Intrinsic Harnack estimates for non-negative local solutions of degenerate parabolic equations

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Abstract

We establish the intrinsic Harnack inequality for non negative solutions of the parabolic p-laplacian equation by a proof that uses neither the comparison principle nor explicit self-similar solutions. The significance is that the proof applies to quasilinear p-laplacian type equations thereby solving a long standing problem in the theory of degenerate parabolic equations.

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1 Main Results

Let E be an open set in $\mathbb{R}^N$ and for $T>0$ let E_T denote the cylindrical domain $E\times(0,T]$. Consider quasi–linear, parabolic differential equations of the form

$$u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = b(x,t,u,Du) \quad \text{weakly in } E_T \tag{1.1}$$

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where the functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $b : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \ge C_o|Du|^p - C^p \\ |\mathbf{A}(x,t,u,Du)| \le C_1|Du|^{p-1} + C^{p-1} \\ |b(x,t,u,Du)| \le C|Du|^{p-1} + C^{p-1} \end{cases} \qquad \text{a.e. in } E_T \tag{1.2}$$

where $p \ge 2$ and C_o and C_1 are given positive constants, and C is a given non–negative constant. A function

$$u \in C_{\rm loc}\big(0,T; L^2_{\rm loc}(E)\big) \cap L^p_{\rm loc}\big(0,T; W^{1,p}_{\rm loc}(E)\big) \tag{1.3}$$

is a local weak solution to (1.1) if for every compact set $K \subset E$ and every sub–interval $[t_1,t_2] \subset (0,T]$

$$\begin{aligned} \int_K u\varphi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \big[-u\varphi_t + \mathbf{A}(x,t,u,Du)\cdot D\varphi\big]dxdt \\ = \int_{t_1}^{t_2}\int_K b(x,t,u,Du)\varphi\, dxdt \end{aligned} \tag{1.4}$$

for all bounded testing functions

$$\varphi \in W^{1,2}_{\rm loc}\big(0,T; L^2(K)\big) \cap L^p_{\rm loc}\big(0,T; W^{1,p}_o(K)\big). \tag{1.5}$$

The parameters $\{N, p, C_o, C_1, C\}$ are the data, and we say that a generic constant $\gamma = \gamma(N,p,C_o,C_1,C)$ depends upon the data, if it can be quantitatively determined a–priori only in terms of the indicated quantities.

For $\rho > 0$ let B_ρ be the ball of center the origin on $\mathbb{R}^N$ and radius ρ and for $y \in \mathbb{R}^N$ let $B_\rho(y)$ denote the homotetic ball centered at y. For $\theta > 0$ set also

$$Q_\rho(\theta) = B_\rho \times (-\theta, \theta]$$

and for $(y,s) \in \mathbb{R}^N \times \mathbb{R}$

$$(y,s) + Q_\rho(\theta) = B_\rho(y) \times (s-\theta, s+\theta].$$

Local weak solutions to (1.1)–(1.5) are locally bounded and locally Hölder continuous in E_T ([3]). This fact was used to prove the Harnack inequality, unfortunately only in some special instances. We can now show that the Harnack estimate actually holds in full generality and independently from the Hölder continuity.

Theorem 1.1 (Intrinsic Harnack Inequality) *Let u be a non–negative weak solution to (1.1)–(1.5). There exist positive constants c and γ depending only upon the data, such that for almost all $(x_o,t_o) \in E_T$ and all cylinders $(x_o,t_o)+Q_{2\rho}(4\theta) \subset E_T$*

$$u(x_o,t_o) \le \gamma\left[\inf_{B_\rho(x_o)} u(x,t_o+\theta) + C\rho\right], \quad \theta = \left(\frac{c}{u(x_o,t_o)}\right)^{p-2}\rho^p \tag{1.6}$$

where C is the same as in (1.2). As a consequence any locally bounded weak solution to (1.1)–(1.2) is locally Hölder continuous in E_T, and thus (1.6) permits an independent proof of the Hölder continuity of solutions established in [3].

In (1.6) the time θ is intrinsic to the solution u and to the geometry of the ball $B_\rho(x_o)$. It would be desirable to have an estimate where the space–time geometry can be prescribed a–priori, independent of $u(x_o, t_o)$: this will be object of a future research.

2 Novelty and Significance

Equation (1.1) with the structure conditions (1.2) is a quasi–linear version of the degenerate, homogeneous equation

$$u_t - \sum_{i,j=1}^{N} D_{x_j}(|Du|^{p-2} a_{ij}(x,t) D_{x_i} u) = 0 \quad \text{weakly in } E_T \tag{2.1}$$

where the coefficients a_{ij} are measurable and locally bounded in E_T and the matrix (a_{ij}) is almost everywhere positive definite in E_T. If $(a_{ij}) = \mathbb{I}$, then (2.1) reduces to the degenerate, prototype parabolic p–Laplace equation

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{weakly in } E_T. \tag{2.2}$$

Both (2.1) and (2.2) satisfy the structure conditions (1.2) with $C = 0$. Accordingly, non–negative weak solutions of these equations satisfy the intrinsic Harnack inequality (1.6) with $C = 0$.

2.1 The Linear Case $p = 2$

The Harnack inequality for local, non–negative solutions of the heat equation $\big((1.6)$, with $p = 2$ and $C = 0\big)$, was established independently by Hadamard ([6]) and Pini ([8]), by local representation of solutions in terms of heat potentials. In [9], Moser established the same Harnack inequality for weak solutions of (2.1) for $p = 2$, by energy based, measure–theoretical arguments. Moser's proof is non–linear in nature, and it can be extended almost verbatim ([10, 1]), to the quasi–linear versions (1.1)–(1.2) with $p = 2$. At almost the same time, Ladyzhenskaja, Solonnikov and Ural'tzeva ([7]), established, by means of DeGiorgi–type measure–theoretical arguments, that weak solutions of such quasi–linear equations (still for $p = 2$), are locally bounded and locally Hölder continuous. It turns out that the Harnack inequality of Moser can be used to establish the Hölder continuity of solutions. On the other hand, it was observed in [2] that the Hölder continuity implies the Harnack inequality for non–negative solutions.

Thus a summary of the quasi–linear theory for the "linear" case $p = 2$, is that Hölder continuity and Harnack inequality for non–negative solutions, are mutually equivalent. However, establishing either of them independently, requires independent measure–theoretical arguments.

2.2 The Degenerate Case $p > 2$

Neither Moser's nor the DeGiorgi's ideas, in the version of [7], seem to apply when $p \neq 2$, even for the prototype case (2.2). Some progress was made by the idea of *time–intrinsic* geometry, by which the time is scaled, roughly speaking by u^{p-2}. This permits to establish that weak solutions of (1.1)–(1.2), for all $p > 1$, are Hölder continuous in E_T ([3], Chapters III and IV). It was also observed that while the Harnack inequality in the Moser form is in general false for $p > 2$, it might hold in this time–intrinsic geometry. Indeed it was shown that (1.6) with $C = 0$, holds for non–negative solutions of (2.2). The proof is based on the maximum principle and comparison functions constructed as variants of the Barenblatt similarity solutions (see [3], Chapter VI, for an account of the theory)

$$\Gamma_p(x,t) = \frac{1}{t^{N/\lambda}} \left[1 - \gamma_p \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}} \qquad t > 0 \tag{2.3}$$

where

$$\gamma_p = \left(\frac{1}{\lambda} \right)^{\frac{1}{p-1}} \frac{p-2}{p}, \qquad \lambda = N(p-2) + p. \tag{2.4}$$

As $p \to 2$ this tends pointwise to the fundamental solution of the heat equation. In this sense Γ_p is some sort of p–heat potential. Thus the approach can be regarded as paralleling that of Hadamard and Pini for the heat equation.

The novelty of Theorem 1.1 lies in producing a proof of the Harnack inequality (1.6) based only on measure–theoretical arguments. This bypasses any notion of maximum principle and potentials, and permits an extension to non–negative solutions of quasi–linear equations of the type of (1.1)–(1.2). Its significance is in paralleling Moser's measure–theoretical, quasi–linear development, following Hadamard and Pini's potential representations for the heat equation. Moreover our approach gets rid of any kind of covering argument and Cross–over Lemma, which were used in Moser's proof but can be considered as rather artificial.

In [5] we will give a detailed proof of Theorem 1.1, built on measure–theoretical facts established in [4]. Particular care will be used in showing how the Harnack inequality implies the Hölder continuity of the solution.

Thus a summary of the quasi–linear theory, for the "degenerate" case $p > 2$, is that Hölder continuity and intrinsic Harnack inequality are mutually equivalent. However, establishing either of them independently, requires independent measure–theoretical arguments. Finally when $p \to 2$ the intrinsic Harnack inequality (1.6) and the corresponding Hölder theory, recover the classical Moser's estimate and the corresponding Hölder estimates of [7].

2.3 Expansion of Positivity

The main technical novelty is illustrated by referring back to the "linear" case $p = 2$.

Let u be a non–negative, local solution of the heat equation in E_T. Let $B_\rho(y) \times (s-\rho^2, s] \subset E_T$ and assume that

$$\text{meas}\{x \in B_\rho(y) \mid u(x, s-\rho^2) < M\} < \alpha \, \text{meas}\{B_\rho\}$$

for some $M > 0$ and some $\alpha \in (0,1)$. Then there exists $\eta = \eta(\alpha) \in (0,1)$, such that for all $x \in B_{2\rho}(y)$

$$u(x, s+4\rho^2) \geq \eta M.$$

Thus information on the measure of the "positivity set" of u at the time level $s-\rho^2$, over the ball $B_\rho(y)$, translates into an expansion of the positivity set both in space $\big($from $B_\rho(y)$ to $B_{2\rho}(y)\big)$, and in time (from $s-\rho^2$ to $s+4\rho^2$). This fact continues to hold for quasi–linear versions of the heat equation and was established in [2].

A similar fact for $p > 2$ is in general false as one can verify from the Barenblatt solution (2.3)–(2.4). The main technical novelty of our investigation is that a similar fact continues to hold for the degenerate equations (1.1)–(1.2), in a time–intrinsic geometry. Precisely

Lemma 2.1 *Let u be a non–negative, local, weak solution of (1.1)–(1.2). There exist positive constants γ and b, and $\eta \in (0,1)$, depending only upon the data and independent of (y,s), ρ and M, such that if*

$$u(x,s) \geq M \quad \text{for all } x \in B_\rho(y) \tag{2.5}$$

then either $M < \gamma \, C \, \rho$, or for a.e. $x \in B_{2\rho}(y)$

$$u(x,t) \geq \eta M \quad \text{with } t = s + \left(\frac{b}{\eta M}\right)^{p-2} (4\rho)^p. \tag{2.6}$$

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