsilon \vec{g}(t, \vec{x}(t), \varepsilon), \quad \text{whenever } \vec{x}(t) \in \Omega^+, \quad (\text{PS}+)$$

eq-disc+

$$\dot{\vec{x}}(t) = \vec{f}^-(\vec{x}(t)) + \varepsilon \vec{g}(t, \vec{x}(t), \varepsilon), \quad \text{whenever } \vec{x}(t) \in \Omega^-. \quad (\text{PS}-)$$

eq-disc-

Moreover, if $\vec{x}(t_0)$ belongs to Ω^0 for some t_0 , then we assume either $\vec{x}(t) \in \Omega^-$ or $\vec{x}(t) \in \Omega^+$ for t in some left neighborhood of t_0 , say $]t_0 - \tau, t_0[$ with $\tau > 0$. In the first case, the left derivative of $\vec{x}(t)$ at $t = t_0$ has to satisfy $\dot{\vec{x}}(t_0^-) = \vec{f}^-(\vec{x}(t_0)) + \varepsilon \vec{g}(t_0, \vec{x}(t_0), \varepsilon)$; while in the second case, $\dot{\vec{x}}(t_0^-) = \vec{f}^+(\vec{x}(t_0)) + \varepsilon \vec{g}(t_0, \vec{x}(t_0), \varepsilon)$. A similar condition is required for the right derivative $\dot{\vec{x}}(t_0^+)$. We stress that, in this paper, we do not consider solutions of equation (PS) that belong to Ω^0 for t in some nontrivial interval, i.e., sliding solutions.

Notation

Throughout the paper we will use the following notation. We denote scalars by small letters, e.g. a , vectors in $\mathbb{R}^2$ with an arrow, e.g. $\vec{a}$, and $n \times n$ matrices by bold letters, e.g. $\mathbf{A}$. By $\vec{a}^*$ and $\mathbf{A}^*$ we mean the transpose of the vector $\vec{a}$ and of the matrix $\mathbf{A}$, resp., so that $\vec{a}^* \vec{b}$ denotes the scalar product of the vectors $\vec{a}$, $\vec{b}$. We denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^2$, while for matrices we use the functional norm $\|\mathbf{A}\| = \sup_{\|\vec{w}\| \leq 1} \|\mathbf{A}\vec{w}\|$. We will use the shorthand notation $\mathbf{f}_x = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ unless this may cause confusion.

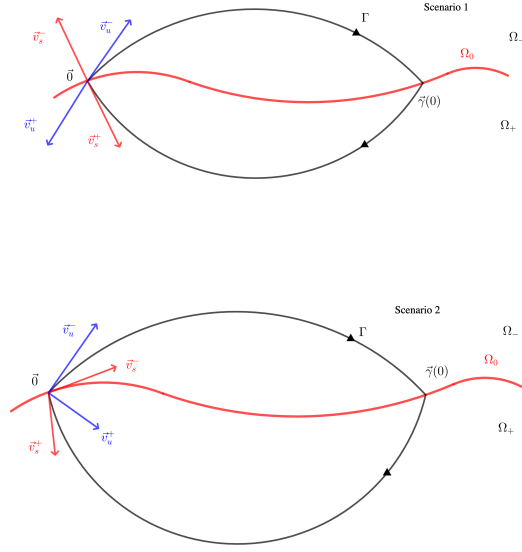


Fig. 2: Scenarios 1 and 2. In both the cases **F2** holds and we can apply our results, Theorems 2.6, 2.8 and 2.14, and we find chaotic patterns. See also §2.3.

scenario123

Further, given $\delta > 0$ we denote by $B(\vec{P}, \delta) := \{\vec{Q} \mid \|\vec{Q} - \vec{P}\| < \delta\}$ and, given a set D and $\delta > 0$, we let

$$B(D, \delta) := \{\vec{Q} \in \mathbb{R}^2 \mid \exists \vec{P} \in D : \|\vec{Q} - \vec{P}\| < \delta\} = \cup \{B(\vec{P}, \delta) \mid \vec{P} \in D\}.$$

We list here some hypotheses which we assume through the whole paper.

F0 We have $\vec{0} \in \Omega^0$, $\vec{f}^\pm(\vec{0}) = \vec{0}$, and the eigenvalues $\lambda_s^\pm, \lambda_u^\pm$ of $\mathbf{f}_x^\pm(\vec{0})$ are such that $\lambda_s^\pm < 0 < \lambda_u^\pm$.

Denote by $\vec{v}_s^\pm, \vec{v}_u^\pm$ the normalized eigenvectors of $\mathbf{f}_x^\pm(\vec{0})$ corresponding to $\lambda_s^\pm, \lambda_u^\pm$. We assume that the eigenvectors $\vec{v}_s^\pm, \vec{v}_u^\pm$ are not orthogonal to $\vec{\nabla}G(\vec{0})$. To fix the ideas we require

$$\mathbf{F1} \quad [\vec{\nabla}G(\vec{0})]^* \vec{v}_u^- < 0 < [\vec{\nabla}G(\vec{0})]^* \vec{v}_u^+, \quad [\vec{\nabla}G(\vec{0})]^* \vec{v}_s^- < 0 < [\vec{\nabla}G(\vec{0})]^* \vec{v}_s^+.$$

Following [13, 14] we require a further condition on the mutual positions of the directions spanned by $\vec{v}_s^\pm, \vec{v}_u^\pm$.

More precisely, set $\mathcal{T}_u^\pm := \{c\vec{v}_u^\pm \mid c \geq 0\}$, and denote by Π_u^1 and Π_u^2 the disjoint open sets in which $\mathbb{R}^2$ is divided by the polyline $\mathcal{T}^u := \mathcal{T}_u^+ \cup \mathcal{T}_u^-$. We require that $\vec{v}_s^+$ and $\vec{v}_s^-$ lie on “opposite sides” with respect to $\mathcal{T}^u$. Hence, to fix the ideas, we assume:

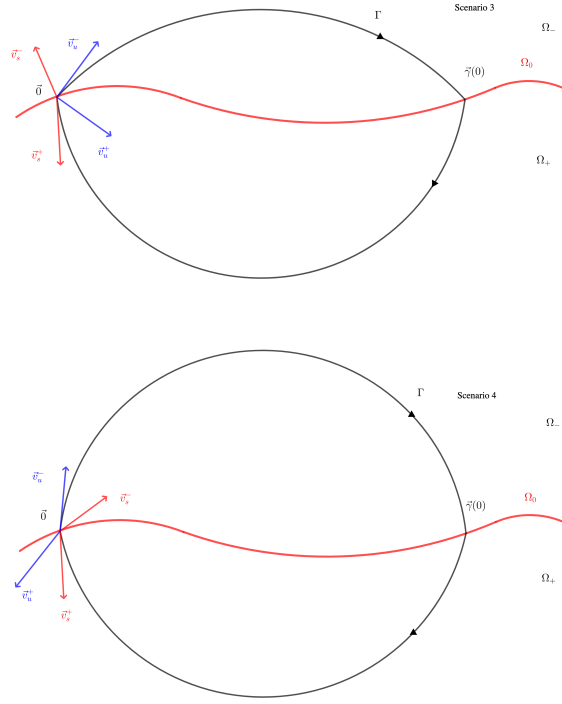


Fig. 3: In these settings **F2** does not hold and sliding may occur close to the origin, so our analysis does not apply directly. Further Melnikov theory guarantees persistence of the homoclinic trajectories [12], but chaos is forbidden generically, [18]. See also §2.3.

scenario45

F2 $\vec{v}_s^+ \in \Pi_u^1$ and $\vec{v}_s^- \in \Pi_u^2$.

In fact it is the *mutual position of the eigenvectors* that plays a role in the argument and all our results hold if $\vec{v}_s^- \in \Pi_u^1$ and $\vec{v}_s^+ \in \Pi_u^2$, too: we assume **F2** for definiteness, see Figure 2, compare also with [13, Remark 2.1].

We emphasize that if **F2** holds, there is no sliding on Ω^0 close to $\vec{0}$. On the other hand, sliding may occur when both $\vec{v}_s^\pm$ lie in Π_u^1 , or they both lie in Π_u^2 , see Figure 3, compare also with [18, §3]. As pointed out in [13, 14], the case $\vec{v}_s^- \in \Pi_u^1$ and $\vec{v}_s^+ \in \Pi_u^2$ can be handled similarly.

We assume further:

K For $\varepsilon = 0$ there is a unique solution $\vec{\gamma}(t)$ of (PS) homoclinic to the origin such that

$$\vec{\gamma}(t) \in \begin{cases} \Omega^-, & t < 0, \\ \Omega^0, & t = 0, \\ \Omega^+, & t > 0. \end{cases}$$

Furthermore, $(\vec{\nabla}G(\vec{\gamma}(0)))^* \vec{f}^\pm(\vec{\gamma}(0)) > 0$.

Recalling the orientation of $\vec{v}_s^\pm, \vec{v}_u^\pm$ chosen in **F1**, we assume w.l.o.g. that

$$\lim_{t \rightarrow -\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|} = \vec{v}_u^- \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|} = -\vec{v}_s^+. \quad (1)$$

ass.scenario

Concerning the perturbation term $\vec{g}$, we assume the following:

G $\vec{g}(t, \vec{x}, \varepsilon)$ is bounded together with its derivatives for any $t \in \mathbb{R}$, $\vec{x} \in B(\Gamma, 1) \cap \Omega$, $\varepsilon \geq 0$ and $\vec{g}(t, \vec{0}, \varepsilon) = \vec{0}$ for any $t \in \mathbb{R}$ $\varepsilon \geq 0$.

Hence, the origin is a critical point for the perturbed problem too.

Let us define the Melnikov function $\mathcal{M} : \mathbb{R} \rightarrow \mathbb{R}$ which, for planar piecewise-smooth systems as **(PS)**, takes the following form. For the computation of the Melnikov function we refer to the Appendix of [14] in which we corrected a small error appeared in [12].

$$\begin{aligned} \mathcal{M}(\alpha) = & c_\perp^- \int_{-\infty}^0 e^{-\int_0^t \text{tr} \vec{F}_\mathbf{x}^-(\vec{\gamma}(s)) ds} \vec{f}^-(\vec{\gamma}(t)) \wedge \vec{g}(t + \alpha, \vec{\gamma}(t), 0) dt \\ & + c_\perp^+ \int_0^{+\infty} e^{-\int_0^t \text{tr} \vec{F}_\mathbf{x}^+(\vec{\gamma}(s)) ds} \vec{f}^+(\vec{\gamma}(t)) \wedge \vec{g}(t + \alpha, \vec{\gamma}(t), 0) dt, \end{aligned} \quad (2)$$

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where

$$c_\perp^\pm = \frac{\|\vec{\nabla}G(\vec{\gamma}(0))\|}{(\vec{\nabla}G(\vec{\gamma}(0)))^* \vec{f}^\pm(\vec{\gamma}(0))} > 0,$$

and by “ $\wedge$ ” we denote the wedge product in $\mathbb{R}^2$ defined by $\vec{a} \wedge \vec{b} = a_1 b_2 - a_2 b_1$ for any vectors $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$. In fact, also in the piecewise-smooth case the function $\mathcal{M}$ is C^r and, following [12], we find

$$\begin{aligned} \mathcal{M}'(\alpha) = & c_\perp^- \int_{-\infty}^0 e^{-\int_0^t \text{tr} \vec{F}_\mathbf{x}^-(\vec{\gamma}(s)) ds} \vec{f}^-(\vec{\gamma}(t)) \wedge \frac{\partial \vec{g}}{\partial t}(t + \alpha, \vec{\gamma}(t), 0) dt \\ & + c_\perp^+ \int_0^{+\infty} e^{-\int_0^t \text{tr} \vec{F}_\mathbf{x}^+(\vec{\gamma}(s)) ds} \vec{f}^+(\vec{\gamma}(t)) \wedge \frac{\partial \vec{g}}{\partial t}(t + \alpha, \vec{\gamma}(t), 0) dt. \end{aligned} \quad (3)$$

melni-disc.diff

Under our assumptions, in [12, Theorem 2.9] the persistence of the homoclinic trajectory was proved (in fact in the general $\Omega \subset \mathbb{R}^n$, $n \geq 2$ case).

TFSF2_2

Theorem A ([12]) *Assume that **F0**, **F1**, **K** and **G** are satisfied, and that there is $\alpha_0 \in \mathbb{R}$ such that $\mathcal{M}(\alpha_0) = 0$ and $\mathcal{M}'(\alpha_0) \neq 0$. Then there is $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$ system **(PS)** admits a unique piecewise C^{r-1} solution $\vec{x}_b(t; \varepsilon)$ bounded on $\mathbb{R}$, and there is a unique C^{r-1} function $\alpha(\varepsilon)$, satisfying $\alpha(0) = \alpha_0$, such that*

$$\sup_{t \in \mathbb{R}} \|\vec{x}_b(t + \alpha(\varepsilon); \varepsilon) - \vec{\gamma}(t)\| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Before stating the main results of the article we collect here for convenience of the reader and future reference, the main constants which will play a role in our argument:

$$\begin{aligned}
\sigma_+^{\text{fwd}} &= \frac{|\lambda_s^+|}{\lambda_u^+ + |\lambda_s^+|}, & \sigma_-^{\text{fwd}} &= \frac{\lambda_u^- + |\lambda_s^-|}{\lambda_u^-}, & \sigma^{\text{fwd}} &= \sigma_+^{\text{fwd}} \sigma_-^{\text{fwd}}, \\
\sigma_+^{\text{bwd}} &= \frac{1}{\sigma_+^{\text{fwd}}}, & \sigma_-^{\text{bwd}} &= \frac{1}{\sigma_-^{\text{fwd}}}, & \sigma^{\text{bwd}} &= \sigma_+^{\text{bwd}} \sigma_-^{\text{bwd}}, \\
\underline{\sigma} &= \min\{\sigma_+^{\text{fwd}}, \sigma_-^{\text{bwd}}\}, & \bar{\sigma} &= \max\{\sigma_+^{\text{fwd}}, \sigma_-^{\text{bwd}}\}, \\
\Sigma_+^{\text{fwd}} &= \frac{1}{\lambda_u^+ + |\lambda_s^+|}, & \Sigma_-^{\text{bwd}} &= \frac{1}{\lambda_u^- + |\lambda_s^-|}, \\
\Sigma^{\text{fwd}} &= \frac{\lambda_u^- + |\lambda_s^-|}{\lambda_u^+ (\lambda_u^+ + |\lambda_s^+|)}, & \Sigma^{\text{bwd}} &= \frac{\lambda_u^- + |\lambda_s^-|}{|\lambda_s^+| (\lambda_u^- + |\lambda_s^-|)}, \\
\underline{\Sigma} &= \min\{\Sigma_+^{\text{fwd}}, \Sigma_-^{\text{bwd}}\}, & \bar{\Sigma} &= \max\{\Sigma_+^{\text{fwd}}, \Sigma_-^{\text{bwd}}\}, \\
\underline{\lambda} &= \min\{\lambda_u^-; \lambda_u^+; |\lambda_s^-|; |\lambda_s^+|\}, & \bar{\lambda} &= \max\{\lambda_u^-; \lambda_u^+; |\lambda_s^-|; |\lambda_s^+|\}.
\end{aligned} \tag{4}$$

defsigma

Notice that, in particular, $\underline{\sigma} \leq \bar{\sigma} < 1$.

Remark 2.1 We will see in Theorem C below that trajectories starting close to $\tilde{\gamma}(0)$ may perform a loop close to Γ and arrive back close to $\tilde{\gamma}(0)$. The constants σ are used to measure the space displacement while the constants Σ are used to measure the time needed to perform a loop or a part of it. We use the “ $\pm$ ” pedices for “staying in $\Omega^\pm$ ” and the “fwd/bwd” apices for “forward and backward time”.

Ronconstants

Remark 2.2 Notice that if $\lambda_u^- = \lambda_u^+ = \lambda_u$ and $\lambda_s^- = \lambda_s^+ = \lambda_s$ as in the smooth case, the constants in (4) simplify as follows:

$$\begin{aligned}
\sigma^{\text{fwd}} &= \frac{|\lambda_s|}{\lambda_u}, & \sigma^{\text{bwd}} &= \frac{\lambda_u}{|\lambda_s|} = (\sigma^{\text{fwd}})^{-1}, \\
\Sigma^{\text{fwd}} &= \frac{1}{\lambda_u}, & \Sigma^{\text{bwd}} &= \frac{1}{|\lambda_s|}.
\end{aligned}$$

In the Hamiltonian case we find $\sigma^{\text{fwd}} = \sigma^{\text{bwd}} = 1$ and $\sigma_+^{\text{fwd}} = \sigma_-^{\text{bwd}} = 1/2$.

We also need to fix the following absolute constants K_0 and ν_0 , which depend only on the eigenvalues in **F0**

$$K_0 := \frac{3\bar{\Sigma}}{2\underline{\sigma}}, \quad \nu_0 := \max\{3\bar{\sigma} - 1; 1\}. \tag{5}$$

defK0-new

Roughly speaking, ν_0 is the lower bound for ν , which is used to tune the size of the sets Σ giving rise to chaotic phenomena and accordingly the time needed by the trajectories to perform a loop: the larger $\nu \geq \nu_0$, the smaller the diameter of Σ , the longer the time needed for a loop; this aspect will be the subject of a forthcoming paper. In fact in this paper the reader can always keep $\nu = \nu_0$. On the other hand, the constant K_0 is needed to measure the minimal time needed to perform a loop, which goes, roughly speaking, as $2K_0(1 + \nu)|\ln(\varepsilon)|$, see (12), (13) below.

data.smooth

Remark 2.3 Let us observe that we can choose $\beta > 0$ small enough so that within $B(\Gamma, \beta)$ no sliding phenomena are possible (for any $0 < \varepsilon \leq \varepsilon_0$, e.g. we can choose $\beta = |\ln(\varepsilon_0)|^{-1}$). Then it follows that local uniqueness of the solutions is ensured within the set $B(\Gamma, \beta)$ where all

our analysis will take place, see Remarks 3.7 and 3.8 in [13]. Further continuous dependence on initial data and parameters of the solutions of (PS) within this set is ensured.

S.results

2.2 Recurrence assumptions and statement of the main results

Now we detail the assumptions on the Melnikov function $\mathcal{M}(\tau)$, defined in (2), hypotheses which replace the standard assumption that $\vec{g}$ is periodic in t (and hence $\mathcal{M}(\cdot)$ is periodic): roughly speaking we ask for infinitely many sign changes of the Melnikov function $\mathcal{M}(\tau)$. We stress once again that the same setting has been considered in [14], where we discussed the existence of trajectories chaotic in the future or chaotic in the past.

P1 There is a constant $\bar{c} > 0$ and an increasing sequence (b_i) , $i \in \mathbb{Z}$, such that $b_{i+1} - b_i \geq 1/10$ and

$$\mathcal{M}(b_{2i}) < -\bar{c} < 0 < \bar{c} < \mathcal{M}(b_{2i+1}),$$

for any $i \in \mathbb{Z}$.

Remark 2.4 Notice that $b_i \rightarrow \pm\infty$ as $i \rightarrow \pm\infty$. Further the assumption $b_{i+1} - b_i \geq 1/10$ could be replaced by $b_{i+1} - b_i \geq c$ where $c > 0$ is a constant independent of i .

If **P1** holds, the intermediate value theorem implies that $\mathcal{M}(\tau)$ has at least one zero in $]b_k, b_{k+1}[$ for any $k \in \mathbb{Z}$.

Before stating our results for (PS), we illustrate what happens replacing **P1** by more restrictive recurrence properties, for clarity and in order to illustrate the novelties introduced by our approach with more usual periodicity assumptions. Namely we consider the following two alternative conditions.

P2 There are b_0, b_1 such that $\mathcal{M}(b_0) < 0 < \mathcal{M}(b_1)$ and $\vec{g}(t, \vec{x}, \varepsilon)$ is almost periodic in t , i.e., for any $\varsigma > 0$ there is $\bar{N} = \bar{N}(\varsigma) > 0$ such that

$$\|\vec{g}(t, \vec{x}, \varepsilon) - \vec{g}(t + \bar{N}, \vec{x}, \varepsilon)\| \leq \varsigma, \quad \text{for any } (t, \vec{x}, \varepsilon) \in \mathbb{R} \times \Omega \times \mathbb{R}.$$

P3 There are b_0, b_1 such that $\mathcal{M}(b_0) < 0 < \mathcal{M}(b_1)$ and $\vec{g}(t, \vec{x}, \varepsilon)$ is 1-periodic in t , i.e

$$\vec{g}(t, \vec{x}, \varepsilon) = \vec{g}(t + 1, \vec{x}, \varepsilon), \quad \text{for any } (t, \vec{x}, \varepsilon) \in \mathbb{R} \times \Omega \times \mathbb{R}.$$

It is classically known that the periodicity and the almost periodicity of $\vec{g}$ imply the periodicity and the almost periodicity of $\mathcal{M}$, so **P3** implies **P2**, which implies **P1**.

We want to consider an increasing sequence of times $\mathcal{T} = (T_j)$, $j \in \mathbb{Z}$ such that $\mathcal{M}(T_{2j}) = 0$ and $T_{j+1} - T_j$ becomes larger and larger as $\varepsilon \rightarrow 0$, see (12) and (13) below. Correspondingly, we find a subsequence $\beta_j := b_{n_j}$ such that

$$\beta_j := b_{n_j} < T_j < \beta'_j := b_{n_j+1}, \quad B_j = \beta'_j - \beta_j = b_{n_j+1} - b_{n_j} > 0, \quad (6)$$

defkj

$j \in \mathbb{Z}$. Concerning the sequence $\mathcal{T}$, when condition **P1** holds we follow two different settings of assumptions; the first is slightly more restrictive but includes the cases of **P2**, **P3** and the case where B_j is bounded: it allows to obtain more precise (and clearer) results, i.e., Theorems 2.6 and 2.8; in the second approach we ask for weaker assumptions but we obtain weaker results, i.e., Theorem 2.14 (where we have a very weak control of the size of $\alpha_j^e(\varepsilon)$ introduced below, and we cannot guarantee the semi-conjugation with the Bernoulli shift, unless B_j is bounded).

We consider now the first setting of assumptions.

We assume first that there may be some $j \in \mathbb{Z}$ such that T_{2j} is an accumulation point of the zeros of $\mathcal{M}(\tau)$, so there are $0 < \delta < 1$, $1/10 > \Lambda^1 > 2\Lambda^0 \geq 0$ and increasing sequences $(a_j^\uparrow)$, $(a_j^\downarrow)$, $j \in \mathbb{Z}$ such that

$$\beta_{2j} < a_j^\uparrow < T_{2j} - \Lambda^0 < T_{2j} < T_{2j} + \Lambda^0 < a_j^\downarrow < \beta'_{2j}$$

and

$$\begin{aligned} |\mathcal{M}(a_j^\uparrow)| &= |\mathcal{M}(a_j^\downarrow)| = \delta \bar{c}, & \mathcal{M}(a_j^\uparrow)\mathcal{M}(a_j^\downarrow) &< 0, \\ \mathcal{M}(\tau) &\neq 0 \quad \forall \tau \in [a_j^\uparrow, a_j^\downarrow] \setminus [T_{2j} - \Lambda^0, T_{2j} + \Lambda^0], \\ 0 &< a_j^\downarrow - a_j^\uparrow \leq \Lambda^1, & \text{for any } j \in \mathbb{Z}. \end{aligned} \tag{7}$$

minimal

We emphasize that both Λ^1 and Λ^0 are independent of $j \in \mathbb{Z}$.

In fact (7) is enough to construct a chaotic pattern, but we obtain sharper results if we assume that there is a strictly monotone increasing and continuous function $\omega_M(h) : [0, \Lambda^1] \rightarrow [0, +\infty[$ (independent of j) such that $\omega_M(0) = 0$ and

$$\begin{aligned} |\mathcal{M}(T_{2j} - \Lambda^0 - h)| &\geq \omega_M(h) \quad \text{for any } (T_{2j} - \Lambda^0 - h) \in [a_j^\uparrow, T_{2j} - \Lambda^0], \\ |\mathcal{M}(T_{2j} + \Lambda^0 + h)| &\geq \omega_M(h) \quad \text{for any } (T_{2j} + \Lambda^0 + h) \in [T_{2j} + \Lambda^0, a_j^\downarrow], \end{aligned} \tag{8}$$

isogen

and any $j \in \mathbb{Z}$.

Notice that (7) and (8) are always satisfied if **P2** or **P3** holds. In fact (7) and (8) can be satisfied even if the sequence (B_j) becomes unbounded, but they require a control from below on the “slope” of the Melnikov function “close to its zeros”.

We stress that in the easier and most significant case where T_{2j} is an isolated zero for $\mathcal{M}(\cdot)$ for any j , we can assume $\Lambda^0 = 0$ so that (7) simplifies as follows

$$\begin{aligned} |\mathcal{M}(a_j^\uparrow)| &= |\mathcal{M}(a_j^\downarrow)| = \delta \bar{c}, & \mathcal{M}(a_j^\uparrow)\mathcal{M}(a_j^\downarrow) &< 0, \\ \mathcal{M}(\tau) &\neq 0 \quad \forall \tau \in [a_j^\uparrow, a_j^\downarrow] \setminus \{T_{2j}\}. \end{aligned} \tag{9}$$

defbetaj

Further in this case (8) reduces to

$$|\mathcal{M}(T_{2j} + h)| \geq \omega_M(|h|) \quad \text{for any } (T_{2j} + h) \in [a_j^\uparrow, a_j^\downarrow], \quad h \neq 0, \tag{10}$$

isolated

and any $j \in \mathbb{Z}$.

When we assume the classical hypothesis that there is $C > 0$ such that

$$\mathcal{M}(T_{2j}) = 0 \quad \text{and} \quad |\mathcal{M}'(T_{2j})| > C \text{ for any } j \in \mathbb{Z}, \quad (11) \quad \boxed{\text{nondeg}}$$

then (7) and (10) hold and we may assume simply $\omega_M(h) = \frac{C}{2}h$ and $\Lambda^1 = 2\frac{\varepsilon}{C}\delta < \frac{1}{10}$.

When **P1** and (7) hold we need the following condition concerning the sequences (T_j) and the time gap $T_{j+1} - T_j$:

$$\begin{aligned} \mathcal{M}(T_{2j}) &= 0, \quad T_0 \in [b_0, b_1], \\ T_{j+1} - T_j &> \Lambda^1 + K_0(1 + \nu)|\ln(\varepsilon)|, \quad \text{for any } j \in \mathbb{Z} \end{aligned} \quad (12) \quad \boxed{\text{TandKnu}}$$

and for any $\nu \geq \nu_0$, where the absolute constants K_0 and ν_0 are as in (5).

Remark 2.5 If **P3** holds and there is t_0 such that $\mathcal{M}(t_0) = 0 \neq \mathcal{M}'(t_0)$, then we can choose $\mathcal{T} = (T_j)$, $T_j = t_0 + j\Theta$, where $\Theta = [K_0(1 + \nu)|\ln(\varepsilon)| + 2]$, where $[\cdot]$ denotes the integer part, so that it satisfies (12) and (11).

In the second approach we drop (7), but we need to ask for a slightly more restrictive condition on the time gap $T_{j+1} - T_j$, i.e., we assume that for a given $\nu \geq \nu_0$ we have:

$$\begin{aligned} \mathcal{M}(T_{2j}) &= 0, \quad T_0 \in [b_0, b_1], \\ T_{j+1} - T_j &> \max\{B_{j+1}; B_j\} + K_0(1 + \nu)|\ln(\varepsilon)|, \quad \text{for any } j \in \mathbb{Z}. \end{aligned} \quad (13) \quad \boxed{\text{TandKnunew}}$$

Now we state our first result concerning the existence of a chaotic pattern which takes place in the whole of $\mathbb{R}$, obtained assuming that the non-degeneracy assumption (11) holds. The proofs of the results of this section are postponed to Section 5.

main.periodic1

Theorem 2.6 Assume $\vec{f}^\pm$ and $\vec{g}$ are C^r , $r \geq 2$ and that **F0**, **F1**, **F2**, **K** and **G** hold true; assume further **P1**. Then we can choose ε_0 small enough so that for any $0 < \varepsilon \leq \varepsilon_0$ and any increasing sequence $\mathcal{T} = (T_j)$, $j \in \mathbb{Z}$ satisfying (12) and (11), there is $\tilde{c}^* > 0$ (independent of $\varepsilon > 0$) for which the following holds.

For any $e = (e_j) \in \mathcal{E}$, there is a compact set $X(e, \mathcal{T})$ such that if $\vec{\xi} \in X(e, \mathcal{T})$ the trajectory $\vec{x}(t, T_0; \vec{\xi})$ satisfies **C**, i.e.

C if $e_j = 1$ then

$$\|\vec{x}(t, T_0; \vec{\xi}) - \vec{\gamma}(t - T_{2j})\| \leq \tilde{c}^* \varepsilon \quad \text{when } t \in [T_{2j-1}, T_{2j+1}], \quad (14) \quad \boxed{e_j=1}$$

while if $e_j = 0$ we have

$$\|\vec{x}(t, T_0; \vec{\xi})\| \leq \tilde{c}^* \varepsilon \quad \text{when } t \in [T_{2j-1}, T_{2j+1}] \quad (15) \quad \boxed{e_j=0}$$

for any $j \in \mathbb{Z}$.

Further, the set $\Sigma(\mathcal{T}) = \cup_{e \in \mathcal{E}} X(e, \mathcal{T}) \subset [B(\vec{\gamma}(0), \tilde{c}^* \varepsilon) \cup B(\vec{0}, \tilde{c}^* \varepsilon)]$.

We emphasize that the flow of (PS) on Σ is semi-conjugated to the Bernoulli shift on $\mathcal{E}$, see Theorem 6.5 for more details.

explainP

Remark 2.7 In Theorem 2.6 we require a lower bound for $T_{j+1} - T_j$, but not an upper bound: this allows a big flexibility in the choice of $\mathcal{T}$. Hence, even if we assume periodicity, i.e. **P3**, we can consider gaps $T_{j+1} - T_j$ which are not constant, e.g. periodic but even aperiodic or unbounded; see Section 6 for more details. In fact this possibility can be obtained also through the approach introduced by the works by Battelli and Fečkan in a similar context, see e.g. [5–7]. However this point is not discussed in those papers.

Now we state some weaker results concerning the case where $\vec{f}^\pm$ and $\vec{g}$ are just C^r , $r > 1$ (i.e., the derivatives are Hölder continuous), and we do not assume the non-degeneracy of the zeroes of $\mathcal{M}$. In the next Theorem 2.8 we drop (11) and we just require (7).

main.weak1

Theorem 2.8 *Assume that $\vec{f}^\pm$ and $\vec{g}$ are C^r with $r > 1$, and that **F0**, **F1**, **F2**, **K** and **G** hold true; assume further **P1**. Then we can choose ε_0 small enough so that for any $0 < \varepsilon \leq \varepsilon_0$ and any increasing sequence $\mathcal{T} = (T_j)$, $j \in \mathbb{Z}$, satisfying (12) and (7), there are $c^* > 0$ (independent of $\varepsilon > 0$) and a continuous increasing function ω satisfying $\omega(0) = 0$ so that we have the following. For any $e = (e_j) \in \mathcal{E}$, there is a compact set $\bar{X}(e, \mathcal{T})$ and a sequence $(\alpha_j^\varepsilon(\varepsilon))$, $j \in \mathbb{Z}$, such that $|\alpha_j^\varepsilon(\varepsilon)| \leq \omega(\varepsilon)$ and for any $\vec{\xi}(e) \in \bar{X}(e, \mathcal{T})$ the trajectory $\vec{x}(t, T_0 + \alpha_0^\varepsilon(\varepsilon); \vec{\xi}(e))$ has the property **C_e**, i.e.*

C_e if $e_j = 1$, then

$$\|\vec{x}(t, T_0 + \alpha_0^\varepsilon(\varepsilon); \vec{\xi}(e)) - \vec{\gamma}(t - T_{2j} - \alpha_j^\varepsilon(\varepsilon))\| \leq c^* \varepsilon$$

when $t \in [T_{2j-1}, T_{2j+1}]$

(16) ej=1w

while if $e_j = 0$, we have

$$\|\vec{x}(t, T_0 + \alpha_0^\varepsilon(\varepsilon); \vec{\xi}(e))\| \leq c^* \varepsilon$$

when $t \in [T_{2j-1}, T_{2j+1}]$

(17) ej=0w

for any $j \in \mathbb{Z}$.

Moreover, we have the following estimates for $\alpha_j^\varepsilon(\varepsilon)$:

- Assume (7), then $|\alpha_j^\varepsilon(\varepsilon)| \leq \Lambda^1$ for any $j \in \mathbb{Z}$.
- Assume (10), then there is a continuous increasing function $\omega_\alpha(\varepsilon)$ (independent of e) satisfying $\omega_\alpha(0) = 0$, such that $|\alpha_j^\varepsilon(\varepsilon)| \leq \omega_\alpha(\varepsilon)$ for any $j \in \mathbb{Z}$.

c.sigma

Corollary 2.9 *Let the assumption of Theorem 2.8 be satisfied and denote by*

$$X(e, \mathcal{T}) := \{\vec{\xi}_0(e) = \vec{x}(T_0, T_0 + \alpha_0^\varepsilon(\varepsilon); \xi(e)) \mid \xi(e) \in \bar{X}(e, \mathcal{T})\},$$

$$\Sigma(\mathcal{T}) := \cup_{e \in \mathcal{E}} X(e, \mathcal{T}).$$
(18) defX

Then $X(e, \mathcal{T})$ is compact for any $e \in \mathcal{E}$. Further,

if we assume (10) then $\Sigma(\mathcal{T}) \subset [B(\vec{\gamma}(0), \omega_\alpha(\varepsilon)) \cup B(\vec{0}, c^ \varepsilon)]$;*

if we assume (7) then $\Sigma(\mathcal{T}) \subset [B(\Gamma^{\Lambda^1}, c^ \varepsilon) \cup B(\vec{0}, c^* \varepsilon)]$ where*

$$\Gamma^{\Lambda^1} := \{\vec{\gamma}(t) \mid |t| \leq \Lambda^1\}.$$
(19) ga1

If the assumptions of either Theorem 2.6 or Theorem 2.8 are satisfied, the flow of (PS) on Σ is semi-conjugated to the Bernoulli shift on $\mathcal{E}$, see Theorem 6.5 for more details.

r.nonunique

Remark 2.10 We stress that in the smooth case (but also in the piecewise-smooth case if $\vec{0} \notin \Omega^0$, see [6]), the classical results found in literature, see e.g. [21, 36, 41, 43, 44], assume that the Melnikov function has non-degenerate simple zeros and state that $X(e, \mathcal{T})$ is a singleton. However in Theorem 2.6 we may have two (or more) different points $\{\vec{\xi}_1, \vec{\xi}_2\} \subset X(e, \mathcal{T})$. We believe this lack of uniqueness is a drawback of the use of the intermediate value theorem in our construction, so we think it is just a technical issue. Possibly, this problem may be overcome by the use of the implicit function theorem as in the approach by Battelli-Fečkan e.g. in [5–7]: the proof of the uniqueness of $\vec{\xi} \in X(e, \mathcal{T})$ will be the object of future investigations.

In fact, this lack of uniqueness is the reason for which in Theorem 6.5 we just get semi-conjugation instead of conjugation as one may wish.

We recall, however, that our approach is most probably optimal for the general case where we do not require $\mathcal{T}$ to satisfy (11): if the function $\mathcal{M}(\tau)$ may be null in some intervals, uniqueness should not be expected. An alternative approach for this degenerate case, based on degree theory, has been developed in [2], already in dimension $n \geq 2$ but just in the smooth case.

regularity2

Remark 2.11 Let us stress, once again, that in Theorem 2.6 we require the regularity assumptions which are standard in this context, i.e. $\vec{f}^\pm$ and $\vec{g}$ need to be C^r with $r \geq 2$, however, in Theorem 2.8 we can ask simply for $r > 1$.

Let us set

$$L^0 = L^0(\delta) := \{\vec{Q} \in \Omega^0 \mid \|\vec{Q} - \vec{\gamma}(0)\| < \delta\}. \quad (20)$$

L0bis

The trajectory $\vec{x}(t, T_0; \vec{\xi})$ crosses Ω^0 close to the origin exactly once between two consecutive 1s of the sequence e . Namely we have the following two results borrowed from [14].

consecutive

Remark 2.12 Let the assumptions of Theorem 2.8 be satisfied. Fix $e \in \mathcal{E}$ and let $j' < j''$ correspond to two consecutive 1s of e , i.e., $e_{j'} = 1 = e_{j''}$ and $e_k = 0$ for any $j' < k < j''$ (if any). Fix $\vec{\xi} \in \bar{X}(e, \mathcal{T})$; then there is $\hat{T}(\vec{\xi}) \in]T_{2j'}, T_{2j''}[$ and the trajectory $\vec{x}(t, T_0 + \alpha_0^e(\varepsilon); \vec{\xi})$ crosses transversely Ω^0 close to $\vec{\gamma}(0)$ (in $L^0(c^*\varepsilon)$) at $t = T_{2j'} + \alpha_{j'}^e(\varepsilon), T_{2j''} + \alpha_{j''}^e(\varepsilon)$, and close to the origin (in $B(\vec{0}, c^*\varepsilon)$) at $t = \hat{T}(\vec{\xi})$; further $\vec{x}(t, T_0 + \alpha_0^e(\varepsilon); \vec{\xi}) \in \Omega^+$ if $t \in]T_{2j'} + \alpha_{j'}^e(\varepsilon), \hat{T}(\vec{\xi})[$ and $\vec{x}(t, T_0 + \alpha_0^e(\varepsilon); \vec{\xi}) \in \Omega^-$ if $t \in]\hat{T}(\vec{\xi}), T_{2j''} + \alpha_{j''}^e(\varepsilon)[$. Finally,

$$\hat{T}(\vec{\xi}) - T_{2j'} > K_0(1 + \nu)|\ln(\varepsilon)| \quad \text{and} \quad T_{2j''} - \hat{T}(\vec{\xi}) > K_0(1 + \nu)|\ln(\varepsilon)|.$$

An analogous remark holds for Theorem 2.6.

r.all

Corollary 2.13 *In Theorem 2.8, if (11) holds, we can suppress the dependence on $\alpha_j^e(\varepsilon)$ of the relations in (16), but paying the price of a loss of precision of the estimate. That is, we can replace (16) by*

$$\|\vec{x}(t, T_0; \vec{\xi}_0) - \vec{\gamma}(t - T_{2j})\| \leq \omega(\varepsilon) \quad \text{when } t \in [T_{2j-1}, T_{2j+1}]. \quad (21)$$

ej=1w.all

We conclude the section with a result in which we drop (7) paying the price of a loss in the precision of the estimates of $\alpha_j^e(\varepsilon)$, which may even be unbounded. In Theorem 2.14 below, condition (12) is replaced by the slightly stronger (13).

veryweak

Theorem 2.14 Assume that $\tilde{f}^\pm$ and $\tilde{g}$ are C^r with $r > 1$, and that **F0**, **F1**, **F2**, **K** and **G** hold true; assume further **P1**. Then we can choose ε_0 small enough so that for any $0 < \varepsilon \leq \varepsilon_0$ and any increasing sequence $\mathcal{T} = (T_j)$, $j \in \mathbb{Z}$, satisfying (13), there is $c^* > 0$ (independent of $\varepsilon > 0$ and of the sequence (T_j)) so that we have the following. For any $e = (e_j) \in \mathcal{E}$, there is a compact set $\tilde{X}(e, \mathcal{T})$ and a sequence $(\alpha_j^\varepsilon(\varepsilon))$, $j \in \mathbb{Z}$, such that $|\alpha_j^\varepsilon(\varepsilon)| \leq B_{2j}$ and for any $\tilde{\xi}(e) \in \tilde{X}(e, \mathcal{T})$ the trajectory $\tilde{x}(t, T_0 + \alpha_0^\varepsilon(\varepsilon); \tilde{\xi}(e))$ has the property **C_e**.

Note that under the assumptions of Theorem 2.14 the flow of (PS) on Σ need not be properly semi-conjugated to the Bernoulli shift on $\mathcal{E}$, see Section 6 for more details.

remP1

Remark 2.15 We stress that **P1** does not require an upper bound on B_j when $|j| \rightarrow +\infty$. Further, the self-similarity required on $\mathcal{M}$ is very weak and we do not need any non-degeneracy of the zeros of the Melnikov function $\mathcal{M}(\tau)$, which may be null in some intervals. Hence with our results we can deal with, e.g.:

$$\begin{aligned}\mathcal{M}_1(\tau) &= 3 \sin(\tau) + k(\tau), \\ \mathcal{M}_2(\tau) &= 3 \sin(\sqrt[3]{1 + \tau^2}) + k(\tau)\end{aligned}\tag{22}$$

e.remP1

where $|k(\tau)| \leq 2$ is arbitrary, and can be regarded as a noise on which we have little information and does not really need to be small.

Notice that $\mathcal{M}_1(\tau)$ satisfies (7) and (8), while $\mathcal{M}_2(\tau)$ does not even satisfy (7). So in the former case we can apply Theorem 2.8 and in the latter just Theorem 2.14.

Further notice that if we assume $k(\tau) \equiv 0$, in the former case we can apply Theorem 2.6, while in the latter we can apply again just Theorem 2.14, since we do not have a uniform lower bound on $|\mathcal{M}'(T_j)|$.

toaddinPrestate

Remark 2.16 As pointed out in [14] classically the recurrence conditions such as periodicity or almost periodicity, are required directly on $\tilde{g}$ and then inherited by $\mathcal{M}$. Here we prefer to ask for conditions on $\mathcal{M}$ (which in general are more difficult to be verified on $\tilde{g}$) in order to include the “large perturbations” as $k(\tau)$ in (22). We think this might be useful for application since it is possible that the perturbation $\tilde{g}$ is made up by a periodic part of which we have a control and a “noise”, possibly not small, of which we just know the size: sometimes this allows to write $\mathcal{M}$ as a periodic part and a “noise” as in (22) which does not need to be small, see Examples 7.3 and 7.5.

S.scenarios

2.3 Some remarks concerning similar scenarios

Now, following again [13], we need to distinguish between several possible scenarios to which our results apply: see Figures 2 and 3. Let us denote by E^{in} the bounded open set enclosed by Γ and by E^{out} the unbounded open set outside $\Gamma \cup E^{\text{in}}$.

Scenario 1 Assume **K** and that there is $\rho > 0$ such that $d\vec{v}_u^+ \in E^{\text{out}}$, $d\vec{v}_s^- \in E^{\text{out}}$ for any $0 < d < \rho$.

Scenario 2 Assume **K** and that there is $\rho > 0$ such that $d\vec{v}_u^+ \in E^{\text{in}}$, $d\vec{v}_s^- \in E^{\text{in}}$ for any $0 < d < \rho$.

Notice that **F2** holds in both Scenarios 1 and 2, see [13] for details.

Theorems 2.6, 2.8 and 2.14 apply in both the scenarios, with no changes. To fix the ideas in the whole paper we focus on Scenario 1, but everything can be adapted without effort to Scenario 2 as well: the main difference is given in Remark 3.1 below.

We emphasize that, if we remove **F2**, i.e. if we allow sliding trajectories along Ω^0 close to the origin, we have persistence of the homoclinic trajectory to perturbation but chaos is forbidden. In fact, as was shown in [18], in this case we have different phenomena which cannot be present in a smooth setting.

Scenario 3 Assume **K** and that there is $\rho > 0$ such that $d\vec{v}_u^+ \in E^{\text{in}}$, $d\vec{v}_s^- \in E^{\text{out}}$ for any $0 < d < \rho$, so **F2** does not hold.

Scenario 4 Assume **K** and that there is $\rho > 0$ such that $d\vec{v}_u^+ \in E^{\text{out}}$, $d\vec{v}_s^- \in E^{\text{in}}$ for any $0 < d < \rho$, so **F2** does not hold.

Finally we emphasize that Scenarios 1, 2 may take place in the smooth case too: in this case we can consider Ω^0 as a generic curve passing through the origin. However, Scenarios 3, 4 require (PS) to be piecewise-smooth and do not have a smooth counterpart.

In this paper we will just consider Scenario 1 for definiteness, although Scenario 2 can be handled in a similar way.

3 Preliminary construction and lemmas

S.prel

Now we define the stable and the unstable leaves $W^s(\tau)$ and $W^u(\tau)$ of (PS). In this section we borrow some facts from [13] and we mainly follow its notation. We refer the reader to [13] for more details.

Assume **F0**, **F1**, **K** and $\tau \in \mathbb{R}$; following [14, §2], see also [13, §2] which is based on [26, Theorem 2.16], we can define the global stable and unstable leaves as follows:

$$\begin{aligned} W^u(\tau) &:= \{\vec{P} \in \mathbb{R}^2 \mid \lim_{t \rightarrow -\infty} \vec{x}(t, \tau; \vec{P}) = \vec{0}\}, \\ W^s(\tau) &:= \{\vec{P} \in \mathbb{R}^2 \mid \lim_{t \rightarrow +\infty} \vec{x}(t, \tau; \vec{P}) = \vec{0}\}. \end{aligned} \tag{23}$$

mm

In fact $W^u(\tau)$ and $W^s(\tau)$ are C^r immersed 1-dimensional manifolds if (PS) is smooth.

Follow $W^u(\tau)$ (respectively $W^s(\tau)$) from the origin towards $L^0(\sqrt{\varepsilon})$: it intersects $L^0(\sqrt{\varepsilon})$ transversely in a point denoted by $\vec{P}_u(\tau)$ (respectively by $\vec{P}_s(\tau)$), see Figure 1. We denote by $\tilde{W}^u(\tau)$ the branch of $W^u(\tau)$ between the origin and $\vec{P}_u(\tau)$ (a path), and by $\tilde{W}^s(\tau)$ the branch of $W^s(\tau)$ between the origin and $\vec{P}_s(\tau)$, in both the cases including the endpoints. Since $\tilde{W}^u(\tau)$ and $\tilde{W}^s(\tau)$ coincide with $\Gamma \cap (\Omega^- \cup \Omega^0)$ and $\Gamma \cap (\Omega^+ \cup \Omega^0)$ if $\varepsilon = 0$, respectively, and vary in a C^r way, we find $\tilde{W}^u(\tau) \subset (\Omega^- \cup \Omega^0)$ and $\tilde{W}^s(\tau) \subset (\Omega^+ \cup \Omega^0)$ for any $\tau \in \mathbb{R}$ and any $0 \leq \varepsilon \leq \varepsilon_0$ (see [13], which is based on [26, Theorem 2.16] or [19, Appendix]). Hence $\tilde{W}^u(\tau)$ and $\tilde{W}^s(\tau)$ are 1-dimensional C^r manifolds and $\vec{P}_u(\tau)$ and $\vec{P}_s(\tau)$ are C^r in ε and τ even when (PS) is just piecewise C^r . Further

$$\begin{aligned} \vec{Q}_u \in \tilde{W}^u(\tau) &\Rightarrow \vec{x}(t, \tau; \vec{Q}_u) \in \tilde{W}^u(t) \subset (\Omega^- \cup \Omega^0) \quad \text{for any } t \leq \tau, \\ \vec{Q}_s \in \tilde{W}^s(\tau) &\Rightarrow \vec{x}(t, \tau; \vec{Q}_s) \in \tilde{W}^s(t) \subset (\Omega^+ \cup \Omega^0) \quad \text{for any } t \geq \tau. \end{aligned} \tag{24}$$

proprieta

We set

$$\tilde{W}(\tau) := \tilde{W}^u(\tau) \cup \tilde{W}^s(\tau), \quad (25)$$

$\tilde{W}$

and we denote by $\tilde{V}(\tau)$ the compact set enclosed by $\tilde{W}(\tau)$ and the branch of Ω^0 connecting $\tilde{P}_u(\tau)$ and $\tilde{P}_s(\tau)$.

Rscenarios12

Remark 3.1 Assume the setting of either one of Theorems 2.6, 2.8 and 2.14 and let $\tilde{\xi} \in \Sigma$ (cf. Corollary 2.9). Then if we are in Scenario 1 the trajectory $\tilde{x}(t, T_0; \tilde{\xi}) \in \tilde{V}(t)$ for any $t \in \mathbb{R}$, while if we are in Scenario 2, then $\tilde{x}(t, T_0; \tilde{\xi}) \in \tilde{\Lambda}(t)$ for any $t \in \mathbb{R}$, where $\tilde{\Lambda}(t)$ is the unbounded closed set defined by $\tilde{\Lambda}(t) = \mathbb{R}^2 \setminus \tilde{V}(t)$.

Recall that we are assuming Scenario 1, and we introduce the following notation.

We denote by $A^{\text{fwd}}(\tau)$ and $B^{\text{fwd}}(\tau)$ the two branches in which $\tilde{P}_s(\tau)$ splits $L^0(\sqrt{\varepsilon})$ and by $A^{\text{bwd}}(\tau)$ and $B^{\text{bwd}}(\tau)$ the two branches in which $\tilde{P}_u(\tau)$ splits $L^0(\sqrt{\varepsilon})$; we assume that the whole of $A^{\text{fwd}}(\tau)$ and $A^{\text{bwd}}(\tau)$ are contained in $\tilde{V}(\tau)$ while at least parts of $B^{\text{fwd}}(\tau)$ and $B^{\text{bwd}}(\tau)$ lie outside $\tilde{V}(\tau)$.

L.loop

Lemma B [13, Lemma 3.6] Assume **F0**, **F1**, **F2**, **K**, **G**. Let $\vec{Q} \in A^{\text{fwd}}(\tau)$. Then there are $\mathcal{T}^{\text{fwd}}(\vec{Q}, \tau) > \tau_1(\vec{Q}, \tau) > \tau$ such that the trajectory $\tilde{x}(t, \tau; \vec{Q})$ crosses transversely Ω^0 at $t \in \{\tau, \tau_1(\vec{Q}, \tau), \mathcal{T}^{\text{fwd}}(\vec{Q}, \tau)\}$. Hence,

$$\mathcal{P}_+^{\text{fwd}}(\vec{Q}, \tau) := \tilde{x}(\tau_1(\vec{Q}, \tau), \tau; \vec{Q}) \in L^{\text{in}},$$

$$\mathcal{P}^{\text{fwd}}(\vec{Q}, \tau) := \tilde{x}(\mathcal{T}^{\text{fwd}}(\vec{Q}, \tau), \tau; \vec{Q}) \in L^0,$$

where $L^{\text{in}} = L^{\text{in}}(\delta) := \{\vec{Q} \in (\Omega^0 \cap \tilde{V}(0)) \mid \|\vec{Q}\| < \delta\}$, and δ is a small parameter.

Analogously, let $\vec{Q} \in A^{\text{bwd}}(\tau)$. Then there are $\mathcal{T}^{\text{bwd}}(\vec{Q}, \tau) < \tau_{-1}(\vec{Q}, \tau) < \tau$ such that $\tilde{x}(t, \tau; \vec{Q})$, crosses transversely Ω^0 at $t \in \{\tau, \tau_{-1}(\vec{Q}, \tau), \mathcal{T}^{\text{bwd}}(\vec{Q}, \tau)\}$. Hence

$$\mathcal{P}_-^{\text{bwd}}(\vec{Q}, \tau) := \tilde{x}(\tau_{-1}(\vec{Q}, \tau), \tau; \vec{Q}) \in L^{\text{in}},$$

$$\mathcal{P}^{\text{bwd}}(\vec{Q}, \tau) := \tilde{x}(\mathcal{T}^{\text{bwd}}(\vec{Q}, \tau), \tau; \vec{Q}) \in L^0.$$

Further the functions $\mathcal{P}_+^{\text{fwd}}(\vec{Q}, \tau)$, $\mathcal{P}^{\text{fwd}}(\vec{Q}, \tau)$, $\mathcal{P}_-^{\text{bwd}}(\vec{Q}, \tau)$ and $\mathcal{P}^{\text{bwd}}(\vec{Q}, \tau)$ are C^r in both the variables.

Let us denote by $\vec{\gamma}^-(s) = \vec{\gamma}(s)$ and by $\vec{\gamma}^+(t) = \vec{\gamma}(t)$ when $s \leq 0 \leq t$ so that $\vec{\gamma}^\pm(\tau) \in (\Omega^\pm \cup \Omega^0)$ for all $\tau \in \mathbb{R}$. We define a directed distance between the points on Ω^0 by arc length, in order to be able to determine their mutual positions: this is possible since Ω^0 is a regular curve. We choose as the positive orientation on Ω^0 the one that goes from the origin to $\vec{\gamma}(0)$. So, for any $\vec{Q} \in L^0$ we define $\ell(\vec{Q}) = \int_{\Omega^0(\vec{0}, \vec{Q})} ds > 0$ where $\Omega^0(\vec{0}, \vec{Q})$ is the (oriented) path of Ω^0 connecting $\vec{0}$ with $\vec{Q}$, and we set

$$\mathcal{D}(\vec{Q}, \vec{P}) := \ell(\vec{P}) - \ell(\vec{Q}) \quad (26)$$

dist

for $\vec{Q}, \vec{P} \in L^0$. Notice that $\mathcal{D}(\vec{Q}, \vec{P}) > 0$ means that $\vec{Q}$ lies on Ω^0 between $\vec{0}$ and $\vec{P}$. Now, we introduce some further crucial notation.

Notation. We denote by $\vec{Q}_s(d, \tau)$ the point in $L^0(\sqrt{\varepsilon})$ such that

$$\mathcal{D}(\vec{Q}_s(d, \tau), \vec{P}_s(\tau)) = d > 0,$$

and by $\vec{Q}_u(d, \tau)$ the point in $L^0(\sqrt{\varepsilon})$ such that

$$\mathcal{D}(\vec{Q}_u(d, \tau), \vec{P}_u(\tau)) = d > 0.$$

Let us set

$$J_0 := \left\{ d \in \mathbb{R} \mid 0 < d \leq \varepsilon^{\frac{1+\nu}{\alpha}} \right\}, \quad \nu \geq \nu_0. \quad (27) \quad \boxed{\text{J0}}$$

As in [13], for any $d \in J_0$, we define the C^r maps

$$\begin{aligned} \mathcal{T}_1(d, \tau) &:= \mathcal{T}^{\text{fwd}}(\vec{Q}_s(d, \tau), \tau), & \mathcal{P}_1(d, \tau) &:= \mathcal{P}^{\text{fwd}}(\vec{Q}_s(d, \tau), \tau), \\ \mathcal{T}_{-1}(d, \tau) &:= \mathcal{T}^{\text{bwd}}(\vec{Q}_u(d, \tau), \tau), & \mathcal{P}_{-1}(d, \tau) &:= \mathcal{P}^{\text{bwd}}(\vec{Q}_u(d, \tau), \tau). \end{aligned} \quad (28) \quad \boxed{\text{Q1T1}}$$

Sometimes we will also make use of the maps

$$\begin{aligned} \mathcal{T}_{\frac{1}{2}}(d, \tau) &:= \tau_1(\vec{Q}_s(d, \tau), \tau), & \mathcal{P}_{\frac{1}{2}}(d, \tau) &:= \mathcal{P}_+^{\text{fwd}}(\vec{Q}_s(d, \tau), \tau), \\ \mathcal{T}_{-\frac{1}{2}}(d, \tau) &:= \tau_{-1}(\vec{Q}_u(d, \tau), \tau), & \mathcal{P}_{-\frac{1}{2}}(d, \tau) &:= \mathcal{P}_-^{\text{bwd}}(\vec{Q}_u(d, \tau), \tau). \end{aligned} \quad (29) \quad \boxed{\text{Q1T1half}}$$

Let us define

$$\mu_0 = \frac{1}{4} \min \{ \Sigma_+^{\text{fwd}}, \Sigma_-^{\text{bwd}}, \underline{\sigma}^2 \}. \quad (30) \quad \boxed{\text{mu0}}$$

We introduce a new parameter $\mu \in]0, \mu_0]$, which gives an upper bound for the estimate of the errors in the evaluations of the maps defined in (28) and (29).

Now we collect some results borrowed from [13] and [14], which will be used in §4.

key

Theorem C [13, Theorem 4.2] Assume **F0**, **F1**, **F2**, **K**, **G** and let $\vec{f}^{\pm}$ and $\vec{g}$ be C^r with $r > 1$. We can find $\varepsilon_0 > 0$, $\delta > 0$, such that for any $0 < \varepsilon \leq \varepsilon_0$, the functions $\mathcal{T}_{\pm 1}(d, \tau)$, $\mathcal{P}_{\pm 1}(d, \tau)$ are C^r when $0 < d \leq \delta$ and $\tau \in \mathbb{R}$. Further, for any $0 < \mu < \mu_0$ we get

$$\begin{aligned} d^{\sigma^{\text{fwd}} + \mu} &\leq \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_u(\mathcal{T}_1(d, \tau))) \leq d^{\sigma^{\text{fwd}} - \mu}, \\ d^{\sigma^{\text{bwd}} + \mu} &\leq \mathcal{D}(\mathcal{P}_{-1}(d, \tau), \vec{P}_s(\mathcal{T}_{-1}(d, \tau))) \leq d^{\sigma^{\text{bwd}} - \mu}, \end{aligned} \quad (31) \quad \boxed{\text{D1T1}}$$

$$\begin{aligned} \|\mathcal{P}_{\frac{1}{2}}(d, \tau)\| &\leq d^{\sigma_+^{\text{fwd}} - \mu}, & \|\mathcal{P}_{-\frac{1}{2}}(d, \tau)\| &\leq d^{\sigma_-^{\text{bwd}} - \mu}, \\ \left[\Sigma^{\text{fwd}} - \mu \right] |\ln(d)| &\leq (\mathcal{T}_1(d, \tau) - \tau) \leq \left[\Sigma^{\text{fwd}} + \mu \right] |\ln(d)|, \\ \left[\Sigma^{\text{bwd}} - \mu \right] |\ln(d)| &\leq \tau - \mathcal{T}_{-1}(d, \tau) \leq \left[\Sigma^{\text{bwd}} + \mu \right] |\ln(d)|, \\ \left[\Sigma_+^{\text{fwd}} - \mu \right] |\ln(d)| &\leq (\mathcal{T}_{\frac{1}{2}}(d, \tau) - \tau) \leq \left[\Sigma_+^{\text{fwd}} + \mu \right] |\ln(d)|, \\ \left[\Sigma_-^{\text{bwd}} - \mu \right] |\ln(d)| &\leq \tau - \mathcal{T}_{-\frac{1}{2}}(d, \tau) \leq \left[\Sigma_-^{\text{bwd}} + \mu \right] |\ln(d)|, \end{aligned} \quad (32) \quad \boxed{\text{T1T-1}}$$

and all the expressions in (31) are uniform with respect to any $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq \varepsilon_0$.

Let us set $\tilde{V}^\pm(\tau) = \tilde{V}(\tau) \cap \Omega^\pm$.

keymissed

Theorem D [13, Theorem 4.3] Assume **F0**, **F1**, **F2**, **K**, **G** and let $\tilde{f}^\pm$ and $\tilde{g}$ be C^r with $r > 1$. We can find $\varepsilon_0 > 0$, $\delta > 0$, such that for any $0 < \varepsilon \leq \varepsilon_0$, $0 < d \leq \delta$, $0 < \mu < \mu_0$ and any $\tau \in \mathbb{R}$ we find

$$\|\tilde{x}(t, \tau; \tilde{Q}_s(d, \tau)) - \tilde{x}(t, \tau; \tilde{P}_s(\tau))\| \leq d^{\sigma_+^{fwd} - \mu} \quad (33)$$

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for any $\tau \leq t \leq \mathcal{T}_{\frac{1}{2}}(d, \tau)$, and

$$\|\tilde{x}(t, \tau; \tilde{Q}_s(d, \tau)) - \tilde{x}(t, \mathcal{T}_1(d, \tau); \tilde{P}_u(\mathcal{T}_1(d, \tau)))\| \leq d^{\sigma_+^{fwd} - \mu} \quad (34)$$

keymissed.es+

for any $\mathcal{T}_{\frac{1}{2}}(d, \tau) \leq t \leq \mathcal{T}_1(d, \tau)$. Further $\tilde{x}(t, \tau; \tilde{Q}_s(d, \tau))$ is in $\tilde{V}^+(t)$ for any $\tau < t < \mathcal{T}_{\frac{1}{2}}(d, \tau)$ and it is in $\tilde{V}^-(t)$ for any $\mathcal{T}_{\frac{1}{2}}(d, \tau) < t < \mathcal{T}_1(d, \tau)$.

Similarly, for any $0 < \varepsilon \leq \varepsilon_0$, $0 < d \leq \delta$, $0 < \mu < \mu_0$ and any $\tau \in \mathbb{R}$ we find

$$\|\tilde{x}(t, \tau; \tilde{Q}_u(d, \tau)) - \tilde{x}(t, \tau; \tilde{P}_u(\tau))\| \leq d^{\sigma_-^{bwd} - \mu} \quad (35)$$

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for any $\mathcal{T}_{-\frac{1}{2}}(d, \tau) \leq t \leq \tau$, and

$$\|\tilde{x}(t, \tau; \tilde{Q}_u(d, \tau)) - \tilde{x}(t, \mathcal{T}_{-1}(d, \tau); \tilde{P}_s(\mathcal{T}_{-1}(d, \tau)))\| \leq d^{\sigma_-^{bwd} - \mu} \quad (36)$$

keymissed.eu-

for any $\mathcal{T}_{-1}(d, \tau) \leq t \leq \mathcal{T}_{-\frac{1}{2}}(d, \tau)$. Further $\tilde{x}(t, \tau; \tilde{Q}_u(d, \tau))$ is in $\tilde{V}^-(t)$ for any $\mathcal{T}_{-\frac{1}{2}}(d, \tau) < t < \tau$ and it is in $\tilde{V}^+(t)$ for any $\mathcal{T}_{-1}(d, \tau) < t < \mathcal{T}_{-\frac{1}{2}}(d, \tau)$.

Recalling that there is $\bar{c}^* > 0$ such that

$$\begin{aligned} \|\tilde{x}(t, \tau; \tilde{P}_u(\tau)) - \tilde{\gamma}^-(t - \tau)\| &\leq \bar{c}^* \varepsilon & \text{for any } t \leq \tau, \\ \|\tilde{x}(t, \tau; \tilde{P}_s(\tau)) - \tilde{\gamma}^+(t - \tau)\| &\leq \bar{c}^* \varepsilon & \text{for any } t \geq \tau. \end{aligned} \quad (37)$$

trajWuWs

we find the following results.

forPropC

Proposition E [14, Proposition 4.7] Assume **F0**, **F1**, **F2**, **K**, **G**, then we can find ε_0 such that for any $0 < \varepsilon \leq \varepsilon_0$ we have the following.

Fix $\tau \in \mathbb{R}$, $d \in J_0$ and $\nu \geq \nu_0$, then there is $c^* > 0$ (independent of d , ν and ε) such that

$$\|\tilde{x}(t, \tau; \tilde{Q}_s(d, \tau)) - \tilde{\gamma}^+(t - \tau)\| \leq \frac{c^*}{2} \varepsilon$$

for any $\tau \leq t \leq \mathcal{T}_{\frac{1}{2}}(d, \tau)$, and

$$\|\tilde{x}(t, \tau; \tilde{Q}_s(d, \tau)) - \tilde{\gamma}^-(t - \mathcal{T}_1(d, \tau))\| \leq \frac{c^*}{2} \varepsilon$$

for any $\mathcal{T}_{\frac{1}{2}}(d, \tau) \leq t \leq \mathcal{T}_1(d, \tau)$.

Further,

$$\|\tilde{x}(t, \tau; \tilde{Q}_u(d, \tau)) - \tilde{\gamma}^-(t - \tau)\| \leq \frac{c^*}{2} \varepsilon$$

for any $\mathcal{T}_{-\frac{1}{2}}(d, \tau) \leq t \leq \tau$, and

$$\|\tilde{x}(t, \tau; \tilde{Q}_u(d, \tau)) - \tilde{\gamma}^+(t - \mathcal{T}_{-1}(d, \tau))\| \leq \frac{c^*}{2} \varepsilon$$

for any $\mathcal{T}_{-1}(d, \tau) \leq t \leq \mathcal{T}_{-\frac{1}{2}}(d, \tau)$.

forgottentimes

Lemma F [14, Lemma 4.9] Let $d \in J_0$ and $c^* > 0$ be as in Proposition E and denote

$$T_a := \frac{1}{\lambda_u} |\ln \varepsilon|, \quad T_b := \frac{1}{|\lambda_s^+|} |\ln \varepsilon|. \quad (38)$$

TaTb

Then

$$\begin{aligned} \tau + T_b &\leq \mathcal{T}_{\frac{1}{2}}(d, \tau) \leq \mathcal{T}_1(d, \tau) - T_a \quad \text{and} \\ \|x(t, \tau; \vec{Q}_s(d, \tau))\| &\leq \frac{3}{4} c^* \varepsilon \quad \text{for any } t \in [\tau + T_b, \mathcal{T}_1(d, \tau) - T_a]. \end{aligned} \quad (39)$$

T.half

Further,

$$\begin{aligned} \mathcal{T}_{-1}(d, \tau) + T_b &\leq \mathcal{T}_{-\frac{1}{2}}(d, \tau) \leq \tau - T_a \quad \text{and} \\ \|x(t, \tau; \vec{Q}_u(d, \tau))\| &\leq \frac{3}{4} c^* \varepsilon \quad \text{for any } t \in [\mathcal{T}_{-1}(d, \tau) + T_b, \tau - T_a]. \end{aligned} \quad (40)$$

T.halfbis

Lemma G [14, Lemma 4.10] Let $d \in J_0$ and $c^* > 0$ be as in Proposition E. If $t \in [\tau + T_b, \mathcal{T}_1(d, \tau) - T_a]$ we get

$$\|x(t, \tau; \vec{Q}_s(d, \tau))\| \leq c^* \varepsilon, \quad (41)$$

$$\|x(t, \tau; \vec{Q}_s(d, \tau)) - \vec{\gamma}(t - \tau)\| \leq c^* \varepsilon, \quad (42)$$

$$\|x(t, \tau; \vec{Q}_s(d, \tau)) - \vec{\gamma}(t - \mathcal{T}_1(d, \tau))\| \leq c^* \varepsilon. \quad (43)$$

The following is a consequence of Theorem C.

key1

Lemma H [14, Lemma 4.11] Assume **F0**, **F1**, **F2**, **K**, **G**. Fix $\nu \geq \nu_0$ and let $d \in J_0$; fix $\tau \in \mathbb{R}$. If there is $c_1 > 0$ such that $\mathcal{M}(\mathcal{T}_{\pm 1}(d, \tau)) \leq -3c_1$ then

$$d_1(d, \tau) := \mathcal{D}(\mathcal{P}_1(d, \tau), \vec{P}_s(\mathcal{T}_1(d, \tau))) > cc_1 \varepsilon > 0,$$

$$d_{-1}(d, \tau) := \mathcal{D}(\mathcal{P}_{-1}(d, \tau), \vec{P}_u(\mathcal{T}_{-1}(d, \tau))) < -cc_1 \varepsilon < 0,$$

while if $\mathcal{M}(\mathcal{T}_{\pm 1}(d, \tau)) \geq 3c_1$ then $\pm d_{\pm 1}(d, \tau) < -cc_1 \varepsilon < 0$.

4 A connection argument

S.connection

Let $e^+ \in \mathcal{E}^+$ and $\tau \in [b_0, b_1]$; following [14] we say that $\vec{x}(t, \tau; \vec{\xi})$ satisfies the property “chaotic in the future”, $C_{e^+}^+$ if there exist $\alpha_j(\varepsilon) = \alpha_j^{e^+}(\varepsilon, \tau, \mathcal{T}^+)$ such that the following conditions are satisfied

$C_{e^+}^+$ $\vec{x}(t, \tau; \vec{\xi}) \in \tilde{V}(t)$ for any $t \geq \tau$, and if $e_j = 1$, then

$$\|\vec{x}(t, \tau; \vec{\xi}) - \vec{\gamma}(t - T_{2j} - \alpha_j(\varepsilon))\| \leq c^* \varepsilon \quad \text{when } t \in [T_{2j-1}, T_{2j+1}], \quad (44)$$

ej+=1w

while if $e_j = 0$, we have

$$\|\vec{x}(t, \tau; \vec{\xi})\| \leq c^* \varepsilon \quad \text{when } t \in [T_{2j-1}, T_{2j+1}] \quad (45)$$

ej+=0w

for any $j \in \mathbb{Z}^+$. Further

$$\|\vec{x}(t, \tau; \vec{\xi}) - \vec{\gamma}(t - \tau)\| \leq c^* \varepsilon \quad \text{when } t \in [\tau, T_1]. \quad (46)$$

ej+=tau

Analogously let $e^- \in \mathcal{E}^-$ and $\tau \in [b_0, b_1]$; we say that $\vec{x}(t, \tau; \vec{\xi})$ satisfies the property “chaotic in the past”, $C_{e^-}^-$ if there exist $\alpha_j(\varepsilon) = \alpha_j^{\varepsilon^-}(\varepsilon, \tau, \mathcal{T}^-)$ such that the following holds:

$C_{e^-}^-$ $\vec{x}(t, \tau; \vec{\xi}) \in \tilde{V}(t)$ for any $t \leq \tau$, and if $e_j = 1$, then

$$\|\vec{x}(t, \tau; \vec{\xi}) - \vec{\gamma}(t - T_{2j} - \alpha_j(\varepsilon))\| \leq c^* \varepsilon \quad \text{when } t \in [T_{2j-1}, T_{2j+1}], \quad (47)$$

ej-1w

while if $e_j = 0$, we have

$$\|\vec{x}(t, \tau; \vec{\xi})\| \leq c^* \varepsilon \quad \text{when } t \in [T_{2j-1}, T_{2j+1}] \quad (48)$$

ej-0w

for any $j \in \mathbb{Z}^-$. Further

$$\|\vec{x}(t, \tau; \vec{\xi}) - \vec{\gamma}(t - \tau)\| \leq c^* \varepsilon \quad \text{when } t \in [T_{-1}, \tau]. \quad (49)$$

ej-tau

From Theorems 3.4 and 3.5 in [14] we see that for any $\tau \in [b_0, b_1]$ there are non-empty compact and connected sets $X^\pm(\tau, \mathcal{T}^\pm, e^\pm)$ such that if $\vec{\xi}^\pm \in X^\pm(\tau, \mathcal{T}^\pm, e^\pm)$ then $\vec{x}(t, \tau; \vec{\xi}^+)$ has property $C_{e^+}^+$ and $\vec{x}(t, \tau; \vec{\xi}^-)$ has property $C_{e^-}^-$. So we can define

$$\begin{aligned} \aleph &= \left\{ (\vec{\xi}, \tau) \mid \vec{\xi} \in L_0(\sqrt{\varepsilon}), \tau \in [b_0, b_1] \right\}; \\ \chi^{\Lambda, +}(e^+) &= \left\{ (\vec{\xi}, \tau) \in \aleph \mid \begin{array}{l} \vec{x}(t, \tau; \vec{\xi}) \text{ has property } C_{e^+}^+ \text{ and} \\ |\alpha_j| \leq a_j^\downarrow - a_j^\uparrow \leq \Lambda^1 \text{ for any } j \in \mathbb{Z}^+ \end{array} \right\}, \\ \chi^{\Lambda, -}(e^-) &= \left\{ (\vec{\xi}, \tau) \in \aleph \mid \begin{array}{l} \vec{x}(t, \tau; \vec{\xi}) \text{ has property } C_{e^-}^- \text{ and} \\ |\alpha_j| \leq a_j^\downarrow - a_j^\uparrow \leq \Lambda^1 \text{ for any } j \in \mathbb{Z}^- \end{array} \right\}. \end{aligned} \quad (50)$$

defX+la

The next result is formulated under the assumptions of Theorems 2.6, 2.8 and gives $|\alpha_j|$ bounded or infinitesimal.

T.connectionLa

Theorem 4.1 Assume that $\vec{f}^\pm$ and $\vec{g}$ are C^r , $r > 1$ and that **F0**, **F1**, **F2**, **K** and **G** hold true; assume further **P1**, and fix $\nu \geq \nu_0$ for ν_0 as in (5) and $\tau \in [b_0, b_1]$; let $e^+ \in \mathcal{E}^+$ and $e^- \in \mathcal{E}^-$. Then we can choose ε_0 small enough so that for any $0 < \varepsilon \leq \varepsilon_0$ and any increasing sequence $\mathcal{T}^+ = (T_j)$ satisfying (12) for $j \in \mathbb{Z}^+$ and

$$T_1 - b_1 > \Lambda^1 + K_0(1 + \nu)|\ln(\varepsilon)| \quad (51)$$

TandKnu_tau+

the set $\chi^{\Lambda, +}(e^+)$ contains a closed connected set $\tilde{\chi}^{\Lambda, +}(e^+)$ which intersects the lines $\tau = b_0$ and $\tau = b_1$.

Analogously, for any $0 < \varepsilon \leq \varepsilon_0$ and any increasing sequence $\mathcal{T}^- = (T_j)$ satisfying (12) for $j \leq -2$ and

$$b_0 - T_{-1} > \Lambda^1 + K_0(1 + \nu)|\ln(\varepsilon)| \quad (52)$$

TandKnu_tau-

the set $\chi^{\Lambda, -}(e^-)$ contains a closed connected set $\tilde{\chi}^{\Lambda, -}(e^-)$ which intersects the lines $\tau = b_0$ and $\tau = b_1$.

Moreover for any $j \in \mathbb{Z} \setminus \{0\}$ we have the following estimates

- a) if (11) holds and $r \geq 2$ there is $c_\alpha > 0$ such that $|\alpha_j^{e^+}(\varepsilon)| \leq c_\alpha \varepsilon$;
b) if (10) holds and $r > 1$ then

$$|\alpha_j^{e^+}(\varepsilon)| \leq \omega_\alpha(\varepsilon),$$

where $\omega_\alpha(\cdot)$ is an increasing continuous function such that $\omega_\alpha(0) = 0$;

- c) if (7) holds and $r > 1$ then $|\alpha_j^{e^+}(\varepsilon)| \leq \Lambda^1$.

The constants ε_0 , c^* , c_α and the function ω_α are independent of ε , ν , τ , $\mathcal{T}^\pm$, $e^\pm$.

Now we consider the case in which (B_j) may become unbounded, as in Theorem 2.14, so the distance from consecutive zeros of $\mathcal{M}$ may become unbounded as well.

Let us set

$$\begin{aligned} \chi^+(e^+) &= \left\{ (\vec{\xi}, \tau) \in \mathbb{N} \left| \begin{array}{l} \vec{x}(t, \tau; \vec{\xi}) \text{ has property } \mathbf{C}_{e^+}^+ \text{ and} \\ |\alpha_j| \leq B_{2j} \text{ for any } j \in \mathbb{Z}^+ \end{array} \right. \right\}, \\ \chi^-(e^-) &= \left\{ (\vec{\xi}, \tau) \in \mathbb{N} \left| \begin{array}{l} \vec{x}(t, \tau; \vec{\xi}) \text{ has property } \mathbf{C}_{e^-}^- \text{ and} \\ |\alpha_j| \leq B_{2j} \text{ for any } j \in \mathbb{Z}^- \end{array} \right. \right\}. \end{aligned} \quad (53) \quad \boxed{\text{defX+}}$$

T.connection

Theorem 4.2 Assume that the hypotheses of Theorem 4.1 hold but replace (12) by (13) and (51) by

$$T_1 - b_1 > \max\{B_1; B_0\} + K_0(1 + \nu)|\ln(\varepsilon)|, \quad (54) \quad \boxed{\text{TandKnunew_tau+}}$$

and (52) by

$$b_0 - T_{-1} > \max\{B_0; B_{-1}\} + K_0(1 + \nu)|\ln(\varepsilon)|. \quad (55) \quad \boxed{\text{TandKnunew_tau-}}$$

Then the set $\chi^+(e^+)$ contains a closed connected set $\tilde{\chi}^+(e^+)$ which intersects the lines $\tau = b_0$ and $\tau = b_1$; analogously the set $\chi^-(e^-)$ contains a closed connected set $\tilde{\chi}^-(e^-)$ which intersects the lines $\tau = b_0$ and $\tau = b_1$. However we just have

$$|\alpha_j^{e^+}(\varepsilon)| \leq B_{2j}, \quad |\alpha_j^{e^-}(\varepsilon)| \leq B_{2j}.$$

4.1 Proof of Theorem 4.2: the $|\alpha_j| \leq B_{2j}$ case

S.theorem2.1

In this section we prove Theorem 4.2 using a topological argument together with the selection scheme developed in [14]: this will be a step necessary to prove Theorem 2.14.

First we need to introduce the following sets

$$\begin{aligned} A_0 &= \left\{ (d, \tau) \mid d \in J_0 = [0, \varepsilon^{(1+\nu)/\alpha}], \tau \in [b_0, b_1] \right\}, \\ S^+ &= \left\{ (d, \tau) \in A_0 \left| \begin{array}{l} \vec{x}(t, \tau; \vec{Q}_s(d, \tau)) \text{ has property } \mathbf{C}_{e^+}^+ \text{ and} \\ |\alpha_j| \leq B_{2j} \text{ for any } j \in \mathbb{Z}^+ \end{array} \right. \right\}, \\ S^- &= \left\{ (d, \tau) \in A_0 \left| \begin{array}{l} \vec{x}(t, \tau; \vec{Q}_u(d, \tau)) \text{ has property } \mathbf{C}_{e^-}^- \text{ and} \\ |\alpha_j| \leq B_{2j} \text{ for any } j \in \mathbb{Z}^- \end{array} \right. \right\}. \end{aligned} \quad (56) \quad \boxed{\text{defS+}}$$

Our argument relies on the following topological result borrowed from [39].

top

Lemma 4.3 [39, Lemma 4] Let $\mathcal{R} = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ be a full rectangle. Let $E \subset \mathcal{R}$ be a closed set such that for any path $\vec{\psi} : [0, 1] \rightarrow \mathcal{R}$, $\vec{\psi}(a) = (x(a), y(a))$ with $x(0) = \alpha_1$ and $x(1) = \beta_1$ there is $\bar{a} \in]0, 1[$ (depending on $\vec{\psi}$) such that $\vec{\psi}(\bar{a}) \in E$. Then E contains a closed connected set $\tilde{E}$ which intersects both $y = \alpha_2$ and $y = \beta_2$.

The core of our argument is made up by the following two propositions. The proof of Proposition 4.4 is postponed by a few lines and the one of Proposition 4.5 is postponed by a few pages.

Sclosed

Proposition 4.4 Let $e^+ \in \mathcal{E}^+$ be fixed; then the set $S^+ \subset A_0$ is closed.

Sintersection

Proposition 4.5 Let $e^+ \in \mathcal{E}^+$ be fixed; then for any continuous path $\vec{\psi} : [0, 1] \rightarrow A_0$, $\vec{\psi}(a) = (d(a), \tau(a))$ such that $d(0) = 0$, $d(1) = \varepsilon^{(1+\nu)/\underline{\alpha}}$ there is $\bar{a} \in]0, 1[$ (depending on $\vec{\psi}$) such that $\vec{\psi}(\bar{a}) \in S^+$.

Once proved Propositions 4.4 and 4.5 we can apply Lemma 4.3 to conclude the following.

topological

Proposition 4.6 Let the hypotheses of Theorem 4.2 be satisfied. Then the set S^+ contains a closed connected set $\tilde{S}^+$ which intersects the lines $\tau = b_0$ and $\tau = b_1$, while the set S^- contains a closed connected set $\tilde{S}^-$ which intersects the lines $\tau = b_0$ and $\tau = b_1$.

Proof The part concerning S^+ and $\tilde{S}^+$, follows from Propositions 4.4 and 4.5, simply applying Lemma 4.3 choosing $\mathcal{R} = A_0$, $E = S^+$.

The part concerning S^- and $\tilde{S}^-$ then follows using an inversion of time argument, see e.g. [14, §5.3]. $\square$

Proof of Proposition 4.4 In this proof, we consider e^+ and ε fixed so we leave the dependence on these variables unsaid. Assume by contradiction that S^+ is not closed. Then there is a sequence $(d_n, \tau_n) \in S^+$ such that $(d_n, \tau_n) \rightarrow (\bar{d}, \bar{\tau}) \notin S^+$. Let us denote by $\alpha_j^n = \alpha_j(d_n, \tau_n)$. By (56), $|\alpha_j^n| \leq B_{2j}$ for any j and n .

Let $\ell \in \mathbb{N}$ be such that the trajectory $\vec{x}(t, \bar{\tau}; \vec{Q}_s(\bar{d}, \bar{\tau}))$ satisfies $\mathbf{C}_{e^+}^+$ with $\alpha_j = \bar{\alpha}_j$, for suitable $\alpha_j = \bar{\alpha}_j$ for any $j \leq \ell - 1$, but $\mathbf{C}_{e^+}^+$ does not hold for $j = \ell$ for any $\alpha_\ell = \bar{\alpha}$ whenever $|\bar{\alpha}| \leq B_{2\ell}$.

We split the argument in the case $e_\ell = 0$ and $e_\ell = 1$.

Assume first $e_\ell = 0$. Then, for any $|\bar{\alpha}| \leq B_{2\ell}$ there is $\bar{t} = \bar{t}(\bar{\alpha}) \in [T_{2\ell-1}, T_{2\ell+1}]$ such that

$$\|\vec{x}(\bar{t}, \bar{\tau}; \vec{Q}_s(\bar{d}, \bar{\tau}))\| > c^* \varepsilon;$$

in fact we may choose $\bar{t} \in]T_{2\ell-1}, T_{2\ell+1}[$. So we can find $\delta > 0$ and a suitable $\mu > 0$ such that

$$\|\vec{x}(t, \bar{\tau}; \vec{Q}_s(\bar{d}, \bar{\tau}))\| > (c^* + 2\mu)\varepsilon$$

for any $t \in [\bar{t} - \delta, \bar{t} + \delta]$. Then, recalling that $\vec{Q}_s(\cdot, \cdot)$ is continuous and using continuous dependence of the solutions of (PS) on data and parameters, see Remark 2.3, we can find $N > 0$ large enough so that

$$\|\vec{x}(t, \tau_n; \vec{Q}_s(d_n, \tau_n))\| > (c^* + \mu)\varepsilon$$

for any $t \in [\bar{t} - \delta, \bar{t} + \delta]$ and any $n \geq N$. But this is a contradiction since $\vec{x}(t, \tau_N; \vec{Q}_s(d_N, \tau_N))$ satisfies $C_{e^+}^+$.

Now assume $e_\ell = 1$, then for any $|\bar{\alpha}| \leq B_{2\ell}$ there is $\bar{t} = \bar{t}(\bar{\alpha}) \in [T_{2\ell-1}, T_{2\ell+1}]$ such that

$$\|\vec{x}(\bar{t}, \bar{\tau}; \vec{Q}_s(\bar{d}, \bar{\tau})) - \vec{\gamma}(\bar{t} - T_{2\ell} - \bar{\alpha})\| > c^* \varepsilon.$$

Again we can assume $\bar{t} \in]T_{2\ell-1}, T_{2\ell+1}[$, so that we can find $\delta > 0$ and a suitable $\mu > 0$ such that

$$\|\vec{x}(t, \bar{\tau}; \vec{Q}_s(\bar{d}, \bar{\tau})) - \vec{\gamma}(t - T_{2\ell} - \bar{\alpha})\| > (c^* + 2\mu)\varepsilon$$

for any $t \in [\bar{t} - \delta, \bar{t} + \delta]$. Then, using again the continuity of $\vec{Q}_s(\cdot, \cdot)$ and the continuous dependence of the solutions of (PS) on data and parameters (see Remark 2.3), we can find $N > 0$ large enough so that

$$\|\vec{x}(t, \tau_n; \vec{Q}_s(d_n, \tau_n)) - \vec{\gamma}(t - T_{2\ell} - \bar{\alpha})\| > (c^* + \mu)\varepsilon$$

for any $t \in [\bar{t} - \delta, \bar{t} + \delta]$ and any $n \geq N$. In particular, if we choose $\bar{\alpha} = \alpha_\ell^N$, we find $\bar{t}^N = \bar{t}(\alpha_\ell^N)$ such that

$$\|\vec{x}(t, \tau_N; \vec{Q}_s(d_N, \tau_N)) - \vec{\gamma}(t - T_{2\ell} - \alpha_\ell^N)\| > (c^* + \mu)\varepsilon$$

for any $t \in [\bar{t}^N - \delta, \bar{t}^N + \delta]$. But this contradicts the fact that the trajectory $\vec{x}(t, \tau_N; \vec{Q}_s(d_N, \tau_N))$ satisfies $C_{e^+}^+$.

Now assume that there is $\bar{T} \in [\tau, T_1]$ such that

$$\|\vec{x}(\bar{T}, \bar{\tau}; \vec{Q}_s(\bar{d}, \bar{\tau})) - \vec{\gamma}(\bar{T} - \bar{\tau})\| > c^* \varepsilon.$$

Arguing as in the $e_\ell = 1$ case, we find a contradiction. $\square$

The Proof of Proposition 4.5 is rather lengthy and requires several lemmas.

We start from some notation borrowed from [14]. We split $\mathcal{E}^+ = \{0, 1\}^{\mathbb{Z}^+}$ in two subsets $\hat{\mathcal{E}}^+$ and $\mathcal{E}_0^+$:

$$\begin{aligned} j_*^+(e^+) &= \sup\{j \mid e_j^+ = 1\}, \\ \hat{\mathcal{E}}^+ &= \{e^+ = (e_j^+) \in \mathcal{E}^+ \mid j_*^+(e^+) = +\infty\}, \\ \mathcal{E}_0^+ &= \{e^+ = (e_j^+) \in \mathcal{E}^+ \mid j_*^+(e^+) < +\infty\}, \\ j_S^+(e^+) &= \begin{cases} \#\{j \mid 1 \leq j \leq j_*^+(e^+), e_j^+ = 1\}, & \text{if } j_*^+(e^+) < \infty, \\ \infty, & \text{if } j_*^+(e^+) = \infty, \end{cases} \end{aligned} \tag{57}$$

efuture

where $\#$ denotes the number of elements. So, $j_*^+(e^+)$ is “the index of the last 1 of e^+ ” and $j_S^+(e^+)$ is “the number of 1s in e^+ ”.

Let us fix $\mathcal{T}^+ = (T_j)$, $j \in \mathbb{Z}^+$ satisfying (13), (54) and $e^+ \in \mathcal{E}^+$; adapting [6] we introduce a new sequence $\mathcal{S} = (\mathcal{S}_j)$, $j = 0, 1, \dots, j_S^+(e^+)$, which is a subsequence of $\mathcal{T}^+$ depending also on e^+ , and we define (Δ_j) , $j = 1, 2, \dots, j_S^+(e^+)$ as follows:

$$\begin{aligned} \mathcal{S}_0 &= \tau, & \mathcal{S}_j &= \min\{T_{2k} > \mathcal{S}_{j-1} \mid e_k = 1\}, & 1 \leq j \leq j_S^+(e^+), \\ \Delta_j &= \mathcal{S}_j - \mathcal{S}_{j-1}, & & \text{if } 1 \leq j \leq j_S^+(e^+). \end{aligned} \tag{58}$$

defSS

Clearly, $\mathcal{S}_j$ has finitely many values if $e^+ \in \mathcal{E}_0^+$, while if $e^+ \in \hat{\mathcal{E}}^+$, then $\mathcal{S}_j$ and Δ_j are defined for any $j \in \mathbb{N}$ and in $\mathbb{Z}^+$, respectively.

We denote by k_j , the subsequence such that $\mathcal{S}_j = T_{2k_j}$, $1 \leq j \leq j_S^+(\mathbf{e}^+)$, and we set $\beta_{2k_0} = b_0$, $\beta_{2k_0}' = b_1$, so that

$$\beta_{2k_j} < \mathcal{S}_j < \beta_{2k_j}', \quad 0 \leq j \leq j_S^+(\mathbf{e}^+) \quad (59)$$

defik

where β_j and β_j' are the ones defined in **P1** and (6). Notice that there is a subsequence n_k such that $\beta_k = b_{n_k}$, so $\beta_{2k_j} = b_{n_{2k_j}}$ and $\beta_{2k_j}' = b_{n_{2k_j}+1}$. Further, setting $B_{2k_0} = B_0 = b_1 - b_0$, by (13) for all $j \geq 1$ we find the estimate

$$\Delta_j \geq 2K_0(1+\nu)|\ln(\varepsilon)| + B_{2k_j} + B_{2k_{j-1}}. \quad (60)$$

defDej

In fact, we will rely mainly on the sequences $(\mathcal{S}_j)$ and (Δ_j) .

We start our argument with the more involved case of $\mathbf{e}^+ \in \hat{\mathcal{E}}^+$, then the case where $\mathbf{e}^+ \in \mathcal{E}_0^+$ will follow more easily.

In the whole proof $\vec{\psi} : [0, 1] \rightarrow A_0$, $\vec{\psi}(a) = (d(a), \tau(a))$ is a generic path such that $d(0) = 0$ and $d(1) = \varepsilon^{(1+\nu)/\sigma}$. Since $\vec{\psi}$, $\mathcal{T}^+$ and $\mathbf{e}^+ \in \mathcal{E}^+$ are fixed in this proof, we leave these dependencies unsaid.

In the proof of [14, Theorem 3.4] we have constructed a sequence of nested intervals $J_n \subset J_{n-1} \subset J_0$ characterized by the following property: if $d \in J_n$ then the trajectory $\vec{x}(t, \tau; \vec{Q}_s(d, \tau))$ has property $\mathbf{C}_{\mathbf{e}^+}^+$ when $t \in [\tau, \mathcal{T}_n(d, \tau)]$ and it will cross (transversely) L^0 at $t = \mathcal{T}_n(d, \tau)$ after performing n loops.

More precisely, following [14] for any $i = 1, \dots, n$ we define the C^r functions

$$\mathcal{T}_i(\cdot, \tau) : J_i \rightarrow [\beta_{2k_i}, \beta_{2k_i}'], \quad d_i(\cdot, \tau) : J_i \rightarrow \mathbb{R},$$

$$\begin{aligned} d_i(d, \tau) &:= d_1(D, A), \quad \text{where } D = d_{i-1}(d, \tau), \text{ and } A = \mathcal{T}_{i-1}(d, \tau), \\ \mathcal{T}_i(d, \tau) &:= \mathcal{T}_1(D, A), \quad \text{where } D = d_{i-1}(d, \tau), \text{ and } A = \mathcal{T}_{i-1}(d, \tau), \\ \mathcal{T}_{i-\frac{1}{2}}(d, \tau) &:= \mathcal{T}_{\frac{1}{2}}(D, A), \quad \text{where } D = d_{i-1}(d, \tau), \text{ and } A = \mathcal{T}_{i-1}(d, \tau) \end{aligned} \quad (61)$$

Tn

with $d_0(d, \tau) = d$, $\mathcal{T}_0(d, \tau) = \tau$ such that, if $d \in J_n$, the trajectory $\vec{x}(t, \tau; \vec{Q}^s(d, \tau))$ performs n loops close to Γ when $t \in [\tau, \mathcal{T}_n(d, \tau)]$, and intersects (transversely) L^0 exactly at $t = \tau$ and at $t = \mathcal{T}_i(d, \tau)$ and L^{in} at $\mathcal{T}_{i-\frac{1}{2}}(d, \tau)$ for $i = 1, \dots, n$.

To prove Proposition 4.5 we just have to repeat the argument of [14, Theorem 3.4] but this time working on the path $\vec{\psi}(a)$ instead of on the horizontal segment $J_0 \times \{\tau\}$. The argument relies mainly on Theorem C and on its uniformity with respect to τ .

So, our aim is to build a sequence of nested compact intervals $\mathcal{J}_{n+1} \subset \mathcal{J}_n \subset \mathcal{J}_1 \subset [0, 1]$ such that, if $a \in \mathcal{J}_n$ then $\vec{x}(t, \tau(a); \vec{Q}_s(d(a), \tau(a)))$ has property $\mathbf{C}_{\mathbf{e}^+}^+$ whenever $t \in [\tau, \mathcal{T}_n(d(a), \tau(a))] \supset [\tau, \beta_{2k_n}]$.

Let $D_A^1 = \exp\left(-\frac{5(\Delta_1 + B_{2k_1})}{2\Sigma^{\text{fwd}}}\right)$ and observe that $D_A^1 < \varepsilon^{(1+\nu)/\sigma}$, cf. (5.5) in [14]. Then set $I_1 = [D_A^1, \varepsilon^{(1+\nu)/\sigma}]$ and

$$\mathcal{I}_1 := d^{-1}(I_1) = \{a \in [0, 1] \mid d(a) \in I_1\}.$$

Notice that $\mathcal{I}_1$ is closed, since I_1 is closed and $d(\cdot)$ is continuous; further $\mathcal{I}_1 \neq \emptyset$ since $d(\cdot) : [0, 1] \rightarrow [0, \varepsilon^{(1+\nu)/\underline{\sigma}}]$ is surjective by construction.

Now we use the stretching property of the function $\mathcal{T}_1(\cdot, \tau)$ as in Theorem C (which is in fact based on the fact that $\mathcal{T}_1(d, \tau)$ grows as $|\ln d|$ as $d \rightarrow 0$, see [14]).

key2bismod

Lemma 4.7 *There are $a', a'' \in \mathcal{I}_1$ such that*

$$\mathcal{T}_1(\vec{\psi}(a')) = \beta_{2k_1}, \quad \mathcal{T}_1(\vec{\psi}(a'')) = \beta'_{2k_1}. \quad (62)$$

goalkey2bis

Hence the image of the function $\mathcal{T}_1(\vec{\psi}(\cdot)) : \mathcal{I}_1 \rightarrow \mathbb{R}$ contains the closed interval $[\beta_{2k_1}, \beta'_{2k_1}]$.

Proof Following the notation of [14, Lemma 5.1], we denote by $D_B^1 = \varepsilon^{(1+\nu)/\underline{\sigma}}$. Let $A' < A''$ be such that $d(A') = D_A^1$, $d(A'') = D_B^1$; we can assume w.l.o.g. that $[A', A''] \subset \mathcal{I}_1$. As in the proof of [14, Lemma 5.1], using Theorem C we see that

$$\mathcal{T}_1(\vec{\psi}(A')) > \mathcal{S}_1 + B_{2k_1}, \quad \mathcal{T}_1(\vec{\psi}(A'')) < \mathcal{S}_1 - B_{2k_1}.$$

Since $[\beta_{2k_1}, \beta'_{2k_1}] \subset [\mathcal{S}_1 - B_{2k_1}, \mathcal{S}_1 + B_{2k_1}]$, from the continuity of $\mathcal{T}_1(\vec{\psi}(\cdot))$ we see that there are $a', a'' \in \mathcal{I}_1$ such that (62) holds. $\square$

Then we set

$$\hat{\mathcal{I}}_1 := \left\{ a \in \mathcal{I}_1 \mid \beta_{2k_1} \leq \mathcal{T}_1(\vec{\psi}(a)) \leq \beta'_{2k_1} \right\}.$$

Observe that $\hat{\mathcal{I}}_1$ is closed but possibly disconnected, so we choose $\check{\mathcal{I}}_1 \subset \hat{\mathcal{I}}_1$ so that $\check{\mathcal{I}}_1$ is a compact connected interval with the following property

$$\mathcal{T}_1(\cdot, \tau) : \check{\mathcal{I}}_1 \rightarrow [\beta_{2k_1}, \beta'_{2k_1}] \quad \text{is surjective.} \quad (63)$$

prop1bis

closest

Remark 4.8 For definiteness we choose $\check{\mathcal{I}}_1$ to be the “interval closest to 0 satisfying property (63)”. Namely notice that by construction there are $A < B$, $A, B \in \hat{\mathcal{I}}_1$ such that $\mathcal{T}_1(\vec{\psi}(A)) = \beta'_{2k_1}$ and $\mathcal{T}_1(\vec{\psi}(B)) = \beta_{2k_1}$, then we set

$$B'_1 = \min\{a \in \hat{\mathcal{I}}_1 \mid \mathcal{T}_1(\vec{\psi}(a)) = \beta_{2k_1}\},$$

$$A'_1 = \max\{a \in \hat{\mathcal{I}}_1 \mid a < B'_1, \mathcal{T}_1(\vec{\psi}(a)) = \beta'_{2k_1}\}$$

and we define $\check{\mathcal{I}}_1 = [A'_1, B'_1]$ (so it is the smallest interval with this property).

Then from property (63), **P1** and Lemma H we see that there are $\mathcal{A}^-, \mathcal{A}^+ \in \check{\mathcal{I}}_1$ such that $d_1(\vec{\psi}(\mathcal{A}^-)) = -\varepsilon^{(1+\nu)/\underline{\sigma}}$ and $d_1(\vec{\psi}(\mathcal{A}^+)) = \varepsilon^{(1+\nu)/\underline{\sigma}}$. Assume to fix the ideas $\mathcal{A}^- < \mathcal{A}^+$, then we set

$$\mathcal{J}_1 := [\mathcal{A}^-, \mathcal{A}^+] \subset \mathcal{I}_1 \quad (64)$$

J1ebis

so we have the following lemma.

Lemma 4.9 *The function $\mathcal{T}_1(\vec{\psi}(\cdot)) : \mathcal{J}_1 \rightarrow [\beta_{2k_1}, \beta'_{2k_1}]$ is well defined and C^r , while the function*

$$d_1(\vec{\psi}(\cdot)) : \mathcal{J}_1 \rightarrow \mathbb{R}$$

is C^r and its image contains $[-\varepsilon^{(1+\nu)/\underline{\sigma}}, \varepsilon^{(1+\nu)/\underline{\sigma}}]$.

Now we keep in mind the definition of $\mathcal{T}_n$ and d_n given in (61), and we proceed by induction to prove the following result.

inductionbis

Proposition 4.10 *Let $e^+ \in \hat{\mathcal{E}}^+$; then there is an infinite sequence of nested compact intervals $\mathcal{J}_n \subset \mathcal{J}_{n-1} \subset \mathcal{J}_1$ such that the function*

$$\mathcal{T}_n(\vec{\psi}(\cdot)) : \mathcal{J}_n \rightarrow [\beta_{2k_n}, \beta'_{2k_n}]$$

is well defined and C^r , while the image of the function

$$d_n(\vec{\psi}(\cdot)) : \mathcal{J}_n \rightarrow \mathbb{R}$$

contains $[-\varepsilon^{(1+\nu)/\sigma}, \varepsilon^{(1+\nu)/\sigma}]$.

Proof We proceed by induction: we have already shown the $n = 1$ step, so we assume the existence of $\mathcal{J}_{n-1}$.

Following the argument in the proof of Proposition 5.7 in [14], we define

$$D_{\#}^n = \exp\left(\frac{-5(\Delta_n + B_{2k_n} + B_{2k_{(n-1)}})}{2\Sigma^{\text{fwd}}}\right) \leq \varepsilon^{(1+\nu)/\sigma},$$

$$\mathcal{I}_n = \left\{a \in \mathcal{J}_{n-1} \mid D_{\#}^n \leq d_n(\vec{\psi}(a)) \leq \varepsilon^{(1+\nu)/\sigma}\right\}.$$

Let $A'_n < A''_n$ be such that $d_n(\vec{\psi}(A'_n)) = D_{\#}^n$, $d_n(\vec{\psi}(A''_n)) = \varepsilon^{(1+\nu)/\sigma}$; we can assume w.l.o.g. that $[A'_n, A''_n] \subset \mathcal{I}_n$.

Then, from the proof of [14, Lemma 5.8] (and the uniformity of the estimates in Theorem C with respect to $\tau \in [b_0, b_1]$), we see that

$$\mathcal{T}_n(\vec{\psi}(A'_n)) \geq \mathcal{S}_n + B_{2k_n} > \beta'_{2k_n}, \quad \mathcal{T}_n(\vec{\psi}(A''_n)) \leq \mathcal{S}_n - B_{2k_n} < \beta_{2k_n}.$$

Hence, using the continuity of $\mathcal{T}_n(\vec{\psi}(\cdot))$, we see that there are a'_n, a''_n such that $A'_n < a'_n < a''_n < A''_n$ and

$$\mathcal{T}_n(\vec{\psi}(a'_n)) = \beta'_{2k_n}, \quad \mathcal{T}_n(\vec{\psi}(a''_n)) = \beta_{2k_n}. \quad (65)$$

hh

Then, following the argument of [14, §5.1.1], we define

$$\hat{\mathcal{I}}_n = \hat{\mathcal{I}}_n^{e^+}(\tau, \mathcal{T}^+) := \left\{a \in \mathcal{I}_n \mid \beta_{2k_n} \leq \mathcal{T}_n(\vec{\psi}(a)) \leq \beta'_{2k_n}\right\}$$

which is a closed nonempty set. Further, possibly shrinking $\hat{\mathcal{I}}_n$, we can assume that

$$\mathcal{T}_n(\vec{\psi}(\cdot)) : \hat{\mathcal{I}}_n \rightarrow [\beta_{2k_n}, \beta'_{2k_n}] \quad \text{is surjective.} \quad (66)$$

calpropn

Then, reasoning as in Remark 4.8, we denote by $\check{\mathcal{I}}_n = \check{\mathcal{I}}_n^{e^+}(\tau, \mathcal{T}^+)$ the “closed interval closest to 0” having property (66).

Now, using P1, (66) and Lemma H we see that there are $\mathcal{A}_n^-, \mathcal{A}_n^+ \in \check{\mathcal{I}}_n$ such that $d_n(\vec{\psi}(\mathcal{A}_n^-)) = -\varepsilon^{(1+\nu)/\sigma}$ and $d_n(\vec{\psi}(\mathcal{A}_n^+)) = \varepsilon^{(1+\nu)/\sigma}$.

Finally, we set

$$\begin{aligned} \check{\mathcal{A}}_n^- &:= \min\{a \in \check{\mathcal{I}}_n \mid d_n(\vec{\psi}(a)) = -\varepsilon^{(1+\nu)/\sigma}\}, \\ \check{\mathcal{A}}_n^+ &:= \min\{a \in \check{\mathcal{I}}_n \mid d_n(\vec{\psi}(a)) = \varepsilon^{(1+\nu)/\sigma}\}, \\ \underline{\mathcal{A}}_n &:= \min\{\check{\mathcal{A}}_n^-, \check{\mathcal{A}}_n^+\}, \quad \overline{\mathcal{A}}_n = \max\{\check{\mathcal{A}}_n^-, \check{\mathcal{A}}_n^+\}, \end{aligned}$$

and we denote by

$$\mathcal{J}_n = \mathcal{J}_n^{e^+}(\tau, \mathcal{T}^+) := [\underline{\mathcal{A}}_n, \overline{\mathcal{A}}_n] \subset \check{\mathcal{I}}_n \subset \mathcal{J}_{n-1}, \quad (67)$$

calJne

so the proposition is proved. $\square$

Now, following again [14, §5.1.1], for any $a \in \mathcal{J}_n$ we set

$$\alpha_{k_j} = \alpha_{k_j}^{e^+}(\varepsilon, \vec{\psi}(a), \mathcal{T}^+) = \mathcal{J}_j(\vec{\psi}(a)) - T_{2k_j} \quad \text{for any } j = 1, \dots, n. \quad (68)$$

caldefaln

Repeating word by word the argument of the proofs of the results in Lemmas 5.10 and 5.12 in [14], but replacing the assumption $(d, \tau) \in J_n \times [b_0, b_1]$ by $a \in \mathcal{J}_n$ one can reprove in this context the analogous results to get the following.

call.stepn

Lemma 4.11 *Let $e^+ \in \mathcal{E}^+$; then for any $1 \leq n \leq j_S^+ = j_S^+(e^+)$ (where $j_S^+ = +\infty$ if $e^+ \in \hat{\mathcal{E}}^+$, and $j_S^+ < +\infty$ if $e^+ \in \mathcal{E}_0^+$), and any $a \in \mathcal{J}_n$, the trajectory $\vec{x}(t, \tau(a); \vec{Q}_s(\vec{\psi}(a)))$ satisfies $C_{e^+}^+$ for any $\tau(a) \leq t \leq \mathcal{T}_n(\vec{\psi}(a))$, i.e., for any $\tau(a) \leq t \leq T_{2k_n} + \alpha_{k_n}$. Further, $|\alpha_{k_n}| \leq B_{2k_n}$.*

If $e^+ \in \hat{\mathcal{E}}^+$, we set

$$\begin{aligned} \mathcal{J}_{+\infty} &= \mathcal{J}_{+\infty}^{e^+}(\tau, \mathcal{T}^+) := \cap_{n=1}^{+\infty} \mathcal{J}_n^{e^+}(\tau, \mathcal{T}^+), \\ a_* &= a_*^{e^+}(\tau, \mathcal{T}^+) = \min\{a \in \mathcal{J}_{+\infty}\}. \end{aligned} \quad (69)$$

Jinftypsi

Notice that we are intersecting infinitely many nested compact intervals, so $\mathcal{J}_{+\infty}$ is a non-empty compact and connected set: whence the minimum a_* exists.

Sintersectionhat

Proposition 4.12 *Proposition 4.5 and Theorem 4.2 hold true for any $e^+ \in \hat{\mathcal{E}}^+$.*

Proof Let $e^+ \in \hat{\mathcal{E}}^+$, then from Lemma 4.11 we immediately obtain a_* such that $\vec{\psi}(a_*) \in S^+$. Then by construction $\vec{x}(t, \tau; \vec{Q}_s(\vec{\psi}(a_*)))$ has property $C_{e^+}^+$. $\square$

Proof of Proposition 4.5 If $e^+ \in \hat{\mathcal{E}}^+$ the result follows from Proposition 4.12.

So, let $e^+ \in \mathcal{E}_0^+$ and consider the set $\mathcal{J}_{j_S^+} = \cap_{n=1}^{j_S^+} \mathcal{J}_n$ constructed via Lemma 4.11, which is again a compact interval. The function

$$d_{j_S^+}(\vec{\psi}(\cdot)) : \mathcal{J}_{j_S^+} \rightarrow \mathbb{R}$$

is C^r and its image contains the interval $[-\varepsilon^{(1+\nu)/\sigma}, \varepsilon^{(1+\nu)/\sigma}]$; further the function

$$\mathcal{J}_{j_S^+}(\vec{\psi}(\cdot)) : \mathcal{J}_{j_S^+} \rightarrow [\beta_{2k_{j_S^+}}, \beta'_{2k_{j_S^+}}]$$

is well defined and C^r .

So the set

$$\mathcal{I}_{+\infty} = \mathcal{I}_{+\infty}^{e^+}(\mathcal{T}^+) := \{a \in \mathcal{J}_{j_S^+} \mid d_{j_S^+}(\vec{\psi}(a)) = 0\}$$

is a non-empty compact set. Let us denote by $a_* = \min \mathcal{I}_{+\infty}$ and by $\mathcal{J}_{+\infty} = \mathcal{J}_{+\infty}^{e^+}(\tau, \mathcal{T}^+)$ the largest connected component of $\mathcal{I}_{+\infty}$ containing a_* . Then $\vec{x}(t, \tau; \vec{Q}_s(\vec{\psi}(a)))$ has property $C_{e^+}^+$ for any $t \in [\tau, \mathcal{T}_{j_S^+}(\vec{\psi}(a))]$, whenever $a \in \mathcal{J}_{+\infty}$.

Further $\vec{x}(\mathcal{T}_{j_S^+}(\vec{\psi}(a)), \tau(a); \vec{Q}_s(\vec{\psi}(a))) = \vec{P}_s(\mathcal{T}_{j_S^+}(\vec{\psi}(a)))$, hence $\vec{x}(t, \tau(a); \vec{Q}_s(\vec{\psi}(a))) \in \tilde{W}^s(t) \subset \Omega^+$ for any $t \geq \mathcal{T}_{j_S^+}(\vec{\psi}(a))$. So from Proposition E, recalling that $\alpha_{k_j} = \mathcal{T}_j(\vec{\psi}(a)) - T_{2k_j}$ for any $1 \leq j \leq j_S^+$, we see that

$$\|\vec{x}(t, \tau(a); \vec{Q}_s(\vec{\psi}(a))) - \vec{\gamma}^+(t - T_{2k_{j_S^+}} - \alpha_{k_{j_S^+}})\| \leq \frac{c^*}{2}\varepsilon$$

for any $t \geq T_{2k_{j_S^+}} + \alpha_{k_{j_S^+}}$. Moreover from Lemma F we see that

$$\|\vec{x}(t, \tau(a); \vec{Q}_s(\vec{\psi}(a)))\| \leq c^*\varepsilon \quad \text{for any } t \geq T_{2k_{j_S^+}+1} \geq \mathcal{T}_{j_S^+}(\vec{\psi}(a)) + T_b.$$

Hence we see that $\vec{x}(t, \tau(a); \vec{Q}_s(\vec{\psi}(a)))$ has property $C_{e^+}^+$ for any $t \geq \tau(a)$, whenever $a \in J_{+\infty}$ and we conclude the proof of Proposition 4.5. $\square$

Now, the assertion of Theorem 4.2 immediately follows from Proposition 4.6 setting

$$\tilde{\chi}^+(e^+) = \left\{ (\vec{Q}_s(d, \tau), \tau) \mid (d, \tau) \in \tilde{S}^+ \right\}, \quad \tilde{\chi}^-(e^-) = \left\{ (\vec{Q}_u(d, \tau), \tau) \mid (d, \tau) \in \tilde{S}^- \right\}.$$

proof.connection2

4.2 Proof of Theorem 4.1

In this section we assume that the hypotheses of Theorem 4.1 are satisfied. Further $\varepsilon \in]0, \varepsilon_0[$, $\nu \geq \nu_0$ are fixed and $\mathcal{T}^+ = (T_j)$, $j \in \mathbb{Z}^+$ and $\mathcal{T}^- = (T_j)$, $j \in \mathbb{Z}^-$ are fixed sequences satisfying (12) for $j \in \mathbb{Z}^+$ and (51), and (12) for $j \leq -2$ and (52), respectively; moreover we assume (7). Finally we let $e^+ \in \mathcal{E}^+$ and $e^- \in \mathcal{E}^-$, then we set

$$\begin{aligned} S^{\Lambda,+} &= \left\{ (d, \tau) \in A_0 \left| \begin{array}{l} \vec{x}(t, \tau; \vec{Q}_s(d, \tau)) \text{ has property } C_{e^+}^+ \text{ and} \\ |\alpha_j| \leq a_j^\downarrow - a_j^\uparrow \leq \Lambda^1 \text{ for any } j \in \mathbb{Z}^+ \end{array} \right. \right\}, \\ S^{\Lambda,-} &= \left\{ (d, \tau) \in A_0 \left| \begin{array}{l} \vec{x}(t, \tau; \vec{Q}_u(d, \tau)) \text{ has property } C_{e^-}^- \text{ and} \\ |\alpha_j| \leq a_j^\downarrow - a_j^\uparrow \leq \Lambda^1 \text{ for any } j \in \mathbb{Z}^- \end{array} \right. \right\}. \end{aligned} \quad (70)$$

defS+1a

Repeating word by word the proof of Propositions 4.4 and 4.5 in Section 4.1, but replacing B_j by $a_j^\downarrow - a_j^\uparrow$ for any $j \in \mathbb{Z}$, we obtain the following.

Sclosedbis

Proposition 4.13 *Let $e^+ \in \mathcal{E}^+$ be fixed; then the set $S^{\Lambda,+} \subset A_0$ is closed.*

Sintersectionbis

Proposition 4.14 *Let $e^+ \in \mathcal{E}^+$ be fixed; then for any continuous path $\vec{\psi} : [0, 1] \rightarrow A_0$, $\vec{\psi}(a) = (d(a), \tau(a))$ such that $d(0) = 0$; $d(1) = \varepsilon^{(1+\nu)/\varrho}$ there is $\bar{a} \in]0, 1[$ (depending on $\vec{\psi}$) such that $\vec{\psi}(\bar{a}) \in S^{\Lambda,+}$.*

Then, as in the proof of Proposition 4.5, we can apply Lemma 4.3, choosing $\mathcal{R} = A_0$, $E = S^{\Lambda,+}$, to conclude the following.

topologicalbis

Proposition 4.15 *Let the hypotheses of Theorem 4.1 be satisfied. Then the set $S^{\Lambda,+}$ contains a closed connected set $\tilde{S}^{\Lambda,+}$ which intersects the lines $\tau = b_0$ and $\tau = b_1$, while the set $S^{\Lambda,-}$ contains a closed connected set $\tilde{S}^{\Lambda,-}$ which intersects the lines $\tau = b_0$ and $\tau = b_1$.*

Proof of Theorem 4.1 Let us focus first on forward time, so let $\mathcal{T}^+ = (T_j)$, $j \in \mathbb{Z}^+$, be a fixed sequence satisfying (12) for $j \in \mathbb{Z}^+$ and (51) for any $e^+ \in \mathcal{E}^+$. Then the existence of the compact connected subset $\tilde{\chi}^{\Lambda,+}(e^+) \subset \chi^{\Lambda,+}(e^+)$ is obtained simply by setting

$$\tilde{\chi}^{\Lambda,+}(e^+) = \left\{ (\vec{Q}_s(d, \tau), \tau) \mid (d, \tau) \in \tilde{S}^{\Lambda,+} \right\}.$$

The estimates on the sequence $\alpha_j^{\varepsilon^+}(\varepsilon)$ are obtained repeating word by word the proof of Lemma 5.27 and Remark 5.28 in [14]. The part of the proof concerning backward time, i.e., $\mathcal{T}^- = (T_j)$, $j \in \mathbb{Z}^-$ and $e^- \in \mathcal{E}^-$, is then obtained with an inversion of time argument as in [14, §5.3]. $\square$

From Proposition 5.30 in [14] we find the following.

closeW

Proposition 4.16 *Let $\vec{x}(t, \tau; \vec{\xi})$ be a trajectory constructed using the first part of either of Theorems 4.1 or 4.2, i.e. a trajectory satisfying property $C_{e^+}^+$. Then*

$$\vec{x}(t, \tau; \vec{\xi}) \in [B(\vec{W}(t), \varepsilon^{(1+\nu)/2}) \cap \vec{V}(t)] \quad (71)$$

e.closeW

for any $t \geq \tau$.

Analogously let $\vec{x}(t, \tau; \vec{\xi})$ be a trajectory constructed using the second part of either of Theorems 4.1 or 4.2, i.e. a trajectory satisfying property $C_{e^-}^-$. Then (71) holds for any $t \leq \tau$.

5 Proof of the main results – Theorems 2.6, 2.8, 2.14

S.mainproof

In this section we glue together trajectories chaotic in the past with the ones chaotic in the future, obtained in the results of Section 4. We start by proving Theorem 5.1 below, which is stated in the setting of Theorem 2.14. Namely, we look for some pairs $(d_\star^\pm, \tau_\star)$ such that $\vec{Q}_s(d_\star^+, \tau_\star) = \vec{Q}_u(d_\star^-, \tau_\star)$ and $(d_\star^+, \tau_\star) \in \tilde{S}^+(\mathcal{T}^+, e^+)$, $(d_\star^-, \tau_\star) \in \tilde{S}^-(\mathcal{T}^-, e^-)$: see Proposition 4.6. For this purpose we rely on the intermediate value theorem and on the sign changes of $\mathcal{M}(\tau)$. We focus on the following stripe:

$$\hat{A} := \{(d, \tau) \mid d \in [-\sqrt{\varepsilon}, \sqrt{\varepsilon}], \tau \in [b_0, b_1]\}.$$

We recall that d^+ and d^- measure respectively the distance of $\vec{Q}_s(d^+, \tau)$ from $\vec{P}_s(\tau)$ and of $\vec{Q}_u(d^-, \tau)$ from $\vec{P}_u(\tau)$ and that they are both $o(\varepsilon)$ (in fact smaller than $\varepsilon^{(1+\nu)/\sigma}$ with $\nu \geq \nu_0$, see (27)). Further the distance between $\vec{P}_s(\tau)$ and $\vec{P}_u(\tau)$ is $O(\varepsilon)$ and it is measured by the Melnikov function, up to the first order in ε . For this reason we need to work on the stripe $\hat{A}$, which is a bit larger than $O(\varepsilon)$.

We are ready to prove the following result.

theoremkey

Theorem 5.1 *Assume that the hypotheses of Theorem 2.14 hold. Let $e \in \mathcal{E}$ be a sequence satisfying $e_0 = 1$. Then there is a compact connected set $X(e, \mathcal{T}) \subset L^0$ and a sequence $(\alpha_j^e(\varepsilon))$, $j \in \mathbb{Z}$, such that if $\vec{\xi} \in X(e, \mathcal{T})$, then $\vec{x}(t, T_0 + \alpha_0^e(\varepsilon); \vec{\xi})$ has property C_e . Further $|\alpha_0| \leq B_0$.*

Before starting the proof, we collect some facts for future reference.
Let us introduce the set

$$\hat{S}^- = \hat{S}^-(\mathcal{T}^-, e^-) := \{(d + \mathcal{D}(\vec{P}_u(\tau), \vec{P}_s(\tau)), \tau) \mid (d, \tau) \in \tilde{S}^-(\mathcal{T}^-, e^-)\}, \quad (72)$$

defhatS

where $\tilde{S}^-$ is as in Proposition 4.6. Notice that by construction $\hat{S}^- \subset \hat{A}$. Further, $\hat{S}^-$ is connected, see Proposition 4.6, and splits $\hat{A}$ in two relatively open and connected components say E^r and E^l , see Figure 4. We can assume that E^r contains the line $d = \sqrt{\varepsilon}$, while E^l contains the line $d = -\sqrt{\varepsilon}$.

Let us set

$$\begin{aligned} \tilde{J}_{+\infty}(\tau) &= \{d \in \mathbb{R} \mid (d, \tau) \in \tilde{S}^+\}, \\ \tilde{J}_{-\infty}(\tau) &= \{d \in \mathbb{R} \mid (d, \tau) \in \tilde{S}^-\}, \end{aligned}$$

$\tilde{S}^+$ being as in Proposition 4.6.

Remark 5.2 If $\hat{d} \in \tilde{J}_{+\infty}(\tau) \cap \tilde{J}_{-\infty}(\tau)$ and $\hat{d} = d^- + \mathcal{D}(\vec{P}_u(\tau), \vec{P}_s(\tau))$ then

$$\hat{d} = \mathcal{D}(\vec{Q}_s(\hat{d}, \tau), \vec{P}_s(\tau)) = \mathcal{D}(\vec{Q}_u(d^-, \tau), \vec{P}_s(\tau)).$$

Hence we find $\vec{Q}_s(\hat{d}, \tau) = \vec{Q}_u(d^-, \tau)$. In fact $\hat{d}$ measures the directed distance on Ω^0 between $\vec{Q}_u$ and $\vec{P}_s$.

To prove Theorem 5.1 we need a further classical result, again borrowed from [14], see e.g. the nice introduction of [27] or [23, §4.5] for a proof.

rMeln

Remark 5.3 Assume that $\vec{f}^\pm$ and $\vec{g}$ are C^r , $r > 1$. Observe that if $0 < \varepsilon \ll 1$ then, for any fixed $\tau \in \mathbb{R}$, there are $c > 0$ (independent of ε and τ) and a monotone increasing and continuous function $\bar{\omega}(\varepsilon)$ such that $\bar{\omega}(0) = 0$ and

$$\mathcal{D}(\vec{P}_s(\tau), \vec{P}_u(\tau)) = c\varepsilon\mathcal{M}(\tau) + \varepsilon\omega(\tau, \varepsilon), \quad |\omega(\tau, \varepsilon)| \leq \bar{\omega}(\varepsilon).$$

Further, if $r \geq 2$, we can find $C_M > 0$ such that $\bar{\omega}(\varepsilon) \leq C_M\varepsilon$.

Proof of Theorem 5.1 We claim that

- there is $(d_\star, \tau_\star) \in (\hat{S}^-(\mathcal{T}^-, e^-) \cap \hat{S}^+(\mathcal{T}^+, e^+))$.

In fact for any $d_i^+ \in \tilde{J}_{+\infty}(b_i)$ and any $\hat{d}_i^- = d_i^- + \mathcal{D}(\vec{P}_u(b_i), \vec{P}_s(b_i)) \in \tilde{J}_{-\infty}(b_i)$ we have

$$d_i^+ - \hat{d}_i^- = d_i^+ - \mathcal{D}(\vec{P}_u(b_i), \vec{P}_s(b_i)) - d_i^-$$

for $i = 0, 1$. From **P1** we see that $\mathcal{M}(b_0) < -\bar{c} < 0 < \bar{c} < \mathcal{M}(b_1)$, so, recalling that $d_i^\pm \in [0, \varepsilon^{(1+\nu)/\sigma}]$ for $i = 0, 1$, using Remark 5.3, for any $0 < \varepsilon \leq \varepsilon_0$ we find

$$\begin{aligned} d_0^+ - \hat{d}_0^- &\leq \frac{c}{2}\mathcal{M}(b_0)\varepsilon + \varepsilon^{1+\bar{\nu}} \leq \frac{c}{4}\mathcal{M}(b_0)\varepsilon < -\frac{c\bar{c}}{4}\varepsilon, \\ d_1^+ - \hat{d}_1^- &\geq \frac{c}{2}\mathcal{M}(b_1)\varepsilon - \varepsilon^{1+\bar{\nu}} \geq \frac{c}{4}\mathcal{M}(b_1)\varepsilon > \frac{c\bar{c}}{4}\varepsilon \end{aligned} \quad (73)$$

d+id-i

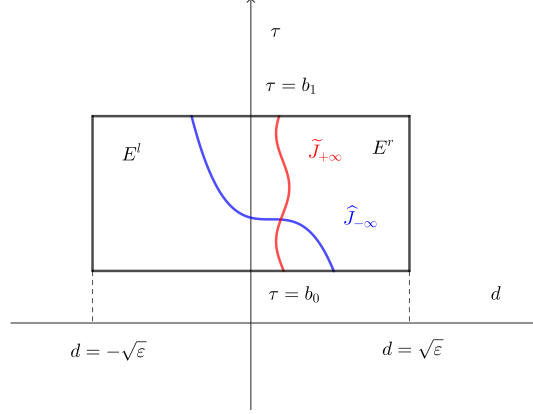


Fig. 4: In this picture we illustrate the set $\hat{A}$ and the proof of Theorem 5.1.

intersection

where $\bar{\nu} = \frac{1-\sigma+\nu}{\sigma} > 1$. Since (73) holds true for any $d_i^+ \in \tilde{J}_{+\infty}(b_i)$ and any $\hat{d}_i^- \in \hat{J}_{-\infty}(b_i)$, we see that $\tilde{J}_{+\infty}(b_0) \times \{b_0\} \subset E^l$ while $\tilde{J}_{+\infty}(b_1) \times \{b_1\} \subset E^r$. Then recalling that $\tilde{J}_{+\infty}(b_i) \times \{b_i\} \subset \tilde{S}^+$ for $i = 0, 1$ and that $\tilde{S}^+$ is connected we prove the claim.

Now let $d_\star^- = d_\star - \mathcal{D}(\vec{P}_u(\tau_\star), \vec{P}_s(\tau_\star))$; then $(d_\star^-, \tau_\star) \in \tilde{S}^- = \tilde{S}^-(\mathcal{T}^-, e^-)$ and

$$\vec{\xi}_\star = \vec{Q}_s(d_\star, \tau_\star) = \vec{Q}_u(d_\star^-, \tau_\star)$$

is such that $\vec{x}(t, \tau_\star; \vec{\xi}_\star)$ has property C_e for any $t \in \mathbb{R}$.

Finally, define

$$\alpha_0 = \alpha_0^{\mathcal{T}, e}(\varepsilon) = \tau_\star - T_0; \quad (74)$$

alpha0

then by construction $\vec{x}(t, T_0 + \alpha_0; \vec{\xi}_\star)$ has property C_e . Further $|\tau_\star - T_0| < b_1 - b_0 = B_0$ so the result is proved. $\square$

uselesspos

Remark 5.4 Note that $\vec{\xi}_\star$ lies on Ω^0 but it is not placed between $\vec{P}_u(\tau_\star)$, $\vec{P}_s(\tau_\star)$: indeed $\mathcal{D}(\vec{\xi}_\star, \vec{P}_u(\tau_\star)) = d_\star^- > 0$ and $\mathcal{D}(\vec{\xi}_\star, \vec{P}_s(\tau_\star)) = d_\star > 0$.

5.1 Setting of Theorems 2.6, 2.8: $|\alpha_j|$ bounded or infinitesimal

S.theorem2.2

In this section we work in the setting of Theorems 2.6 and 2.8. Here we glue together the trajectories chaotic in the past and in the future obtained via Theorem 4.1, so we will always assume **P1**, (7) and we consider a sequence $\mathcal{T}$ satisfying (12) for $\nu \geq \nu_0$. In fact we just need to adapt slightly the argument of Section 4.1 in order to obtain better estimates on $|\alpha_j|$. Therefore we replace the set $\hat{A}$ by the set $\hat{A}^\Lambda$ defined as

$$\hat{A}^\Lambda := \{(d, \tau) \mid d \in [-\sqrt{\varepsilon}, \sqrt{\varepsilon}], \tau \in [a_0^\uparrow, a_0^\downarrow]\},$$

further we replace $\tilde{S}^\pm$ defined in Proposition 4.6 by the sets $\tilde{S}^{\Lambda, \pm}$ defined in Proposition 4.15, and the set $\hat{S}^-$ by the set $\hat{S}^{\Lambda, -}$ defined as

$$\begin{aligned} \hat{S}^{\Lambda, -} &= \hat{S}^{\Lambda, -}(\mathcal{T}^-, e^-) \\ &:= \{(d + \mathcal{D}(\vec{P}_u(\tau), \vec{P}_s(\tau)), \tau) \mid (d, \tau) \in \tilde{S}^{\Lambda, -}(\mathcal{T}^-, e^-)\}. \end{aligned} \quad (75)$$

defhatSbis

So we can prove the following.

theoremkeybis

Theorem 5.5 *Assume that the hypotheses of Theorem 2.8 hold; in particular let $\mathcal{T}$ be a sequence satisfying (12) and (7). Let $e \in \mathcal{E}$ be a sequence satisfying $e_0 = 1$. Then there is a compact connected set $X(e, \mathcal{T}) \subset L^0$ and a sequence $(\alpha_j^e(\varepsilon))$, $j \in \mathbb{Z}$, such that if $\vec{\xi} \in X(e, \mathcal{T})$, then $\vec{x}(t, T_0 + \alpha_0^e(\varepsilon); \vec{\xi})$ has property C_e . Further $|\alpha_0| \leq \Lambda^1$.*

Proof We just need to adapt the argument of the proof of Theorem 5.1.

We claim that

- there is $(d_\star^\Lambda, \tau_\star^\Lambda) \in (\hat{S}^{\Lambda, -}(\mathcal{T}^-, e^-) \cap \tilde{S}^{\Lambda, +}(\mathcal{T}^+, e^+))$.

In fact let $d_\star^\pm \in \tilde{J}_{\pm\infty}(a_0^\uparrow)$, $d_\star^\pm \in \tilde{J}_{\pm\infty}(a_0^\downarrow)$, and $\hat{d}_\star^- = d_\star^- + \mathcal{D}(\vec{P}_u(a_0^\uparrow), \vec{P}_s(a_0^\uparrow)) \in \hat{J}_{-\infty}(a_0^\uparrow)$, $\hat{d}_\star^- = d_\star^- + \mathcal{D}(\vec{P}_u(a_0^\downarrow), \vec{P}_s(a_0^\downarrow)) \in \hat{J}_{-\infty}(a_0^\downarrow)$; from (7) and (9) we can assume $\mathcal{M}(a_0^\uparrow) = -\bar{c}\delta < 0 < \bar{c}\delta = \mathcal{M}(a_0^\downarrow)$ (or we have opposite sign conditions but the argument goes through without significant changes); hence using Remark 5.3, for any $0 < \varepsilon \leq \varepsilon_0$ we find

$$\begin{aligned} d_\star^+ - \hat{d}_\star^- &\leq \frac{c}{2}\mathcal{M}(a_0^\uparrow)\varepsilon + \varepsilon^{1+\bar{\nu}} \leq \frac{c}{4}\mathcal{M}(a_0^\uparrow)\varepsilon = -\frac{c\bar{c}}{4}\delta\varepsilon, \\ d_\star^+ - \hat{d}_\star^- &\geq \frac{c}{2}\mathcal{M}(a_0^\downarrow)\varepsilon - \varepsilon^{1+\bar{\nu}} \geq \frac{c}{4}\mathcal{M}(a_0^\downarrow)\varepsilon = \frac{c\bar{c}}{4}\delta\varepsilon, \end{aligned} \quad (76)$$

d+id-ibis

with $\bar{\nu} = \frac{1-\sigma+\nu}{\sigma} > 1$. Then, arguing as in the proof of Theorem 5.1, we prove the claim.

Now let $d_\star^{\Lambda, -} = d_\star^\Lambda - \mathcal{D}(\vec{P}_u(\tau_\star^\Lambda), \vec{P}_s(\tau_\star^\Lambda))$; then $(d_\star^{\Lambda, -}, \tau_\star^\Lambda) \in \tilde{S}^{\Lambda, -} = \tilde{S}^{\Lambda, -}(\mathcal{T}^-, e^-)$ and

$$\vec{\xi}_\star = \vec{Q}_s(d_\star^{\Lambda, -}, \tau_\star^\Lambda) = \vec{Q}_u(d_\star^{\Lambda, -}, \tau_\star^\Lambda)$$

is such that $\vec{x}(t, \tau_\star^\Lambda; \vec{\xi}_\star)$ has property C_e for any $t \in \mathbb{R}$ or, in other words, $\vec{x}(t, T_0 + \alpha_0, \vec{\xi}_\star)$ with $\alpha_0 = \tau_\star^\Lambda - T_0$ has property C_e for any $t \in \mathbb{R}$.

Further by construction $|\alpha_0| = |\tau_\star^\Lambda - T_0| \leq a_0^\downarrow - a_0^\uparrow \leq \Lambda^1$, so the theorem is proved. $\square$

c.alpha0

Corollary 5.6 *Let the assumptions of Theorem 5.5 be satisfied but replace (7) by (8). Then there is a continuous increasing function $\bar{\omega}_\alpha(\varepsilon)$ (independent of $e \in \mathcal{E}$) such that $\bar{\omega}_\alpha(0) = 0$ and $|\alpha_0^e| \leq \bar{\omega}_\alpha(\varepsilon) + \Lambda^0$. In particular if (10) holds then $|\alpha_0^e| \leq \bar{\omega}_\alpha(\varepsilon)$.*

Now replace (7) by (11) and assume that $\vec{f}$ and $\vec{g}$ are C^r with $r \geq 2$; then we find $c_\alpha > 0$ such that $|\alpha_0^e| \leq c_\alpha \varepsilon$, where c_α depends neither on ε nor on $e \in \mathcal{E}$.

Proof With the same notation as in the proof of Theorem 5.5, observe first that

$$\begin{aligned} |\mathcal{D}(\vec{P}_s(\tau_\star^\Lambda), \vec{P}_u(\tau_\star^\Lambda))| &= |c\varepsilon\mathcal{M}(\tau_\star^\Lambda) + \varepsilon\omega(\tau_\star^\Lambda, \varepsilon)| \\ &= |\mathcal{D}(\vec{P}_s(\tau_\star^\Lambda), \vec{\xi}_\star) + \mathcal{D}(\vec{\xi}_\star, \vec{P}_u(\tau_\star^\Lambda))| \leq \varepsilon^{\frac{1+\bar{\nu}}{\sigma}} = \varepsilon^{1+\bar{\nu}} \end{aligned} \quad (77)$$

estDDDbis

with $\bar{\nu} = \frac{1-\sigma+\nu}{\sigma} > 1$, where we have used Remark 5.4.

We recall that $\vec{f}^{\pm}$ and $\vec{g}$ are C^r with $r > 1$, and that the Melnikov function $\mathcal{M}$ inherits the same regularity. Hence there is $c_r > 0$ such that $|\omega(\tau_\star^\Lambda, \varepsilon)| \leq c_r(\varepsilon + \varepsilon^{r-1})$. Assume first (8), so that $|\mathcal{M}(\tau)| \geq \omega_M(|\tau - T_0 - \Lambda^0|)$ when $\tau \in [T_0 + \Lambda^0, a_0^\downarrow]$.

Then from (77), we find

$$|\omega_M(|\tau_\star^\Lambda - T_0 - \Lambda^0|)| \leq |\mathcal{M}(\tau_\star^\Lambda)| \leq \frac{c_r(\varepsilon + \varepsilon^{r-1}) + \varepsilon^{\bar{\nu}}}{c}, \quad (78)$$

xxi12bis

when $\tau_\star^\Lambda \in [T_0 + \Lambda^0, a_0^\downarrow]$. Hence there is $\bar{\omega}_\alpha(\varepsilon)$ as in the statement such that

$$|\alpha_0| = |\tau_\star^\Lambda - T_0| \leq \Lambda^0 + \omega_M^{-1} \left(\frac{c_r(\varepsilon + \varepsilon^{r-1}) + \varepsilon^{\bar{\nu}}}{c} \right) = \Lambda^0 + \bar{\omega}_\alpha(\varepsilon), \quad (79)$$

tau12

when $\tau_\star^\Lambda \in [T_0 + \Lambda^0, a_0^\downarrow]$. The estimate when $\tau_\star^\Lambda \in [a^\uparrow, T_0 - \Lambda^0]$ is analogous and it is omitted; when $\tau_\star^\Lambda \in [T_0 - \Lambda^0, T_0 + \Lambda^0]$, so $|\alpha_0| = |\tau_\star^\Lambda - T_0| \leq \Lambda^0$, whence the result is proved. When (10) holds we obtain the estimates by setting $\Lambda^0 = 0$.

Now we assume $r \geq 2$ and that (11) holds so $|\mathcal{M}'(T_0)| \geq C > 0$. Then, from (78), we find $C_M > 0$ such that

$$\begin{aligned} \frac{|\mathcal{M}'(T_0)|}{2} |\tau_\star^\Lambda - T_0| &\leq |\mathcal{M}(\tau_\star^\Lambda)| \leq \frac{2c_r\varepsilon + \varepsilon^{\bar{\nu}}}{c} \leq C_M\varepsilon, \\ |\alpha_0| = |\tau_\star^\Lambda - T_0| &\leq \frac{2C_M\varepsilon}{C} =: c_\alpha\varepsilon. \end{aligned} \quad (80)$$

tau12*

So the proof is concluded. $\square$

Remark 5.7 Assume that $\mathcal{M}^{(j)}(T_0) = 0$ for $j = 0, \dots, k-1$ and $\mathcal{M}^{(k)}(T_0) \neq 0$: then $|\mathcal{M}(T_0 + h)| \geq C_k|h|^k$ for some $C_k > 0$ when $|h|$ is small enough. If $\tilde{f}^\pm$ and $\tilde{g}$ are C^r with $r \geq k \geq 2$, we can adapt slightly the previous argument in order to set $\omega_M(\varepsilon) = C_k\varepsilon^k$. Then we get

$$|\alpha_0| = |\tau_\star^\Lambda - T_0| \leq \left[\frac{2C_M\varepsilon}{C_k} \right]^{\frac{1}{k}} = O(\varepsilon^{1/k}).$$

foralpha0

Remark 5.8 Let the assumptions of any of Theorems 2.8 and 2.14 be satisfied. Assume now that $|\mathcal{M}'(T_0)| = C > 0$, then there is $c_\alpha > 0$ such that $|\alpha_0^e| < c_\alpha\varepsilon$, uniformly for any $e \in \mathcal{E}$. To prove this fact, we can repeat word by word the proof of Corollary 5.6.

5.2 Proof of Theorems 2.6, 2.8, 2.14

Here we complete the proof of our main results. Actually, we just consider the setting of Theorem 2.8 (and 2.6): the proof of Theorem 2.14 follows exactly the same lines and it is omitted.

Proof of Theorem 2.8 Let $\mathcal{T} = (T_j)$, $j \in \mathbb{Z}$ be a fixed sequence satisfying (12). Then, via Theorem 5.5, for any $\bar{e} \in \mathcal{E}$ such that $\bar{e}_0 = 1$, we construct a (compact) set $X(\bar{e}) \subset L^0$ such that if $\tilde{\xi}(\bar{e}) \in X(\bar{e})$ then there is a sequence $\alpha_j^{\bar{e}}$ such that the trajectory $\vec{x}(t, T_0 + \alpha_0^{\bar{e}}; \tilde{\xi}(\bar{e}))$ exhibits property $\mathcal{C}_{\bar{e}}$. Let us denote by

$$\Sigma' := \bigcup \{X(e) \mid e_0 = 1\}. \quad (81)$$

defsigma0

Then fix $k \in (\mathbb{Z} \setminus \{0\})$ and denote by $\tilde{\mathcal{T}}^k = (\tilde{T}_j^k)$, $j \in \mathbb{Z}$, the sequence where $\tilde{T}_j^k = T_{2k+j}$. Notice that $\tilde{T}_0^k \in [a_k^\uparrow, a_k^\downarrow] \subset [\beta_{2k}, \beta'_{2k}]$ by (9). Now let $\bar{e}$ be such that $\bar{e}_k = 1$, we denote by $\bar{e}^k = (\bar{e}_{k+j})_{j \in \mathbb{Z}}$, so that $\bar{e}_0^k = \bar{e}_k = 1$: we can apply Theorem 5.5 on the shifted sequence to construct the set $X^k(\bar{e}^k) \subset L^0$ such that if $\tilde{\xi}^k \in X^k(\bar{e}^k)$ then the trajectory $\vec{x}(t, \tilde{T}_0^k + \tilde{\alpha}_0^{\bar{e}^k}; \tilde{\xi}^k)$ has property $\mathcal{C}_{\bar{e}^k}$ using the time sequence $\tilde{\mathcal{T}}^k$.

Let us denote by $\tilde{\xi}^0 = \vec{x}(T_0 + \tilde{\alpha}_{-k}^{\bar{e}^k}, \tilde{T}_0^k + \tilde{\alpha}_0^{\bar{e}^k}; \tilde{\xi}^k)$ and by $\bar{\alpha}_j^{\bar{e}} = \tilde{\alpha}_{j-k}^{\bar{e}^k}$, so that

$$\vec{x}(t, T_0 + \bar{\alpha}_0^{\bar{e}}; \tilde{\xi}^0) \equiv \vec{x}(t, \tilde{T}_0^k + \tilde{\alpha}_0^{\bar{e}^k}; \tilde{\xi}^k),$$

for any $t \in \mathbb{R}$; then by construction $\vec{x}(t, T_0 + \vec{\alpha}_0^{\vec{e}}; \vec{\xi}^0)$ has property $C_{\vec{e}}$. So we have constructed $X(e)$ so that $\vec{x}(t, T_0 + \alpha_0^e; \vec{\xi}^0)$ has property C_e for any e such that $e_k = 1$. Let us define

$$\tilde{\Sigma}_k := \bigcup \{X(e) \mid e_k = 1\}, \quad \Sigma = \left(\bigcup_{k \in \mathbb{Z} \setminus \{0\}} \tilde{\Sigma}_k \right) \cup \Sigma' \cup \{\vec{0}\}. \quad (82)$$

defsigmak

Then the set $\Sigma = \Sigma(\mathcal{T})$ in (82) has the required properties, i.e., for any $e \in \mathcal{E}$ there is $X(e) \subset \Sigma$ such that, for any $\xi \in X(e)$, the trajectory $\vec{x}(t, T_0 + \alpha_0^e; \xi)$ has property C_e . Clearly the initial condition $\vec{\xi} = \vec{0}$ corresponds to the null sequence $e_j = 0$ for any $j \in \mathbb{Z}$. $\square$

r.disjointX

Remark 5.9 We emphasize that $\tilde{\Sigma}_k \cap \tilde{\Sigma}_j$ is nonempty. However $X(e) \cap X(e') = \emptyset$ if $e \neq e'$.

Now we use Theorem 2.8 to prove Theorem 2.6.

Proof of Theorem 2.6. Let us set

$$\tilde{c} = 2 \sup \{ \|\dot{\vec{\gamma}}^+(t)\|; \|\dot{\vec{\gamma}}^-(s)\| \mid s < 0 < t \}, \quad (83)$$

ctilde

and observe that $\tilde{c} > 0$ is finite. From the proof of Corollary 3.7 in [14] we see that for any $j \in \mathbb{Z}$, $t \in \mathbb{R}$, we get

$$\|\vec{\gamma}(t + \alpha_j(\varepsilon)) - \vec{\gamma}(t)\| \leq \tilde{c}|\alpha_j(\varepsilon)|; \quad (84)$$

easy

further

- assume (11), then for any $j \in \mathbb{Z}$ we find

$$\sup_{t \in \mathbb{R}} \|\vec{\gamma}(t - T_{2j} - \alpha_j(\varepsilon)) - \vec{\gamma}(t - T_{2j})\| \leq \tilde{c}c_\alpha \varepsilon \quad (85)$$

easier

where $c_\alpha > 0$ is given in (80) and it is independent of $j \in \mathbb{Z}$;

- assume (10), then for any $j \in \mathbb{Z}$ we find

$$\sup_{t \in \mathbb{R}} \|\vec{\gamma}(t - T_{2j} - \alpha_j(\varepsilon)) - \vec{\gamma}(t - T_{2j})\| \leq \tilde{c}\bar{\omega}_\alpha(\varepsilon) =: \omega_\alpha(\varepsilon)$$

where $\bar{\omega}_\alpha(\varepsilon)$ is given in (79) and it is independent of $j \in \mathbb{Z}$.

Theorem 2.6 follows from Theorem 2.8 simply choosing

$$\tilde{c}^* = c^* + \tilde{c}c_\alpha. \quad (86)$$

deftildecstar

$\square$

Corollary 2.13 is proved with the same idea.

6 Semi-conjugacy with the Bernoulli shift

proof.Bernoulli

Let $\sigma : \mathcal{E} \rightarrow \mathcal{E}$ be the (forward) Bernoulli shift, defined by $\sigma(e) := (e_{m+1})_{m \in \mathbb{Z}}$ where $e = (e_m) \in \mathcal{E}$. In this section, using a classical argument, we show that the action of the flow of (PS) on the set Σ constructed either via Theorem 2.6 or via Corollary 2.9 is semi-conjugated to the Bernoulli shift. In the whole section we follow quite closely

[6, §6] and we borrow some results from [14, §6]. For the analogous sets constructed via Theorem 2.14 we are able to obtain just partial results. Set

$$\begin{aligned}\hat{\mathcal{E}} &:= \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} = -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} = \infty\}, \\ \mathcal{E}_0^+ &:= \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} = \infty\}, \\ \mathcal{E}_0^- &:= \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} = -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty\}, \\ \mathcal{E}_0 &:= \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty\}.\end{aligned}$$

Note that $\hat{\mathcal{E}}$, $\mathcal{E}_0^-$, $\mathcal{E}_0^+$, $\mathcal{E}_0$ are invariant under the Bernoulli shift.

The set $\mathcal{E}$ becomes a totally disconnected compact metric space with the distance

$$d(e', e'') = \sum_{m \in \mathbb{Z}} \frac{|e'_m - e''_m|}{2^{|m|+1}}. \quad (87) \quad \boxed{\text{metric}}$$

Let $\mathcal{T} = (T_m)$, $m \in \mathbb{Z}$ be a fixed sequence of values satisfying (12). Following [6, §6] we set $\mathcal{T}^{(k)} = (T_{m+2k})$ for any $k \in \mathbb{Z}$. Further let us denote by

$$\sigma^k = \overbrace{\sigma \circ \dots \circ \sigma}^{k \text{ times}}.$$

Let $X(e, \mathcal{T})$ and $\Sigma(\mathcal{T})$ be the sets constructed via Theorem 2.6, for any $k \in \mathbb{Z}$ we introduce the sets

$$\Sigma_k := \left\{ \vec{\xi}_k = \vec{x}(T_k, T_0; \vec{\xi}_0) \mid \vec{\xi}_0 \in X(e, \mathcal{T}), e \in \mathcal{E} \right\}. \quad (88) \quad \boxed{\text{defSk}}$$

notunique

Remark 6.1 Note that we might have $\vec{\xi}_0 \neq \vec{\xi}'_0$, such that $\vec{\xi}_0, \vec{\xi}'_0 \in X(e, \mathcal{T})$, while in the classical case $X(e, \mathcal{T})$ is a singleton.

Notice that the set $\Sigma(\mathcal{T})$ constructed via Theorem 2.6 satisfies $\Sigma(\mathcal{T}) = \Sigma_0$, where Σ_0 is as in (88). Further, using (14) and (15) we get the following.

remtwo

Remark 6.2 Assume the hypotheses of Theorem 2.6, then for any $k \in \mathbb{Z}$ we find

$$\Sigma_{2k} \subset [B(\vec{\gamma}(0), \tilde{c}^* \varepsilon) \cup B(\vec{0}, \tilde{c}^* \varepsilon)].$$

Similarly in the setting of Corollary 2.9, if (10) holds for any $k \in \mathbb{Z}$ we find

$$\Sigma_{2k} \subset [B(\vec{\gamma}(0), \omega_\alpha(\varepsilon)) \cup B(\vec{0}, c^* \varepsilon)].$$

Finally in the setting of Corollary 2.9, if (7) holds for any $k \in \mathbb{Z}$ we find

$$\Sigma_{2k} \subset [B(\mathbf{\Gamma}^{\Lambda^1}, c^* \varepsilon) \cup B(\vec{0}, c^* \varepsilon)]$$

where $\mathbf{\Gamma}^{\Lambda^1} = \{\vec{\gamma}(t) \mid |t| \leq \Lambda^1\}$.

Further $\vec{x}(t, T_0; \vec{\xi}_0) \in B(\mathbf{\Gamma}, \sqrt{\varepsilon})$ for any $t \in \mathbb{R}$; hence from Remark 2.3 we see that local uniqueness and continuous dependence on initial data is ensured for any trajectory $\vec{x}(t, T_0; \vec{\xi}_0)$ such that $\vec{\xi}_0 \in \Sigma_0$. So we get the following.

remone

Remark 6.3 By construction, if $\vec{\xi}_0 \in \Sigma_0$ then $\vec{\xi}_k = \vec{x}(T_{2k}, T_0; \vec{\xi}_0) \in \Sigma_{2k}$ and we have

$$\vec{x}(t, T_0; \vec{\xi}_0) \equiv \vec{x}(t, T_{2k}; \vec{\xi}_k), \quad \text{for any } t \in \mathbb{R}. \quad (89)$$

identity

Viceversa, if $\vec{\xi}_k \in \Sigma_{2k}$ then $\vec{\xi}_0 = \vec{x}(T_0, T_{2k}; \vec{\xi}_k) \in \Sigma_0$ and (89) holds. Then $\vec{\xi}_0 \in X(\bar{e}, \mathcal{T})$ if and only if $\vec{\xi}_k \in X(\sigma^k(\bar{e}), \mathcal{T}^{(k)})$.

Let $\vec{\xi}_0 \in X(e, \mathcal{T})$, then we set $\Psi_0(\vec{\xi}_0) = e$, so that $\Psi_0 : \Sigma_0 \rightarrow \mathcal{E}$ is well defined and onto.

Similarly, let $\vec{\xi}_k \in \Sigma_{2k}$, then there is a uniquely defined $\vec{\xi}_0 \in \Sigma_0$ such that $\vec{x}(T_0, T_{2k}; \vec{\xi}_k) = \vec{\xi}_0$; further there is a uniquely defined $e \in \mathcal{E}$ such that $\vec{\xi}_0 \in X(e, \mathcal{T})$. Let $e^k = \sigma^k(e)$, then we set $\Psi_k(\vec{\xi}_k) = e^k$, so that $\Psi_k : \Sigma_{2k} \rightarrow \mathcal{E}$ is well defined.

Notice that Ψ_k is onto by construction but it might be not injective, see Remark 6.1.

Adapting [14, Proposition 6.4] we find the following

continuityine

Proposition 6.4 *Let the assumptions of Theorem 2.6 be satisfied, then the map $\Psi_k : \Sigma_{2k} \rightarrow \mathcal{E}$ is continuous.*

Now let us fix $\rho > 0$ small enough, independent of ε , so that in $B(\Gamma, \rho)$ no sliding phenomena may take place. Then from Remark 2.3 we see that for any $k \in \mathbb{Z}$ the function $F_k : B(\Gamma, \rho) \rightarrow \Omega$, $F_k(\vec{\xi}) = \vec{x}(T_{2k+2}, T_{2k}; \vec{\xi})$ is a homomorphism onto its image; however, notice that F_k is not a diffeomorphism, since the flow of (PS) is continuous in the domain but not smooth. Hence by construction $F_k : \Sigma_{2k} \rightarrow \Sigma_{2(k+1)}$ is a well-defined homeomorphism too; so adapting [14, Theorem 6.5] we find the following.

thm.bernoulli

Theorem 6.5 *Let the assumptions of Theorem 2.6 be satisfied. Then for any $0 < \varepsilon \leq \varepsilon_0$, we find*

$$\Psi_{k+1} \circ F_k(\vec{\xi}) = \sigma \circ \Psi_k(\vec{\xi}), \quad k \in \mathbb{Z}, \vec{\xi} \in \Sigma_{2k}, \quad (90)$$

eq.com

i.e., for all $k \in \mathbb{Z}$ the following diagram commutes:

$$\begin{array}{ccc} \Sigma_{2k} & \xrightarrow{F_k} & \Sigma_{2(k+1)} \\ \Psi_k \downarrow & & \downarrow \Psi_{k+1} \\ \mathcal{E} & \xrightarrow{\sigma} & \mathcal{E} \end{array}$$

Moreover, for all $k \in \mathbb{Z}$ the map Ψ_k is continuous and onto.

Proof We borrow the argument from the proof of Theorem 6.1 in [6].

Note that, by Proposition 6.4 the map Ψ_k is continuous and onto. Let us show that the diagram commutes. Let $\vec{\xi}_k \in \Sigma_{2k}$ and let $\vec{\xi}_{k+1} = F_k(\vec{\xi}_k) = \vec{x}(T_{2k+2}, T_{2k}; \vec{\xi}_k)$, and $\vec{\xi}_0 = \vec{x}(T_0, T_{2k}; \vec{\xi}_k) \in \Sigma_0$. Let us denote by $e = \Psi_0(\vec{\xi}_0)$, $e' = \Psi_k(\vec{\xi}_k)$, $e'' = \Psi_{k+1}(\vec{\xi}_{k+1})$; then by construction $e'' = \sigma^{k+1}(e)$ and $e' = \sigma^k(e)$ so that $e'' = \sigma(e')$. Hence

$$\Psi_{k+1} \circ F_k(\vec{\xi}_k) = \Psi_{k+1}(\vec{\xi}_{k+1}) = e'' = \sigma(e') = \sigma \circ \Psi_k(\vec{\xi}_k),$$

so the diagram commutes. $\square$

Now we briefly consider the setting of Theorem 2.14 following [14, §6]. In this case we can still define in a similar way as above the sets Σ_k and the mappings Ψ_k and F_k , where Ψ_k is onto and F_k is a homeomorphism. However Remark 6.2 is replaced by the following weaker result.

remthree

Remark 6.6 Assume the hypotheses of Theorem 2.14, then

$$\Sigma_{2k} \subset [B(\mathbf{\Gamma}^{B_{2k}}, \tilde{c}^* \varepsilon) \cup B(\vec{0}, \tilde{c}^* \varepsilon)] \subset B(\mathbf{\Gamma}, \tilde{c}^* \varepsilon), \quad \text{for any } k \in \mathbb{Z},$$

where $\mathbf{\Gamma}^{B_{2k}} = \{\vec{\gamma}(t) \mid |t| \leq B_{2k}\}$, $\tilde{c}^*$ is given by (86), Σ_k is given by (88) and $X(e, \mathcal{T}) = \{\vec{x}(T_0, T_0 + \alpha_0^e(\varepsilon); \vec{\xi}) \mid \vec{\xi} \in \bar{X}(e, \mathcal{T})\}$.

Notice that if (B_k) is unbounded, then the two sets $B(\mathbf{\Gamma}^{B_{2k}}, \tilde{c}^* \varepsilon)$ and $B(\vec{0}, \tilde{c}^* \varepsilon)$ may intersect, so Ψ_k might not be continuous.

In conclusion, Ψ_k is onto but possibly discontinuous, so we do not have a real semi-conjugation with the Bernoulli shift even if the diagram in Theorem 6.5 still commutes.

7 Examples

s.ex

In this section we present several examples of simple piecewise-smooth systems satisfying the assumptions of Theorems 2.6, 2.8 and 2.14: the examples are meant mainly to illustrate assumptions **P1**, **P2** or **P3**.

Let us consider the following system:

$$\begin{aligned} \dot{x} &= y + x^2 + \varepsilon g_1(t, x, y) & \text{if } y > 0, \\ \dot{y} &= x - 2xy - 2x^3 \\ \dot{x} &= y - x^2 + \varepsilon g_1(t, x, y) & \text{if } y < 0 \\ \dot{y} &= x + 2xy - 2x^3 \end{aligned} \tag{91}$$

eq_ex_ps

on $\Omega =]-2, 2[\times]-2, 2[$. We let $G(x, y) = -y$, so $\Omega^- =]-2, 2[\times]0, 2[$, $\Omega^+ =]-2, 2[\times]-2, 0[$, $\Omega^0 =]-2, 2[\times \{0\}$. Moreover, $\vec{g} = (g_1, g_2)$ with $g_2 \equiv 0$ and we will choose $g_1(t, x, y)$ so that $g \in C_b^r(\mathbb{R} \times \Omega, \mathbb{R}^2)$. Note that, if $\varepsilon = 0$, system (91) has the homoclinic solution (see Figure 5)

$$\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t)) = \begin{cases} (e^t, e^t - e^{2t}), & t \leq 0, \\ (e^{-t}, -e^{-t} + e^{-2t}), & t > 0 \end{cases} \tag{92}$$

eqexhom

corresponding to the homoclinic point $\vec{0}$. Note that for $t > 0$, we have

$$\begin{aligned} \gamma_1(-t) &= e^{-t} = \gamma_1(t), \\ \gamma_2(-t) &= e^{-t} - e^{-2t} = -\gamma_2(t). \end{aligned}$$

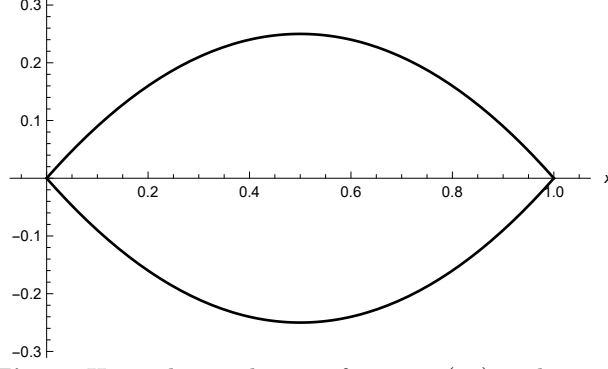


Fig. 5: Homoclinic solution of system (91) with $\varepsilon = 0$.

figEx-hom

Further, $\lambda_u^\pm = 1$, $\lambda_s^\pm = -1$, $\vec{v}_u^\pm = \mp \frac{1}{\sqrt{2}}(1, 1)$, $\vec{v}_s^\pm = \pm \frac{1}{\sqrt{2}}(1, -1)$, so **F0**, **F1**, **F2**, **K** are verified. Notice that $\underline{\sigma} = \bar{\sigma} = 1/2$. Since $\text{tr } \vec{f}_x^\pm(x, y) = 0$ for any $(x, y) \in \Omega^\pm$ and $c_\perp^\pm = 1$, the corresponding Melnikov function given by (2) has the form

$$\begin{aligned} \mathcal{M}(\theta) &= - \int_{-\infty}^0 (\gamma_1(t) - 2\gamma_1(t)\gamma_2(t) - 2\gamma_1^3(t)) g_1(t + \theta, \vec{\gamma}(t)) dt \\ &\quad - \int_0^{+\infty} (\gamma_1(t) + 2\gamma_1(t)\gamma_2(t) - 2\gamma_1^3(t)) g_1(t + \theta, \vec{\gamma}(t)) dt \\ &= \int_{-\infty}^0 (2e^{2t} - e^t) g_1(t + \theta, \vec{\gamma}(t)) dt + \int_0^{+\infty} (2e^{-2t} - e^{-t}) g_1(t + \theta, \vec{\gamma}(t)) dt. \end{aligned} \tag{93}$$

eq_ex_Meln

In the examples below the perturbation term $\vec{g} = (g_1, 0)$ will always be of the form $g_1(t, x, y) = xh(t)$ with different choices of $h(t)$, so that (93) reads

$$\begin{aligned} \mathcal{M}(\theta) &= \int_{-\infty}^0 (2e^{2t} - e^t) e^t h(t + \theta) dt + \int_0^{+\infty} (2e^{-2t} - e^{-t}) e^{-t} h(t + \theta) dt \\ &= 2A(3, \theta) - A(2, \theta) \end{aligned} \tag{94}$$

eq_ex_M

where we denote

$$A(k, \theta) = \int_{-\infty}^0 e^{kt} h(t + \theta) dt + \int_0^{\infty} e^{-kt} h(t + \theta) dt. \tag{95}$$

eq_ex_A

ex1

Example 7.1 In this first example we choose $g_1(t, x, y) = xh(t)$, $h(t) = \sin(2\pi t)$.

Plugging the given perturbation into (95), we get by direct computation

$$A(k, \theta) = \frac{2k}{4\pi^2 + k^2} \sin(2\pi\theta),$$

which gives $\mathcal{M}(\theta) = C_1 \sin(2\pi\theta)$ where $C_1 = \frac{8\pi^2+3}{(4\pi^2+9)(\pi^2+1)} > \frac{2}{13} > 0$. So in this case assumption **P3** is satisfied.

Observe that $\mathcal{M}(j/2) = 0$ and $|\mathcal{M}'(j/2)| = 2\pi C_1 > 0$ for any $j \in \mathbb{Z}$ so (11) holds. Therefore we can apply Theorem 2.6 to find an ε_0 such that we have the following. For any $0 < \varepsilon \leq \varepsilon_0$ and any $\nu \geq \nu_0 = 1$ (see (5)) we can define

$$\Theta = [K_0(1 + \nu)|\ln(\varepsilon)| + 2] = [3(1 + \nu)|\ln(\varepsilon)| + 2]$$

where $[\cdot]$ denotes the integer part; then we see that any sequence $\mathcal{T} = (T_j)$, $j \in \mathbb{Z}$ where $T_j = j\Theta$ verifies (12) (in fact also (13) since $b_j = -1/4 + j/2$ and $B_j = 1/2$).

So we can apply Theorem 2.6 and we find a set of initial conditions $\Sigma(\mathcal{T})$ such that $\vec{x}(t, T_0; \vec{Q})$ has property **C** for any $Q \in \Sigma(\mathcal{T})$.

Note that using a sequence of the form $\tilde{\mathcal{T}} = (\tilde{T}_k)$, $\tilde{T}_k = k\Theta + 1/2$, $k \in \mathbb{Z}$, we find a different chaotic pattern starting from a new set, say $\Sigma(\tilde{\mathcal{T}})$. Further, starting from sequences jumping between T_j and $\tilde{T}_j$ (even randomly), e.g. $\tilde{\mathcal{T}} = (\tilde{T}_k)$ where

$$\tilde{T}_k = \begin{cases} k\Theta & \text{if } k \text{ is odd,} \\ k\Theta + 1/2 & \text{if } k \text{ is even,} \end{cases}$$

we find a different chaotic pattern $\Sigma(\tilde{\mathcal{T}})$. Moreover we can choose also sequences with non constant gaps, e.g. we can allow $T_{j+1} - T_j$ to vary randomly in the semi-integers between Θ and 2Θ .

Analogously, one can verify the next example.

Example 7.2 System (91) with $g_1(t, x, y) = x(\sin t + \sin(2\pi t))$ satisfies **P2** with $\mathcal{M}(\theta) = \frac{2}{5} \sin \theta + C_1 \sin(2\pi\theta)$.

Notice that $\theta = 0$ is a non-degenerate zero of $\mathcal{M}$, further $\mathcal{M}$ is quasi periodic, hence almost periodic: it follows that in each quasi period $\tilde{N}$ (cf. **P2**) $\mathcal{M}$ admits a non degenerate zero. So we can find a sequence $\mathcal{T} = (T_j)$ of non-degenerate zeros of $\mathcal{M}$ such that $T_{j+1} - T_j \geq \Theta$, for any $j \in \mathbb{Z}$: thus (12) and (11) are satisfied and we can apply Theorem 2.6.

exgen0

Example 7.3 Let us consider system (91) with $g_1(t, x, y) = xh(t)$, and $h(t) = 3 \sin(2\pi t) + R(t)$, where $R(t)$ is a C^1 function such that $|R(t)| \leq 1$.

In this case $R(t)$ can be regarded as a noise or a disturbance on which we have little information. Repeating the previous computation we see that in this case we get

$$\mathcal{M}(\theta) = 3C_1 \sin(2\pi\theta) + M_r(t), \quad \text{where } |M_r(t)| \leq 1/3.$$

The estimate on $M_r(t)$ is obtained by plugging $|R(t)| \leq 1$ in (94). Note that $\mathcal{M}(\theta)$ admits at least one zero in each interval of length 1, since $3C_1 > 6/13 > 1/3$ but we have no information on the non-degeneracy of these zeros and in fact we do not know their precise locations. Further g , and hence $\mathcal{M}$ do not satisfy any classical recurrence property.

However (91) satisfies **P1** and also (7), so we deduce the existence of a sequence of zeros of $\mathcal{M}(\theta)$, say $\bar{T} = (\bar{T}_j)$ where $\bar{T}_j \in [\Theta j, \Theta j + 1[$, which satisfies (12) and (13), so that we can apply Theorem 2.8.

Hence for any $e \in \mathcal{E}$ we find (a not uniquely defined) $\vec{\xi}(e)$ and a sequence $\alpha_j^e = \alpha_j^e(\varepsilon)$ such that the trajectory $\vec{x}(t, T_0 + \alpha_0^e; \vec{\xi}(e))$ has property **C_e**, but we just know that $|\alpha_j^e(\varepsilon)| \leq 1$.

Finally notice that if we happen to know that $R(t)$ is such that $M_r(0) = M'_r(t) \equiv 0$ in an arbitrarily small neighborhood of the origin, then we can apply Remark 5.8 and we find that $\alpha_0^e = O(\varepsilon)$.

exgen

Example 7.4 Let us consider system (91) with $g_1(t, x, y) = xh(t)$, $h(t) = \text{sign}(t) \sin^{2r+1} \sqrt{|t|}$, $r \in \mathbb{Z}^+$, $r \geq 2$, where $\text{sign}(t) = 1$ if $t > 0$, $\text{sign}(t) = -1$ if $t < 0$, and $\text{sign}(t) = 0$ if $t = 0$.

It can be shown that $h \in C^r(\mathbb{R})$ so $\vec{g} \in C_b^r(\mathbb{R} \times \Omega, \mathbb{R}^2)$. Notice that h is odd, and consequently $\mathcal{M}$ is odd. So, in what follows, we assume that $\theta > 0$.

Once again plugging the given perturbation into (95), we get

$$\begin{aligned} A(k, \theta) = & - \int_{-\infty}^{-\theta} e^{kt} \sin^{2r+1} \sqrt{-(t+\theta)} dt + \int_{-\theta}^0 e^{kt} \sin^{2r+1} \sqrt{t+\theta} dt \\ & + \int_0^{\infty} e^{-kt} \sin^{2r+1} \sqrt{t+\theta} dt. \end{aligned}$$

Now, we estimate these three integrals:

$$\begin{aligned} I_1 = & - \int_{-\infty}^{-\theta} e^{kt} \sin^{2r+1} \sqrt{-(t+\theta)} dt = - \int_0^{\infty} e^{-k(t+\theta)} \sin^{2r+1} \sqrt{t} dt, \\ |I_1| \leq & e^{-k\theta} \int_0^{\infty} e^{-kt} dt \rightarrow 0 \end{aligned}$$

as $\theta \rightarrow \infty$. Next, integrating by parts, we derive

$$\begin{aligned} I_2 = & \int_{-\theta}^0 e^{kt} \sin^{2r+1} \sqrt{t+\theta} dt = 2 \int_0^{\sqrt{\theta}} t e^{k(t^2-\theta)} \sin^{2r+1} t dt \\ = & \frac{1}{k} \sin^{2r+1} \sqrt{\theta} - \frac{2r+1}{k} \int_0^{\sqrt{\theta}} e^{k(t^2-\theta)} \sin^{2r} t \cos t dt. \end{aligned}$$

Similarly,

$$\begin{aligned} I_3 &= \int_0^\infty e^{-kt} \sin^{2r+1} \sqrt{t+\theta} dt = 2 \int_{\sqrt{\theta}}^\infty t e^{-k(t^2-\theta)} \sin^{2r+1} t dt \\ &= \frac{1}{k} \sin^{2r+1} \sqrt{\theta} + \frac{2r+1}{k} \int_{\sqrt{\theta}}^\infty e^{-k(t^2-\theta)} \sin^{2r} t \cos t dt. \end{aligned}$$

So we have

$$\begin{aligned} A(k, \theta) &= - \int_0^\infty e^{-k(t+\theta)} \sin^{2r+1} \sqrt{t} dt + \frac{2}{k} \sin^{2r+1} \sqrt{\theta} \\ &+ \frac{2r+1}{k} \left[\int_{\sqrt{\theta}}^\infty e^{-k(t^2-\theta)} \sin^{2r} t \cos t dt - \int_0^{\sqrt{\theta}} e^{k(t^2-\theta)} \sin^{2r} t \cos t dt \right]. \end{aligned}$$

For the integrals in the bracket, we have the following estimates

$$\left| \int_{\sqrt{\theta}}^\infty e^{-k(t^2-\theta)} \sin^{2r} t \cos t dt \right| \leq e^{k\theta} \int_{\sqrt{\theta}}^\infty e^{-kt^2} dt,$$

where

$$\lim_{\theta \rightarrow \infty} \frac{\int_{\sqrt{\theta}}^\infty e^{-kt^2} dt}{e^{-k\theta}} = \lim_{\theta \rightarrow \infty} \frac{e^{-k\theta}}{2k\sqrt{\theta} e^{-k\theta}} = \lim_{\theta \rightarrow \infty} \frac{1}{2k\sqrt{\theta}} = 0$$

by de l'Hopital's rule, and analogously

$$\left| \int_0^{\sqrt{\theta}} e^{k(t^2-\theta)} \sin^{2r} t \cos t dt \right| \leq e^{-k\theta} \int_0^{\sqrt{\theta}} e^{kt^2} dt,$$

where

$$\lim_{\theta \rightarrow \infty} \frac{\int_0^{\sqrt{\theta}} e^{kt^2} dt}{e^{k\theta}} = \lim_{\theta \rightarrow \infty} \frac{e^{k\theta}}{2k\sqrt{\theta} e^{k\theta}} = \lim_{\theta \rightarrow \infty} \frac{1}{2k\sqrt{\theta}} = 0.$$

In conclusion, for $\theta > 0$ we find

$$\mathcal{M}(\theta) = \frac{1}{3} \sin^{2r+1} \sqrt{\theta} + R_1(\theta),$$

where $R_1(\theta) \rightarrow 0$ as $\theta \rightarrow +\infty$.

Since $\mathcal{M}$ is odd, it follows

$$\mathcal{M}(\theta) = \text{sign}(\theta) \frac{1}{3} \sin^{2r+1} \sqrt{|\theta|} + \text{sign}(\theta) R_1(|\theta|).$$

This means that the Melnikov function has a doubly infinite sequence of negative minima $\mathcal{M}(b_{2i})$ and positive maxima $\mathcal{M}(b_{2i+1})$ such that

$$\begin{aligned} \mathcal{M}(b_{2i}) &< -\frac{1}{6} < 0 < \frac{1}{6} < \mathcal{M}(b_{2i+1}), \quad i \in \mathbb{Z}, \\ b_j &= \text{sign}(2j-1) \left(\frac{\pi}{2} (2j-1) \right)^2, \quad j \in \mathbb{Z} \end{aligned} \tag{96}$$

MMadd

so assumption **P1** holds with $\bar{c} = 1/6$ (see also Remark 2.15). Moreover, note that the zeros are getting further and further away from each other as θ tends to $+\infty$ or $-\infty$.

In fact $\mathcal{M}(0) = 0$ since $\mathcal{M}$ is odd. Further we find

$$\mathcal{M}'(\theta) = 2A_\theta(3, \theta) - A_\theta(2, \theta),$$

$$A_\theta(k, \theta) = \int_{-\infty}^{+\infty} e^{-k|t|} h'(t + \theta) dt$$

for $A_\theta(k, \theta) = \frac{\partial}{\partial \theta} A(k, \theta)$, where we used the exponential behaviour of the integrand and the Lebesgue's dominated convergence theorem. So when $\theta = 0$ we find

$$A_\theta(k, 0) = 2(2r + 1) \int_0^{+\infty} e^{-ks^2} \sin^{2r} s \cos s ds,$$

$$\mathcal{M}'(0) = 2A_\theta(3, 0) - A_\theta(2, 0) \tag{97}$$

aprimo

taking into account that the integrand is even and we set $t = s^2$.

One can compute numerically $\mathcal{M}'(0)$ for some specific values of r ; for instance, we find the following:

$$\text{for } r = 2 : \mathcal{M}'(0) \cong 0.023;$$

$$\text{for } r = 3 : \mathcal{M}'(0) \cong -0.020.$$

Note that in both cases $\mathcal{M}'(0) \neq 0$. However in this case, (11) and even (8) are not satisfied because we do not have a uniform control of the slope of $\mathcal{M}(\theta)$ in its zeroes. So we can just apply Theorem 2.14, and we obtain a chaotic pattern similar to the one of Example 7.1. That is, for any sequence of zeros of $\mathcal{M}(\cdot)$, say $\mathcal{T} = (T_j)$, $j \in \mathbb{Z}$, such that $T_0 = 0$, $T_{j+1} - T_j \geq \Theta$, we obtain a set $\Sigma(\mathcal{T})$ such that for any $e \in \mathcal{E}$ there is a $\tilde{\xi}(e) \in \Sigma(\mathcal{T})$, and a sequence $(\alpha_j^e(\varepsilon))$, $j \in \mathbb{Z}$ such that $\vec{x}(t, T_0 + \alpha_0^e(\varepsilon); \tilde{\xi}(e))$ has property $\mathbf{C}_e$.

Further $\mathcal{M}'(T_0) \neq 0$, so we can apply Remark 5.8 to find $|\alpha_0^e(\varepsilon)| \leq C_\alpha \varepsilon$. However for $k \neq 0$ we just have the a priori estimate

$$|\alpha_k^e(\varepsilon)| \leq B_{2k} = b_{n_{2k}+1} - b_{n_{2k}} = \begin{cases} 2\pi^2 |n_{2k}|, & n_{2k} \in \mathbb{Z} \setminus \{0\}, \\ \frac{\pi^2}{2}, & n_{2k} = 0. \end{cases} \tag{98}$$

aBk

As far as we are aware, even in the smooth case, it was not possible to apply the results available in literature to this type of Melnikov functions (with distance between consecutive zeros going to infinity).

We can go even further with a more general aperiodic perturbation satisfying **P1**:

exgen2

Example 7.5 Let us consider system (91) with $g_1(t, x, y) = xh(t)$ and odd function $h \in C^r(\mathbb{R})$ such that $|h(t)| \leq H_0$, $|h'(t)| \leq H_1$ for all $t \in \mathbb{R}$ and some constants $H_0, H_1 > 0$; further we assume that there is a sequence $(\theta_j)_{j \in \mathbb{N}}$, $\theta_j > 0$ for all $j \in \mathbb{N}$, $\theta_j \rightarrow +\infty$ as $j \rightarrow +\infty$ such that $h(\theta_{2j}) < -c$, $h(\theta_{2j+1}) > c$ for some $c > 0$ and $h'(t) \rightarrow 0$ as $t \rightarrow +\infty$.

As before we find that

$$\mathcal{M}(\theta) = 2A(3, \theta) - A(2, \theta),$$

$$\mathcal{M}'(\theta) = 2A_\theta(3, \theta) - A_\theta(2, \theta).$$

(99) Mprimo

We assume firstly $\theta > 0$ and we investigate the integrals:

$$\left| \int_{-\infty}^{-\theta} e^{kt} h(-(t+\theta)) dt \right| \leq \int_0^\infty e^{-k(t+\theta)} |h(t)| dt \leq H_0 e^{-k\theta} \int_0^\infty e^{-kt} dt$$

which tends to 0 as $\theta \rightarrow \infty$. Integrating by parts we get

$$\begin{aligned} \int_{-\theta}^0 e^{kt} h(t+\theta) dt &= \int_0^\theta e^{k(t-\theta)} h(t) dt \\ &= \frac{h(\theta)}{k} - \frac{e^{-k\theta} h(0)}{k} - \frac{e^{-k\theta}}{k} \int_0^\theta e^{kt} h'(t) dt. \end{aligned}$$

Here if $e^{kt} h'(t) \in L^1(0, \infty)$, we can find $K > 0$ such that

$$\left| \frac{e^{-k\theta} h(0)}{k} + \frac{e^{-k\theta}}{k} \int_0^\theta e^{kt} h'(t) dt \right| \leq \frac{K}{k} e^{-k\theta};$$

otherwise, if $e^{kt} h'(t) \notin L^1(0, \infty)$, using de l'Hopital's rule, as $\theta \rightarrow \infty$ we find

$$\frac{e^{-k\theta} h(0)}{k} + \frac{e^{-k\theta}}{k} \int_0^\theta e^{kt} h'(t) dt \sim \frac{h(0)}{k} e^{-k\theta} + \frac{h'(\theta)}{k^2}.$$

Finally similarly,

$$\int_0^\infty e^{-kt} h(t+\theta) dt = \int_\theta^\infty e^{-k(t-\theta)} h(t) dt = \frac{h(\theta)}{k} + \frac{e^{k\theta}}{k} \int_\theta^\infty e^{-kt} h'(t) dt$$

with, for θ large enough,

$$\left| \frac{e^{k\theta}}{k} \int_\theta^\infty e^{-kt} h'(t) dt \right| \leq 2 \frac{|h'(\theta)|}{k^2}.$$

In conclusion,

$$A(k, \theta) = \frac{2h(\theta)}{k} + R(\theta, k), \quad |R(\theta, k)| \leq \frac{\bar{K} e^{-k\theta}}{k} + \frac{3|h'(\theta)|}{k^2}$$

for $\theta > 0$ large and a suitable $\bar{K} > 0$. Hence, using the fact that $\mathcal{M}$ is odd, and since by assumption $h'(\theta) \rightarrow 0$ as $\theta \rightarrow +\infty$, we obtain

$$\mathcal{M}(\theta) = \frac{h(\theta)}{3} + R(\theta), \quad |R(\theta)| \rightarrow 0 \text{ as } \theta \rightarrow +\infty.$$

This means that $\mathcal{M}(\theta_{2j}) < -\frac{\varepsilon}{6} < 0$, $\mathcal{M}(\theta_{2j+1}) > \frac{\varepsilon}{6} > 0$ and $\mathcal{M}(-\theta_{2j}) > \frac{\varepsilon}{6} > 0$, $\mathcal{M}(-\theta_{2j+1}) < -\frac{\varepsilon}{6} < 0$ if $|j|$ is large enough, so assumption **P1** is fulfilled. Further from the assumptions on h it follows that the distance between consecutive zeros of $\mathcal{M}$ is unbounded.

Hence we find a sequence of zeros of $\mathcal{M}(\cdot)$, say $\mathcal{T} = (T_j)$, such that $T_0 = 0$, $T_{j+1} - T_j \geq \Theta$, but we do not have control on $\mathcal{M}'(T_k)$ so we can just apply Theorem 2.14 and we obtain results analogous to the ones of Example 7.4, in particular we just find $|\alpha_k| \leq B_{2k}$ and (B_{2k}) may become unbounded. Notice that in this result the function h can again be regarded also as some kind of noise on which we have no information but the one given in the statement.

However, if we assume $\mathcal{M}'(0) \neq 0$ (which is in fact an assumption on h , see (99) and (97)) we can apply Remark 5.8 and we find $\alpha_0(\varepsilon) = O(\varepsilon)$.

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