THE ASYMPTOTIC PROFILE OF SOLUTIONS OF A CLASS OF DOUBLY NONLINEAR EQUATIONS

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ABSTRACT. Let Ω be a bounded domain of $\mathbf{R}^N$ and let u_0 belong to $L^{\frac{m+2p-3}{p-1}}(\Omega)$, where $p > 1$, $2 < m + p < 3$ and $p\frac{m+2p-3}{m-3+(4-m)p-p^2} > N$. We study the asymptotic behaviour of the solutions of the initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \Omega\times]0,\infty[, \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

under Dirichlet, variational and mixed boundary conditions.

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1. Introduction and main results.

In the last few years several authors studied the regularity and the asymptotic behaviour of nonnegative weak solutions of the boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = 0 & \text{in } \Omega\times]0,T[, \\ u \in C(0,T;L^2(\Omega)), \quad u^m \in L^2(0,T;W_0^{1,2}(\Omega)) & \\ u(x,0) = u_0(x) \in L^{1+m}(\Omega) & \end{cases} \tag{1,1}$$

with $0 < m < 1$.

Among them, we quote [2] where the asymptotic behaviour for separable classical solutions is studied (we refer the reader also to [1] where the physical motivation of (1,1) is pointed out).

In [12] the result of [2] is extended to the case of weak solutions. In [8] it is proved that if $\frac{(N-2)+}{N+2} < m < 1$ then the solution is analytic in the interior of Ω with respect to the space variables and Lipschitz continuous with respect to t up to the extinction time.

The aim of this note is to extend the previous results to a larger class of equations under more general boundary conditions dropping the hypothesis of nonnegativity of the initial datum and weakening the regularity assumptions on the boundary of Ω (in the above mentioned papers $\partial\Omega$ is assumed to be of class C^2).

We consider the case of a bounded domain $\Omega \subset \mathbf{R}^N$ with boundary of class $C^{1,\alpha}$ for some $\alpha > 0$. The initial datum u_0 belongs to $L^{\frac{m+2p-3}{p-1}}(\Omega)$, where:

$$p > 1, \quad 2 < m + p < 3, \quad p\frac{m+2p-3}{m-3+(4-m)p-p^2} > N \tag{1,2}$$

We study consider the weak solutions of:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \Omega\times]0,\infty[, \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u \in C(0,\infty;L^2(\Omega)), \quad u^{\frac{m-1}{p-1}}Du \in L^p(\Omega\times]0,\infty[) & \end{cases} \tag{1,3}$$

satisfying Dirichlet, variational or mixed conditions.

Here the notion of weak solution is standard and we refer the reader to [7] for details.

Equations of the type (1,3) are classified as "doubly nonlinear" ([14]) or "with implicit nonlinearity" ([11]). Many authors studied this kind of equations on account of their physical and mathematical interest. We refer the reader to the review paper [11] and to the introduction of [16].

Before stating the main results, let us make some remarks about the range of m and p. If $m + p \leq 2$, it is not known whether L^∞−estimates and regularity results hold. See the detailed discussion in [16].

If $m+p<3$, then the solution extinguishes after a finite time (see [5]), while this does not happen if $m+p\geq 3$. Different techniques are required in the latter case, which will be studied in a forthcoming paper.

The condition $p\frac{m+2p-3}{m-3+(4-m)p-p^2}>N$ implies that a Sobolev imbedding inequality holds.

Main results.

1.1 THEOREM. *Let $u_0\not\equiv 0$, and let u be a weak solution of $(1,3)$ satisfying Dirichlet or mixed boundary conditions, and let $T>0$ be the extinction time. Let $\bar{u}(x,t)=u(x,t)(T-t)^{\frac{-1}{3-m-p}}$. Then there exists a sequence $t_n\nearrow T$ such that $\bar{u}(x,t)\to w(x)$, where w is a nontrivial solution of the equation:*

$$\operatorname{div}(|w|^{m-1}|Dw|^{p-2}Dv)=\frac{1}{3-m-p}w,\quad \text{in } \Omega \tag{1,4}$$

satisfying Dirichlet or mixed boundary conditions, respectively. ∎

1.2 THEOREM. *Let u be a weak solution of $(1,3)$ satisfying variational boundary conditions.*
If $\int_\Omega u_0(x)\,dx\neq 0$, then:

$$\lim_{t\to\infty}u(x,t)=|\Omega|^{-1}\int_\Omega u_0(x)\,dx,\qquad \text{in } W^{1,p}(\Omega) \tag{1,5}$$

If $\int_\Omega u_0(x)\,dx=0$ and $u_0\not\equiv 0$ then the conclusion of Theorem 1.1 holds. ∎

1.3 REMARK. Result $(1,5)$ holds for $2<m+p<3$ and $p>1$.

1.4 REMARK. Theorem 1.1 above implies the existence of a nontrivial weak solution of the problem $(1,4)$ satisfying Dirichlet or mixed boundary conditions. A straightforward consequence of theorem 1.2 is that the average of any solution of $(1,4)$ satisfying variational boundary conditions is zero. Indeed if $u(x,t)$ solves $(1,3)$ under variational boundary conditions then $\int_\Omega u(x,t)\,dx$ is constant. Now, assume the existence of a solution $w(x)$ of (1.4) satisfying variational boundary conditions with $\int_\Omega w(x)\,dx\neq 0$. Set $u(x,t)=w(x)(1-t)^{\frac{1}{3-m-p}}$. The function $u(x,t)$ solves $(1,3)$, it satisfies variational boundary conditions and its average is not constant in time. Contradiction.

If u is a nonnegative local solution of $(1,3)$, i.e.:

$$\begin{cases} u\in C_{loc}(0,T;L^2_{loc}(\Omega)),\quad u^{\frac{m-1}{p-1}}Du\in L^p_{loc}(\Omega\times[0,T[) \\ \dfrac{\partial u}{\partial t}=\operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) \qquad \text{in } \Omega\times]0,T[\end{cases}$$

where $p > 1$, $m + p > 2$, $m + p + \frac{p}{N} > 3$, we can state more precise results.

Fix any $(x_0, t_0) \in \Omega_T = \Omega \times]0, T[$, assume that $u(x_0, t_0) > 0$ and let $1 > \rho > 0$ be so small such that the cylindrical domain:

$$\{|x - x_0| < \rho\} \times (t_0 - u(x_0, t_0)^{3-m-p}\rho^p, t_0 + u(x_0, t_0)^{3-m-p}\rho^p)$$

is contained in Ω_T.

1.5 THEOREM. *There exists a constant C depending only upon N, m and p and independent of u such that:*

$$|Du(x_0, t_0)| \leq C\rho^{-1}u(x_0, t_0) \quad \blacksquare \tag{1,6}$$

1.6 REMARK. Note that estimate (1,6) is false when $m + p = 3$. For instance, the function:

$$u(x,t) = ct^{-\frac{N}{p}} \exp\left[(p-1)^2\left(\frac{|x|t^{\frac{-1}{p}}}{p}\right)^{\frac{p-1}{p}}\right]$$

solves the equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{2-p}|Du|^{p-2}Du) & \text{in } \mathbf{R}^N \times]0, \infty[\\ u(0) = c_1\delta(0) \end{cases}$$

The counterexample can be easily built choosing $t = 1$ and a sequence of points x_n such that $|x_n| \to \infty$.

Now we focus our attention on the case $u_0 \geq 0$ and Dirichlet boundary condition.

1.7 THEOREM. *For any $\epsilon \in (0, T)$, there exist constants γ_i, $i = 1, 2, 3$, depending only upon $N, m, p, \|u_0\|_{L^{\frac{m+2p-3}{p-1}}(\Omega)}, \Omega$ and ϵ such that for all $(x,t) \in \Omega_T$, $t > \epsilon$:*

$$\gamma_1 \operatorname{dist}(x, \partial\Omega)^{\frac{1-m}{m+p-2}}(T-t)^{\frac{1}{3-m-p}} \leq u(x,t) \leq \gamma_2 \operatorname{dist}(x, \partial\Omega)^{\frac{1-m}{m+p-2}}(T-t)^{\frac{1}{3-m-p}} \tag{1,7}$$

$$|Du(x,t)| \leq \gamma_3 \operatorname{dist}(x, \partial\Omega)^{\frac{1-m}{m+p-2}}(T-t)^{\frac{1}{3-m-p}} \quad \blacksquare \tag{1,8}$$

1.8 REMARK. As $p \frac{m+2p-3}{m=3+(4-m)p-p^2} \searrow N$ the constants γ_i blow up.

1.9 REMARK. Note that (1,7) and (1,8) imply that any nontrivial nonnegative solution of:

$$\begin{cases} \operatorname{div}(w^{m-1}|Dw|^{p-2}Dw) = \frac{1}{3-m-p}w & \text{in } \Omega, \\ w|_{\partial\Omega} = 0, \end{cases}$$

satisfies:

$$\gamma_1 \operatorname{dist}(x,\partial\Omega)^{\frac{1-m}{m+p-3}} \leq w(x,t) \leq \gamma_2 \operatorname{dist}(x,\partial\Omega)^{\frac{1-m}{m+p-3}}; \tag{1,9}$$

$$|Du(x,t)| \leq \gamma_3 \operatorname{dist}(x,\partial\Omega)^{\frac{1-m}{m+p-3}}. \tag{1,10}$$

2. Asymptotic results in the general case.

As in [2], we suppose that u is a regular solution. The general case can be studied by approximating the equations with nonsingular ones (see [12]).

Let

$$a = \frac{p-1}{m+p-2}, \quad a > p-1 > 0; \qquad u = v^a.$$

2.1 LEMMA. *The quotient:*

$$\mathcal{E}[v](t) = \left(\frac{\|Dv(t)\|_{L^p(\Omega)}}{\|v(t)\|_{L^{a+1}(\Omega)}}\right)^p \tag{2,1}$$

is non increasing in time.

PROOF. Note that v satisfies the equation:

$$\frac{\partial}{\partial t}(v^a) = a^{p-1}\,\mathrm{div}(|Dv|^{p-2}Dv) \tag{2,2}$$

in $\Omega\times]0,\infty[$; multiplying by v and integrating over Ω we obtain:

$$\frac{1}{a+1}\frac{d}{dt}\int_\Omega v^{a+1}\,dx + a^{p-2}\int_\Omega |Dv|^p\,dx = 0. \tag{2,3}$$

Setting $\Delta_p v = \mathrm{div}(|Dv|^{p-2}Dv)$, we get:

$$\int_\Omega |Dv|^p\,dx = -\int_\Omega v\Delta_p v\,dx \le \left(\int_\Omega v^{a+1}\,dx\right)^{1/2}\left(\int_\Omega \frac{|\Delta_p v|^2}{v^{a-1}}\,dx\right)^{1/2}. \tag{2,4}$$

On the other hand,

$$\frac{d}{dt}\int_\Omega |Dv|^p\,dx = p\int_\Omega |Dv|^{p-2}\,Dv\cdot Dv_t\,dx = -p\int_\Omega v_t\Delta_p v\,dx, \tag{2,5}$$

and from equation (2.2) we have:

$$v_t = a^{p-2}\frac{\Delta_p v}{v^{a-1}}.$$

Hence:

$$\frac{d}{dt}\int_\Omega |Dv|^p\,dx = -pa^{p-2}\int_\Omega \frac{|\Delta_p v|^2}{v^{a-1}}\,dx, \tag{2,6}$$

which gives, together with (2,4):

$$\frac{d}{dt}\int_\Omega |Dv|^p\,dx \le -pa^{p-2}\frac{\left(\int_\Omega |Dv|^p\,dx\right)^2}{\int_\Omega v^{a+1}\,dx}. \tag{2,7}$$

Finally, by (2,2)

$$\frac{1}{p}\frac{\frac{d}{dt}\int_\Omega |Dv|^p\,dx}{\int_\Omega |Dv|^p\,dx} \le \frac{1}{a+1}\frac{\frac{d}{dt}\int_\Omega v^{a+1}\,dx}{\int_\Omega v^{a+1}\,dx}$$

and:

$$\frac{d}{dt}\log\left(\int_\Omega |Dv|^p\,dx\right)^{1/p} \le \frac{d}{dt}\log\left(\int_\Omega v^{a+1}\,dx\right)^{\frac{1}{a+1}}$$

which means that the function $\mathcal{E}[v](t)$ does not increase in time. ∎

2.2 REMARK. By (2,3) we get

$$\frac{d}{dt}\int_\Omega v^{a+1}\,dx = -(a+1)a^{p-2}\int_\Omega |Dv|^p\,dx = -(a+1)a^{p-2}\mathcal{E}[v]\left(\int_\Omega v^{a+1}\right)^{\frac{p}{a+1}}$$

that is

$$\frac{d}{dt}\|v\|^{a+1-p}_{L^{a+1}(\Omega)} = -(a+1-p)a^{p-2}\mathcal{E}[v|, \qquad a+1-p>0.$$

Integrating the O.D.E. we have:

$$\|v(t)\|^{a+1-p}_{L^{a+1}(\Omega)} = -(a+1-p)a^{p-2}\int_0^t \mathcal{E}[v](s)\,ds + \|v_0\|^{a+1-p}_{L^{a+1}(\Omega)}. \tag{2,8}$$

2.3 REMARK. Assume that the solution v satisfies a Sobolev type inequality, i. e. :

$$\exists B = B_{m,p}(\Omega) > 0: \ \left(\int_\Omega v^{a+1}\,dx\right)^{\frac{p}{a+1}} \le B\int_\Omega |Dv|^p\,dx \tag{2,9}$$

Then: $\mathcal{E}[v](0) \ge \mathcal{E}[v](t) \ge B$ and by (2,8) we get that u becomes extinct after a finite time T, with rate of extinction $(T-t)^{\frac{1}{a+1-p}}$.

We make the substitution $t = T - Te^{-\tau}$ and consider the function $\tilde{v}$:

$$\tilde{v}(\cdot,\tau) = \frac{v(\cdot, T-Te^{-\tau})}{(Te^{-\tau})^{\frac{1}{a+1-p}}}$$

that satisfies the equation:

$$\frac{\partial}{\partial t}(\tilde{v}^a) - a^{p-1}\Delta_p\tilde{v} = \frac{a}{a+1-p}\tilde{v}^a \tag{2,10}$$

2.4 LEMMA. *Under the hypotheses of lemma* 2.1, *the function:*

$$F[\tilde{v}](\tau) = a^{p-2}\int_\Omega |D\tilde{v}|^p\,dx - \frac{p}{(a+1)(a+1-p)}\int_\Omega \tilde{v}^{a+1}\,dx \tag{2,11}$$

is not increasing.

From $(2,5)$ and $(2,10)$, we get:

$$a^{p-2}\frac{d}{dt}\int_\Omega |D\tilde{v}|^p\,dx = -\,p\int_\Omega (\tilde{v}_t)^2\tilde{v}^{a-1}\,dx + \frac{p}{a+1-p}\int_\Omega \tilde{v}_t\tilde{v}^a\,dx =$$
$$= -\,p\int_\Omega (\tilde{v}_t)^2\tilde{v}^{a-1}\,dx + \frac{p}{(a+1)(a+1-p)}\frac{d}{dt}\int_\Omega \tilde{v}^{a+1}\,dx \quad \blacksquare$$

2.5 COROLLARY. *There exists a sequence $\tau_n \to \infty$ such that:*

$$\lim_{n\to\infty}\frac{d}{d\tau}F[\tilde{v}](\tau_n) = 0$$

PROOF. It is sufficient to remark that $F(\tau)$ is bounded from below. $\blacksquare$

Proof of Theorem 1.1.

In the cases of mixed or Dirichlet conditions a Sobolev inequality holds and the results following remark 2.3 are verified. Therefore, arguing as in [2], we get that there is a sequence $t_n \nearrow T$ such that:

$$\frac{u(\cdot,t_n)}{(T-t_n)^{\frac{1}{3-m-p}}} \to w$$

where w is a solution of the equation:

$$\operatorname{div}\left(|w|^{m-1}|Dw|^{p-2}Dw\right) = \frac{1}{3-m-p}w$$

satisfying the corresponding boundary conditions. On the other hand, from $(2,8)$ we get:

$$c_1 \le \frac{\|v(t_n)\|_{L^{a+1}(\Omega)}}{(T-t_n)^{\frac{1}{a+1-p}}} \le c_2$$

Hence by lemma 2.1 and remark 2.3 we have

$$Bc_1 \le \frac{\|Dv(t_n)\|_{L^p(\Omega)}}{(T-t_n)^{\frac{1}{a+1-p}}} \le \mathcal{E}[v](0)c_2$$

Therefore a suitable subsequence of $\left\{\frac{v(t_n)}{(T-t_n)^{\frac{1}{a+1-p}}}\right\}_{n\in\mathbb{N}}$ strongly converges in $L^{a+1}(\Omega)$ to a function $w^{\frac{1}{a}} \not\equiv 0$. $\blacksquare$

Proof of Theorem 1.2.

If $\int_\Omega u_0\,dx = 0$, then a Sobolev inequality holds for each $t \geq 0$ and we may repeat the previous arguments.

In the case $\int_\Omega u_0\,dx \neq 0$, there is not extinction time. Recalling that $\|v\|_{L^{a+1}(\Omega)}$ is non increasing by Remark 2.2, from (2,7) we deduce:

$$\frac{d}{dt}\int_\Omega |Dv|^p\,dx \leq -pa^{p-2}\frac{\left(\int_\Omega |Dv|^p\,dx\right)^2}{\int_\Omega v_0^{a+1}\,dx} \tag{2,12}$$

which implies

$$\int_\Omega |Dv(x,t)|^p\,dx \leq \frac{c}{t} \tag{2,13}$$

On the other hand we have:

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0\,dx \tag{2,14}$$

Therefore

$$u(\cdot,t) \to |\Omega|^{-1}\int_\Omega u_0\,dx \quad \text{in } W^{1,p}(\Omega) \quad \blacksquare$$

3. Regularity results for nonnegative local solutions.

In order to prove theorem 1.5, let us recall a result of [16].

3.1 PROPOSITION. *Let u be a nonnegative local solution of (1,3). Fix $(x_0,t_0)\in\Omega_T$ and set $u_0=u(x_0,t_0)$; choose $\rho>0$ so small that the cylindrical domain:*

$$C_{4\rho}=\{|x-x_0|<4\rho\}\times(t_0-u_0^{3-m-p}(4\rho)^p,t_0+u_0^{3-m-p}(4\rho)^p)$$

is contained in Ω_T. There exist constants γ,γ_1 depending only upon N,m,p such that:

$$\inf_{|x-x_0|<\rho}u(x,t_0)\geq\gamma\sup_{|x-x_0|<\rho}u(x,t_0) \tag{3,1}$$

and:

$$\inf_{C_\rho}u(x,t)\geq\gamma_1\sup_{C_\rho}u(x,t) \tag{3,2}$$

where $m+p<3$ and $m+p+\frac{p}{N}>3$. ∎

3.2 REMARK. As pointed out in [17], inequalities (3,1) and (3,2) do not hold in the case $m+p=3$.

Another tool is the following regularity result:

3.3 PROPOSITION. *Let $Q_s=\{|x|<s\}\times\{|t|<s\}$ and let u be a nonnegative local solution of the equation:*

$$\frac{\partial u}{\partial t}=\operatorname{div}(|u|^{m-1}|Du|^{p-2}Du)\quad\text{in } Q_1 \tag{3,3}$$

Moreover assume $\frac{1}{2}\leq u\leq 2$ in Q_1; then there exists $\alpha>0$ such that $u\in C^{1,\alpha}(\overline{Q_{1/2}})$ and:

$$\|Du\|_{L^\infty(Q_{1/2})}\leq K \tag{3,4}$$

where K depends only upon N,m and p.

PROOF. Let $(x_0,t_0)\in Q_{1/2}$, $u_0=u(x_0,t_0)$ and for $r<1/2$ denote by Q_r' the cylinder:

$$\{|x-x_0|<r\}\times]t_0-r,t_0[\subset Q_1.$$

Let v the solution of:

$$\begin{cases}\dfrac{\partial v}{\partial t}=\operatorname{div}(u_0^{m-1}|Dv|^{p-2}Dv) & \text{in } Q_r'\\ v=u & \text{on the parabolic boundary of } Q_r'\end{cases}$$

In [13] (see also [9]) it is proved that there is $\alpha > 0$ such that for each $0 < \rho < r$:

$$\iint_{Q'_\rho} |Dv - (Dv)_\rho|^p \, dx\, dt \le c_\alpha \left(\frac{\rho}{r}\right)^{N+\alpha} \iint_{Q'_r} |Dv - (Dv)_r|^p \, dx\, dt$$

where $(Dv)_s = \frac{1}{|Q'_s|} \iint_{Q'_s} Dv \, dx\, dt$.

Hence:

$$\frac{\partial}{\partial t}(v-u) - \operatorname{div}[u_0^{m-1}(|Dv|^{p-2}Dv - |Du|^{p-2}Du)] =$$
$$= \operatorname{div}[(u^{m-1} - u_0^{m-1})|Du|^{p-2}Du]$$

Therefore, since u is Hölder continuous ([15], [16]), arguing as in [5] one gets:

$$\iint_{Q'_r} |Dv - Du|^p \, dx\, dt \le \gamma r^\delta \iint_{Q'_r} |Du|^p \, dx\, dt$$

and consequently:

$$\iint_{Q'_\rho} |Du - (Du)_\rho|^p \, dx\, dt \le$$
$$\le c \iint_{Q'_\rho} \{|Dv - (Dv)_\rho|^p + |Du - Dv|^p + |(Du)_\rho - (Dv)_\rho|^p\} \, dx\, dt \le$$
$$\le c\gamma r^\delta \iint_{Q'_r} |Du|^p \, dx\, dt + cc_\alpha \left(\frac{\rho}{r}\right)^{N+\alpha} \iint_{Q'_r} |Dv - (Dv)_r|^p \, dx\, dt \le$$
$$\le Kr^\delta \iint_{Q'_r} |Du|^p \, dx\, dt + K \left(\frac{\rho}{r}\right)^{N+\alpha} \iint_{Q'_r} |Du - (Du)_r|^p \, dx\, dt$$

The last inequality implies that the gradient of u is Hölder continuous (see [4], [10]). ∎

Proof of theorem 1.5.

Fix $(x_0, t_0) \in \Omega_T$, $u_0 = u(x_0, t_0)$ and for $\theta > 0$ consider the cylinder

$$C = \{|x - x_0| < 8r\} \times (t_0 - u_0^{3-m-p} 8\theta r^p, t_0 + u_0^{3-m-p} 8\theta r^p) \subset \Omega_T.$$

The change of variables

$$x \to \frac{x - x_0}{r}, \qquad t \to \frac{t - t_0}{u_0^{3-m-p} r^p}, \qquad v = \frac{u}{u_0}$$

maps C in $B_8 \times (-8\theta, 8\theta)$ and v satisfies:

$$\begin{cases} \frac{\partial}{\partial t} v - \operatorname{div}(v^{m-1}|Dv|^{p-2}Dv) = 0 & \text{in } B_8 \times (-8\theta, 8\theta) \\ v(0,0) = 1 \end{cases}$$

By (3,2):

$$\sup_{B_4 \times (-4\theta, 4\theta)} v \le \gamma_1^{-1}, \qquad \inf_{B_4 \times (-4\theta, 4\theta)} v \ge \gamma_1$$

Hence by proposition 3.3 $v \in C^{1,\alpha}(B_4 \times (-4\theta, 4\theta))$ and $|Dv(0,0)| \le K$.

Estimate (1,6) follows by recalling the definition of v ∎

4. Asymptotic results for nonnegative Dirichlet solutions.

Here we follow the approach introduced in [8]. Our main goal is to weaken the regularity assumptions on the boundary. The proof of Theorem 1.7 is based on the following two proposition.

4.1 PROPOSITION. *Let u a nonnegative bounded solution of:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) & \text{in } \Omega\times]s,T[, \\ u \in C(s,T;L^2(\Omega)), \quad u^{\frac{m-1}{p-1}}Du \in L^p(\Omega\times]s,T[) \\ u \le M \end{cases} \tag{4,1}$$

for some $s \in]0,T[$ and some $M > 0$. For every $\nu > 0$, there exists a constant γ depending only upon N, m, p, ν and $\|\partial\Omega\|_{1,\alpha}$ such that, for all $t - s \ge \nu M^{3-m-p}$:

$$u(x,t) \le \gamma M[\operatorname{dist}(x,\partial\Omega)]^a, \quad a = \frac{p-1}{m+p-2} \tag{4,2}$$

4.2 PROPOSITION. *Let u be a non-negative bounded solution of $(4,1)$ for some $s \in (0,T)$ and some $M > 0$. For $r > 0$ let:*

$$\Omega^r = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \ge r\}, \quad \Omega^{r,t} = \Omega^r \times (s,t), \quad \mu(r) = \inf_{\Omega^{r,t}} u \tag{4,3}$$

For every $\nu > 0$ there exist constants r_0 and γ depending only upon N, m, p, ν and Ω such that for all $t - s > \nu M^{3-m-p}$:

$$u(x,t) \ge \gamma\mu(r_0)[d(x)]^a, \quad a = \frac{p-1}{m+p-2} \ge 1 \tag{4,4}$$

Denote by $d(x)$ the distance $\operatorname{dist}(x,\partial\Omega)$ and by Ω_s, $s > 0$, the subset of Ω where $d(x) < s$. For a point $x \in \Omega$ denote by Γ_x the set of the elements of $\partial\Omega$ closest to x: $\Gamma_x = \{y \in \partial\Omega : |x - y| = d(x)\} \ne \emptyset$. For every point $y \in \Gamma_x$ set:

$$y_\lambda = y + \lambda\frac{y-x}{|y-x|}, \quad d_\lambda = d(y_\lambda), \quad \lambda \ge 0$$

For $C^{1,\alpha}$ open sets, we have the following geometrical lemma, whose easy proof is left to the reader.

4.3 LEMMA. *Let us fix two constants $\beta \in]0,\alpha[$ and $\epsilon > 0$. Then there exist constants $\bar{s}, \bar{\lambda} > 0$ depending only upon Ω such that for $x \in \Omega_s$, $y \in \Gamma_x$, $\lambda \in]0,\bar{\lambda}[$ we have:*

$$\lambda - d_\lambda \leq \epsilon \lambda^{1+\beta}, \qquad d_\lambda \geq \frac{\lambda}{2} \quad \blacksquare \tag{4,5}$$

Consider now the function $v = u^{\frac{1}{a}}$ solution of (2,2), set $\tilde{a} = (a^{-1} - 1)^+$ and fix a number $\nu > 0$. We call I_ν the interval $]s + \nu M^{3-m-p}, T[$ and we set

$$M_r = \sup_{\Omega_r \times I_\nu} v(x,t) \leq M^{\frac{1}{a}}.$$

We prove the following:

4.4 LEMMA. *Let us fix $\beta \in]0,\alpha[$; there exist $c, \bar{r} > 0$ such that for $0 < r < \bar{r}$ we have*

$$M_r \leq r^{\frac{\beta}{1+\beta}} M_{cr^{\frac{1}{1+\beta}}} \tag{4,6}$$

PROOF. Let $(x_0,t_0) \in \Omega_r \times I_\nu$, $y \in \Gamma_{x_0}$ $I_\nu(t_0) =]t_0 - \nu M^{3-m-p}, t_0[\subset]0,T[$.

Following [8], we construct a barrier $\psi_{k,\lambda}$ of the type:

$$\psi_{k,\lambda}(x,t) = CM_{1/k}(1 - \eta_{k,\lambda}(x,t))$$

$$\eta_{k,\lambda}(x,t) = \exp[-k(|x - y_\lambda| - d_\lambda] \exp[M^{p+m-3}(t - t_0)^{1+\tilde{a}}] \tag{4,7}$$

We impose conditions on k, λ in order to have $v \leq \psi_{k,\lambda}$ in the set

$$\mathcal{N}_{k,\lambda} = \{x \in \Omega : \ |x - y_\lambda| - d_\lambda \leq \frac{1}{k}\} \times I_\nu(t_0) \subset \Omega_{1/k} \times I_\nu(t_0). \tag{4,8}$$

In view of the maximum principle, it is sufficient that

$$v(x,t) \leq \psi_{k,\lambda}(x,t), \qquad \text{on the parabolic boundary of } \mathcal{N}_{k,\lambda}, \tag{4,9}$$

$$\frac{\partial}{\partial t}[\psi^a_{k,\lambda}] - \Delta_p \psi \geq 0, \qquad \text{in } \mathcal{N}_{k,\lambda}. \tag{4,10}$$

Inequality (4,9) is verified if

$$C = \max\{(1 - e^{-1})^{-1}, (1 - e^{-\nu})^{-1}\}.$$

Indeed, $\psi_{k,\lambda} \geq 0$ in $\mathcal{N}_{k,\lambda}$. Moreover in the set

$$\{x \in \Omega : \ |x - y_\lambda| - d_\lambda = 1/k\} \times I_\nu(t_0)$$

it holds $\psi_{k,\lambda} \geq CM_{1/k}(1-e^{-1}) \geq C(1-e^{-1})v$. Finally, in

$$\Omega_{1/k} \times \{t_0 - \nu M^{3-m-p}\},$$

we have $\psi_{k,\lambda} \geq CM_{1/k}(1-e^{-\nu})$.

By direct calculations, (4,10) is equivalent to:

$$(p-1)k^p - \frac{N-1}{|x-y_\lambda|}k^{p-1} - (1+\tilde{a})a^{2-p}(CM_{1/k})^{a+1-p}M^{m+p-3}(1-\eta_{k,\lambda})^{a-1}(t-t_0)^{\tilde{a}}\eta^{2-p} \geq 0 \tag{4,11}$$

By recalling that

$$\inf_{x\in\Omega}|x-y_\lambda| = d_\lambda, \quad M_{1/k}^{a+1-p} \leq M^{\frac{a+1-p}{a}} = M^{3-m-p}, \quad 0 \leq \eta_{k,\lambda} \leq 1,$$

inequality (4,11) holds if:

$$k \geq C_{m,p,\nu} + 2\frac{N-1}{(p-1)d_\lambda} \tag{4,12}$$

Let $1/k = 2Cr^{\frac{1}{1+\beta}}$, $\lambda = 8C\left(\frac{N-1}{p-1}\right)r^{\frac{1}{1+\beta}}$ and $(x_0,t_0) \in \Omega_r \times I_\nu$. Choose $\bar{r}$ small enough (depending only on m,p,N,ν,r). By the maximum principle

$$v(x_0,t_0) \leq \psi_{k,\lambda}(x_0,t_0).$$

Hence:

$$v(x_0,t_0) \leq CM_{\frac{1}{k}}k(r+\lambda-d_\lambda) \leq M_{2Cr^{\frac{1}{1+\beta}}}r^{\frac{\beta}{1+\beta}} \tag{4,13}$$

Estimate (4,13) implies (4,6) because the constants are independent of the choice of (x_0,t_0). ∎

PROOF OF PROPOSITION 4.1.

It is sufficient to prove (4,1) for $x \in \Omega_{\bar{r}}$ with $\bar{r}$ given by the previous lemma. Moreover, (4,1) is equivalent to:

$$\forall r < \bar{r}: \qquad M_r \leq \gamma M r \tag{4,14}$$

Let $\delta = \frac{1}{1+\beta}$, $C' = (2C)^{\frac{\beta}{\beta+1}}$ and let $n \in \mathbf{N}$ be such that:

$$C'r^{(\delta^n)} \leq \bar{r} \leq C'r^{(\delta^{n+1})}, \quad \text{for } n \in \mathbf{N} \tag{4,15}$$

By (4,13), we get:

$$M_r \leq \left(M_{C'r^{\delta^n}}\right)r^{\beta\delta\frac{1-\delta^n}{1-\delta}} \leq C'M_{\bar{r}}\,r\frac{1}{\bar{r}^{\frac{1}{1+\beta}}} \leq \frac{C'}{\bar{r}^{\frac{1}{1+\beta}}}Mr \quad ∎$$

Before proving theorem 1.7 let us introduce some notation. Let $x \in \Omega$, $y \in \Gamma_x$ and denote by y^λ the point

$$y^\lambda = x - \lambda\frac{y-x}{|y-x|}, \qquad d^\lambda = d(y^\lambda)$$

As before, the following statement can be easily proved:

4.5 LEMMA. *Let us fix two constants* $\beta \in]0,\alpha[$ *and* $\epsilon > 0$. *Then there exist constants* $\bar{s}, \bar{\lambda} > 0$ *depending only upon* Ω *such that for* $x \in \Omega^{\bar{s}}$, $y \in \Gamma_x$, $\lambda \in]0,\bar{\lambda}[$ *we have*

$$d(x) + \lambda - d^\lambda \leq \epsilon\lambda^{1+\beta}, \qquad d^\lambda \geq d(x) + \frac{\lambda}{2} \quad \blacksquare \tag{4,16}$$

PROOF OF PROPOSITION 4.2.

Let (x_0, t_0) be a point in $\Omega^r \times I_\nu$, let y belong to Γ_{x_0}, and let $I_\nu(t_0)$ be equal to $]s + \nu M^{3-m-p}, t_0[$.

Consider the barrier:

$$\tilde{\psi}_{k,\lambda} = e^{-1}\mu_{1/k}(\tilde{\eta}_{k,\lambda} - 1), \quad \tilde{\eta}_{k,\lambda}(x,t) = \exp[-k(|x - y_\lambda| - d_\lambda)] \exp\left(\frac{t - t_0}{t_0 - s}\right)^{1+\tilde{\alpha}}$$

in:

$$\tilde{\mathcal{N}}_{k,\lambda} = \{x \in \Omega : \ d^\lambda - 1/k \leq |x - y^\lambda| \leq d^\lambda\} \times I_\nu(t_0).$$

$v \geq u$ in $\tilde{\mathcal{N}}_{k,\lambda}$ by means of the comparison principle as in lemma 4.3.

Actually, in the set

$$\{|x - y^\lambda| = d^\lambda\} \times I_\nu(t_0)$$

we have $\tilde{\psi}_{k,\lambda} \leq 0$, in the set

$$\{|x - y^\lambda| = d^\lambda - 1/k\} \times I_\nu(t_0) \subset \Omega^{1/k} \times I_\nu(t_0)$$

we have $\tilde{\psi}_{k,\lambda} \leq \frac{e-1}{e}\mu_{1/k} \leq v$ and in the bottom of $\tilde{\mathcal{N}}_{k,\lambda}$ we have $\tilde{\eta}_{k,\lambda} \leq 1$ and $\tilde{\psi}_{k,\lambda} \leq 0$.

Moreover, by direct calculations it is possible to check that:

$$\frac{\partial}{\partial t}[\psi^a_{k,\lambda}] - \Delta_{t^*}\psi \leq 0, \qquad \text{in } \mathcal{N}_{k,\lambda}$$

if:

$$k \geq C_{m,p,\nu} + 2\frac{N-1}{(p-1)d^\lambda}$$

where C depends only on m, p, ν. Choosing $1/k = \frac{1}{2e}r^{\frac{1}{1+\beta}}$, $\lambda = 2\frac{N-1}{p-1}r^{\frac{1}{1+\beta}}$ and $\bar{r}$ sufficiently small, we get

$$v(x_0,t_0) \geq e^{-1}\mu_{1/k}k|d(x_0) - (d(x_0) + \lambda - d^\lambda)| \geq 2\mu_{1/k}r^{\frac{-1}{1+\beta}}[r - \epsilon\lambda^{1+\beta}] \geq \mu_{2er^{\frac{1}{1+\beta}}}\, r^{\frac{\beta}{1+\beta}}$$

The last inequality implies (4, 15) through arguments similar to the ones of Proposition 4.1. ∎

PROOF OF THEOREM 1.7.

As the proof is similar to the one followed in [8], we focus our attention only on what is really new and we refer the reader to section 6 of [8].

By the L^∞−estimates proved in section 3 of [17], and by the Raleigh quotient one may deduce that $\forall \epsilon < t < T$:

$$c_0(T-t)^{\frac{1}{3-m-p}} \leq \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_0(\epsilon)(T-t)^{\frac{1}{3-m-p}} \tag{4,17}$$

where C_0 depends on the initial datum and on ϵ.

Consider the change of coordinates: $T-t=(T-t_0)e^{-\tau}$. Let

$$w(\cdot,\tau) = \frac{u(\cdot, T-(T-t_0)e^{-\tau})}{(T-t_0)^{\frac{1}{3-m-p}} e^{\frac{\tau}{3-m-p}}}$$

The function w is a non-negative bounded weak solution of:

$$\begin{cases} \frac{\partial}{\partial t} w - \operatorname{div}(w^{m-1}|Dw|^{p-2}Dw) = \frac{1}{3-m-p} w, & \text{in } \Omega \times (0,\infty) \\ c_0 \leq \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq C_0 \end{cases} \tag{4,18}$$

The next step is to show that for every $\sigma > 0$ there exist two positive constants $\lambda < \Lambda$ depending only upon $N, m, p, \epsilon, \|\partial\Omega\|_{1,\alpha}$ and $\sigma > 0$ such that $\forall t > \sigma$, $\forall x \in \Omega$:

$$\lambda d(x)^a \leq w(x,t) \leq \Lambda d(x)^a \tag{4,19}$$

Now by (4,2)

$$w(x,t) \leq \Lambda d(x)^\eta \tag{4,20}$$

Note that (4,18) and (4,20) imply that the maximum is achieved at some point (x_0,t_0) satisfying:

$$d(x_0) \geq \left(\frac{c_0}{\Lambda}\right)^{\frac{1}{a}} = r$$

Estimate (4,19) follows by arguing as in lemma 6.1 of [8] and by choosing as a subsolution:

$$\psi = \frac{c v^{\frac{1}{m+p-1}}}{\mathcal{F}^{\frac{\theta}{3-m-p}}}$$

where $\mathcal{F} = 1 + c^{\frac{3-m-p}{p-1}} b \left(\frac{|x|^p}{t}\right)^{\frac{1}{p-1}}$, $v = (R^\beta - |x|^\beta)^p$, c is a positive constant and θ, β, b are positive numbers that can be determined a priori only in terms of N, m, p and c.

Now estimates (1,7) and (1,8) follow by (4,19) by means of the regularity result of section 3.

4.6 REMARK. All the results of this section are based on the compact imbedding of $W^{1,p}(\Omega)$ in $L^{\frac{m+2p-3}{m+p-2}}(\Omega)$; by the Rellich-Kondrachov theorem, this holds true if:

$$\frac{m+p-2}{m+2p-3} > \frac{1}{p} - \frac{1}{N} \quad \Leftrightarrow \quad \frac{p(m+2p-3)}{m-3+(4-m)p-p^2} > N$$

Theorem 1.7 does not hold if the above condition on m, p is not satisfied. Actually, let $\Omega = \{x \in \mathbf{R}^N : |x| \leq 1\}$, $p = 2$, $N \geq 3$ and $m = \frac{N-2}{N+2}$. Assume that Theorem 1.7 holds. Then there exists a nontrivial nonnegative solution of the equation:

$$\begin{cases} \Delta u = u^{\frac{N+2}{N-2}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4,21}$$

This is a contradiction because problem (4.21) admits only the trivial solution (see for instance [3]).

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