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Coprime action and principal blocks

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Abstract. In the situation where a finite group A acts coprimely by automorphisms on another finite group G, we characterize when the principal p-block of G contains a unique A-invariant irreducible character, in terms of the subgroup $\mathbf{C}_G(A)$ of fixed points under such action.

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1. Introduction. Let A and G be finite groups. Suppose that A acts coprimely by automorphisms on G. Then there exists a canonical bijection $\rho(G, A)$ from the set $\mathrm{Irr}_A(G)$ of irreducible A-invariant characters of G onto $\mathrm{Irr}(\mathbf{C}_G(A))$, known as the Glauberman–Isaacs correspondence (see [5, 11, 26]). The importance of the Glauberman–Isaacs correspondence in group representation theory cannot be overestimated. It implies that the actions of A on irreducible characters and conjugacy classes of G are permutation isomorphic. Moreover, its properties when A is a p-group are key for proving the p-solvable case, and therefore also the reduction theorems, of the McKay conjecture and its many refinements, as well as the Alperin weight conjecture (see [20] for more information). One immediate consequence of the Glauberman–Isaacs correspondence is that the group G contains a unique A-invariant character, namely 1_G the

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trivial one, if and only if A acts without fixed points on G. These equivalent conditions imply the solvability of G by [1, Thm. B].

Let p be a prime divisor of the order of G and let B be a Brauer p-block of G. [20, Prob. 6] asked whether an A-invariant block B would always contain A-invariant characters, both modular and ordinary. This problem was positively solved in [17, Thm. A]. However, the Glauberman–Isaacs correspondence does not exhibit a good behavior with respect to block theory (outside the case where the action of A fixes a defect group pointwise [9, 25]). For instance, the images under the Glauberman–Isaacs correspondence of A-invariant characters in a p-block of G do not generally belong to the same p-block of $\mathbf{C}_G(A)$. It is easy to find examples of this misbehavior. One is $G = \mathsf{D}_{14}$ the dihedral group of 14 elements, A of order 3 acting faithfully on G, and $p = 7$. In this case, G has a unique p-block while $\mathbf{C}_G(A)$ has two. Whether it is true that irreducible characters of $\mathbf{C}_G(A)$ lying in the same p-block correspond to A-invariant irreducible characters in the same p-block of G seems more difficult to understand, as noted by Gabriel Navarro in private communication with the authors of this note. For instance, it is not hard to show that counterexamples cannot arise when G is a p-solvable group.

Given that every A-invariant block contains at least one A-invariant irreducible character, it is natural to study those blocks B that contain exactly one A-invariant irreducible character. In this note, we make progress on this problem by characterizing when the principal p-block of G contains a unique A-invariant irreducible character in group-theoretical terms. In the following, $\mathbf{O}_{p'}(G)$ denotes the largest normal subgroup of order coprime to p of G.

Theorem A. *Let A and G be finite groups. Suppose that A acts coprimely by automorphisms on G. Let p be a prime and denote by B_0 the principal p-block of G. Then the only A-invariant irreducible character in B_0 is 1_G if and only if $\mathbf{C}_G(A) \subseteq \mathbf{O}_{p'}(G)$.*

Our proof of Theorem A depends on the fact that both conditions imply the p-solvability of G, and we use the classification of finite simple groups to prove it.

It remains a challenge to understand whether it is possible to characterize the groups G that possess a unique A-invariant irreducible Brauer character in the principal block. Unlike in the ordinary case, there are simple non-abelian groups of order divisible by p that satisfy this property. For instance, if $G = {}^2\mathsf{B}_2(8)$ and $A \leq \mathrm{Aut}(G)$ has order 3, then $\mathbf{C}_G(A)$ is a Frobenius group of order 20 [6, Thm. 2.2.7 (a)]. In this case, for $p = 2$, there are only two A-invariant Brauer characters of G, the trivial one 1_G and a defect zero character; hence the set $\mathrm{IBr}_A(B_0(G))$ of A-invariant Brauer characters in the principal 2-block of G contains only one character. For every other prime p dividing the order of G, we see that $|\mathrm{IBr}_A(B_0(G))| \geq 2$. In fact, one could be tempted to conjecture that a necessary condition for $|\mathrm{IBr}_A(B_0(G))| = 1$ is that $\mathbf{C}_G(A)$ has a normal p-complement. However, we refrain from making such a conjecture here due to a lack of evidence. We note that it is easy to find examples where $\mathbf{C}_G(A)$

has a normal p-complement but for which $|\mathrm{IBr}_A(B_0(G))| \geq 2$, for instance $G = \mathsf{D}_{14} \times \mathsf{C}_7$, $A = \mathsf{C}_3$, and $p = 7$.

Structure of the paper. In Sect. 2, we show that it is possible to find an A-invariant character in the principal block lying over a fixed A-invariant character lying in the principal block of a normal subgroup. This is proven in further generality in Theorem 2.1 following the proof [17, Thm. 6.1 (a)]. In Sect. 3, we prove Theorem A by first reducing its statement to show that every simple non-abelian group S of order divisible by p contains an A-invariant irreducible character $\chi \neq 1_S$ in the principal p-block whenever A acts coprimely on S, and later showing this result by invoking the classification of finite simple groups.

2. Above an *A*-invariant character. In [17, Thm. A], it was shown that an A-invariant block B of G does always contain some A-invariant $\chi \in \mathrm{Irr}(B)$ whenever A acts coprimely on G. In order to do so, Malle, Navarro, and Späth used Dade's ramification group as in [19] and proved results on character triples with good compatibility with A-action and blocks.

For proving Theorem A from the introduction, we will use a *relative* version of the Malle–Navarro–Späth result mentioned above, where by relative we mean that we will be interested in characters of G lying above a fixed character of a normal subgroup of G. The precise statement we need is Theorem 2.1 below in the case of principal p-blocks. We care to mention that techniques introduced in [17] do suffice to prove such a relative statement, although it is not written explicitly anywhere there. We include a proof here for completeness.

Our notation for characters and blocks is standard as in [12, 21]. We recall that whenever A acts coprimely on G by automorphisms and both groups act compatibly on a non-empty set Ω, if G acts transitively on Ω, then A fixes some element of Ω (and any two A-fixed elements are $\mathbf{C}_G(A)$-conjugate). This is Glauberman's lemma [13, Lemma 3.24] that we shall frequently use in the next proof. For the definition and properties of Dade's ramification group, we refer the reader to [17, §4]. For the definition and properties of character triple isomorphisms, we refer the reader to [12, Ch. 11] or [22, Ch. 5].

Theorem 2.1. *Let A and G be finite groups. Suppose that A acts coprimely by automorphisms on G. Let N be a normal subgroup of G, and let $\theta \in \mathrm{Irr}_A(N)$. Let p be a prime. Write $b = \mathrm{Bl}(\theta) \in \mathrm{Bl}(N)$. If $B \in \mathrm{Bl}(G)$ is A-invariant and covers b, then there is some A-invariant $\chi \in \mathrm{Irr}(B|\theta)$.*

Proof. We proceed by induction on $|G : N|$.

Claim 1: We can assume that $G_b = G$.

Suppose that G_b, the stabilizer of the block b in G, is strictly contained in G. Notice that G_b is A-invariant. Let $B' \in \mathrm{Bl}(G_b|b)$ be the Fong–Reynolds correspondent of B as in [21, Thm. 9.14]. By [21, Prob. 4.4], the block B' is A-invariant. By induction, let $\psi \in \mathrm{Irr}_A(B'|\theta)$ be an irreducible A-invariant character of B' lying over θ. Then $\chi = \psi^G \in \mathrm{Irr}_A(B|\theta)$, as wanted.

Claim 2: We can assume that G/N is a minimal normal subgroup of GA/N.

Let $K \lhd GA$. Suppose that $N \subsetneq K \subsetneq G$. Notice that A acts transitively on the (non-empty) set of blocks in $\mathrm{Bl}(K|b)$ covered by B. By Glauberman's lemma, we can choose some A-invariant $B' \in \mathrm{Bl}(K|b)$ covered by B. By using induction twice, we first find $\psi \in \mathrm{Irr}_A(B'|\theta)$ and then $\chi \in \mathrm{Irr}_A(B|\psi)$. Since χ lies over θ, we are also done in this case.

Claim 3: We can assume that $G[b] = G$ where $G[b]$ is Dade's ramification group as described in [17, Thm. 4.2.(a)].

Recall that $N \subseteq G[b] \lhd G$. We first prove that $G[b]$ is A-invariant. In fact, let D be an A-invariant defect group of b by Glauberman's lemma. The blocks of $D\mathbf{C}_N(D)$ that induce b are called the roots of b (see the comment after the proof of [21, Thm. 9.7]). Notice that $\mathbf{N}_N(D)$ acts transitively on the roots of b by the extended Brauer's first main theorem [21, Thm. 9.7]. By using Glauberman's lemma again, we can choose an A-invariant root $b_0 \in \mathrm{Bl}(D\mathbf{C}_N(D))$ of b. In particular, the corresponding canonical character η_0, as in [21, p 204], is A-invariant. Now, it is not difficult to prove that the group $H^\perp \cap K$ as in [17, Thm. 4.2 (a)] is A-invariant (where the bilinear form $\langle\langle\, ,\, \rangle\rangle$ is computed with respect to η_0). Hence, also $G[b] = N(H^\perp \cap K) \lhd G$ is A-invariant.

By the second claim, we have that $G[b]$ is either N or G. In the case where $G[b] = N$, we have that $b^G = B$ by [19, Thm. 3.5]. In particular, B is the only block of G covering b. By [12, Thm. 13.28 and Cor. 13.30], we can find some $\chi \in \mathrm{Irr}_A(G|\psi)$. Then χ lies in B and above θ, and the statement follows. We can therefore assume that $G[b] = G$.

Thanks to the second claim, we know that G/N is a chief factor of GA. There are three mutually exclusive options for the structure of G/N: either (i) G/N is a p-group, (ii) G/N is a group of order not divisible by p, or (iii) G/N is semisimple of order divisible by p.

In case (i), there is a unique block of G covering b by [21, Cor. 9.6]. In particular, $\mathrm{Irr}_A(B|\theta) = \mathrm{Irr}_A(G|\theta)$. In this case, the result follows from [12, Thm. 13.28 and Cor. 13.30].

In order to deal with cases (ii) and (iii), we use character triple isomorphisms as in [17, §5]. By [17, Prop. 5.1 and 5.2], and using the third claim, there exists a character triple (G^*, N^*, θ^*) and an action of A on G^* stabilizing both N^* and θ^* and an isomorphism of character triples

$$(\imath, \sigma)\colon (G, N, \theta) \to (G^*, N^*, \theta^*)$$

such that $N^* \subseteq \mathbf{Z}(G^*)$ and both $\imath$ and σ_G are A-equivariant maps. Moreover, there exists a block B^* of G^* which is A-invariant and covers the block b^* of N^* to which θ^* belongs such that

$$\sigma_G(\mathrm{Irr}_A(B|\theta)) = \mathrm{Irr}_A(B^*|\theta^*).$$

By working in G^*, we may assume without loss of generality that $N \subseteq \mathbf{Z}(G)$.

In case (ii), we have that B has a central defect $D \subseteq N$. Then $\mathrm{Irr}(B) = \{\eta_\xi \mid \xi \in \mathrm{Irr}(D)\}$ as in [21, Thm. 9.12], where η is the canonical character of B. Since η^0 is the only irreducible Brauer character in B and B is A-invariant, we deduce that η is A-invariant. Now θ lies under some $\chi = \eta_\xi$, where $\xi \in \mathrm{Irr}(D)$. By the definition of η_ξ, we see that θ extends ξ, and therefore ξ is A-invariant because θ also is. By the definition of η_ξ, we gather that η_ξ is A-invariant and we are done in this case. (Actually, in this case, $\mathrm{Irr}(B|\theta) = \{\chi\}$.)

In case (iii), we notice that $N = \mathbf{Z}(G)$. We let $Z \in \mathrm{Syl}_p(N)$ and write $\nu = \theta_Z$. By [17, Cor. 2.6], there is some A-invariant $\chi \in \mathrm{Irr}(B|\nu)$. Write $N = Z \times X$, and $\lambda = \theta_X$. Let $c = \mathrm{Bl}(\lambda)$. Since B covers c, and c is G-invariant, we have that c is the only block covered by B by [21, Cor. 9.3]. Then χ_X has an irreducible constituent in c by [21, Thm. 9.2]. As $\mathrm{Irr}(c) = \{\lambda\}$, we get that χ lies over λ. Since χ lies over a unique irreducible character of N, we get that χ lies over $\theta = \nu \times \lambda$, and we are also done in this case. □

3. Proof of Theorem A. Our aim is to prove Theorem A from the introduction, which we restate here for the reader's convenience.

Theorem 3.1. *Let A and G be finite groups. Suppose that A acts coprimely by automorphisms on G. Let p be a prime and denote by B_0 the principal p-block of G. Then $|\mathrm{Irr}_A(B_0)| = 1$ if and only if $\mathbf{C}_G(A) \subseteq \mathbf{O}_{p'}(G)$.*

We first prove that the statement of Theorem 3.1 holds for all groups provided it does hold for the finite simple non-abelian groups.

Theorem 3.2. *The statement of Theorem 3.1 holds for every finite group G if whenever S is a finite simple non-abelian group of order divisible by p and A acts coprimely by automorphisms on S, there exists some A-invariant $1_S \neq \chi \in \mathrm{Irr}(S)$, lying in the principal p-block of S.*

Proof. Write $N = \mathbf{O}_{p'}(G)$. By [13, Cor. 3.8], $\mathbf{C}_{G/N}(A) = \mathbf{C}_G(A)N/N$.

First suppose that $\mathbf{C}_G(A) \subseteq N$. Then $\mathbf{C}_{G/N}(A) = 1$. By [1, Thm. B], this implies that G/N is solvable, and therefore G is p-solvable.

We claim that, under our hypotheses, the condition $|\mathrm{Irr}_A(B_0(G))| = 1$ also implies the p-solvability of G. In order to prove our claim, we proceed by induction on $|G|$. Let N be a minimal normal subgroup of GA contained in G. Then both $\mathrm{Irr}_A(B_0(G/N))$ and $\mathrm{Irr}_A(B_0(N))$ have cardinality one, where we are using that $\mathrm{Irr}(B_0(G/N) \subseteq \mathrm{Irr}(B_0(G)))$ by block domination [21, p. 199] and Theorem 2.1. If $N < G$, by induction, we get that G/N and N are p-solvable groups, then so is G.

Hence we can assume that G is a direct product of copies of some simple group S transitively permuted by A. If G is a p-group or has order not divisible by p, then there is nothing to prove. In particular, we may assume that S is non-abelian simple of order divisible by p. Write $G = S_1 \times \cdots \times S_t$, where $S_i \cong S$, and A permutes the S_i's transitively. Consider $B = \mathbf{N}_A(S_1)$. Then $1 = |\mathrm{Irr}_A(B_0(G))| = |\mathrm{Irr}_B(B_0(S_1))|$ by [24, Lemma 6.6(a)–(b)], which contradicts our hypotheses.

Now we know that both our conditions imply the p-solvability of G. By Fong's theorem [21, Thm. 10.20], the sets $\mathrm{Irr}(B_0(G))$ and $\mathrm{Irr}(G/N)$ can be identified. Hence, the Glauberman–Isaacs correspondence for G/N tells us that $|\mathrm{Irr}_A(B_0(G))| = |\mathrm{Irr}_A(G/N)| = |\mathrm{Irr}(\mathbf{C}_G(A)N/N)|$, and the statement follows. □

In order to complete the proof of Theorem 3.1, it remains to show that, whenever a group A acts coprimely by automorphisms on a non-abelian simple group S, and for any prime p dividing the order of S, there exists a non-trivial

A-invariant character lying in the principal p-block of S. Since the alternating and sporadic simple groups do not admit non-trivial coprime actions, and by the classification of finite simple groups, the remainder of this note will focus on the representation theory of finite groups of Lie type.

Theorem 3.3. *Let $\mathbf{G}$ be a simple algebraic group of simply connected type over an algebraic closure of a finite field of characteristic r, and $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism some power of which is a Frobenius endomorphism defining a rational structure on $\mathbf{G}$ over $\mathbb{F}_q$, where q is a power of the prime r. Assume that $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is an abstract simple group. Let p be a prime dividing the order of $\mathbf{G}^F$. Then there is a non-trivial unipotent character lying in the principal p-block of $\mathbf{G}^F$ unless when $\mathbf{G}^F = \mathrm{SL}_2(q)$ (with $q > 3$) and $p \mid q$.*

Proof. First let $p \mid q$. Then by [10, §8], except for the Steinberg character, all the other unipotent characters lie in the principal p-block of $\mathbf{G}^F$. So there is a non-trivial unipotent character lying in the principal p-block of $\mathbf{G}^F$ if and only if $\mathbf{G}^F$ possesses at least three unipotent characters. If $\mathbf{G}^F$ has exactly two unipotent characters, then case by case calculation (see for example [3, Chapter 13] for the lists of unipotent characters of groups of Lie type) shows that $\mathbf{G}^F = \mathrm{SL}_2(q)$ (with $q > 3$) and $p \mid q$.

Now we assume that $p \nmid q$. Hiss [8, Thm.] presented the list of groups, whose Steinberg character lies in the principal p-block for all primes $p \nmid q$. From this, we assume that $(\mathbf{G}, F)$ is of classical type or of the exceptional type E_6, ${}^2\mathsf{E}_6$, E_7, or E_8, and F is a Frobenius endomorphism.

Let e be the multiplicative order of q modulo p or 4 according to p being odd or $p = 2$. Let $\mathbf{L}$ be a Sylow e-split Levi subgroup of $\mathbf{G}$. Recall from [4, Cor. 7.3.4] that ${}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(1_{\mathbf{G}^F}) = 1_{\mathbf{L}^F}$, so $1_{\mathbf{G}^F}$ is an irreducible constituent of $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(1_{\mathbf{L}^F})$. By [15, Thm. 3.4], all the irreducible constituents of $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(1_{\mathbf{L}^F})$ lie in the principal p-block of $\mathbf{G}^F$. Moreover, by [2, Thm. 3.2], the number of the irreducible constituents of $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(1_{\mathbf{L}^F})$ is equal to the number of the irreducible characters of the relative Weyl group $W_{\mathbf{G}^F}(\mathbf{L}, 1_{\mathbf{L}^F}) = \mathbf{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$. Thus to finish the proof of this theorem, it suffices to show $\mathbf{L}^F < \mathbf{N}_{\mathbf{G}^F}(\mathbf{L})$.

We claim that $\mathbf{L}^F = \mathbf{N}_{\mathbf{G}^F}(\mathbf{L})$ occurs only when $\mathbf{L} = \mathbf{G}$, which is equivalent to the statement that the Sylow e-tori of $\mathbf{G}$ are trivial. If $(\mathbf{G}, F)$ is of classical type, then the structure of the relative Weyl groups $W_{\mathbf{G}^F}(\mathbf{L}, 1_{\mathbf{L}^F})$ are given in the proof of [2, Thm. 3.2], and we can check this claim directly. If $\mathbf{L}$ is a maximal torus, then $\mathbf{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ is a Weyl group, and thus [18, Lemma 23.3] yields that $\mathbf{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F = (\mathbf{N}_{\mathbf{G}}(\mathbf{L})/\mathbf{L})^F \neq 1$. If $(\mathbf{G}, F)$ is of the exceptional type E_6, ${}^2\mathsf{E}_6$, E_7, or E_8 and $\mathbf{L}$ is not a torus, then the groups $\mathbf{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$ are listed in [23, Table 3], whence $\mathbf{L}^F < \mathbf{N}_{\mathbf{G}^F}(\mathbf{L})$ holds for the situation that the Sylow e-tori of $\mathbf{G}$ are not trivial. So the claim holds.

If $p = 2$, then $\mathbf{G}$ has a non-trivial Sylow 1-torus and a non-trivial Sylow 2-torus by checking the generic order of $(\mathbf{G}, F)$ (see, for example, [18, Table 24.1]). According to [16, Cor. 5.4], if p is odd, then the assumption p dividing the order of $\mathbf{G}^F$ implies that $\mathbf{G}$ has a non-trivial Sylow e-torus. Therefore, we get $\mathbf{L}^F < \mathbf{N}_{\mathbf{G}^F}(\mathbf{L})$, which completes the proof. □

Theorem 3.4. *Let S be a finite simple non-abelian group of order divisible by p. Let A act on S coprimely by automorphisms. Then $|\operatorname{Irr}_A(B_0))| > 1$, where B_0 is the principal p-block of S.*

Proof. We assume that $A > 1$, as otherwise this assertion holds automatically. Without loss of generality, we may assume that the action of A on S is faithful, that is, $A \leq \operatorname{Aut}(S)$.

According to the classification of finite simple groups and the structure of automorphism groups of simple groups (see, e.g., [6, §2.5, §5.2, §5.3]), S has to be a simple group of Lie type and A is $\operatorname{Aut}(S)$-conjugate to some group of field automorphisms of S (see also [7, p. 192]). In particular, A is cyclic. As the Tits group ${}^2\mathsf{F}_4(2)'$ does not possess coprime automorphisms, one has that $S = \mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$, where $\mathbf{G}$ is a simple algebraic group of simply connected type over an algebraic closure of a finite field of characteristic r, and $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism some power of which is a Frobenius endomorphism defining a rational structure on $\mathbf{G}$ over $\mathbb{F}_q$, where q is a power of the prime r. The irreducible characters in the principal p-block B_0 of S are just the irreducible characters in the principal p-block of $\mathbf{G}^F$ whose kernel contains $\mathbf{Z}(\mathbf{G}^F)$.

We claim that $|\operatorname{Irr}_A(B_0))| > 1$ if there is a non-trivial unipotent character of $\mathbf{G}^F$ lying in the principal block. Note that $\mathbf{Z}(\mathbf{G}^F)$ is contained in the kernel of any unipotent character of $\mathbf{G}^F$, and every unipotent character is invariant under the action of field automorphisms (see [16, Prop. 3.7 and 3.9]). Let $A = \langle \alpha F_1 \alpha^{-1} \rangle$ where F_1 is a field automorphism and $\alpha \in \operatorname{Aut}(\mathbf{G}^F)$. Then for any unipotent character χ, we have $\chi^{\alpha F_1 \alpha^{-1}} = \chi$ since χ^{α} is also a unipotent character. Thus this claim holds.

Thanks to Theorem 3.3, we are left to deal with the case that $\mathbf{G}^F = \mathrm{SL}_2(q)$ (with $q > 3$) and $p \mid q$. An irreducible character of $\mathbf{G}^F$ different from the Steinberg character belongs to the principle block if and only if the center $\mathbf{Z}(\mathbf{G}^F)$ is contained in its kernel. We will make use of the notation for the irreducible characters of $\mathbf{G}^F$ as in [14, Table 1]. First let $4 \mid (q+1)$. Then the characters χ_a^+ and χ_a^- lie in the principal p-block of $\mathbf{G}^F$. By [14, Lemma 15.1], the characters χ_a^+ and χ_a^- are both invariant under the action of field automorphisms, and the set $\{\chi_a^+, \chi_a^-\}$ is $\operatorname{Aut}(\mathbf{G}^F)$-stable. Thus both χ_a^+ and χ_a^- are A-invariant. The proof for the case $4 \mid (q-1)$ is completely analogous, with $\chi_a^{\pm}$ replaced by $\chi_b^{\pm}$. Finally we let $q = 2^f$ be a power of 2, in which case we have $S = \mathbf{G}^F$. It follows from, for example, [14, Table 1] that $|\operatorname{Irr}(S)| = q+1$. Let F_0 denote the field automorphism of $\mathbf{G}$ such that $(a_{ij}) \mapsto (a_{ij}^2)$. Then $\operatorname{Aut}(S) \cong S \rtimes \langle F_0 \rangle$. Let $B = \langle F_0^k \rangle$ be the subgroup of $\langle F_0 \rangle$ which is $\operatorname{Aut}(S)$-conjugate to A, whence $\mathbf{C}_S(A) \cong \mathbf{C}_S(B)$. Recall that, except for the Steinberg character, all the other irreducible characters lie in the principal p-block of S. So by the Glauberman–Isaacs correspondence,

$$|\operatorname{Irr}_A(B_0)| = |\operatorname{Irr}_A(S)| - 1 = |\operatorname{Irr}(\mathbf{C}_S(A))| - 1 = |\operatorname{Irr}(\mathbf{C}_S(B))| - 1.$$

Now $\mathbf{C}_S(B) = \mathbf{G}^{F_0^k} \cong \mathrm{SL}_2(q_0)$ for some $q_0 \mid q$, which implies that $|\operatorname{Irr}(\mathbf{C}_S(B))| = q_0 + 1 \geq 3$ as $q_0 \geq 2$. Thus $|\operatorname{Irr}_A(B_0)| \geq 2$, which completes the proof. $\square$

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