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Stress boundedness and existence of radial minimizers in constrained nonlinear elasticity

Paolo Maria Mariano [a,*], Domenico Mucci [b]

[a] DICEA, Università di Firenze, via Santa Marta 3, Firenze, I-50139, Italy
[b] DSMFI, Università di Parma, Parco Area delle Scienze 53/A, Parma, I-43100, Italy

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ABSTRACT

For simple elastic bodies in small strain regime, the convexity of the energy allows one to discuss equilibrium problems under stress constraints (the specification of an admissible convex region in the stress space) in terms of the complementary energy. In the presence of large strains, the necessary lack of energy convexity does not allow one to retrace the same path. A significant concept of complementary energy in large strain regime rests on the Legendre transform of the energy with respect to the deformation gradient, its cofactor and determinant. The related minimum problem necessarily requires that constraints be assigned to the derivatives of the energy density with respect to the variables already listed (these derivatives are required to be in a convex subset of an appropriate linear space). A problem is to characterize the related stresses in terms of a constrained energy. We tackle this problem for radially symmetric simple bodies under radial deformations and show how the resulting stress is bounded. We also prove the existence of radially symmetric minimizers for the constrained elastic energy under Dirichlet boundary conditions.

1. Introduction

Real materials sustain stress until a certain threshold beyond which there is failure. But thresholds may also be referred to other phase transitions such as the one from rigid to plastic behavior, from elastic to plastic, etc. Generically, a stress threshold is defined by conditions $\mathfrak{f}(\boldsymbol{\sigma}) \le 0$ or $\mathfrak{f}(P) \le 0$, where $\boldsymbol{\sigma}$ is the Cauchy stress and P the first Piola-Kirchhoff one, while $\mathfrak{f}$ is a real-valued function.

In a common description of elastoplastic processes, when $\mathfrak{f}$ is differentiable, the Clausius-Duhem inequality and the acceptance of a maximum dissipation principle impose that the admissibility region be convex (Simo and Hughes, 1998). When $\mathfrak{f}$ is not so or at least is piecewise differentiable, as in the case of Tresca's criterion, the convexity of the admissibility region is commonly postulated a priori (Eve et al., 1990; He et al., 2005). For materials with elastic range (see, e.g., Lucchesi et al., 1992; Mariano, 1998 and references therein), a problem is to find energy minimizers with associated stress in the admissibility region. In the small strain regime, the analysis reduces to a saddle point problem that involves the complementary energy. Its density is the Legendre transform of the elastic energy with respect to the small strain tensor $\boldsymbol{\varepsilon}$. Related existence theorems have been proven in Anzellotti (1983), Anzellotti and Giaquinta (1980, 1982).

In the presence of large strains, the energy depends on the deformation gradient F in addition to possibly the place x; it cannot be convex with respect to F because convexity is physically incompatible with the objectivity of energy, that is, with the requirement of invariance under changes of observers of rigid-body type in the physical space (Coleman and Noll, 1959). From an analytical point

* Corresponding author.
E-mail addresses: paolomaria.mariano@unifi.it (P.M. Mariano), domenico.mucci@unipr.it (D. Mucci).

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of view, it would be convenient to choose the energy as a quasiconvex function of F. In this case, however, we could not ensure that the energy minimizers preserve the orientation of the space, that is, they do not necessarily satisfy the physical constraint $\det F > 0$. Thus, to ensure positiveness of the determinant of F, a natural choice is to consider the energy density W to be a polyconvex function of F, that is,

$$W(F) = e(F, \operatorname{cof} F, \det F),$$

where $e(\cdot)$ is a convex function of the triple $(F, \operatorname{cof} F, \det F)$ (Ball, 1976), and $\operatorname{cof} F$ is the cofactor of F.

To construct a counterpart of the minimum problem discussed in small strain ambient setting in Anzellotti (1983), Anzellotti and Giaquinta (1980, 1982), since a polyconvex energy density is a convex function of all minors of F, we essentially need to impose constraints on the triplet of derivatives

$$\left(\frac{\partial e}{\partial F}, \frac{\partial e}{\partial \operatorname{cof} F}, \frac{\partial e}{\partial \det F} \right),$$

prescribing that this triplet lies in a convex set of the pertinent tensor space, as shown in Giaquinta et al. (2015), where the existence of minimizers for the pertinent minimum energy problem has been proven under Dirichlet boundary conditions.

In fact, the partial derivatives of e reconstruct the first Piola-Kirchhoff stress, whose properties must be further specified in terms of the constrained energy to be determined.

In what follows, we tackle this problem for radially symmetric simple bodies under radial deformations, determine the constrained energy, and prove that the related stress is bounded. In addition, we prove the existence of radially symmetric minimizers of the constrained energy under Dirichlet boundary conditions. Our work differs from the existential analysis of radial minimizers of the unconstrained elastic energy presented in Stuart (1985, 1993) in two essential aspects: (1) we consider the above mentioned constraints unlike the quoted papers; (2) we thus adopt a different technique in the proof.

The paper is organized as sketched below. In Section 2 we collect a few notations. In Section 3 we present an overview of the problem considered. In Section 4 we discuss properties of constrained elastic energies of radially symmetric bodies in the small strain regime; in particular, we show that if we constrain the small strain tensor ε to lie in the convex region, the corresponding set of bounded stresses is not necessarily convex. Then, we move to the large strain regime in Section 5 and analyze the connection between constraints on the energy derivatives and the set of bounded stresses. In Section 6 we prove the existence of radial minimizers of the elastic energy subject to stress constraints. Final remarks are given in Section 7.

2. Notations

An interposed dot, $\bullet$, will indicate the scalar product in $\mathbb{R}^N$, with $N = 1, 2, \ldots$. The symbol $\mathbf{1}_C$ will indicate the indicator function of a set $C \subset \mathbb{R}^N$, so that $\mathbf{1}_C(\tau) = 0$ if $\tau \in C$ and $\mathbf{1}_C(\tau) = +\infty$ if $\tau \in \mathbb{R}^N \setminus C$. We will often be interested in the case $N \geq 2$.

$B^N_{\hat{R}}$ will indicate the open unit ball of radius $\hat{R}$ centered at the origin.

With $\mathcal{U} \subset \mathbb{R}^N$ a convex set and $f : \mathcal{U} \longrightarrow \mathbb{R}$ a strictly convex function, for $\omega \in \mathbb{R}^N$ the symbol f^* will indicate the *conjugate* of f, given by

$$f^*(\tau) = \sup_{x \in A} \{\tau \bullet x - f(x)\}.$$

For simplicity, we refer the analysis to orthonormal frames of reference, so that we will identify covariant and contravariant components of the tensors involved.

3. An overview on the scenario considered

The general framework we consider deals with the energy

$$\mathcal{E}(u) = \int_{B^N} W(Du(x))\, dx \tag{3.1}$$

associated with sufficiently smooth deformations $u : B^N \to \mathbb{R}^N$ given by

$$u(x) = v(|x|)\, \frac{x}{|x|}, \qquad 0 < |x| < 1, \tag{3.2}$$

where B^N is the open unit ball in $\mathbb{R}^N$, centered at the origin and subject to the boundary displacement condition $u(x) = \lambda x$ for $|x| = 1$ with $\lambda \geq 1$. The ball B^N is considered embedded in $\bar{\mathbb{R}}^N$, an identical copy of the N-dimensional physical space $\mathbb{R}^N$.

The radially symmetric deformation u given by (3.2) is uniquely determined by its *profile function* $v : (0,1) \to \mathbb{R}$, which we assume to be strictly increasing and satisfying $0 = v(0) < v(1) = \lambda$ where, here and in the sequel, $v(0) = v(0^+)$ and $v(1) = v(1^-)$ are the right and left limits at the end-points of the interval $(0, 1)$.

The strain energy density is an isotropic function $W : \mathbb{M}^{N\times N}_+ \to \mathbb{R}$ defined in the set $\mathbb{M}^{N\times N}_+$ of real valued $N \times N$ matrices with positive determinant such that $W(A) \to +\infty$ as $\det A \to 0^+$ or $\det A \to +\infty$. Therefore, W can be written as a symmetric function of the singular values $\lambda_n(A)$ $(n = 1, \ldots, N)$ of A, that is,

$$W(A) = \Phi(\lambda_1(A), \cdots, \lambda_N(A)), \qquad A \in \mathbb{M}^{N\times N}_+,$$

where Φ is a real-valued function invariant under interchange of the entries and defined on the set of vectors of $\mathbb{R}^N$ with positive components and, for profile functions v as above, the corresponding deformations u in (3.2) are differentiable almost everywhere in B^N with gradients $Du(x)$ given by

$$Du(x) = \frac{v(|x|)}{|x|}\mathbb{I}_N + \left(v'(|x|) - \frac{v(|x|)}{|x|}\right)\frac{x \otimes x}{|x|^2} \qquad \text{for a.e. } x \in B^N \tag{3.3}$$

(see Ball, 1982 or Stuart, 1985), where $\mathbb{I}_N$ is the identity matrix $N \times N$, and with singular values or principal stretches of $Du(x)$ given for a.e. $x \in B^N$ by

$$\lambda_1(Du(x)) = v'(|x|) \quad \text{and} \quad \lambda_n(Du(x)) = \frac{v(|x|)}{|x|}, \qquad n = 2, \dots, N, \tag{3.4}$$

so that

$$W(Du(x)) = \Phi\big(v'(x), v(|x|)/|x|, \cdots, v(|x|)/|x|\big) \qquad \text{for a.e. } x \in B^N. \tag{3.5}$$

More precisely, in view of (3.4), setting

$$\xi := \lambda_1, \quad \eta := \lambda_n, \quad n = 2, \dots, N, \quad \text{and} \quad \Phi(\xi, \eta) := \Phi(\lambda_1, \lambda_2, \cdots, \lambda_N),$$

for simplicity, by a change of variable referring to the radial coordinate r, we obtain

$$\mathcal{E}(u) = \mathcal{F}(v) := \sigma_N \int_0^1 r^{N-1}\Phi\big(v'(r), v(r)/r\big)\, dr \tag{3.6}$$

where σ_N is the $(N-1)$-dimensional measure of the unit sphere in $\mathbb{R}^N$. Dimensions of strict physical interest are typically $1, 2, 3$. Thus, in what follows, we focus on $N = 3$. In this case, the area of the graph of u is given by

$$\mathfrak{A}(u) = \int_{B^3} \left\{1 + |\nabla u|^2 + |\text{cof } Du|^2 + (\det Du)^2\right\}^{1/2} dx\,,$$

where, as already mentioned, cof F is the cofactor of $F \in \mathbb{R}^{3\times 3}$. In the radially symmetric case, we have the identities

$$\begin{array}{ll} |Du| = J_1(v'(r), v(r)/r)\,, & J_1(\xi, \eta) := \sqrt{\xi^2 + 2\eta^2}\,, \\ |\text{cof } Du| = J_2(v'(r), v(r)/r)\,, & J_2(\xi, \eta) := \sqrt{2\xi^2\eta^2 + \eta^4}\,, \\ \det Du = J_3(v'(r), v(r)/r)\,, & J_3(\xi, \eta) := \xi\,\eta^2\,. \end{array} \tag{3.7}$$

Therefore, using that

$$1 + \sum_{\ell=1}^{3} J_\ell(\xi, \eta)^2 = (1 + \eta^2)^2(1 + \xi^2)\,,$$

by the area formula we obtain the equivalent expression

$$\mathfrak{A}(u) = 4\pi \int_0^1 r^2\left\{1 + \sum_{\ell=1}^{3} J_\ell(v'(r), v(r)/r)^2\right\}^{1/2} dr = 4\pi \int_0^1 (r^2 + v(r)^2)\sqrt{1 + v'(r)^2}\, dr\,. \tag{3.8}$$

Examples of concrete elastic energies that refer to the case analyzed here are listed below (Ericksen, 1998). They refer to a sphere with a reference radius $\hat{R}$ and an after-deformation radius $\hat{r} := \tilde{r}(\hat{R})$.

- Neo-Hoookean materials: the total energy of the ball is given by

  $$\mathcal{E}(\lambda) = a(2\lambda^2 + \lambda^{-4} - 3)\,,$$

  with $a > 0$ and $\lambda = \frac{\hat{r}}{\hat{R}}$.

- Mooney-Rivlin materials: the total energy of the ball is given by

  $$\mathcal{E}(\lambda) = a\left(2\lambda^2 + \lambda^{-4} - 3 + \frac{2\lambda^{-2} + \lambda^4 - 3}{k}\right),$$

  with $4 \le k \le 8$.

4. Stress constraints in small strain regime

In this section, the analysis is not restricted to radial symmetry. The results establish a comparison context with what is discussed in the large strain regime in the rest of this paper.

As is well-known, the small strain regime is defined by the condition $|Du| \ll 1$, where $|\cdot|$ indicates the Frobenius norm. In this setting, reference and current configurations are not distinguished and the first Piola-Kirchhoff stress approximates the Cauchy stress to the leading terms. The energy density is a convex function of the small strain deformation tensor $\varepsilon := \text{sym}Du(x)$.

We perform the Euler variations $u(x) + \epsilon\, \phi(x)$ of the energy functional

$$\mathcal{E}(u) = \int_{B^3} f(\varepsilon)\, dx \tag{4.1}$$

with test functions $\phi \in C_c^{\infty}(B^3, \mathbb{R}^3)$, that is, smooth functions compactly supported in B^3; they vanish at the ball boundary ∂B^3. We thus obtain

$$\delta\mathcal{E} = \int_{B^3} \sum_{i,j=1}^{3} \boldsymbol{\sigma}(\varepsilon)_i^j\, (\mathrm{sym} D\phi)_i^j\, dx$$

where summation over repeated indices is left understood here and in what follows, and the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} := Df(\varepsilon),$$

where D indicates here the derivative with respect to the argument.

We now assume that ε belongs to a convex set in the space $\mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3)$ of linear symmetric operators. Then, we look at the geometric structure of the space containing bounded stresses that correspond to these strains. Regarding notation, in what follows B_K^{Sym} will indicate the ball with radius $K \in \mathbb{R}^+$ in the linear space $\mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3)$.

Proposition 1. *Let $f : \mathcal{U} \to \mathbb{R}$ be a strictly convex function of class C^2 defined on a convex set $\mathcal{U} \subset \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3)$, and assume that the gradient function $Df : \mathcal{U} \to \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3)$ is surjective. For any $K > 0$, let $f_K : \mathcal{U} \to \overline{\mathbb{R}}$ be the conjugate of the function $\tau \mapsto f^*(\tau) + \mathbf{1}_{B_K^{Sym}}(\tau)$, where $f^* : \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3) \to \overline{\mathbb{R}}$ is the conjugate function to f. Then $f_K(\varepsilon) = f(\varepsilon)$ for every $\varepsilon \in \bar{\mathcal{U}}_K$, where*

$$\mathcal{U}_K = \{\varepsilon \in \mathcal{U} \;:\; |Df(\varepsilon)| < K\}.$$

Furthermore, if $\varepsilon \notin \bar{\mathcal{U}}_K$, there is a unique point $\varepsilon_K \in \partial\mathcal{U}_K$ such that

$$\varepsilon - \varepsilon_K = \frac{|\varepsilon - \varepsilon_K|}{K}\, Df(\varepsilon_K),$$

and we have

$$f_K(\varepsilon) = f(\varepsilon_K) + K\, |\varepsilon - \varepsilon_K|.$$

Proof. Since f is smooth and strictly convex, its gradient function is injective on $\mathcal{U}$. As a consequence, $\partial\mathcal{U}_K \subset \mathcal{U}$ and hence the open set $\mathcal{U}_K$ is strictly contained in $\mathcal{U}$. Moreover, since the gradient of $|Df|^2$ never vanishes on $\partial\mathcal{U}_K = \{\varepsilon \in \mathcal{U} \;:\; |Df(\varepsilon)| = K\}$, by Dini's theorem $\mathcal{U}_K$ is a smooth set topologically equivalent to B_K^{Sym}. Finally, denoting by $\hat{n}(\varepsilon)$ the outward unit normal to $\mathcal{U}_K$, since the Hessian matrix $D^2 f$ is positive semidefinite everywhere, it turns out that $\hat{n}(\varepsilon) \bullet Df(x) \geq 0$ for every $\varepsilon \in \partial\mathcal{U}_K$. Of course, being B_K^{Sym} a ball in the tensor space of second-rank tensors, $\hat{n}(\varepsilon)$ is a second-rank symmetric tensor too.

Recalling that for every $\boldsymbol{\sigma} \in \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3)$

$$f^*(\boldsymbol{\sigma}) = \sup_{\varepsilon \in \mathcal{U}} \{\boldsymbol{\sigma} \bullet \varepsilon - f(\varepsilon)\}$$

we clearly have

$$f^*(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \bullet g(\boldsymbol{\sigma}) - f(g(\boldsymbol{\sigma})) \quad \forall\, \boldsymbol{\sigma} \in \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3),$$

where $g : \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3) \to \mathcal{U}$ is the inverse of the gradient function Df. Moreover,

$$\nabla f^*(\boldsymbol{\sigma}) = g(\boldsymbol{\sigma}) + Dg(\boldsymbol{\sigma})\,(\boldsymbol{\sigma} - Df(g(\boldsymbol{\sigma})) = g(\boldsymbol{\sigma}) \quad \forall\, \boldsymbol{\sigma} \in \mathrm{Sym}(\mathbb{R}^3, \mathbb{R}^3).$$

Therefore, for any $\varepsilon \in \mathcal{U}$ we can write

$$f_K(\varepsilon) = \sup_{\boldsymbol{\sigma} \in B_K^{Sym}} \left\{\varepsilon \bullet \boldsymbol{\sigma} - f^*(\boldsymbol{\sigma})\right\}.$$

We first consider the maximum of the function $h_\varepsilon(\boldsymbol{\sigma}) = \varepsilon \bullet \boldsymbol{\sigma} - f^*(\boldsymbol{\sigma})$ that is constrained to the set $|\boldsymbol{\sigma}|^2 = K^2$. Since $Dh_\varepsilon(\boldsymbol{\sigma}) = \varepsilon - g(\boldsymbol{\sigma})$, by the Lagrange multipliers rule we have to solve the system

$$\varepsilon - g(\boldsymbol{\sigma}) = \lambda\, \boldsymbol{\sigma}, \quad |\boldsymbol{\sigma}|^2 = K^2.$$

Of course, the Lagrange multiplier is intended as endowed with dimensions such that the above equation is dimensionless. With $\bar{\varepsilon} = g(\boldsymbol{\sigma})$, the above system is equivalent to

$$\varepsilon - \bar{\varepsilon} = \lambda\, Df(\bar{\varepsilon}), \quad \bar{\varepsilon} \in \partial\mathcal{U}_K. \tag{4.2}$$

If $\varepsilon \in \mathcal{U}_K$, for any solution $\bar{\varepsilon} \in \partial\mathcal{U}_K$ we obtain

$$h_\varepsilon(\bar{\varepsilon}) = f(\bar{\varepsilon}) + Df(\bar{\varepsilon}) \bullet (\varepsilon - \bar{\varepsilon}).$$

However, h_ε has one critical point in B_K^{Sym} at $\boldsymbol{\sigma} = Df(\varepsilon)$, so that by the convexity of f we conclude that $f_K(\varepsilon) = f(\varepsilon)$. By continuity, equation $f_K(\varepsilon) = f(\varepsilon)$ holds true for every $\varepsilon \in \bar{\mathcal{U}}_K$.

If $\varepsilon \notin \bar{\mathcal{U}}_K$, we find two solutions $\bar{\varepsilon}^{\pm} \in \partial\mathcal{U}_K$ of the system (4.2), with the corresponding multipliers $\lambda^+ > 0$ and $\lambda^- < 0$. The maximum is attained in correspondence to the positive multiplier, and setting $\boldsymbol{\varepsilon}_K = \bar{\varepsilon}^+$, the maximum point is $\boldsymbol{\sigma}_{\varepsilon,K} = \nabla f(\boldsymbol{\varepsilon}_K)$ and we have

$$f_K(\boldsymbol{\varepsilon}) = h(\boldsymbol{\sigma}_{\varepsilon,K}) = \boldsymbol{\varepsilon} \bullet \boldsymbol{\sigma}_{\varepsilon,K} + f(g(\boldsymbol{\sigma}_{\varepsilon,K})) - \boldsymbol{\sigma}_{\varepsilon,K} \bullet g(\boldsymbol{\sigma}_{\varepsilon,K}).$$

Using $g(\boldsymbol{\sigma}_{\varepsilon,K}) = \boldsymbol{\varepsilon}_K$, we obtain

$$f_K(\boldsymbol{\varepsilon}) = f(\boldsymbol{\varepsilon}_K) + Df(\boldsymbol{\varepsilon}_K) \bullet (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_K)$$

and the assertion follows since $\lambda^+ = K^{-1}|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_K|$ and hence $(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_K) = K^{-1}|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_K|\, \nabla f(\boldsymbol{\varepsilon}_K)$. $\square$

Proposition 2. *Under the previous hypotheses, we have*

$$Df_K(\boldsymbol{\varepsilon}) = K\, \frac{\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_K}{|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_K|} \quad \in \boldsymbol{\varepsilon} \notin \bar{\mathcal{U}}_K .$$

Therefore, we have $\sup_{\mathcal{U}} |Df_K| \le K$.

Proof. Let $h_K : \mathcal{U} \to \bar{\mathcal{U}}_K$ be the smooth function such that

$$f_K(\boldsymbol{\varepsilon}) = f(h_K(\boldsymbol{\varepsilon})) + K\,|\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})|\,, \quad \boldsymbol{\varepsilon} \in \mathcal{U}\,,$$

so that $h_K(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}$ if $\boldsymbol{\varepsilon} \in \bar{\mathcal{U}}_K$. With the previous notation, for $\boldsymbol{\varepsilon} \notin \bar{\mathcal{U}}_K$ we have

$$\begin{aligned} Df_K(\boldsymbol{\varepsilon}) = {} & Df(h_K(\boldsymbol{\varepsilon}))Dh_K(\boldsymbol{\varepsilon}) + K\, \frac{\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})}{|\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})|} \big(\mathbb{I} - Dh_K(\boldsymbol{\varepsilon})\big) \\ = {} & K\, \frac{\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})}{|\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})|} + \left(Df(h_K(\boldsymbol{\varepsilon})) - K\, \frac{\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})}{|\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})|} \right) Dh_K(\boldsymbol{\varepsilon}) \end{aligned}$$

where $\mathbb{I}$ is the identity in the linear space of fourth-rank tensors, and the second-rank tensor $h_K(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}_K \in \bar{\mathcal{U}}_K$ satisfies the equation

$$\frac{\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})}{|\boldsymbol{\varepsilon} - h_K(\boldsymbol{\varepsilon})|} = \frac{1}{K}\, Df(h_K(\boldsymbol{\varepsilon}))\,.$$

The assertion follows. $\square$

Indeed, the same conclusions hold without assuming the surjectivity of the gradient function Df at the cost of increasing the complexity of the proof. The set $\mathcal{U}_K$ is not convex, in general.

Thus, from the convex energy density $f(\boldsymbol{\varepsilon})$ we can obtain an energy density $f_K(\boldsymbol{\varepsilon})$ whose gradient (that is, the Cauchy stress) is bounded by the given positive constant K.

5. Constraints in large strain regime

As usual, F will indicate $Du(x)$. In the large strain regime, the convexity of the energy density with respect to F is not compatible with the objectivity of the energy, that is, the invariance under changes of observers of rigid-body type. Then, we assume polyconvexity, that is, we consider a structure of the energy given by

$$W(F) = e(F, \operatorname{cof} F, \det F)\,, \quad F \in \mathbb{M}^{3\times 3}_+\,,$$

where e is a convex function of the triplet $(F, \operatorname{cof} F, \det F)$. Thus, the first Piola-Kirchhoff stress P is given by

$$P = \frac{\mathrm{d}W(F)}{\mathrm{d}F} = \frac{\partial e}{\partial F} + \frac{\partial e}{\partial \operatorname{cof} F}\frac{\partial \operatorname{cof} F}{\partial F} + \frac{\partial e}{\partial \det F}\frac{\partial \det F}{\partial F}\,, \tag{5.1}$$

where

$$\frac{\partial \det F}{\partial F} = \operatorname{cof} F\,.$$

The approach presented in Giaquinta et al. (2015) consists in prescribing constraints not directly on P, but on the coefficients

$$\frac{\partial e}{\partial F}\,, \quad \frac{\partial e}{\partial \operatorname{cof} F}\,, \quad \frac{\partial e}{\partial \det F}\,.$$

However, this time a bound on those coefficients in terms of a positive constant K does not imply a bound on the norm of P, due, for example, to the presence of $\operatorname{cof} Du$ in the third addendum of the expression of P from (5.1).

We follow a similar approach in the case of radially symmetric simple bodies, so that we consider the energy $\mathcal{F}(v)$ in (3.6), where $N = 3$. We therefore set $\phi(x) = \varphi(|x|)\,\frac{x}{|x|}$, where $\varphi \in C^\infty_c((0,1))$ is strictly increasing, and Euler variations are given by $v(r) + \epsilon\,\varphi(r)$. We correspondingly have

$$\delta\mathcal{F} = 4\pi \int_0^1 r^2 \left[\partial_\xi \Phi\big(v'(r), v(r)/r\big)\, \varphi'(r) + \partial_\eta \Phi\big(v'(r), v(r)/r\big)\, \varphi(r)/r\right] dr\,.$$

Therefore, based on Eq. (3.3) for $D\Phi(x)$, it turns out that the role of the Piola-Kirchhoff stress tensor is played by the vector

$$T_R(r, v(r), v'(r)) := r^2 \big(\partial_\xi \Phi\big(v'(r), v(r)/r\big),\, \partial_\eta \Phi\big(v'(r), v(r)/r\big)\big)\,, \quad r \in (0,1)\,,$$

that we may call *radially symmetric Piola-Kirchhoff stress vector*.

Therefore, we let $\mathsf{A} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0, +\infty)$. If the energy density of a radially symmetric convex body is polyconvex, we can write

$$\Phi(\xi,\eta) = f(J_1(\xi,\eta), J_2(\xi,\eta), J_3(\xi,\eta)),$$

where $f : \mathsf{A} \to \mathbb{R}^+$ is separately convex in each variable, and the terms $J_\ell(\xi,\eta)$ are given by formula (3.7).

For the sake of simplicity, we now assume that

$$f(J_1, J_2, J_3) = h_1(J_1) + h_2(J_2) + w(J_3), \tag{5.2}$$

where $h_\ell : \mathbb{R}^+ \to \mathbb{R}^+$, for $\ell = 1, 2$, is a strictly convex and increasing smooth function such that $h_\ell(J) \to 0$ and $h'_\ell(J) \to 0$ as $J \to 0^+$, while $h_\ell(J) \to +\infty$ and $h'_\ell(J) \to +\infty$ as $J \to +\infty$, the model case being $h_\ell(J) = J^p$ for some $p > 1$. This choice encompasses as a special case what has been discussed in Section 4.

We also assume that $w : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly convex smooth function such that $w(J) \to +\infty$ as $J \to 0^+$ and $J \to +\infty$, while $w'(J) \to w^\infty \in \mathbb{R}^+ \cup \{+\infty\}$ as $J \to +\infty$. Notice that $w'(J) \to -\infty$ as $J \to 0^+$ and $J^{-1}w(J) \to w^\infty$ as $J \to +\infty$. In the case of a linear growth, that is, when $w^\infty \in \mathbb{R}^+$, we also assume that $\big(J\, w'(J) - w(J)\big) \to 0$ for $J \to +\infty$.

The model cases are given for any exponent $\alpha > 0$ by $w(J) = J^2 + J^{-\alpha}$, so that $w^\infty = +\infty$, and by $w(J) = w^\infty J + J^{-\alpha}$, in the case $w^\infty \in \mathbb{R}^+$.

For any $K > 0$, we denote

$$Q_K = \{\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3 \;:\; |\tau_\ell| \le K \quad \forall \ell = 1, 2, 3\}.$$

Arguing as in the previous results, we readily obtain the following

Proposition 3. *According to the previous hypotheses, let $f_K : \mathsf{A} \to \overline{\mathbb{R}}$ be the conjugate of the function $f^*(\tau) + \mathbf{1}_{Q_K}(\tau)$, where $\tau \in \mathbb{R}^3$ and $f^* : \mathbb{R}^3 \to \overline{\mathbb{R}}$ is the conjugate of the function f given by (5.2). Then, we have*

$$f_K(J_1, J_2, J_3) = h_{1,K}(J_1) + h_{2,K}(J_2) + w_K(J_3), \tag{5.3}$$

where $J_\ell \in \mathbb{R}^+$, $\ell = 1, 2, 3$. Moreover, for $\ell = 1, 2$ we have

$$h_{\ell,K}(J) = \begin{cases} h_\ell(J) & \text{if} \quad h'_\ell(J) \le K \\ K\, J + h_\ell(g_\ell(K)) - K\, g_\ell(K) & \text{if} \quad h'_\ell(J) > K, \end{cases}$$

$g_\ell : \mathbb{R}^+ \to \mathbb{R}^+$ being the inverse of the derivative function h'_ℓ, and

$$w_K(J) = \begin{cases} -K\, J + w(g_w(-K)) + K\, g_w(-K) & \text{if} \quad w'(J) < -K \\ w(J) & \text{if} \quad |w'(J)| \le K \\ K\, J + w(g_w(K)) - K\, g_w(K) & \text{if} \quad w'(J) > K, \end{cases}$$

$g_w : (-\infty, w^\infty[\to \mathbb{R}^+$ being the inverse of the derivative function w'.

Notice that the third case in the formula of w_K is verified if and only if $K < w^\infty$.

According to (5.2), and denoting for simplicity $J_\ell = J_\ell(\xi,\eta)$ in (3.7), we have

$$\partial_\xi \Phi = h'_1(J_1)\,\frac{\xi}{J_1} + h'_2(J_2)\,\frac{2\xi\eta^2}{J_2} + w'(J_3)\,\eta^2,$$

$$\partial_\eta \Phi = h'_1(J_1)\,\frac{2\eta}{J_1} + h'_2(J_2)\,\frac{2\xi^2\eta + 2\eta^3}{J_2} + w'(J_3)\,2\xi\eta.$$

As a consequence, a bound on the derivatives h'_ℓ and w', as we obtained in the previous result, does not imply a bound on the radially symmetric Piola-Kirchhoff stress vector.

In fact, the term $2\xi\eta$ in the last addendum reads as the factor $2rv(r)v'(r)$ in the second entry of the vector $T_R(r, v(r), v'(r))$, which explicitly depends on the derivative of the profile function v. However, notice that the other five factors are equibounded; in particular, so is the term $r^2\eta^2 = v(r)^2$ by the combination of the Dirichlet condition $v(1) \le \lambda$ and the monotonicity of v.

6. Existence of energy minimizers with radial symmetry

The stress constraint imposed this way leads us to consider a minimization problem among radially symmetric deformations as in (3.2). We consider prescribed Dirichlet boundary conditions $v(1) = \lambda$ to a functional $\mathcal{E}(u)$ as in (3.1), whose energy density $W_K(F)$ corresponds to the energy $\mathcal{F}(v)$ in (3.6), where $N = 3$ and the integrand $\Phi = \Phi_K$ is given by

$$\Phi_K(\xi,\eta) = f_K(J_1(\xi,\eta), J_2(\xi,\eta), J_3(\xi,\eta)),$$

where f_K is given by (5.3). Thus, we define

$$\mathcal{F}_K(v) := 4\pi \int_0^1 r^2 \Phi_K\big(v'(r), v(r)/r\big)\, dr,$$

and explore the existence of energy minimizers by adopting direct methods in the calculus of variations.

We are thus led to consider profile functions v in the class $\mathcal{A}(\lambda)$ of non-decreasing and non negative functions $v : I \to \mathbb{R}^+$, where $I = (0,1)$, such that $v(0) \geq 0$ and $v(1) \leq \lambda$, for some given $\lambda > 0$.

For example, consider the class $\mathcal{A}_\infty(\lambda)$ of smooth (or, more generally, absolutely continuous) and strictly increasing profile functions $v : I \to \mathbb{R}^+$ such that $v(0) = 0$ and $v(1) = \lambda$. Let $\{v_h\}$ be a sequence in $\mathcal{A}_\infty(\lambda)$ such that

$$\lim_{h\to\infty} \mathcal{F}_K(v_h) = \inf\{\mathcal{F}_K(v) \mid v \in \mathcal{A}_\infty(\lambda)\} < \infty .$$

Due to the linear growth of the three terms $h_{1,K}$, $h_{2,K}$, and w_K, in general we can only find a subsequence not relabeled and a function $v \in \mathcal{A}(\lambda)$ such that $v_h \to v$ in $L^1(I)$. We thus proceed by relaxation.

By continuity, we set

$$\Phi_K(\xi, 0) = \lim_{\eta\to 0^+} \Phi_K(\xi, \eta)$$

for every $\xi > 0$ and $\Phi_K(\xi,\eta) = +\infty$ if $\xi \leq 0$ or $\eta < 0$. It turns out that $\Phi_K(\cdot,\eta)$ is a convex function in $\mathbb{R}$ for every $\eta \in \mathbb{R}$.

We then introduce the parametric extension $\widehat{\Phi}_K : \mathbb{R}^2 \times [0,+\infty) \to \overline{\mathbb{R}}$

$$\widehat{\Phi}_K((\zeta,\xi),\eta) := \begin{cases} \zeta \bullet \Phi_K\left(\dfrac{\xi}{\zeta}, \eta\right) & \text{if} \quad \zeta > 0,\, \xi > 0 \\ +\infty & \text{elsewhere} \end{cases}, \tag{6.1}$$

and, for any given $\eta \geq 0$, we define the convex lower semicontinuous envelope

$$\overline{\Phi}_K((\zeta,\xi),\eta) := \sup\left\{ g(\zeta,\xi), \quad g : \mathbb{R}^2 \to \mathbb{R} \quad \text{convex}, \quad g(z) \leq \widehat{\Phi}_K(z,\eta), \quad \forall\, z \in \mathbb{R}^2 \right\} \tag{6.2}$$

where $(\zeta,\xi) \in \mathbb{R}^2$. Since we deal with a 1D problem, the recession of the function $\Phi_K(\cdot,\eta)$ comes into play. More precisely, for every $\eta > 0$ we let

$$\Phi^\infty_{K(\eta)} := \lim_{\xi \longrightarrow +\infty} \frac{\Phi_K(\xi,\eta)}{\xi} .$$

For $\ell = 1,2$, we have $h'_\ell(J) \geq K$ for every $J > c$, where the positive constant c depends on K and h_ℓ, and for every $\eta > 0$ we have $J_\ell(\xi,\eta) > c$ if ξ is large enough. Recalling the formulas in (3.7), we have the following:

$$\lim_{\xi\to+\infty} \frac{h_{1,K}(\xi,\eta)}{\xi} = \lim_{\xi\to+\infty} \frac{K\sqrt{\xi^2 + 2\eta^2}}{\xi} = K$$

and

$$\lim_{\xi\to+\infty} \frac{h_{2,K}(\xi,\eta)}{\xi} = \lim_{\xi\to+\infty} \frac{K\sqrt{2\xi^2\eta^2 + \eta^4}}{\xi} = K\sqrt{2}\eta .$$

Similarly, for every $\eta > 0$ we obtain the following:

$$\lim_{\xi\to+\infty} \frac{w_K(\xi,\eta)}{\xi} = \lim_{\xi\to+\infty} \frac{K\,\xi\eta^2}{\xi} = K\eta^2 ,$$

if $K < w^\infty$, whereas if $K \geq w^\infty$,

$$\lim_{\xi\to+\infty} \frac{w_K(\xi,\eta)}{\xi} = \lim_{\xi\to+\infty} \frac{w(\xi\eta^2)}{\xi} = w^\infty \eta^2 .$$

We definitely obtain for every $\eta \geq 0$

$$\Phi^\infty_{K(\eta)} = K(1 + \sqrt{2}\eta) + \min\{w^\infty, K\}\,\eta^2 , \tag{6.3}$$

and the explicit formula of the extension becomes for every $\eta \geq 0$

$$\overline{\Phi}_K((\zeta,\xi),\eta) := \begin{cases} \zeta \cdot \Phi_K\left(\dfrac{\xi}{\zeta}, \eta\right) & \text{if} \quad \zeta > 0,\, \xi > 0 \\ \Phi^\infty_K(\eta) & \text{if} \quad \zeta = 0,\, \xi > 0 \\ +\infty & \text{elsewhere}\,. \end{cases} \tag{6.4}$$

In the relaxation process, we deal with linear functionals $T = T_v$ corresponding to profile functions $v \in \mathcal{A}(\lambda)$. In particular, we focus on the space $\mathcal{A}_\infty(\lambda)$ of smooth profile functions. If $v \in \mathcal{A}_\infty(\lambda)$, we consider linear functionals G_v carried by the graph of v as integration on the naturally oriented graph of v of compactly supported and smooth 1-forms τ, whose space is indicated by $\mathcal{D}^1(\mathbb{R}^2)$. For $T = G_v$, with $v \in \mathcal{A}_\infty(\lambda)$, we thus have

$$\partial T = \delta_1 \times \delta_\lambda - \delta_0 \times \delta_0 , \quad \mathbf{M}(T) \leq 1 + \lambda , \tag{6.5}$$

where δ_a is the unit Dirac measure at the point $a \in \mathbb{R}$, ∂T is the (so-called) *boundary* of T, given in this case by integration of Dirac measures, and $\mathbf{M}(T)$ is the (so-called) *mass* of T given by the integration over the entire graph of v.

In total, we analyze the structure of the class $\mathcal{T}(\lambda)$ that naturally arises as the limit points (in the sequential weak* topology) of a sequence $\{G_{v_h}\}$, where $v_h \in \mathcal{A}_\infty(\lambda)$ for every h.

For any $v \in \mathcal{A}(\lambda)$, we let

$$\mathcal{SG}_v = \{(r, v) \in I \times \mathbb{R} \mid v < v(r)\},$$

where at a discontinuity point of v we identify $v(r)$ with the average

$$v(r) := \frac{1}{2}\left(v(r^-) + v(r^+)\right)$$

between the right and left limits of v at r. Then we have

$$T = T_v = \delta_0 \times [\![0, v(0)]\!] - (\partial [\![\mathcal{SG}_v]\!])\llcorner(I \times \mathbb{R}) + \delta_0 \times [\![v(1), \lambda]\!],$$

where $v \in \mathcal{A}(\lambda)$ is the L^1-limit of the sequence $\{v_h\}$. Note also that $T_v = G_v$ if $v \in \mathcal{A}_{\not{\!\!\beta}}(\lambda)$.

We thus denote by $\mathcal{T}(\lambda)$ the class of linear functionals $T = T_v$ defined above for some $v \in \mathcal{A}(\lambda)$. Notice that T in $\mathcal{T}(\lambda)$ satisfies the properties in (6.5); its support is contained in $[0, 1] \times [0, \lambda]$. Moreover, we can find a 1-rectifiable set Γ contained in $[0, 1] \times [0, \lambda]$, and a unit vector field $\tau : \Gamma \to \mathbb{R}^2$ that is measurable with respect to the one-dimensional Hausdorff measure $\mathcal{H}^1$ restricted to Γ and provides an orientation to the tangent line to Γ at $\mathcal{H}^1$-almost every point $(r, v) \in \Gamma$, such that we have

$$\langle T_v, \tau\rangle = \int_\Gamma \langle \tau, \tau\rangle \, d\mathcal{H}^1 \quad \forall\, \tau \in \mathcal{D}^1(\mathbb{R}^2),$$

where $\langle \cdot, \cdot \rangle$ indicates the duality product. In addition, both components of the unit vector τ are always non-negative.

The energy functional $\overline{\mathcal{F}}_K(T)$ is defined in $T \in \mathcal{T}(\lambda)$ letting

$$\overline{\mathcal{F}}_K(T) = \int_\Gamma r^2\, \overline{\Phi}_K(\tau, v/r)\, d\mathcal{H}^1(r, v).$$

Therefore, if $T = G_v$ for some $v \in \mathcal{A}_\infty(\lambda)$ we obtain the identity

$$\overline{\mathcal{F}}_K(G_v) = \mathcal{F}_K(v).$$

Assume, for example, that v is absolutely continuous on $(0, r_0)$ and on $(r_0, 1)$ for some $r_0 \in (0, 1)$, and that $v(r_0^-) < v(r_0^+)$. In that case, we have

$$-([\![\mathcal{SG}_v]\!])\llcorner(I \times \mathbb{R}) = G_v + \delta_{r_0} \times [\![v(r_0^-), v(r_0^+)]\!],$$

and

$$\overline{\mathcal{F}}_K(T_v) = \mathcal{F}_K(v) + \psi(0; 0, v(0)) + \psi(r_0; v(r_0^-), v(r_0^+)) + \psi(1; v(1), \lambda)$$

where the term $\mathcal{F}_K(v)$ is defined as before and the non-negative real number $\psi(r; \alpha, \beta)$ corresponds to the contribution of $\delta_r \times [\![\alpha, \beta]\!]$ to the $\overline{\mathcal{F}}_K$-energy, with $r \in [0, 1]$ and $0 \le \alpha \le \beta \le \lambda$. More precisely, on account of the expressions in (6.3) and (6.4), we have

$$\psi(r; \alpha, \beta) := \overline{\mathcal{F}}_T(\delta_r \times [\![\alpha, \beta]\!]) = 4\pi\left(K\, r^2(\beta - \alpha) + K\, r\, \sqrt{2}\, \frac{\beta^2 - \alpha^2}{2} + \min\{w^\infty, K\}\, \frac{\beta^3 - \alpha^3}{3}\right).$$

If the distributional derivative Dv of v has no Cantor part, and $\{r_i\}_{i=1}^{n_0}$, where $n_0 \in \mathbb{N} \cup \{+\infty\}$, is the countable set of discontinuity points of v, we have

$$\overline{\mathcal{F}}_K(T_v) = \mathcal{F}_K(v) + \psi(0; 0, v(0)) + \sum_{i=1}^{n_0} \psi(r_i; v(r_i^-), v(r_i^+)) + \psi(1; v(1), \lambda).$$

The general expression of the energy $\overline{\mathcal{F}}_K(T_v)$ contains an additional term depending on the Cantor component of Dv. Two borderline situations occur in the cases $v_0(r) \equiv 0$ and $v_\lambda(r) \equiv \lambda$, where the rectifiable set Γ is given by the union of two segments. For $v = v_0$, it is the union of the horizontal segment from $(0, 0)$ to $(1, 0)$, joined with the vertical segment from $(1, 0)$ to $(1, \lambda)$. For $v = v_\lambda$, it is the union of the vertical segment from $(0, 0)$ to $(0, \lambda)$, joined with the horizontal segment from $(0, \lambda)$ to $(1, \lambda)$.

The class $\mathcal{T}(\lambda)$ is sequentially weakly closed by the Federer and Fleming classical theorem (Federer and Fleming, 1960), and the functional $T \mapsto \overline{\mathcal{F}}_K(T)$ is sequentially weakly lower semicontinuous in $\mathcal{T}(\lambda)$. Thus, we obtain the following existence result:

Theorem 1. *With the previous notation, the minimum of the problem*

$$\inf\{\overline{\mathcal{F}}_K(T) \mid T \in \mathcal{T}(\lambda)\}$$

is achieved for every $\lambda > 0$.

In general, we do not have information regarding the optimal profile and its regularity. We only remark here that there exists a positive constant μ, depending on f and K, such that

$$\left\{1 + \sum_{\ell=1}^{3} J_\ell^{\,2}\right\}^{1/2} \le \mu\left(1 + f_K(J_1, J_2, J_3)\right) \quad \forall (J_1, J_2, J_3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+.$$

As a consequence, if for example $v \in \mathcal{A}_\infty(\lambda)$ and $u : B^3 \to \mathbb{R}^3$ is the corresponding radially symmetric deformation map, on the basis of (3.8), we have

$$\mathfrak{A}(u) \le \mu\left(|B^3| + \mathcal{F}_K(v)\right).$$

Generally, a linear functional $\overline{T}$, radially symmetric, defined on the graph of u in $B^3 \times \mathbb{R}^3$, whose mass is controlled by the energy $\mathcal{F}_K(T)$, corresponds to $T \in \mathcal{T}(\lambda)$. Specifically, if $v \in \mathcal{A}_\infty(\lambda)$, the graph $\mathcal{G}_u$ in $B^3 \times \mathbb{R}^3$ of the corresponding deformation map u is obtained by taking the image

$$\mathcal{G}_u = h(\mathcal{G}_v \times \mathbb{S}^2),$$

where $\mathcal{G}_v$ is the graph of v in $I \times \mathbb{R}^+$ and $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ is the map defined by

$$h(r, v, z) := (r\, z, v\, z), \quad (r, v, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 .$$

The radially symmetric linear functional $\overline{T}$ in $\mathbb{R}^3 \times \mathbb{R}^3$ is given by the integration of compactly supported and smooth 3-forms along the $SO(3) \times SO(3)$-invariant and oriented 3D set given by the image $h(\Gamma \times \mathbb{S}^2)$, where Γ is the 1-rectifiable set defining T_v. If, for example, $v \in \mathcal{A}(\lambda)$ is smooth on I, with $0 < \varepsilon = v(0) \leq v(1) = \lambda$, we obtain

$$\overline{T} = G_u + \delta_{0_{\mathbb{R}^3}} ⟜ [\![B^3(0_{\mathbb{R}^3}, \varepsilon)]\!] ,$$

where G_u is a linear functional on the graph of a radially symmetric map u in (3.2)—in terms of geometric measure theory, G_u is the 3-current carried by the graph of u.

7. Final remarks

The results in Section 4 do not depend on radial symmetry. In addition, they can be generalized to constraints of type $\mathfrak{f}(\sigma) \leq 0$, provided that the function $\mathfrak{f}$ is differentiable. Thus, classical strength criteria such as the Huber-von Mises-Hencky criterion or Beltrami's one can be adopted in this analysis.

Stress constraints can commonly be intended as indicators of a threshold between two different phases. They do not give direct information on the post-critical behavior, unless we do not derive it from some first principle or postulate directly flow rules as it occurs in the description of plastic flows. In that case, the differentiability presumed for $\mathfrak{f}$, the dissipation inequality, and the acceptance of a maximum dissipation principle imply associated flow rules and the convexity of the stress admissibility region for the elastic phase (see, e.g., the analysis in Simo and Hughes, 1998). The choice of assigning constraints in large strain regime in terms of the derivatives

$$\left(\frac{\partial e}{\partial F}, \frac{\partial e}{\partial \mathrm{cof}\, F}, \frac{\partial e}{\partial \det F} \right),$$

rather than the first Piola-Kirchhoff stress, as adopted in Giaquinta et al. (2015), seems to be the only alternative to have a minimization problem that can be tackled. Write $M(F)$ for the linear operator (a third-rank tensor) that has as entries all the components of F, $\mathrm{cof}\, F$, $\det F$ (see Giaquinta et al., 1998). The polyconvex energy density of a simple hyperelastic body can thus be written as $e(M(F))$; it is convex with respect to $M(F)$. The first Piola-Kirchhoff stress is thus

$$P = \frac{\partial e}{\partial M(F)} \frac{\mathrm{d}M(F)}{\mathrm{d}F},$$

where $\mathrm{d}M(F)/\mathrm{d}F$ is a fifth-rank tensor. It maps $\partial e/\partial M(F)$, a third-rank tensor, to the space of two-point tensors P such that $PF^\top$ is symmetric. When we constrain $\partial e/\partial M(F)$ to be in a convex set, the projector $\mathrm{d}M(F)/\mathrm{d}F$ does not necessarily grant in general that P lies itself in a convex set.

When we explore the post-critical behavior, we face energies that include further state variables. For example, referring to crystals undergoing slips, we may choose the Burgers tensor as an additional state variable. The minimality of these energies under stress constraints is an interesting problem to explore. However, already in the absence of stress constraints we find nontrivial difficulties in both static (see, e.g., Mariano and Mucci, 2023) and evolutionary (see, e.g., Mielke, 2005) circumstances.

CRediT authorship contribution statement

Paolo Maria Mariano: Conceptualization, Formal analysis, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing, Investigation; **Domenico Mucci:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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