



UNIVERSITÀ
DEGLI STUDI
FIRENZE

FLORE

Repository istituzionale dell'Università degli Studi di Firenze

Self-Similar Solutions for the Heat Equation with a Signum-Type Source Term

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

Original Citation:

Self-Similar Solutions for the Heat Equation with a Signum-Type Source Term / Roberto Gianni, A.F.. - In: QUARTERLY OF APPLIED MATHEMATICS. - ISSN 0033-569X. - ELETTRONICO. - (In corso di stampa), pp. 0-0.

Availability:

The webpage <https://hdl.handle.net/2158/1488913> of the repository was last updated on 2026-09-16T12:45:26Z

Terms of use:

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

Publisher copyright claim:

Conformità alle politiche dell'editore / Compliance to publisher's policies

Questa versione della pubblicazione è conforme a quanto richiesto dalle politiche dell'editore in materia di copyright.

This version of the publication conforms to the publisher's copyright policies.

La data sopra indicata si riferisce all'ultimo aggiornamento della scheda del Repository FloRe - The above-mentioned date refers to the last update of the record in the Institutional Repository FloRe

(Article begins on next page)

Self-Similar Solutions for the Heat Equation with a Signum-Type Source Term

Angiolo Farina^{*,a}, Roberto Gianni^a, Fabio Rosso^a

^a*Dipartimento di Matematica e Informatica “Ulisse Dini”, Università degli Studi di Firenze, Viale Morgagni 67/a, Firenze, 50134, Italy*

Abstract

We study the existence and multiplicity of self-similar solutions for the semilinear heat equation

$$u_t = u_{xx} + S(u),$$

where S is a signum-type nonlinearity. Such reaction term, characterized by discontinuous and non-Lipschitz behavior, gives rise to qualitative features that sharply differ from the classical theory, including the loss of uniqueness and the emergence of sign-changing profiles.

By exploiting the invariance under parabolic rescaling, we reduce the problem to the analysis of a nonlinear ODE for the similarity profile and show that the resulting boundary value problem may admit at most two sign-changing solutions. A key point of the analysis is the characterization of the parameter domains in which sign-changing self-similar solutions exist, which is achieved through a detailed study of an algebraic constraint arising from the matching of the similarity solutions with the initial condition.

These results highlight how weak regularity of the source/sink term can induce nonstandard dynamical effects in semilinear parabolic equations, including

^{*}Corresponding author

Email address: angiolo.farina@unifi.it (Angiolo Farina)

17 multiplicity of solutions and the onset of sign-changing structures in problems
18 where monotone behavior would normally be expected.
19 **Keywords:** semilinear heat equation, nonlinear ODE, non-uniqueness, self-
20 similar solutions.

21 1. Introduction

22 We consider the initial value problem

$$\begin{cases} u_t = u_{xx} + S(u), & x \in \mathbb{R}, \quad 0 < t, \\ u(x, 0) = -cx^2, & x \in \mathbb{R}, \end{cases} \quad (1)$$

23 where $c > 0$, and where S is a signum-type graph

$$\begin{cases} S(\xi) = 1, & \text{if } \xi > 0, \\ S(\xi) \in [\varepsilon, 1], & \text{if } \xi = 0, \\ S(\xi) = \varepsilon, & \text{if } \xi < 0, \end{cases} \quad (2)$$

24 with $\varepsilon < 0$. It is obvious that the form (2) corresponds to the signum graph
25 for $\varepsilon = -1$ and that any graph for which $S(\xi) = a$ for $\xi < 0$, $S(0) \in [a, b]$ and
26 $S(\xi) = b$ for $\xi > 0$, with $a < 0$ and $b > 0$, can be reduced to the form (2) by a
27 suitable scaling of the function u . The case $\varepsilon = 0$, corresponding to $S(\xi)$ equal
28 to the Heaviside graph, has been analyzed in [1] and [2].

29 Reaction terms of this type arise naturally in a variety of physical and biolog-
30 ical models, including threshold-type activation mechanisms, singular reaction
31 kinetics, combustion and effective descriptions of diffusion-driven processes in
32 heterogeneous media (see, e.g., [3–16]). The lack of regularity of the source/sink
33 term plays a central role in producing mathematical behavior that strongly de-
34 parts from the classical theory of semilinear parabolic equations. The central
35 aim of this study is to establish a detailed existence and multiplicity theory for
36 self-similar solutions of the form

$$u(x, t) = t^\gamma f\left(\frac{x}{\sqrt{t}}\right), \quad (3)$$

where γ is a suitable scaling exponent determined by the structure of the nonlinearity, and where the similarity profile f solves a nonlinear ordinary differential equation obtained via the invariance of the problem under parabolic rescaling. Self-similar solutions are of particular interest because they often describe intermediate asymptotic of more general solutions and can shed light on the long-time and singularity formation behavior of the system (see, e.g., [17–20]).

Our analysis proves the existence of multiple self-similar profiles. In particular, we establish the presence of one strictly negative self-similar solution, as well as the existence of two sign-changing profiles. These sign-changing solutions highlight a striking contrast with the classical case of smooth, Lipschitz continuous reaction terms, where the regularity of the source/sink term typically precludes oscillations and rules out the existence of sign-changing structures.

A key aspect of the problem is the phenomenon of *non-uniqueness*. The failure of the Lipschitz condition for $S(u)$ enables the emergence of distinct regular solutions from the same initial configuration, a phenomenon that is, here, rigorously demonstrated through the construction of at most three self-similar profiles. This behavior stands in sharp contrast with the standard theory for parabolic equations with smooth nonlinearities, where uniqueness is usually guaranteed by classical Picard-type arguments.

An additional and particularly intriguing consequence of the non-Lipschitz structure being the appearance of *sign-changing self-similar solutions* despite the initial datum is non-positive for $x \in (-\infty, \infty)$, which implies that, except at $t = 0$, $S(u)$ acts as sink term. Their existence indicates that the interplay between diffusion and a weakly regular reaction term can generate complex dynamics, including the possibility of transient or persistent changes of sign that are not directly dictated by the sign of the source. These complex dynamics are determined not only by the structure of the source/sink term, i.e., by $S(u)$, but are also strongly dependent on the profile of the initial datum. Indeed, if $u(x, 0)$ is monotone and there exists x_0 such that $u(x_0, 0) = 0$, then the solution to problem $(1)_1$ coupled with such an initial datum is unique [21].

Taken together, our findings reveal a rich variety of behaviors for non-

68 Lipschitz semilinear heat equations. They also provide a framework for the
69 systematic study of self-similar structures in problems where the failure of stan-
70 dard regularity assumptions leads to non-unique evolutions, oscillatory patterns,
71 and unconventional solution profiles. These results contribute to a growing body
72 of literature highlighting the subtle and often counterintuitive effects of low reg-
73 ularity in nonlinear parabolic equations.

74 The remainder of this work is organized as follows. In Sect. 2 we introduce
75 the self-similar framework, derive the associated similarity ODE, and describe
76 the matching procedure across the curve where the sign of the solution changes.
77 Section 3 contains the core of our analysis: we establish the existence of at
78 most two sign-changing similarity profiles, characterize their qualitative prop-
79 erties, and determine the ε parameters interval in which such solutions may
80 arise. In the same section we also analyze the algebraic condition linking the
81 similarity profiles to the initial datum and identify the critical thresholds gov-
82 erning existence and multiplicity. Finally, Sect. 4 summarizes the main results
83 and discusses their relevance to the dynamics of reaction-diffusion systems with
84 signum-type source terms.

85 2. Similarity solutions

86 We seek solutions of problem (1) in a self-similar form and which change
87 sign (i.e. solutions which are positive in a neighborhood of the origin). These
88 are of particular interest from a physical point of view. In fact, if u represents
89 the temperature, $u > 0$ means that the reaction has been activated. In other
90 words, an initial negative value everywhere except at one point is sufficient to
91 trigger a positive reaction, despite the term $S(u)$, which, at least at the initial
92 time, acts as a negative source.

93 The sign-changing similarity solutions will be added to the “classical” one
94 (see, e.g., [22]), i.e.

$$u(x, t) = -(2c + |\varepsilon|)t - cx^2 = t \left[-(2c + |\varepsilon|) - c \left(\frac{x}{\sqrt{t}} \right)^2 \right],$$

95 which is clearly in the form (3) but which never attains positive value (and for
 96 which $S(u)$ is identically ε).

97 Let us now focus on the similarity solutions to (1), searching among them for
 98 those that are positive for $t > 0$ and x within a suitable interval, and negative
 99 outside this interval. Considering

$$\eta = \frac{x}{\sqrt{t}},$$

100 we look for u in the form (3) with $\gamma = 1$, namely

$$u(x, t) = tf\left(\frac{x}{\sqrt{t}}\right). \quad (4)$$

101 Upon substitution in (1)₁, we find

$$f'' + \frac{\eta}{2}f' - f = -S(f), \quad (5)$$

102 where the prime denotes the derivative with respect to η . Because of symmetry,
 103 we solve (5) for $\eta \geq 0$. As for the “initial” conditions to be coupled to (5), we
 104 take

$$f'(0) = 0, \quad f(0) = \alpha > 0. \quad (6)$$

105 The first one is due to the symmetry of problem (1): u must be an even function
 106 in the space variable x and so f has to be even. The second one is motivated
 107 by the fact that we look for a solution u which is positive in a neighborhood of
 108 $x = 0$. Hence, recalling (2), equation (5) reduces to

$$f'' + \frac{\eta}{2}f' - f + 1 = 0, \quad \text{if } f > 0, \quad (7)$$

109 since $S(f) = 1$, because f is positive. If we take

$$f(\eta) = \alpha - \frac{1-\alpha}{2}\eta^2, \quad (8)$$

110 we have that both (7) and (6) are fulfilled. Moreover $f > 0$ provided

$$0 \leq \eta < \sqrt{\frac{2\alpha}{1-\alpha}}, \quad \text{and} \quad 0 < \alpha < 1. \quad (9)$$

111 Setting

$$\beta = \sqrt{\frac{2\alpha}{1-\alpha}}, \quad \Leftrightarrow \quad \alpha = \frac{\beta^2}{2+\beta^2}, \quad 0 < \beta < +\infty, \quad (10)$$

112 (8) rewrites as

$$f(\eta) = \frac{\beta^2}{2+\beta^2} \left(1 - \frac{\eta^2}{\beta^2}\right), \quad 0 \leq \eta \leq \beta, \quad (11)$$

113 while condition (9), understood on the (x, t) plane, means that u is positive

114 when $0 \leq x < \beta\sqrt{t}$, and $u = 0$ for $\eta = \beta$, i.e. $x = \beta\sqrt{t}$. In particular

$$f'(\eta)|_{\eta=\beta} = -\frac{2\beta}{2+\beta^2} = -\sqrt{2\alpha(1-\alpha)}.$$

115 If $\eta > \beta$, the Cauchy problem (7), (6) has to be replaced with the following one

116 (which holds if $f < 0$)

$$\begin{cases} f'' + \frac{\eta}{2}f' - f + \varepsilon = 0, & \eta > \beta, \\ f(\beta) = 0, \quad f'(\beta) = -\frac{2\beta}{2+\beta^2}, \end{cases} \quad (12)$$

117 where the two conditions on $\eta = \beta$ are the matching conditions (i.e. $f|_{\eta=\beta-} =$

118 $f|_{\eta=\beta+}$ and $f'|_{\eta=\beta-} = f'|_{\eta=\beta+}$) with (11). The solution to (12) is obtained by

119 setting $\varphi(\eta) = f(\eta) - \varepsilon$, and then proceeding as in [1], Appendix 1. We thus

120 have

$$f(\eta) = -2\beta e^{\beta^2/4} (1-\varepsilon) (\eta^2 + 2) \int_{\beta}^{\eta} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds - \varepsilon \left(\frac{\eta^2 + 2}{\beta^2 + 2} - 1 \right), \quad \beta < \eta < +\infty,$$

121 which, when $\varepsilon = 0$, coincides with formula (20) of [2].

122 3. Existence of similarity solutions

123 Summarizing, we have found the following self-similar profile

$$f(\eta) = \begin{cases} \frac{\beta^2 - \eta^2}{2 + \beta^2}, & 0 \leq \eta < \beta, \\ -2\beta e^{\beta^2/4} (1 - \varepsilon) (\eta^2 + 2) \int_{\beta}^{\eta} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds & \\ -\varepsilon \left(\frac{\eta^2 + 2}{\beta^2 + 2} - 1 \right), & \beta < \eta < +\infty, \end{cases} \quad (13)$$

124 with $\beta \in (0, \infty)$ linked to $\alpha \in (0, 1)$ via formula (10). However f given by (13)
 125 is not the self-similar solution of problem (1), since we still have to impose the
 126 initial condition (1)₂. To this end, we have to require an additional condition
 127 for f . Since (1)₂ entails

$$\frac{u(x, t)}{t} \bigg/ \frac{x^2}{t} \longrightarrow -c, \quad \text{as } t \rightarrow 0,$$

128 then f has to satisfy

$$\lim_{\eta \rightarrow +\infty} \frac{f(\eta)}{\eta^2} = -c. \quad (14)$$

129 Hence, exploiting (13)₂ we define

$$F(\beta, \varepsilon) = -\lim_{\eta \rightarrow \infty} \frac{f(\eta)}{\eta^2},$$

130 getting

$$F(\beta, \varepsilon) = 2\beta e^{\beta^2/4} (1 - \varepsilon) \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds + \frac{\varepsilon}{\beta^2 + 2}, \quad (15)$$

131 defined in the domain $\{0 < \beta < \infty, -\infty < \varepsilon < 0\}$.

132 Condition (14) therefore turns into the following algebraic equation

$$F(\beta, \varepsilon) = c. \quad (16)$$

133 It is thus a matter of determining for which values of $\varepsilon < 0$ and $c > 0$ the
 134 above equation admits solutions $\beta > 0$, and how many such solutions there are.

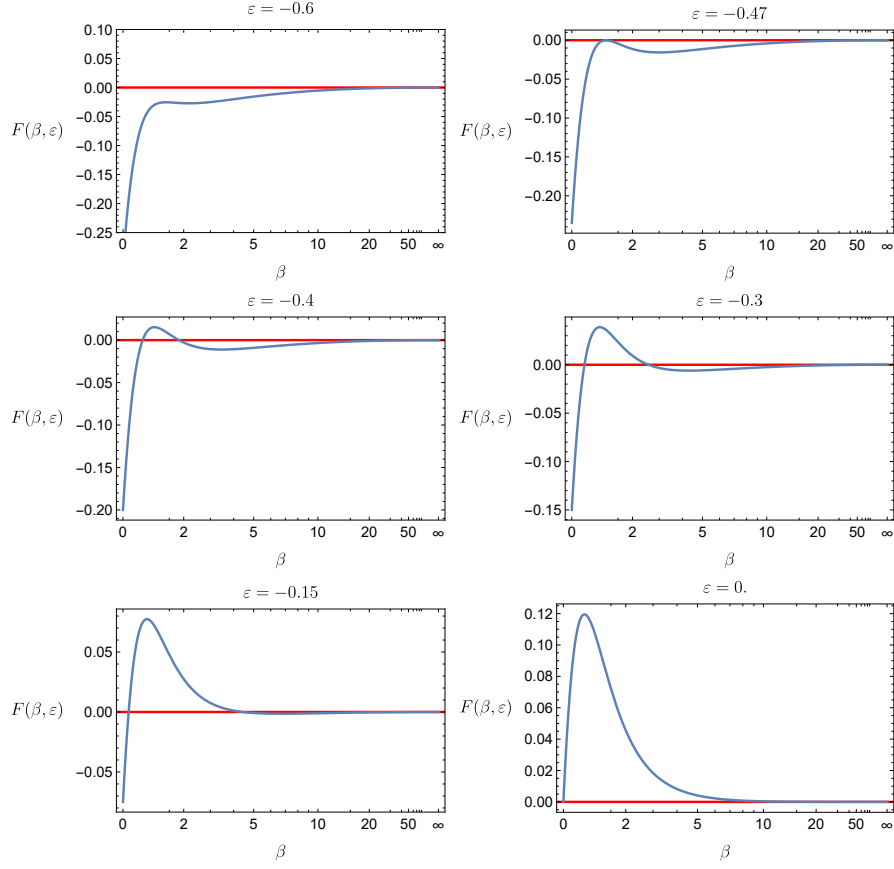


Figure 1: Effective behavior of $F(\beta, \varepsilon)$ for $\varepsilon \leq 0$. We can see that for ε sufficiently small $F(\beta, \varepsilon)$ is always negative so that equation (16) does not have any solution. We note that the case $\varepsilon = 0$ reproduces exactly Fig. 1 in [2].

Figure 1, showing six possible behaviors of F , clearly highlights that there can be values of ε for which (16) has no solutions.

Introducing

$$D(\beta, \varepsilon) = 2(1 - \varepsilon) \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds + \frac{\varepsilon e^{-\beta^2/4}}{\beta(\beta^2 + 2)}, \quad (17)$$

so that

$$F(\beta, \varepsilon) = \beta e^{\beta^2/4} D(\beta, \varepsilon), \quad (18)$$

equation (16) rewrites as

$$\beta e^{\beta^2/4} D(\beta, \varepsilon) = c. \quad (19)$$

So the existence of self-similar solutions with a region of positivity is linked to the positivity of $D(\beta, \varepsilon)$ in the domain $\{0 < \beta < \infty, -\infty < \varepsilon < 0\}$.

In the sequel we prove that, for suitable $\varepsilon < 0$, equation (16) admits at most two solutions, provided the parameter c lies in a suitable interval. This fact, which we emphasize once again, implies that problem (1) admits two sign-changing self-similar solutions. The proof goes through the following steps:

1. We first focus on the (β, ε) -plane and show that, in the set $\{0 < \beta, \varepsilon < 0\}$, there exists a non-empty domain, denoted by Q , with the property that if $(\beta, \varepsilon) \in Q$, then the function D is positive and hence equation (19) admits a solution for some positive value of c . Conversely, if $(\beta, \varepsilon) \notin \bar{Q}$, then D is strictly negative and therefore equation (19) has no solutions.
- In particular, we prove that the region Q has nonempty intersection with line of constant positive β , and is bounded above by the line $\varepsilon = 0$ and below by a continuous curve $\varepsilon_0(\beta)$. Moreover, $\varepsilon_0(\beta)$ is shown to attain a negative minimum $\varepsilon_{0\min}$. This allows us to conclude that if $\varepsilon < \varepsilon_{0\min}$, then equation (19) has no solutions and, consequently, the problem admits no changing sign self-similar solutions.

- 157 **2.** Still working in the set $\{0 < \beta, \varepsilon < 0\}$, we consider the intersection of the
158 domain Q with a horizontal line, that is, a line parallel to the β -axis, at
159 height ε , with $\varepsilon_{0\min} < \varepsilon < 0$. We show that this intersection is an interval.
- 160 **3.** We now focus the analysis to this interval (where we know that F is
161 positive) and show (and this is the key point) that F has a unique positive
162 maximum (which we denote by $c_{\text{crit}}(\varepsilon)$) and vanishes at the endpoints,
163 that is $F = 0$, at both interval extremes. This immediately implies that,
164 for each ε , with $\varepsilon_{0\min} < \varepsilon \leq 0$, equation (16) admits at most two solutions,
165 provided that the parameter c is smaller than the maximum of F , i.e.
166 provided that $0 < c < c_{\text{crit}}(\varepsilon)$.
- 167 **4.** The final step consists in showing that $c_{\text{crit}}(\varepsilon)$ is a monotonically increas-
168 ing function of ε .

169 **Point 1.**

170 Let us take an arbitrary positive value of β . We can state the following propo-
171 sition.

172 **Proposition 1.** *Given an arbitrary $\beta > 0$, there exists $\varepsilon_0 < 0$ such that*
173 *$D(\beta, \varepsilon) > 0$ for $\varepsilon_0 < \varepsilon \leq 0$, $D(\beta, \varepsilon_0) = 0$ and $D(\beta, \varepsilon) < 0$ for $\varepsilon < \varepsilon_0$.*

174 **Proof.** Given $\beta > 0$, we consider the lines

$$r_{1,\beta}(\varepsilon) = (\varepsilon - 1)m_1(\beta), \quad r_{2,\beta}(\varepsilon) = \varepsilon m_2(\beta),$$

175 with

$$m_1(\beta) = 2 \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds, \quad m_2(\beta) = \frac{e^{-\beta^2/4}}{\beta(\beta^2 + 2)}, \quad \text{with } \beta > 0,$$

176 so that we can write definition (17) as

$$D(\beta, \varepsilon) = -r_{1,\beta}(\varepsilon) + r_{2,\beta}(\varepsilon) = -(\varepsilon - 1)m_1(\beta) + \varepsilon m_2(\beta).$$

177 Hence $D(\beta, \varepsilon) > 0$ for $\varepsilon < 0$ entails $r_{2,\beta}(\varepsilon) > r_{1,\beta}(\varepsilon)$. Now the slope of the
 178 line $r_{2,\beta}(\varepsilon)$ is greater than the one of $r_{1,\beta}(\varepsilon)$, as we will show shortly. As a
 179 consequence there exists $\varepsilon_0 < 0$ such that $r_{1,\beta}(\varepsilon_0) = r_{2,\beta}(\varepsilon_0)$ and $r_{1,\beta}(\varepsilon) <$
 180 $r_{2,\beta}(\varepsilon)$ for $\varepsilon \in (\varepsilon_0, 0]$. The corresponding geometrical interpretation in terms of
 181 the lines $r_{1,\beta}(\varepsilon)$ and $r_{2,\beta}(\varepsilon)$ is illustrated in Fig. 2.

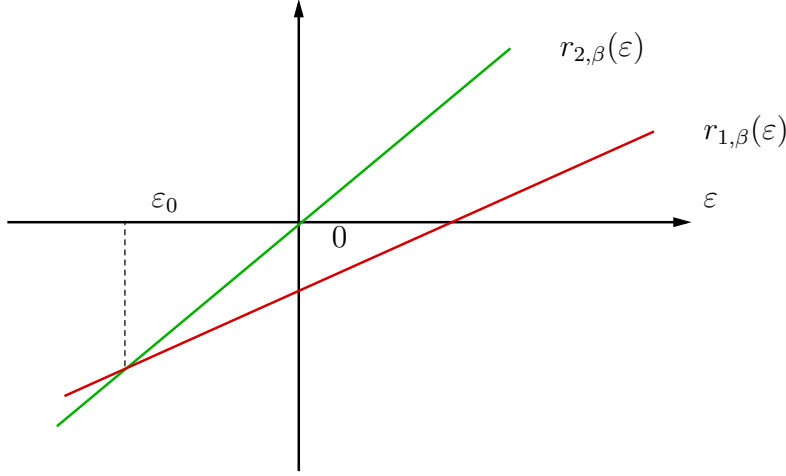


Figure 2: Behavior of the lines $r_{1,\beta}(\varepsilon)$ and $r_{2,\beta}(\varepsilon)$. Since r_2 is steeper than r_1 , relation $r_1 < r_2$ can be satisfied only in a limited portion of $(-\infty, 0)$, i.e. only in the interval $(\varepsilon_0, 0)$.

182 To show that the slope of $r_{2,\beta}(\varepsilon)$ is greater than the one of $r_{1,\beta}(\varepsilon)$, we consider

$$M(\beta) = m_2(\beta) - m_1(\beta), \quad (20)$$

183 for $\beta > 0$. Since

$$\lim_{\beta \rightarrow 0^+} M(\beta) = +\infty, \quad \lim_{\beta \rightarrow \infty} M(\beta) = 0, \quad (21)$$

184 and

$$M'(\beta) = -\frac{e^{-\beta^2/4}}{2\beta^2} < 0,$$

185 we have $M(\beta) > 0$, for any $\beta > 0$.

186 ■

187 A direct consequence of this proposition is that we can define a mapping which
 188 associates to each $\beta \in (0, \infty)$ a unique $\varepsilon_0 < 0$, such that

$$(\varepsilon_0 - 1) m_1(\beta) = \varepsilon_0 m_2(\beta).$$

189 We therefore define the function, denoted by $\varepsilon_0(\beta)$, as

$$\varepsilon_0(\beta) = -\frac{m_1(\beta)}{m_2(\beta) - m_1(\beta)} \stackrel{(20)}{=} -\frac{m_1(\beta)}{M(\beta)}. \quad (22)$$

190 In other words, the curve $\varepsilon_0(\beta)$ is such that $D(\beta, \varepsilon_0(\beta)) = 0$, and i.e. it corre-
 191 sponds to the zero level set of D in the quadrant $\{0 < \beta, \varepsilon < 0\}$. In particular,
 192 based on the properties of $M(\beta)$ discussed above, it follows that $\varepsilon_0(\beta)$ is a
 193 continuous function for all $\beta \in (0, \infty)$, is always negative and

$$\lim_{\beta \rightarrow 0^+} \varepsilon_0(\beta) = \lim_{\beta \rightarrow \infty} \varepsilon_0(\beta) = 0.$$

194 As for the computation of the first limit, it suffices to recall (21). Indeed

$$\lim_{\beta \rightarrow 0^+} \varepsilon_0(\beta) = -\lim_{\beta \rightarrow 0^+} \left(\frac{m_1(\beta)}{M(\beta)} \right) = -\lim_{\beta \rightarrow 0^+} \frac{2 \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2+2)^2} ds}{M(\beta)}.$$

195 The numerator clearly converges to a finite value (which can also be easily com-
 196 puted numerically and is approximately 0.44) while, by (21), the denominator
 197 tends to $+\infty$. Therefore,

$$\lim_{\beta \rightarrow 0^+} \varepsilon_0(\beta) = 0.$$

198 The second limit, i.e. $\lim_{\beta \rightarrow \infty} \varepsilon_0(\beta)$, is computed by evaluating $\lim_{\beta \rightarrow \infty} \frac{m_2(\beta)}{m_1(\beta)}$,
 199 since, by recalling (22), we have

$$\lim_{\beta \rightarrow \infty} \varepsilon_0(\beta) = -\lim_{\beta \rightarrow \infty} \frac{1}{\frac{m_2(\beta)}{m_1(\beta)} - 1}.$$

200 By applying l'Hôpital's rule, one easily shows that $\lim_{\beta \rightarrow \infty} \frac{m_2(\beta)}{m_1(\beta)} = \infty$, and
 201 thus $\lim_{\beta \rightarrow \infty} \varepsilon_0(\beta) = 0$.

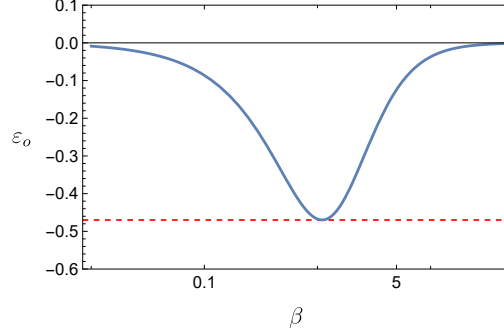


Figure 3: Graph of the function $\varepsilon_0(\beta)$. All values remain between ≈ -0.47 and 0.

Therefore there exists a finite minimum of $\varepsilon_0(\beta)$, which we denote as $\varepsilon_{0,\min}$.
In particular, $\varepsilon_{0,\min} \approx -0.47$ (see Fig. 3).

In the (β, ε) plane we define the domain

$$Q = \{(\beta, \varepsilon) \in \mathbb{R}^2 : 0 < \beta < \infty, \varepsilon_0(\beta) < \varepsilon \leq 0\},$$

which lies within the strip $\{0 < \beta < \infty, \varepsilon_{0,\min} < \varepsilon \leq 0\}$. We can therefore state
the following

Proposition 2. *Given $(\beta, \varepsilon) \in Q$, then the algebraic equation (16) is certainly fulfilled for some positive c .*

Proof. Let $(\beta, \varepsilon) \in Q$, i.e. $\beta > 0$ and $\varepsilon_0(\beta) < \varepsilon \leq 0$, and hence $D(\beta, \varepsilon)$ is positive. Moreover, D is continuous with respect to ε , since it is a linear combination of the continuous functions m_1 and m_2 . Therefore, the intermediate value Theorem implies that, for sufficiently small positive values of c , the algebraic equation (16) is certainly fulfilled.

■

Point 2.

Now, in order to determine the number of self-similar solutions, we section the domain Q with lines parallel to the β -axis. Let us indeed fix ε , $\varepsilon_{0,\min} < \varepsilon \leq 0$,

218 and consider the line parallel to the β -axis at ordinate ε and, moving along this
 219 line, we analyze the behavior of F (that is, we regard $F(\beta, \varepsilon)$ as a function of
 220 β only, $\beta \in \mathbb{R}^+$). We have

$$F(0, \varepsilon) = -\frac{|\varepsilon|}{2}, \quad (23)$$

221 and

$$\lim_{\beta \rightarrow \infty} F(\beta, \varepsilon) = 0, \quad (24)$$

222 since, by invoking l'Hôpital's rule, we obtain

$$\lim_{\beta \rightarrow \infty} F(\beta, \varepsilon) = (1 - \varepsilon) \lim_{\beta \rightarrow \infty} \frac{\int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds}{(2\beta e^{\beta^2/4})^{-1}} = 4(1 - \varepsilon) \lim_{\beta \rightarrow \infty} \frac{\beta^2}{(\beta^2 + 2)^3} = 0.$$

223 Moreover $F(\beta, \varepsilon) > 0$ only on those portions of the line that lie within the
 224 domain Q . However, in order to fully analyze the behavior of F along this line,
 225 it is necessary to show that the intersection between Q and any line parallel to
 226 the β -axis, at height ε , with $\varepsilon_{0\min} < \varepsilon < 0$, is an interval (which becomes the
 227 entire half-line $\beta > 0$, if $\varepsilon = 0$). We indeed have the following:

228 **Proposition 3.** *The intersection between the line parallel to the β -axis with*
 229 *ordinate ε , $\varepsilon_{0\min} < \varepsilon \leq 0$, with the domain Q is a finite interval.*

230 **Proof.** Every line parallel to the β -axis at level ε , with $\varepsilon_{0\min} < \varepsilon \leq 0$, has a
 231 non-empty intersection with the domain Q . We show that the intersection of
 232 this line with the domain Q is actually an interval. Focussing on D we have

$$\lim_{\beta \rightarrow 0} D(\beta, \varepsilon) = -\infty, \quad (25)$$

$$\lim_{\beta \rightarrow \infty} D(\beta, \varepsilon) = 0, \quad (26)$$

234 and computing $\frac{\partial D}{\partial \beta}$ we get

$$\begin{aligned}
\frac{\partial D}{\partial \beta} &= \frac{e^{-\beta^2/4}}{2\beta^2(\beta^2+2)^2} [-\varepsilon\beta^4 - 4\beta^2\varepsilon - 4\varepsilon - 4\beta^2] \\
&= \frac{e^{-\beta^2/4}}{2\beta^2(\beta^2+2)^2} [|\varepsilon|(\beta^2+2)^2 - 4\beta^2] \\
&= \frac{e^{-\beta^2/4}}{2\beta^2(\beta^2+2)^2} [\sqrt{|\varepsilon|}(\beta^2+2) + 2\beta] [\sqrt{|\varepsilon|}(\beta^2+2) - 2\beta].
\end{aligned}$$

235 Since we restrict ourselves to $\beta > 0$, we have that $\frac{\partial D}{\partial \beta} \geq 0$ if

$$\beta^2 - \frac{2\beta}{\sqrt{|\varepsilon|}} + 2 \geq 0, \quad \Rightarrow \quad \beta \leq \beta_1 \quad \text{and} \quad \beta_2 \leq \beta,$$

236 with

$$\begin{cases} \beta_1 = \frac{1}{\sqrt{|\varepsilon|}} - \sqrt{\frac{1}{|\varepsilon|} - 2}, \\ \beta_2 = \frac{1}{\sqrt{|\varepsilon|}} + \sqrt{\frac{1}{|\varepsilon|} - 2}. \end{cases}$$

237 Let first $\varepsilon_0^{\min} < \varepsilon < 0$. From the above computation of $\partial D / \partial \beta$, we have

$$\frac{\partial D}{\partial \beta} > 0, \quad \text{for} \quad 0 < \beta < \beta_1,$$

238

$$\frac{\partial D}{\partial \beta} < 0, \quad \text{for} \quad \beta_1 < \beta < \beta_2,$$

239 and

$$\frac{\partial D}{\partial \beta} > 0, \quad \text{for} \quad \beta > \beta_2.$$

240 Since the intersection of the line at height ε with Q is non-empty, D is positive
241 for some $\beta > 0$, and therefore $D(\beta_1, \varepsilon) > 0$. On the other hand, since D is
242 increasing for $\beta > \beta_2$ and (26) holds true, we necessarily have $D(\beta_2, \varepsilon) < 0$.
243 Recalling also (25), the intermediate value theorem, together with the above
244 monotonicity properties, implies that D has exactly two zeros, one in $(0, \beta_1)$
245 and the other in (β_1, β_2) . Hence D is positive exactly between these two zeros,
246 and the intersection of the line at height ε with Q is a finite interval. This
247 behavior is illustrated in Fig. 4.

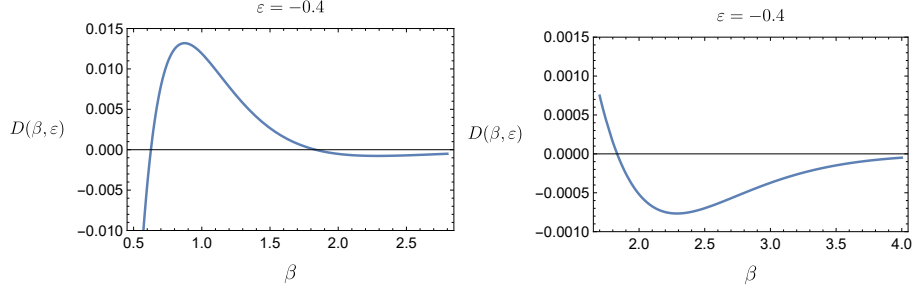


Figure 4: Effective behavior of $D(\beta, \varepsilon)$ for $\varepsilon_{0,\min} < \varepsilon \leq 0$. The function values vary over a very small scale: we have focused over different intervals for visualization purposes. The right panel shows a magnified view of the left panel over the interval $(1.5, 4)$. It is included to emphasize the presence of a negative minimum of D .

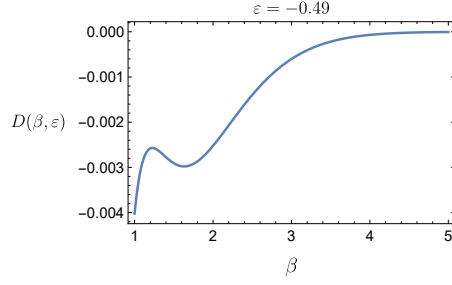


Figure 5: Effective behavior of $D(\beta, \varepsilon)$ for $-1/2 < \varepsilon < \varepsilon_{0,\min}$. In such a case D never attains positive values.

248 For $\varepsilon = 0$, it follows directly from (17) that $D(\beta, 0) > 0$ for every $\beta > 0$, so that
 249 the intersection with Q is the whole half-line $\beta > 0$.
 250 Incidentally, we have also provided a lower bound for $\varepsilon_{0,\min}$. Indeed, D has
 251 neither maxima nor minima if $|\varepsilon| \geq 1/2$ (i.e. $\varepsilon \leq -1/2$, since $\varepsilon < 0$). We thus
 252 have $\varepsilon_{0,\min} > -1/2$.

253 ■

254 Figure 5 illustrates the behavior of $D(\beta, \varepsilon)$, for $-1/2 < \varepsilon < \varepsilon_{0,\min}$. In this range,
 255 the maximum of D , still attained at β_1 , is negative, so that D remains negative
 256 throughout. In particular, this confirms that the intersection of a line parallel
 257 to the β -axis with Q is a finite interval, on which D attains a unique positive
 258 maximum.

259 **Point 3.**

Now, by analyzing the behavior of F on this segment, we can state the following result:

Proposition 4. *For every $\varepsilon \in (\varepsilon_{0,\min}, 0]$, the algebraic equation (16) is fulfilled by at most two values of β .*

Proof. We restrict ourselves to the segment given by the intersection of the line parallel to the β -axis, at height $\varepsilon \in (\varepsilon_{0,\min}, 0]$, with the domain Q . We have that F , given by (18), is positive on such an interval. To show that equation (16) has at most two solutions, we need to prove that F , viewed as a function of β , possesses only one maximum in such an interval. We start writing $\frac{\partial F}{\partial \beta}$, i.e.

$$\frac{\partial F}{\partial \beta} = (2 + \beta^2) (1 - \varepsilon) e^{\beta^2/4} \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds - \frac{2\beta}{(\beta^2 + 2)^2}. \quad (27)$$

It is clear that determining the zeros of $\frac{\partial F}{\partial \beta}$ is equivalent to finding the zeros of

$$G(\varepsilon, \beta) = \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds - \frac{2\beta e^{-\beta^2/4}}{(1 - \varepsilon)(\beta^2 + 2)^3}, \quad (28)$$

since, for $\beta > 0$, $\text{sgn} G = \text{sgn} \left(\frac{\partial F}{\partial \beta} \right)$. We first observe that $G(\varepsilon, 0) > 0$ and that $\lim_{\beta \rightarrow \infty} G = 0$. Next,

$$\frac{\partial G}{\partial \beta} = \frac{e^{-\beta^2/4}}{(\beta^2 + 2)^2} \left[-1 + \frac{1}{(1 - \varepsilon)(\beta^2 + 2)} \left(-2 + \beta^2 + \frac{12\beta^2}{\beta^2 + 2} \right) \right].$$

Hence, setting $s = \beta^2 + 2$, we have that $\frac{\partial G}{\partial \beta} = 0$, provided

$$|\varepsilon| s^2 - 8s + 24 = 0,$$

namely

$$s = \begin{cases} \frac{4 - \sqrt{16 - 24|\varepsilon|}}{|\varepsilon|}, \\ \frac{4 + \sqrt{16 - 24|\varepsilon|}}{|\varepsilon|}. \end{cases}$$

Before proceeding, it is appropriate to point out a couple of remarks. The first is that $\frac{\partial G}{\partial \beta}$ can vanish only if $|\varepsilon| < 2/3$ (which is fully consistent with what has already been noted, namely $|\varepsilon| < |\varepsilon_{0 \min}| < 1/2$). The second is that

$$s - 2 = \begin{cases} \frac{4 - 2|\varepsilon| + \sqrt{16 - 24|\varepsilon|}}{|\varepsilon|}, \\ \frac{4 - 2|\varepsilon| - \sqrt{16 - 24|\varepsilon|}}{|\varepsilon|}, \end{cases}$$

is certainly non-negative, so that the equation $\beta^2 = s - 2$, does not lead to any contradiction. Thus, considering $\beta > 0$, $\frac{\partial G}{\partial \beta}$ vanishes for $\beta = \beta_{\pm}$, where

$$\beta_{\pm} = \sqrt{\frac{4 - 2|\varepsilon| \pm \sqrt{16 - 24|\varepsilon|}}{|\varepsilon|}}.$$

In particular, since $\left. \frac{\partial G}{\partial \beta} \right|_{\beta=0} < 0$, we have that β_- corresponds to a local minimum while β_+ a local maximum. The possible behaviors of G are those represented in Fig. 6.

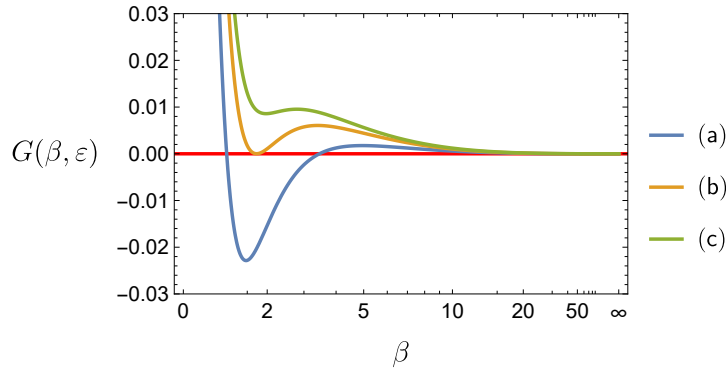


Figure 6: Possible behavior of the function G given by (28).

283 If G behaved as in cases (b) or (c) of Fig. 6, then G would be non-negative for
 284 every $\beta > 0$. Since, for $\beta > 0$, the sign of $\partial F/\partial\beta$ coincides with the sign of G ,
 285 the function F would be non-decreasing. Recalling (23) and (24), this would
 286 imply that F is always non-positive, in contradiction with the fact that F is
 287 positive for some β . Therefore, only case (a) can occur.

288 More precisely, β_- and β_+ are respectively a local minimum and a local
 289 maximum of G . Since $G(\varepsilon, 0) > 0$ and $G(\varepsilon, \beta_-) < 0$, while $G(\varepsilon, \beta_+) > 0$, the
 290 continuity of G implies the existence of exactly two positive zeros, say $\hat{\beta}_-$ and
 291 $\hat{\beta}_+$, such that

$$0 < \hat{\beta}_- < \beta_- < \hat{\beta}_+ < \beta_+.$$

292 Consequently,

$$\frac{\partial F}{\partial \beta} > 0, \quad \text{for } 0 < \beta < \hat{\beta}_-,$$

293

$$\frac{\partial F}{\partial \beta} < 0 \quad \text{for } \hat{\beta}_- < \beta < \hat{\beta}_+,$$

294 and

$$\frac{\partial F}{\partial \beta} > 0 \quad \text{for } \beta > \hat{\beta}_+.$$

295 Hence F attains a unique local maximum at $\hat{\beta}_-$ and a unique local minimum at
 296 $\hat{\beta}_+$. Therefore, taking (23) into account, the possible behaviors of F are those
 297 shown in Fig. 7.

298 Cases (ii) and (iii) are not possible because F is positive for some β , provided
 299 $\varepsilon \in (\varepsilon_{0,\min}, 0]$. Therefore, the only possible case is (i). Consequently, by the
 300 monotonicity properties of F established above, equation (16) admits at most
 301 two solutions.

302 ■

303 So, every $\varepsilon \in (\varepsilon_{0,\min}, 0]$ we denote by $c_{\text{crit}}(\varepsilon)$ the maximum of F , i.e.

$$c_{\text{crit}}(\varepsilon) = \max_{\beta > 0} F(\beta, \varepsilon).$$

304 Therefore c_{crit} represents the “critical” value of c below which two intersections
 305 exist. To determine $c_{\text{crit}}(\varepsilon)$ we use formula (18). Differentiating with respect
 306 to β we get

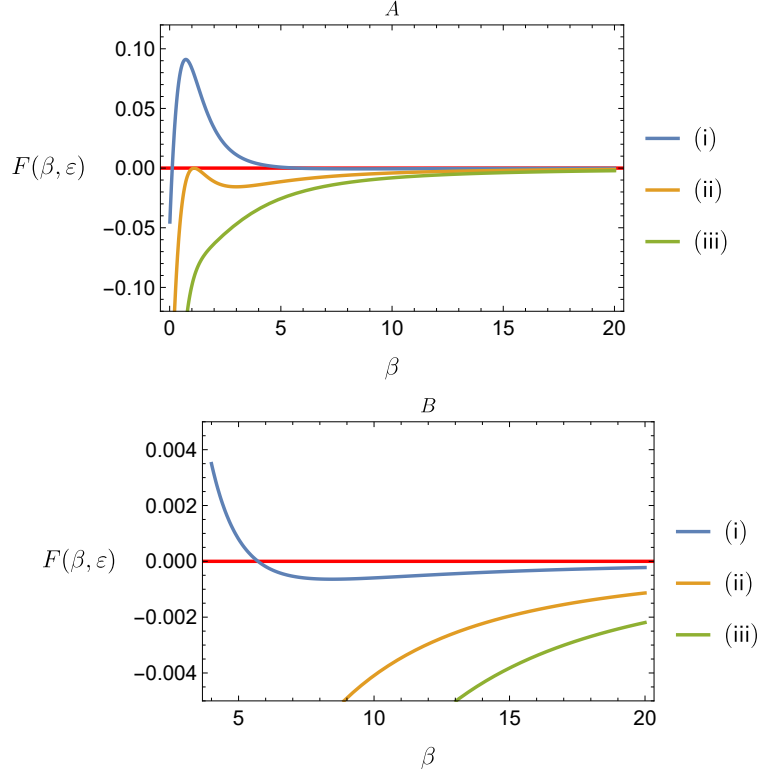


Figure 7: Possible behavior of the function F given by (15). Figure B presents a magnification of Fig. A over the interval $\beta \in (5, 20)$, clearly highlighting that, for $\varepsilon \in (\varepsilon_{0 \min}, 0)$, i.e. blue line, F indeed attains a negative minimum.

$$D(\beta, \varepsilon) = -\frac{\partial D(\beta, \varepsilon)}{\partial \beta} \frac{\beta}{1 + \beta^2/2}. \quad (29)$$

307 Then, denoting by $\beta^*(\varepsilon)$ the solution of (29) corresponding to the local maxi-
 308 mum, we have

$$c_{\text{crit}}(\varepsilon) = D(\beta^*, \varepsilon) \beta^* e^{\beta^{*2}/4} \underset{(18)}{=} F(\beta^*, \varepsilon). \quad (30)$$

309

310 **Point 4.**

311 Figure 8 shows c_{crit} in the allowable range of ε while Tab. 1 provides some of
 312 its numerically calculated values.

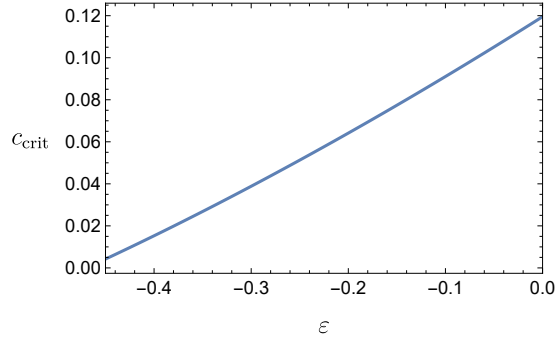


Figure 8: Function $c_{\text{crit}}(\varepsilon)$ in the allowable range of ε

Table 1: Calculated values of $c_{\text{crit}}(\varepsilon)$

ε	c_{crit}
-0.45	0.0042
-0.4	0.0152
-0.35	0.0268
-0.3	0.0388
-0.25	0.0512
-0.2	0.0641
-0.15	0.0773
-0.1	0.0910
-0.05	0.1050
0.0	0.1195

313 We actually can show that $c_{\text{crit}}(\varepsilon)$ is a monotonically increasing function. We
 314 indeed have the following:

315 **Proposition 5.** *The function $c_{\text{crit}}(\varepsilon)$ is a monotonic increasing for $\varepsilon \in (\varepsilon_{0,\min}, 0]$.*

316 **Proof.** We start recalling that $\beta^*(\varepsilon)$ is the solution of (29), namely

$$\frac{\partial}{\partial \beta} \left[\beta e^{\beta^2/4} D(\beta, \varepsilon) \right] \Big|_{\beta^*(\varepsilon)} = 0. \quad (31)$$

317 Hence $\beta^*(\varepsilon)$ gives the value β at which F attains its maximum, i.e. $c_{\text{crit}}(\varepsilon)$. We
 318 therefore exploit equation (30) to differentiate with respect to ε the function
 319 $c_{\text{crit}}(\varepsilon)$, namely

$$\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} = \frac{\partial}{\partial \varepsilon} [D(\beta^*, \varepsilon)] \beta^* e^{\beta^{*2}/4} + \frac{\partial}{\partial \beta^*} [D(\beta^*, \varepsilon) \beta^* e^{\beta^{*2}/4}] \frac{\partial \beta^*}{\partial \varepsilon}.$$

320 The second term vanishes because of (31) and, exploiting (17), we obtain

$$\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} = H(\beta^*) \beta^* e^{\beta^{*2}/4},$$

321 where

$$H(\beta) = \frac{e^{-\beta^2/4}}{\beta(\beta^2 + 2)} - 2 \int_{\beta}^{\infty} \frac{e^{-s^2/4}}{(s^2 + 2)^2} ds.$$

322 We thus have $\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} \geq 0$ provided $H(\beta) \geq 0$ for $0 < \beta < \infty$. We now remark
323 that

$$\lim_{\beta \rightarrow 0} H(\beta) = +\infty, \quad \lim_{\beta \rightarrow +\infty} H(\beta) = 0,$$

324 and that

$$\frac{dH(\beta)}{d\beta} = -\frac{e^{-\beta^2/4}}{2(\beta^2 + 2)} - \frac{e^{-\beta^2/4}}{\beta^2(\beta^2 + 2)} < 0, \quad \forall \beta > 0.$$

325 Hence $H(\beta)$ is positive for $0 < \beta < \infty$, and consequently $\frac{dc_{\text{crit}}(\varepsilon)}{d\varepsilon} > 0$.

326 ■

327 4. Conclusions

328 In the context of physical models, the present study is motivated by the
329 fundamental question of whether a reaction diffusion process governed by equation
330 (1), with $S(u)$ specified by (2), can be initiated under the condition that
331 the initial datum $u(x, 0)$ is negative everywhere (thus rendering the source term
332 inactive) except at a single point $x = 0$, where $u = 0$. A key characteristic of
333 model (1)₁ is that the positive source term becomes active exclusively for $u > 0$.
334 From this standpoint, sign-changing solutions of (1) are particularly relevant to
335 elucidating the underlying dynamics of the system.

336 The findings of this study reveal an unexpected outcome. The source/sink
337 term $S(u)$, which may assume both positive and negative values, can effectively

338 act as a source thereby initiating, for instance, a chemical reaction, provided that
 339 the initial condition exhibits small curvature (i.e., limited values of the constant
 340 c) and that the sink effect is sufficiently moderate, specifically for $|\varepsilon| < |\varepsilon_{0,\min}|$.
 341 It has been demonstrated that problem (1) admits at most two self-similar,
 342 sign-changing solutions for $\varepsilon \in (\varepsilon_{0,\min}, 0]$, and only for adequately small values
 343 of the parameter c . Moreover, for a given $\varepsilon \in (\varepsilon_{0,\min}, 0]$, the corresponding
 344 critical value of c , that is, the threshold beyond which problem (1) no longer
 345 admits self-similar sign-changing solutions, has been determined numerically.
 346 The behavior of $c_{\text{crit}}(\varepsilon)$ is shown in Fig. 8.

347 **Acknowledgments**

348 This research was partially supported by GNFM of Italian INDAM.

349 References

- 350 [1] R. Gianni and J. Hulshof. The semilinear heat equation with a heaviside
351 source term. *European Journal of Applied Mathematics*, 3(4):367–379, 1992.
- 352 [2] A. Farina and R. Gianni. Self-similar solutions for the heat equation with a
353 positive non-lipschitz continuous, semilinear source term. *Nonlinear Anal-
354 ysis: Real World Applications*, 79:104121, 2024.
- 355 [3] D. G. Aronson. Nonlinear diffusion in population genetics, combustion,
356 and nerve pulse propagation. *Partial Differential Equations and Related
357 Topics*, pages 1–23, 1975.
- 358 [4] J. Keener. Propagation and its failure in coupled systems of discrete ex-
359 citable cells. *SIAM Journal on Applied Mathematics*, 47(3):556–572, 1987.
- 360 [5] M. A. Herrero and J. J. L. Velázquez. Blow-up in parabolic equations
361 with nonlinearities of arbitrary growth. *Journal of Differential Equations*,
362 70(2):356–374, 1987.
- 363 [6] F. Merle and H. Zaag. Blow-up solutions of semilinear heat equations
364 with optimal rates. *Communications on Pure and Applied Mathematics*,
365 58(11):1597–1654, 2005.
- 366 [7] V. A. Galaktionov and J. L. Vázquez. A unified approach to blow-up
367 analysis for nonlinear heat equations. *Journal of Differential Equations*,
368 127(1):1–40, 1995.
- 369 [8] M. Chipot and B. Lovat. On the analysis of some reaction–diffusion systems
370 with singular reaction terms. *Nonlinear Analysis*, 30(6):3801–3816, 1997.
- 371 [9] G. A. Pavliotis and A. M. Stuart. Multiscale methods: averaging and
372 homogenization. *Springer*, 2008.
- 373 [10] L. A. Brown and D. R. Smith. Thermal ignition in heterogeneous media.
374 *Proceedings of the Royal Society A*, 451(1941):579–607, 1995.

- [11] V. Volpert and S. Petrovskii. Traveling waves for reaction–diffusion systems: A review. *Physics of Life Reviews*, 2(1):1–67, 2005.
- [12] J. D. Murray. *Mathematical Biology I: An Introduction*. Springer, 3rd edition, 2002.
- [13] A. Volpert, Vi. Volpert, and Vl. Volpert. *Traveling Wave Solutions of Parabolic Systems*. American Mathematical Society, 1994.
- [14] J. Norbury and A. M. Stuart. Parabolic free boundary problems arising in porous medium combustion. *IMA journal of applied mathematics*, 39(3):241–257, 1987.
- [15] J. Norbury and A. M. Stuart. A model for porous-medium combustion. *The Quarterly Journal of Mechanics and Applied Mathematics*, 42(1):159–178, 1989.
- [16] A. Friedman and A. E. Tzavaras. Combustion in a porous medium. *SIAM Journal on Mathematical Analysis*, 19(3):509–519, 1988.
- [17] G. I. Barenblatt. *Scaling, Self-Similarity, and Intermediate Asymptotics*. Cambridge University Press, 1996.
- [18] Ya. B. Zel’dovich and Yu. P. Raizer. *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*. Dover, 2002.
- [19] S. Kamin and L. A. Peletier. Large time behaviour of solutions of the porous medium equation. *Annales de l’IHP Analyse Non Linéaire*, 2(3):299–308, 1985.
- [20] A. Friedman and S. Kamin. The asymptotic behavior of gas in an expanding universe. *Transactions of the AMS*, 262(2):551–563, 1980.
- [21] R. Gianni and R. Ricci. Classical solvability of some free boundary problems through the geometry of the level lines. *Advances in Mathematical Sciences and Applications*, 5:557–567, 1995.

401 [22] J. R. Cannon. *The One-Dimensional Heat Equation*. Encyclopedia of Math-
402 ematics and its Applications. Cambridge University Press, Cambridge,
403 1984.