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Identification of two memory kernels and the time dependence of the heat source for a parabolic conserved phase-field model

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SUMMARY

In this paper we consider a system of two integro-differential evolution equations coming from a conservative phase-field model in which the principal part of the elliptic operators, involved in the two evolution equations, have different orders. The inverse problem consists in finding the evolution of: the temperature, the phase-field, the two memory kernels and the time dependence of the heat source when we suppose to know additional measurements of the temperature on some part of the body Ω . Our results are set within the framework of Hölder continuous function spaces with values in a Banach space X . We prove that the inverse problem admits a local in time solution, but we are also able to prove a global in time uniqueness result. Our setting, when we choose for example $X = C(\bar{\Omega})$, allows us to make additional measurements of the temperature on the boundary of the body Ω . Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: phase-field system with memory; heat equation; Cahn–Hilliard type equation; inverse problem

1. INTRODUCTION

In the last two decades there has been an increasing interest in the study of phase-field models. Many important problems in this area have been proposed and solved. Among them, without claiming completeness, we mention References [1–10] and, for phase-field models with memory, References [11–15] and the literature therein.

Some of the models consist of a system of equations describing the evolution of the temperature θ and of the phase-field χ , that may stand for the local proportion of one of the two phases, solid or liquid, in a three-dimensional body Ω .

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We point out that only in the recent literature, see References [16–19], we can find contributions related to inverse problems for phase-field models, in fact most of the papers deal with direct problems in which the only unknowns are θ and χ . In particular, in Reference [17] the author considers a class of quasilinear systems with memory kernels that includes, as a particular case, the quasilinear version of the model introduced in Reference [16]; the paper [18] analyses a variation of the class of problems studied in this paper, while Reference [19] contains an interesting global in time result in the framework of Hilbert spaces.

One of the main motivations in the study of identification problems arises in the applications. In fact, any physical equation, because of the constitutive laws, contains parameters and functions that are supposed to be known (cf., e.g. (2)–(4)). On the other hand, what leads to inverse problems is the remark that such elements entering the constitutive laws are sometimes impossible to be measured directly. What we can do in this case is, for example, to measure other variables and determine such unknown parameters and functions indirectly, coupling the evolution equations with additional measurements maps. The reader interested in an overview on such problems, in further motivations and to some techniques for solving them, can consult the book [20] and the references therein. If we consider phase-field models with memory, the thermal memory is taken into account modifying the Fourier's heat conduction law by the introduction of a convolution kernel k . Unfortunately, k cannot be measured directly, but it can be recovered, for instance, by means of additional measurements on the temperature. Consequently, we have to face the problem to determine θ , χ and k simultaneously, i.e. we have to solve an identification problem. In this paper we will also identify a convolution kernel h related to the internal energy and the time dependence of the heat source f . Before we introduce the phase-field model and the related inverse problem, let us observe that we will reformulate it in an abstract setting. Such reformulation leads us to a more general problem that contains as a particular case the concrete problem we want to investigate. We will precisely discuss all its advantages in the sequel.

The model and the identification problem: In this article, we study the model introduced in Reference [15], which is recalled here for completeness. Let Ω be an open bounded set in \mathbb{R}^3 with sufficiently regular boundary $\partial\Omega$, occupied by an isotropic, rigid and homogeneous heat conductor. We consider only small variations of the absolute temperature and its gradient. A material, which exhibits phase transitions due to the temperature variations, is described by two state variables: the absolute temperature Θ and the phase-field χ at each point $x \in \Omega$ and $t \in [0, T]$, for $T > 0$, where χ takes approximately value -1 in the liquid and $+1$ in the solid. In our model we prefer to work with the *temperature variational field* θ defined by

$$\theta = \frac{\Theta - \Theta_c}{\Theta_c} \quad (1)$$

where Θ_c denotes the reference temperature at which the transition should occur. The energy balance equation is

$$\partial_t e + \nabla \cdot \mathbf{J} = \mathcal{F} \quad (2)$$

where e is the internal energy, \mathbf{J} is the heat flux and \mathcal{F} is the external heat supply. Taking into account a linearized version of the Coleman–Gurtin theory, we assume, as in

References [8,14], the constitutive equations

$$e(t,x) = e_c + c_v \Theta_c \theta(t,x) + \int_0^t h(s) \theta(t-s,x) ds + \Theta_c \chi(t,x) \tag{3}$$

$$\mathbf{J} = -\delta_1 \nabla \theta(t,x) - \int_0^t k(s) \nabla \theta(t-s,x) ds \tag{4}$$

where h and k account the memory effects, e_c , c_v and δ_1 are the internal energy at equilibrium, the specific heat and a conductivity coefficient, respectively. By (2)–(4) we get

$$\begin{aligned} & \partial_t \left(e_c + c_v \Theta_c \theta(t,x) + \int_0^t h(s) \theta(t-s,x) ds + \Theta_c \chi(t,x) \right) \\ & - \nabla \cdot \left(\delta_1 \nabla \theta(t,x) + \int_0^t k(s) \nabla \theta(t-s,x) ds \right) = \mathcal{F}(t,x) \end{aligned} \tag{5}$$

In the following, we will assume that h and k are unknown functions of time that, for the same physical reasons, cannot be measured directly. For more details on h and k we refer the reader to Reference [21].

Following Reference [22], we couple the evolution equation (5) with the Cahn–Hilliard type equation which rules the phase evolution (see References [5,23,24])

$$\partial_t \chi(t,x) = \Delta[-\Delta \chi(t,x) + \chi^3(t,x) + \gamma'(\chi(t,x)) - \theta(t,x)], \quad t \in [0, T], \quad x \in \Omega \tag{6}$$

where γ is a given smooth function. For the sake of simplicity, we set the physical constants equal to 1. Then, we associate with the evolution equations for θ and χ a set of initial and no flux boundary conditions, so that the phase-field model under consideration is one of the so-called *conserved* model, that is, the integral of χ on the space domain Ω does not vary. We get

$$\left\{ \begin{aligned} & \partial_t(\theta(t,x) + \chi(t,x) + (h * \theta)(t,x)) \\ & \quad = \Delta \theta(t,x) + (k * \Delta \theta)(t,x) + \mathcal{F}(t,x), \quad t \in [0, T], \quad x \in \Omega \\ & \partial_t \chi(t,x) = -\Delta^2 \chi(t,x) \\ & \quad \quad + \Delta[\chi^3(t,x) + \gamma'(\chi(t,x)) - \theta(t,x)], \quad t \in [0, T], \quad x \in \Omega \\ & \theta(0,x) = \theta_0(x), \quad x \in \Omega \\ & \chi(0,x) = \chi_0(x), \quad x \in \Omega \\ & D_\nu \theta(t,x) = 0, \quad t \in [0, T], \quad x \in \partial \Omega \\ & D_\nu \chi(t,x) = D_\nu \Delta \chi(t,x) = 0, \quad t \in [0, T], \quad x \in \partial \Omega \end{aligned} \right. \tag{7}$$

where D_ν denotes the derivative with respect to the outward unit normal vector to $\partial \Omega$ and the symbol $*$ denotes the convolution with respect to the time. We suppose that $\theta_0, \chi_0 : \Omega \rightarrow \mathbb{R}$ are known functions.

A further difficulty arises when the heat source \mathcal{F} is placed in a fixed position of the material but the time dependence is not known. We assume that

$$\mathcal{F}(t, x) = f(t)g(x) \tag{8}$$

where $g(x)$ is a given data but $f(t)$ has to be considered a further unknown of the problem. To determine the functions h, k and f , we suppose to know additional information on the temperature which can be represented in integral form as

$$\int_{\Omega} \theta(t, x) \mu_j(dx) = b_j(t), \quad \forall t \in [0, T], \quad j = 1, 2, 3 \tag{9}$$

where μ_j are given measures depending on the type of thermometer used and $b_j : [0, T] \rightarrow \mathbb{R}$ represents the result of the additional measurements on θ . The inverse problem (IP) we are going to investigate in this paper is the following.

Problem 1.1 (Inverse Problem (IP))

Determine five continuous functions θ, χ, f, h, k satisfying system (7)–(9).

The precise regularity we require for the unknowns are indicated in (31)–(35).

Abstract reformulation of the inverse problem: We prove our main results for an abstract reformulation of Problem 1.1, in the setting of Hölder continuous functions with values in a Banach space X . The precise regularity and compatibility conditions on the data are in Section 2. Here we just want to point out the advantage of the abstract setting. So the problem we solve consists in determining (θ, χ, f, h, k) which satisfy the system

$$\begin{cases} \theta'(t) + \chi'(t) + (h * \theta)'(t) = A\theta(t) + k * A\theta(t) + f(t)g \\ \chi'(t) = B\chi(t) + F(\chi(t)) - A\theta(t) \\ \theta(0) = \theta_0 \\ \chi(0) = \chi_0 \\ \Phi_j[\theta(t)] = b_j(t), \quad \forall t \in [0, \tau], \quad \tau \in (0, T], \quad j = 1, 2, 3 \end{cases} \tag{10}$$

where $A : \mathcal{D}(A) \rightarrow X$ and $B : \mathcal{D}(B) \rightarrow X$ are linear closed non commuting operators, and $\theta_0, \chi_0, g, F, b_j$ ($j = 1, 2, 3$) are given data whose properties will be listed in the sequel and Φ_j are bounded linear functionals on X .

We prove that a local in time solution (θ, χ, f, h, k) exists for system (10) and we also show a *global* in time uniqueness results for its solution (θ, χ, f, h, k) .

Among the assumptions we make, we have the inclusions $\mathcal{D}(B) \hookrightarrow \mathcal{D}_B(\delta, \infty) \hookrightarrow \mathcal{D}(A) \hookrightarrow X$, for some $\delta \in (0, 1)$ and we require that the non-linear function F is defined on the interpolation space $\mathcal{D}_B(\delta, \infty)$, i.e. $F : \mathcal{D}_B(\delta, \infty) \rightarrow X$.

This leads to the fact that if B , in the concrete case, is an elliptic operator of order $2m$, for $m \in \mathbb{N}$, we are allowed to treat non-linear terms with derivatives of order up to $2m - 1$.

One of our mathematical tools are optimal regularity results. We have proved in Corollary 4.1 an optimal regularity result for the linear equations associated to the first and the second equations in problem (10), when the operators A and B do not commute. So we are allowed to choose elliptic operators A and B of any kind, which means that we are not restricted to the case of operators with constant coefficients.

The novelties with respect to the existing literature: In this paper, we focus our attention on the Hölder setting, while the case of Sobolev spaces has been studied in Reference [25]. Let us point out the difference between this paper and Reference [25]. In both cases the temperature, the phase-field and the two convolution memory kernels are assumed to be unknown, but the problems differ as follows: here the heat source is considered (partially) unknown, while in Reference [25] the fifth unknown is the instantaneous conductivity, and the heat source is given.

What is more important, the continuous space setting used here has the advantage to allow additional measurements of the temperature also on the boundary of Ω , while in the L^p setting, for technical reasons, one is compelled to make further measurements inside the body. This can be seen comparing the additional conditions (9) (continuous functions spaces) and the conditions of type

$$\int_{\Omega} \phi(x)\theta(t,x) dx = b(t), \quad \forall t \in [0, T] \tag{11}$$

given in Reference [25] (L^p spaces with $p \in (1, \infty)$): in the first case, μ_j ($1 \leq j \leq 3$) is a Borel function in $\bar{\Omega}$, for example a surface integral in $\partial\Omega$, while, in the second case, ϕ is an element of $L^{p'}(\Omega)$ with $1/p + 1/p' = 1$ (and b is still a given datum).

In Reference [16] the evolution equation for the temperature is coupled with the following law, ruling the phase field dynamics:

$$D_t\chi(t,x) = \Delta\chi(t,x) - \chi^3(t,x) + \chi(t,x) + \theta(t,x), \quad (t,x) \in (0, T] \times \Omega \tag{12}$$

We observe that the Laplace operator applied to χ is the same operator applied to θ in the heat equation and the non-linearities in (12) do not involve the derivatives of the unknowns, while, in this paper, we have (spatial) operators of different orders and the evolution equation for χ has a more difficult structure. In fact the non-linear term is $\Delta[\chi^3(t,x) + \gamma'(\chi(t,x)) - \theta(t,x)]$, where $\gamma'(\chi(t,x))$ is any regular function of class C^4 (cf. K2 in Section 3.2).

In Reference [18] the author considers a model which is similar to the one studied here but with only the kernel k . Moreover, an existence and uniqueness local in time theorem is proved (in a closed ball that depends on the data) for the solution (θ, χ, k) . We obtain the same results for the inverse problem involving (θ, χ, k, h, f) and, in addition, we are also able to prove a *global* in time uniqueness result.

Let us compare the possible choice of the non-linear function F when $B = -\Delta^2$ and $A = \Delta$. In Reference [18], the non-linearities are defined only on the domain of operator A , so the author is allowed to consider a non-linear function that involves (spatial) derivatives of order at most 2. Our abstract theorems allow us to deal with more general non-linearities, since F is defined on the interpolation spaces $\mathcal{D}_B(\delta, \infty)$, so that we can treat space derivatives of order up to 3 (cf. H2 and H3 in the sequel).

The recent paper [19] is concerned with the above-mentioned model: an existence and uniqueness result global in time is proved in a Hilbert space, for the inverse problem involving the unknowns (θ, χ, k) . Such a result is shown under the condition that the non-linear function of χ is bounded in $L^2(0, T; L^1(\Omega))$. This is trivially satisfied if one is able to prove that χ takes values in $[0, 1]$. Unfortunately, as the authors point out in Remark 2.14 of Reference [19], it is still an open question to find such a bound, since a proper maximum principle for a fourth-order Caginalp-type system is not known.

We point out that in our case one of the problems to get existence and uniqueness global in time is to find an *a priori* estimate for χ and its space derivatives involved in the non-linear function F . In other words, we must find an *a priori* estimate which works in the interpolation space $\mathcal{D}_B(\delta, \infty)$, while in Reference [19], the authors require a bound in the space $L^2(0, T; L^1(\Omega))$, that seems to be much easier to find. The result in Reference [19] is the first existence and uniqueness global in time result appearing in the literature for non-linear inverse problems of this type.

The plan of the paper:

- In Section 2 we introduce the functional setting and we reformulate Problem 1.1 within this framework.
- In Section 3 we state our main abstract results and we give an application in the case of the continuous functions space.
- Section 4 contains the preliminary lemmas which will be necessary to prove our main results.
- In Section 5 we give an equivalent reformulation of the abstract version of inverse problem in terms of fixed point equations.
- Finally, in Section 6, we prove our main abstract theorems by fixed point arguments, thanks to the preliminary results of Sections 4 and 5.

2. ABSTRACT SETTING FOR THE INVERSE PROBLEM

Let X be a Banach space with the norm $\|\cdot\|$ and let $T > 0$. We denote by $C([0, T]; X)$ the space of continuous functions with values in X equipped with the norm

$$\|u\|_{0, T, X} = \sup_{t \in [0, T]} \|u(t)\| \quad (13)$$

For $\beta \in (0, 1)$ we define

$$C^\beta([0, T]; X) = \left\{ u \in C([0, T]; X) : |u|_{\beta, T, X} = \sup_{0 \leq s < t \leq T} (t-s)^{-\beta} \|u(t) - u(s)\| < \infty \right\} \quad (14)$$

and we endow it with the norm

$$\|u\|_{\beta, T, X} = \|u(0)\| + |u|_{\beta, T, X} \quad (15)$$

If $k \in \mathbb{N}$, we set

$$C^{k+\beta}([0, T]; X) := \{u \in C^k([0, T]; X) : u^{(k)} \in C^\beta([0, T]; X)\} \quad (16)$$

If $f \in C^{k+\beta}([0, T]; X)$, we define

$$\|u\|_{k+\beta, T, X} := \sum_{j=0}^k \|u^{(j)}(0)\| + |u^{(k)}|_{\beta, T, X} \quad (17)$$

For any pair of Banach spaces X and X_1 , $\mathcal{L}(X; X_1)$ denotes the space of all bounded linear operators from X to X_1 equipped with the sup-norm. When $X = X_1$, we set $\mathcal{L}(X) = \mathcal{L}(X; X)$

and we denote by X' the space of all bounded and linear functionals on X . We now give the definition of sectorial operator in order to introduce the analytic semigroups.

Definition 2.1

Let $A: \mathcal{D}(A) \subset X \rightarrow X$ be a linear operator, possibly with $\overline{\mathcal{D}(A)} \neq X$. The operator A is said to be sectorial if it satisfies the following assumptions:

1. there exist two constants $R > 0$ and $\phi \in (\pi/2, \pi)$ such that any $\lambda \in \mathbb{C}$ with $|\lambda| \geq R$ and $|\arg \lambda| \leq \phi$ belongs to the resolvent set of A ;
2. there exists $M > 0$ such that $\|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \geq R, |\arg \lambda| \leq \phi$.

The fact that the resolvent set of A is not void implies that A is closed, so that $\mathcal{D}(A)$, endowed with the graph norm

$$\|x\|_{\mathcal{D}(A)} := \|x\| + \|Ax\| \tag{18}$$

becomes a Banach space.

If A satisfies Definition 2.1, it is possible to define the semigroup $\{e^{tA}\}_{t \geq 0}$ of bounded linear operators in $\mathcal{L}(X)$ (so that $t \rightarrow e^{tA}$ is an analytic function from $(0, \infty)$ to $\mathcal{L}(X)$, for more details see References [26–28]). Let us define the family of interpolation spaces $\mathcal{D}_A(\beta, \infty)$, $\beta \in (0, 1)$, between $\mathcal{D}(A)$ and X by

$$\mathcal{D}_A(\beta, \infty) = \left\{ x \in X : |x|_{\mathcal{D}_A(\beta, \infty)} := \sup_{0 < t < 1} t^{1-\beta} \|Ae^{tA}x\| < \infty \right\} \tag{19}$$

equipped with the norms

$$\|x\|_{\mathcal{D}_A(\beta, \infty)} = \|x\| + |x|_{\mathcal{D}_A(\beta, \infty)} \tag{20}$$

that make $\mathcal{D}_A(\beta, \infty)$ Banach spaces. The continuous inclusions

$$\mathcal{D}(A) \subseteq \mathcal{D}_A(\beta, \infty) \subseteq X \tag{21}$$

hold for every $\beta \in (0, 1)$. One can also show that, if $\lambda \in \mathbb{C}$ with $|\lambda| \geq R$ and $|\arg \lambda| \leq \phi$, for some $C > 0$ independent of λ , we have (see References [26,29]):

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X, \mathcal{D}_A(\beta, \infty))} \leq C|\lambda|^{\beta-1} \tag{22}$$

We are now in the position to state the inverse problem in the desired abstract setting

Problem 2.1 (Inverse Abstract Problem (IAP))

Let X be a Banach space. Determine five continuous functions $\theta: [0, T] \rightarrow X$, $\chi: [0, T] \rightarrow X$, $f: [0, T] \rightarrow \mathbb{R}$, $h: [0, T] \rightarrow \mathbb{R}$ and $k: [0, T] \rightarrow \mathbb{R}$ satisfying the system

$$\begin{cases} \theta'(t) + \chi'(t) + (h * \theta)'(t) = A\theta(t) + k * A\theta(t) + f(t)g \\ \chi'(t) = B\chi(t) + F(\chi(t)) - A\theta(t) \\ \theta(0) = \theta_0 \\ \chi(0) = \chi_0 \end{cases} \tag{23}$$

with the additional conditions

$$\Phi_j[\theta(t)] = b_j(t), \quad \forall t \in [0, \tau], \quad \tau \in (0, T], \quad j = 1, 2, 3 \tag{24}$$

where $A : \mathcal{D}(A) \rightarrow X$ and $B : \mathcal{D}(B) \rightarrow X$ are linear closed non-commuting operators, and $\theta_0, \chi_0, g, F, \Phi_j, b_j$ ($j = 1, 2, 3$) are given data whose properties are listed in the sequel.

We study the IAP under the following assumptions on the data.

H1: $X, \mathcal{D}(A)$ and $\mathcal{D}(B)$ are Banach spaces with the following continuous embedding:
 $\mathcal{D}(B) \hookrightarrow \mathcal{D}_B(\delta, \infty) \hookrightarrow \mathcal{D}(A) \hookrightarrow X$, for some $\delta \in (0, 1)$.

H2: $A : \mathcal{D}(A) \rightarrow X$ and $B : \mathcal{D}(B) \rightarrow X$ are linear, sectorial operators in X .

H3: $F \in C^2(\mathcal{D}_B(\delta, \infty), X)$ and F'' is uniformly Lipschitz continuous from $\mathcal{D}_B(\delta, \infty)$ to $\mathcal{L}(\mathcal{D}_B(\delta, \infty), \mathcal{L}(\mathcal{D}_B(\delta, \infty), X))$ on every bounded subset of $\mathcal{D}_B(\delta, \infty)$.

H4:

$$\begin{aligned} \theta_0 &\in \mathcal{D}(A), \quad \chi_0 \in \mathcal{D}(B) \\ v_0 &:= B\chi_0 + F(\chi_0) - A\theta_0 \in \mathcal{D}(B) \end{aligned} \tag{25}$$

H5: $g, A\theta_0 \in \overline{\mathcal{D}(A)}$.

H6: $\Phi_j \in X', \quad j = 1, 2, 3$.

H7: $b_j \in C^{2+\beta}([0, T])$, $\beta \in (0, 1)$, for $j = 1, 2, 3$.

H8: $\text{rank}(N) = \text{rank}(\tilde{N}) = 2$, where

$$N := \begin{bmatrix} \Phi_1(\theta_0) & -\Phi_1(g) \\ \Phi_2(\theta_0) & -\Phi_2(g) \\ \Phi_3(\theta_0) & -\Phi_3(g) \end{bmatrix} \tag{26}$$

$$\tilde{N} := \begin{bmatrix} \Phi_1(\theta_0) & -\Phi_1(g) & \Phi_1[A\theta_0] - \Phi_1[v_0] - b'_1(0) \\ \Phi_2(\theta_0) & -\Phi_2(g) & \Phi_2[A\theta_0] - \Phi_2[v_0] - b'_2(0) \\ \Phi_3(\theta_0) & -\Phi_3(g) & \Phi_3[A\theta_0] - \Phi_3[v_0] - b'_3(0) \end{bmatrix} \tag{27}$$

As a consequence of H5 and H8, system

$$\Phi_j(\theta_0)h_0 - \Phi_j[g]f_0 = \Phi_j[A\theta_0] - \Phi_j[v_0] - b'_j(0), \quad j = 1, 2, 3$$

has a unique solution (h_0, f_0) .

H9: $u_0 := A\theta_0 + f_0g - v_0 - h_0\theta_0 \in \mathcal{D}(A)$.

H10: $\text{Det } M \neq 0$ where

$$M := \begin{bmatrix} \Phi_1(\theta_0) & -\Phi_1(A\theta_0) & -\Phi_1(g) \\ \Phi_2(\theta_0) & -\Phi_2(A\theta_0) & -\Phi_2(g) \\ \Phi_3(\theta_0) & -\Phi_3(A\theta_0) & -\Phi_3(g) \end{bmatrix} \tag{28}$$

Now we set

$$v_1 := [B + F'(\chi_0)]v_0 - Au_0 \tag{29}$$

As a consequence of H5 and H10, the linear system

$$b'_j(0) + \Phi_j[v_1] = \Phi_j[Au_0 + k_0A\theta_0] + \Phi_j[z_0g - h_0u_0 - w_0\theta_0], \quad j = 1, 2, 3 \tag{30}$$

has a unique solution (w_0, k_0, z_0) .

H11: $[2A - h_0]u_0 - [B + F'(\chi_0)]v_0 - w_0\theta_0 + k_0A\theta_0 + z_0g \in \mathcal{D}_A(\delta, \infty)$.

H12: $v_1 \in \mathcal{D}_B(\delta, \infty)$.

H13: $\Phi_j[\theta_0] = b_j(0)$, $\Phi_j[u_0] = b'_j(0)$, $j = 1, 2, 3$.

3. THE MAIN RESULTS

In this section we state a *local in time* existence result for the IAP and a *global in time* uniqueness result for the same problem. More exactly:

- in Theorem 3.1 we claim that: *there exists $\tau \in (0, T]$ such that problem (23)–(24) has a solution (θ, χ, h, k, f) of domain $[0, \tau]$, satisfying the regularity conditions (31)–(35), the uniqueness, which appears in its proof, is related to a closed ball that depends on the data;*
- in Theorem 3.2 we claim that, *if $(\theta_1, \chi_1, h_1, k_1, f_1)$ and $(\theta_2, \chi_2, h_2, k_2, f_2)$ are solutions of (23)–(24) of domain $[0, T_1]$, for some $T_1 \in (0, T]$, with the declared regularity, then they coincide in $[0, T_1]$.*

Remark 3.1

We could also prove that a continuous dependence result hold. We omit the statement and the relative proof just for sake of brevity, since the calculations are long and tedious, but they can be deduced by the estimates obtained in the proofs of Theorems 3.1 and 3.2.

Since the proofs of the following Theorems 3.1 and 3.2 are quite long we postpone them in the next sections.

3.1. The main abstract results

Theorem 3.1 (Existence local in time)

Let the assumptions H1–H13 hold for $\beta \in (0, 1)$ and $T > 0$. Then there exists $\tau \in (0, T]$ such that problem (23)–(24) has a solution (θ, χ, h, k, f) , with

$$\theta \in C^{2+\beta}([0, \tau]; X) \cap C^{1+\beta}([0, \tau]; \mathcal{D}(A)) \tag{31}$$

$$\chi \in C^{2+\beta}([0, \tau]; X) \cap C^{1+\beta}([0, \tau]; \mathcal{D}(B)) \tag{32}$$

$$h \in C^{1+\beta}([0, \tau]) \tag{33}$$

$$k \in C^\beta([0, \tau]) \tag{34}$$

$$f \in C^{1+\beta}([0, \tau]) \tag{35}$$

Proof

See Section 6. □

Theorem 3.2 (Uniqueness global in time)

Let the assumptions H1–H13 hold for $\beta \in (0, 1)$ and $T > 0$. If $(\theta_1, \chi_1, h_1, k_1, f_1)$ and $(\theta_2, \chi_2, h_2, k_2, f_2)$ are solutions of (23)–(24) of domain $[0, T_1]$, with $0 < T_1 \leq T$, both satisfying the regularity conditions

$$\theta_j \in C^{2+\beta}([0, T_1]; X) \cap C^{1+\beta}([0, T_1]; \mathcal{D}(A)) \tag{36}$$

$$\chi_j \in C^{2+\beta}([0, T_1]; X) \cap C^{1+\beta}([0, T_1]; \mathcal{D}(B)) \tag{37}$$

$$h_j \in C^{1+\beta}([0, T_1]) \tag{38}$$

$$k_j \in C^\beta([0, T_1]) \tag{39}$$

$$f_j \in C^{1+\beta}([0, T_1]) \tag{40}$$

for $j = 1, 2$, then they coincide.

Proof

See Section 6. □

3.2. An application

We wish to apply Theorems 3.1 and 3.2 in the case X is the continuous functions space, so we take:

$$X = C(\bar{\Omega}) \tag{41}$$

We define

$$\left\{ \begin{aligned} \mathcal{D}(A) &= \left\{ \theta \in \bigcap_{1 \leq p < +\infty} W^{2,p}(\Omega) : \Delta\theta \in C(\bar{\Omega}), \quad D_\nu\theta|_{\partial\Omega} = 0 \right\} \\ A\theta &:= \Delta\theta, \quad \forall \theta \in \mathcal{D}(A) \end{aligned} \right. \tag{42}$$

and

$$\left\{ \begin{aligned} \mathcal{D}(B) &= \left\{ \chi \in \bigcap_{1 \leq p < +\infty} W^{4,p}(\Omega) : \Delta^2\chi \in C(\bar{\Omega}), \quad D_\nu\chi|_{\partial\Omega} = D_\nu\Delta\chi|_{\partial\Omega} = 0 \right\} \\ B\chi &:= -\Delta^2\chi, \quad \forall \chi \in \mathcal{D}(B) \end{aligned} \right. \tag{43}$$

We recall the following characterizations concerning the interpolation spaces related to A and B (see Reference [30, Theorem 3.6]).

We consider the Besov spaces $B_{\infty,\infty}^\alpha(\Omega)$ ($0 \leq \alpha < 4$). We recall that

$$B_{\infty,\infty}^\alpha(\Omega) = C^\alpha(\bar{\Omega})$$

whenever $\alpha \notin (0, 4) \setminus \mathbb{Z}$. Now, we fix two operators \tilde{B}_1 and \tilde{B}_2 , of order (respectively) one and three and coefficients of class $C^3(\bar{\Omega})$, such that they coincide in $\partial\Omega$ with the derivatives D_ν and $D_\nu\Delta$. Next, we set

$$B_{0,\infty,\infty}^0(\Omega) := \{f|_\Omega : f \in B_{\infty,\infty}^0(\mathbb{R}^3), \text{ supp}(f) \subseteq \bar{\Omega}\}$$

so we have

$$\mathcal{D}_A(\alpha, \infty) = \begin{cases} C^{2\alpha}(\bar{\Omega}) & \text{if } 0 < \alpha < \frac{1}{2} \\ \{f \in B^1_{\infty, \infty}(\Omega) : \tilde{B}_1 f \in B^0_{\infty, \infty}(\Omega)\} & \text{if } \alpha = \frac{1}{2} \\ \{f \in C^{2\alpha}(\bar{\Omega}) : D_\nu f|_{\partial\Omega} = 0\} & \text{if } \frac{1}{2} < \alpha < 1 \end{cases} \quad (44)$$

$$\mathcal{D}_B(\delta, \infty) = \begin{cases} C^{4\delta}(\bar{\Omega}) & \text{if } 0 < \delta < \frac{1}{4} \\ \{f \in B^1_{\infty, \infty}(\Omega) : \tilde{B}_1 f \in B^0_{\infty, \infty}(\Omega)\} & \text{if } \delta = \frac{1}{4} \\ \{u \in C^{4\delta}(\bar{\Omega}) : D_\nu u|_{\partial\Omega} = 0\} & \text{if } \frac{1}{4} < \delta < \frac{1}{2} \\ \{f \in B^1_{\infty, \infty}(\bar{\Omega}) : D_\nu f|_{\partial\Omega} = 0\} & \text{if } \delta = \frac{1}{2} \\ \{u \in C^{4\delta}(\bar{\Omega}) : D_\nu u|_{\partial\Omega} = 0\} & \text{if } \frac{1}{2} < \delta < \frac{3}{4} \\ \{f \in B^3_{\infty, \infty}(\Omega) : D_\nu f|_{\partial\Omega} = 0, \tilde{B}_2 f \in B^0_{\infty, \infty}(\Omega)\} & \text{if } \delta = \frac{3}{4} \\ \{f \in C^{4\delta}(\bar{\Omega}) : D_\nu f|_{\partial\Omega} = D_\nu \Delta f|_{\partial\Omega} = 0\} & \text{if } \frac{3}{4} < \delta < 1 \end{cases} \quad (45)$$

We consider the following system:

$$\begin{cases} \partial_t(\theta + \chi + (h * \theta))(t, x) \\ \quad = \Delta\theta(t, x) + (k * \Delta\theta)(t, x) + f(t)g(x), \quad t \in [0, T], \quad x \in \Omega \\ \partial_t \chi(t, x) = -\Delta^2 \chi(t, x) + \Delta[\phi \circ \chi - \theta](t, x), \quad t \in [0, T], \quad x \in \Omega \\ \theta(0, x) = \theta_0(x), \quad x \in \Omega \\ \chi(0, x) = \chi_0(x), \quad x \in \Omega \\ D_\nu \theta(t, x) = 0, \quad t \in [0, T], \quad x \in \partial\Omega \\ D_\nu \chi(t, x) = D_\nu \Delta \chi(t, x) = 0, \quad t \in [0, T], \quad x \in \partial\Omega \end{cases} \quad (46)$$

which, for $\phi \circ \chi =: \chi^3 + \gamma'(\chi)$, becomes system (7)–(8). Here D_ν denotes the derivative with respect to the outward unit normal vector to $\partial\Omega$ and the symbol $*$ denotes the convolution with respect to the time. The functions: $\phi : \mathbb{R} \rightarrow \mathbb{R}$, and $\theta_0, \chi_0 : \Omega \rightarrow \mathbb{R}$ are known and their properties will be specified in the sequel. To apply the abstract theorems we have to verify that conditions H1–H13 are satisfied in this case. We make the following assumptions:

- K1: Ω is an open bounded set in \mathbb{R}^3 , lying on one side of its boundary $\partial\Omega$, which is a sub-manifold of \mathbb{R}^3 of class C^4 .
- K2: We set

$$F(\chi) := \Delta(\phi \circ \chi) \quad (47)$$

and we assume that $\phi \in C^4(\mathbb{R})$ with $\phi^{(4)}$ Lipschitz continuous on bounded subsets of \mathbb{R} .

So, if K1 and K2 hold, we have

- H1 is satisfied, if $\delta \in (\frac{1}{2}, 1)$.
- H2 holds, owing to the generation results proved by Stewart in Reference [31].
- From K2 we have that $G: \chi \rightarrow \phi \circ \chi$ is of class C^2 from $C^2(\bar{\Omega})$ into itself and G'' is Lipschitz continuous on bounded subsets of $C^2(\bar{\Omega})$. In fact, it is easily seen that, if χ_1, χ_2, χ_3 are arbitrary elements of $C^2(\bar{\Omega})$, we have

$$G''(\chi_1)(\chi_2)(\chi_3) = (\phi'' \circ \chi_1)\chi_2\chi_3$$

It follows that F is of class C^2 from $C^2(\bar{\Omega})$ to $C(\bar{\Omega})$ and F'' is Lipschitz continuous on bounded subsets of $C^2(\bar{\Omega})$. So, if K1 and K2 are satisfied, H3 holds taking $\delta \in (\frac{1}{2}, 1)$.

Moreover, we assume what follows:

K3: $\theta_0 \in \mathcal{D}(A)$ (defined in (42)), $\chi_0 \in \mathcal{D}(B)$ (defined in (43)).

K4: $v_0 := -\Delta^2 \chi_0 + \Delta(\phi \circ \chi_0) - \Delta \theta_0 \in \mathcal{D}(B)$.

Note that K3 and K4 imply H4.

K5: $g, A\theta_0 \in C(\bar{\Omega})$ since $\overline{\mathcal{D}(A)} = C(\bar{\Omega})$ so H5 is satisfied.

K6: μ_j is a Borel measure in $\bar{\Omega}$, for $j=1,2,3$, so that H6 is satisfied.

K7: $b_j \in C^{2+\beta}([0, T])$, $\beta \in (0, 1)$, for $j=1,2,3$, so we get H7.

K8: Suppose H8 holds.

Set

$$\Phi_j[g] := \int_{\bar{\Omega}} g \mu_j(dx) \quad \text{for } j=1,2,3 \quad (48)$$

Now, taking into account (48), K3, K5, we can consider the matrices N and \tilde{N} defined in (26) and (27), respectively.

Thanks to assumptions K7 and H8 we are allowed to introduce h_0, f_0 as in H8.

K9: Define $u_0 := A\theta_0 + f_0 g - v_0 - h_0 \theta_0 \in \mathcal{D}(A)$ for problem (46).

Next, we can consider the matrix M as in (28), and we require that

K10: Condition H10 holds.

Now we can introduce w_0, k_0, z_0 as in H10.

K11: Suppose that H11 and H12 hold. Define $v_1 := [B + F'(\chi_0)]v_0 - Au_0 \in \mathcal{D}_B(\delta, \infty)$ for problem (46) (cf. H12 and (29)).

K12: For $j=1,2,3$, we set

$$\int_{\bar{\Omega}} \theta_0(x) \mu_j(dx) = b_j(0), \quad j=1,2,3 \quad (49)$$

$$\int_{\bar{\Omega}} u_0(x) \mu_j(dx) = b'_j(0), \quad j=1,2,3 \quad (50)$$

so that conditions H13 are satisfied.

So, applying Theorems 3.1 and 3.2, we can conclude that:

Theorem 3.3 (Existence local in time)

Suppose that K1–K12 hold for $\beta \in (0, 1)$ and $T > 0$. Then there exists $\tau \in (0, T]$, such that the inverse problem (46) and (9), has a solution (θ, χ, h, k, f) , with

$$\theta \in C^{2+\beta}([0, \tau]; C(\bar{\Omega})) \cap C^{1+\beta}([0, \tau]; \mathcal{D}(A)) \tag{51}$$

$$\chi \in C^{2+\beta}([0, \tau]; C(\bar{\Omega})) \cap C^{1+\beta}([0, \tau]; \mathcal{D}(B)) \tag{52}$$

$$h \in C^{1+\beta}([0, \tau]) \tag{53}$$

$$k \in C^\beta([0, \tau]) \tag{54}$$

$$f \in C^{1+\beta}([0, \tau]) \tag{55}$$

Theorem 3.4 (Uniqueness global in time)

Suppose that K1–K12 hold for $\beta \in (0, 1)$ and $T > 0$. If $(\theta_1, \chi_1, h_1, k_1, f_1)$ and $(\theta_2, \chi_2, h_2, k_2, f_2)$ are solutions of the inverse problem (46) and (9), of domain $[0, T_1]$, with $0 < T_1 \leq T$, both satisfying the regularity conditions

$$\theta_j \in C^{2+\beta}([0, T_1]; C(\bar{\Omega})) \cap C^{1+\beta}([0, T_1]; \mathcal{D}(A)) \tag{56}$$

$$\chi_j \in C^{2+\beta}([0, T_1]; C(\bar{\Omega})) \cap C^{1+\beta}([0, T_1]; \mathcal{D}(B)) \tag{57}$$

$$h_j \in C^{1+\beta}([0, T_1]) \tag{58}$$

$$k_j \in C^\beta([0, T_1]) \tag{59}$$

$$f_j \in C^{1+\beta}([0, T_1]) \tag{60}$$

$j = 1, 2$, then they coincide.

Remark 3.2

What we have observed at the beginning of this section for the abstract Theorems 3.1 and 3.2 holds obviously for Theorems 3.3 and 3.4.

Remark 3.3

With our approach we can replace operators $-\Delta^2$ (applied to χ) and Δ (applied to θ) by operators that do not commute. For example by $-\eta_1 \Delta^2$ and $\eta_2 \Delta$, respectively, where the coefficients $\eta_1 > 0$, $\eta_2 > 0$ are functions of the space variables, in fact, Corollary 4.1 that does not require commutation conditions on the operators.

4. PRELIMINARY LEMMAS

In this section we collect some preliminary results which will be necessary to prove our main theorems. In this and in the next sections the symbols $C(\cdot)$, $C(\cdot, \cdot, \dots)$, sometimes with an

index, will denote positive constants continuously depending on the arguments pointed out. Let us begin by stating an optimal regularity result for the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \end{cases} \quad (61)$$

Theorem 4.1

Let $A: \mathcal{D}(A) \subset X \rightarrow X$ be a sectorial operator in X . Then, for $\beta \in (0, 1)$, and for any $f \in C^\beta([0, T]; X)$, $u_0 \in \mathcal{D}(A)$ with $Au_0 + f(0) \in \mathcal{D}_A(\beta, \infty)$, problem (61) admits a unique solution $u \in C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; \mathcal{D}(A))$ represented by the formula

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s) ds := e^{tA}u_0 + (e^{tA} * f)(t) \quad (62)$$

Moreover, if $T \leq T_0$, with $T_0 \in \mathbb{R}^+$, then

$$\begin{aligned} & \|u\|_{1+\beta, T, X} + \|u\|_{\beta, T, \mathcal{D}(A)} \\ & \leq C(T_0)(\|f\|_{\beta, T, X} + \|u_0\|_{\mathcal{D}(A)} + \|Au_0 + f(0)\|_{\mathcal{D}_A(\beta, \infty)}) \end{aligned} \quad (63)$$

Proof

The proof of this theorem can be found in Reference [28]. \square

Lemma 4.1

Let X be a Banach space and let A be a sectorial operator in X . Suppose that y_0 belongs to $\mathcal{D}(A)$, $h \in C^\beta([0, \tau])$, with $h(0) = 0$ and $\tau \in \mathbb{R}^+$ and set

$$z(t) = \int_0^t e^{(t-s)A} h(s) y_0 ds, \quad t \in [0, \tau]$$

Then

$$\|z\|_{\beta, \tau, \mathcal{D}(A)} \leq \eta(\tau) \|h\|_{\beta, \tau, \mathbb{R}}$$

with $\eta(\tau)$ independent of h and $\lim_{\tau \rightarrow 0} \eta(\tau) = 0$.

Proof

It is Lemma 4.7 in Reference [32]. \square

In the sequel we need the following particular case of Proposition 2.4.1 in Reference [26].

Lemma 4.2

Let A be a sectorial operator in the Banach space X , $\delta \in (0, 1)$ and $P \in \mathcal{L}(\mathcal{D}_A(\delta, \infty), X)$. Then, the operator $A + P$, with domain $\mathcal{D}(A)$, is sectorial.

We can now prove an optimal regularity result for a particular linear system (see Corollary 4.1 in the following). First, we assume that the conditions H1–H3 hold and we

introduce the following linear operator \mathcal{A} in the space $X \times X$:

$$\mathcal{D}(\mathcal{A}) := \mathcal{D}(A) \times \mathcal{D}(B) \tag{64}$$

$$\mathcal{A}(u, v) := ([2A - h_0I]u - [B + F'(\chi_0)]v, -Au + [B + F'(\chi_0)]v) \tag{65}$$

with $h_0 \in \mathbb{C}$.

Lemma 4.3

If H1–H3 hold and $h_0 \in \mathbb{C}$, then the operator \mathcal{A} is sectorial in $X \times X$.

Proof

Let us consider the system

$$\begin{cases} \lambda u - (2A - h_0I)u + [B + F'(\chi_0)]v = \psi_1 \\ \lambda v + Au - [B + F'(\chi_0)]v = \psi_2 \end{cases} \tag{66}$$

with $\lambda \in \mathbb{C}$, $\psi_j \in X$, $j = 1, 2$ and define the operator

$$\tilde{B} := B + F'(\chi_0) \tag{67}$$

Since (cf. H3) $F'(\chi_0) \in \mathcal{L}(\mathcal{D}_B(\delta, \infty), X)$, by Lemma 4.2, \tilde{B} is sectorial in X . So we can write (66) in the form

$$\begin{cases} \lambda u - (2A - h_0I)u + \tilde{B}v = \psi_1 \\ \lambda v + Au - \tilde{B}v = \psi_2 \end{cases} \tag{68}$$

Setting

$$\mathcal{U} = u + v \quad (\in \mathcal{D}(A))$$

we obtain the equivalent system

$$\begin{cases} \lambda \mathcal{U} - [A - h_0I]\mathcal{U} + (A - h_0I)v = \psi_1 + \psi_2 \\ \lambda v - (\tilde{B} + A)v = \psi_2 - A\mathcal{U} \end{cases} \tag{69}$$

Now, $\tilde{B} + A$ is sectorial by Lemma 4.2, so there exist $\alpha_0 \in (\pi/2, \pi)$ and $r_0 > 0$ so that, if $|\lambda| \geq r_0$ and $|\text{Arg } \lambda| \leq \alpha_0$, then $\forall \mathcal{U} \in \mathcal{D}(A)$, the second equation in (69) has a unique solution v in $\mathcal{D}(B)$: we have

$$v = (\lambda I - \tilde{B} - A)^{-1} \psi_2 - (\lambda I - \tilde{B} - A)^{-1} A \mathcal{U}$$

Moreover, for some $C_1 > 0$ independent of λ , ψ_2 and \mathcal{U} , we have

$$|\lambda| \|v\| + \|v\|_{\mathcal{D}(B)} \leq C_1 [\|\psi_2\| + \|A\mathcal{U}\|] \tag{70}$$

Replacing v in the first equation in (69), we get

$$\lambda \mathcal{U} - (A - h_0I)\mathcal{U} = \psi_1 + \psi_2 - (A - h_0I)(\lambda I - \tilde{B} - A)^{-1} \psi_2 + (A - h_0I)(\lambda I - \tilde{B} - A)^{-1} A \mathcal{U} \tag{71}$$

Since $A - h_0I$ is sectorial, we can modify (if necessary) $|\lambda| \geq r_0$ and $|\text{Arg } \lambda| \leq \alpha_0$, in such a way that operator $\lambda I + h_0I - A$ is invertible and

$$\|(\lambda I + h_0I - A)^{-1}\|_{\mathcal{L}(X)} \leq C_2/|\lambda|$$

Setting

$$x := (\lambda I + h_0I - A)\mathcal{U}$$

we obtain

$$x = \psi_1 + \psi_2 - (A - h_0I)(\lambda I - \tilde{B} - A)^{-1}\psi_2 + (A - h_0I)(\lambda I - \tilde{B} - A)^{-1}A(\lambda I + h_0I - A)^{-1}x \quad (72)$$

The estimate

$$\begin{aligned} & \| (A - h_0I)(\lambda I - \tilde{B} - A)^{-1}A(\lambda I + h_0I - A)^{-1} \|_{\mathcal{L}(X)} \\ & \leq C_3 \| (A - h_0I)(\lambda I - \tilde{B} - A)^{-1} \|_{\mathcal{L}(X)} \\ & \leq C_4 \| (\lambda I - \tilde{B} - A)^{-1} \|_{\mathcal{L}(X, \mathcal{D}_B(\delta, \infty))} \leq C_5 |\lambda|^{\delta-1} \end{aligned}$$

follows from (22). So, if $|\lambda|$ is sufficiently large, in such a way that $C_5|\lambda|^{\delta-1} < 1$, there exists a unique $x \in X$ solving equation (72). From (72), it also follows that there exists $C_6 > 0$, independent of λ, ψ_1, ψ_2 , such that

$$\|x\| \leq C_6(\|\psi_1\| + \|\psi_2\|) \quad (73)$$

putting $\mathcal{U} := (\lambda I + h_0I - A)^{-1}x$, we have

$$|\lambda| \|\mathcal{U}\| + \|\mathcal{U}\|_{\mathcal{D}(A)} \leq C_7(\|\psi_1\| + \|\psi_2\|) \quad (74)$$

From (70) and (74), we also obtain

$$|\lambda| \|v\| + \|v\|_{\mathcal{D}(B)} \leq C_8[\|\psi_1\| + \|\psi_2\|] \quad (75)$$

thus we deduce

$$\|v\| \leq \frac{C_1}{|\lambda|} [\|\psi_1\| + \|\psi_2\|]$$

From the position

$$u = \mathcal{U} - v$$

we get the desired estimates also for u , recalling that, for some $C_9 > 0$ we have (cf. H1)

$$\|v\|_{\mathcal{D}(A)} \leq C_9 \|v\|_{\mathcal{D}(B)} \quad \square$$

An immediate consequence of Lemma 4.3 is the following result that we will use to prove our main abstract theorems.

Corollary 4.1

Let the assumptions of Lemma 4.3 be satisfied, let $\beta \in (0, 1)$ and $T > 0$. Consider the Cauchy problem

$$\begin{cases} u'(t) = [2A - h_0I]u(t) - [B + F'(\chi_0)]v(t) + f_1(t) \\ v'(t) = [B + F'(\chi_0)]v(t) - Au(t) + f_2(t) \\ u(0) = u_0 \\ v(0) = v_0 \end{cases} \tag{76}$$

with the regularity conditions

- (1) $f_1, f_2 \in C^\beta([0, T]; X)$,
- (2) $u_0 \in \mathcal{D}(A)$,
- (3) $v_0 \in \mathcal{D}(B)$,
- (4) $[2A - h_0I]u_0 - [B + F'(\chi_0)]v_0 + f_1(0) \in \mathcal{D}_A(\beta, \infty)$,
- (5) $[B + F'(\chi_0)]v_0 - Au_0 + f_2(0) \in \mathcal{D}_B(\beta, \infty)$.

Then the Cauchy problem (76) has a unique solution

$$\begin{aligned} u &\in C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; \mathcal{D}(A)) \\ v &\in C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; \mathcal{D}(B)) \end{aligned}$$

Moreover, if we consider the semigroup $e^{t\mathcal{A}}$, where \mathcal{A} is defined as in (64) and (65), we have

$$e^{t\mathcal{A}} = \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix} \tag{77}$$

and the variation of parameter formulas

$$u(t) = S_{11}(t)u_0 + S_{12}v_0 + \int_0^t [S_{11}(t-s)f_1(s) + S_{12}(t-s)f_2(s)] ds \tag{78}$$

$$v(t) = S_{21}(t)u_0 + S_{22}v_0 + \int_0^t [S_{21}(t-s)f_1(s) + S_{22}(t-s)f_2(s)] ds \tag{79}$$

We now state some lemmas which are necessary to study the non-linear perturbations of the linear problem.

Lemma 4.4

Suppose that

- $\mathcal{X}, \mathcal{Y}, \mathcal{V}$ are three Banach spaces,
- $\pi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{V}$ is a bilinear and continuous map,
- $G \in C^1(\mathcal{Y}, \mathcal{X})$, with G' uniformly Lipschitz continuous on the bounded subsets of \mathcal{Y} and with $G(0) = 0$,

- $\beta \in (0, 1)$, $T \in \mathbb{R}^+$, $\tau \in (0, T]$,
- $y_0 \in \mathcal{Y}$,
- V, V_1, V_2 elements of $C^\beta([0, \tau], \mathcal{Y})$,
- $V(0) = V_1(0) = V_2(0) = y_0 \in \mathcal{Y}$,
- let $R > 0$, be such that

$$\max\{\|V\|_{\beta, \tau, \mathcal{Y}}, \|V_1\|_{\beta, \tau, \mathcal{Y}}, \|V_2\|_{\beta, \tau, \mathcal{Y}}\} \leq R$$

Then we have:

- (I) $\|\pi(G(1 * V), V)\|_{\beta, \tau, \mathcal{Y}} \leq C(R, T)\tau^{1-\beta}$,
 (II) $\|\pi(G(1 * V_1), V_1) - \pi(G(1 * V_2), V_2)\|_{\beta, \tau, \mathcal{Y}} \leq \tau C(R, T)|V_1 - V_2|_{\beta, \tau, \mathcal{Y}}$,

where C denotes different positive constants.

Proof

(I) As $G(0) = 0$, we have

$$\|\pi(G(1 * V), V)\|_{\beta, \tau, \mathcal{Y}} = |\pi(G(1 * V), V)|_{\beta, \tau, \mathcal{Y}}$$

and clearly, for some $C \in \mathbb{R}^+$, it follows that

$$|\pi(G(1 * V), V)|_{\beta, \tau, \mathcal{Y}} \leq C(|G(1 * V)|_{\beta, \tau, \mathcal{X}} \|V\|_{0, \tau, \mathcal{Y}} + R \|G(1 * V)\|_{0, \tau, \mathcal{X}})$$

Observe first that

$$\|V\|_{0, \tau, \mathcal{Y}} \leq \|y_0\|_{\mathcal{Y}} + \tau^\beta |V|_{\beta, \tau, \mathcal{Y}} \leq R(1 \vee T^\beta) \quad (80)$$

so we get

$$\begin{aligned} \|1 * V\|_{0, \tau, \mathcal{X}} &\leq \tau \|V\|_{0, \tau, \mathcal{Y}} \\ &\leq R(1 \vee T^\beta)\tau \end{aligned} \quad (81)$$

from which it follows

$$\begin{aligned} \|G(1 * V)\|_{0, \tau, \mathcal{X}} &\leq C(R) \|1 * V\|_{0, \tau, \mathcal{X}} \\ &\leq C(R, T)\tau \end{aligned} \quad (82)$$

Now we estimate $|G(1 * V)|_{\beta, \tau, \mathcal{X}}$. Clearly, by (81) we deduce

$$\begin{aligned} |G(1 * V)|_{\beta, \tau, \mathcal{X}} &\leq C(R, T) |1 * V|_{\beta, \tau, \mathcal{Y}} \\ &\leq C(R, T)\tau^{1-\beta} \|V\|_{0, \tau, \mathcal{Y}} \\ &\leq C(R, T)\tau^{1-\beta} \end{aligned} \quad (83)$$

Therefore (I) follows from (80), (82), (83).

We show (II). We consider the chain of inequalities

$$\begin{aligned}
 \|\pi(G(1 * V_1), V_1) - \pi(G(1 * V_2), V_2)\|_{\beta, \tau, \mathcal{V}} &= |\pi(G(1 * V_1), V_1) - \pi(G(1 * V_2), V_2)|_{\beta, \tau, \mathcal{V}} \\
 &\leq |\pi(G(1 * V_1) - G(1 * V_2), V_1)|_{\beta, \tau, \mathcal{V}} \\
 &\quad + |\pi(G(1 * V_2), V_1 - V_2)|_{\beta, \tau, \mathcal{V}} \\
 &\leq C(|G(1 * V_1) - G(1 * V_2)|_{\beta, \tau, \mathcal{X}} \|V_1\|_{0, \tau, \mathcal{Y}} \\
 &\quad + \|G(1 * V_1) - G(1 * V_2)\|_{0, \tau, \mathcal{X}} |V_1|_{\beta, \tau, \mathcal{Y}} \\
 &\quad + |G(1 * V_2)|_{\beta, \tau, \mathcal{X}} \|V_1 - V_2\|_{0, \tau, \mathcal{Y}} \\
 &\quad + \|G(1 * V_2)\|_{0, \tau, \mathcal{X}} |V_1 - V_2|_{\beta, \tau, \mathcal{Y}}) \\
 &:= C(I_1 + I_2 + I_3 + I_4)
 \end{aligned}$$

Owing to (82), we get

$$I_4 \leq C(R, T)\tau |V_1 - V_2|_{\beta, \tau, \mathcal{Y}} \tag{84}$$

and, on account of (83), we obtain

$$I_3 \leq C(R, T)\tau^{1-\beta} \|V_1 - V_2\|_{0, \tau, \mathcal{Y}} \leq C(R, T)\tau |V_1 - V_2|_{\beta, \tau, \mathcal{Y}} \tag{85}$$

Moreover,

$$\begin{aligned}
 \|G(1 * V_1) - G(1 * V_2)\|_{0, \tau, \mathcal{X}} &\leq C(R) \|1 * (V_1 - V_2)\|_{0, \tau, \mathcal{Y}} \\
 &\leq C(R)\tau \|V_1 - V_2\|_{0, \tau, \mathcal{Y}} \\
 &\leq C(R)\tau^{1+\beta} |V_1 - V_2|_{\beta, \tau, \mathcal{Y}}
 \end{aligned}$$

and we conclude that

$$I_2 \leq C(R)\tau^{1+\beta} |V_1 - V_2|_{\beta, \tau, \mathcal{Y}} \tag{86}$$

Consider now $0 \leq s < t \leq \tau$, so we can write

$$G(1 * V_1(t)) - G(1 * V_2(t)) = \int_0^1 G'[1 * V_2(t) + \rho[1 * (V_1 - V_2)(t)]] [1 * (V_1 - V_2)(t)] d\rho$$

from which we deduce

$$\begin{aligned}
& \|G(1 * V_1(t)) - G(1 * V_2(t)) - G(1 * V_1(s)) + G(1 * V_2(s))\|_{\mathcal{X}} \\
& \leq \left\| \int_0^1 \{G'[1 * V_2(t) + \rho[1 * (V_1 - V_2)(t)]] - G'[1 * V_2(s) \right. \\
& \quad \left. + \rho[1 * (V_1 - V_2)(s)]]\}[1 * (V_1 - V_2)(t)] \, d\rho \right\|_{\mathcal{X}} \\
& \quad + \left\| \int_0^1 G'[1 * V_2(s) + \rho[1 * (V_1 - V_2)(s)]] [1 * (V_1 - V_2)(t) - 1 * (V_1 - V_2)(s)] \, d\rho \right\|_{\mathcal{X}} \\
& := I_{11} + I_{12}
\end{aligned}$$

Then we have

$$\begin{aligned}
\|1 * (V_1 - V_2)(t)\|_{\mathcal{Y}} & \leq \tau \|V_1 - V_2\|_{0, \tau, \mathcal{Y}} \\
& \leq \tau^{1+\beta} |V_1 - V_2|_{\beta, \tau, \mathcal{Y}}
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
\|1 * (V_1 - V_2)(t) - 1 * (V_1 - V_2)(s)\|_{\mathcal{Y}} & \leq (t - s) \|V_1 - V_2\|_{0, \tau, \mathcal{Y}} \\
& \leq (t - s) \tau^\beta |V_1 - V_2|_{\beta, \tau, \mathcal{Y}}
\end{aligned} \tag{88}$$

so we deduce that

$$\begin{aligned}
I_{12} & \leq C(R) \|1 * (V_1 - V_2)(t) - 1 * (V_1 - V_2)(s)\|_{\mathcal{Y}} \\
& \leq C(R) (t - s) \tau^\beta |V_1 - V_2|_{\beta, \tau, \mathcal{Y}} \\
& \leq C(R) (t - s)^\beta \tau |V_1 - V_2|_{\beta, \tau, \mathcal{Y}}
\end{aligned} \tag{89}$$

Finally, we get

$$\begin{aligned}
I_{11} & \leq C(R) (\|1 * V_2(t) - 1 * V_2(s)\|_{\mathcal{Y}} + \|1 * (V_1 - V_2)(t) - 1 * (V_1 - V_2)(s)\|_{\mathcal{Y}}) \\
& \quad \times \|1 * (V_1 - V_2)(t)\|_{\mathcal{Y}} \\
& \leq C(R) [(t - s) \|V_2\|_{0, \tau, \mathcal{Y}} + (t - s) \tau^\beta |V_1 - V_2|_{\beta, \tau, \mathcal{Y}}] \tau^{1+\beta} |V_1 - V_2|_{\beta, \tau, \mathcal{Y}} \\
& \leq C(R, T) (t - s)^\beta \tau^2 |V_1 - V_2|_{\beta, \tau, \mathcal{Y}}
\end{aligned} \tag{90}$$

From (89) and (90), we obtain

$$I_1 \leq C(R, T)\tau|V_1 - V_2|_{\beta, \tau, \mathcal{Y}} \tag{91}$$

So (II) follows from (91), (86), (85), (84). □

Corollary 4.2

Let the assumptions (H1)–(H3) hold. Suppose that, for $\beta, \delta \in (0, 1)$

- $\chi_0, v_0 \in \mathcal{D}_B(\delta, \infty)$,
- $T \in \mathbb{R}^+, \tau \in (0, T]$,
- V, V_1, V_2 belong to $C^\beta([0, \tau], \mathcal{D}_B(\delta, \infty))$,
- $V(0) = V_1(0) = V_2(0) = v_0$,
- $R > 0$ is such that

$$\max\{\|V\|_{\beta, \tau, \mathcal{D}_B(\delta, \infty)}, \|V_1\|_{\beta, \tau, \mathcal{D}_B(\delta, \infty)}, \|V_2\|_{\beta, \tau, \mathcal{D}_B(\delta, \infty)}\} \leq R$$

Then we have the estimates

- (I) $\|[F'(\chi_0 + 1 * V) - F'(\chi_0)]V\|_{\beta, \tau, X} \leq C(R, T)\tau^{1-\beta}$,
- (II) $\|[F'(\chi_0 + 1 * V_1) - F'(\chi_0)]V_1 - [F'(\chi_0 + 1 * V_2) - F'(\chi_0)]V_2\|_{\beta, \tau, X} \leq C(R, T)\tau|V_1 - V_2|_{\beta, \tau, \mathcal{D}_B(\delta, \infty)}$.

Proof

If we set:

- $X := \mathcal{L}(\mathcal{D}_B(\delta, \infty), X)$,
- $\mathcal{Y} := \mathcal{D}_B(\delta, \infty)$,
- $\mathcal{V} := X$,
- $\pi : \mathcal{L}(\mathcal{D}_B(\delta, \infty), X) \times \mathcal{D}_B(\delta, \infty) \rightarrow X$,
- $\pi(S, z) = Sz$,
- $G := F'(\chi_0 + \cdot) - F'(\chi_0)$,
- $y_0 = v_0$,

the statement follows immediately from Lemma 4.4. □

The following result is easy to prove and it is left to the reader.

Lemma 4.5

The convolution operator

$$w * h(t) := \int_0^t h(t-s)w(s) ds, \quad t \in [0, \tau] \tag{92}$$

maps $C([0, \tau]; X) \times C^\beta([0, \tau]; \mathbb{R})$ into $C^\beta([0, \tau]; X)$, for $\beta \in (0, 1)$, and satisfies the following estimate:

$$\|w * h\|_{\beta, \tau, X} \leq C(\beta, \tau)\tau^{1-\beta}\|h\|_{\beta, \tau, \mathbb{R}}\|w\|_{0, \tau, X} \tag{93}$$

5. AN EQUIVALENT FIXED POINT PROBLEM

Assume that the conditions H1–H13 are satisfied and a solution (θ, χ, h, k, f) to problem (23)–(24), satisfying the regularity assumptions (31)–(35), in some interval $[0, \tau]$, exists. We observe that, from the second equation in (23), we get

$$\chi'(0) = v_0 \quad (94)$$

Applying Φ_j (for $j = 1, 2, 3$) to the first equation in (23) and using the last for $t = 0$, we get

$$b'_j(0) + \Phi_j(v_0) + h(0)\Phi_j(\theta_0) = \Phi_j(A\theta_0) + f(0)\Phi_j(g), \quad j = 1, 2, 3$$

From H8 it follows that

$$h(0) = h_0, \quad f(0) = f_0 \quad (95)$$

Using again the first equation in (23) for $t = 0$, we get

$$\theta'(0) = u_0 \quad (96)$$

Now we set, for $\beta \in (0, 1)$,

$$\begin{aligned} u &:= \theta' \in C^{1+\beta}([0, \tau], X) \cap C^\beta([0, \tau], \mathcal{D}(A)) \\ v &:= \chi' \in C^{1+\beta}([0, \tau], X) \cap C^\beta([0, \tau], \mathcal{D}(B)) \\ w &:= h' \in C^\beta([0, \tau]) \\ z &:= f' \in C^\beta([0, \tau]) \end{aligned} \quad (97)$$

so that, differentiating system (23), we obtain

$$\begin{cases} u'(t) + v'(t) = (A - h_0 I)u(t) + k(t)A\theta_0 + k * Au(t) \\ \quad + z(t)g - w(t)\theta_0 - w * u(t) \\ v'(t) = Bv(t) + F'(\chi_0 + 1 * v(t))v(t) - Au(t) \\ u(0) = u_0 \\ v(0) = v_0 \end{cases} \quad (98)$$

From (98), we have that

$$v'(0) = v_1 \quad (99)$$

Applying Φ_j (for $j = 1, 2, 3$) to the first equation in (98) and using $\Phi_j[u'] = b'_j$, we obtain, for $t = 0$

$$b''_j(0) + \Phi_j[v_1] = \Phi_j[(A - h_0 I)u_0] + k(0)\Phi_j[A\theta_0] + z(0)\Phi_j[g] - w(0)\Phi_j[\theta_0] \quad (100)$$

Because of condition H10, we get $w(0) = w_0$, $z(0) = z_0$ and $k(0) = k_0$. Using now the second equation in (98), we get

$$\begin{cases} u'(t) = (2A - h_0I)u(t) - [B + F'(\chi_0)]v(t) - [F'(\chi_0 + 1 * v(t)) - F'(\chi_0)]v(t) \\ \quad - w(t)\theta_0 + k(t)A\theta_0 + k * Au(t) + z(t)g - w * u(t) \\ v'(t) = [B + F'(\chi_0)]v(t) - Au(t) + [F'(\chi_0 + 1 * v(t)) - F'(\chi_0)]v(t) \\ u(0) = u_0 \\ v(0) = v_0 \end{cases} \tag{101}$$

Now we consider the system, for $t \in [0, T]$:

$$\begin{cases} \mathcal{U}'_0(t) = (2A - h_0I)\mathcal{U}_0(t) - [B + F'(\chi_0)]\mathcal{V}'_0(t) - w_0\theta_0 + k_0A\theta_0 + z_0g \\ \mathcal{V}'_0(t) = [B + F'(\chi_0)]\mathcal{V}_0(t) - A\mathcal{U}_0(t) \\ \mathcal{U}(0) = u_0 \\ \mathcal{V}(0) = v_0 \end{cases} \tag{102}$$

Then, owing to Corollary 4.1 and the assumptions H4, H9, H11, H12, (102) has a unique solution $(\mathcal{U}_0, \mathcal{V}_0)$ belonging to $(C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; \mathcal{D}(A))) \times (C^{1+\beta}([0, T]; X) \cap C^\beta([0, T]; \mathcal{D}(B)))$. Define the operators

$$\begin{aligned} \mathcal{N}_1(u, v, w, k, z)(t) &:= \int_0^t S_{11}(t-s) \{ (w_0 - w(s))\theta_0 + (k(s) - k_0)A\theta_0 + (z(s) - z_0)g \\ &\quad - [F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s) + k * Au(s) - w * u(s) \} ds \\ &\quad + \int_0^t S_{12}(t-s) [F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s) ds \end{aligned} \tag{103}$$

$$\begin{aligned} \mathcal{N}_2(u, v, w, k, z)(t) &:= \int_0^t S_{21}(t-s) \{ (w_0 - w(s))\theta_0 + (k(s) - k_0)A\theta_0 + (z(s) - z_0)g \\ &\quad - [F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s) + k * Au(s) - w * u(s) \} ds \\ &\quad + \int_0^t S_{22}(t-s) [F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s) ds \end{aligned} \tag{104}$$

Then, from (101) and Corollary 4.1, we get (for $t \in [0, \tau]$)

$$\begin{cases} u(t) = \mathcal{U}_0(t) + \mathcal{N}_1(u, v, w, k, z)(t) \\ v(t) = \mathcal{V}_0(t) + \mathcal{N}_2(u, v, w, k, z)(t) \end{cases} \tag{105}$$

We now set, for sake of simplicity

$$\tilde{B} := B + F'(\chi_0) \tag{106}$$

Applying Φ_j (for $j = 1, 2, 3$) to the first equation in (101), we have also

$$\begin{aligned} b_j''(t) &= \Phi_j[(2A - h_0I)u(t)] - \Phi_j[\tilde{B}v(t)] - \Phi_j[[F'(\chi_0 + 1 * v(t)) - F'(\chi_0)]v(t)] \\ &\quad - w(t)\Phi_j[\theta_0] + k(t)\Phi_j[A\theta_0] + \Phi_j[k * Au](t) + z(t)\Phi_j[g] \\ &\quad - \Phi_j[w * u](t), \quad j = 1, 2, 3 \end{aligned} \quad (107)$$

which implies

$$w(t)\Phi_j[\theta_0] - k(t)\Phi_j[A\theta_0] - z(t)\Phi_j[g] = \Gamma_{0j}(t) + \Gamma_j(u, v, w, k, z)(t) \quad (108)$$

where we have set

$$\Gamma_{0j}(t) := -b_j''(t) + \Phi_j[[2A - h_0I]\mathcal{U}_0(t)] - \Phi_j[\tilde{B}(\mathcal{V}_0)(t)] \quad (109)$$

$$\begin{aligned} \Gamma_j(u, v, w, k, z)(t) &:= \Phi_j[[2A - h_0I]\mathcal{N}_1(u, v, w, k, z)(t)] - \Phi_j[\tilde{B}\mathcal{N}_2(u, v, w, k, z)(t)] \\ &\quad - \Phi_j[[F'(\chi_0 + 1 * v(t)) - F'(\chi_0)]v(t)] + \Phi_j[k * Au](t) \\ &\quad + \Phi_j[w * u](t), \quad j = 1, 2, 3 \end{aligned} \quad (110)$$

From the assumption H10, we obtain

$$\begin{bmatrix} w(t) \\ k(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \tilde{\Gamma}_{01}(t) \\ \tilde{\Gamma}_{02}(t) \\ \tilde{\Gamma}_{03}(t) \end{bmatrix} + \begin{bmatrix} \tilde{\Gamma}_1(u, v, w, k, z)(t) \\ \tilde{\Gamma}_2(u, v, w, k, z)(t) \\ \tilde{\Gamma}_3(u, v, w, k, z)(t) \end{bmatrix} \quad (111)$$

with

$$\begin{bmatrix} \tilde{\Gamma}_1(w, k, z, u, v) \\ \tilde{\Gamma}_2(w, k, z, u, v) \\ \tilde{\Gamma}_3(w, k, z, u, v) \end{bmatrix} := M^{-1} \begin{bmatrix} \Gamma_1(w, k, z, u, v) \\ \Gamma_2(w, k, z, u, v) \\ \Gamma_3(w, k, z, u, v) \end{bmatrix} \quad (112)$$

$$\begin{bmatrix} \tilde{\Gamma}_{01}(t) \\ \tilde{\Gamma}_{02}(t) \\ \tilde{\Gamma}_{03}(t) \end{bmatrix} := M^{-1} \begin{bmatrix} \Gamma_{01}(t) \\ \Gamma_{02}(t) \\ \Gamma_{03}(t) \end{bmatrix} \quad (113)$$

We point out that, for $j = 1, 2, 3$, $\Gamma_{0j} \in C^\beta([0, \tau])$. Now we are able to state and prove the following result of equivalence.

Proposition 5.1

Suppose that H1–H13 are fulfilled. Set $\beta \in (0, 1)$ and let (θ, χ, h, k, f) be a solution of system (23)–(24), satisfying the regularity assumptions (31)–(35). Define $u := \theta'$, $v := \chi'$, $w := h'$, $z = f'$, then

- (I) $u \in C^\beta([0, \tau]; \mathcal{D}(A))$
- (II) $v \in C^\beta([0, \tau]; \mathcal{D}(B))$
- (III) $w, k, z \in C^\beta([0, \tau])$
- (IV) $w(0) = w_0, k(0) = k_0, z(0) = z_0$
- (V) (u, v, w, k, z) satisfies Equations (105) and (111), for every $t \in [0, \tau]$.

On the other hand, assume that the conditions (I)–(V) are satisfied. Set $\theta := \theta_0 + 1 * u$, $\chi := \chi_0 + 1 * v$, $h := h_0 + 1 * w$, $f := f_0 + 1 * z$. Then (θ, χ, h, k, f) satisfies the regularity conditions (31)–(35) and is a solution of system (23)–(24) in $[0, \tau]$.

Proof

We have already seen that, if (θ, χ, h, k, f) is a solution of the system (23)–(24), satisfying the regularity assumptions (31)–(35), then (u, v, w, k, z) satisfies the conditions (I)–(V).

On the other hand, assume that the conditions (I)–(V) are fulfilled.

From the assumptions H4, H9, H11, H12, owing to Corollary 4.1, we have

$$\begin{aligned} \mathcal{U}_0 &\in C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(A)) \\ \mathcal{V}_0 &\in C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(B)) \end{aligned}$$

Applying Corollaries 4.1 and 4.2 and Lemma 4.5, we also obtain that

$$\begin{aligned} \mathcal{N}_1(u, v, w, k, z) &\in C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(A)) \\ \mathcal{N}_2(u, v, w, k, z) &\in C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(B)) \end{aligned}$$

so we have

$$\begin{aligned} u &\in C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(A)) \\ v &\in C^{1+\beta}([0, \tau]; X) \cap C^\beta([0, \tau]; \mathcal{D}(B)) \end{aligned}$$

and (u, v) solves system (101) and so, also system (98). The two first equations in (98) can be written in the form

$$\begin{cases} D_t(\theta' + \chi' + (h * \theta)')(t) = D_t[A\theta + k * A\theta + fg] \\ D_t[\chi'](t) = D_t[B\chi + F \circ \chi - A\theta](t) \end{cases} \tag{114}$$

So the definitions of u_0 and v_0 (in H9 and (25)) imply that (θ, χ) solves the system (23). It remains to show that even the conditions (24) are fulfilled. It is easy to see that (108) and (107) follow from (111). Using the second equation in (98), from (107)

we obtain

$$\begin{aligned}
 b_j''(t) + \Phi_j[v'(t)] &= \Phi_j[(A - h_0I)u(t) + k(t)A\theta_0 + k * Au(t) \\
 &\quad + z(t)g - w(t)\theta_0 - w * u(t)], \quad t \in [0, \tau], \quad j = 1, 2, 3
 \end{aligned}
 \tag{115}$$

Applying Φ_j to the first equation in (98), we obtain

$$\begin{aligned}
 \Phi_j[u'(t)] + \Phi_j[v'(t)] &= \Phi_j[(A - h_0I)u(t) + k(t)A\theta_0 + k * Au(t) \\
 &\quad + z(t)g - w(t)\theta_0 - w * u(t)], \quad t \in [0, \tau], \quad j = 1, 2, 3
 \end{aligned}
 \tag{116}$$

which, together with (115), gives

$$\Phi_j[u'(t)] = b_j''(t), \quad t \in [0, \tau], \quad j = 1, 2, 3
 \tag{117}$$

Finally, combining (117) with H13 we get (24). □

6. PROOFS OF THE MAIN RESULTS

We are now in position to give the proofs of the main abstract theorems using the preliminary results of Section 4 and the reformulation of the identification problem of Section 5.

6.1. Proof of Theorem 3.1

We consider, for $\tau \in \mathbb{R}^+$ and $\beta \in (0, 1)$, the linear space

$$X(\tau) := C^\beta([0, \tau]; \mathcal{D}(A)) \times C^\beta([0, \tau]; \mathcal{D}(B)) \times [C^\beta([0, \tau])]^3
 \tag{118}$$

equipped with the norm

$$\|(u, v, w, k, z)\|_{X(\tau)} := \|u\|_{\beta, \tau, \mathcal{D}(A)} + \|v\|_{\beta, \tau, \mathcal{D}(B)} + \|w\|_{\beta, \tau, \mathbb{R}} + \|k\|_{\beta, \tau, \mathbb{R}} + \|z\|_{\beta, \tau, \mathbb{R}}
 \tag{119}$$

It is easily seen that $X(\tau)$ is a Banach space with the norm $\|\cdot\|_{X(\tau)}$. Let $\rho \in \mathbb{R}^+$ and set

$$\begin{aligned}
 Z(\tau, \rho) &:= \{(u, v, w, k, z) \in X(\tau) : u(0) = u_0, \quad v(0) = v_0, \quad w(0) = w_0, \quad k(0) = k_0 \\
 &\quad z(0) = z_0, \quad \|(u - \mathcal{U}_0, v - \mathcal{V}_0, w - \tilde{\Gamma}_{01}, k - \tilde{\Gamma}_{02}, z - \tilde{\Gamma}_{03})\|_{X(\tau)} \leq \rho\}
 \end{aligned}
 \tag{120}$$

then we observe that $Z(\tau, \rho)$ is a closed subset of $X(\tau)$. We define the operator

$$\begin{aligned}
 \mathbf{N}(u, v, w, k, z) &:= (\mathcal{U}_0 + \mathcal{N}_1(u, v, w, k, z), \mathcal{V}_0 + \mathcal{N}_2(u, v, w, k, z), \tilde{\Gamma}_{01} + \tilde{\Gamma}_1(u, v, w, k, z), \tilde{\Gamma}_{02} \\
 &\quad + \tilde{\Gamma}_2(u, v, w, k, z), \tilde{\Gamma}_{03} + \tilde{\Gamma}_3(u, v, w, k, z))
 \end{aligned}
 \tag{121}$$

Owing to Corollaries 4.1 and 4.2 and Lemma 4.5, \mathbf{N} maps $Z(\tau, \rho)$ into

$$\{(u, v, w, k, z) \in X(\tau) : u(0) = u_0, \quad v(0) = v_0, \quad w(0) = w_0, \quad k(0) = k_0, \quad z(0) = z_0\}$$

We shall show that for every $\rho \in \mathbb{R}^+$ there exists $\tau(\rho) \in (0, T]$, such that, if $\tau \in (0, \tau(\rho))$, \mathbf{N} has a unique fixed point in $Z(\tau, \rho)$.

In the following we shall use the letter ω to indicate real valued functions converging to 0 for $\tau \rightarrow 0$.

Let $(u, v, w, k, z) \in Z(\tau, \rho)$, then, owing to Lemma 4.1, H13, Corollary 4.2(I), Lemma 4.5, we have

$$\|\mathcal{N}_1(u, v, w, k, z)\|_{\beta, \tau, \mathcal{D}(A)} + \|\mathcal{N}_2(u, v, w, k, z)\|_{\beta, \tau, \mathcal{D}(B)} \leq C(R, T)\omega_1(\tau)$$

where

$$R := \|(\mathcal{U}_0, \mathcal{V}_0, \tilde{\Gamma}_0, \tilde{\Gamma}_2, \tilde{\Gamma}_3)\|_{X(\tau)} + \rho \tag{122}$$

which implies

$$\sum_{j=1}^3 \|\tilde{\Gamma}_{0j}\|_{\beta, \tau} \leq C(R, T)\omega_2(\tau)$$

So we have

$$\|\mathbf{N}(u, v, w, k, z) - (\mathcal{U}_0, \mathcal{V}_0, \tilde{\Gamma}_0, \tilde{\Gamma}_2, \tilde{\Gamma}_3)\|_{X(\tau)} \leq C(R, T)\omega_3(\tau) \tag{123}$$

Analogously, we can show that, if $(u_1, v_1, w_1, k_1, z_1)$ and $(u_2, v_2, w_2, k_2, z_2)$ are arbitrary elements of $Z(\tau, \rho)$, we have

$$\begin{aligned} & \|\mathbf{N}(u_1, v_1, w_1, k_1, z_1) - \mathbf{N}(u_2, v_2, w_2, k_2, z_2)\|_{X(\tau)} \\ & \leq C(R, T)\omega_4(\tau)\|(u_1 - u_2, v_1 - v_2, w_1 - w_2, k_1 - k_2, z_1 - z_2)\|_{X(\tau)} \end{aligned} \tag{124}$$

So, if we choose τ such that

$$C(R, T)\omega_3(\tau) \leq \rho, \quad C(R, T)\omega_4(\tau) < 1 \tag{125}$$

the existence of a unique fixed point for \mathbf{N} in $Z(\tau, \rho)$ follows from the contraction mapping theorem.

In force of Proposition 5.1, this proves the existence, local in time, of solutions to (23)–(24) with the regularity (31)–(35). \square

6.2. Proof of Theorem 3.2

Now we show that, if $(\theta_1, \chi_1, h_1, k_1, f_1)$ and $(\theta_2, \chi_2, h_2, k_2, f_2)$ are solutions of (23)–(24) both satisfying the regularity conditions (36)–(40), for some $T_1 \in (0, T]$, then they coincide. We set, for $i \in \{1, 2\}$,

$$u_i := D_t \theta_i \tag{126}$$

$$v_i := D_t \chi_i \tag{127}$$

$$w_i := D_t h_i \tag{128}$$

$$z_i := D_t f_i \tag{129}$$

As, for a fixed $\rho \in \mathbb{R}^+$, \mathbf{N} has a unique fixed point in $Z(\sigma, \rho)$ if σ is sufficiently small (cf. proof of Theorem 3.1), necessarily $(u_1, v_1, w_1, k_1, z_1)$ and $(u_2, v_2, w_2, k_2, z_2)$ coincide in some right neighbourhood of 0. So we set

$$\tau_1 := \inf\{t \in [0, T_1] : \|(u_1 - u_2, v_1 - v_2, w_1 - w_2, k_1 - k_2, z_1 - z_2)\|_{X(t)} > 0\} \quad (130)$$

Of course, we assume that the set in the right-hand side of (130) is not empty, so that $\tau_1 \in (0, T_1)$. We observe that

$$\|(u_1 - u_2, v_1 - v_2, w_1 - w_2, k_1 - k_2, z_1 - z_2)\|_{X(\tau_1)} = 0 \quad (131)$$

which also implies

$$u_1(\tau_1) = u_2(\tau_1), \quad v_1(\tau_1) = v_2(\tau_1), \quad w_1(\tau_1) = w_2(\tau_1), \quad k_1(\tau_1) = k_2(\tau_1), \quad z_1(\tau_1) = z_2(\tau_1) \quad (132)$$

For $t \in [0, T_1 - \tau_1]$, we define

$$U(t) := u_1(\tau_1 + t) - u_2(\tau_1 + t) \quad (133)$$

$$V(t) := v_1(\tau_1 + t) - v_2(\tau_1 + t) \quad (134)$$

$$W(t) := w_1(\tau_1 + t) - w_2(\tau_1 + t) \quad (135)$$

$$K(t) := k_1(\tau_1 + t) - k_2(\tau_1 + t) \quad (136)$$

$$Z(t) := z_1(\tau_1 + t) - z_2(\tau_1 + t) \quad (137)$$

If $t \in [0, T_1 - \tau_1]$ we have

$$\begin{aligned} (w_1 * u_1)(\tau_1 + t) - (w_2 * u_2)(\tau_1 + t) &= [(w_1 - w_2) * u_1](\tau_1 + t) + [w_2 * (u_1 - u_2)](\tau_1 + t) \\ &= \int_0^t [w_1(\tau_1 + t - s) - w_2(\tau_1 + t - s)] u_1(s) \, ds \\ &\quad + \int_0^t w_2(t - s) [u_1(\tau_1 + s) - u_2(\tau_1 + s)] \, ds \\ &= (W * u_1)(t) + (w_2 * U)(t) \end{aligned} \quad (138)$$

Moreover, for $i \in \{1, 2\}$, $t \in [0, T_1 - \tau_1]$, we consider

$$\begin{aligned} \chi_0 + (1 * v_i)(\tau_1 + t) &= \chi_0 + \int_0^{\tau_1} v_i(s) \, ds + \int_0^t v_i(s + \tau_1) \, ds \\ &= \chi_1 + (1 * v_i(\cdot + \tau_1)) \end{aligned} \quad (139)$$

with

$$\chi_1 := \chi_0 + \int_0^{\tau_1} v_1(s) \, ds = \chi_0 + \int_0^{\tau_1} v_2(s) \, ds \quad (140)$$

We observe that $\chi_1 \in \mathcal{D}(B)$. Consequently, we have, for $t \in [0, T_1 - \tau_1]$

$$\left\{ \begin{array}{l} U'(t) = (2A - h_0)U(t) - [B + F'(\chi_1)]V(t) \\ \quad - \{F'[\chi_1 + (1 * v_1(\cdot + \tau_1))(t)] - F'(\chi_1)\}v_1(t + \tau_1) \\ \quad + \{F'[\chi_1 + (1 * v_2(\cdot + \tau_1))(t)] - F'(\chi_1)\}v_2(t + \tau_1) \\ \quad - W(t)\theta_0 + K(t)A\theta_0 + K * Au_1(t) + k_2 * AU(t) + Z(t)g \\ \quad - W * u_1(t) - w_2 * U(t) \\ V'(t) = [B + F'(\chi_1)]V(t) - AU(t) + \{F'[\chi_1 + (1 * v_1(\cdot + \tau_1))(t)] \\ \quad - F'(\chi_1)\}v_1(t + \tau_1) - \{F'[\chi_1 + (1 * v_2(\cdot + \tau_1))(t)] \\ \quad - F'(\chi_1)\}v_2(t + \tau_1) \\ U(0) = 0 \\ V(0) = 0 \end{array} \right. \tag{141}$$

Let $\sigma \in (0, T_1 - \tau_1]$. Then, applying Corollary 4.1 (with χ_1 replacing χ_0), Corollary 4.2, Lemma 4.1, Lemma 4.5, from (141) we get

$$\|U\|_{\beta, \sigma, \mathcal{D}(A)} + \|V\|_{\beta, \sigma, \mathcal{D}(B)} \leq C(R', T_1 - \tau_1)\omega_5(\sigma)\|(U, V, W, K, Z)\|_{X(\sigma)} \tag{142}$$

with

$$R' := \|v_1\|_{\beta, T_1, \mathcal{D}(B)} + \|v_2\|_{\beta, T_1, \mathcal{D}(B)} + \|u_1\|_{0, T_1, \mathcal{D}(A)} + \|k_2\|_{0, T_1, \mathbb{R}} + \|w_2\|_{0, T_1, \mathbb{R}} \tag{143}$$

Inequality (142) implies that, for some $\sigma_0 \in (0, T_1 - \tau_1]$, if $0 < \sigma < \sigma_0$

$$\|U\|_{\beta, \sigma, \mathcal{D}(A)} + \|V\|_{\beta, \sigma, \mathcal{D}(B)} \leq C(R', T_1 - \tau_1)\omega_5(\sigma)(\|W\|_{\beta, \sigma, \mathbb{R}} + \|K\|_{\beta, \sigma, \mathbb{R}} + \|Z\|_{\beta, \sigma, \mathbb{R}}) \tag{144}$$

If $t \in [0, T_1 - \tau_1]$, we also have

$$\begin{aligned} & W(t)\Phi_j[\theta_0] - K(t)\Phi_j[A\theta_0] - Z(t)\Phi_j[g] \\ &= \Phi_j[(2A - h_0)U(t)] - \Phi_j[B + F'(\chi_1)]V(t) \\ & \quad - \Phi_j\{\{F'[\chi_1 + (1 * v_1(\cdot + \tau_1))(t)] - F'(\chi_1)\}v_1(t + \tau_1)\} \\ & \quad + \Phi_j\{\{F'[\chi_1 + (1 * v_2(\cdot + \tau_1))(t)] \\ & \quad - F'(\chi_1)\}v_2(t + \tau_1)\} + \Phi_j[K * Au_1](t) \\ & \quad + \Phi_j[k_2 * AU](t) - \Phi_j[W * u_1(t) + w_2 * U(t)] \end{aligned} \tag{145}$$

which implies

$$\begin{aligned} \|W\|_{\beta,\sigma,\mathbb{R}} + \|K\|_{\beta,\sigma,\mathbb{R}} + \|Z\|_{\beta,\sigma,\mathbb{R}} &\leq C(R', T_1 - \tau_1)[\|U\|_{\beta,\sigma,\mathcal{D}(A)} + \|V\|_{\beta,\sigma,\mathcal{D}(B)} \\ &\quad + \omega_4(\sigma)(\|W\|_{\beta,\sigma,\mathbb{R}} + \|K\|_{\beta,\sigma,\mathbb{R}})] \end{aligned} \quad (146)$$

On account of (144), we obtain

$$\begin{aligned} \|W\|_{\beta,\sigma,\mathbb{R}} + \|K\|_{\beta,\sigma,\mathbb{R}} + \|Z\|_{\beta,\sigma,\mathbb{R}} &\leq C(R', T_1 - \tau_1)\omega_5(\sigma)(\|W\|_{\beta,\sigma,\mathbb{R}} \\ &\quad + \|K\|_{\beta,\sigma,\mathbb{R}} + \|Z\|_{\beta,\sigma,\mathbb{R}}) \end{aligned} \quad (147)$$

implying that W , K and Z vanish in some right neighbourhood of 0. So, using again (144), we conclude that even U and V vanish in some right neighbourhood of 0, in contradiction with the definition of τ_1 . \square

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