

Approval Voting with Fixed Electorates, Fixed Agendas and No Ballot Assumptions

Raffaele Berzi, Stéphane Gonzalez

▶ **To cite this version:**

Raffaele Berzi, Stéphane Gonzalez. Approval Voting with Fixed Electorates, Fixed Agendas and No Ballot Assumptions. 2026. ⟨hal-05717382⟩

HAL Id: hal-05717382

https://hal.science/hal-05717382v1

Preprint submitted on 14 Aug 2026

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

Approval Voting with Fixed Electorates, Fixed Agendas and No Ballot Assumptions*

Raffaele Berzi†
Dipartimento di Scienze per l'Economia e l'Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127 Firenze, Italy

Stéphane Gonzalez
GATE Lyon Saint-Étienne,
Université Jean Monnet,
CNRS, Université de Lyon, France

August 14, 2026

Abstract

We provide an axiomatic characterization of approval voting that does not impose any restriction on the structure of ballots. Voters submit abstract signals whose approval content is identified endogenously through the characterization itself. In contrast to the standard approach in the literature, we consider a fixed electorate and a fixed set of alternatives, and our analysis does not rely on consistency axioms or population-variation arguments. Approval voting is characterized by four axioms: (i) a richness condition requiring that every nonempty subset of alternatives can arise as the outcome under some unanimous profile; (ii) a unanimity axiom guaranteeing that alternatives supported by all voters are selected; and (iii) two axioms that adapt the classical principle of positive responsiveness to our abstract environment. Our results demonstrate that the distinctive properties of approval voting can be identified without presupposing any particular ballot format.

Keywords: Approval voting; axiomatic characterization; social choice; fixed electorate; positive responsiveness

JEL Classification: D71

1 Introduction

Approval voting is a voting rule under which each voter indicates, for every available alternative, whether she finds it acceptable, and the alternative(s) receiving the highest number of approvals are selected. Since the seminal contributions of Brams and Fishburn (1978) and Fishburn (1978), approval voting has been the subject of extensive axiomatic analysis.[1]

*This research was funded by the French National Research Agency (ANR) under the Condorcet project "ANR-24-EXMA-0001". Financial support from MODMAD is also gratefully acknowledged. Raffaele Berzi has also been supported by local funding from the Università degli Studi di Firenze. This article contributes to the doctoral research of Raffaele Berzi, who gratefully acknowledges the guidance of his PhD supervisor, Michele Gori. We would also like to thank Ulle Endriss and Vincent Merlin for their valuable comments. All remaining errors are our own.

†Corresponding author

[1] Comprehensive surveys of axiomatic characterizations are provided by Xu (2010) and Brandl and Peters (2022), the latter also providing new characterization results. Other recent contributions include Maniquet and Mongin

The vast majority of these characterizations are developed in frameworks with a variable electorate. This modeling choice is closely related to the central role played by consistency axioms (also known as reinforcement axioms), which link the outcome of a voting rule in disjoint electorates to its outcome in their union and thereby capture the additive structure of scoring rules. As documented by Brandl and Peters (2022), virtually all existing axiomatic characterizations of approval voting rely on some form of consistency and therefore exploit population variability in an essential way. A small number of contributions also allow variation in the set of alternatives (Vorsatz, 2007; Sato, 2014, 2019). Both Vorsatz (2007) and Sato (2019) also obtain characterization results in which the electorate and the set of alternatives are fixed.

More broadly, these axiomatic studies follow a common methodological pattern. The informational format through which voters express their opinions is fixed ex ante—whether in the form of linear orders, weak orders, numerical scores, or dichotomous preferences—and the analysis then focuses on how such information should be aggregated into a collective decision.

This modeling choice is far from innocuous. The ballot format is itself an institutional choice. Different formats allow voters to express different kinds of information, and their design has therefore been widely debated. Linear orders provide a simple ordinal language but do not capture preference intensity. Scoring ballots are richer in this respect, yet they raise difficult issues concerning the meaning of numerical scales and their interpersonal comparability. Approval ballots have progressively emerged as an attractive intermediate format, combining relative simplicity with substantial expressive power. As a result, they have generated a large literature studying their properties in single-winner settings (Laslier and Sanver, 2010; Alcalde-Unzu and Vorsatz, 2009; Bardal and Endriss, 2025), multi-winner environments (Lackner and Skowron, 2023), and other collective choice contexts (Bogomolnaia and Moulin, 2004; Bogomolnaia et al., 2005; Alós-Ferrer et al., 2025; Cato et al., 2025).

At the same time, the informational content of approval ballots is not entirely transparent, since the set of non-approved alternatives may reflect indifference, opposition, or a mixture of both. This ambiguity has motivated the study of richer formats that distinguish explicitly between approval, neutrality, and disapproval (Alcantud and Laruelle, 2014; Gonzalez et al., 2019). These debates highlight that ballot design is not merely technical but raises substantive informational and institutional questions. They also strengthen the case for an approach that does not take the ballot format as primitive, but instead seeks to recover its informational meaning from the behavior of the rule itself.

When the informational format is fixed exogenously, axiomatic comparisons across voting rules that rely on qualitatively different types of information become difficult. Approval voting is typically studied on domains of approval ballots or dichotomous preferences, whereas rules such as the Borda count aggregate profiles of rankings. Axiomatic conclusions are therefore tied to the informational environment on which the rule is defined. This restricts cross-rule comparisons and prevents a systematic assessment of whether the choice of the balloting procedure itself has normative content. An illustration is provided by Bubboloni and Gori (2025), who show that Young's axiomatic characterization of the Borda count extends to any domain satisfying certain structural properties. Since both linear orders and dichotomous preferences satisfy these properties, the same axioms characterize the Borda count on rankings and approval voting on dichotomous preferences. Hence,

(2015) and Sato (2019).

axioms imposed solely on the social choice function cannot distinguish between the two rules when each is evaluated on its natural informational domain.

A small number of contributions depart from the standard approach by characterizing voting rules on abstract or unrestricted domains, without committing to a specific ballot format. Myerson (1995) and Saari (1990) develop geometric approaches to derive scoring rules, while Pivato (2013) provides an algebraic derivation based on group-theoretic techniques. Closer in spirit to the present paper, Ceron and Gonzalez (2021) characterize approval-type rules without imposing any restriction on ballot formats and derive approval ballots endogenously. Despite their methodological differences, these approaches share a common feature: they rely on variable-population frameworks and make essential use of consistency axioms. Population variability serves as an identification device. The meaning of an individual ballot is inferred from how the rule behaves when the electorate changes; in particular, examining the one-voter case reveals the interpretation of individual messages.

A similar identification strategy can be implemented through agenda variation while keeping the electorate fixed. Informational content is then reconstructed from the rule's behavior across different feasible sets. This logic parallels revealed-preference analysis, where preferences are recovered from observed choices over varying opportunity sets (Sen, 1969, 1971, 1993). Just as variation in budgets allows the economist to identify underlying preferences, variation in the agenda can reveal the structure of individual inputs.

Once both population and agenda variation are eliminated, however, these external identification devices disappear. The informational structure of admissible messages can no longer be anchored through inter-population or inter-menu comparisons and must instead be recovered from the rule's behavior within a single, fixed institutional environment.

We study approval voting precisely in such a framework: both the electorate and the set of alternatives are fixed, and no axiom exploits population or agenda variability. In this environment, unrestricted signal spaces give rise to an identification problem, since distinct interpretations of signals may generate identical observable outcomes. The central methodological question is therefore how to recover the informational content of ballots without relying on external variation.

Our answer is to introduce an internal identification device based on unanimous profiles. Intuitively, a voter's signal is interpreted by asking what would happen if every voter submitted that same signal. The outcome of the corresponding unanimous profile serves as a benchmark that reveals which alternatives the signal supports. This internal preference-revelation mechanism allows us to reconstruct approval information directly from the rule's behavior within the fixed environment.

We obtain our main result: an axiomatic characterization of approval voting (Theorem 3.1) in this fixed-electorate, fixed-agenda framework. Voting rules are modeled as mappings from profiles of abstract signals to nonempty sets of alternatives. The characterization relies on four axioms: a richness condition ensuring that unanimous profiles can generate any nonempty outcome; a unanimity principle; and two responsiveness-type axioms governing how collective outcomes react to marginal changes in individual support. Importantly, "marginal support" is defined internally, by comparing outcomes when a single voter changes her signal while all others remain fixed, using unanimous benchmarks to interpret the direction of change. The first responsiveness axiom is a direct analogue of May's Positive Responsiveness (May, 1952);[2] the second, No Overreaction, prevents disproportionate reactions to minimal support expansions. Together, these principles pin

[2]See also (Horan et al., 2019; Goodin and List, 2006).

down an endogenous score representation and yield approval voting without presupposing approval ballots.

2 Setup of the model

We consider a voting environment in which individuals send abstract signals whose informational content is not specified a priori. Let X be a (possibly infinite) set of signals.

We fix two finite nonempty sets: a set of voters V, with $|V| = n \geqslant 2$, and a set of alternatives A, with $|A| = k \geqslant 2$. A profile is a function $p : V \to X$.

A selection rule is a mapping

$$\varphi : X^V \to P_0(A),$$

where $P_0(A)$ denotes the collection of nonempty subsets of A.

In standard models, the ballot format is specified independently of the aggregation rule: voters submit rankings, scores, or dichotomous judgments, and the rule operates on this given informational structure. In contrast, we treat the collection of information as part of the rule itself. The way signals are interpreted by the decision procedure is not neutral and cannot be separated from the voting method.

Approval voting corresponds to a particularly simple informational design. Each signal is interpreted as a set of acceptable alternatives. Formally, this amounts to the existence of an interpretation function $\psi : X \to P_0(A)$ translating abstract signals into approval ballots.

A selection rule φ is an approval rule if there exists a surjective mapping $\psi : X \to P_0(A)$ such that for every profile $p \in X^V$,

$$\varphi(p) = \arg\max_{a \in A} \left|\{v \in V : \ a \in \psi(p(v))\}\right|.$$

We restrict attention to nonempty approval sets. This entails no loss of generality. In standard approval voting, submitting an empty ballot or approving all alternatives affects all scores symmetrically and therefore does not change the set of maximizers. Allowing both would introduce signals that are distinct at the level of X but behaviorally equivalent under the rule. Excluding the empty set eliminates this redundancy.

For notational convenience, define for each $a \in A$, $p \in X^V$ and each $\psi : X \to P_0(A)$,

$$s_p^{\psi}(a) = \left|\{v \in V : \ a \in \psi(p(v))\}\right|.$$

An approval rule φ with interpretation function ψ can then be written equivalently as

$$\varphi(p) = \arg\max_{a \in A} s_p^{\psi}(a).$$

To avoid a heavy notation, we do not express the dependence of φ from ψ.

For any profile p and alternative a, we have $s_p^{\psi}(a) \in \{0, \ldots, n\}$. If $s_p^{\psi}(a) = n$, then a is unanimously approved and thus belongs to $\varphi(p)$. If $s_p^{\psi}(a) = 0$, then a receives no approval and cannot be selected, since $\psi(x) \neq \varnothing$ for all $x \in X$.

3 An Axiomatization of Approval Voting

We now turn to the axiomatization of approval voting in the fixed-electorate, fixed-agenda framework introduced above. The axioms are imposed directly on the selection rule φ, without presupposing any interpretation function. In particular, no primitive notion of approval is assumed: the informational content of signals will be inferred entirely from the behavior of the rule itself.

We say that a profile is unanimous if in that profile every voter submits the same signal. Given $p \in X^V$ and $v \in V$, define the unanimous profile $p_v \in X^V$ by $p_v(w) = p(v)$ for all $w \in V$. Thus, p_v is the profile in which all voters submit the same signal that voter v submits in p.

We begin with a minimal expressiveness requirement.

Sovereignty (S). For every $B \in P_0(A)$, there exists a unanimous profile $p \in X^V$ such that $\varphi(p) = B$.

Sovereignty ensures that unanimous profiles are sufficiently expressive to generate any feasible collective outcome. Even in the absence of a predefined ballot format, the rule must allow every nonempty subset of alternatives to arise when voters are perfectly aligned. This axiom is satisfied by virtually any rule defined on weak orders or numerical scores, since the top indifference class—or the set of score-maximizing alternatives—can range over all nonempty subsets of A as the common signal varies.

The next axiom links the behavior of the rule under unanimous profiles to its behavior at arbitrary profiles.

Consider a profile p and a voter v. If, under the unanimous profile p_v, an alternative a is selected, then the signal $p(v)$ reveals support for a. In this way, unanimous profiles provide an internal benchmark for interpreting individual signals.

Unanimity requires that if every voter sends a signal revealing support for a in this sense, then a must be selected at the original profile.

Unanimity (U). For every $a \in A$ and $p \in X^V$, if $a \in \varphi(p_v)$ for all $v \in V$, then $a \in \varphi(p)$.

This axiom is the natural adaptation of the standard unanimity principle in voting theory. Once signals are interpreted as ballots, it coincides with the usual requirement that unanimous support for an alternative guarantees its selection. It is a standard condition satisfied by a wide range of ballot formats. In particular, it holds for every positional scoring rule—including Borda defined on weak or strict orders—as well as for every Condorcet-consistent rule and every quota or threshold rule.

For the next two axioms, we continue to exploit unanimous profiles as a device for revealing the direction of individual signals.

Suppose that at a profile p, an alternative a is selected. Consider now a voter v whose signal does not reveal support for a, that is, $a \notin \varphi(p_v)$. Suppose this voter changes her signal so that, under unanimity, a becomes selected, while all other voters keep their signals unchanged. The collective outcome should then react in a predictable manner.

This requirement captures the idea that a marginal increase in support for a currently selected alternative should strengthen its collective position. It is the natural analogue, in our abstract

environment, of May's Positive Responsiveness principle (May, 1952).

Positive Responsiveness (PR). A selection rule φ satisfies Positive Responsiveness if, for every $p, p' \in X^V$, whenever $a \in \varphi(p)$ and there exists $v \in V$ such that $a \notin \varphi(p_v)$ and $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$, while $p'_w = p_w$ for every $w \in V \backslash v$, then $\varphi(p') = \{a\}$.

May's axiom requires that if an alternative is selected and receives additional support, it becomes the unique winner. Positive Responsiveness extends this idea to our general signal environment. It is likewise satisfied by a broad class of standard voting rules. Any argmax scoring rule defined on signals that can generate non-singleton unanimous outcomes satisfies PR. This includes Borda on weak orders, range voting, and dis&approval voting, which assigns scores in $\{-1, 0, 1\}$ to each alternative (Alcantud and Laruelle, 2014). Moreover, standard rules defined on linear orders also satisfy PR vacuously, since their unanimous outcomes are always singletons. As a result, the antecedent of PR—which requires $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$—can never be satisfied.

Positive Responsiveness constrains how the rule reacts to an increase in support for a currently selected alternative. A complementary requirement is needed to prevent excessive reactions in the opposite case. If an alternative that is not selected receives a mild increase in revealed support—namely, if a single voter modifies her signal so as to add exactly that alternative under unanimity—this limited favorable change should not be sufficient to make it the unique winner.

This condition can be viewed as a fixed-electorate analogue of the "no single voter overrides" principle used by Ceron and Gonzalez (2021) to characterize approval voting in variable-population frameworks. However, it is strictly weaker in spirit. Rather than preventing any individual deviation from drastically altering the outcome,[3] it only restricts monotone and minimal expansions of support. The axiom therefore rules out disproportionate reactions to small positive perturbations, while remaining compatible with approval voting.

No Overreaction (NOR). A selection rule φ satisfies No Overreaction if, for every $p, p' \in X^V$, if $a \notin \varphi(p)$ and there exists $v \in V$ such that $a \notin \varphi(p_v)$ and $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$, and $p'_w = p_w$ for every $w \in V \backslash \{v\}$, then $\varphi(p') \neq \{a\}$.

Observe that, like PR, this axiom is trivially satisfied by standard rules defined on linear orders. Beyond approval voting, NOR is also satisfied by a broad class of quota-based selection rules.[4]

Together, PR and NOR restrict how collective outcomes respond to local perturbations of individual signals, ruling out both perverse losses from increased support and disproportionate gains. Importantly, these restrictions are imposed entirely at the level of the selection rule itself and do not presuppose any particular informational structure on the signal space.

More generally, none of the axioms introduced above imposes any restriction on the nature of individual signals. Each of them is compatible with a wide variety of voting environments and ballot formats, including weak orders, linear orders, numerical evaluations, threshold-based rules, and other non-dichotomous domains. The signal space X itself remains entirely unrestricted: signals may consist of rankings, numerical scores, linguistic messages, or arbitrary symbols. The axioms constrain only the behavior of the selection rule φ, not the internal structure of the domain X. It is only through their joint interaction that approval-type information emerges endogenously.

[3] Which, in fact, is possible in a fixed-electorate framework.

[4] To fix ideas, consider for example a rule defined on weak orders that selects all candidates whose Borda score exceeds 2 whenever such candidates exist, and selects the whole set A otherwise.

We now state our main result.

Theorem 3.1. *Let φ be a selection rule. φ satisfies* ***Sovereignty****,* ***Unanimity****,* ***Positive Responsiveness*** *and* ***No Overreaction*** *if and only if it is an approval rule.*

Proof. **Step 1. If φ is an approval rule then it satisfies S.** By definition, there exists a surjective function $\psi : X \to P_0(A)$ such that, for every $p \in X^V$,

$$\varphi(p) = \arg\max_{a \in A} s_p^{\psi}(a).$$

Let $B \in P_0(A)$. By surjectivity, pick $x \in X$ with $\psi(x) = B$ and consider the unanimous profile p with $p(v) = x$ for all $v \in V$. Then $s_p^{\psi}(b) = n$ for all $b \in B$ and $s_p^{\psi}(a) = 0$ for all $a \in A \backslash B$, hence $\varphi(p) = B$.

Step 2. Let φ be a selection rule. Then φ is an approval if and only if the following conditions are satisfied:

(a) for every $p \in X^V$,

$$\varphi(p) = \operatorname*{argmax}_{a \in A} |\{v \in V : a \in \varphi(p_v)\}|; \tag{1}$$

(b) φ satisfies S.

Suppose that φ is an approval rule, so that there exists a surjective function $\psi : X \to P_0(A)$ such that, for every $p \in X^V$, we have

$$\varphi(p) = \operatorname*{argmax}_{a \in A} s_p^{\psi}(a).$$

By Step 1, we know that (b) is satisfied. To prove (a), we show that, for every $a \in A$, we have $a \in \psi(p(v))$ if and only if $a \in \varphi(p_v)$. Indeed, for every $v \in V$, if we have $a \in \psi(p(v))$, then we have $a \in \psi(p_v(w))$ for every $w \in V$, that implies $s_{p_v}^{\psi}(a) = n$, thus $a \in \varphi(p_v)$; if otherwise we have $a \notin \psi(p(v))$, then for every $w \in V$ we have $a \notin \psi(p_v(w))$, that is $s_{p_v}^{\psi}(a) = 0$, that implies $a \notin \varphi(p_v)$. Hence, for every $p \in X^V$, we have

$$\operatorname*{argmax}_{a \in A} |\{v \in V : a \in \psi(p(v))\}| = \operatorname*{argmax}_{a \in A} |\{v \in V : a \in \varphi(p_v)\}|,$$

thus

$$\varphi(p) = \operatorname*{argmax}_{a \in A} |\{v \in V : a \in \varphi(p_v)\}|.$$

If otherwise (a) and (b) are satisfied, we need to prove that there exists a surjective function ψ such that, for every $p \in X^V$,

$$\varphi(p) = \operatorname*{argmax}_{a \in A} s_p^{\psi}(a).$$

Consider the function $\alpha : X \to X^V$ that associates with every $x \in X$ the profile $\alpha(x) \in X^V$ defined, for every $v \in V$, by $\alpha(x)(v) = x$. We set $\psi = \varphi \circ \alpha$. For every $x \in X$, $\psi(x)$ is the outcome of φ for the profile $\alpha(x)$, that is the profile in which every voter sends the signal x. To prove that ψ is surjective, we need to show that φ is surjective as a function from $\alpha(X)$ to $P_0(A)$, or in other words we need to prove that, for every $B \in P_0(A)$, there exists a profile p in which every voter sends the

same signal and such that $\varphi(p) = B$. That is equivalent to say that φ satisfies **S**, that is (b). Take now $v \in V$ and $p \in X^V$. We have $\psi(p(v)) = \varphi(\alpha(p(v))) = \varphi(p_v)$. Thus, by (a),

$$\varphi(p) = \operatorname*{argmax}_{a \in A} |\{v \in V : a \in \psi(p(v))\}|.$$

Hence φ is an approval rule.

From the next steps, we will use the following convenient notation: for any selection rule φ, and for any profile $p \in X^V$ and alternative $a \in A$, we denote by $s_p^\varphi(a)$ the number of voters whose signal, when made unanimous, leads the rule to select a. Formally,

$$s_p^\varphi(a) = |\{v \in V : \ a \in \varphi(p_v)\}| .$$

Step 3. If φ is an approval rule then it satisfies U, PR and NOR.

U. Let $a \in A$ and $p \in X^V$. If $a \in \varphi(p_v)$ for all $v \in V$, then $s_p^\varphi(a) = n$, hence $s_p^\varphi(a) \geqslant s_p^\varphi(b)$ for all $b \in A$ and therefore $a \in \varphi(p)$.

PR. Let $p, p' \in X^V$ be such that $a \in \varphi(p)$ and there exists $v \in V$ with $a \notin \varphi(p_v)$, $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$, and $p'_w = p_w$ for all $w \neq v$. Then $s_{p'}^\varphi(a) = s_p^\varphi(a) + 1$ and $s_{p'}^\varphi(b) = s_p^\varphi(b)$ for all $b \neq a$. Since $a \in \varphi(p)$, we have $s_p^\varphi(a) \geqslant s_p^\varphi(b)$ for all $b \in A$, hence $s_{p'}^\varphi(a) > s_{p'}^\varphi(b)$ for all $b \neq a$, and therefore $\varphi(p') = \{a\}$.

NOR. Let $p, p' \in X^V$ be such that $a \notin \varphi(p)$ and there exists $v \in V$ with $a \notin \varphi(p_v)$, $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$, and $p'_w = p_w$ for all $w \neq v$. Then $s_{p'}^\varphi(a) = s_p^\varphi(a) + 1$ and $s_{p'}^\varphi(b) = s_p^\varphi(b)$ for all $b \neq a$. Since $a \notin \varphi(p)$, there exists $b \in A$ such that $s_p^\varphi(b) > s_p^\varphi(a)$, hence $s_{p'}^\varphi(b) \geqslant s_{p'}^\varphi(a)$ and therefore $\varphi(p') \neq \{a\}$.

Step 4 (Sufficiency). Assume that φ satisfies **S**, **U**, **PR** and **NOR**. By Step 2, it suffices to prove that (1) holds for every $p \in X^V$, i.e.,

$$\varphi(p) = \arg\max_{a \in A} s_p^\varphi(a).$$

Fix $p \in X^V$. We prove the two inclusions

$$\arg\max_{a \in A} s_p^\varphi(a) \subseteq \varphi(p) \qquad \text{and} \qquad \varphi(p) \subseteq \arg\max_{a \in A} s_p^\varphi(a).$$

First inclusion. Let $a \in \arg\max_{b \in A} s_p^\varphi(b)$ and set $m := s_p^\varphi(a)$. We show that $a \in \varphi(p)$ by descending induction on m.

If $m = n$, then $a \in \varphi(p_v)$ for all $v \in V$, hence $a \in \varphi(p)$ by **U**. Assume now $m < n$ and that the claim holds for $m + 1$. Since $m < n$, there exists $v \in V$ such that $a \notin \varphi(p_v)$. By **S**, choose a profile $p' \in X^V$ such that

$$\varphi(p'_v) = \varphi(p_v) \cup \{a\} \quad \text{and} \quad p'_w = p_w \text{ for all } w \neq v.$$

Then $s_{p'}^\varphi(a) = m + 1$ and $s_{p'}^\varphi(b) = s_p^\varphi(b)$ for all $b \neq a$. In particular, $a \in \arg\max_{b \in A} s_{p'}^\varphi(b)$, so by the induction hypothesis we have $a \in \varphi(p')$.

We claim that $\varphi(p') = \{a\}$. Otherwise, pick $a' \in \varphi(p')$ with $a' \neq a$. Since $s_{p'}^\varphi(a') < s_{p'}^\varphi(a) \leqslant n$, there exists $v' \in V$ such that $a' \notin \varphi(p'_{v'})$. By **S**, choose $p'' \in X^V$ such that

$$\varphi(p''_{v'}) = \varphi(p'_{v'}) \cup \{a'\} \quad \text{and} \quad p''_w = p'_w \text{ for all } w \neq v'.$$

By **PR**, we get $\varphi(p'') = \{a'\}$. Yet the score of a is unchanged, i.e., $s^{\varphi}_{p''}(a) = s^{\varphi}_{p'}(a) = m+1$, and a remains a score maximizer at p''. Applying the induction hypothesis at level $m+1$ yields $a \in \varphi(p'')$, a contradiction. Hence $\varphi(p') = \{a\}$.

Finally, since p' differs from p only at v, $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$ and $\varphi(p') = \{a\}$, axiom **NOR** implies $a \in \varphi(p)$.

Second inclusion. Conversely, suppose that $a \in \varphi(p)$. Let $m := s^{\varphi}_{p}(a)$.

If $m = n$, then $a \in \arg\max_{b \in A} s^{\varphi}_{p}(b)$ holds trivially. Assume $m < n$. Then there exists $v \in V$ such that $a \notin \varphi(p_v)$.

By **S**, let p' be a profile such that

$$\varphi(p'_v) = \varphi(p_v) \cup \{a\} \quad \text{and} \quad p'_w = p_w \text{ for all } w \in V \backslash \{v\}.$$

Since $a \in \varphi(p)$ and $a \in \varphi(p'_v)$ while $p'_w = p_w$ for all $w \neq v$, **PR** implies that $\varphi(p') = \{a\}$.

We claim that a is the unique maximizer of the score at p'. Indeed, if there existed $b \neq a$ with $s^{\varphi}_{p'}(b) \geqslant s^{\varphi}_{p'}(a)$, then $b \in \arg\max_{c \in A} s^{\varphi}_{p'}(c)$, and by the first inclusion (applied to p') we would have $b \in \varphi(p')$, contradicting $\varphi(p') = \{a\}$. Hence,

$$s^{\varphi}_{p'}(a) > s^{\varphi}_{p'}(b) \quad \text{for all } b \neq a.$$

Finally, by construction we have $s^{\varphi}_{p'}(a) = s^{\varphi}_{p}(a) + 1 = m + 1$ and $s^{\varphi}_{p'}(b) = s^{\varphi}_{p}(b)$ for all $b \neq a$. Therefore, for every $b \neq a$,

$$s^{\varphi}_{p}(a) = s^{\varphi}_{p'}(a) - 1 \geqslant s^{\varphi}_{p'}(b) - 1 = s^{\varphi}_{p}(b),$$

which shows that $a \in \arg\max_{b \in A} s^{\varphi}_{p}(b)$. □

The theorem shows that the distinctive features of approval voting can be identified even when the ballot format is left completely unrestricted. In particular, the characterization does *not* rely on the two standard identification devices of the literature, namely population variation (consistency / reinforcement axioms) and agenda variation. The key methodological move is to replace these external sources of identification with an *internal* one that operates entirely within the fixed institutional environment: unanimous profiles p_v provide a behavioral benchmark that endogenizes the meaning of each individual signal.

More precisely, the event $a \in \varphi(p_v)$ is interpreted as "voter v's signal reveals support for a," and this yields the endogenous score $s^{\varphi}_{p}(a) = |\{v \in V : \ a \in \varphi(p_v)\}|$, counting how many signals, when made unanimous, lead the rule to select a. From this perspective, **Sovereignty** plays a dual role. First, it is a minimal expressiveness requirement: any nonempty subset of alternatives must be attainable under perfect agreement. Second, it is the richness condition that allows one, in the sufficiency proof, to *construct* one-voter deviations that add exactly one alternative to the unanimous outcome. These deviations are the technical substitute for inter-population comparisons.

The axioms **PR** and **NOR** are then natural adaptations of positive responsiveness to an unordered signal space. They do not appeal to an exogenous notion of "increasing" a ballot, but to a *revealed* marginal gain in support defined through unanimous benchmarks. **PR** requires that a marginal

gain in support for a currently selected alternative makes it the unique winner (a direct analogue of May's axiom in binary settings), while **NOR** prevents a disproportionate reaction in the opposite direction: a marginal gain in support for a previously unselected alternative cannot by itself turn it into the unique winner. Together, these two axioms pin down the local response of outcomes to minimal support expansions, ruling out both perverse losses and excessive jumps. The conclusion is an endogenous score representation, hence an approval rule: $\varphi(p) = \arg\max_{a\in A} s_p^{\varphi}(a)$.

4 Independence of the axioms

Our goal in this section is to prove the independence of the axioms of Theorem 3.1. For each axiom we define a selection rule that does not satisfy that axiom but satisfies the others.

We denote by $L(A)$ the set of all the linear orders on A, that is, the set of all the complete, antisymmetric and transitive relations on A. A selection rule φ is a Borda rule if there exists a surjective mapping $\psi_B : X \to L(A)$ such that, for every $p \in X^V$,

$$\varphi(p) = \operatorname*{argmax}_{a\in A} \sum_{v\in V} \beta_a(p(v)),$$

where, for each $p \in X^V$, $a \in A$ and $v \in V$, $\beta_a(p(v)) = |\{b \in A : a\psi_B(p(v))b\}|$.

The idea of Borda rules is to translate the signals into linear orders, that can be seen as rankings of the alternatives. For each ranking, the alternatives get a score that is the number of alternatives in the same of lower positions in the ranking. In the outcome of a profile we find the alternatives with the higher scores for the signals in that profile.

Proposition 4.1. *Let φ_B be a Borda rule . Then φ_B satisfies **U**, **PR**, **NOR** and does not satisfy **S**.*

Proof. **S**: By definition, for every $p \in X^V$ and $v \in V$, $\psi_B(p(v))$ is a linear order, hence there exists a unique $a \in A$ such that $a\psi_B(p(v))b$ for every $b \in A$. Thus, if $p(w) = p(v)$ for every $w \in V$, we have $\varphi_B(p) = \{a\}$. We have shown that the outcome of a unanimous profile is a set with a single alternative, thus it is impossible, for instance, to obtain A as outcome of a unanimous profile.

U: Let $p \in X^V$ and $a \in A$ be such that $a \in \varphi_B(p_v)$ for every $v \in V$. We have that, for every $v \in V$ and $b \in A$, $\beta_a(p(v)) \geqslant \beta_b(p(v))$. Thus, for every $b \in A$, we have $\sum_{v\in V} \beta_a(p(v)) \geqslant \sum_{v\in V} \beta_b(p(v))$, that is, $a \in \varphi_B(p)$.

PR and **NOR**: As we have seen, the outcomes of unanimous profiles for the Borda rule are singletons. Thus, it is impossible to have two profiles p and p', $a \in A$ and $v \in V$ such that $a \notin \varphi_B(p_v)$ and $\varphi_B(p'_v) = \varphi_B(p_v) \cup \{a\}$. For this reason, the two axioms are trivially satisfied by φ_B. □

A selection rule φ is a union rule if there exists a surjective mapping $\psi_U : X \to P_0(A)$ such that, for every $p \in X^V$,

$$\varphi(p) = \{a \in A : \exists v \in V \text{ st } a \in \psi_U(p(v))\}.$$

Like in the case of Approval, for the union rules the signals are translated into nonempty sets of alternatives. In the outcome of a profile we find the alternatives that are in at least one set corresponding to a signal in that profile.

Proposition 4.2. *Let φ_U be a union rule. Then φ_U satisfies **S**, **U**, **NOR** and does not satisfy **PR**.*

Proof. **S**: Let $B \in P_0(A)$ and consider $x \in X$ such that $\psi_U(x) = B$. Such x exists since, by definition, ψ_U is surjective. If p is the profile such that for every $v \in V$, $p(v) = x$ then we have $\varphi_U(p) = \{a \in A : \exists v \in V \text{ st } a \in \psi_U(p(v))\} = B$.

U: Let $p \in X^V$ and $a \in A$ be such that $a \in \varphi_U(p_v)$ for every $v \in V$. Then we have $a \in \psi_U(p(v))$ for every $v \in V$, hence $a \in \varphi_U(p)$.

NOR: Let $p, p' \in X^V$ and $a \in A$ be such that $a \notin \varphi(p)$ and there exists $v \in V$ such that $a \notin \varphi(p_v)$, $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$, and $p'_w = p_w$ for every $w \in V\backslash\{v\}$. Since $\varphi_U(p) \neq \varnothing$, there exists $b \in A\backslash\{a\}$ such that $b \in \varphi_U(p)$, and by construction we also have $b \in \varphi(p')$. Thus $\varphi(p') \neq \{a\}$.

PR: Since φ_U satisfies **S**, we know that we can obtain any nonempty subset of A as outcome of a unanimous profile. Fix $a, b \in A$, $v \in V$ and consider $p \in X^V$ such that $\varphi_U(p_w) = \{a, b\}$ for every $w \in V\backslash\{v\}$ and $\varphi_U(p_v) = \{b\}$. We have $\varphi_U(p) = \{a, b\}$, and if we consider $p' \in X^V$ such that $p_w = p'_w$ for every $w \in V\backslash\{v\}$ and $\varphi_U(p'_v) = \varphi_U(p_v) \cup \{a\} = \{a, b\}$ we still have $\varphi_U(p') = \{a, b\}$, thus $\varphi(p') \neq \{a\}$. □

We denote by N the set of all functions $\nu : A \to \{0, 1, 2\}$. A selection rule φ is a 3-level range rule if there exists a surjective mapping $\psi_R : X \to N$ such that, for every $p \in X^V$,

$$\varphi(p) = \operatorname*{argmax}_{a \in A} \sum_{v \in V} \psi_R(p(v))(a).$$

For the 3-level range rules we translate the signals into functions that assign a score for each alternatives. Given a profile, we sum the scores correspondent to each signal and take the alternatives with the higher scores.

Proposition 4.3. *Let φ_R be a 3-level range rule. Then φ_R satisfies **S**, **U**, **PR** and does not satisfy **NOR**.*

Proof. **S**: Let $B \in P_0(A)$ and consider $x \in X$ such that $\psi_R(x)(b) = 2$ for every $b \in B$ and $\psi_R(x)(a) = 0$ for every $a \in A\backslash B$. Such x exists since, by definition, ψ_R is surjective. If p is the profile such that for every $v \in V$, $p(v) = x$ then we have $\varphi_R(p) = B$.

U: Let $p \in X^V$ and $a \in A$ be such that $a \in \varphi_R(p_v)$ for every $v \in V$. Then we have, for every $v \in V$, that $\psi_R(p(v))(a) \geqslant \psi_R(p(v))(b)$ for every $b \in A$. Thus, $\sum_{v \in V} \psi_R(p(v))(a) \geqslant \sum_{v \in V} \psi_R(p(v))(b)$ for every $b \in A$, that is, $a \in \varphi_R(p)$.

PR: Let $p, p' \in X^V$ and $a \in A$ be such that $a \in \varphi(p)$ and there exists $v \in V$ such that $a \notin \varphi(p_v)$, $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$, and $p'_w = p_w$ for every $w \in V\backslash\{v\}$. Since $a \in \varphi(p)$ we have that $\sum_{v \in V} \psi_R(p(v))(a) \geqslant \sum_{v \in V} \psi_R(p(v))(b)$ for every $b \in A$. We have $a \notin \varphi(p_v)$ and $\varphi(p'_v) = \varphi(p_v) \cup \{a\}$, thus $\psi_R(p'(v))(a) > \psi_R(p(v))(a)$, while $\psi_R(p'(v))(b) = \psi_R(p(v))(b)$ for every $b \in A\backslash\{a\}$, and $\psi_R(p'(w))(b) = \psi_R(p(w))(b)$ for every $w \in V\backslash\{v\}$ and $b \in A$. Thus $\sum_{v \in V} \psi_R(p'(v))(a) \geqslant \sum_{v \in V} \psi_R(p'(v))(b)$ for every $b \in A\backslash\{b\}$, that is, $\varphi(p') = \{a\}$.

NOR: Since φ_R satisfies **S**, we know that we can obtain any nonempty subset of A as outcome of a unanimous profile. Fix $a, b \in A$, $v_1, v_2 \in V$ and consider $p \in X^V$ such that $\psi_R(p(v_1))(a) = 0$,

$\psi_R(p(v_1))(b) = 2$, $\psi_R(p(v_2))(a) = 1$, $\psi_R(p(v_2))(b) = 0$, $\psi_R(p(w))(a) = \psi_R(p(w))(b) = 2$ for every $w \in V\backslash\{v_1, v_2\}$ and $\psi_R(p(v))(c) = 0$ for every $c \in A\backslash\{a, b\}$ and $v \in V$. We have $\varphi_R(p) = \{b\}$, $a \notin \varphi_R(p_{v_1})$ and $\sum_{v\in V} \psi_R(p(v))(b) = \sum_{v\in V} \psi_R(p(v))(a) + 1$. Consider now the profile p' that coincides with p except for the fact that $\psi_R(p'(v_1))(a) = 2$. We have $\varphi_R(p'_{v_1}) = \varphi_R(p_{v_1}) \cup \{a\} = \{a, b\}$, and $\sum_{v\in V} \psi_R(p'(v))(a) = \sum_{v\in V} \psi_R(p'(v))(b) + 1$. Thus $\varphi_R(p') = \{a\}$. □

To define the last rule we set $n = 2$ and $k = 3$, so that we have two voters, $V = \{v, w\}$, and three alternatives, $A = \{a_1, a_2, a_3\}$. Like in the case of an approval rule, let $\psi_Q : X \to P_0(A)$ be a surjective function. We need to define a partition of the set of all profiles X^V: for each $(t_1, t_2, t_3) \in \{0, 1, 2\}^3$, we set $Q_{(t_1,t_2,t_3)} = \{p \in X^V : \forall i \in \{1, 2, 3\}, s_p^{\psi_Q}(a_i) = t_i\}$. Each profile belongs to exactly one set of the form $Q_{(t_1,t_2,t_3)}$. For example, if we have $p(v) \in \psi^{-1}(\{a_1, a_2\})$ and $p(w) \in \psi^{-1}(\{a_1\})$, we have $p \in Q_{(2,1,0)}$. Let φ be the approval rule defined by ψ_Q. We define the rule φ_Q as follows: for each $p \in Q_{(1,1,0)}$, $\varphi_Q(p) = \{a_2\}$; for each $p \in Q_{(1,1,1)}$, $\varphi_Q(p) = \{a_2, a_3\}$; for each $p \in Q_{(2,1,0)}$, $\varphi_Q(p) = \{a_3\}$; for each $p \in Q_{(2,1,1)}$, $\varphi_Q(p) = \{a_3\}$; for each $p \in Q_{(2,1,2)}$, $\varphi_Q(p) = \{a_3\}$ and for all the remaining $p \in X^V$, $\varphi_Q(p) = \varphi(p)$. The rule φ_Q is a modified version of the approval rule, where we identify some special cases with different outcomes.

Proposition 4.4. *The rule φ_Q satisfies **S**, **PR**, **NOR** and does not satisfy **U**.*

Proof. **S**: Since the unanimous profiles does not fall under the special cases that we have identified, in those cases φ_Q behaves like an approval rule, that satisfies **S**.

U: If we have $p \in Q_{(2,1,0)}$, by definition we have $\varphi_Q(p) = \{a_3\}$. Since we have $a_1 \notin \varphi_Q(p)$, φ_Q does not satisfy **U**.

PR and **NOR**: In this case, the proof is a simple case-by-case check of all the possible transitions between the profiles. The details are omitted for brevity, but it is possible to verify that the transition from the special cases to the approval cases does not violate **PR** and **NOR**, thus the axioms are satisfied. □

5 Conclusion

We have provided an axiomatic characterization of approval voting in a framework with a fixed electorate, a fixed agenda, and no prior restriction on the structure of ballots. The result complements the variable-population approach of Ceron and Gonzalez (2021): where ballot meaning is identified by varying the electorate and invoking consistency requirements, our setting forecloses both strategies and compels identification to proceed entirely from within a single institutional environment.

The characterization also clarifies the informational basis of approval voting. Once its axioms are accepted, the instrumental content of a ballot is fully captured by the set of alternatives it reveals as supported—approval ballots are not primitives but objects recovered by the rule's internal logic. Two ballots inducing the same approval set therefore have identical consequences for the collective outcome. This instrumental equivalence does not, however, exhaust what ballot design may matter for. Experimental evidence suggests that labeling and format can influence voter behavior even when they carry no additional aggregative weight (Baujard et al., 2018); legal or political considerations may likewise justify richer formats that preserve expressive freedom or secrecy, and the design of expressive ballot formats raises substantive questions of its own (Delemazure, 2025). Our result

isolates the minimal informational content required to implement approval voting, while leaving open the possibility that richer structures remain desirable for reasons beyond aggregation.

More broadly, the paper suggests a reorientation of perspective. Rather than specifying a ballot format ex ante and imposing axioms on the aggregation rule defined over it, one may ask which informational structures are implied by basic behavioral principles on collective choice. Approval voting then emerges not only as a scoring rule, but as the minimal informational structure consistent with local responsiveness and unanimity. Whether similar internal identification strategies can recover the foundations of other voting rules is a natural direction for future work.

Declarations of competing interest

The authors have no competing interests to declare that are relevant to the content of this article.

References

Jorge Alcalde-Unzu and Marc Vorsatz. Size approval voting. *Journal of Economic Theory*, 144(3): 1187–1210, 2009.

José Carlos R Alcantud and Annick Laruelle. Dis&approval voting: a characterization. *Social Choice and Welfare*, 43(1):1–10, 2014.

Carlos Alós-Ferrer, Michele Garagnani, and Jaume García-Segarra. The framing of elections: cooperation vs. competition. *Social Choice and Welfare*, 65(2):363–385, 2025.

Tuva Bardal and Ulle Endriss. Axiomatic analysis of approval-based scoring rules. *Games and Economic Behavior*, 2025.

Antoinette Baujard, Frédéric Gavrel, Herrade Igersheim, Jean-François Laslier, and Isabelle Lebon. How voters use grade scales in evaluative voting. *European Journal of Political Economy*, 55(C): 14–28, 2018.

Anna Bogomolnaia and Hervé Moulin. Random matching under dichotomous preferences. *Econometrica*, 72(1):257–279, 2004.

Anna Bogomolnaia, Hervé Moulin, and Richard Stong. Collective choice under dichotomous preferences. *Journal of Economic Theory*, 122(2):165–184, 2005.

Steven J Brams and Peter C Fishburn. Approval voting. *American Political Science Review*, 72(3): 831–847, 1978.

Florian Brandl and Dominik Peters. Approval voting under dichotomous preferences: A catalogue of characterizations. *Journal of Economic Theory*, 205, 2022.

Daniela Bubboloni and Michele Gori. A generalization to networks of young's characterization of the borda rule. *Annals of Operations Research*, 349:1501–1552, 2025.

Susumu Cato, Stéphane Gonzalez, Eric Rémila, and Philippe Solal. An axiomatic characterization of the proportional threshold methods: a geometric approach: S. cato et al. *Social Choice and Welfare*, pages 1–27, 2025.

Federica Ceron and Stéphane Gonzalez. Approval voting without ballot restrictions. *Theoretical Economics*, 16(3):759–775, 2021.

Théo Delemazure. *Expressive ballots for voting systems and political analysis.* PhD thesis, Université Paris sciences et lettres, 2025.

Peter C Fishburn. Symmetric and consistent aggregation with dichotomous voting. *Aggregation and revelation of preferences. North Holland, Amsterdam*, pages 201–218, 1978.

Stéphane Gonzalez, Annick Laruelle, and Philippe Solal. Dilemma with approval and disapproval votes. *Social Choice and Welfare*, pages 1–21, 2019.

Robert E Goodin and Christian List. A conditional defense of plurality rule: generalizing may's theorem in a restricted informational environment. *American Journal of Political Science*, 50(4): 940–949, 2006.

Sean Horan, Martin J Osborne, and M Remzi Sanver. Positively responsive collection choice rules and majority rule: a generalization of may's theorem to many alternatives. *International Economic Review*, forthcoming, 2019.

Martin Lackner and Piotr Skowron. *Multi-winner voting with approval preferences*. Springer, 2023.

Jean-François Laslier and M Remzi Sanver. Introduction to the handbook on approval voting. In *Handbook on Approval Voting*, pages 1–12. Springer, 2010.

François Maniquet and Philippe Mongin. Approval voting and arrow's impossibility theorem. *Social Choice and Welfare*, 44(3):519–532, 2015.

Kenneth O May. A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica: Journal of the Econometric Society*, pages 680–684, 1952.

Roger B Myerson. Axiomatic derivation of scoring rules without the ordering assumption. *Social Choice and Welfare*, 12(1):59–74, 1995.

Marcus Pivato. Variable-population voting rules. *Journal of Mathematical Economics*, 49(3):210–221, 2013.

Donald G Saari. Consistency of decision processes. *Annals of Operations Research*, 23(1):103–137, 1990.

N. Sato. A characterization result for approval voting with a variable set of alternatives. *Social Choice and Welfare*, 43:809–825, 2014.

N. Sato. Approval voting and fixed electorate with dichotomous preferences. *Mathematical Social Sciences*, 97:51–60, 2019.

Amartya Sen. Quasi-transitivity, rational choice and collective decisions. *The Review of Economic Studies*, 36(3):381–393, 1969.

Amartya Sen. Internal consistency of choice. *Econometrica*, 61(3):495–521, 1993.

Amartya K Sen. Choice functions and revealed preference. *The Review of Economic Studies*, 38 (3):307–317, 1971.

M. Vorsatz. Approval voting on dichotomous preferences. *Social Choice and Welfare*, 28(1):127–141, 2007.

Yongsheng Xu. Axiomatizations of approval voting. In *Handbook on Approval Voting*, pages 91–102. Springer, 2010.