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Bertrand Toën
Gabriele Vezzosi



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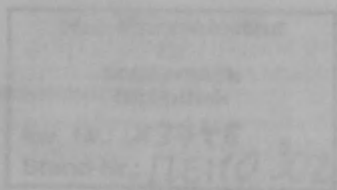
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Abstract

¹ This is the second part of a series of papers called “HAG”, and devoted to develop the foundations of *homotopical algebraic geometry*. We start by defining and studying generalizations of standard notions of linear algebra in an abstract monoidal model category, such as derivations, étale and smooth morphisms, flat and projective modules, etc. We then use our theory of stacks over model categories, introduced in [HAGI], in order to define a general notion of geometric stack over a base symmetric monoidal model category C , and prove that this notion satisfies the expected properties.

The rest of the paper consists in specializing C in order to give various examples of applications in several different contexts. First of all, when $C = k\text{-Mod}$ is the category of k -modules with the trivial model structure, we show that our notion gives back the algebraic n -stacks of C. Simpson. Then we set $C = sk\text{-Mod}$, the model category of simplicial k -modules, and obtain this way a notion of geometric D^- -stack which are the main geometric objects of *derived algebraic geometry*. We give several examples of derived version of classical moduli stacks, as the D^- -stack of local systems on a space, the D^- -stack of algebra structures over an operad, the D^- -stack of flat bundles on a projective complex manifold, etc. We also present the cases where $C = C(k)$ is the model category of unbounded complexes of k -modules, and $C = Sp^\Sigma$ the model category of symmetric spectra. In these two contexts we give some examples of geometric stacks such as the stack of associative dg-algebras, the stack of dg-categories, and a geometric stack constructed using topological modular forms.

There are more things in heaven and earth, Horatio,
than are dreamt of in our philosophy. But come...

W. Shakespeare, *Hamlet*, Act 1, Sc. 5.

Mon cher Cato, il faut en convenir, les forces de l'éther nous pénètrent,
et ce fait délirant il nous faut l'appréhender coûte que coûte.

P. Sellers, *Quand la panthère rose s'en mêle*.

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1.1.3 This is the second part of a series of papers called "Geometric interpretation of modules". It develops the foundations of homotopy algebraic geometry. It starts by defining and studying generalizations of standard notions of linear algebra in an abstract noncommutative model category, such as derivations, free and smooth noncommutative, and projective modules, etc. We then use our theory of stacks over model categories introduced in [HAG1] in order to define a general notion of geometric stack over a base symmetric monoidal model category C , and prove that this notion satisfies the expected properties.

1.1.4 The rest of the paper consists in specializing C in order to give various examples of applications in several different contexts. First of all, when $C = \mathbf{Set}$, this is the category of k -modules with the trivial model structure, two obvious subcategories give back the algebraic stacks of C (Simpson). Then we recall, using that the model category of simplicial k -modules, and obtain this way, a noncommutative model category of simplicial k -modules, objects of which are algebraic stacks. The k -stack which are the main geometric objects of derived algebraic geometry, give several examples of derived version of classical model stacks, as the D -stack of local systems on a space, the D -stack of algebraic structures over k (space), the D -stack of flat bundles on a projective complex manifold, etc. We also present the case where $C = C(k)$ is the model category of unbounded complexes of k -modules, and $C = \mathcal{S}p^2$ the model category of simplicial spectra. In these two contexts we give some examples of geometric stacks such as the stack of associative algebras, the stack of dg-categories, and a geometric stack constructed using topological methods.

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Introduction

This is the second part of a series of papers called "HAG", devoted to start the development of *homotopical algebraic geometry*. The first part [HAGI] was concerned with the homotopical generalization of sheaf theory, and contains the notions of model topologies, model sites, stacks over model sites and model topoi, all of these being homotopical versions of the usual notions of Grothendieck topologies, sites, sheaves and topoi. The purpose of the present work is to use these new concepts in some specific situations, in order to introduce a very general notion of *geometric stacks*, a far reaching homotopical generalization of the notion of algebraic stacks introduced by P. Deligne, D. Mumford and M. Artin. This paper includes the general study and the standard properties of geometric stacks, as well as various examples of applications in the contexts of algebraic geometry and algebraic topology.

Reminders on abstract algebraic geometry. A modern point of view on algebraic geometry consists of viewing algebraic varieties and schemes through their functors of points. In this functorial point of view, schemes are certain sheaves of sets on the category of commutative rings endowed with a suitable topology (e.g. the Zariski topology). Keeping this in mind, it turns out that the whole theory of schemes can be completely reconstructed starting from the symmetric monoidal category $\mathbf{Z} - \mathbf{Mod}$ of \mathbf{Z} -modules alone. Indeed, the category of commutative rings is reconstructed by taking the category of commutative monoids in $\mathbf{Z} - \mathbf{Mod}$. Flat morphisms can be recognized via the exactness property of the base change functor on the category of modules. Finitely presented morphisms are recognized via the usual categorical characterization in terms of commutation of mapping sets with respect to filtered colimits. Finally, Zariski open immersion can be defined as flat morphisms of finite presentation $A \rightarrow B$ such that $B \simeq B \otimes_A B$. Schemes are then reconstructed as certain Zariski sheaves on the opposite category of commutative rings, which are obtained by gluing affine schemes via Zariski open immersions (see for example the first chapters of [Dem-Gab]).

The fact that the notion of schemes has such a purely categorical interpretation has naturally lead to the theory of *relative algebraic geometry*, in which the base symmetric monoidal category $\mathbf{Z} - \mathbf{Mod}$ is replaced by an abstract base symmetric monoidal category \mathcal{C} , and under reasonable assumptions on \mathcal{C} the notion of *schemes over \mathcal{C}* can be made meaningful as well as useful (see for example [Dell, Ha] for some applications).

The key observation of this work is that one can generalize further the theory of relative algebraic geometry by requiring \mathcal{C} to be endowed with an additional *model category* structure, compatible with its monoidal structure (relative algebraic geometry is then recovered by taking the trivial model structure), in such a way that the notions of *schemes* and more generally of *algebraic spaces* or *algebraic stacks* still have a natural and useful meaning, compatible with the homotopy theory carried

by \mathcal{C} . In this work, we present this general theory, and show how this enlarges the field of applicability by investigating several examples not covered by the standard theory of relative algebraic geometry. The most important of these applications is the existence of foundations for *derived algebraic geometry*, a global counter part of the derived deformation theory of V. Drinfel'd, M. Kontsevich and al.

The setting. Our basic datum is a symmetric monoidal model category \mathcal{C} (in the sense of [Ho1]), on which certain conditions are imposed (see assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4). We briefly discuss these requirements here. The model category \mathcal{C} is assumed to satisfy some reasonable additional properties (as for example being proper, or that cofibrant objects are flat for the monoidal structure). These assumptions are only made for the convenience of certain constructions, and may clearly be omitted. The model category \mathcal{C} is also assumed to be *combinatorial* (see e.g. [Du2]), making it reasonably behaved with respect to localization techniques. The first really important assumption on \mathcal{C} states that it is pointed (i.e. that the final and initial object coincide) and that its homotopy category $\mathrm{Ho}(\mathcal{C})$ is additive. This makes the model category \mathcal{C} *homotopically additive*, which is a rather strong condition, but is used all along this work and seems difficult to avoid (see however [To-Va2]). Finally, the last condition we make on \mathcal{C} is also rather strong, and states that the theory of commutative monoids in \mathcal{C} , and the theory of modules over them, both possess reasonable model category structures. This last condition is of course far from being satisfied in general (as for example it is not satisfied when \mathcal{C} is the model category of complexes over some ring which is not of characteristic zero), but all the examples we have in mind can be treated in this setting². The model categories of simplicial modules, of complexes over a ring of characteristic zero, and of symmetric spectra are three important examples of symmetric monoidal model category satisfying all our assumptions. More generally, the model categories of sheaves with values in any of these three fundamental categories provide additional examples.

Linear and commutative algebra in a symmetric monoidal model category. An important consequence of our assumptions on the base symmetric monoidal model category \mathcal{C} is the existence of reasonable generalizations of general constructions and results from standard linear and commutative algebra. We have gathered some of these notions (we do not claim to be exhaustive) in §1.1. For example, we give definitions of derivations as well as of cotangent complexes representing them, we define formally étale morphisms, flat morphisms, open Zariski immersions, formally unramified morphisms, finitely presented morphism of commutative monoids and modules, projective and flat modules, Hochschild cohomology, etc. They are all generalizations of the well known notions in the sense that when applied to the case where $\mathcal{C} = \mathbf{Z} - \mathbf{Mod}$ with the trivial model structure we find back the usual notions. However, there are sometimes several nonequivalent generalizations, as for example there exist at least two, nonequivalent reasonable generalizations of smooth morphisms which restrict to the usual one when $\mathcal{C} = \mathbf{Z} - \mathbf{Mod}$. This is why we have tried to give an overview of several possible generalizations, as we think all definitions could have their own interest depending both on the context and on what one wants to do with them. Also we wish to mention that all these notions depend heavily on the base model category \mathcal{C} , in the sense that the same object in \mathcal{C} , when considered

²Alternatively, one could switch to E_∞ -algebras (and modules over them) for which useful model and semi-model structures are known to exist, thanks to the work of M. Spitzweck [Sp], in much more general situations than for the case of commutative monoids (and modules over them).

in different model categories structures on \mathcal{C} , might not behave the same way. For example, a commutative ring can also be considered as a simplicial commutative ring, and the notion of finitely presented morphisms is not the same in the two cases. We think that keeping track of the base model category \mathcal{C} is rather important, since playing with the change of base categories might be very useful, and is also an interesting feature of the theory.

The reader will immediately notice that several notions behave in a much better way when the base model category satisfies certain stability assumptions (e.g. is a stable model category, or when the suspension functor is fully faithful, see for example Prop. 1.2.6.5, Cor. 1.2.6.6). We think this is one of the main features of homotopical algebraic geometry: linear and commutative algebra notions tend to be better behaved as the base model category tend to be “more” stable. We do not claim that everything becomes simpler in the stable situation, but that certain difficulties encountered can be highly simplified by enlarging the base model category to a more stable one.

Geometric stacks. In §1.3 we present the general notions of geometric stacks relative to our base model category \mathcal{C} . Of course, we start by defining $\text{Aff}_{\mathcal{C}}$, the model category of affine objects over \mathcal{C} , as the opposite of the model category $\text{Comm}(\mathcal{C})$ of commutative monoids in \mathcal{C} . We assume we are given a model (pre-)topology τ on $\text{Aff}_{\mathcal{C}}$, in the sense we have given to this expression in [HAGI, Def. 4.3.1] (see also Def. 1.3.1.1). We also assume that this model topology satisfies certain natural assumptions, as quasi-compactness and the descent property for modules. The model category $\text{Aff}_{\mathcal{C}}$ together with its model topology τ is a model site in the sense of [HAGI, Def. 4.3.1] or Def. 1.3.1.1, and it gives rise to a model category of stacks $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$. The homotopy category of $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ will simply be denoted by $\text{St}(\mathcal{C}, \tau)$. The Yoneda embedding for model categories allows us to embed the homotopy category $\text{Ho}(\text{Aff}_{\mathcal{C}})$ into $\text{St}(\mathcal{C}, \tau)$, and this gives a notion of representable stack, our analog of the notion of affine scheme. Geometric stacks will result from a certain kind of gluing representable stacks.

Our notion of geometric stack is relative to a class of morphisms \mathbf{P} in $\text{Aff}_{\mathcal{C}}$, satisfying some compatibility conditions with respect to the topology τ , essentially stating that the notion of morphisms in \mathbf{P} is local for the topology τ . With these two notions, τ and \mathbf{P} , we define by induction on n a notion of n -geometric stack (see 1.3.3.1). The precise definition is unfortunately too long to be reproduced here, but one can roughly say that n -geometric stacks are stacks F whose diagonal is $(n-1)$ -representable (i.e. its fibers over representable stacks are $(n-1)$ -geometric stacks), and which admits a covering by representable stacks $\coprod_i U_i \rightarrow F$, such that all morphisms $U_i \rightarrow F$ are in \mathbf{P} .

The notion of n -geometric stack satisfies all the expected basic properties. For example, geometric stacks are stable by (homotopy) fiber products and disjoint unions, and being an n -geometric stack is a local property (see Prop. 1.3.3.3, 1.3.3.4). We also present a way to produce n -geometric stacks as certain quotients of groupoid actions, in the same way that algebraic stacks (in groupoids) can always be presented as quotients of a scheme by a smooth groupoid action (see Prop. 1.3.4.2). When a property \mathbf{Q} of morphisms in $\text{Aff}_{\mathcal{C}}$ satisfies a certain compatibility with both \mathbf{P} and τ , there exists a natural notion of \mathbf{Q} -morphism between n -geometric stacks, satisfying all the expected properties (see Def. 1.3.6.4, Prop. 1.3.6.3). We define the stack of quasi-coherent modules, as well its sub-stacks of vector bundles and of perfect modules (see Thm. 1.3.7.2, Cor. 1.3.7.3). These also behave as expected, and for example

the stack of vector bundles is shown to be 1-geometric as soon as the class \mathbf{P} contains the class of smooth morphisms (see Cor. 1.3.7.12).

Infinitesimal theory. In §1.4, we investigate the infinitesimal properties of geometric stacks. For this we define a notion of derivation of a stack F with coefficients in a module, and the notion of cotangent complex is defined via the representability of the functor of derivations (see Def. 1.4.1.4, 1.4.1.5, 1.4.1.7). The object representing the derivations, the cotangent complex, is not in general an object in the base model category \mathcal{C} , but belongs to the stabilization of \mathcal{C} (this is of course related to the well known fact that cotangent spaces of algebraic stacks are not vector spaces but rather complexes of vector spaces). This is why these notions will be only defined when the suspension functor of \mathcal{C} is fully faithful, or equivalently when the stabilization functor from \mathcal{C} to its stabilization is fully faithful (this is again an incarnation of the fact, mentioned above, that homotopical algebraic geometry seems to prefer stable situations). In a way, this explains from a conceptual point of view the fact that the infinitesimal study of usual algebraic stacks in the sense of Artin is already part of homotopical algebraic geometry, and does not really belong to standard algebraic geometry. We also define stacks having an *obstruction theory* (see Def. 1.4.2.1), a notion which controls obstruction to lifting morphisms along a first order deformation in terms of the cotangent complex. Despite its name, having an obstruction theory is a property of a stack and not an additional structure. Again, this notion is really well behaved when the suspension functor of \mathcal{C} is fully faithful, and this once again explains the relevance of derived algebraic geometry with respect to infinitesimal deformation theory. Finally, in the last section we give sufficient conditions (that we called *Artin's conditions*) insuring that any n -geometric stack has an obstruction theory (Thm. 1.4.3.2). This last result can be considered as a far reaching generalization of the existence of cotangent complexes for algebraic stacks as presented in [La-Mo].

Higher Artin stacks (after C. Simpson). As a first example of application, we show how our general notion of geometric stacks specializes to C. Simpson's algebraic n -stacks introduced in [S3]. For this, we let $\mathcal{C} = k\text{-Mod}$, be the symmetric monoidal category of k -modules (for some fixed commutative ring k), endowed with its trivial model structure. The topology τ is chosen to be the étale (ét) topology, and \mathbf{P} is chosen to be the class of smooth morphisms. We denote by $\text{St}(k)$ the corresponding homotopy category of ét-stacks. Then, our definition of n -geometric stack gives back the notion of algebraic n -stack introduced in [S3] (except that the two n 's might differ); these stacks will be called *Artin n -stacks* as they contain the usual algebraic stacks in the sense of Artin as particular cases (see Prop. 2.1.2.1). However, all our infinitesimal study (cotangent complexes and obstruction theory) does not apply here as the suspension functor on $k\text{-Mod}$ is the zero functor. This should not be viewed as a drawback of the theory; on the contrary we rather think this explains why deformation theory and obstruction theory in fact already belong to the realm of derived algebraic geometry, which is our next application.

Derived algebraic geometry: D^- -stacks. Our second application is the so-called *derived algebraic geometry*. The base model category \mathcal{C} is chosen to be $sk\text{-Mod}$, the symmetric monoidal model category of simplicial commutative k -modules, k being some fixed base ring. The category of affine objects is $k\text{-}D^-Aff$, the opposite model category of the category of commutative simplicial k -algebras. In this setting,

our general notions of flat, étale, smooth morphisms and Zariski open immersions all have explicit descriptions in terms of standard notions (see Thm. 2.2.2.6). More precisely, we prove that a morphism of simplicial commutative k -algebras $A \rightarrow B$ is flat (resp. smooth, resp. étale, resp. a Zariski open immersion) in the general sense we have given to these notions in §1.2, if and only if it satisfies the following two conditions

- The induced morphism of affine schemes

$$\mathrm{Spec} \pi_0(B) \rightarrow \mathrm{Spec} \pi_0(A)$$

is flat (resp. smooth, resp. étale, resp. a Zariski open immersion) in the usual sense.

- The natural morphism

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_*(B)$$

is an isomorphism.

We endow $k - D^-Aff$ with the étale model topology, a natural extension of the étale topology for affine schemes; the corresponding homotopy category of D^- -stacks is simply denoted by $D^-St(k)$. The class \mathbf{P} is taken to be the class of smooth morphisms. The n -geometric stacks in this context will be called *n -geometric D^- -stacks*, where the notation D^- is meant to bring to mind the negative bounded derived category³. An important consequence of the above descriptions of étale and smooth morphisms is that the natural inclusion functor from the category of k -modules to the category of simplicial k -modules, induces a full embedding of the category of Artin n -stacks into the category of n -geometric D^- -stack. This inclusion functor i has furthermore a right adjoint, called the truncation functor t_0 (see Def. 2.2.4.3), and the adjunction morphism $it_0(F) \rightarrow F$ provides a closed embedding of the classical Artin n -stack $it_0(F)$ to its derived version F , which behaves like a formal thickening (see Prop. 2.2.4.7). This is a global counterpart of the common picture of derived deformation theory of a formal classical moduli space sitting as a closed sub-space in the corresponding formal derived moduli space.

We also prove that our general conditions for the existence of an obstruction theory are satisfied, and so any n -geometric D^- -stack has an obstruction theory (see Prop. 2.2.3.3). An important particular case is when this result is applied to the image of an Artin n -stack via the natural inclusion functor i ; we obtain in this way the existence of an obstruction theory for *any* Artin n -stack, and in particular the existence of a cotangent complex. This is a very good instance of our principle that things simplify when the base model category \mathcal{C} becomes more stable: the infinitesimal study of classical objects of algebraic geometry (such as schemes, algebraic stacks or Artin n -stacks) becomes conceptually clearer and behaves much better when we consider these objects as geometric D^- -stacks.

Finally, we give several examples of D^- -stacks being derived versions of some well known classical moduli problems. First of all the D^- -stack of *local systems* on a topological space, and the D^- -stack of *algebra structures* over a given operad are shown to be 1-geometric (see Lem. 2.2.6.3, Prop. 2.2.6.8). We also present derived versions of the *scheme of morphisms* between two projective schemes, and of the moduli stack of *flat bundles* on a projective complex manifold (Cor. 2.2.6.14 and Cor. 2.2.6.15). The proofs that these last two stacks are geometric rely on a special version of J. Lurie's

³Recall that the homotopy theory of simplicial k -modules is equivalent to the homotopy theory of negatively graded cochain complexes of k -modules. Therefore, derived algebraic geometry can also be considered as algebraic geometry over the category of negatively graded complexes.

representability theorem (see [Lu1] and Appendix C).

Complicial algebraic geometry: D -stacks. What we call *complicial algebraic geometry* is an unbounded version of derived algebraic geometry in which the base model category is $C(k)$ the category of complexes over some commutative ring k (of characteristic zero), and is presented in §2.3. It turns out that linear algebra over $C(k)$ behaves rather differently than over the category of simplicial k -modules (corresponding to complexes in non-positive degrees). Indeed, the smooth, étale and Zariski open immersion can not be described using a simple description on homotopy groups anymore. For example, a usual ring A may have Zariski open localizations $A \rightarrow A'$ in the context of complicial algebraic geometry such that A' is not cohomologically concentrated in degree 0 anymore. Also, a usual non affine scheme might be affine when considered as a scheme over $C(k)$: for example any quasi-compact open subscheme of a usual affine scheme is representable by an affine scheme over $C(k)$ (see example 2.3.1.5).

This makes the complicial theory rather different from derived algebraic geometry for which the geometric intuition was instead quite close to the usual one, and constitutes a very interesting new feature of complicial algebraic geometry. The category Aff_C is here the opposite of the category of unbounded commutative differential graded algebras over k . It is endowed with a *strong étale topology*, and the corresponding homotopy category of D -stacks is simply denoted by $DSt(k)$. A new feature is here the existence of several interesting choices for the class \mathbf{P} . We will present two of them, one for which \mathbf{P} is taken to be the class of perfect morphisms, a rather weak notion of smoothness, and a second one for which \mathbf{P} is taken to be the class of *fip-smooth* morphism, a definitely stronger notion behaving similarly to usual smooth morphisms with respect to lifting properties. We check that such choices satisfy the required properties in order for n -geometric stacks (called *weakly n -geometric D -stacks* and *n -geometric D -stacks*, according to the choice of \mathbf{P}) to make sense. Furthermore, for our second choice, we prove that Artin's conditions are satisfied, and thus that n -geometric D -stacks have a good infinitesimal theory. We give several examples of weakly geometric D -stacks, the first one being the D -stack of *perfect modules* \mathbf{Perf} . We also show that the D -stack of *associative algebra structures* \mathbf{Ass} is a weakly 1-geometric D -stack. Finally, the D -stack of *connected dg-categories* \mathbf{Cat}_* is shown to be weakly 2-geometric. It is important to note that these D -stacks can not be reasonably described as geometric D^- -stacks, and provide examples of truly "exotic" geometric objects.

Suitable slight modifications of the D -stacks \mathbf{Perf} , \mathbf{Ass} and \mathbf{Cat}_* are given and shown to be geometric. This allows us to study their tangent complexes, and show in particular that the infinitesimal theory of a certain class of dg-algebras and dg-categories is controlled by derivations and Hochschild cohomology, respectively (see Cor. 2.3.5.9 and Cor. 2.3.5.12). We also show that Hochschild cohomology does not control deformations of general dg-categories in any reasonable sense (see Cor. 2.3.5.13 and Rem. 2.3.5.14). This has been a true surprise, as it contradicts some of the statements one finds in the existing literature, including some made by the authors themselves (see e.g. [To-Ve2, Thm. 5.6]).

Brave new algebraic geometry: S -stacks. Our last context of application, briefly presented in §2.4, is the one where the base symmetric monoidal model category is $C = Sp^{\Sigma}$, the model category of symmetric spectra ([HSS, Shi]), and gives rise to what we call, after F. Waldhausen, *brave new algebraic geometry*. Like in the

complicial case, the existence of negative homotopy groups makes the general theory of flat, smooth, étale morphisms and of Zariski open immersions rather different from the corresponding one in derived algebraic geometry. Moreover, typical phenomena coming from the existence of Steenrod operations makes the notion of smooth morphism even more exotic and rather different from algebraic geometry; to give just a striking example, $\mathbb{Z}[T]$ is *not* smooth over \mathbb{Z} in the context of brave new algebraic geometry. Once again, we do not think this is a drawback of the theory, but rather an interesting new feature one should contemplate and try to understand, as it might reveal interesting new insights also on classical objects. In brave new algebraic geometry, we also check that the strong étale topology and the class \mathbf{P} of fip-smooth morphisms satisfy our general assumptions, so that n -geometric stacks exists in this context. We call them n -geometric S -stacks, while the homotopy category of S -stacks for the strong étale topology is simply denoted by $\mathrm{St}(S)$. As an example, we give a construction of a 1-geometric S -stack starting from the “sheaf” of topological modular forms (Thm. 2.4.2.1).

Relations with other works. It would be rather long to present all related works, and we apologize in advance for not mentioning all of them.

The general fact that the notion of geometric stack only depends on a topology and a choice of the class of morphisms \mathbf{P} has already been stressed by Carlos Simpson in [S3], who attributes this idea to C. Walter. Our general definition of geometric n -stacks is a straightforward generalization to our abstract context of the definitions found in [S3].

Originally, derived algebraic geometry have been approached using the notion of dg -schemes, as introduced by M. Kontsevich, and developed by I. Ciocan-Fontanine and M. Kapranov. We have not tried to make a full comparison with our theory. Let us only mention that there exists a functor from dg -schemes to our category of 1-geometric D^- -stacks (see [To-Ve2, §3.3]). Essentially nothing is known about this functor: we tend to believe that it is not fully faithful, though this question does not seem very relevant. On the contrary, the examples of dg -schemes constructed in [Ci-Ka1, Ci-Ka2] do provide examples of D^- -stacks and we think it is interesting to look for derived moduli-theoretic interpretations of these (i.e. describe their functors of points).

We would like to mention that an approach to formal derived algebraic geometry has been settled down by V. Hinich in [Hin2]. As far as we know, this is the first functorial point of view on derived algebraic geometry that appeared in the literature.

There is a big overlap between our Chapter 2.2 and Jacob Lurie’s thesis [Lu1]. The approach to derived algebraic geometry used by J. Lurie is different from ours as it is based on a notion of ∞ -category, whereas we are working with model categories. The simplicial localization techniques of Dwyer and Kan provide a way to pass from model categories to ∞ -categories, and the “strictification” theorem of [To-Ve1, Thm. 4.2.1] can be used to see that our approach and Lurie’s approach are in fact equivalent (up to some slight differences, for instance concerning the notion of descent). We think that the present work and [Lu1] can not be reasonably considered as totally independent, as their authors have been frequently communicating on the subject since the spring 2002. It seems rather clear that we all have been influenced by these communications and that we have greatly benefited from the reading of the first drafts of [Lu1]. We have to mention however that a huge part of the material of the present paper had been already announced in earlier papers (see e.g. [To-Ve4, To-Ve2]), and have been worked out since the summer 2000 at the time were our project has

started. The two works are also rather disjoint and complementary, as [Lu1] contains much more materials on derived algebraic geometry than what we have included in §2.2 (e.g. a wonderful generalization of Artin's representability theorem, to state only the most striking result). On the other hand, our "HAG" project has also been motivated by rather exotic contexts of applications, as the ones exposed for example in §2.3 and §2.4, and which are not covered by the framework of [Lu1].

The work of K. Behrend on dg -schemes and dg -stacks [Be1, Be2] has been done while we were working on our project, and therefore §2.2 also has some overlaps with his work. However, the two approaches are rather different and nonequivalent, as K. Behrend uses a 2-truncated version of our notions of stacks, with the effect of killing some higher homotopical information. We have not investigated a precise comparison between these two approaches in this work, but we would like to mention that there exists a functor from our category of D^- -stacks to K. Behrend's category of dg -sheaves. This functor is extremely badly behaved: it is not full, nor faithful, nor essentially surjective, nor even injective on isomorphism classes of objects. The only good property is that it sends 1-geometric Deligne-Mumford D^- -stacks to Deligne-Mumford dg -stack in Behrend's sense. However, there are non geometric D^- -stacks that become geometric objects in Behrend's category of dg -sheaves.

Some notions of étale and smooth morphisms of commutative S -algebras have been introduced in [MCM], and they seem to be related to the general notions we present in §1.2. However a precise comparison is not so easy. Moreover, [MCM] contains some wrong statements like the fact that thh -smoothness generalizes smoothness for discrete algebras (right after Definition 4.2) or like Lemma 4.2 (2). The proof of Theorem 6.1 also contains an important gap, since the local equivalences at the end of the proof are not checked to glue together.

Very recently, J. Rognes has proposed a brave new version of Galois theory, including brave new notions of étaleness which are very close to our notions (see [Ro]).

A construction of the moduli of dg -algebras and dg -categories appears in [Ko-So]. These moduli are only formal moduli by construction, and we propose our D -stacks **Ass** and **Cat**, as their global geometrical counterparts.

We wish to mention the work of M. Spitzweck [Sp], in which he proves the existence of model category structures for E_∞ -algebras and modules in a rather general context. This work can therefore be used in order to suppress our assumptions on the existence of model category of commutative monoids. Also, a nice symmetric monoidal model category of motivic complexes is defined in [Sp], providing a new interesting context to investigate. It has been suggested to us to consider this example of *algebraic geometry over motives* by Yu. Manin, already during spring 2000, but we do not have at the moment interesting things to say on the subject.

Finally, J. Gorski has recently constructed a D^- -stack version of the Quot functor (see [Go]), providing this way a functorial interpretation of the derived Quot scheme of [Ci-Kal]. A geometric D^- -stack classifying objects in a dg -category has been recently constructed by the first author and M. Vaquié in [To-Va1].

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We are very grateful to J. Lurie for various communications on the subject, and for sharing with us his work [Lu1]. We have learned a lot about derived algebraic geometry from him.

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Notations and conventions. We will use the word *universe* in the sense of [SGA4-I, Exp.I, Appendice]. Universes will be denoted by $U \in \mathbb{V} \in W \dots$. For any universe U we will assume that $N \in U$. The category of sets (resp. simplicial sets, resp. ...) belonging to a universe U will be denoted by Set_U (resp. $SSet_U$, resp. ...). The objects of Set_U (resp. $SSet_U$, resp. ...) will be called U -sets (resp. U -simplicial sets, resp. ...). We will use the expression U -small set (resp. U -small simplicial set, resp. ...) to mean a set isomorphic to a set in U (resp. a simplicial set isomorphic to a simplicial set in U , resp. ...). A unique exception concerns categories. The expression U -category refers to the usual notion of [SGA4-I, IDef.1.2], and denotes a category \mathcal{C} such that for any two objects x and y in \mathcal{C} the set $Hom_{\mathcal{C}}(x, y)$ is U -small. In the same way, a category \mathcal{C} is U -small if it is isomorphic to some element in U .

Our references for model categories are [Ho1] and [Hi]. By definition, our model categories will always be *closed* model categories, will have all *small* limits and colimits and the functorial factorization property. The word *equivalence* will always mean *weak equivalence* and will refer to a model category structure.

The homotopy category of a model category M is $W^{-1}M$ (see [Ho1, Def. 1.2.1]), where W is the subcategory of equivalences in M , and it will be denoted as $Ho(M)$. The sets of morphisms in $Ho(M)$ will be denoted by $[-, -]_M$, or simply by $[-, -]$ when the reference to the model category M is clear. We will say that two objects in a model category M are equivalent if they are isomorphic in $Ho(M)$. We say that two model categories are *Quillen equivalent* if they can be connected by a finite string of Quillen adjunctions each one being a Quillen equivalence. The mapping space of morphisms between two objects x and y in a model category M is denoted by $Map_M(x, y)$ (see [Ho1, §5]), or simply $Map(x, y)$ if the reference to M is clear. The simplicial set depends on the choice of cofibrant and fibrant resolution functors, but is well defined as an object in the homotopy category of simplicial sets $Ho(SSet)$. If M is a U -category, then $Map_M(x, y)$ is a U -small simplicial set.

The homotopy fiber product (see [Hi, 13.3, 19.5], [DHK, Ch. XIV] or [DS, 10]) of a diagram $x \longrightarrow z \longleftarrow y$ in a model category M will be denoted by $x \times_z^h y$. In the same way, the homotopy push-out of a diagram $x \longleftarrow z \longrightarrow y$ will be denoted by $x \amalg_z^h y$. For a pointed model category M , the suspension and loop functors functor will be denoted by

$$S : Ho(M) \longrightarrow Ho(M) \quad Ho(M) \longleftarrow Ho(M) : \Omega.$$

Recall that $S(x) := * \amalg_x^L *$, and $\Omega(x) := * \times_x^h *$.

When a model category M is a simplicial model category, its simplicial sets of morphisms will be denoted by $\underline{Hom}_M(-, -)$, and their derived functors by $\mathbb{R}\underline{Hom}_M$ (see [Ho1, 1.3.2]), or simply $\underline{Hom}(-, -)$ and $\mathbb{R}\underline{Hom}(-, -)$ when the reference to M is clear. When M is a symmetric monoidal model category in the sense of [Ho1, §4], the derived monoidal structure will be denoted by \otimes^L .

For the notions of U -cofibrantly generated, U -combinatorial and U -cellular model category, we refer to [Ho1, Hi, Du2], or to [HAGI, Appendix], where the basic definitions and crucial properties are recalled in a way that is suitable for our needs.

As usual, the standard simplicial category will be denoted by Δ . The category of simplicial objects in a category \mathcal{C} will be denoted by $s\mathcal{C} := \mathcal{C}^{\Delta^{op}}$. In the same way, the category of co-simplicial objects in \mathcal{C} will be denoted by $cs\mathcal{C}$. For any simplicial object $F \in s\mathcal{C}$ in a category \mathcal{C} , we will use the notation $F_n := F([n])$. Similarly, for any co-simplicial object $F \in \mathcal{C}^{\Delta}$, we will use the notation $F_n := F([n])$. Moreover, when \mathcal{C} is a model category, we will use the notation

$$|X_*| := \operatorname{Hocolim}_{[n] \in \Delta^{op}} X_n$$

for any $X_* \in s\mathcal{C}$.

A sub-simplicial set $K \subset L$ will be called *full* if K is a union of connected components of L . We will also say that a morphism $f : K \rightarrow L$ of simplicial sets is *full* if it induces an equivalence between K and a full sub-simplicial set of L . In the same way, we will use the expressions *full sub-simplicial presheaf*, and *full morphisms of simplicial presheaves* for the levelwise extension of the above notions to presheaves of simplicial sets.

For a Grothendieck site (\mathcal{C}, τ) in a universe \mathbf{U} , we will denote by $Pr(\mathcal{C})$ the category of presheaves of \mathbf{U} -sets on \mathcal{C} , $Pr(\mathcal{C}) := \mathcal{C}^{Set_{\mathbf{U}}^{op}}$. The subcategory of sheaves on (\mathcal{C}, τ) will be denoted by $Sh_{\tau}(\mathcal{C})$, or simply by $Sh(\mathcal{C})$ if the topology τ is unambiguous.

All complexes will be cochain complexes (i.e. with differential increasing the degree by one) and therefore will look like

$$\cdots \longrightarrow E^n \xrightarrow{d_n} E^{n+1} \longrightarrow \cdots \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

The following notations concerning various homotopy categories of stacks are defined in the main text, and recalled here for readers' convenience (see also the Index at the end of the book).

$$\operatorname{St}(\mathcal{C}, \tau) := \operatorname{Ho}(Aff_{\mathcal{C}}^{\sim, \tau})$$

$$\operatorname{St}(k) := \operatorname{Ho}(k - Aff^{\sim, \acute{e}t})$$

$$D^{-}\operatorname{St}(k) := \operatorname{Ho}(k - D^{-}Aff^{\sim, \acute{e}t})$$

$$D\operatorname{St}(k) := \operatorname{Ho}(k - DAff^{\sim, s\acute{e}t})$$

$$\operatorname{St}(S) := \operatorname{Ho}(SAff^{\sim, s\acute{e}t})$$

Introduction to Part 1

In this first part we will study the general theory of stacks and geometric stacks over a base symmetric monoidal model category \mathcal{C} . For this, we will start in §1.1 by introducing the notion of a *homotopical algebraic context* (HA context for short), which consists of a triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ where \mathcal{C} is our base monoidal model category, \mathcal{C}_0 is a sub-category of \mathcal{C} , and \mathcal{A} is a sub-category of the category $\text{Comm}(\mathcal{C})$ of commutative monoids in \mathcal{C} ; we also require that the triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ satisfies certain compatibility conditions. Although this might look like a rather unusual and complicated definition, it will be shown in §1.2 that this data precisely allows us to define abstract versions of standard notions such as derivations, unramified, étale, smooth and flat morphisms. In other words a HA context provides an abstract context in which the basic notions of linear and commutative algebra can be developed.

Part 1

General theory of geometric stacks

Most of the notions of Grothendieck topology and of Grothendieck topoi and which will be used all along this work. Next, we introduce a notion of a *homotopical algebraic geometric context* (HAG context for short), consisting of a HA context together with two additional data, τ and P , satisfying some compatibility conditions. The first datum τ is a model topology on \mathcal{A}/\mathcal{C} , the opposite model category of commutative monoids in \mathcal{C} . The second datum P consists of a class of morphisms in \mathcal{A}/\mathcal{C} which behaves well with respect to τ . The model topology τ gives a category of stacks over \mathcal{A}/\mathcal{C} (a homotopical generalization of the category of sheaves on affine schemes) in which everything is going to be embedded by means of a Yoneda lemma. The class of morphisms P will then be used in order to define geometric stacks and more generally n -geometric stacks, by considering successive quotient stacks of objects of \mathcal{A}/\mathcal{C} by actions of groups whose structural morphisms are in P . The compatibility axioms between τ and P will insure that this action of geometricity behaves well, and will also insure the basic expected properties (stability by homotopy pullbacks, giving and certain cofibrations).

In §1.4, the last chapter of part I, we will go more deeply into the study of geometric stacks by introducing infinitesimal constructions such as derivations, cotangent complexes and other related notions. The main result of this last chapter states that the geometric stack has an obstruction theory (including a cotangent complex) as well as the HAG context satisfies certain additional conditions.

Introduction to Part 1

In this first part we will study the general theory of stacks and geometric stacks over a base symmetric monoidal model category \mathcal{C} . For this, we will start in §1.1 by introducing the notion of a *homotopical algebraic context* (HA context for short), which consists of a triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ where \mathcal{C} is our base monoidal model category, \mathcal{C}_0 is a sub-category of \mathcal{C} , and \mathcal{A} is a sub-category of the category $\text{Comm}(\mathcal{C})$ of commutative monoids in \mathcal{C} ; we also require that the triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ satisfies certain compatibility conditions. Although this might look like a rather unnatural and complicated definition, it will be shown in §1.2 that this data precisely allows us to define abstract versions of standard notions such as derivations, unramified, étale, smooth and flat morphisms. In other words a HA context describes an abstract context in which the basic notions of linear and commutative algebra can be developed.

The first two chapters are only concerned with purely algebraic notions and the geometry only starts in the third one, §1.3. We start by some reminders on the notions of model topology and of model topos (developed in [HAGI]), which are homotopical versions of the notions of Grothendieck topology and of Grothendieck topos and which will be used all along this work. Next, we introduce a notion of a *homotopical algebraic geometry context* (HAG context for short), consisting of a HA context together with two additional data, τ and \mathbf{P} , satisfying some compatibility conditions. The first datum τ is a model topology on $\text{Aff}_{\mathcal{C}}$, the opposite model category of commutative monoids in \mathcal{C} . The second datum \mathbf{P} consists of a class of morphisms in $\text{Aff}_{\mathcal{C}}$ which behaves well with respect to τ . The model topology τ gives a category of stacks over $\text{Aff}_{\mathcal{C}}$ (a homotopical generalization of the category of sheaves on affine schemes) in which everything is going to be embedded by means of a Yoneda lemma. The class of morphisms \mathbf{P} will then be used in order to define *geometric stacks* and more generally *n-geometric stacks*, by considering successive quotient stacks of objects of $\text{Aff}_{\mathcal{C}}$ by action of groupoids whose structural morphisms are in \mathbf{P} . The compatibility axioms between τ and \mathbf{P} will insure that this notion of geometricity behaves well, and satisfies the basic expected properties (stability by homotopy pullbacks, gluing and certain quotients).

In §1.4, the last chapter of part I, we will go more deeply into the study of geometric stacks by introducing infinitesimal constructions such as derivations, cotangent complexes and obstruction theories. The main result of this last chapter states that any geometric stack has an obstruction theory (including a cotangent complex) as soon as the HAG context satisfies suitable additional conditions.

Introduction to Part I

In the first part we will study the general theory of stacks and geometric stacks over a base symmetric monoidal model category \mathcal{C} . For this we will start in §1.1 by introducing the notion of a homotopy algebraic context (HAC context for short) which consists of a triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ where \mathcal{C} is our base monoidal model category, \mathcal{C}_0 is a sub-category of \mathcal{C} , and \mathcal{A} is a sub-category of the category $\mathcal{C}^{(\mathcal{C}_0, \mathcal{A})}$ of commutative monoids in \mathcal{C} ; we also require that the triple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ satisfies certain compatibility conditions. Although this might look like a rather artificial and complicated definition, it will be shown in §1.2 that this data precisely allows us to define a natural version of standard notions such as geometric, unimodular, étale, smooth and flat morphisms. In later words a HAC context describes an abstract context in which the basic notions of linear and commutative algebras can be developed.

The first two chapters are only concerned with purely algebraic notions and the geometry only starts in the third one, §1.3. We start by some reminders on the notions of model topology and of model topos (developed in [MAG1]), which are topological versions of the notions of Grothendieck topology and of Grothendieck topos and which will be used all along this work. Next, we introduce a notion of a homotopy algebraic geometric context (HAG context for short), consisting of a HAC context together with two additional data, τ and \mathcal{F} , satisfying some compatibility conditions. The first datum τ is a model topology on \mathcal{A} , the opposite model category of commutative monoids in \mathcal{C} . The second datum \mathcal{F} consists of a class of morphisms in \mathcal{A} , which behaves well with respect to τ . The model topology τ gives a category of stacks over \mathcal{A} (a homotopy generalization of the category of sheaves on étale schemes) in which everything is going to be modeled by means of a Zariski topology. The class of morphisms \mathcal{F} will then be used in order to define geometric stacks and more generally τ -geometric stacks by representing geometric presheaves of objects of \mathcal{A} by actions of groupsoids whose étale local morphisms are in \mathcal{F} . The compatibility axioms between τ and \mathcal{F} will insure that this notion of geometricity behaves well and satisfies the basic expected properties (stability by homotopy, cofiber, fibration and certain quotients).

In §1.4, the last chapter of part I, we will go more deeply into the study of geometric stacks by introducing infinitesimal extensions and as differential extensions complexes and obstruction theories. The main result of this last chapter states that any geometric stack has an obstruction theory (including a universal complex) as soon as the HAG context satisfies suitable additional conditions.

Homotopical algebraic context

The purpose of this chapter is to fix once for all our base model category as well as several general assumptions it should satisfy.

All along this chapter, we refer to [Ho1] for the general definition of monoidal model categories, and to [Schw-Shi] for general results about monoids and modules in monoidal model categories.

From now on, and all along this work, we fix three universes $U \in V \in W$ (see, e.g. [SGA4-I, Exp.I, Appendice]). We also let $(C, \otimes, 1)$ be a symmetric monoidal model category in the sense of [Ho1, §4]. We assume that C is a V -small category, and that it is U -combinatorial in the sense of [HAGI, Appendix].

We make a first assumption on the base model category C , making it closer to an additive category. Recall that we denote by Q a cofibrant replacement functor and by R a fibrant replacement functor in M .

ASSUMPTION 1.1.0.1. (1) *The model category C is proper, pointed (i.e. the final object is also an initial object) and for any two object X and Y in C the natural morphisms*

$$QX \amalg QY \longrightarrow X \amalg Y \longrightarrow RX \times RY$$

are all equivalences.

(2) *The homotopy category $\mathrm{Ho}(C)$ is an additive category.*

Assumption 1.1.0.1 implies in particular that finite homotopy coproducts are also finite homotopy products in C . It is always satisfied when C is furthermore a stable model category in the sense of [Ho1, §7]. Note that 1.1.0.1 implies in particular that for any two objects x and y in C , the set $[x, y]$ has a natural abelian group structure.

As $(C, \otimes, 1)$ is a symmetric monoidal category, which is closed and has U -small limits and colimits, all the standard notions and constructions of linear algebra makes sense in C (e.g. monoids, modules over monoids, operads, algebra over an operad ...). The category of all associative, commutative and unital monoids in C will be denoted by $\mathrm{Comm}(C)$. Objects of $\mathrm{Comm}(C)$ will simply be called *commutative monoids in C* , or *commutative monoids* if C is clear. In the same way, one defines $\mathrm{Comm}_{\mathrm{nu}}(C)$ to be the category of non-unital commutative monoids in C . Therefore, our convention will be that monoids are unital unless the contrary is specified.

The categories $\mathrm{Comm}(C)$ and $\mathrm{Comm}_{\mathrm{nu}}(C)$ are again U -categories which are V -small categories, and possess all U -small limits and colimits. They come equipped with natural forgetful functors

$$\mathrm{Comm}(C) \longrightarrow C \quad \mathrm{Comm}_{\mathrm{nu}}(C) \longrightarrow C,$$

possessing left adjoints

$$F : \mathcal{C} \longrightarrow \text{Comm}(\mathcal{C}) \quad F_{\text{nu}} : \mathcal{C} \longrightarrow \text{Comm}_{\text{nu}}(\mathcal{C})$$

sending an object of \mathcal{C} to the free commutative monoid it generates. We recall that for $X \in \mathcal{C}$ one has

$$F(X) = \coprod_{n \in \mathbb{N}} X^{\otimes n} / \Sigma_n$$

$$F_{\text{nu}}(X) = \coprod_{n \in \mathbb{N} - \{0\}} X^{\otimes n} / \Sigma_n,$$

where $X^{\otimes n}$ is the n -tensor power of X , Σ_n acts on it by permuting the factors and $X^{\otimes n} / \Sigma_n$ denotes the quotient of this action in \mathcal{C} .

Let $A \in \text{Comm}(\mathcal{C})$ be a commutative monoid. We will denote by $A - \text{Mod}$ the category of unital left A -modules in \mathcal{C} . The category $A - \text{Mod}$ is again a \mathbf{U} -category which is a \mathbf{V} -small category, and has all \mathbf{U} -small limits and colimits. The objects in $A - \text{Mod}$ will simply be called A -modules. It comes equipped with a natural forgetful functor

$$A - \text{Mod} \longrightarrow \mathcal{C},$$

possessing a left adjoint

$$A \otimes - : \mathcal{C} \longrightarrow A - \text{Mod}$$

sending an object of \mathcal{C} to the free A -module it generates. We also recall that the category $A - \text{Mod}$ has a natural symmetric monoidal structure $- \otimes_A -$. For two A -modules X and Y , the object $X \otimes_A Y$ is defined as the coequalizer in \mathcal{C} of the two natural morphisms

$$X \otimes A \otimes Y \longrightarrow X \otimes Y \quad X \otimes A \otimes Y \longrightarrow X \otimes Y.$$

This symmetric monoidal structure is furthermore closed, and for two A -modules X and Y we will denote by $\underline{\text{Hom}}_A(X, Y)$ the A -module of morphisms. One has the usual adjunction isomorphisms

$$\text{Hom}(X \otimes_A Y, Z) \simeq \text{Hom}(X, \underline{\text{Hom}}_A(Y, Z)),$$

as well as isomorphisms of A -modules

$$\underline{\text{Hom}}_A(X \otimes_A Y, Z) \simeq \underline{\text{Hom}}(X, \underline{\text{Hom}}_A(Y, Z)).$$

We define a morphism in $A - \text{Mod}$ to be a *fibration* or an *equivalence* if it is so on the underlying objects in \mathcal{C} .

ASSUMPTION 1.1.0.2. *Let $A \in \text{Comm}(\mathcal{C})$ be any commutative monoid in \mathcal{C} . Then, the above notions of equivalences and fibrations makes $A - \text{Mod}$ into a \mathbf{U} -combinatorial proper model category. The monoidal structure $- \otimes_A -$ makes furthermore $A - \text{Mod}$ into a symmetric monoidal model category in the sense of [Ho1, §4].*

Using the assumption 1.1.0.2 one sees that the homotopy category $\text{Ho}(A - \text{Mod})$ has a natural symmetric monoidal structure \otimes_A^L , and derived internal Hom 's associated to it $\mathbb{R}\underline{\text{Hom}}_A$, satisfying the usual adjunction rule

$$[X \otimes_A^L Y, Z] \simeq [X, \mathbb{R}\underline{\text{Hom}}_A(Y, Z)].$$

ASSUMPTION 1.1.0.3. *Let A be a commutative monoid in \mathcal{C} . For any cofibrant object $M \in A - \text{Mod}$, the functor*

$$- \otimes_A M : A - \text{Mod} \longrightarrow A - \text{Mod}$$

preserves equivalences.

Let us still denote by A a commutative monoid in \mathcal{C} . We have categories of commutative monoids in $A - \text{Mod}$, and non-unital commutative monoids in $A - \text{Mod}$, denoted respectively by $A - \text{Comm}(\mathcal{C})$ and $A - \text{Comm}_{\text{nu}}(\mathcal{C})$, and whose objects will be called *commutative A -algebras* and *non-unital commutative A -algebras*. They come equipped with natural forgetful functors

$$A - \text{Comm}(\mathcal{C}) \longrightarrow A - \text{Mod} \quad A - \text{Comm}_{\text{nu}}(\mathcal{C}) \longrightarrow A - \text{Mod},$$

possessing left adjoints

$$F_A : A - \text{Mod} \longrightarrow A - \text{Comm}(\mathcal{C}) \quad F_A^{\text{nu}} : A - \text{Mod} \longrightarrow A - \text{Comm}_{\text{nu}}(\mathcal{C})$$

sending an object of $A - \text{Mod}$ to the free commutative monoid it generates. We recall that for $X \in A - \text{Mod}$ one has

$$F_A(X) = \coprod_{n \in \mathbb{N}} X^{\otimes_A n} / \Sigma_n$$

$$F_A^{\text{nu}}(X) = \coprod_{n \in \mathbb{N} - \{0\}} X^{\otimes_A n} / \Sigma_n,$$

where $X^{\otimes_A n}$ is the n -tensor power of X in $A - \text{Mod}$, Σ_n acts on it by permuting the factors and $X^{\otimes_A n} / \Sigma_n$ denotes the quotient of this action in $A - \text{Mod}$.

Finally, we define a morphism in $A - \text{Comm}(\mathcal{C})$ or in $A - \text{Comm}_{\text{nu}}(\mathcal{C})$ to be a *fibration* (resp. an *equivalence*) if it is so as a morphism in the category \mathcal{C} (or equivalently as a morphism in $A - \text{Mod}$).

ASSUMPTION 1.1.0.4. *Let A be any commutative monoid in \mathcal{C} .*

- (1) *The above classes of equivalences and fibrations make the categories $A - \text{Comm}(\mathcal{C})$ and $A - \text{Comm}_{\text{nu}}(\mathcal{C})$ into \mathbb{U} -combinatorial proper model categories.*
- (2) *If B is a cofibrant object in $A - \text{Comm}(\mathcal{C})$, then the functor*

$$B \otimes_A - : A - \text{Mod} \longrightarrow B - \text{Mod}$$

preserves equivalences.

REMARK 1.1.0.5. One word concerning non-unital algebras. We will not really use this notion in the sequel, except at one point in order to prove the existence of a cotangent complex (so the reader is essentially allowed to forget about this unfrequently used notion). In fact, by our assumptions, the model category of non-unital commutative A -algebras is Quillen equivalent to the model category of augmented commutative A -algebra. However, the categories themselves are not equivalent, since the category \mathcal{C} is not assumed to be strictly speaking additive, but only additive up to homotopy (e.g. it could be the model category of symmetric spectra of [HSS]). Therefore, we do not think that the existence of the model structure on $A - \text{Comm}(\mathcal{C})$ implies the existence of the model structure on $A - \text{Comm}_{\text{nu}}(\mathcal{C})$; this explains why we had to add condition (1) on $A - \text{Comm}_{\text{nu}}(\mathcal{C})$ in Assumption 1.1.0.4. Furthermore, the passage from augmented A -algebras to non-unital A -algebras will be in any case necessary to construct a certain Quillen adjunction during the proof of Prop. 1.2.1.2, because such a Quillen adjunction does not exist from the model category of augmented A -algebras (as it is a composition of a left Quillen functor by a right Quillen equivalence).

An important consequence of assumption 1.1.0.4 (2) is that for A a commutative monoid in \mathcal{C} , and B, B' two commutative A -algebras, the natural morphism in $\text{Ho}(A -$

$Mod)$

$$B \coprod_A^L B' \longrightarrow B \otimes_A^L B'$$

is an isomorphism (here the object on the left is the homotopy coproduct in $A - Comm(\mathcal{C})$, and the one on the right is the derived tensor product in $A - Mod$).

An important remark we will use implicitly very often in this paper is that the category $A - Comm(\mathcal{C})$ is naturally equivalent to the comma category $A/Comm(\mathcal{C})$, of objects under A . Moreover, the model structure on $A - Comm(\mathcal{C})$ coincides through this equivalence with the comma model category $A/Comm(\mathcal{C})$.

We will also fix a full subcategory \mathcal{C}_0 of \mathcal{C} , playing essentially the role of a kind of “ t -structure” on \mathcal{C} (i.e. essentially defining which are the “non-positively graded objects”, keeping in mind that in this work we use the cohomological grading when concerned with complexes). More precisely, we will fix a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ satisfying the following conditions.

- ASSUMPTION 1.1.0.6. (1) $1 \in \mathcal{C}_0$.
 (2) The full subcategory \mathcal{C}_0 of \mathcal{C} is stable by equivalences and by \mathbb{U} -small homotopy colimits.
 (3) The full subcategory $Ho(\mathcal{C}_0)$ of $Ho(\mathcal{C})$ is stable by the monoidal structure $- \otimes^L -$ (i.e. for X and Y in $Ho(\mathcal{C}_0)$ we have $X \otimes^L Y \in Ho(\mathcal{C}_0)$).

Recall that as \mathcal{C} is a pointed model category one can define its *suspension functor*

$$\begin{aligned} S: Ho(\mathcal{C}) &\longrightarrow Ho(\mathcal{C}) \\ x &\longmapsto * \coprod_x^L * \end{aligned}$$

left adjoint to the *loop functor*

$$\begin{aligned} \Omega: Ho(\mathcal{C}) &\longrightarrow Ho(\mathcal{C}) \\ x &\longmapsto := * \times_x^h *. \end{aligned}$$

We set \mathcal{C}_1 to be the full subcategory of \mathcal{C} consisting of all objects equivalent to the suspension of some object in \mathcal{C}_0 . The full subcategory \mathcal{C}_1 of \mathcal{C} is also closed by equivalences, homotopy colimits and the derived tensor structure. We will denote by $Comm(\mathcal{C})_0$ the full subcategory of $Comm(\mathcal{C})$ consisting of commutative monoids whose underlying \mathcal{C} -object lies in \mathcal{C}_0 . In the same way, for $A \in Comm(\mathcal{C})$ we denote by $A - Mod_0$ (resp. $A - Mod_1$, resp. $A - Comm(\mathcal{C})_0$) the full subcategory of $A - Mod$ consisting of A -modules whose underlying \mathcal{C} -object lies in \mathcal{C}_0 (resp. of $A - Mod$ consisting of A -modules whose underlying \mathcal{C} -object lies in \mathcal{C}_1 , resp. of $A - Comm(\mathcal{C})$ consisting of commutative A -algebras whose underlying \mathcal{C} -object lies in \mathcal{C}_0).

An important consequence of Assumption 1.1.0.6 is that for any morphism $A \longrightarrow B$ in $Comm(\mathcal{C})_0$, and any $M \in A - Mod_0$, we have $B \otimes_A^L M \in Ho(B - Mod_0)$. Indeed, any such A -module can be written as a homotopy colimit of A -modules of the form $A \otimes^{L_n} M$, for which we have

$$B \otimes_A^L A \otimes^{L_n} M \simeq A \otimes^{L(n-1)} M.$$

In particular the full subcategory $Comm(\mathcal{C})_0$ is closed by homotopy push-outs in $Comm(\mathcal{C})$. Passing to the suspension we also see that for any $M \in A - Mod_1$, one also has $B \otimes_A^L M \in Ho(B - Mod_1)$.

REMARK 1.1.0.7. (1) The reason for introducing the subcategory \mathcal{C}_0 is to be able to consider reasonable infinitesimal lifting properties; these infinitesimal lifting properties will be used to develop the abstract obstruction

theory of geometric stacks in §1.4. It is useful to keep in mind that C_0 plays a role analogous to a kind of t -structure on \mathcal{C} , in that it morally defines which are the non-positively graded objects (a typical example will appear in §2.3 where \mathcal{C} will be the model category of unbounded complexes and C_0 the subcategory of complexes with vanishing positive cohomology). Different choices of C_0 will then give different notions of formal smoothness (see §1.2.8), and thus possibly different notions of geometric stacks. We think that playing with C_0 as a degree of freedom is an interesting feature of our abstract infinitesimal theory.

- (2) Assumptions 1.1.0.1, 1.1.0.2 and 1.1.0.3 are not really serious, and are only useful to avoid taking too many fibrant and cofibrant replacements. With some care, they can be omitted. On the other hand, the careful reader will probably be surprised by assumption 1.1.0.4, as it is known not to be satisfied in several interesting examples (e.g. when \mathcal{C} is the model category of complexes over some commutative ring k , not of characteristic zero). Also, it is well known that in some situations the notion of commutative monoid is too strict and it is often preferable to use the weaker notion of E_∞ -monoid. Two reasons has led us to assume 1.1.0.4. First of all, for all contexts of application of the general theory we will present in this work, there is always a base model category \mathcal{C} for which this condition is satisfied and gives rise to the correct theory. Moreover, if one replaces commutative monoids by E_∞ -monoids then assumption 1.1.0.4 is almost always satisfied, as shown in [Sp], and we think that translating our general constructions should then be a rather academic exercise. Working with commutative monoids instead of E_∞ -monoids simplifies a lot the notations and certain constructions, and in our opinion this theory already captures the real essence of the subject.

Finally, in partial defense of our choice, let us also mention that contrary to what one could think at first sight, working with E_∞ -monoids would not strictly speaking increase the degree of generality of the theory. Indeed, one of our major application is to the category of simplicial k -modules, whose category of commutative monoids is the category of simplicial commutative k -algebras. However, if k has non-zero characteristic, the homotopy theory of simplicial commutative k -algebras is *not* equivalent to the homotopy theory of E_∞ -monoids in simplicial k -modules (the latter is equivalent to the homotopy theory of E_∞ - k -algebras in non positive degrees). Therefore, using E_∞ -monoids throughout would prevent us from developing derived algebraic geometry as presented in §2.2.

We list below some important examples of symmetric monoidal model categories \mathcal{C} satisfying the four above assumptions, and of crucial importance for our applications.

- (1) Let k be any commutative ring, and $\mathcal{C} = k\text{-Mod}$ be the category of U - k -modules, symmetric monoidal for the tensor product \otimes_k , and endowed with its trivial model structure (i.e. equivalences are isomorphisms and all morphisms are fibrations and cofibrations). Then assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4 are satisfied. For C_0 one can take the whole \mathcal{C} for which Assumption 1.1.0.6 is clearly satisfied.
- (2) Let k be a commutative ring of characteristic 0, and $\mathcal{C} = C(k)$ be the category of (unbounded) complexes of U - k -modules, symmetric monoidal for the tensor product of complexes \otimes_k , and endowed with its projective model structure for which equivalences are quasi-isomorphisms and fibrations are epimorphisms (see [Ho1]). Then, the category $Comm(\mathcal{C})$ is the

category of commutative differential graded k -algebras belonging to \mathcal{U} . For $A \in \text{Comm}(\mathcal{C})$, the category $A - \text{Mod}$ is then the category of differential graded A -modules. It is well known that as k is of characteristic zero then assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4 are satisfied (see e.g. [Hin1]). For \mathcal{C}_0 one can take either the whole \mathcal{C} , or the full subcategory of \mathcal{C} consisting of complexes E such that $H^i(E) = 0$ for any $i > 0$, for which Assumption 1.1.0.6 is satisfied.

A similar example is given by non-positively graded complexes $\mathcal{C} = \mathcal{C}^-(k)$.

- (3) Let k be any commutative ring, and $\mathcal{C} = s\text{Mod}_k$ be the category of \mathcal{U} -simplicial k -modules, endowed with the levelwise tensor product and the usual model structure for which equivalences and fibrations are defined on the underlying simplicial sets (see e.g. [Goe-Ja]). The category $\text{Comm}(\mathcal{C})$ is then the category of simplicial commutative k -algebras, and for $A \in s\text{Mod}_k$, $A - \text{Mod}$ is the category of simplicial modules over the simplicial ring A . Assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4 are again well known to be satisfied. For \mathcal{C}_0 one can take the whole \mathcal{C} .

Dually, one could also let $\mathcal{C} = cs\text{Mod}_k$ be the category of co-simplicial k -modules, and our assumptions would again be satisfied. In this case, \mathcal{C}_0 could be for example the full subcategory of co-simplicial modules E such that $\pi_{-i}(E) = H^i(E) = 0$ for any $i > 0$. This subcategory is stable under homotopy colimits as the functor $E \mapsto H^0(E)$ is right Quillen and right adjoint to the inclusion functor $k - \text{Mod} \rightarrow cs\text{Mod}_k$.

- (4) Let Sp^Σ be the category of \mathcal{U} -symmetric spectra and its smash product, endowed with the positive stable model structure of [Shi]. Then, $\text{Comm}(\mathcal{C})$ is the category of commutative symmetric ring spectra, and assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4 are known to be satisfied (see [Shi, Thm. 3.1, Thm. 3.2, Cor. 4.3]). The two canonical choices for \mathcal{C}_0 are the whole \mathcal{C} or the full subcategory of connective spectra.

It is important to note that one can also take for \mathcal{C} the category of Hk -modules in Sp^Σ for some commutative ring k . This will give a model for the homotopy theory of E_∞ - k -algebras that were not provided by our example 2.

- (5) Finally, the above three examples can be sheafified over some Grothendieck site, giving the corresponding relative theories over a base Grothendieck topos.

In few words, let \mathcal{S} be a \mathcal{U} -small Grothendieck site, and \mathcal{C} be one of the three symmetric monoidal model categories $\mathcal{C}(k)$, $s\text{Mod}_k$, Sp^Σ discussed above. One considers the corresponding categories of presheaves on \mathcal{S} , $\text{Pr}(\mathcal{S}, \mathcal{C}(k))$, $\text{Pr}(\mathcal{S}, s\text{Mod}_k)$, $\text{Pr}(\mathcal{S}, Sp^\Sigma)$. They can be endowed with the projective model structures for which fibrations and equivalences are defined levelwise. This first model structure does not depend on the topology on \mathcal{S} , and will be called the *strong model structure*: its (co)fibrations and equivalences will be called *global (co)fibrations* and *global equivalences*.

The next step is to introduce notions of *local equivalences* in the model categories $\text{Pr}(\mathcal{S}, \mathcal{C}(k))$, $\text{Pr}(\mathcal{S}, s\text{Mod}_k)$, $\text{Pr}(\mathcal{S}, Sp^\Sigma)$. This notion is defined by first defining reasonable *homotopy sheaves*, as done for the notion of local equivalences in the theory of simplicial presheaves, and then define a morphism to be a local equivalence if it induces isomorphisms on all homotopy sheaves (for various choice of base points, see [Jol, Jal] for more details). The final model structures on $\text{Pr}(\mathcal{S}, \mathcal{C}(k))$, $\text{Pr}(\mathcal{S}, s\text{Mod}_k)$, $\text{Pr}(\mathcal{S}, Sp^\Sigma)$ are

the one for which equivalences are the local equivalences, and cofibrations are the global cofibrations. The proof that this indeed defines model categories is not given here and is very similar to the proof of the existence of the local projective model structure on the category of simplicial presheaves (see for example [HAGI]).

Finally, the symmetric monoidal structures on the categories $C(k)$, $sMod_k$ and Sp^Σ induces natural symmetric monoidal structures on the categories $Pr(\mathcal{S}, C(k))$, $Pr(\mathcal{S}, sMod_k)$, $Pr(\mathcal{S}, Sp^\Sigma)$. These symmetric monoidal structures make them into symmetric monoidal model categories when \mathcal{S} has finite products. One can also check that the symmetric monoidal model categories $Pr(\mathcal{S}, C(k))$, $Pr(\mathcal{S}, sMod_k)$, $Pr(\mathcal{S}, Sp^\Sigma)$ constructed that way all satisfy the assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4.

An important example is the following. Let \mathcal{O} be a sheaf of commutative rings on the site \mathcal{S} , and let $H\mathcal{O} \in Pr(\mathcal{S}, Sp^\Sigma)$ be the presheaf of symmetric spectra it defines. The object $H\mathcal{O}$ is a commutative monoid in $Pr(\mathcal{S}, Sp^\Sigma)$ and one can therefore consider the model category $H\mathcal{O} - Mod$, of $H\mathcal{O}$ -modules. The category $H\mathcal{O} - Mod$ is a symmetric monoidal model category and its homotopy category is equivalent to the unbounded derived category $D(\mathcal{S}, \mathcal{O})$ of \mathcal{O} -modules on the site \mathcal{S} . This gives a way to define all the standard constructions as derived tensor products, derived internal Hom 's etc., in the context of unbounded complexes of \mathcal{O} -modules.

Let $f : A \rightarrow B$ be a morphism of commutative monoids in \mathcal{C} . We deduce an adjunction between the categories of modules

$$f^* : A - Mod \rightarrow B - Mod \quad A - Mod \leftarrow B - Mod : f_*$$

where $f^*(M) := B \otimes_A M$, and f_* is the forgetful functor that sees a B -module as an A -module through the morphism f . Assumption 1.1.0.2 tells us that this adjunction is a Quillen adjunction, and assumption 1.1.0.3 implies it is furthermore a Quillen equivalence when f is an equivalence (this is one of the main reasons for assumption 1.1.0.3).

The morphism f induces a pair of adjoint derived functors

$$Lf^* : Ho(A - Mod) \rightarrow Ho(B - Mod) \quad Ho(A - Mod) \leftarrow Ho(B - Mod) : Rf_* \simeq f_*$$

and, as usual, we will also use the notation

$$Lf^*(M) =: B \otimes_A^L M \in Ho(B - Mod).$$

Finally, let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ p \downarrow & & \downarrow p' \\ A' & \xrightarrow{f'} & B' \end{array}$$

be a homotopy cofiber square in $Comm(\mathcal{C})$. Then, for any A' -module M we have the well known base change morphism

$$Lf^*p_*(M) \rightarrow (p')_*L(f')^*(M).$$

PROPOSITION 1.1.0.8. *Let us keep the notations as above. Then, the morphism*

$$Lf^*p_*(M) \rightarrow (p')_*L(f')^*(M)$$

is an isomorphism in $Ho(B - Mod)$ for any A' -module M .

PROOF. As the homotopy categories of modules are invariant under equivalences of commutative monoids, one can suppose that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ p \downarrow & & \downarrow p' \\ A' & \xrightarrow{f'} & B' \end{array}$$

is cocartesian in $Comm(\mathcal{C})$, and consists of cofibrations in $Comm(\mathcal{C})$. Then, using 1.1.0.3 and 1.1.0.4 (2) one sees that the natural morphisms

$$M \otimes_A^L B \longrightarrow M \otimes_A B \quad M \otimes_{A'}^L B' \longrightarrow M \otimes_{A'} B',$$

are isomorphism in $\mathrm{Ho}(B\text{-Mod})$. Therefore, the proposition follows from the natural isomorphism of B -modules

$$M \otimes_{A'} B' \simeq M \otimes_{A'} (A' \otimes_A B) \simeq M \otimes_A B.$$

□

REMARK 1.1.0.9. The above base change formula will be extremely important in the sequel, and most often used implicitly. It should be noticed that it implies that the homotopy coproduct in the model category of commutative monoids is given by the derived tensor product. This last property is only satisfied because we have used commutative monoids, and is clearly wrong for simply associative monoids. This is one major reason why our setting cannot be used, at least without some modifications, to develop truly non-commutative geometries. Even partially commutative structures, like E_n -monoids for $n > 1$ would not satisfy the base change formula, and one really needs E_∞ -monoids at least. This fact also prevents us to generalize our setting by replacing the category of commutative monoids by more general categories, like some category of algebras over more general operads.

We now consider $A \in Comm(\mathcal{C})$ and the natural inclusion $A\text{-Mod}_0 \longrightarrow A\text{-Mod}$. We consider the restricted Yoneda embedding

$$\mathbb{R}h_0^- : \mathrm{Ho}(A\text{-Mod}_0^{op}) \longrightarrow \mathrm{Ho}((A\text{-Mod}_0^{op})^\wedge),$$

sending an A -module M to the functor

$$Map(M, -) : A\text{-Mod}_0^{op} \longrightarrow SSet_v.$$

We recall here from [HAGI, §4.1] that for a model category M , and a full subcategory stable by equivalences $M_0 \subset M$, M_0^\wedge is the left Bousfield localization of $SPr(M_0)$ along equivalences in M_0 .

DEFINITION 1.1.0.10. We will say that $A \in Comm(\mathcal{C})$ is good with respect to \mathcal{C}_0 (or simply \mathcal{C}_0 -good) if the functor

$$\mathrm{Ho}(A\text{-Mod}_0^{op}) \longrightarrow \mathrm{Ho}((A\text{-Mod}_0^{op})^\wedge)$$

is fully faithful.

In usual category theory, a full subcategory $D \subset C$ is called *dense* if the restricted Yoneda functor $C \longrightarrow Pr(D) := \underline{Hom}(D^{op}, Ens)$ is fully faithful (in [SGA4-I] this notion is equivalent to the fact that D generates C through strict epimorphisms). This implies for example that any object of C is the colimit of objects of D , but is a slightly stronger condition because any object $x \in C$ is in fact isomorphic to the colimit of the canonical diagram $D/x \longrightarrow C$ (see e.g. [SGA4-I, ExpI-Prop. 7.2]).

Our notion of being good (Def. 1.1.0.10) essentially means that $A - \text{Mod}_0^{\text{op}}$ is *homotopically dense* in $A - \text{Mod}^{\text{op}}$. This of course implies that any object in $A - \text{Mod}$ is equivalent to a homotopy limit of objects in $A - \text{Mod}_0$, and is equivalent to the fact that any cofibrant object $M \in A - \text{Mod}$ is equivalent to the homotopy limit of the natural diagram $(M/A - \text{Mod}_0)^c \rightarrow A - \text{Mod}$, where $(M/A - \text{Mod}_0)^c$ denotes the category of cofibrations under M . Dually, one could say that A being good with respect to C_0 means that $A - \text{Mod}_0$ *cogenerates* $A - \text{Mod}$ through strict monomorphisms in a homotopical sense.

We finish this first chapter by the following definition, gathering our assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3, 1.1.0.4 and 1.1.0.6 all together.

DEFINITION 1.1.0.11. A Homotopical Algebraic context (or simply HA context) is a triplet (C, C_0, \mathcal{A}) , consisting of a symmetric monoidal model category C , two full sub-categories stable by equivalences

$$C_0 \subset C \quad \mathcal{A} \subset \text{Comm}(C),$$

such that any $A \in \mathcal{A}$ is C_0 -good, and assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3, 1.1.0.4, 1.1.0.6 are satisfied.

Let $A \in \text{Comm}(C)$ be a commutative monoid in C and M be an A -module. We define a new commutative monoid $A \amalg M$ in the following way. The underlying object of $A \amalg M$ is the coproduct $A \amalg M$. The multiplicative structure is defined by the map

$$(A \amalg M) \otimes (A \amalg M) \rightarrow A \otimes A \amalg A \otimes M \amalg M \otimes A \amalg M \otimes M \rightarrow A \amalg M$$

given by the three equations

$$a \amalg a' \mapsto A \otimes A \rightarrow A \amalg M,$$

$$a \amalg m \mapsto A \otimes M \rightarrow A \amalg M$$

$$m \amalg m' \mapsto M \otimes M \rightarrow M$$

where $a, a' \mapsto A \otimes A \rightarrow A$ is the multiplicative structure of A , and $a \amalg m \mapsto A \otimes M \rightarrow M$ is the multiplication of M . This monoid $A \amalg M$ is commutative and initial, and defines an object of $\text{Comm}(C)$. To obtain the coproduct with a natural morphism of commutative monoids $A \amalg A' \rightarrow A \amalg M' \rightarrow A'$, which has a natural section $A \amalg A' \rightarrow A \amalg M'$.

Since $A \amalg A' \rightarrow A'$ is a morphism in $\text{Comm}(C)$, and M' is a left module for the monoid $A' \amalg A' \rightarrow A'$, we have also a morphism of commutative A' -algebras. In other words, $A \amalg M' \rightarrow A'$ is a morphism of A' -modules. This defines the commutative monoid category $A - \text{Comm}(C) \cup B$.

PROPOSITION 1.2.1.1. Let $A \rightarrow B$ be a morphism of commutative monoids, and M be a B -module. The coproduct of A with M is a commutative monoid $A \amalg M$, and the map

$$\text{Hom}_A(M, M') \rightarrow \text{Hom}_{A \amalg M}(M, M') \quad (f \mapsto f \circ \text{incl}_M)$$

is an isomorphism. The map $A \amalg M \rightarrow B \amalg M'$ defines a natural transformation $A \amalg M \rightarrow B \amalg M'$. This defines the homotopy category of commutative monoids $\text{Ho}(\text{Comm}(C))$. When passing to the homotopy category of mapping spaces, we obtain that any map $f: A \rightarrow B$ in $\text{Ho}(\text{Comm}(C))$ can be extended to a unique map

$$\text{Ho}(\text{Comm}(C)) \rightarrow \text{Ho}(\text{Comm}(C)) \quad (f \mapsto f \circ \text{incl}_M)$$

CHAPTER 1.2

Preliminaries on linear and commutative algebra in an HA context

All along this chapter we fix once for all a HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$, in the sense of Def. 1.1.0.11. The purpose of this chapter is to show that the assumptions of the last chapter imply that many general notions of linear and commutative algebra generalize in some reasonable sense in our base category \mathcal{C} .

1.2.1. Derivations and the cotangent complex

This section is nothing else than a rewriting of the first pages of [Ba], which stay valid in our general context.

Let $A \in \text{Comm}(\mathcal{C})$ be a commutative monoid in \mathcal{C} , and M be an A -module. We define a new commutative monoid $A \oplus M$ in the following way. The underlying object of $A \oplus M$ is the coproduct $A \amalg M$. The multiplicative structure is defined by the morphism

$$(A \amalg M) \otimes (A \amalg M) \simeq A \otimes A \amalg A \otimes M \amalg A \otimes M \amalg M \otimes M \longrightarrow A \amalg M$$

given by the three morphisms

$$\mu \amalg * : A \otimes A \longrightarrow A \amalg M$$

$$* \amalg \rho : A \otimes M \longrightarrow A \amalg M$$

$$* : M \otimes M \longrightarrow M,$$

where $\mu : A \otimes A \longrightarrow A$ is the multiplicative structure of A , and $\rho : A \otimes M \longrightarrow M$ is the module structure of M . The monoid $A \oplus M$ is commutative and unital, and defines an object in $\text{Comm}(\mathcal{C})$. It comes furthermore with a natural morphism of commutative monoids $\text{id} \amalg * : A \oplus M \longrightarrow A$, which has a natural section $\text{id} \amalg * : A \longrightarrow A \oplus M$.

Now, if $A \longrightarrow B$ is a morphism in $\text{Comm}(\mathcal{C})$, and M is a B -module, the morphism $B \oplus M \longrightarrow B$ can be seen as a morphism of commutative A -algebras. In other words, $B \oplus M$ can be seen as an object of the double comma model category $A\text{-Comm}(\mathcal{C})/B$.

DEFINITION 1.2.1.1. *Let $A \longrightarrow B$ be a morphism of commutative monoids, and M be a B -module. The simplicial set of derived A -derivations from B to M , is the object*

$$\text{Der}_A(B, M) := \text{Map}_{A\text{-Comm}(\mathcal{C})/B}(B, B \oplus M) \in \text{Ho}(\text{SSet}_{\mathcal{U}}).$$

Clearly, $M \mapsto \text{Der}_A(B, M)$ defines a functor from the homotopy category of B -module $\text{Ho}(B\text{-Mod})$ to the homotopy category of simplicial sets $\text{Ho}(\text{SSet})$. More precisely, the functoriality of the construction of mapping spaces implies that one can also construct a genuine functor

$$\text{Der}_A(B, -) : B\text{-Mod} \longrightarrow \text{SSet}_{\mathcal{U}},$$

lifting the previous functor on the homotopy categories. This last functor will be considered as an object in the model category of pre-stacks $(B - Mod^{op})^\wedge$ as defined in [HAGI, §4.1]. Recall from [HAGI, §4.2] that there exists a Yoneda embedding

$$\mathrm{Ho}(B - Mod)^{op} \longrightarrow \mathrm{Ho}((B - Mod^{op})^\wedge)$$

sending a B -module M to the simplicial presheaf $N \mapsto \mathrm{Map}_{B-Mod}(M, N)$, and objects in the essential image will be called *co-representable*.

PROPOSITION 1.2.1.2. *For any morphism $A \rightarrow B$ in $\mathrm{Comm}(\mathcal{C})$, there exists a B -module $\mathbb{L}_{B/A}$, and an element $d \in \pi_0(\mathrm{Der}_A(B, \mathbb{L}_{B/A}))$, such that for any B -module M , the natural morphism obtained by composing with d*

$$d^* : \mathrm{Map}_{B-Mod}(\mathbb{L}_{B/A}, M) \longrightarrow \mathrm{Der}_A(B, M)$$

is an isomorphism in $\mathrm{Ho}(S\mathrm{Set})$.

PROOF. The proof is the same as in [Ba], and uses our assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3 and 1.1.0.4. We will reproduce it for the reader convenience.

We first consider the Quillen adjunction

$$-\otimes_A B : A - \mathrm{Comm}(\mathcal{C})/B \longrightarrow B - \mathrm{Comm}(\mathcal{C})/B \quad A - \mathrm{Comm}(\mathcal{C})/B \longleftarrow B - \mathrm{Comm}(\mathcal{C})/B : F,$$

where F is the forgetful functor. This induces an adjunction on the level of homotopy categories

$$-\otimes_A^{\mathbb{L}} B : \mathrm{Ho}(A - \mathrm{Comm}(\mathcal{C})/B) \longrightarrow \mathrm{Ho}(B - \mathrm{Comm}(\mathcal{C})/B)$$

$$\mathrm{Ho}(A - \mathrm{Comm}(\mathcal{C})/B) \longleftarrow \mathrm{Ho}(B - \mathrm{Comm}(\mathcal{C})/B) : F.$$

We consider a second Quillen adjunction

$$K : B - \mathrm{Comm}_{nu}(\mathcal{C}) \longrightarrow B - \mathrm{Comm}(\mathcal{C})/B \quad B - \mathrm{Comm}_{nu}(\mathcal{C}) \longleftarrow B - \mathrm{Comm}(\mathcal{C})/B : I,$$

where $B - \mathrm{Comm}_{nu}(\mathcal{C})$ is the category of non-unital commutative B -algebras (i.e. non-unital commutative monoids in $B - Mod$). The functor I takes a diagram of commutative monoids $B \xrightarrow{s} C \xrightarrow{p} B$ to the kernel of p computed in the category of non-unital commutative B -algebras. In the other direction, the functor K takes a non-unital commutative B -algebra C to the trivial extension of B by C (defined as our $B \oplus M$ but taking into account the multiplication on C). Clearly, I is a right Quillen functor, and the adjunction defines an adjunction on the homotopy categories

$$\mathrm{LK} : \mathrm{Ho}(B - \mathrm{Comm}_{nu}(\mathcal{C})) \longrightarrow \mathrm{Ho}(B - \mathrm{Comm}(\mathcal{C})/B)$$

$$\mathrm{Ho}(B - \mathrm{Comm}_{nu}(\mathcal{C})) \longleftarrow \mathrm{Ho}(B - \mathrm{Comm}(\mathcal{C})/B) : \mathbb{R}I.$$

LEMMA 1.2.1.3. *The adjunction $(\mathrm{LK}, \mathbb{R}I)$ is an equivalence.*

PROOF. This follows easily from our assumption 1.1.0.1. Indeed, it implies that for any fibration in \mathcal{C} , $f : X \rightarrow Y$, which has a section $s : Y \rightarrow X$, the natural morphism

$$i \coprod s : F \coprod Y \longrightarrow X,$$

where $i : F \rightarrow X$ is the fiber of f , is an equivalence. It also implies that the homotopy fiber of the natural morphism $id \coprod * : X \coprod Y \rightarrow X$ is naturally equivalent to Y . These two facts imply the lemma. \square

Finally, we consider a third adjunction

$$Q : B - \text{Comm}_{\text{nu}}(\mathcal{C}) \longrightarrow B - \text{Mod} \quad B - \text{Comm}_{\text{nu}}(\mathcal{C}) \longleftarrow B - \text{Mod} : Z,$$

where Q of an object $C \in B - \text{Comm}_{\text{nu}}(\mathcal{C})$ is the push-out of B -modules

$$\begin{array}{ccc} C \otimes_B C & \xrightarrow{\mu} & C \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & Q(C), \end{array}$$

and Z sends a B -module M to the non-unital B -algebra M endowed with the zero multiplication. Clearly, (Q, Z) is a Quillen adjunction and gives rise to an adjunction on the homotopy categories

$$\text{LQ} : \text{Ho}(B - \text{Comm}_{\text{nu}}(\mathcal{C})) \longrightarrow \text{Ho}(B - \text{Mod})$$

$$\text{Ho}(B - \text{Comm}_{\text{nu}}(\mathcal{C})) \longleftarrow \text{Ho}(B - \text{Mod}) : Z.$$

We can now conclude the proof of the proposition by chaining up the various adjunction to get a string of isomorphisms in $\text{Ho}(\mathcal{S}\text{Set})$

$$\begin{aligned} \text{Der}_A(B, M) &\simeq \text{Map}_{A - \text{Comm}(\mathcal{C})/B}(B, F(B \oplus M)) \simeq \text{Map}_{B - \text{Comm}(\mathcal{C})/B}(B \otimes_A^{\mathbb{L}} B, B \oplus M) \\ &\simeq \text{Map}_{B - \text{Comm}_{\text{nu}}(\mathcal{C})}(\mathbb{R}I(B \otimes_A^{\mathbb{L}} B), \mathbb{R}I(B \oplus M)) \simeq \text{Map}_{B - \text{Comm}_{\text{nu}}(\mathcal{C})}(\mathbb{R}I(B \otimes_A^{\mathbb{L}} B), Z(M)) \\ &\simeq \text{Map}_{B - \text{Mod}}(\text{LQRI}(B \otimes_A^{\mathbb{L}} B), M). \end{aligned}$$

Therefore, $\mathbb{L}_{B/A} := \text{LQRI}(B \otimes_A^{\mathbb{L}} B)$ and the image of $\text{id} \in \text{Map}_{B - \text{Mod}}(\mathbb{L}_{B/A}, \mathbb{L}_{B/A})$ gives what we were looking for. \square

REMARK 1.2.1.4. Proposition 1.2.1.2 implies that the two functors

$$M \mapsto \text{Map}_{B - \text{Mod}}(\mathbb{L}_{B/A}, M) \quad M \mapsto \text{Der}_A(B, M)$$

are isomorphic as objects in $\text{Ho}((B - \text{Mod}^{\text{op}})^{\wedge})$. In other words, Prop. 1.2.1.2 implies that the functor $\text{Der}_A(B, -)$ is *co-representable* in the sense of [HAGI].

DEFINITION 1.2.1.5. Let $A \longrightarrow B$ be a morphism in $\text{Comm}(\mathcal{C})$.

- (1) The B -module $\mathbb{L}_{B/A} \in \text{Ho}(B - \text{Mod})$ is called the cotangent complex of B over A .
- (2) When $A = 1$, we will use the following notation

$$\mathbb{L}_B := \mathbb{L}_{B/1},$$

and \mathbb{L}_B will be called the cotangent complex of B .

Using the definition and proposition 1.2.1.2, it is easy to check the following facts.

PROPOSITION 1.2.1.6. (1) Let $A \longrightarrow B \longrightarrow C$ be two morphisms in $\text{Comm}(\mathcal{C})$. Then, there is a homotopy cofiber sequence in $\mathcal{C} - \text{Mod}$

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C \longrightarrow \mathbb{L}_{C/A} \longrightarrow \mathbb{L}_{C/B}.$$

- (2) Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a homotopy cofiber square in $\text{Comm}(\mathcal{C})$. Then, the natural morphism

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} B' \longrightarrow \mathbb{L}_{B'/A'}$$

is an isomorphism in $\text{Ho}(B' - \text{Mod})$. Furthermore, the natural morphism

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} B' \coprod \mathbb{L}_{A'/A} \otimes_{A'}^{\mathbb{L}} B' \longrightarrow \mathbb{L}_{B'/A}$$

is an isomorphism in $\text{Ho}(B' - \text{Mod})$.

(3) Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a homotopy cofiber square in $\text{Comm}(\mathcal{C})$. Then the following square is homotopy cocartesian in $B' - \text{Mod}$

$$\begin{array}{ccc} \mathbb{L}_A \otimes_A^{\mathbb{L}} B' & \longrightarrow & \mathbb{L}_B \otimes_B^{\mathbb{L}} B' \\ \downarrow & & \downarrow \\ \mathbb{L}_{A'} \otimes_{A'}^{\mathbb{L}} B' & \longrightarrow & \mathbb{L}_{B'} \end{array}$$

(4) For any commutative monoid A and any A -module M , one has a natural isomorphism in $\text{Ho}(A - \text{Mod})$

$$\mathbb{L}_{A \oplus M} \otimes_{A \oplus M}^{\mathbb{L}} A \simeq \mathbb{L}_A \coprod \mathbb{L}QZ(M),$$

where

$$Q : A - \text{Comm}_{\text{nu}}(\mathcal{C}) \longrightarrow A - \text{Mod} \quad A - \text{Comm}_{\text{nu}}(\mathcal{C}) \longleftarrow A - \text{Mod} : Z$$

is the Quillen adjunction used during the proof of 1.2.1.2.

PROOF. (1) to (3) are simple exercises, using the definitions and that for any morphism of commutative monoids $A \longrightarrow B$, and any B -module M , the following square is homotopy cartesian in $\text{Comm}(\mathcal{C})$ (because of our assumption 1.1.0.1)

$$\begin{array}{ccc} A \oplus M & \longrightarrow & B \oplus M \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

(4) We note that for any commutative monoid A , and any A -modules M and N , one has a natural homotopy fiber sequence

$$\begin{aligned} \text{Map}_{A - \text{Comm}(\mathcal{C})/A}(A \oplus M, A \oplus N) &\longrightarrow \text{Map}_{\text{Comm}(\mathcal{C})/A}(A \oplus M, A \oplus N) \longrightarrow \\ &\longrightarrow \text{Map}_{\text{Comm}(\mathcal{C})/A}(A, A \oplus N), \end{aligned}$$

or equivalently using lemma 1.2.1.3

$$\text{Map}_{A - \text{Mod}}(\mathbb{L}QZ(M), N) \longrightarrow \text{Der}(A \oplus M, N) \longrightarrow \text{Der}(A, N).$$

This implies the existence of a natural homotopy cofiber sequence of A -modules

$$\mathbb{L}_A \longrightarrow \mathbb{L}_{A \oplus M} \otimes_{A \oplus M}^{\mathbb{L}} A \longrightarrow \mathbb{L}QZ(M).$$

Clearly this sequence splits in $\text{Ho}(A - \text{Mod})$ and gives rise to a natural isomorphism

$$\mathbb{L}_{A \oplus M} \otimes_{A \oplus M}^{\mathbb{L}} A \simeq \mathbb{L}_A \coprod \mathbb{L}QZ(M).$$

The importance of derivations come from the fact that they give rise to infinitesimal extensions in the following way. Let $A \rightarrow B$ be a morphism of commutative monoids in \mathcal{C} , and M be a B -module. Let $d : \mathbb{L}_{B/A} \rightarrow M$ be a morphism in $\mathrm{Ho}(B\text{-Mod})$, corresponding to a derivation $d \in \pi_0(\mathrm{Der}_A(B, M))$. This derivation can be seen as a section $d : B \rightarrow B \oplus M$ of the morphism of commutative A -algebras $B \oplus M \rightarrow B$. We consider the following homotopy cartesian diagram in the category of commutative A -algebras

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow d \\ B & \xrightarrow{s} & B \oplus M \end{array}$$

where $s : B \rightarrow B \oplus M$ is the natural section corresponding to the zero morphism $\mathbb{L}_{B/A} \rightarrow M$. Then, $C \rightarrow B$ is a morphism of commutative A -algebras such that its fiber is a non-unital commutative A -algebra isomorphic in $\mathrm{Ho}(A\text{-Comm}_{\mathrm{nu}}(\mathcal{C}))$ to the loop A -module $\Omega M := * \times_M^h *$ with the zero multiplication. In other words, C is a *square zero extension* of B by ΩM . It will be denoted by $B \oplus_d \Omega M$. The most important case is of course when $A = B$, and we make the following definition.

DEFINITION 1.2.1.7. Let A be a commutative monoid, M and A -module and $d \in \pi_0 \mathrm{Der}(A, M)$ be a derivation given by a morphism in $d : A \rightarrow A \oplus M$ in $\mathrm{Ho}(\mathrm{Comm}(\mathcal{C})/A)$. The square zero extension associated to d , denoted by $A \oplus_d \Omega M$, is defined as the homotopy pullback diagram of commutative monoids

$$\begin{array}{ccc} A \oplus_d \Omega M & \longrightarrow & A \\ \downarrow & & \downarrow d \\ A & \xrightarrow{s} & A \oplus M, \end{array}$$

where s is the natural morphism corresponding to the zero derivation. The top horizontal morphism $A \oplus_d \Omega M \rightarrow A$ will be called the *natural projection*.

1.2.2. Hochschild homology

For a commutative monoid $A \in \mathrm{Comm}(\mathcal{C})$, we set

$$THH(A) := S^1 \otimes^{\mathbb{L}} A \in \mathrm{Ho}(\mathrm{Comm}(\mathcal{C})),$$

where $S^1 \otimes^{\mathbb{L}}$ denotes the derived external product of object in $\mathrm{Comm}(\mathcal{C})$ by the simplicial circle $S^1 := \Delta^1 / \partial \Delta^1$. Presenting the circle S^1 has the homotopy push-out

$$\begin{array}{c} * \\ \prod^{\mathbb{L}} \\ * \end{array}$$

one gets that

$$THH(A) \simeq A \otimes_{A \otimes^{\mathbb{L}} A}^{\mathbb{L}} A.$$

The natural point $*$ $\rightarrow S^1$ induces a natural morphism in $\mathrm{Ho}(\mathrm{Comm}(\mathcal{C}))$

$$A \rightarrow THH(A)$$

making $THH(A)$ as a commutative A -algebra, and as a natural object in $\mathrm{Ho}(A\text{-Comm}(\mathcal{C}))$.

DEFINITION 1.2.2.1. Let A be a commutative monoid in \mathcal{C} . The topological Hochschild homology of A (or simply Hochschild homology) is the commutative A -algebra $THH(A) := S^1 \otimes^{\mathbb{L}} A$.

More generally, if $A \rightarrow B$ is a morphism of commutative monoids in \mathcal{C} , the relative topological Hochschild homology of B over A (or simply relative Hochschild homology) is the commutative A -algebra

$$THH(B/A) := THH(B) \otimes_{THH(A)}^L A.$$

By definition, we have for any commutative monoid B ,

$$Map_{Comm(\mathcal{C})}(THH(A), B) \simeq Map_{SSet}(S^1, Map_{Comm(\mathcal{C})}(A, B)).$$

This implies that if $f : A \rightarrow B$ is a morphism of commutative monoids in \mathcal{C} , then we have

$$Map_{A-Comm(\mathcal{C})}(THH(A), B) \simeq \Omega_f Map_{Comm(\mathcal{C})}(A, B),$$

where $\Omega_f Map_{Comm(\mathcal{C})}(A, B)$ is the loop space of $Map_{Comm(\mathcal{C})}(A, B)$ at the point f . More generally, if B and C are commutative A -algebras, then

$$Map_{A-Comm(\mathcal{C})}(THH(B/A), C) \simeq Map_{SSet}(S^1, Map_{A-Comm(\mathcal{C})}(B, C)),$$

and for a morphism $f : B \rightarrow C$ of commutative A -algebras

$$Map_{B-Comm(\mathcal{C})}(THH(B/A), C) \simeq \Omega_f Map_{A-Comm(\mathcal{C})}(B, C).$$

PROPOSITION 1.2.2.2. (1) Let $A \rightarrow B \rightarrow C$ be two morphisms in $Comm(\mathcal{C})$. Then, the natural morphism

$$THH(C/A) \otimes_{THH(B/A)}^L B \rightarrow THH(C/B)$$

is an isomorphism in $Ho(B-Comm(\mathcal{C}))$.

(2) Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a homotopy cofiber square in $Comm(\mathcal{C})$. Then, the natural morphism

$$THH(B/A) \otimes_A^L THH(A'/A) \rightarrow THH(B'/A)$$

is an isomorphism in $Ho(A-Comm(\mathcal{C}))$.

PROOF. Exercise. □

1.2.3. Finiteness conditions

We present here two different finiteness conditions for objects in model categories. The first one is valid in any model category, and is a homotopy analog of the notion of finitely presented object in a category. The second one is only valid for symmetric monoidal categories, and is a homotopy generalization of the notion of rigid objects in monoidal categories.

DEFINITION 1.2.3.1. A morphism $x \rightarrow y$ in a proper model category M is finitely presented (we also say that y is finitely presented over x) if for any filtered diagram of objects under x , $\{z_i\}_{i \in I} \in x/M$, the natural morphism

$$Hocolim_{i \in I} Map_{x/M}(y, z_i) \rightarrow Map_{x/M}(y, Hocolim_{i \in I} z_i)$$

is an isomorphism in $Ho(SSet)$.

REMARK 1.2.3.2. If the model category M is not proper the definition 1.2.3.1 has to be modified by replacing $Map_{x/M}$ with $Map_{Qx/M}$, where Qx is a cofibrant model for x . By our assumption 1.1.0.1 all the model categories we will use are proper.

PROPOSITION 1.2.3.3. Let M be a proper model category.

- (1) *Finitely presented morphisms in M are stable by equivalences. In other words, if one has a commutative diagram in M*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ p \downarrow & & \downarrow q \\ x' & \xrightarrow{f'} & y' \end{array}$$

such that p and q are equivalences, then f is finitely presented if and only if f' is finitely presented.

- (2) *Finitely presented morphisms in M are stable by compositions and retracts.*
 (3) *Finitely presented morphisms in M are stable by homotopy push-outs. In other words, if one has a homotopy push-out diagram in M*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ p \downarrow & & \downarrow q \\ x' & \xrightarrow{f'} & y' \end{array}$$

then f' is finitely presented if f is so.

PROOF. (1) is clear as $\text{Map}_{x/M}(a, b)$ only depends on the isomorphism class of a and b as objects in the homotopy category $\text{Ho}(x/M)$.

(2) Let $x \rightarrow y \rightarrow z$ be two finitely presented morphisms in M , and let $\{z_i\}_{i \in I} \in x/M$ be a filtered diagrams of objects. Then, one has for any object $t \in x/M$ a fibration sequence of simplicial sets

$$\text{Map}_{y/M}(z, t) \rightarrow \text{Map}_{x/M}(z, t) \rightarrow \text{Map}_{x/M}(y, t).$$

As fibration sequences are stable by filtered homotopy colimits, one gets a morphism of fibration sequences

$$\begin{array}{ccccc} \text{Hocolim}_{i \in I} \text{Map}_{y/M}(z, z_i) & \longrightarrow & \text{Hocolim}_{i \in I} \text{Map}_{x/M}(z, z_i) & \longrightarrow & \text{Hocolim}_{i \in I} \text{Map}_{x/M}(y, z_i) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Map}_{y/M}(z, \text{Hocolim}_{i \in I} z_i) & \longrightarrow & \text{Map}_{x/M}(z, \text{Hocolim}_{i \in I} z_i) & \longrightarrow & \text{Map}_{x/M}(y, \text{Hocolim}_{i \in I} z_i), \end{array}$$

and the five lemma tells us that the vertical arrow in the middle is an isomorphism in $\text{Ho}(SSet)$. This implies that z is finitely presented over x .

The assertion concerning retracts is clear since, if $x \rightarrow y$ is a retract of $x' \rightarrow y'$, for any $z \in x/M$ the simplicial set $\text{Map}_{x/M}(y, z)$ is a retract of $\text{Map}_{x'/M}(y', z)$.

- (3) This is clear since we have for any object $t \in x'/M$, a natural equivalence $\text{Map}_{x'/M}(y', t) \simeq \text{Map}_{x/M}(y, t)$. \square

Let us now fix I , a set of generating cofibrations in M .

DEFINITION 1.2.3.4. (1) *An object X is a strict finite I -cell object, if there exists a finite sequence*

$$X_0 = \emptyset \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n = X,$$

and for any $0 \leq i < n$ a push-out square

$$\begin{array}{ccc} X_i & \longrightarrow & X_{i+1} \\ \uparrow & & \uparrow \\ A & \xrightarrow{u_i} & B, \end{array}$$

with $u_i \in I$.

- (2) An object X is a finite I -cell object (or simply a finite cell object when I is clear) if it is equivalent to a strict finite I -cell object.
- (3) The model category M is compactly generated if it satisfies the following conditions.
 - (a) The model category M is cellular (in the sense of [Hi, §12]).
 - (b) There exists a set of generating cofibrations I , and generating trivial cofibrations J whose domains and codomains are cofibrant, ω -compact (in the sense of [Hi, §10.8]) and ω -small with respect to the whole category M .
 - (c) Filtered colimits commute with finite limits in M .

The following proposition identifies finitely presented objects when M is compactly generated.

PROPOSITION 1.2.3.5. *Let M be a compactly generated model category, and I be a set of generating cofibrations whose domains and codomains are cofibrant, ω -compact and ω -small with respect to the whole category M .*

- (1) A filtered colimit of fibrations (resp. trivial fibrations) is a fibration (resp. a trivial fibration).
- (2) For any filtered diagram X_i in M , the natural morphism

$$\operatorname{Hocolim}_i X_i \longrightarrow \operatorname{Colim}_i X_i$$
 is an isomorphism in $\operatorname{Ho}(M)$.
- (3) Any object X in M is equivalent to a filtered colimit of strict finite I -cell objects.
- (4) An object X in M is finitely presented if and only if it is equivalent to a retract, in $\operatorname{Ho}(M)$, of a strict finite I -cell object.

PROOF. (1) By assumption the domain and codomain of morphisms of I are ω -small, so M is finitely generated in the sense of [Ho1, §7]. Property (1) is then proved in [Ho1, §7].

- (2) For a filtered category A , the colimit functor

$$\operatorname{Colim} : M^A \longrightarrow M$$

is a left Quillen functor for the levelwise projective model structure on M . By (1) we know that Colim preserves trivial fibrations, and thus that it also preserves equivalences. We therefore have isomorphisms of functors $\operatorname{Hocolim} \simeq \operatorname{LColim} \simeq \operatorname{Colim}$.

(3) The small object argument (e.g. [Ho1, Thm. 2.1.14]) gives that any object X is equivalent to a I -cell complex $Q(X)$. By ω -compactness of the domains and codomains of I , $Q(X)$ is the filtered colimit of its finite sub- I -cell complexes. This implies that X is equivalent to a filtered colimit of strict finite I -cell objects.

(4) Let A be a filtered category, and $Y \in M^A$ be a A -diagram. Let $c(Y) \longrightarrow R_*(Y)$ be a Reedy fibrant replacement of the constant simplicial object $c(Y)$ with

values Y (in the model category of simplicial objects in M^A , see [Ho1, §5.2]). By (2), the induced morphism

$$c(\operatorname{Colim}_{a \in A} Y_a) \longrightarrow \operatorname{Colim}_{a \in A} R_*(Y_a)$$

is an equivalence of simplicial objects in M^A . Moreover, (1) and the exactness of filtered colimits implies that $\operatorname{Colim}_{a \in A} R_*(Y_a)$ is a Reedy fibrant object in the model category of simplicial objects in M (as filtered colimits commute with matching objects for the Reedy category Δ^{op} , see [Ho1, §5.2]). This implies that for any cofibrant and ω -small object K in M , we have

$$\begin{aligned} \operatorname{Hocolim}_{a \in A} \operatorname{Map}(K, Y_a) &\simeq \operatorname{Colim}_{a \in A} \operatorname{Map}(K, Y_a) \simeq \operatorname{Colim}_{a \in A} \operatorname{Hom}(K, R_*(Y_a)) \simeq \\ &\operatorname{Hom}(K, \operatorname{Colim}_{a \in A} R_*(Y_a)) \simeq \operatorname{Map}(K, \operatorname{Colim}_{a \in A} Y_a). \end{aligned}$$

This implies that the domains and codomains of I are homotopically finitely presented.

As filtered colimits of simplicial sets preserve homotopy pull-backs, we deduce that any finite cell objects is also homotopically finitely presented, as they are constructed from domains and codomains of I by iterated homotopy push-outs (we use here that domains and codomains of I are cofibrant). This implies that any retract of a finite cell object is homotopically finitely presented. Conversely, let X be a homotopically finitely presented object in $\operatorname{Ho}(M)$, and by (3) let us write it as $\operatorname{Colim}_i X_i$, where X_i is a filtered diagram of finite cell objects. Then, $[X, X] \simeq \operatorname{Colim}_i [X, X_i]$, which implies that this identity of X factors through some X_i , or in other words that X is a retract in $\operatorname{Ho}(M)$ of some X_i . \square

Now, let M be a symmetric monoidal model category in the sense of [Ho1, §4]. We remind that this implies in particular that the monoidal structure on M is closed, and therefore possesses Hom 's objects $\underline{\operatorname{Hom}}_M(x, y) \in M$ satisfying the usual adjunction rule

$$\operatorname{Hom}(x, \underline{\operatorname{Hom}}_M(y, z)) \simeq \operatorname{Hom}(x \otimes y, z).$$

The internal structure can be derived, and gives on one side a symmetric monoidal structure $- \otimes^L -$ on $\operatorname{Ho}(M)$, as well as Hom 's objects $\mathbb{R}\underline{\operatorname{Hom}}_M(x, y) \in \operatorname{Ho}(M)$ satisfying the derived version of the previous adjunction

$$[x, \mathbb{R}\underline{\operatorname{Hom}}_M(y, z)] \simeq [x \otimes^L y, z].$$

In particular, if $\mathbf{1}$ is the unit of the monoidal structure of M , then

$$[\mathbf{1}, \mathbb{R}\underline{\operatorname{Hom}}_M(x, y)] \simeq [x, y],$$

and more generally

$$\operatorname{Map}_M(\mathbf{1}, \mathbb{R}\underline{\operatorname{Hom}}_M(x, y)) \simeq \operatorname{Map}_M(x, y).$$

Moreover, the adjunction between $- \otimes^L$ and $\mathbb{R}\underline{\operatorname{Hom}}_M$ extends naturally to an adjunction isomorphism

$$\mathbb{R}\underline{\operatorname{Hom}}_M(x, \mathbb{R}\underline{\operatorname{Hom}}_M(y, z)) \simeq \mathbb{R}\underline{\operatorname{Hom}}_M(x \otimes^L y, z).$$

The derived dual of an object $x \in M$ will be denoted by

$$x^\vee := \mathbb{R}\underline{\operatorname{Hom}}_M(x, \mathbf{1}).$$

DEFINITION 1.2.3.6. Let M be a symmetric monoidal model category. An object $x \in M$ is called perfect if the natural morphism

$$x \otimes^L x^\vee \longrightarrow \mathbb{R}\underline{\operatorname{Hom}}_M(x, x)$$

is an isomorphism in $\operatorname{Ho}(M)$.

PROPOSITION 1.2.3.7. Let M be a symmetric monoidal model category.

- (1) If x and y are perfect objects in M , then so is $x \otimes^L y$.
 (2) If x is a perfect object in M , then for any objects y and z , the natural morphism

$$\mathbf{R}\underline{\mathbf{Hom}}_M(y, x \otimes^L z) \longrightarrow \mathbf{R}\underline{\mathbf{Hom}}_M(y \otimes^L x^\vee, z)$$

is an isomorphism in $\mathbf{Ho}(M)$.

- (3) If x is perfect in M , and $y \in M$, then the natural morphism

$$\mathbf{R}\underline{\mathbf{Hom}}_M(x, y) \longrightarrow x^\vee \otimes^L y$$

is an isomorphism in $\mathbf{Ho}(M)$.

- (4) If $\mathbf{1}$ is finitely presented in M , then so is any perfect object.
 (5) If M is furthermore a stable model category then perfect objects are stable by homotopy push-outs and homotopy pullbacks. In other words, if $x \longrightarrow y \longrightarrow z$ is a homotopy fiber sequence in M , and if two of the objects x , y , and z are perfect then so is the third.
 (6) Perfect objects are stable by retracts in $\mathbf{Ho}(M)$.

PROOF. (1), (2) and (3) are standard, as perfect objects are precisely the strongly dualizable objects of the closed monoidal category $\mathbf{Ho}(M)$ (see for example [May2]).

(4) Let x be a perfect object in M , and $\{z_i\}_{i \in I}$ be a filtered diagram of objects in M . Let $x^\vee := \mathbf{R}\underline{\mathbf{Hom}}(x, \mathbf{1})$ the dual of x in $\mathbf{Ho}(M)$. Then, we have

$$\begin{aligned} \mathrm{Map}_M(x, \mathrm{Hocolim}_i z_i) &\simeq \mathrm{Map}_M(\mathbf{1}, x^\vee \otimes^L \mathrm{Hocolim}_i z_i) \simeq \mathrm{Map}_M(\mathbf{1}, \mathrm{Hocolim}_i x^\vee \otimes^L z_i) \\ &\simeq \mathrm{Hocolim}_i \mathrm{Map}_M(\mathbf{1}, x^\vee \otimes^L z_i) \simeq \mathrm{Hocolim}_i \mathrm{Map}_M(x, z_i). \end{aligned}$$

(5) Let $x \longrightarrow y \longrightarrow z$ be a homotopy fiber sequence in M . It is enough to prove that if y and z are perfect then so is x . For this, let x^\vee , y^\vee and z^\vee the duals of x , y and z .

One has a morphism of homotopy fiber sequences

$$\begin{array}{ccccc} x \otimes^L z^\vee & \longrightarrow & x \otimes^L y^\vee & \longrightarrow & x \otimes^L x^\vee \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{R}\underline{\mathbf{Hom}}_M(z, x) & \longrightarrow & \mathbf{R}\underline{\mathbf{Hom}}_M(y, x) & \longrightarrow & \mathbf{R}\underline{\mathbf{Hom}}_M(x, x). \end{array}$$

The five lemma and point (3) implies that the last vertical morphism is isomorphism, and that x is perfect.

- (6) If x is a retract of y , then the natural morphism

$$x \otimes^L x^\vee \longrightarrow \mathbf{R}\underline{\mathbf{Hom}}_M(x, x)$$

is a retract of

$$y \otimes^L y^\vee \longrightarrow \mathbf{R}\underline{\mathbf{Hom}}_M(y, y).$$

□

The following corollary gives a condition under which perfect and finitely presented objects are the same.

COROLLARY 1.2.3.8. Suppose that M is a stable and compactly generated symmetric monoidal model category with $\mathbf{1}$ being ω -compact and cofibrant. We assume that the set I of morphisms of the form

$$S^n \otimes \mathbf{1} \longrightarrow \Delta^{n+1} \otimes \mathbf{1}$$

is a set of generating cofibration for M . Then an object x in M is perfect if and only if it is finitely presented, and if and only if it is a retract of a finite I -cell object.

PROOF. This essentially follows from Prop. 1.2.3.5 and Prop. 1.2.3.7, the only statement which remains to be proved is that retract of finite I -cell objects are perfect. But this follows from the fact that $\mathbf{1}$ is always perfect, and from Prop. 1.2.3.7 (5) and (6). \square

REMARK 1.2.3.9. The notion of finitely presented morphisms and perfect objects depend on the model structure and not only on the underlying category M . They specialize to the corresponding usual categorical notions when M is endowed with the trivial model structure.

1.2.4. Some properties of modules

In this Section we give some general notions of flatness and projectiveness of modules over a commutative monoid in \mathcal{C} .

DEFINITION 1.2.4.1. Let $A \in \text{Comm}(\mathcal{C})$ be a commutative monoid, and M be an A -module.

- (1) The A -module M is flat if the functor

$$-\otimes_A^L M : \text{Ho}(A - \text{Mod}) \longrightarrow \text{Ho}(A - \text{Mod})$$

preserves homotopy pullbacks.

- (2) The A -module M is projective if it is a retract in $\text{Ho}(A - \text{Mod})$ of $\coprod_E^L A$ for some \mathbf{U} -small set E .

PROPOSITION 1.2.4.2. Let $A \rightarrow B$ be a morphism of commutative monoids in \mathcal{C} .

- (1) The free A -module A^n of rank n is flat. Moreover, if infinite direct sums in $\text{Ho}(A - \text{Mod})$ commute with homotopy pull-backs, then for any \mathbf{U} -small set E , the free A -module $\coprod_E^L A$ is flat.
- (2) Flat modules in $\text{Ho}(A - \text{Mod})$ are stable by derived tensor products, finite coproducts and retracts.
- (3) Projective modules in $\text{Ho}(A - \text{Mod})$ are stable by derived tensor products, finite coproducts and retracts.
- (4) If M is a flat (resp. projective) A -module, then $M \otimes_A^L B$ is a flat (resp. projective) B -module.
- (5) A perfect A -module is flat.
- (6) Let us suppose that $\mathbf{1}$ is a finitely presented object in \mathcal{C} . Then, a projective A -module is finitely presented if and only if it is a retract of $\coprod_E^L A$ for some finite set E .
- (7) Let us assume that $\mathbf{1}$ is a finitely presented object in \mathcal{C} . Then, a projective A -module is perfect if and only if it is finitely presented.

PROOF. (1) to (4) are easy and follow from the definitions.

(5) Let M be a perfect A -module, and $M^\vee := \mathbb{R}\underline{\text{Hom}}_A(M, A)$ be its dual. Then, for any homotopy cartesian square of A -modules

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ P' & \longrightarrow & Q' \end{array}$$

the diagram

$$\begin{array}{ccc} P \otimes_A^L M & \longrightarrow & Q \otimes_A^L M \\ \downarrow & & \downarrow \\ P' \otimes_A^L M & \longrightarrow & Q' \otimes_A^L M \end{array}$$

is equivalent to

$$\begin{array}{ccc} \mathbb{R}Hom_A(M^\vee, P) & \longrightarrow & \mathbb{R}Hom_A(M^\vee, Q) \\ \downarrow & & \downarrow \\ \mathbb{R}Hom_A(M^\vee, P') & \longrightarrow & \mathbb{R}Hom_A(M^\vee, Q'), \end{array}$$

which is again homotopy cartesian by the general properties of derived internal Hom 's. This shows that $- \otimes_A^L M$ preserves homotopy pullbacks, and hence that M is flat.

(6) Clearly, if E a finite set, then $\coprod_E^L A$ is finitely presented object, as for any A -module M one has

$$Map_{A-Mod}(\coprod_E^L A, M) \simeq Map_{\mathcal{C}}(1, M)^E.$$

Therefore, a retract of $\coprod_E^L A$ is also finitely presented.

Conversely, let M be a projective A -module which is also finitely presented. Let $i : M \rightarrow \coprod_E^L A$ be a morphism which admits a retraction. As $\coprod_E^L A$ is the colimit of $\coprod_{E_0}^L A$, for E_0 running over the finite subsets of E , the morphism i factors as

$$M \rightarrow \coprod_{E_0}^L A \rightarrow \coprod_E^L A$$

for some finite subset $E_0 \subset E$. This shows that M is in fact a retract of $\coprod_E^L A$.

(7) Using Prop. 1.2.3.7 (4) one sees that if M is perfect then it is finitely presented. Conversely, let M be a finitely presented projective A -module. By (6) we know that M is a retract of $\coprod_E^L A$ for some finite set E . But, as E is finite, $\coprod_E^L A$ is perfect, and therefore so is M as a retract of a perfect module. \square

1.2.5. Formal coverings

The following notion will be highly used, and is a categorical version of faithful morphisms of affine schemes.

DEFINITION 1.2.5.1. A family of morphisms of commutative monoids $\{f_i : A \rightarrow B_i\}_{i \in I}$ is a formal covering if the family of functors

$$\{Lf_i^* : \mathrm{Ho}(A - \mathrm{Mod}) \rightarrow \mathrm{Ho}(B_i - \mathrm{Mod})\}_{i \in I}$$

is conservative (i.e. a morphism u in $\mathrm{Ho}(A - \mathrm{Mod})$ is an isomorphism if and only if all the $Lf_i^*(u)$ are isomorphisms).

The formal covering families are stable by equivalences, homotopy push-outs and compositions and therefore do form a model topology in the sense of [HAGI, Def. 4.3.1] (or Def. 1.3.1.1).

PROPOSITION 1.2.5.2. Formal covering families form a model topology (Def. 1.3.1.1) on the model category $\mathrm{Comm}(\mathcal{C})$.

PROOF. Stability by equivalences and composition is clear. The stability by homotopy push-outs is an easy consequence of the transfer formula Prop. 1.1.0.8. \square

1.2.6. Some properties of morphisms

In this part we review several classes of morphisms of commutative monoids in \mathcal{C} , generalizing the usual notions of Zariski open immersions, unramified, étale, smooth and flat morphisms of affine schemes. It is interesting to notice that these notions make sense in our general context, but specialize to very different notions in various specific cases (see the examples at the beginning of each section §2.1, §2.2, §2.3 and §2.4).

DEFINITION 1.2.6.1. Let $f : A \rightarrow B$ be a morphism of commutative monoids in \mathcal{C} .

- (1) The morphism f is an epimorphism if for any commutative A -algebra C the simplicial set $\text{Map}_{A\text{-Comm}(\mathcal{C})}(B, C)$ is either empty or contractible.
- (2) The morphism f is flat if the induced functor

$$\mathbb{L}f^* : \text{Ho}(A\text{-Mod}) \rightarrow \text{Ho}(B\text{-Mod})$$

commutes with finite homotopy limits.

- (3) The morphism f is a formal Zariski open immersion if it is flat and if the functor

$$f_* : \text{Ho}(B\text{-Mod}) \rightarrow \text{Ho}(A\text{-Mod})$$

is fully faithful.

- (4) The morphism f is formally unramified if $\mathbb{L}_{B/A} \simeq 0$ in $\text{Ho}(B\text{-Mod})$.
- (5) The morphism f is formally étale if the natural morphism

$$\mathbb{L}_A \otimes_A^{\mathbb{L}} B \rightarrow \mathbb{L}_B$$

is an isomorphism in $\text{Ho}(B\text{-Mod})$.

- (6) The morphism f is formally thh-étale if the natural morphism

$$B \rightarrow \text{THH}(B/A)$$

is an isomorphism in $\text{Ho}(\text{Comm}(\mathcal{C}))$.

REMARK 1.2.6.2. One remark concerning our notion of epimorphism of commutative monoids is in order. First of all, in a category \mathcal{C} (without any model structure) having fiber products, a morphism $x \rightarrow y$ is a monomorphism if and only if the diagonal morphism $x \rightarrow x \times_y x$ is an isomorphism. A natural generalization of this fact gives a notion of monomorphism in any model category M , as a morphism $x \rightarrow y$ whose diagonal $x \rightarrow x \times_y^h x$ is an isomorphism in $\text{Ho}(M)$. Equivalently, the morphism $x \rightarrow y$ is a monomorphism if and only if for any $z \in M$ the induced morphism of simplicial sets $\text{Map}_M(z, x) \rightarrow \text{Map}_M(z, y)$ is a monomorphism in the model category $S\text{Set}$. Furthermore, it is easy to check that a morphism $f : K \rightarrow L$ is a monomorphism in the model category $S\text{Set}$ if and only if for any $s \in L$ the homotopy fiber of f at s is either empty or contractible. Therefore we see that a morphism $A \rightarrow B$ in $\text{Comm}(\mathcal{C})$ is an epimorphism in the sense of Def. 1.2.6.1 (1) if and only if it is a monomorphism when considered as a morphism in the model category $\text{Comm}(\mathcal{C})^{\text{op}}$, or equivalently if the induced morphism $B \otimes_A^{\mathbb{L}} B \rightarrow B$ is an isomorphism in $\text{Ho}(\text{Comm}(\mathcal{C}))$. This justifies our terminology, and moreover shows that when the model structure on the category M is trivial, an epimorphism in the sense of our definition is nothing else than an epimorphism in M in the usual categorical sense.

PROPOSITION 1.2.6.3. *Epimorphisms, flat morphisms, formal Zariski open immersions, formally unramified morphisms, formally étale morphisms, and formally thh-étale morphisms, are all stable by compositions, equivalences and homotopy push-outs.*

PROOF. This is a simple exercise using the definitions and Propositions 1.1.0.8, 1.2.1.6, 1.2.2.2, and 1.2.3.7. \square

The relations between all these notions are given by the following proposition.

PROPOSITION 1.2.6.4. (1) *A morphism $f : A \rightarrow B$ is an epimorphism if and only if the functor*

$$f_* : \mathrm{Ho}(B - \mathrm{Mod}) \rightarrow \mathrm{Ho}(A - \mathrm{Mod})$$

is fully faithful.

(2) *A formal Zariski open immersion is an epimorphism. A flat epimorphism is a formal Zariski open immersion.*

(3) *A morphism $f : A \rightarrow B$ of commutative monoids is formally thh-étale if and only if for any commutative A -algebra C the simplicial set*

$$\mathrm{Map}_{A - \mathrm{Comm}(C)}(B, C)$$

is discrete (i.e. equivalent to a set).

(4) *A formally étale morphism is formally unramified.*

(5) *An epimorphism is formally unramified and formally thh-étale.*

(6) *A morphism $f : A \rightarrow B$ in $\mathrm{Comm}(C)$ is formally unramified if and only if the morphism*

$$B \otimes_A^L B \rightarrow B$$

is formally étale.

PROOF. (1) Let $f : A \rightarrow B$ be a morphism such that the right Quillen functor $f_* : B - \mathrm{Mod} \rightarrow A - \mathrm{Mod}$ induces a fully faithful functor on the homotopy categories. Therefore, the adjunction morphism

$$M \otimes_A^L B \rightarrow M$$

is an isomorphism for any $M \in \mathrm{Ho}(B - \mathrm{Mod})$. In particular, the functor

$$f_* : \mathrm{Ho}(B - \mathrm{Comm}(C)) \rightarrow \mathrm{Ho}(A - \mathrm{Comm}(C))$$

is also fully faithful. Let C be a commutative A -algebra, and let us suppose that $\mathrm{Map}_{A - \mathrm{Comm}(C)}(B, C)$ is not empty. This implies that C is isomorphic in $\mathrm{Ho}(A - \mathrm{Comm}(C))$ to some $f_*(C')$ for $C' \in \mathrm{Ho}(B - \mathrm{Comm}(C))$. Therefore, we have

$$\mathrm{Map}_{A - \mathrm{Comm}(C)}(B, C) \simeq \mathrm{Map}_{A - \mathrm{Comm}(C)}(f_*(B), f_*(C')) \simeq \mathrm{Map}_{B - \mathrm{Comm}(C)}(B, C') \simeq *,$$

showing that f is a epimorphism.

Conversely, let $f : A \rightarrow B$ be epimorphism. For any commutative A -algebra C , we have

$$\mathrm{Map}_{A - \mathrm{Comm}(C)}(B, C) \simeq \mathrm{Map}_{A - \mathrm{Comm}(C)}(B, C) \times \mathrm{Map}_{A - \mathrm{Comm}(C)}(B, C),$$

showing that the natural morphism $B \otimes_A^L B \rightarrow B$ is an isomorphism in $\mathrm{Ho}(A - \mathrm{Comm}(C))$. This implies that for any B -module M , we have

$$M \otimes_A^L B \simeq M \otimes_B^L (B \otimes_A^L B) \simeq M,$$

or in other words, that the adjunction morphism $M \rightarrow f_* L f^*(M) \simeq M \otimes_A^L B$ is an isomorphism in $\mathrm{Ho}(B - \mathrm{Mod})$. This means that $f_* : \mathrm{Ho}(B - \mathrm{Mod}) \rightarrow \mathrm{Ho}(A - \mathrm{Mod})$ is fully faithful.

(2) This is clear by (1) and the definitions.

(3) For any morphism of commutative A -algebras, $f : B \rightarrow C$ we have

$$\mathrm{Map}_{B-\mathrm{Comm}(\mathcal{C})}(\mathrm{THH}(B/A), C) \simeq \Omega_f \mathrm{Map}_{A-\mathrm{Comm}(\mathcal{C})}(B, C).$$

Therefore, $B \rightarrow \mathrm{THH}(B/A)$ is an equivalence if and only if for any such $f : B \rightarrow C$, the simplicial set $\Omega_f \mathrm{Map}_{A-\mathrm{Comm}(\mathcal{C})}(B, C)$ is contractible. Equivalently, f is formally thh-étale if and only if $\mathrm{Map}_{A-\mathrm{Comm}(\mathcal{C})}(B, C)$ is discrete.

(4) Let $f : A \rightarrow B$ be a formally étale morphism of commutative monoids in \mathcal{C} . By Prop. 1.2.1.6 (1), there is a homotopy cofiber sequence of B -modules

$$\mathbb{L}_A \otimes_A^{\mathbb{L}} B \rightarrow \mathbb{L}_B \rightarrow \mathbb{L}_{B/A},$$

showing that if the first morphism is an isomorphism then $\mathbb{L}_{B/A} \simeq *$ and therefore that f is formally unramified.

(5) Let $f : A \rightarrow B$ be an epimorphism. By definition and (3) we know that f is formally thh-étale. Let us prove that f is also formally unramified. Let M be a B -module. As we have seen before, for any commutative B -algebra C , the adjunction morphism $C \rightarrow C \otimes_A^{\mathbb{L}} B$ is an isomorphism. In particular, the functor

$$f_* : \mathrm{Ho}(B - \mathrm{Comm}(\mathcal{C})/B) \rightarrow \mathrm{Ho}(A - \mathrm{Comm}(\mathcal{C})/B)$$

is fully faithful. Therefore we have

$$\mathrm{Der}_A(B, M) \simeq \mathrm{Map}_{A-\mathrm{Comm}(\mathcal{C})/B}(B, B \oplus M) \simeq \mathrm{Map}_{B-\mathrm{Comm}(\mathcal{C})/B}(B, B \oplus M) \simeq *,$$

showing that $\mathrm{Der}_A(B, M) \simeq *$ for any B -module M , or equivalently that $\mathbb{L}_{B/A} \simeq *$.

(6) Finally, Prop. 1.2.1.6 (2) shows that the morphism $B \otimes_A^{\mathbb{L}} B \rightarrow B$ is formally étale if and only if the natural morphism

$$\mathbb{L}_{B/A} \coprod \mathbb{L}_{B/A} \rightarrow \mathbb{L}_{B/A}$$

is an isomorphism in $\mathrm{Ho}(B - \mathrm{Mod})$. But this is equivalent to $\mathbb{L}_{B/A} \simeq *$. \square

In order to state the next results we recall that as \mathcal{C} is a pointed model category one can define a suspension functor (see [Ho1, §7])

$$\begin{aligned} S : \mathrm{Ho}(\mathcal{C}) &\rightarrow \mathrm{Ho}(\mathcal{C}) \\ X &\mapsto S(X) := * \coprod_X^{\mathbb{L}} *. \end{aligned}$$

This functor possesses a right adjoint, the loop functor

$$\begin{aligned} \Omega : \mathrm{Ho}(\mathcal{C}) &\rightarrow \mathrm{Ho}(\mathcal{C}) \\ X &\mapsto \Omega(X) := * \times_X^h *. \end{aligned}$$

PROPOSITION 1.2.6.5. *Assume that the base model category \mathcal{C} is such that the suspension functor $S : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C})$ is fully faithful.*

- (1) *A morphism of commutative monoids in \mathcal{C} is formally étale if and only if it is formally unramified.*
- (2) *A formally thh-étale morphism of commutative monoids in \mathcal{C} is a formally étale morphism.*
- (3) *An epimorphism of commutative monoids in \mathcal{C} is formally étale.*

PROOF. (1) By the last proposition we only need to prove that a formally unramified morphism is also formally étale. Let $f : A \rightarrow B$ be such a morphism. By Prop. 1.2.1.6 (1) there is a homotopy cofiber sequence of B -modules

$$\mathbb{L}_A \otimes_A^{\mathbb{L}} B \rightarrow \mathbb{L}_B \rightarrow *$$

This implies that for any B -module M , the homotopy fiber of the morphism

$$\mathrm{Der}(B, M) \rightarrow \mathrm{Der}(A, M)$$

is contractible, and in particular that this morphism induces isomorphisms on all higher homotopy groups. It remains to show that this morphism induces also an isomorphism on π_0 . For this, we can use the hypothesis on \mathcal{C} which implies that the suspension functor on $\mathrm{Ho}(B\text{-Mod})$ is fully faithful. Therefore, we have

$$\begin{aligned} \pi_0(\mathrm{Der}(B, M)) &\simeq \pi_0(\mathrm{Der}(B, \Omega S(M))) \simeq \pi_1(\mathrm{Der}(B, SM)) \simeq \\ &\simeq \pi_1(\mathrm{Der}(A, SM)) \simeq \pi_0(\mathrm{Der}(A, M)). \end{aligned}$$

(2) Let $A \rightarrow B$ be a formally *thh*-étale morphism in $\mathrm{Comm}(\mathcal{C})$.

As $\mathrm{Map}_{A\text{-}\mathrm{Comm}(\mathcal{C})}(B, C)$ is discrete for any commutative A -algebra C , the simplicial set $\mathrm{Der}_A(B, M)$ is discrete for any B -module M . Using the hypothesis on \mathcal{C} we get that for any B -module M

$$\pi_0(\mathrm{Der}_A(B, M)) \simeq \pi_0(\mathrm{Der}_A(B, \Omega S(M))) \simeq \pi_1(\mathrm{Der}_A(B, SM)) \simeq 0,$$

showing that $\mathrm{Der}_A(B, M) \simeq *$, and therefore that $\mathbb{L}_{B/A} \simeq *$. This implies that f is formally unramified, and therefore is formally étale by the first part of the proposition.

(3) This follows from (2) and Prop. 1.2.6.4 (5). □

The hypothesis of Proposition 1.2.6.5 saying that the suspension is fully faithful will appear in many places in the sequel. It is essentially equivalent to saying that the homotopy theory of \mathcal{C} can be embedded in a stable homotopy theory in such a way that homotopy colimits are preserved (it could be in fact called *left semi-stable*). It is a very natural condition as many statements will then simplify, as shown for example by our infinitesimal theory in §3.

COROLLARY 1.2.6.6. *Assume furthermore that \mathcal{C} is a stable model category, and let $f : A \rightarrow B$ be a morphism of commutative monoids in \mathcal{C} . The following are equivalent.*

- (1) *The morphism f is a formal Zariski open immersion.*
- (2) *The morphism f is an epimorphism.*
- (3) *The morphism f is a formally étale epimorphism.*

PROOF. Indeed, in the stable case all base change functors commute with limits, so Prop. 1.2.6.4 (1) and (2) shows that formal Zariski open immersions are exactly the epimorphisms. Furthermore, by Prop. 1.2.6.4 (5) and Prop. 1.2.6.5 all epimorphisms are formally étale. □

DEFINITION 1.2.6.7. *A morphism of commutative monoids in \mathcal{C} is a Zariski open immersion (resp. unramified, resp. étale, resp. *thh*-étale) if it is finitely presented (as a morphism in the model category $\mathrm{Comm}(\mathcal{C})$) and is a formal Zariski open immersion (resp. formally unramified, resp. formally étale, resp. formally *thh*-étale).*

Clearly, using what we have seen before, Zariski open immersions, unramified morphisms, étale morphisms, and *thh*-étale morphisms are all stable by equivalences, compositions and push-outs.

1.2.7. Smoothness

We define two general notions of smoothness, both different generalizations of the usual notion, and both useful in certain contexts. A third, and still different, notion of smoothness will be given in the next section.

DEFINITION 1.2.7.1. Let $f : A \rightarrow B$ be a morphism of commutative algebras.

- (1) The morphism f is formally perfect (or simply fp) if the B -module $L_{B/A}$ is perfect (in the sense of Def. 1.2.3.6).
- (2) The morphism f is formally smooth if the B -module $L_{B/A}$ is projective (in the sense of Def. 1.2.4.1) and if the morphism

$$L_A \otimes_A^L B \rightarrow L_B$$

has a retraction in $\mathrm{Ho}(B - \mathrm{Mod})$.

DEFINITION 1.2.7.2. Let $f : B \rightarrow C$ be a morphism of commutative algebras. The morphism f is perfect, or simply p, (resp. smooth) if it is finitely presented (as a morphism in the model category $\mathrm{Comm}(\mathcal{C})$) and is fp (resp. formally smooth).

Of course, (formally) étale morphisms are (formally) smooth morphisms as well as (formally) perfect morphisms.

PROPOSITION 1.2.7.3. The fp, perfect, formally smooth and smooth morphisms are all stable by compositions, homotopy push outs and equivalences.

PROOF. Exercise. □

1.2.8. Infinitesimal lifting properties

While the first two notions of smoothness only depend on the underlying symmetric monoidal model category \mathcal{C} , the third one, to be defined below, will also depend on the HA context we are working in.

Recall from Definition 1.1.0.11 that an HA context is a triplet $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$, consisting of a symmetric monoidal model category \mathcal{C} , two full sub-categories stable by equivalences

$$\mathcal{C}_0 \subset \mathcal{C} \quad \mathcal{A} \subset \mathrm{Comm}(\mathcal{C}),$$

such that:

- $1 \in \mathcal{C}_0$, \mathcal{C}_0 is closed under by U-small homotopy colimits, and $X \otimes^L Y \in \mathrm{Ho}(\mathcal{C}_0)$ if X and Y in $\mathrm{Ho}(\mathcal{C}_0)$.
- any $A \in \mathcal{A}$ is \mathcal{C}_0 -good (i.e. the functor

$$\mathrm{Ho}(A - \mathrm{Mod}) \rightarrow \mathrm{Ho}((A - \mathrm{Mod}_0^{cp})^\wedge)$$

is fully faithful);

- assumptions 1.1.0.1, 1.1.0.2, 1.1.0.3, 1.1.0.4, 1.1.0.6 are satisfied.

Recall also that \mathcal{C}_1 is the full subcategory of \mathcal{C} consisting of all objects equivalent to suspensions of objects in \mathcal{C}_0 , $\mathrm{Comm}(\mathcal{C})_0$ the full subcategory of $\mathrm{Comm}(\mathcal{C})$ consisting of commutative monoids whose underlying \mathcal{C} -object is in \mathcal{C}_0 , and, for $A \in \mathrm{Comm}(\mathcal{C})$, $A - \mathrm{Mod}_0$ (resp. $A - \mathrm{Mod}_1$, resp. $A - \mathrm{Comm}(\mathcal{C})_0$) is the full subcategory of $A - \mathrm{Mod}$ consisting of A -modules whose underlying \mathcal{C} -object is in \mathcal{C}_0 (resp. of $A - \mathrm{Mod}$ consisting of A -modules whose underlying \mathcal{C} -object is in \mathcal{C}_1 , resp. of $A - \mathrm{Comm}(\mathcal{C})$ consisting of commutative A -algebras whose underlying \mathcal{C} -object is in \mathcal{C}_0).

DEFINITION 1.2.8.1. Let $f : A \rightarrow B$ be a morphism in $\mathrm{Comm}(\mathcal{C})$.

- (1) The morphism f is called formally infinitesimally smooth relative to the HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ (or simply formally i -smooth when the HA context is clear) if for any $R \in \mathcal{A}$, any morphism $A \rightarrow R$ of commutative monoids, any $M \in R - \text{Mod}_1$, and any $d \in \pi_0(\text{Der}_A(R, M))$, the natural morphism
- $$\pi_0(\text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R \oplus_d \Omega M)) \rightarrow \pi_0(\text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R))$$
- is surjective.
- (2) The morphism f is called i -smooth if it is formally i -smooth and finitely presented.

The following proposition is immediate from the definition.

PROPOSITION 1.2.8.2. *Formally i -smooth and i -smooth morphisms are stable by equivalences, composition and homotopy push-outs.*

The next result provides a criterion for formally i -smooth morphisms in terms of their cotangent complexes.

PROPOSITION 1.2.8.3. *A morphism $f : A \rightarrow B$ is formally i -smooth if and only if for any morphism $B \rightarrow R$ with $R \in \mathcal{A}$, and any R -module $M \in R - \text{Mod}_1$, the natural morphism*

$$[\mathbb{L}_{R/A}, M] \simeq \pi_0(\text{Der}_A(R, M)) \rightarrow \pi_0(\text{Der}_A(B, M)) \simeq [\mathbb{L}_{B/A}, M]_{B-\text{Mod}}$$

is zero.

PROOF. Let us first assume that the condition of the proposition is satisfied. Let us consider $R \in \mathcal{A}$, a morphism $A \rightarrow R$, an R -module $M \in R - \text{Mod}_1$ and $d \in \pi_0(\text{Der}_A(R, M))$. The homotopy fiber of the natural morphism

$$\text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R \oplus_d \Omega M) \rightarrow \text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R)$$

taken at some morphism $B \rightarrow R$ in $\text{Ho}(\text{Comm}(\mathcal{C}))$, can be identified with the path space $\text{Path}_{0, d'} \text{Der}_A(B, M)$ from 0 to d' in $\text{Der}_A(B, M)$, where d' is the image of d under the natural morphism

$$\pi_0(\text{Der}_A(R, M)) \rightarrow \pi_0(\text{Der}_A(B, M)).$$

By assumption, 0 and d' belong to the same connected component of $\text{Der}_A(B, M)$, and thus we see that $\text{Path}_{0, d'} \text{Der}_A(B, M)$ is non-empty. We have thus shown that

$$\text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R \oplus_d \Omega M) \rightarrow \text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R)$$

has non-empty homotopy fibers and therefore that

$$\pi_0(\text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R \oplus_d \Omega M)) \rightarrow \pi_0(\text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R))$$

is surjective. The morphism f is therefore formally i -smooth.

Conversely, let $R \in \mathcal{A}$, $A \rightarrow R$ a morphism, $M \in R - \text{Mod}_1$ and $d \in \pi_0(\text{Der}_A(R, M))$. Let $A \rightarrow R$ be a morphism of commutative monoids. We consider the diagram

$$\begin{array}{ccccc} A & \longrightarrow & R[\Omega_d M] & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & R & \xrightarrow{s} & R \oplus M. \end{array}$$

The homotopy fiber of

$$\text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R \oplus_d \Omega M) \rightarrow \text{Map}_{A-\text{Comm}(\mathcal{C})}(B, R)$$

is non-empty because f is formally i -smooth. By definition this means that the image of d by the morphism

$$\pi_0(\mathrm{Der}_A(R, M)) \longrightarrow \pi_0(\mathrm{Der}_A(B, M))$$

is zero. As this is true for any d , this finishes the proof of the proposition. \square

COROLLARY 1.2.8.4. (1) *Any formally unramified morphism is formally i -smooth.*

(2) *Assume that the suspension functor S is fully faithful, that $\mathcal{C}_1 = \mathcal{C}$ (so that in particular \mathcal{C} is stable), and $\mathcal{A} = \mathrm{Comm}(\mathcal{C})$. Then formally i -smooth morphisms are precisely the formally étale morphisms.*

PROOF. It follows immediately from 1.2.8.3. \square

COROLLARY 1.2.8.5. *We assume that for any $M \in \mathcal{C}_1$ one has $[1, M] = 0$. Then any formally smooth morphism is formally i -smooth.*

PROOF. By Prop. 1.2.8.3 and definition of formal smoothness, it is enough to show that for any commutative monoid $A \in \mathcal{C}$, any A -module $M \in A\text{-Mod}_1$ and any projective A -module P we have $[P, M] = 0$. By assumption this is true for $P = A$, and thus also true for free A -modules and their retracts. \square

1.2.9. Standard localizations and Zariski open immersions

For any object $A \in \mathcal{C}$, we can define its *underlying space* as

$$|A| := \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, A) \in \mathrm{Ho}(\mathcal{S}\mathrm{Set}_{\mathrm{U}}).$$

The model category \mathcal{C} being pointed, the simplicial set $|A|$ has a natural base point $*$ $\in |A|$, and one can therefore define the homotopy groups of A

$$\pi_i(A) := \pi_i(|A|, *).$$

When A is the underlying object of a commutative monoid $A \in \mathrm{Comm}(\mathcal{C})$ one has by adjunction

$$\pi_0(A) = \pi_0(\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, A)) \simeq [1, A]_{\mathcal{C}} \simeq [A, A]_{A\text{-Mod}}.$$

Since A is the unit of the monoidal structure on the additive category $\mathrm{Ho}(A\text{-Mod})$, the composition of endomorphisms endows $\pi_0(A)$ with a multiplicative structure making it into a commutative ring. More generally, the category $\mathrm{Ho}(A\text{-Mod})$ has a natural structure of a graded category (i.e. has a natural enrichment into the symmetric monoidal category of \mathbb{N} -graded abelian groups), defined by

$$[M, N]_* := \bigoplus_i [S^i(N), M] \simeq \bigoplus_i [N, \Omega^i(M)],$$

where S^i is the i -fold iterated suspension functor, and Ω^i the i -fold iterated loop functor. Therefore, the graded endomorphism ring of the unit A has a natural structure of a graded commutative ring. As this endomorphism ring is naturally isomorphic to

$$\pi_*(A) := \bigoplus_i \pi_i(A),$$

we obtain this way a natural structure of a graded commutative ring on $\pi_*(A)$. This clearly defines a functor

$$\pi_* : \mathrm{Ho}(\mathrm{Comm}(\mathcal{C})) \longrightarrow G\mathrm{Comm},$$

from $\mathrm{Ho}(\mathrm{Comm}(\mathcal{C}))$ to the category of \mathbb{N} -graded commutative rings.

In the same way, for a commutative monoid $A \in \mathrm{Comm}(\mathcal{C})$, and a A -module M , one can define

$$\pi_*(M) := \pi_*(|M|) = \pi_*(\mathrm{Map}_{A\text{-Mod}}(A, M)),$$

which has a natural structure of a graded $\pi_*(A)$ -module. This defines a functor

$$\pi_* : \text{Ho}(A - \text{Mod}) \longrightarrow \pi_*(A) - \text{GMod},$$

from $\text{Ho}(A - \text{Mod})$ to the category of \mathbb{N} -graded $\pi_*(A)$ -modules.

PROPOSITION 1.2.9.1. *Let $A \in \text{Comm}(\mathcal{C})$ be a commutative monoid in \mathcal{C} , and $a \in \pi_0(A)$. There exists an epimorphism $A \longrightarrow A[a^{-1}]$, such that for any commutative A -algebra C , the simplicial set $\text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], C)$ is non-empty (and thus contractible) if and only if the image of a in $\pi_0(C)$ by the morphism $\pi_0(A) \rightarrow \pi_0(C)$ is an invertible element.*

PROOF. We represent the element a as a morphism in $\text{Ho}(A - \text{Mod})$ $a : A \longrightarrow A$. Taking the image of a by the left derived functor of the free commutative A -algebra functor (which is left Quillen)

$$F_A : A - \text{Mod} \longrightarrow A - \text{Comm}(\mathcal{C})$$

we find a morphism in $\text{Ho}(A - \text{Comm}(\mathcal{C}))$

$$a : \mathbb{L}F_A(A) \longrightarrow \mathbb{L}F_A(A).$$

As the model category $A - \text{Comm}(\mathcal{C})$ is a \mathbb{U} -combinatorial model category, one can apply the localization techniques of in order to invert any \mathbb{U} -small set of morphisms in $A - \text{Comm}(\mathcal{C})$ (see [Sm, Du2]). We let $L_a A - \text{Comm}(\mathcal{C})$ be the left Bousfield localization of $A - \text{Comm}(\mathcal{C})$ along the set of morphisms (with one element)

$$S_a := \{a : \mathbb{L}F_A(A) \longrightarrow \mathbb{L}F_A(A)\}.$$

We define $A[a^{-1}] \in \text{Ho}(A - \text{Comm}(\mathcal{C}))$ as a local model of A in $L_a A - \text{Comm}(\mathcal{C})$, the left Bousfield localization of $A - \text{Comm}(\mathcal{C})$ along S_a .

First of all, the S_a -local objects are the commutative A -algebras B such that the induced morphism

$$a^* : \text{Map}_{A - \text{Mod}}(A, B) \longrightarrow \text{Map}_{A - \text{Mod}}(A, B)$$

is an equivalence. Equivalently, the multiplication by $a \in \pi_0(A)$

$$\times a : \pi_*(B) \longrightarrow \pi_*(B)$$

is an isomorphism. This shows that the S_a -local objects are the commutative A -algebras $A \longrightarrow B$ such that the image of a by $\pi_0(A) \longrightarrow \pi_0(B)$ is invertible.

Suppose now that $C \in A - \text{Comm}(\mathcal{C})$ is such that $\text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], C)$ is not empty. The morphism $\pi_0(A) \longrightarrow \pi_0(C)$ then factors through $\pi_0(A[a^{-1}])$, and thus the image of a is invertible in $\pi_0(C)$. Therefore, C is a S_a -local object, and thus

$$\text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], C) \simeq \text{Map}_{A - \text{Comm}(\mathcal{C})}(A, C) \simeq *.$$

This implies that $A \longrightarrow A[a^{-1}]$ is an epimorphism. It only remain to prove that if $C \in A - \text{Comm}(\mathcal{C})$ is such that the image of a is invertible in $\pi_0(C)$, then $\text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], C)$ is non-empty. But such a C is a S_a -local object, and therefore

$$* \simeq \text{Map}_{A - \text{Comm}(\mathcal{C})}(A, C) \simeq \text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], C).$$

□

DEFINITION 1.2.9.2. *Let $A \in \text{Comm}(\mathcal{C})$ and $a \in \pi_0(A)$. The commutative A -algebra $A[a^{-1}]$ is called the standard localization of A with respect to a .*

A useful property of standard localizations is given by the following corollary of the proof of Prop. 1.2.9.1. In order to state it, we will use the following notations. For any $A \in \text{Comm}(\mathcal{C})$ and $a \in \pi_0(A)$, we represent a as a morphism in $\text{Ho}(A - \text{Mod})$, $A \rightarrow A$. Tensoring this morphism with M gives a morphism in $\text{Ho}(A - \text{Mod})$, denoted by

$$\times a : M \rightarrow M.$$

COROLLARY 1.2.9.3. *Let $A \in \text{Comm}(\mathcal{C})$, $a \in \pi_0(A)$ and let $f : A \rightarrow A[a^{-1}]$ be as in Prop. 1.2.9.1.*

(1) *The functor*

$$\text{Ho}(A[a^{-1}] - \text{Comm}(\mathcal{C})) \rightarrow \text{Ho}(A - \text{Comm}(\mathcal{C}))$$

is fully faithful, and its essential image consists of all commutative B -algebras such that the image of a is invertible in $\pi_0(B)$.

(2) *The functor*

$$f_* : \text{Ho}(A[a^{-1}] - \text{Mod}) \rightarrow \text{Ho}(A - \text{Mod})$$

is fully faithful and its essential image consists of all A -modules M such that the multiplication by a

$$\times a : M \rightarrow M$$

is an isomorphism in $\text{Ho}(A - \text{Mod})$.

PROOF. (1) The fact that the functor f_* is fully faithful is immediate as f is an epimorphism. The fact that the functor f_* takes its values in the required subcategory is clear by functoriality of the construction π_* .

Let B be a commutative A -algebra such that the image of a is invertible in $\pi_0(B)$. Then, we know that B is a S_a -local object. Therefore, one has

$$\text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], B) \simeq \text{Map}_{A - \text{Comm}(\mathcal{C})}(A, B) \simeq *,$$

showing that B is in the image of f_* .

(2) The fact that f_* is fully faithful follows from Prop. 1.2.6.4 (1). Let $M \in A[a^{-1}] - \text{Mod}$ and let us prove that the morphism $\times a : M \rightarrow M$ is an isomorphism in $\text{Ho}(A - \text{Mod})$. Using that $A \rightarrow A[a^{-1}]$ is an epimorphism, one finds $M \simeq M \otimes_A^L A[a^{-1}]$, which reduces the problem to the case where $M = A[a^{-1}]$. But then, the morphism $\times a : A[a^{-1}] \rightarrow A[a^{-1}]$, as a morphism in $\text{Ho}(A[a^{-1}] - \text{Mod})$ lives in $[A[a^{-1}], A[a^{-1}]] \simeq \pi_0(A[a^{-1}])$, and correspond to the image of a by the morphism $\pi_0(A) \rightarrow \pi_0(A[a^{-1}])$, which is then invertible. In other words, $\times a : A[a^{-1}] \rightarrow A[a^{-1}]$ is an isomorphism.

Conversely, let M be an A -module such that the morphism $\times a : M \rightarrow M$ is an isomorphism in $\text{Ho}(A - \text{Mod})$. We need to show that the adjunction morphism

$$M \rightarrow M \otimes_A^L A[a^{-1}]$$

is an isomorphism. For this, we use that the morphism $A \rightarrow A[a^{-1}]$ can be constructed using a small object argument with respect to the horns over the morphism $a : \text{LF}_A(A) \rightarrow \text{LF}_A(A)$ (see [Hi, 4.2]). Therefore, a transfinite induction argument shows that it is enough to prove that the morphism induced by tensoring

$$a \otimes \text{Id} : \text{LF}_A(A) \otimes_A^L M \simeq \text{LF}_A(M) \rightarrow \text{LF}_A(A) \otimes_A^L M \simeq \text{LF}_A(M)$$

is an isomorphism in $\text{Ho}(A - \text{Mod})$. But this morphism is the image by the functor LF_A of the morphism $\times a : M \rightarrow M$, and is therefore an isomorphism. \square

PROPOSITION 1.2.9.4. *Let $A \in \text{Comm}(\mathcal{C})$, $a \in \pi_0(A)$ and $A \rightarrow A[a^{-1}]$ the standard localization with respect to a . Assume that the model category \mathcal{C} is finitely generated (in the sense of [Hol]).*

- (1) *The morphism $A \rightarrow A[a^{-1}]$ is a formal Zariski open immersion.*
- (2) *If 1 is a finitely presented object in \mathcal{C} , then $A \rightarrow A[a^{-1}]$ is a Zariski open immersion.*

PROOF. (1) It only remains to show that the morphism $A \rightarrow A[a^{-1}]$ is flat. In other words, we need to prove that the functor $M \mapsto M \otimes_A^L A[a^{-1}]$, preserves homotopy fiber sequences of A -modules.

The model category $A - \text{Mod}$ is \mathbb{U} -combinatorial and finitely generated. In particular, there exists a \mathbb{U} -small set G of ω -small cofibrant objects in $A - \text{Mod}$, such that a morphism $N \rightarrow P$ is an equivalence in $A - \text{Mod}$ if and only if for any $X \in G$ the induced morphism $\text{Map}_{A - \text{Mod}}(X, N) \rightarrow \text{Map}_{A - \text{Mod}}(X, P)$ is an isomorphism in $\text{Ho}(\text{SSet})$. Furthermore, filtered homotopy colimits preserve homotopy fiber sequences. For any A -module M , we let M_a be the transfinite homotopy colimit

$$M \xrightarrow{\times a} M \xrightarrow{\times a} \dots$$

where the morphism $\times a$ is composed with itself ω -times.

The functor $M \mapsto M_a$ commutes with homotopy fiber sequences. Therefore, it only remain to show that M_a is naturally isomorphic in $\text{Ho}(A - \text{Mod})$ to $M \otimes_A^L A[a^{-1}]$. For this, it is enough to check that the natural morphism $M \rightarrow M_a$ induces an isomorphism in $\text{Ho}(A - \text{Mod})$

$$M \otimes_A^L A[a^{-1}] \simeq (M_a) \otimes_A^L A[a^{-1}],$$

and that the natural morphism

$$(M_a) \otimes_A^L A[a^{-1}] \rightarrow M_a$$

is an isomorphism in $\text{Ho}(A - \text{Mod})$. The first assumption follows easily from the fact that $-\otimes_A^L A[a^{-1}]$ commutes with homotopy colimits, and the fact that $\times a : A[a^{-1}] \rightarrow A[a^{-1}]$ is an isomorphism in $\text{Ho}(A - \text{Mod})$. For the second assumption we use Cor. 1.2.9.3 (2), which tells us that it is enough to check that the morphism

$$\times a : M_a \rightarrow M_a$$

is an isomorphism in $\text{Ho}(A - \text{Mod})$. For this, we need to show that for any $X \in G$ the induced morphism

$$\text{Map}_{A - \text{Mod}}(X, M_a) \rightarrow \text{Map}_{A - \text{Mod}}(X, M_a)$$

is an isomorphism in $\text{Ho}(\text{SSet})$. But, as the objects in G are cofibrant and ω -small, $\text{Map}_{A - \text{Mod}}(X, -)$ commutes with ω -filtered homotopy colimits, and the morphism

$$\text{Map}_{A - \text{Mod}}(X, M_a) \rightarrow \text{Map}_{A - \text{Mod}}(X, M_a)$$

is then obviously an isomorphism in $\text{Ho}(\text{SSet})$ by the construction of M_a .

(2) When 1 is finitely presented, one has for any filtered diagram of commutative A -algebras B_i an isomorphism

$$\text{Colim}_i \pi_*(B_i) \simeq \pi_*(\text{Hocolim}_i B_i).$$

Using that $\text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], B)$ is either empty or contractible, depending whether or not a goes to a unit in $\pi_*(B)$, we easily deduce that

$$\text{Hocolim}_i \text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], B_i) \simeq \text{Map}_{A - \text{Comm}(\mathcal{C})}(A[a^{-1}], \text{Hocolim}_i B_i).$$

□

We can also show that the natural morphism $A \rightarrow A[a^{-1}]$ is a formally étale morphism in the sense of Def. 1.2.6.1.

PROPOSITION 1.2.9.5. *Let $A \in \text{Comm}(\mathcal{C})$ and $a \in \pi_0(A)$. Then, the natural morphism $A \rightarrow A[a^{-1}]$ is formally étale.*

PROOF. Let M be any $A[a^{-1}]$ -module. We need to show that the natural morphism

$$\text{Map}_{A-\text{Comm}(\mathcal{C})/A[a^{-1}]}(A[a^{-1}], A[a^{-1}] \oplus M) \rightarrow \text{Map}_{A-\text{Comm}(\mathcal{C})/A[a^{-1}]}(A, A[a^{-1}] \oplus M)$$

is an isomorphism in $\text{Ho}(\mathcal{S}\text{Set})$. Using the universal property of $A \rightarrow A[a^{-1}]$ given by Prop. 1.2.9.1 we see that it is enough to prove that for any $B \in \text{Comm}(\mathcal{C})$, and any B -module M the natural projection $\pi_0(B \oplus M) \rightarrow \pi_0(B)$ reflects invertible elements (i.e. an element in $\pi_0(B \oplus M)$ is invertible if and only its image in $\pi_0(B)$ is so). But, clearly, $\pi_0(B \oplus M)$ can be identified with the trivial square zero extension of the commutative ring $\pi_0(B)$ by $\pi_0(M)$, which implies the required result. \square

COROLLARY 1.2.9.6. *Assume that the model category \mathcal{C} is finitely presented, and that the unit $\mathbf{1}$ is finitely presented in \mathcal{C} . Then for any $A \in \text{Comm}(\mathcal{C})$ and $a \in \pi_0(A)$, the morphism $A \rightarrow A[a^{-1}]$ is an étale, flat epimorphism.*

PROOF. Put 1.2.9.4 and 1.2.9.5 together. \square

1.2.10. Zariski open immersions and perfect modules

Let A be a commutative monoid in \mathcal{C} and K be a perfect A -module in the sense of Def. 1.2.3.6. We are going to define a Zariski open immersion $A \rightarrow A_K$, which has to be thought as the complement of the support of the A -module K .

PROPOSITION 1.2.10.1. *Assume that \mathcal{C} is stable model category. Then there exists a formal Zariski open immersion $A \rightarrow A_K$, such that for any commutative A -algebra C , the simplicial set*

$$\text{Map}_{A-\text{Comm}(\mathcal{C})}(A_K, C)$$

*is non-empty (and thus contractible) if and only if $K \otimes_A^L C \simeq *$ in $\text{Ho}(\mathcal{C} - \text{Mod})$. If the unit $\mathbf{1}$ is furthermore finitely presented, then $A \rightarrow A_K$ is finitely presented and thus is a Zariski open immersion.*

PROOF. The commutative A -algebra is constructed using a left Bousfield localization of the model category $A - \text{Comm}(\mathcal{C})$, as done in the proof of Prop. 1.2.9.1.

We let I be a generating \mathcal{U} -small set of cofibrations in $A - \text{Comm}(\mathcal{C})$, and $K^\vee := \mathbb{R}\text{Hom}_{A-\text{Mod}}(K, A)$ be the dual of K in $\text{Ho}(A - \text{Mod})$. For any morphism $X \rightarrow Y$ in I , we consider the morphism of free commutative A -algebras

$$\text{LF}_A(K^\vee \otimes_A^L X) \rightarrow \text{LF}_A(K^\vee \otimes_A^L Y),$$

where $F_A : A - \text{Mod} \rightarrow A - \text{Comm}(\mathcal{C})$ is the left Quillen functor sending an A -module to the free commutative A -algebra it generates. When $X \rightarrow Y$ varies in I this gives a \mathcal{U} -small set of morphisms denoted by S_K in $A - \text{Comm}(\mathcal{C})$. We consider $L_K A - \text{Comm}(\mathcal{C})$, the left Bousfield localization of $A - \text{Comm}(\mathcal{C})$ along the set S_K . By definition, $A \rightarrow A_K$ is an S_K -local model of A in the localized model category $L_K A - \text{Comm}(\mathcal{C})$.

LEMMA 1.2.10.2. *The S_K -local objects in $L_K A - \text{Comm}(\mathcal{C})$ are precisely the commutative A -algebras B such that $K \otimes_A^L B \simeq *$ in $\text{Ho}(A - \text{Mod})$.*

PROOF. First of all, one has an adjunction isomorphism in $\text{Ho}(S\text{Set})$

$$\text{Map}_{A-\text{Comm}(\mathcal{C})}(\mathbb{L}F_A(K^\vee \otimes_A^{\mathbb{L}} X), B) \simeq \text{Map}_{A-\text{Mod}}(X, K \otimes_A^{\mathbb{L}} B).$$

This implies that an object $B \in A - \text{Comm}(\mathcal{C})$ is S_K -local if and only if for all morphism $X \rightarrow Y$ in I the induced morphism

$$\text{Map}_{A-\text{Mod}}(Y, K \otimes_A^{\mathbb{L}} B) \rightarrow \text{Map}_{A-\text{Mod}}(X, K \otimes_A^{\mathbb{L}} B)$$

is an isomorphism in $\text{Ho}(S\text{Set})$. As I is a set of generating cofibrations in $A - \text{Mod}$ this implies that B is S_K -local if and only if $K \otimes_A^{\mathbb{L}} C \simeq *$ in $\text{Ho}(A - \text{Mod})$. \square

We now finish the proof of proposition 1.2.10.1. First of all, Lem. 1.2.10.2 implies that for any commutative A -algebra B , if the mapping space $\text{Map}_{A-\text{Comm}(\mathcal{C})}(A_K, B)$ is non-empty then

$$B \otimes_A^{\mathbb{L}} K \simeq B \otimes_{A_K}^{\mathbb{L}} (A_K \otimes_A^{\mathbb{L}} K) \simeq *,$$

and thus B is an S_K -local object. This shows that if $\text{Map}_{A-\text{Comm}(\mathcal{C})}(A_K, B)$ is non-empty then one has

$$\text{Map}_{A-\text{Comm}(\mathcal{C})}(A_K, B) \simeq \text{Map}_{A-\text{Comm}(\mathcal{C})}(A, B) \simeq *.$$

In other words $A \rightarrow A_K$ is a formal Zariski open immersion by Cor. 1.2.6.6 and the stability assumption on \mathcal{C} . It only remains to prove that $A \rightarrow A_K$ is also finitely presented when $\mathbf{1}$ is a finitely presented object in \mathcal{C} .

For this, let $\{C_i\}_{i \in I}$ be a filtered diagram of commutative A -algebras and C be its homotopy colimit. By the property of $A \rightarrow A_K$, we need to show that if $K \otimes_A^{\mathbb{L}} C \simeq *$ then there is an $i \in I$ such that $K \otimes_A^{\mathbb{L}} C_i \simeq *$. For this, we consider the two elements Id and $*$ in $[K \otimes_A^{\mathbb{L}} C, K \otimes_A^{\mathbb{L}} C]_{C-\text{Mod}}$. As K is perfect and $\mathbf{1}$ is a finitely presented object, K is a finitely presented A -module by Prop. 1.2.3.7 (4). Therefore, one has

$$* \simeq [K \otimes_A^{\mathbb{L}} C, K \otimes_A^{\mathbb{L}} C] \simeq \text{Colim}_{i \in I} [K, K \otimes_A^{\mathbb{L}} C_i]_{A-\text{Mod}}.$$

As the two elements Id and $*$ becomes equal in the colimit, there is an i such that they are equal as elements in

$$[K, K \otimes_A^{\mathbb{L}} C_i]_{A-\text{Mod}} \simeq [K \otimes_A^{\mathbb{L}} C_i, K \otimes_A^{\mathbb{L}} C_i]_{C_i-\text{Mod}},$$

showing that $K \otimes_A^{\mathbb{L}} C_i \simeq *$ in $\text{Ho}(C_i - \text{Mod})$. \square

COROLLARY 1.2.10.3. Assume that \mathcal{C} is a stable model category, and let $f : A \rightarrow A_K$ be as in Prop. 1.2.10.1.

(1) The functor

$$\text{Ho}(A_K - \text{Comm}(\mathcal{C})) \rightarrow \text{Ho}(A - \text{Comm}(\mathcal{C}))$$

is fully faithful, and its essential image consists of all commutative B -algebras such that $B \otimes_A^{\mathbb{L}} K \simeq *$.

(2) The functor

$$f_* : \text{Ho}(A_K - \text{Mod}) \rightarrow \text{Ho}(A - \text{Mod})$$

is fully faithful and its essential image consists of all A -modules M such that $M \otimes_A^{\mathbb{L}} K \simeq *$.

PROOF. As the morphism $A \rightarrow A_K$ is an epimorphism, we know by Prop. 1.2.6.4 that the functors

$$\text{Ho}(A_K - \text{Comm}(\mathcal{C})) \rightarrow \text{Ho}(A - \text{Comm}(\mathcal{C})) \quad \text{Ho}(A_K - \text{Mod}) \rightarrow \text{Ho}(A - \text{Mod})$$

are both fully faithful. Furthermore, a commutative A -algebra B is in the essential image of the first one if and only if $\text{Map}_{A\text{-Comm}(\mathcal{C})}(A_K, B) \simeq *$, and therefore if and only if $B \otimes_A^L K \simeq *$. For the second functor, it's clear that if M is a A_K -module, then

$$M \otimes_A^L K \simeq M \otimes_{A_K}^L A_K \otimes_A^L K \simeq *.$$

Conversely, let M be an A -module such that $M \otimes_A^L K \simeq *$.

LEMMA 1.2.10.4. *For any A -module M , $K \otimes_A^L M \simeq *$ if and only if $K^\vee \otimes_A^L M \simeq *$.*

PROOF. Indeed, as K is perfect, K^\vee is a retract of $K^\vee \otimes_A^L K \otimes_A^L K^\vee$ in $\text{Ho}(A\text{-Mod})$. This implies that $K^\vee \otimes_A^L M$ is a retract of $K^\vee \otimes_A^L K \otimes_A^L K^\vee \otimes_A^L M$, showing that

$$(K \otimes_A^L M \simeq *) \Rightarrow (K^\vee \otimes_A^L M \simeq *).$$

By symmetry this proves the lemma. \square

We need to prove that the adjunction morphism

$$M \longrightarrow M \otimes_A^L A_K$$

is an isomorphism in $\text{Ho}(A\text{-Mod})$. For this, we use the fact that the morphism $A \longrightarrow A_K$ can be constructed using a small object argument on the set of horns on the set S_K (see [Hi, 4.2]). By a transfinite induction we are therefore reduced to show that for any morphism $X \longrightarrow Y$ in \mathcal{C} , the natural morphism

$$\text{LF}_A(K^\vee \otimes_A^L X) \otimes_A^L M \longrightarrow \text{LF}_A(K^\vee \otimes_A^L Y) \otimes_A^L M$$

is an isomorphism in $\text{Ho}(\mathcal{C})$. But, using that $M \otimes_A^L K \simeq *$ and lemma 1.2.10.4, this is clear by the explicit description of the functor F_A . \square

1.2.11. Stable modules

We recall that as \mathcal{C} is a pointed model category one can define a suspension functor (see [Ho1, §7])

$$\begin{aligned} S: \text{Ho}(\mathcal{C}) &\longrightarrow \text{Ho}(\mathcal{C}) \\ X &\longmapsto S(X) := * \coprod_X^L *. \end{aligned}$$

This functor possesses a right adjoint, the loop functor

$$\begin{aligned} \Omega: \text{Ho}(\mathcal{C}) &\longrightarrow \text{Ho}(\mathcal{C}) \\ X &\longmapsto \Omega(X) := * \times_X^h *. \end{aligned}$$

We fix, once for all an object $S_C^1 \in \mathcal{C}$, which is a cofibrant model for $S(1) \in \text{Ho}(\mathcal{C})$. For any commutative monoid $A \in \text{Comm}(\mathcal{C})$, we let

$$S_A^1 := S_C^1 \otimes A \in A\text{-Mod}$$

be the free A -module on S_C^1 . It is a cofibrant object in $A\text{-Mod}$, which is a model for the suspension $S(A)$ (note that S_A^1 is cofibrant in $A\text{-Mod}$, but not in \mathcal{C} unless A is itself cofibrant in \mathcal{C}). The functor

$$S_A^1 \otimes_A -: A\text{-Mod} \longrightarrow A\text{-Mod}$$

has a right adjoint

$$\text{Hom}_A(S_A^1, -): A\text{-Mod} \longrightarrow A\text{-Mod}.$$

Furthermore, assumption 1.1.0.2 implies that $S_A^1 \otimes_A -$ is a left Quillen functor. We can therefore apply the general construction of [Ho2] in order to produce a model category $\text{Sp}^{S_A^1}(A\text{-Mod})$, of spectra in $A\text{-Mod}$ with respect to the left Quillen endofunctor $S_A^1 \otimes_A -$.

DEFINITION 1.2.11.1. Let $A \in \text{Comm}(\mathcal{C})$ be a commutative monoid in \mathcal{C} . The model category of stable A -modules is the model category $\text{Sp}^{S_A^1}(A - \text{Mod})$, of spectra in $A - \text{Mod}$ with respect to the left Quillen endo-functor

$$S_A^1 \otimes_A - : A - \text{Mod} \longrightarrow A - \text{Mod}.$$

It will simply be denoted by $\text{Sp}(A - \text{Mod})$, and its objects will be called stable A -modules.

Recall that objects in the category $\text{Sp}(A - \text{Mod})$ are families of objects $M_n \in A - \text{Mod}$ for $n \geq 0$, together with morphisms $\sigma_n : S_A^1 \otimes_A M_n \longrightarrow M_{n+1}$. Morphisms in $\text{Sp}(A - \text{Mod})$ are simply families of morphisms $f_n : M_n \rightarrow N_n$ commuting with the morphisms σ_n . One starts by endowing $\text{Sp}(A - \text{Mod})$ with the levelwise model structure, for which equivalences (resp. fibrations) are the morphisms $f : M_* \rightarrow N_*$ such that each morphism $f_n : M_n \rightarrow N_n$ is an equivalence in $A - \text{Mod}$ (resp. a fibration). The definitive model structure, called the stable model structure, is the left Bousfield localization of $\text{Sp}(A - \text{Mod})$ whose local objects are the stable modules $M_* \in \text{Sp}(A - \text{Mod})$ such that each induced morphism

$$M_n \longrightarrow \mathbb{R}\underline{\text{Hom}}_A(S_A^1, M_{n+1})$$

is an isomorphism in $\text{Ho}(A - \text{Mod})$. These local objects will be called Ω -stable A -modules. We refer to [Ho2] for details concerning the existence and the properties of this model structure.

There exists an adjunction

$$S_A : A - \text{Mod} \longrightarrow \text{Sp}(A - \text{Mod}) \quad A - \text{Mod} \longleftarrow \text{Sp}(A - \text{Mod}) : (-)_0,$$

where the right adjoint sends a stable A -module M_* to M_0 . The left adjoint is defined by $S_A(M)_n := (S_A^1)^{\otimes_A n} \otimes_A M$, with the natural transition morphisms. This adjunction is a Quillen adjunction, and can be derived into an adjunction on the level of homotopy categories

$$\text{LS}_A \simeq S_A : \text{Ho}(A - \text{Mod}) \longrightarrow \text{Ho}(\text{Sp}(A - \text{Mod}))$$

$$\text{Ho}(A - \text{Mod}) \longleftarrow \text{Ho}(\text{Sp}(A - \text{Mod})) : \mathbb{R}(-)_0.$$

Note that by 1.1.0.2, S_A preserves equivalences, so $\text{LS}_A \simeq S_A$. On the contrary, the functor $(-)_0$ does not preserve equivalences and must be derived on the right. In particular, the functor $S_A : \text{Ho}(A - \text{Mod}) \longrightarrow \text{Ho}(\text{Sp}(A - \text{Mod}))$ is not fully faithful in general.

LEMMA 1.2.11.2. (1) Assume that the suspension functor

$$S : \text{Ho}(\mathcal{C}) \longrightarrow \text{Ho}(\mathcal{C})$$

is fully faithful. Then, for any commutative monoid $A \in \text{Comm}(\mathcal{C})$, the functor

$$S_A : \text{Ho}(A - \text{Mod}) \longrightarrow \text{Ho}(\text{Sp}(A - \text{Mod}))$$

is fully faithful.

(2) If furthermore, \mathcal{C} is a stable model category then S_A is a Quillen equivalence.

PROOF. (1) As the adjunction morphism $M \longrightarrow S_A(M)_0$ is always an isomorphism in $A - \text{Mod}$, it is enough to show that for any $M \in A - \text{Mod}$ the stable A -module $S_A(M)$ is a Ω -stable A -module. For this, it is enough to show that for any $M \in A - \text{Mod}$, the adjunction morphism

$$M \longrightarrow \mathbb{R}\underline{\text{Hom}}_A(S_A^1, S_A^1 \otimes_A M)$$

is an isomorphism in $\text{Ho}(\mathcal{C})$. But, one has natural isomorphisms in $\text{Ho}(\mathcal{C})$

$$S_A^1 \otimes_A M \simeq S_C^1 \otimes M \simeq S(M)$$

$$\mathbb{R}\text{Hom}_A(S_A^1, S_A^1 \otimes_A M) \simeq \mathbb{R}\text{Hom}_1(S^1, S(M)) \simeq \Omega(S(M)).$$

This shows that the above morphism is in fact isomorphic in $\text{Ho}(\mathcal{C})$ to the adjunction morphism

$$M \longrightarrow \Omega(S(M))$$

which is an isomorphism by hypothesis on \mathcal{C} .

(2) As \mathcal{C} is a stable model category, the functor $S_C^1 \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ is a Quillen equivalence. This also implies that for any $A \in \text{Comm}(\mathcal{C})$, the functor $S_A^1 : A - \text{Mod} \longrightarrow A - \text{Mod}$ is a Quillen equivalence. We know by [Ho2] that $S_A : A - \text{Mod} \longrightarrow \text{Sp}(A - \text{Mod})$ is a Quillen equivalence. \square

The Quillen adjunction

$$S_A : A - \text{Mod} \longrightarrow \text{Sp}(A - \text{Mod}) \quad A - \text{Mod} \longleftarrow \text{Sp}(A - \text{Mod}) : (-)_0,$$

is furthermore functorial in A . Indeed, for $A \longrightarrow B$ a morphism in $\text{Comm}(\mathcal{C})$, one defines a functor

$$- \otimes_A B : \text{Sp}(A - \text{Mod}) \longrightarrow \text{Sp}(B - \text{Mod})$$

defined by

$$(M_* \otimes_A B)_n := M_n \otimes_A B.$$

The transitions morphisms are given by

$$S_B^1 \otimes_B (M_n \otimes_A B) \simeq (S_A^1 \otimes_A M_n) \otimes_A B \longrightarrow M_{n+1} \otimes_A B.$$

Clearly, the square of left Quillen functors

$$\begin{array}{ccc} A - \text{Mod} & \xrightarrow{S_A} & \text{Sp}(A - \text{Mod}) \\ \downarrow - \otimes_A B & & \downarrow - \otimes_A B \\ B - \text{Mod} & \xrightarrow{S_B} & \text{Sp}(B - \text{Mod}) \end{array}$$

commutes up to a natural isomorphism. So does the square of right Quillen functors

$$\begin{array}{ccc} B - \text{Mod} & \longrightarrow & \text{Sp}(B - \text{Mod}) \\ \downarrow & & \downarrow \\ A - \text{Mod} & \longrightarrow & \text{Sp}(A - \text{Mod}). \end{array}$$

Finally, using techniques of symmetric spectra, as done in [Ho2], it is possible to show that the homotopy category of stable A -modules inherits from $\text{Ho}(A - \text{Mod})$ a symmetric monoidal structure, still denoted by $- \otimes_A^L -$. This makes the homotopy category $\text{Ho}(\text{Sp}(A - \text{Mod}))$ into a closed symmetric monoidal category. In particular, for two stable A -modules M_* and N_* , one can define a stable A -modules of morphisms

$$\mathbb{R}\text{Hom}_A^{\text{Sp}}(M_*, N_*) \in \text{Ho}(\text{Sp}(A - \text{Mod})).$$

We now consider the category $(A - \text{Mod}_0^{\text{op}})^{\wedge}$ of pre-stacks over $A - \text{Mod}_0^{\text{op}}$, as defined in [HAGI]. Recall it is the category of \mathbb{V} -simplicial presheaves on $A - \text{Mod}_0^{\text{op}}$, and that its model structure is obtained from the projective levelwise model structure by a left Bousfield localization inverting the equivalences in $A - \text{Mod}$. The homotopy

category $\mathrm{Ho}((A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge)$ can be naturally identified with the full subcategory of $\mathrm{Ho}(\mathrm{Spr}(A - \mathrm{Mod}_0^{\mathrm{op}}))$ consisting of functors

$$F : A - \mathrm{Mod}_0 \longrightarrow \mathrm{SSet}_\forall,$$

sending equivalences of A -modules to equivalences of simplicial sets.

We define a functor

$$\begin{array}{ccc} \underline{h}_s^- : \mathrm{Sp}(A - \mathrm{Mod})^{\mathrm{op}} & \longrightarrow & (A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge \\ M_* & \mapsto & \underline{h}_s^{M_*} \end{array}$$

by

$$\begin{array}{ccc} \underline{h}_s^{M_*} : A - \mathrm{Mod}_0 & \longrightarrow & \mathrm{SSet}_\forall \\ N & \mapsto & \mathrm{Hom}(M_*, \Gamma_*(S_A(N))), \end{array}$$

where Γ_* is a simplicial resolution functor on the model category $\mathrm{Sp}(A - \mathrm{Mod})$.

Finally we will need some terminology. A stable A -module $M_* \in \mathrm{Ho}(\mathrm{Sp}(A - \mathrm{Mod}))$ is called *0-connective*, if it is isomorphic to some $S_A(M)$ for an A -module $M \in \mathrm{Ho}(A - \mathrm{Mod}_0)$. By induction, for an integer $n > 0$, a stable A -module $M_* \in \mathrm{Ho}(\mathrm{Sp}(A - \mathrm{Mod}))$ is called *$(-n)$ -connective*, if it is isomorphic to $\Omega(M'_*)$ for some $(-n-1)$ -connective stable A -module M'_* (here Ω is the loop functor on $\mathrm{Ho}(\mathrm{Sp}(A - \mathrm{Mod}))$). Note that if the suspension functor is fully faithful, connective stable modules are exactly connective objects with respect to the natural t -structure on A -modules.

PROPOSITION 1.2.11.3. *For any $A \in \mathrm{Comm}(\mathcal{C})$, the functor \underline{h}_s^- has a total right derived functor*

$$\mathbb{R}\underline{h}_s^- : \mathrm{Ho}(\mathrm{Sp}(A - \mathrm{Mod}))^{\mathrm{op}} \longrightarrow \mathrm{Ho}((A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge),$$

which commutes with homotopy limits¹. If the suspension functor

$$S : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

is fully faithful, and if $A \in \mathcal{A}$, then for any integer $n \geq 0$, the functor $\mathbb{R}\underline{h}_s^-$ is fully faithful when restricted to the full subcategory of $(-n)$ -connective objects.

PROOF. As $S_A : A - \mathrm{Mod}_0 \longrightarrow \mathrm{Sp}(A - \mathrm{Mod})$ preserves equivalences, one checks easily that $\underline{h}_s^{M_*}$ is a fibrant object in $(A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge$ when M_* is cofibrant in $\mathrm{Sp}(A - \mathrm{Mod})$. This easily implies that $M_* \mapsto \underline{h}_s^{Q M_*}$ is a right derived functor for \underline{h}_s^- , and the standard properties of mapping spaces imply that it commutes with homotopy limits.

We now assume that the suspension functor

$$S : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

is fully faithful, and that $A \in \mathcal{A}$. Let $n \geq 0$ be an integer.

Let $S^n : A - \mathrm{Mod} \longrightarrow A - \mathrm{Mod}$ be a left Quillen functor which is a model for the suspension functor iterated n times (e.g. $S^n(N) := S_A^n \otimes_A N$, where $S_A^n := (S_A^1)^{\otimes_A n}$). There is a pullback functor

$$(S^n)^* : \mathrm{Ho}((A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge) \longrightarrow \mathrm{Ho}((A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge)$$

defined by $(S^n)^*(F)(N) := F(S^n(N))$ for any $N \in A - \mathrm{Mod}_0$ (note that S^n stabilizes the subcategory $A - \mathrm{Mod}_0$ because of our assumption 1.1.0.6). For an $(-n)$ -connective object M_* , we claim there exists a natural isomorphism in $\mathrm{Ho}((A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge)$ between

¹This makes sense as the functor $\mathbb{R}\underline{h}_s^-$ is naturally defined on the level of the Dwyer-Kan simplicial localizations with respect to equivalences

$$L\mathrm{Sp}(A - \mathrm{Mod})^{\mathrm{op}} \longrightarrow L((A - \mathrm{Mod}_0^{\mathrm{op}})^\wedge).$$

$\mathbb{R}h_s^{M*}$ and $(S^n)^*(\mathbb{R}h^M)$, where $\mathbb{R}h^M$ is the value at M of the restricted Yoneda embedding

$$\mathbb{R}h_0^M : \text{Ho}(A - \text{Mod}) \longrightarrow \text{Ho}((A - \text{Mod}_0^{\text{op}})^\wedge).$$

Indeed, let us write M as $\Omega^n(S_A(M))$ for some object $M \in \text{Ho}(A - \text{Mod})$, where Ω^n is the loop functor of $\text{Sp}(A - \text{Mod})$, iterated n times. Then, using our lemma Lem. 1.2.11.2, for any $N \in \text{Ho}(A - \text{Mod})$, we have natural isomorphisms in $\text{Ho}(\text{SSet})$

$$\begin{aligned} \text{Map}_{\text{Sp}(A - \text{Mod})}(M_*, S_A(N)) &\simeq \text{Map}_{\text{Sp}(A - \text{Mod})}(S_A(M), S^n(S_A(N))) \simeq \\ &\simeq \text{Map}_{A - \text{Mod}}(M, S^n(N)), \end{aligned}$$

where S^n denotes the suspension functor iterated n -times. Using that A is \mathcal{C}_0 -good, this shows that

$$\mathbb{R}h_s^{M*} \simeq (S^n)^*(\mathbb{R}h_0^M).$$

Moreover, for any $(-n)$ -connective objects M_* and N_* in $\text{Ho}(\text{Sp}(A - \text{Mod}))$ one has natural isomorphisms in $\text{Ho}(\text{SSet})$

$$\text{Map}_{\text{Sp}(A - \text{Mod})}(M_*, N_*) \simeq \text{Map}_{A - \text{Mod}}(M, N)$$

where $M_* \simeq \Omega^n(S_A(M))$ and $N_* \simeq \Omega^n(S_A(N))$. We are therefore reduced to show that for any A -modules M and N in $\text{Ho}(A - \text{Mod})$, the natural morphism in $\text{Ho}(\text{SSet})$

$$\text{Map}_{A - \text{Mod}}(M, N) \longrightarrow \text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}((S^n)^*(\mathbb{R}h_0^N), (S^n)^*(\mathbb{R}h_0^M))$$

is an isomorphism. To see this, we define a morphism in the opposite direction in the following way. Taking the n -th loop functor on each side gives a morphism

$$\begin{aligned} &\text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}((S^n)^*(\mathbb{R}h_0^N), (S^n)^*(\mathbb{R}h_0^M)) \rightarrow \\ &\longrightarrow \text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}(\Omega^n(S^n)^*(\mathbb{R}h_0^N), \Omega^n(S^n)^*(\mathbb{R}h_0^M)). \end{aligned}$$

Moreover, there are isomorphisms

$$\Omega^n \text{Map}_{A - \text{Mod}}(M, S^n(P)) \simeq \text{Map}_{A - \text{Mod}}(M, \Omega^n(S^n(P))) \simeq \text{Map}_{A - \text{Mod}}(M, P) \simeq \mathbb{R}h_0^M(P),$$

showing that there exists a natural isomorphism in $\text{Ho}((A - \text{Mod}_0^{\text{op}})^\wedge)$ between $\Omega^n(S^n)^*(\mathbb{R}h_0^M)$ and $\mathbb{R}h_0^M$. One therefore gets a morphism

$$\begin{aligned} &\text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}((S^n)^*(\mathbb{R}h_0^N), (S^n)^*(\mathbb{R}h_0^M)) \rightarrow \\ &\longrightarrow \text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}(\Omega^n(S^n)^*(\mathbb{R}h_0^N), \Omega^n(S^n)^*(\mathbb{R}h_0^M)) \simeq \\ &\simeq \text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}(\mathbb{R}h_0^N, \mathbb{R}h_0^M). \end{aligned}$$

Using that A is \mathcal{C}_0 -good we get the required morphism

$$\text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}((S^n)^*(\mathbb{R}h_0^N), (S^n)^*(\mathbb{R}h_0^M)) \longrightarrow \text{Map}_{A - \text{Mod}}(M, N),$$

and it is easy to check it is an inverse in $\text{Ho}(\text{SSet})$ to the natural morphism

$$\text{Map}_{A - \text{Mod}}(M, N) \longrightarrow \text{Map}_{(A - \text{Mod}_0^{\text{op}})^\wedge}((S^n)^*(\mathbb{R}h_0^N), (S^n)^*(\mathbb{R}h_0^M)).$$

□

1.2.12. Descent for modules and stable modules

In this last section we present some definitions concerning descent for modules and stable modules in our general context. In a few words, a co-augmented co-simplicial object $A \rightarrow B_*$ in $\text{Comm}(\mathcal{C})$, is said to *have the descent property for modules* (resp. *for stable modules*) if the homotopy theory of A -modules (resp. of stable A -modules) is equivalent to the homotopy theory of certain co-simplicial B_* -modules (resp. stable B_* -modules). From the geometric, dual point of view, the object $A \rightarrow B_*$ should be thought as an augmented simplicial space $Y_* \rightarrow X$, and having the descent property essentially means that the theory of sheaves on X is equivalent to the theory of certain sheaves on Y_* (see [SGA4-II, Exp. V^{bis}] for more details on this point of view).

The purpose of this section is only to introduce the basic set-up for descent that will be used later to state that the theory of quasi-coherent modules is local with respect to the topology, or in other words that hypercovers have the descent property for modules. This is a very important property allowing local-to-global arguments.

Let A_* be a co-simplicial object in the category $\text{Comm}(\mathcal{C})$ of commutative monoids. Therefore, A_* is given by a functor

$$\begin{array}{ccc} \Delta & \longrightarrow & \text{Comm}(\mathcal{C}) \\ [n] & \longmapsto & A_n. \end{array}$$

A co-simplicial A_* -module M_* is by definition the following datum.

- A A_n -module $M_n \in A_n - \text{Mod}$ for any $n \in \Delta$.
- For any morphism $u : [n] \rightarrow [m]$ in Δ , a morphism of A_n -modules $\alpha_u : M_n \rightarrow M_m$, such that $\alpha_v \circ \alpha_u = \alpha_{v \circ u}$ for any $[n] \xrightarrow{u} [m] \xrightarrow{v} [p]$ in Δ .

In the same way, a morphism of co-simplicial A_* -modules $f : M_* \rightarrow N_*$ is the data of morphisms $f_n : M_n \rightarrow N_n$ for any n , commuting with the α 's

$$\alpha_u^N \circ f_n = f_m \circ \alpha_u^M$$

for any $u : [n] \rightarrow [m]$.

The co-simplicial A_* -modules and morphisms of A_* -modules form a category, denoted by $csA_* - \text{Mod}$. It is furthermore a U-combinatorial model category for which the equivalences (resp. fibrations) are the morphisms $f : M_* \rightarrow N_*$ such that each $f_n : M_n \rightarrow N_n$ is an equivalence (resp. a fibration) in $A_n - \text{Mod}$.

Let A be a commutative monoid, and B_* be a co-simplicial commutative A -algebra. We can also consider B_* as a co-simplicial commutative monoid together with a co-augmentation

$$A \rightarrow B_*$$

which is a morphism of co-simplicial objects when A is considered as a constant co-simplicial object. As a co-simplicial commutative monoid A possesses a category of co-simplicial A_* -modules $csA - \text{Mod}$ which is nothing else than the model category of co-simplicial objects in $A - \text{Mod}$ with its projective levelwise model structure.

For a co-simplicial A -module M_* , we define a co-simplicial B_* -module $B_* \otimes_A M_*$ by the formula

$$(B_* \otimes_A M_*) := B_n \otimes_A M_n,$$

and for which the transitions morphisms are given by the one of B_* and of M_* . This construction defines a functor

$$B_* \otimes_A - : csA - \text{Mod} \rightarrow csB_* - \text{Mod},$$

which has a right adjoint

$$csB_* - Mod \longrightarrow csA - Mod.$$

This right adjoint, sends a B_* -module M_* to its underlying co-simplicial A -module. Clearly, this defines a Quillen adjunction. There exists another adjunction

$$ct : Ho(A - Mod) \longrightarrow Ho(csA - Mod) \quad Ho(A - Mod) \longleftarrow Ho(csA - Mod) : Holim,$$

where $Holim$ is defined as the total right derived functor of the functor \lim (see [Hi, 8.5], [DS, 10.13]).

Composing these two adjunctions gives a new adjunction

$$B_* \otimes_A^L - : Ho(A - Mod) \longrightarrow Ho(csB_* - Mod) \quad Ho(A - Mod) \longleftarrow Ho(csB_* - Mod) : \int.$$

DEFINITION 1.2.12.1. (1) Let B_* be a co-simplicial commutative monoid and M_* be a co-simplicial B_* -module. We say that M_* is homotopy cartesian if for any $u : [n] \rightarrow [m]$ in Δ the morphism, induced by α_u ,

$$M_n \otimes_{B_n}^L B_m \longrightarrow M_m$$

is an isomorphism in $Ho(B_m - Mod)$.

(2) Let A be a commutative monoid and B_* be a co-simplicial commutative A -algebra. We say that the co-augmentation morphism $A \rightarrow B_*$ satisfies the descent condition, if in the adjunction

$$B_* \otimes_A^L - : Ho(A - Mod) \longrightarrow Ho(csB_* - Mod) \quad Ho(A - Mod) \longleftarrow Ho(csB_* - Mod) : \int$$

the functor $B_* \otimes_A^L -$ is fully faithful and induces an equivalence between $Ho(A - Mod)$ and the full subcategory of $Ho(csB_* - Mod)$ consisting of homotopy cartesian objects.

REMARK 1.2.12.2. If a morphism $A \rightarrow B_*$ satisfies the descent condition, then so does any co-simplicial object equivalent to it (as a morphism in the model category of co-simplicial commutative monoids).

LEMMA 1.2.12.3. Let $A \rightarrow B_*$ be a co-augmented co-simplicial commutative monoid in $Comm(\mathcal{C})$ which satisfies the descent condition. Then, the natural morphism

$$A \longrightarrow Holim_n B_n$$

is an isomorphism in $Ho(Comm(\mathcal{C}))$.

PROOF. This is clear as $\int B_* = Holim_n B_n$. □

We now pass to descent for stable modules. For this, let again A_* be a co-simplicial object in the category $Comm(\mathcal{C})$ of commutative monoids. A co-simplicial stable A_* -module M_* is by definition the following datum.

- A stable A_n -module $M_n \in Sp(A_n - Mod)$ for any $n \in \Delta$.
- For any morphism $u : [n] \rightarrow [m]$ in Δ , a morphism of stable A_n -modules $\alpha_u : M_n \rightarrow M_m$, such that $\alpha_v \circ \alpha_u = \alpha_{v \circ u}$ for any $[n] \xrightarrow{u} [m] \xrightarrow{v} [p]$ in Δ .

In the same way, a morphism of co-simplicial stable A_* -modules $f : M_* \rightarrow N_*$ is the data of morphisms $f_n : M_n \rightarrow N_n$ in $Sp(A_n - Mod)$ for any n , commuting with the α 's

$$\alpha_u^N \circ f_n = f_m \circ \alpha_u^M$$

for any $u : [n] \rightarrow [m]$.

The co-simplicial stable A_* -modules and morphisms of A_* -modules form a category, denoted by $Sp(csA_* - Mod)$. It is furthermore a U-combinatorial model category for which the equivalences (resp. fibrations) are the morphisms $f : M_* \rightarrow N_*$ such that each $f_n : M_n \rightarrow N_n$ is an equivalence (resp. a fibration) in $Sp(A_n - Mod)$. We note that $Sp(csA_* - Mod)$ is also naturally equivalent as a category to the category of S^1_A -spectra in $csA_* - Mod$, hence the notation $Sp(csA_* - Mod)$ is not ambiguous.

Let A be a commutative monoid, and B_* be a co-simplicial commutative A -algebra. We can also consider B_* as a co-simplicial commutative monoid together with a co-augmentation

$$A \rightarrow B_*,$$

which is a morphism of co-simplicial objects when A is considered as a constant co-simplicial object. As a co-simplicial commutative monoid A possesses a category of co-simplicial stable A_* -modules $Sp(csA - Mod)$ which is nothing else than the model category of co-simplicial objects in $Sp(A - Mod)$ with its projective levelwise model structure.

For a co-simplicial stable A -module M_* , we define a co-simplicial stable B_* -module $B_* \otimes_A M_*$ by the formula

$$(B_* \otimes_A M_*) := B_n \otimes_A M_n,$$

and for which the transitions morphisms are given by the one of B_* and of M_* . This construction defines a functor

$$B_* \otimes_A - : Sp(csA - Mod) \rightarrow Sp(csB_* - Mod),$$

which has a right adjoint

$$Sp(csB_* - Mod) \rightarrow Sp(csA - Mod).$$

This right adjoint, sends a co-simplicial stable B_* -module M_* to its underlying co-simplicial stable A -module. Clearly, this is a Quillen adjunction. One also has an adjunction

$$ct : Ho(Sp(A - Mod)) \rightarrow Ho(Sp(csA - Mod))$$

$$Ho(Sp(A - Mod)) \leftarrow Ho(Sp(csA - Mod)) : Holim,$$

where $Holim$ is defined as the total right derived functor of the functor \lim (see [Hi, 8.5], [DS, 10.13]).

Composing these two adjunctions gives a new adjunction

$$B_* \otimes_A^L - : Ho(Sp(A - Mod)) \rightarrow Ho(Sp(csB_* - Mod))$$

$$Ho(Sp(A - Mod)) \leftarrow Ho(Sp(csB_* - Mod)) : \int.$$

DEFINITION 1.2.12.4. (1) Let B_* be a co-simplicial commutative monoid and M_* be a co-simplicial stable B_* -module. We say that M_* is homotopy cartesian if for any $u : [n] \rightarrow [m]$ in Δ the morphism, induced by α_u ,

$$M_n \otimes_{B_n}^L B_m \rightarrow M_m$$

is an isomorphism in $Ho(Sp(B_m - Mod))$.

(2) Let A be a commutative monoid and B_* be a co-simplicial commutative A -algebra. We say that the co-augmentation morphism $A \rightarrow B_*$ satisfies the stable descent condition, if in the adjunction

$$B_* \otimes_A^L - : Ho(Sp(A - Mod)) \rightarrow Ho(Sp(csB_* - Mod))$$

$$Ho(Sp(A - Mod)) \leftarrow Ho(Sp(csB_* - Mod)) : \int$$

the functor $B_* \otimes_A^L -$ is fully faithful and induces an equivalence between $\text{Ho}(Sp(A - \text{Mod}))$ and the full subcategory of $\text{Ho}(Sp(cs B_* - \text{Mod}))$ consisting of homotopy cartesian objects.

The following proposition insures that the unstable descent condition implies the stable one under certain conditions.

PROPOSITION 1.2.12.5. *Let A be a commutative monoid and B_* be a co-simplicial commutative A -algebra, such that $A \rightarrow B_*$ satisfies the descent condition. Assume that the two following conditions are satisfied.*

- (1) *The suspension functor $S : \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{C})$ is fully faithful.*
- (2) *For any n , the morphism $A \rightarrow B_n$ is flat in the sense of Def. 1.2.4.1.*

Then $A \rightarrow B_$ satisfies the stable descent property.*

PROOF. The second condition insures that the diagram

$$\begin{array}{ccc} \text{Ho}(Sp(A - \text{Mod})) & \longrightarrow & \text{Ho}(Sp(cs B_* - \text{Mod})) \\ \downarrow & & \downarrow \\ \text{Ho}(A - \text{Mod}) & \longrightarrow & \text{Ho}(cs B_* - \text{Mod}) \end{array}$$

commutes up to a natural isomorphism (here the vertical functors are the right adjoint to the suspensions inclusion functors, and send a spectrum to its 0-th level). So, for any $M \in Sp(A - \text{Mod})$, the adjunction morphism

$$M \rightarrow \int (B_* \otimes_A^L M)$$

is easily seen to induces an isomorphism on the 0-th level objects in $\text{Ho}(A - \text{Mod})$. As this is true for any M , and in particular for the suspensions of M , we find that this adjunction morphism induces isomorphisms on each n -th level objects in $\text{Ho}(A - \text{Mod})$. Therefore it is an isomorphism. In the same way, one proves that for any cartesian object $M_* \in \text{Ho}(Sp(cs B_* - \text{Mod}))$ the adjunction morphism

$$B_* \otimes_A \int (M_*) \rightarrow M_*$$

is an isomorphism. □

1.2.13. Comparison with the usual notions

In this last section we present what the general notions introduced before give, when \mathcal{C} is the model category of k -modules with the trivial model structure. The other non trivial examples will be given at the beginning of the various chapters §2.2, §2.3 and §2.4 where the case of simplicial modules, complexes and symmetric spectra will be studied.

Let k be a commutative ring in \mathcal{U} , and we let \mathcal{C} be the category of k -modules belonging to the universe \mathcal{U} , endowed with its trivial model structure for which equivalences are isomorphisms and all morphisms are fibrations and cofibrations. The category \mathcal{C} is then a symmetric monoidal model category for the tensor product $- \otimes_k -$. Furthermore, all of our assumption 1.1.0.1, 1.1.0.3, 1.1.0.2 and 1.1.0.4 are satisfied. The category $\text{Comm}(\mathcal{C})$ is of course the category of commutative k -algebras in \mathcal{U} . The category $A - \text{Mod}$ is the usual category of A -modules in \mathcal{U} together with its trivial model structure. In the same way, for any commutative k -algebra A , the category $A - \text{Mod}$ is the usual category of A -modules in \mathcal{U} together with its trivial model structure. We set $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{A} = \text{Comm}(\mathcal{C})$. The notions we

have presented before restrict essentially to the usual notions of algebraic geometry with some remarkable caveats.

- For any morphism of commutative k -algebras $A \rightarrow B$ and any B -module M , the simplicial set $\mathrm{Der}_A(B, M)$ is discrete and naturally isomorphic to the set $\mathrm{Der}_A(B, M)$ of derivations from B to M over A . Equivalently, the B -module $\mathbb{L}_{B/A}$ (defined in Def. 1.2.1.5) is simply the usual B -module of Kähler differentials $\Omega_{B/A}^1$. Note that this $\mathbb{L}_{B/A}$ is *not* the Quillen-Illusie cotangent complex of $A \rightarrow B$.
- For any morphism of commutative k -algebras $A \rightarrow B$ the natural morphism $B \rightarrow \mathrm{THH}(B/A)$ is always an isomorphism. Indeed

$$\mathrm{THH}(A) \simeq A \otimes_{A \otimes_k A} A \simeq A.$$

- A morphism of commutative k -algebras $A \rightarrow B$ is finitely presented in the sense of Def. 1.2.3.1 if and only if it is a finitely presented A -algebra in the usual sense. In the same way, finitely presented objects in $A\text{-Mod}$ in the sense of Def. 1.2.3.1 are the finitely presented A -modules. Also, perfect objects in $A\text{-Mod}$ are the projective A -modules of finite type.
- A morphism $A \rightarrow B$ of commutative k -algebras is a formal covering if and only if it is a faithful morphism of rings.
- Let $f: A \rightarrow B$ be a morphism of commutative k -algebras, and $\mathrm{Spec} B \rightarrow \mathrm{Spec} A$ the associated morphism of schemes. We have the following comparison board.

In the sense of Def. 1.2.6.1, 1.2.6.7, 1.2.7.1	As a morphism of affine schemes
---	--

<i>epimorphism</i>	<i>monomorphism</i>
<i>flat</i>	<i>flat</i>
<i>formal Zariski open immersion</i>	<i>flat monomorphism</i>
<i>Zariski open immersion</i>	<i>open immersion</i>
<i>(formally) unramified</i>	<i>(formally) unramified</i>
<i>formally thh – étale</i>	<i>always satisfied</i>
<i>formally i – smooth</i>	<i>always satisfied</i>
<i>thh – étale</i>	<i>finitely presented</i>

The reader will immediately notice the absence of (formally) smooth and (formally) étale maps in the previous table. This is essentially due to the fact that there are no easy *general* characterizations of this maps in terms of the module of Kähler differentials alone (which in this trivial model structure context is our cotangent complex). On the contrary such characterizations do exist in terms of the “correct” cotangent complex which is the Quillen-Illusie one. But this correct cotangent complex of a morphism of usual commutative k -algebras $A \rightarrow B$ will appear as the cotangent complex according to our definition 1.2.1.5 only if we consider this morphism in the category of simplicial k -modules, i.e. if we replace the category \mathcal{C} of k -modules with the category of simplicial k -modules. In other words our definitions of (formally) smooth and (formally) étale maps reduce to the usual ones (e.g. [EGAIV]) between commutative k -algebras only if we consider them in the context of *derived algebraic geometry* (see §2.2). This is consistent with the general philosophy that some aspects of usual algebraic

geometry, especially those related to infinitesimal lifting properties and deformation theory, are conceptually more transparent in (and actually already a part of) derived algebraic geometry. See also Remark 2.1.2.2 for another instance of this point of view.

- For commutative monoid A , one has $\pi_*(A) = A$, and for any $a \in A$ the commutative A -algebra $A \rightarrow A[a^{-1}]$ is the usual localization of A inverting a .
- One has that $A_K \simeq 0$ for any projective A -module K of constant finite rank $n > 0$ (for example if the scheme $\text{Spec } A$ is connected and K is non zero).

In this work, the notion of a *geometric stack* (the definition we present here is a generalization of the original notion of geometric n -stack introduced by G. Laumon in [83]). As already remarked in [83], the notion of a *geometric stack* only depends on a topology on the opposite category of *crystalline stacks* $\text{Cris}(C)^{\text{op}}$, and on a class \mathcal{P} of morphisms. Roughly speaking, *geometric stacks* are the stacks obtained by taking quotients of representable stacks by some equivalence relations in \mathcal{P} . By choosing different classes \mathcal{P} one gets different notions of geometric stacks. For example, in the classical situation where $C = \mathbb{Z} = \text{Mod}$, and the topology is étale, to be the étale topology (Deligne-Mumford algebras), stacks correspond to the case where \mathcal{P} is the class of étale morphisms, whereas Artin algebras correspond to the case where \mathcal{P} is the class of smooth morphisms. We think it is important to stress the choice of the class \mathcal{P} open in the general definition, as that it can be specified differently depending of the kind of objects one is willing to consider.

From the second section on, we will fix a HA context $(\mathcal{C}, \text{Co}, A)$, in the sense of Def. 1.1.3.11.

1.3.1. Reminders on model topol

To make the paper essentially self-contained, we briefly summarize in this subsection the basic notions and results of the theory of stacks in homotopical contexts as exposed in [HAG-I]. We will limit ourselves to recall only the topics that will be needed in the sequel; the reader is addressed to [HAG-I] for further details and the proofs.

Let \mathcal{M} be a (small) model category, and let $\mathcal{W}(\mathcal{M})$ be class of weak equivalences. We let $\text{SPr}(\mathcal{M}) := \text{SPr}^{\text{cof}}(\mathcal{M})$ be the category of presheaves on \mathcal{M} with the projective model structure, i.e. with cofibrations and fibrations defined objectwise.

The model category of presheaves \mathcal{M}^{cof} on \mathcal{M} is the model category obtained as the $\mathcal{W}(\mathcal{M})$ -localization of $\text{SPr}(\mathcal{M})$ at $\{h_1, \dots, h_n : \mathcal{M} \rightarrow \mathcal{P}(\mathcal{M}) \mid \mathcal{P}(\mathcal{M}) := \text{cof}(\mathcal{M})\}$ is the (cofibrant) Yoneda embedding. The homotopy category $\text{Ho}(\mathcal{M}^{\text{cof}})$ can be identified with the full subcategory of $\text{Ho}(\text{SPr}(\mathcal{M}))$ consisting of those simplicial presheaves F on \mathcal{M} that preserve weak equivalences ([HAG-I, Def. 4.1.4]); say such simplicial presheaves (i.e. any object in $\text{Ho}(\mathcal{M}^{\text{cof}})$) will be called a *presheaf* on \mathcal{M} . \mathcal{M}^{cof} is cofibrant and U-combinatorial ([HAG-I, App. A]); simplicial model category and its derived simplicial Hom's will be denoted simply by $\mathcal{H}om$ (instead of $\mathcal{R}\mathcal{H}om$ in [HAG-I, §4.1]).

If $T_* : \mathcal{M} \rightarrow \mathcal{M}^{\text{cof}}$ is a cofibrant resolution functor for \mathcal{M} ([HAG-I, §4.1]), and we define

$$T : \mathcal{M} \rightarrow \mathcal{M}^{\text{cof}} : x \mapsto (T_* x \rightarrow \mathcal{H}om_{\mathcal{M}}(T_*(x), x))$$

CHAPTER 1.3

Geometric stacks: Basic theory

In this chapter, after a brief reminder of [HAGI], we present the key definition of this work, the notion of *n-geometric stack*. The definition we present here is a generalization of the original notion of *geometric n-stack* introduced by C. Simpson in [S3]. As already remarked in [S3], the notion of *n-geometric stack* only depends on a topology on the opposite category of commutative monoids $Comm(\mathcal{C})^{op}$, and on a class \mathbf{P} of morphisms. Roughly speaking, geometric stacks are the stacks obtained by taking quotient of representable stacks by some equivalence relations in \mathbf{P} . By choosing different classes \mathbf{P} one gets different notions of geometric stacks. For example, in the classical situation where $\mathcal{C} = \mathbb{Z} - Mod$, and the topology is chosen to be the étale topology, Deligne-Mumford algebraic stacks correspond to the case where \mathbf{P} is the class of étale morphisms, whereas Artin algebraic stacks correspond to the case where \mathbf{P} is the class of smooth morphisms. We think it is important to leave the choice of the class \mathbf{P} open in the general definition, so that it can be specialized differently depending of the kind of objects one is willing to consider.

From the second section on, we will fix a HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$, in the sense of Def. 1.1.0.11.

1.3.1. Reminders on model topoi

To make the paper essentially self-contained, we briefly summarize in this subsection the basic notions and results of the theory of stacks in homotopical contexts as exposed in [HAGI]. We will limit ourselves to recall only the topics that will be needed in the sequel; the reader is addressed to [HAGI] for further details and for proofs.

Let M be a \mathbf{U} -small model category, and $W(M)$ its class of weak equivalences. We let $SPr(M) := SSets^{M^{op}}$ be the category of simplicial presheaves on M with its projective model structure, i.e. with equivalences and fibrations defined objectwise.

The model category of *prestacks* M^\wedge on M is the model category obtained as the left Bousfield localization of $SPr(M)$ at $\{h_u \mid u \in W(M)\}$, where $h : M \rightarrow Pr(M) \hookrightarrow SPr(M)$ is the (constant) Yoneda embedding. The homotopy category $Ho(M^\wedge)$ can be identified with the full subcategory of $Ho(SPr(M))$ consisting of those simplicial presheaves F on M that preserve weak equivalences ([HAGI, Def. 4.1.4]); any such simplicial presheaf (i.e. any object in $Ho(M^\wedge)$) will be called a *prestack on M* . M^\wedge is a \mathbf{U} -cellular and \mathbf{U} -combinatorial ([HAGI, App. A]) simplicial model category and its derived simplicial *Hom*'s will be denoted simply by $\mathbb{R}Hom$ (denoted as $\mathbb{R}_w Hom$ in [HAGI, §4.1]).

If $\Gamma_* : M \rightarrow M^\Delta$ is a cofibrant resolution functor for M ([Hi, 16.1]), and we define

$$\underline{h} : M \rightarrow M^\wedge : x \longmapsto (\underline{h}_x : y \mapsto Hom_M(\Gamma_*(y), x)),$$

we have that \underline{h} preserves fibrant objects and weak equivalences between fibrant objects ([HAGI, Lem. 4.2.1]). Therefore we can right-derive \underline{h} to get a functor $\mathbb{R}\underline{h} := \underline{h} \circ R : \text{Ho}(M) \rightarrow \text{Ho}(M^\wedge)$, where R is a fibrant replacement functor in M ; $\mathbb{R}\underline{h}$ is in fact fully faithful ([HAGI, Thm. 4.2.3]) and is therefore called the (model) *Yoneda embedding* for the model category M ($\mathbb{R}\underline{h}$, as opposed to \underline{h} , does not depend, up to a unique isomorphism, on the choice of the cofibrant resolution functor Γ_*).

We also recall that the canonical morphism $h_x \rightarrow \mathbb{R}h_x$ is always an isomorphism in $\text{Ho}(M^\wedge)$ ([HAGI, Lem. 4.2.2]), and that with the notations introduced above for the derived simplicial *Hom*'s in M^\wedge , the model Yoneda lemma ([HAGI, Cor. 4.2.4]) is expressed by the isomorphisms in $\text{Ho}(SSets)$

$$\mathbb{R}\underline{\text{Hom}}(\mathbb{R}h_x, F) \simeq \mathbb{R}\underline{\text{Hom}}(h_x, F) \simeq F(x)$$

for any fibrant object F in M^\wedge .

A convenient homotopical replacement of the notion of a Grothendieck topology in the case of model categories, is the following ([HAGI, Def. 4.3.1])

DEFINITION 1.3.1.1. *A model (pre-)topology τ on a \mathbb{U} -small model category M , is the datum for any object $x \in M$, of a set $\text{Cov}_\tau(x)$ of subsets of objects in $\text{Ho}(M)/x$, called τ -covering families of x , satisfying the following three conditions*

- (1) (Stability) *For all $x \in M$ and any isomorphism $y \rightarrow x$ in $\text{Ho}(M)$, the one-element set $\{y \rightarrow x\}$ is in $\text{Cov}_\tau(x)$.*
- (2) (Composition) *If $\{u_i \rightarrow x\}_{i \in I} \in \text{Cov}_\tau(x)$, and for any $i \in I$, $\{v_{ij} \rightarrow u_i\}_{j \in J_i} \in \text{Cov}_\tau(u_i)$, the family $\{v_{ij} \rightarrow x\}_{i \in I, j \in J_i}$ is in $\text{Cov}_\tau(x)$.*
- (3) (Homotopy base change) *Assume the two previous conditions hold. For any $\{u_i \rightarrow x\}_{i \in I} \in \text{Cov}_\tau(x)$, and any morphism in $\text{Ho}(M)$, $y \rightarrow x$, the family $\{u_i \times_x^h y \rightarrow y\}_{i \in I}$ is in $\text{Cov}_\tau(y)$.*

A \mathbb{U} -small model category M together with a model pre-topology τ will be called a \mathbb{U} -small model site.

By [HAGI, Prop. 4.3.5] a model pre-topology τ on M induces and is essentially the same thing as a Grothendieck topology, still denoted by τ , on the homotopy category $\text{Ho}(M)$.

Given a model site (M, τ) we have, as in [HAGI, Thm. 4.6.1], a model category $M^{\sim, \tau}$ (\mathbb{U} -combinatorial and left proper) of *stacks* on the model site, which is defined as the left Bousfield localization of the model category M^\wedge of prestacks on M along a class H_τ of *homotopy τ -hypercovers* ([HAGI, 4.4, 4.5]). To any prestack F we can associate a sheaf π_0 of connected components on the site $(\text{Ho}(M), \tau)$ defined as the associated sheaf to the presheaf $x \mapsto \pi_0(F(x))$. In a similar way ([HAGI, Def. 4.5.3]), for any $i > 0$, any fibrant object $x \in M$, and any $s \in F(x)_0$, we can define a sheaf of homotopy groups $\pi_i(F, s)$ on the induced comma site $(\text{Ho}(M/x), \tau)$. The weak equivalences in $M^{\sim, \tau}$ turn out to be exactly the π_* -sheaves isomorphisms ([HAGI, Thm. 4.6.1]), i.e. those maps $u : F \rightarrow G$ in M^\wedge inducing an isomorphism of sheaves $\pi_0(F) \simeq \pi_0(G)$ on $(\text{Ho}(M), \tau)$, and isomorphisms $\pi_i(F, s) \simeq \pi_i(G, u(s))$ of sheaves on $(\text{Ho}(M/x), \tau)$ for any $i \geq 0$, for any choice of fibrant $x \in M$ and any base point $s \in F(x)_0$.

The left Bousfield localization construction defining $M^{\sim, \tau}$ yields a pair of adjoint Quillen functors

$$\text{Id} : M^\wedge \longrightarrow M^{\sim, \tau} \quad M^\wedge \longleftarrow M^{\sim, \tau} : \text{Id}$$

which induces an adjunction pair at the level of homotopy categories

$$a := \text{LId} : \text{Ho}(M^\wedge) \longrightarrow \text{Ho}(M^{\sim, \tau}) \quad \text{Ho}(M^\wedge) \longleftarrow \text{Ho}(M^{\sim, \tau}) : j := \text{RId}$$

where j is fully faithful.

- DEFINITION 1.3.1.2. (1) A stack on the model site (M, τ) is an object $F \in \text{SPR}(M)$ whose image in $\text{Ho}(M^\wedge)$ is in the essential image of the functor j .
 (2) If F and G are stacks on the model site (M, τ) , a morphism of stacks is a morphism $F \rightarrow G$ in $\text{Ho}(\text{SPR}(M))$, or equivalently in $\text{Ho}(M^\wedge)$, or equivalently in $\text{Ho}(M^{\sim, \tau})$.
 (3) A morphism of stacks $f : F \rightarrow G$ is a covering (or a cover or an epimorphism) if the induced morphism of sheaves

$$\pi_0(f) : \pi_0(F) \rightarrow \pi_0(G)$$

is an epimorphism in the category of sheaves.

Recall that a simplicial presheaf $F : M^{\text{op}} \rightarrow \text{SSet}_V$ is a stack if and only if it preserves weak equivalences and satisfy a τ -hyperdescent condition (descent, i.e. sheaf-like, condition with respect to the class H_τ of homotopy hypercovers): see [HAGI, Def. 4.6.5 and Cor. 4.6.3]. We will always consider $\text{Ho}(M^{\sim, \tau})$ embedded in $\text{Ho}(M^\wedge)$ embedded in $\text{Ho}(\text{SPR}(M))$, omitting in particular to mention explicitly the functor j above. With this conventions, the functor a above becomes an endofunctor of $\text{Ho}(M^\wedge)$, called the *associated stack functor* for the model site (M, τ) . The associated stack functor preserves finite homotopy limits and all homotopy colimits ([HAGI, Prop. 4.6.7]).

DEFINITION 1.3.1.3. A model pre-topology τ on M is sub-canonical if for any $x \in M$ the pre-stack Rh_x is a stack.

$M^{\sim, \tau}$ is a left proper (but not right proper) simplicial model category and its derived simplicial Hom 's will be denoted by $\mathbb{R}_\tau \text{Hom}$ (denoted by $\mathbb{R}_{w, \tau} \text{Hom}$ in [HAGI, Def. 4.6.6]); for F and G prestacks on M , there is always a morphism in $\text{Ho}(\text{SSet})$

$$\mathbb{R}_\tau \text{Hom}(F, G) \rightarrow \mathbb{R} \text{Hom}(F, G)$$

which is an isomorphism when G is a stack ([HAGI, Prop. 4.6.7]).

Moreover $M^{\sim, \tau}$ is a t -complete model topos ([HAGI, Def. 3.8.2]) therefore possesses important exactness properties. For the readers' convenience we collect below (from [HAGI]) the definition of (t -complete) model topoi and the main theorem characterizing them (Giraud-like theorem).

For a \mathbf{U} -combinatorial model category N , and a \mathbf{U} -small set S of morphisms in N , we denote by $L_S N$ the left Bousfield localization of N along S . It is a model category, having N as underlying category, with the same cofibrations as N and whose equivalences are the S -local equivalences ([Hi, Ch. 3]).

- DEFINITION 1.3.1.4. (1) An object x in a model category N is truncated if there exists an integer $n \geq 0$, such that for any $y \in N$ the mapping space $\text{Map}_N(y, x)$ ([Hi, §17.4]) is a n -truncated simplicial set (i.e. $\pi_i(\text{Map}_N(x, y), u) = 0$ for all $i > n$ and for all base point u).
 (2) A model category N is t -complete if a morphism $u : y \rightarrow y'$ in $\text{Ho}(N)$ is an isomorphism if and only if the induced map $u^* : [y', x] \rightarrow [y, x]$ is a bijection for any truncated object $x \in N$.

Recall that for any \mathbf{U} -small S -category T (i.e. a category T enriched over simplicial sets, [HAGI, Def. 2.1.1]), we can define a \mathbf{U} -combinatorial model category

$SPr(T)$ of simplicial functors $T^{op} \rightarrow SSet_U$, in which equivalences and fibrations are defined levelwise ([HAGI, Def. 2.3.2]).

DEFINITION 1.3.1.5. ([HAGI, §3.8]) A U -model topos is a U -combinatorial model category N such that there exists a U -small S -category T and a U -small set of morphisms S in $SPr(T)$ satisfying the following two conditions.

- (1) The model category N is Quillen equivalent to $L_S SPr(T)$.
- (2) The identity functor

$$\text{Id} : SPr(T) \rightarrow L_S SPr(T)$$

preserves homotopy pullbacks.

A t -complete model topos is a U -model topos N which is t -complete as a model category.

We need to recall a few special morphisms in the standard simplicial category Δ . For any $n > 0$, and $0 \leq i < n$ we let

$$\begin{array}{ccc} \sigma_i : [1] & \longrightarrow & [n] \\ 0 & \mapsto & i \\ 1 & \mapsto & i+1. \end{array}$$

DEFINITION 1.3.1.6. Let N be a model category. A Segal groupoid object in N is a simplicial object

$$X_* : \Delta^{op} \rightarrow N$$

satisfying the following two conditions.

- (1) For any $n > 0$, the natural morphism

$$\prod_{0 \leq i < n} \sigma_i : X_n \longrightarrow \underbrace{X_1 \times_{X_0}^h X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1}_{n \text{ times}}$$

is an isomorphism in $\text{Ho}(N)$.

- (2) The morphism

$$d_0 \times d_1 : X_2 \longrightarrow X_1 \times_{d_0, X_0, d_0}^h X_1$$

is an isomorphism in $\text{Ho}(N)$.

The homotopy category of Segal groupoid objects in N is the full subcategory of $\text{Ho}(N^{\Delta^{op}})$ consisting of Segal groupoid objects. It is denoted by $\text{Ho}(\text{SegPd}(N))$.

The main theorem characterizing model topoi is the following analog of Giraud's theorem.

THEOREM 1.3.1.7. ([HAGI, Thm. 4.9.2]) A model category N is a model topos if and only if it satisfies the following conditions.

- (1) The model category N is U -combinatorial.
- (2) For any U -small family of objects $\{x_i\}_{i \in I}$ in N , and any $i \neq j$ in I the following square

$$\begin{array}{ccc} \emptyset & \longrightarrow & x_i \\ \downarrow & & \downarrow \\ x_j & \longrightarrow & \coprod_{k \in I}^L x_k \end{array}$$

is homotopy cartesian.

- (3) For any \mathcal{U} -small category I , any morphism $y \rightarrow z$ and any I -diagram $x : I \rightarrow N/z$, the natural morphism

$$\mathrm{Hocolim}_{i \in I} (x_i \times_z^h y) \longrightarrow (\mathrm{Hocolim}_{i \in I} x_i) \times_z^h y$$

is an isomorphism in $\mathrm{Ho}(N)$.

- (4) For any Segal groupoid object (in the sense of Def. 1.3.1.6)

$$X_* : \Delta^{op} \longrightarrow N,$$

the natural morphism

$$X_1 \longrightarrow X_0 \times_{|X_*|}^h X_0$$

is an isomorphism in $\mathrm{Ho}(N)$.

An important consequence is the following

COROLLARY 1.3.1.8. *For any \mathcal{U} -model topos N and any fibrant object $x \in N$, the category $\mathrm{Ho}(N/x)$ is cartesian closed.*

The exactness properties of model topoi will be frequently used all along this work. For instance, we will often use that for any cover of stacks $p : F \rightarrow G$ (over some model site (M, τ)), the natural morphism

$$|F_*| \longrightarrow G$$

is an isomorphism of stacks, where F_* is the homotopy nerve of p (i.e. the nerve of a fibration equivalent to p , computed in the category of simplicial presheaves). This result is also recalled in Lem. 1.3.4.3.

1.3.2. Homotopical algebraic geometry context

Let us fix a HA context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$.

We denote by $\mathrm{Aff}_{\mathcal{C}}$ the opposite of the model category $\mathrm{Comm}(\mathcal{C})$: this will be our base model category M to which we will apply the [HAGI] constructions recalled in §1.3.1.

An object $X \in \mathrm{Aff}_{\mathcal{C}}$ corresponding to a commutative monoid $A \in \mathrm{Comm}(\mathcal{C})$ will be symbolically denoted by $X = \mathrm{Spec} A$. We will consider the model category $\mathrm{Aff}_{\mathcal{C}}^{\wedge}$ of pre-stacks on $\mathrm{Aff}_{\mathcal{C}}$ as described in §1.3.1 above. By definition, it is the left Bousfield localization of $S\mathrm{Pr}(\mathrm{Aff}_{\mathcal{C}}) := S\mathrm{Set}_{\mathcal{V}}^{\mathrm{Aff}_{\mathcal{C}}^{op}}$ (the model category of \mathcal{V} -simplicial presheaves on $\mathrm{Aff}_{\mathcal{C}}$) along the $(\mathcal{V}$ -small) set of equivalences of $\mathrm{Aff}_{\mathcal{C}}$, and the homotopy category $\mathrm{Ho}(\mathrm{Aff}_{\mathcal{C}}^{\wedge})$ will be naturally identified with the full subcategory of $\mathrm{Ho}(S\mathrm{Pr}(\mathrm{Aff}_{\mathcal{C}}))$ consisting of all functors $F : \mathrm{Aff}_{\mathcal{C}}^{op} \rightarrow S\mathrm{Set}_{\mathcal{V}}$ preserving weak equivalences. Objects in $\mathrm{Ho}(\mathrm{Aff}_{\mathcal{C}}^{\wedge})$ will be called *pre-stacks*, and the derived simplicial *Hom*'s of the simplicial model category $\mathrm{Aff}_{\mathcal{C}}^{\wedge}$ will be denoted by $\mathbb{R}\mathrm{Hom}$.

We will fix once for all a model pre-topology τ on $\mathrm{Aff}_{\mathcal{C}}$ (Def. 1.3.1.1), which induces a Grothendieck topology on $\mathrm{Ho}(\mathrm{Aff}_{\mathcal{C}})$, still denoted by the same symbol. As recalled in §1.3.1.1, one can then consider a model category $\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$, of stacks on the model site $(\mathrm{Aff}_{\mathcal{C}}, \tau)$. A morphism $F \rightarrow G$ of pre-stacks is an equivalence in $\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$ if it induces isomorphisms on all homotopy sheaves (for any choice of $X \in \mathrm{Aff}_{\mathcal{C}}$ and any $s \in F(X)$).

To ease the notation we will write $\mathrm{St}(\mathcal{C}, \tau)$ for the homotopy category $\mathrm{Ho}(\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau})$ of stacks.

The Bousfield localization construction yields an adjunction

$$a : \mathrm{Ho}(\mathrm{Aff}_C^\wedge) \longrightarrow \mathrm{St}(C, \tau) \quad \mathrm{Ho}(\mathrm{Aff}_C^\wedge) \longleftarrow \mathrm{St}(C, \tau) : j$$

where j is fully faithful.

- DEFINITION 1.3.2.1. (1) A stack is an object $F \in \mathrm{SPr}(\mathrm{Aff}_C)$ whose image in $\mathrm{Ho}(\mathrm{Aff}_C^\wedge)$ is in the essential image of the functor j above.
 (2) A morphism of stacks is a morphism between stacks in $\mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_C))$, or equivalently in $\mathrm{Ho}(\mathrm{Aff}_C^\wedge)$, or equivalently in $\mathrm{St}(C, \tau)$.
 (3) A morphism of stacks $f : F \longrightarrow G$ is a covering (or a cover) if the induced morphism of sheaves

$$\pi_0(f) : \pi_0(F) \longrightarrow \pi_0(G)$$

is an epimorphism in the category of sheaves.

We will always omit mentioning the functor j and consider the category $\mathrm{St}(C, \tau)$ as embedded in $\mathrm{Ho}(\mathrm{Aff}_C^\wedge)$, and therefore embedded in $\mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_C))$. With these conventions, the endofunctor a of $\mathrm{Ho}(\mathrm{Aff}_C^\wedge)$ becomes the associated stack functor, which commutes with finite homotopy limits and arbitrary homotopy colimits.

A functor $F : \mathrm{Aff}_C^{\mathrm{op}} \longrightarrow \mathrm{SSet}_\tau$ is a stack (§1.3.1) if and only if it preserves equivalences and possesses the descent property with respect to homotopy τ -hypercovers. The derived simplicial Hom 's in the model category $\mathrm{Aff}_C^{\sim, \tau}$ of stacks will be denoted by $\mathbb{R}_\tau \underline{\mathrm{Hom}}$. The natural morphism

$$\mathbb{R} \underline{\mathrm{Hom}}(F, G) \longrightarrow \mathbb{R}_\tau \underline{\mathrm{Hom}}(F, G)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{SSet})$ when G is a stack.

The model category $\mathrm{Aff}_C^{\sim, \tau}$ is a t -complete model topos (Def. 1.3.1.5). We warn the reader that neither of the model categories Aff_C^\wedge and $\mathrm{Aff}_C^{\sim, \tau}$ is right proper, though they are both left proper. Because of this certain care has to be taken when considering homotopy pullbacks and more generally homotopy limit constructions, as well as comma model categories of objects over a base object. Therefore, even when nothing is specified, adequate fibrant replacement may have been chosen before considering certain constructions.

The model Yoneda embedding (§1.3.1)

$$\underline{h} : \mathrm{Aff}_C \longrightarrow \mathrm{Aff}_C^\wedge$$

has a total right derived functor

$$\mathbb{R} \underline{h} : \mathrm{Ho}(\mathrm{Aff}_C) \longrightarrow \mathrm{Ho}(\mathrm{Aff}_C^\wedge),$$

which is fully faithful. We also have a naive Yoneda functor

$$h : \mathrm{Aff}_C \longrightarrow \mathrm{Aff}_C^\wedge$$

sending an object $X \in \mathrm{Aff}_C$ to the presheaf of sets it represents (viewed as a simplicial presheaf). With these notations, the Yoneda lemma reads

$$\mathbb{R} \underline{\mathrm{Hom}}(\mathbb{R} \underline{h}_X, F) \simeq \mathbb{R} \underline{\mathrm{Hom}}(h_X, F) \simeq F(X)$$

for any fibrant object $F \in \mathrm{Aff}_C^\wedge$. The natural morphism $h_X \longrightarrow \mathbb{R} \underline{h}_X$ is always an isomorphism in $\mathrm{Ho}(\mathrm{Aff}_C^\wedge)$ for any $X \in \mathrm{Aff}_C$.

We will also use the notation $\underline{\mathrm{Spec}} A$ for $\underline{h}_{\mathrm{Spec} A}$. We warn the reader that $\mathrm{Spec} A$ lives in Aff_C whereas $\underline{\mathrm{Spec}} A$ is an object of the model category of stacks $\mathrm{Aff}_C^{\sim, \tau}$.

We will assume the topology τ satisfies some conditions. In order to state them, recall the category $s\mathrm{Aff}_C$ of simplicial objects in Aff_C is a simplicial model category

for the Reedy model structure. Therefore, for any object $X_* \in sAff_C$ and any U-small simplicial set K we can define an object $X_*^{\mathbb{R}K} \in \text{Ho}(sAff_C)$, by first taking a Reedy fibrant model for X_* and then the exponential by K . The zero-th part of $X_*^{\mathbb{R}K}$ will be simply denoted by

$$X_*^{\mathbb{R}K} := (X_*^{\mathbb{R}K})_0 \in \text{Ho}(Aff_C).$$

We also refer the reader to [HAGI, §4.4] for more details and notations.

The following assumption on the pre-topology will be made.

ASSUMPTION 1.3.2.2. (1) *The topology τ on $\text{Ho}(Aff_C)$ is quasi-compact. In other words, for any covering family $\{U_i \rightarrow X\}_{i \in I}$ in Aff_C there exists a finite subset $I_0 \subset I$ such that the induced family $\{U_i \rightarrow X\}_{i \in I_0}$ is a covering.*

(2) *For any finite family of objects $\{X_i\}_{i \in I}$ in Aff_C (including the empty family) the family of morphisms*

$$\{X_i \rightarrow \prod_{j \in I}^L X_j\}_{i \in I}$$

form a τ -covering family of $\prod_{j \in I}^L X_j$.

(3) *Let $X_* \rightarrow Y$ be an augmented simplicial object in Aff_C , corresponding to a co-augmented co-simplicial object $A \rightarrow B_*$ in $\text{Comm}(\mathcal{C})$. We assume that for any n , the one element family of morphisms*

$$X_n \rightarrow X_*^{\mathbb{R}\partial\Delta^n} \times_{Y^{\mathbb{R}\partial\Delta^n}}^h Y$$

form a τ -covering family in Aff_C . Then the morphism

$$A \rightarrow B_*$$

satisfies the descent condition in the sense of Def. 1.2.12.1.

The previous assumption has several consequences on the homotopy theory of stacks. They are subsumed in the following lemma.

LEMMA 1.3.2.3. (1) *For any finite family of objects X_i in Aff_C the natural morphism*

$$\prod_i \mathbb{R}h_{X_i} \rightarrow \mathbb{R}h_{\prod_i^L X_i}$$

is an equivalence in $Aff_C^{\sim, \tau}$.

(2) *Let H be the (\mathcal{V} -small) set of augmented simplicial objects $X_* \rightarrow Y$ in Aff_C such that for any n the one element family of morphisms*

$$X_n \rightarrow X_*^{\mathbb{R}\partial\Delta^n} \times_{Y^{\mathbb{R}\partial\Delta^n}}^h Y$$

is a τ -covering family in Aff_C . Then, the model category $Aff_C^{\sim, \tau}$ is the left Bousfield localization of Aff_C^{\wedge} along the set of morphisms

$$|\mathbb{R}h_{X_*}| \rightarrow \mathbb{R}h_Y \quad \prod_i \mathbb{R}h_{U_i} \rightarrow \mathbb{R}h_{\prod_i^L U_i}$$

where $X_ \rightarrow Y$ runs in H and $\{U_i\}$ runs through the set of all finite families of objects in Aff_C .*

PROOF. (1) The case where the set of indices I is empty follows from our assumption 1.3.2.2 (2) with I empty, as it states that the empty family covers the initial object on $Aff_{\mathcal{C}}$.

Let us assume that the set of indices is not empty. By induction, it is clearly enough to treat the case where the finite family consists of two objects X and Y . Our assumption 1.3.2.2 (2) then implies that the natural morphism

$$p: \mathbb{R}h_X \coprod \mathbb{R}h_Y \longrightarrow \mathbb{R}h_{X \coprod Y}$$

is a covering. Therefore, $\mathbb{R}h_{X \coprod Y}$ is naturally equivalent to the homotopy colimit of the homotopy nerve of the morphism p . Using this remark we see that it is enough to prove that

$$\mathbb{R}h_X \times_{\mathbb{R}h_{X \coprod Y}} \mathbb{R}h_Y \simeq 0,$$

as then the homotopy nerve of p will be a constant simplicial object with values $\mathbb{R}h_X \coprod \mathbb{R}h_Y$. As the functor $\mathbb{R}h$ commutes with homotopy pullbacks, it is therefore enough to check that

$$A \otimes_{A \times B}^L B \simeq 0,$$

for A and B two commutative monoids in \mathcal{C} such that $X = \text{Spec } A$ and $Y = \text{Spec } B$ (here 0 is the final object in $\text{Comm}(\mathcal{C})$). For this we can of course suppose that A and B are fibrant objects in \mathcal{C} .

We define a functor

$$F: A \times B - \text{Comm}(\mathcal{C}) \longrightarrow A - \text{Comm}(\mathcal{C}) \times B - \text{Comm}(\mathcal{C})$$

by the formula

$$F(C) := (C \otimes_{A \times B}^L A, C \otimes_{A \times B}^L B).$$

The functor F is left Quillen for the product model structures on the right hand side, and its right adjoint is given by $G(C, D) := C \times D$ for any $(C, D) \in A - \text{Comm}(\mathcal{C}) \times B - \text{Comm}(\mathcal{C})$. For any $C \in A \times B - \text{Comm}(\mathcal{C})$, one has

$$C \simeq C \otimes_{A \times B}^L (A \times B) \simeq (C \otimes_{A \times B}^L A) \times (C \otimes_{A \times B}^L B),$$

because of our assumptions 1.1.0.1 and 1.1.0.4, which implies that the adjunction morphism

$$C \longrightarrow \mathbf{R}G(\mathbf{L}F(C))$$

is an isomorphism in $\text{Ho}(A \times B - \text{Comm}(\mathcal{C}))$. As the functor G reflects equivalences (because of our assumption 1.1.0.1) this implies that F and G form a Quillen equivalence. Therefore, the functor $\mathbf{R}G$ commutes with homotopy push outs, and we have

$$A \otimes_{A \times B}^L B \simeq \mathbf{R}G(A, 0) \otimes_{\mathbf{R}G(A, B)}^L \mathbf{R}G(0, B) \simeq \mathbf{R}G \left((A, 0) \prod_{(A, B)}^L (0, B) \right) \simeq \mathbf{R}G(0) \simeq 0.$$

(2) We know by [HAGI] that $Aff_{\mathcal{C}}^{\sim, \tau}$ is the left Bousfield localization of $Aff_{\mathcal{C}}^{\wedge}$ along the set of morphisms $|F_*| \longrightarrow h_X$, where $F_* \longrightarrow h_X$ runs in a certain \mathbb{V} -small set of τ -hypercovers. Recall that for each hypercover $F_* \longrightarrow h_X$ in this set, each simplicial presheaf F_n is a coproduct of some h_U . Using the quasi-compactness assumption 1.3.2.2 (1) one sees immediately that one can furthermore assume that each F_n is a finite coproduct of some h_U . Finally, using the part (1) of the present lemma we see that the descent condition of [HAGI] can be stated as two distinct conditions, one concerning finite coproducts and the other one concerning representable hypercovers. From this we deduce part (2) of the lemma. \square

Lemma 1.3.2.3 (2) can be reformulated as follows.

COROLLARY 1.3.2.4. *A simplicial presheaf*

$$F : \text{Comm}(\mathcal{C}) \longrightarrow \text{SSet}_V$$

is a stack if and only if it satisfies the following three conditions.

- For any equivalence $A \longrightarrow B$ in $\text{Comm}(\mathcal{C})$ the induced morphism $F(A) \longrightarrow F(B)$ is an equivalence of simplicial sets.
- For any finite family of commutative monoids $\{A_i\}_{i \in I}$ in \mathcal{C} (including the empty family), the natural morphism

$$F\left(\prod_{i \in I} {}^h A_i\right) \longrightarrow \prod_{i \in I} F(A_i)$$

is an isomorphism in $\text{Ho}(\text{SSet})$.

- For any co-simplicial commutative A -algebra $A \longrightarrow B_*$, corresponding to a τ -hypercover

$$\text{Spec } B_* \longrightarrow \text{Spec } A$$

in $\text{Aff}_{\mathcal{C}}$, the induced morphism

$$F(A) \longrightarrow \text{Holim}_{[n] \in \Delta} F(B_n)$$

is an isomorphism in $\text{Ho}(\text{SSet})$.

Another important consequence of lemma 1.3.2.3 is the following.

COROLLARY 1.3.2.5. *The model pre-topology τ on $\text{Aff}_{\mathcal{C}}$ is sub-canonical in the sense of Def. 1.3.1.3.*

PROOF. We need to show that for any $Z \in \text{Aff}_{\mathcal{C}}$ the object $G := \mathbb{R}h_Z$ is a stack, or in other words is a local object in $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$. For this, we use our lemma 1.3.2.3 (2). The descent property for finite coproducts is obviously satisfied because of the Yoneda lemma. Let $X_* \longrightarrow Y$ be a simplicial object in $\text{Aff}_{\mathcal{C}}$ such that

$$\mathbb{R}h_{X_*} \longrightarrow \mathbb{R}h_Y$$

is a τ -hypercover. By lemma 1.2.12.3 the natural morphism

$$\text{Hocolim}_n X_n \longrightarrow Y$$

is an isomorphism in $\text{Ho}(\text{Aff}_{\mathcal{C}})$. Therefore, the Yoneda lemma implies that one has

$$\mathbb{R}\text{Hom}(h_Y, G) \simeq \text{Map}(Y, Z) \simeq \text{Holim}_n \text{Map}(X_n, Z) \simeq \text{Holim}_n \mathbb{R}\text{Hom}(F_*, G),$$

showing that G is a stack. \square

The corollary 1.3.2.5 implies that $\mathbb{R}h$ provides a fully faithful functor

$$\mathbb{R}h : \text{Ho}(\text{Aff}_{\mathcal{C}}) \longrightarrow \text{St}(\mathcal{C}, \tau).$$

Objects in the essential image of $\mathbb{R}h$ will be called *representable objects*. If such an object corresponds to a commutative monoid $A \in \text{Ho}(\text{Comm}(\mathcal{C}))$, it will also be denoted by $\mathbb{R}\text{Spec } A \in \text{St}(\mathcal{C}, \tau)$. In formula

$$\mathbb{R}\text{Spec } A := \mathbb{R}h_{\text{Spec } A},$$

for any $A \in \text{Comm}(\mathcal{C})$ corresponding to $\text{Spec } A \in \text{Aff}_{\mathcal{C}}$. As $\mathbb{R}h$ commutes with U-small homotopy limits, we see that the subcategory of representable stacks is stable by U-small homotopy limits. The reader should be careful that a V-small homotopy limit of representable stacks is not representable in general. Lemma 1.3.2.3 (1) also implies that a finite coproduct of representable stacks is a representable stack, and we have

$$\coprod_i \mathbb{R}h_{U_i} \simeq \mathbb{R}h_{\coprod_i U_i}.$$

Also, by identifying the category $\mathrm{Ho}(\mathrm{Comm}(\mathcal{C}))^{\mathrm{op}}$ with the full subcategory of $\mathrm{St}(\mathcal{C}, \tau)$ consisting of representable stacks, one can extend the notions of morphisms defined in §1.2 (e.g. (formally) étale, Zariski open immersion, flat, smooth ...) to morphisms between representable stacks. Indeed, they are all invariant by equivalences and therefore are properties of morphisms in $\mathrm{Ho}(\mathrm{Comm}(\mathcal{C}))$. We will often use implicitly these extended notions. In particular, we will use the expression τ -covering families of representable stacks to denote families of morphisms of representable stacks corresponding in $\mathrm{Ho}(\mathrm{Comm}(\mathcal{C}))^{\mathrm{op}}$ to τ -covering families.

We will use the same terminology for the morphisms in $\mathrm{Ho}(\mathrm{Comm}(\mathcal{C}))$ and for the corresponding morphisms of representable stacks, except for the notion of epimorphism (Def. 1.2.6.1), which for obvious reasons will be replaced by *monomorphism* in the context of stacks. This is justified since a morphism $A \rightarrow B$ is an epimorphism in the sense of Def. 1.2.6.1 if and only if the induced morphism of stacks

$$\mathbb{R}\mathrm{Spec} B \rightarrow \mathbb{R}\mathrm{Spec} A$$

is a monomorphism in the model category $\mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$ (see Remark 1.2.6.2).

REMARK 1.3.2.6. The reader should be warned that we will also use the expression *epimorphism of stacks*, which will refer to a morphism of stacks that induces an epimorphism on the sheaves π_0 (see Def. 1.3.1.2 or [HAGI], where they are also called *coverings*). It is important to notice that a τ -covering family of representable stacks $\{X_i \rightarrow X\}$ induces an epimorphism of stacks $\coprod X_i \rightarrow X$. On the contrary, there might very well exist families of morphisms of representable stacks $\{X_i \rightarrow X\}$ such that $\coprod X_i \rightarrow X$ is an epimorphism of stacks, but which are not τ -covering families (e.g. a morphism between representable stacks that admits a section).

COROLLARY 1.3.2.7. Let $\{u_i : X_i = \mathrm{Spec} A_i \rightarrow X = \mathrm{Spec} A\}_{i \in I}$ be a covering family in $\mathrm{Aff}_{\mathcal{C}}$. Then, the family of base change functors

$$\{\mathrm{Lu}_i^* : \mathrm{Ho}(A - \mathrm{Mod}) \rightarrow \mathrm{Ho}(A_i - \mathrm{Mod})\}_{i \in I}$$

is conservative. In other words, a τ -covering family in $\mathrm{Aff}_{\mathcal{C}}$ is a formal covering in the sense of Def. 1.2.5.1.

PROOF. By the quasi-compactness assumption on τ we can assume that the covering family is finite. Also, the morphism $A \rightarrow \prod_i^h A_i = B$ is a covering. Therefore, the descent assumption 1.3.2.2 (3) implies that the base change functor

$$B \otimes_A^{\mathbb{L}} - : \mathrm{Ho}(A - \mathrm{Mod}) \rightarrow \mathrm{Ho}(B - \mathrm{Mod})$$

is conservative. Finally, we have seen during the proof of 1.3.2.3 (1) that the product of the base change functors

$$\prod_i A_i \otimes_B^{\mathbb{L}} - : \mathrm{Ho}(B - \mathrm{Mod}) \rightarrow \prod_i \mathrm{Ho}(A_i - \mathrm{Mod})$$

is an equivalence. Therefore, the composition

$$\prod_i A_i \otimes_A^{\mathbb{L}} - : \mathrm{Ho}(A - \mathrm{Mod}) \rightarrow \prod_i \mathrm{Ho}(A_i - \mathrm{Mod})$$

is a conservative functor. □

We also recall the Yoneda lemma for stacks, stating that for any $A \in \mathrm{Comm}(\mathcal{C})$, and any fibrant object $F \in \mathrm{Aff}_{\mathcal{C}}^{\sim, \tau}$, there is a natural equivalence of simplicial sets

$$\mathbb{R}\mathrm{Hom}(\mathbb{R}\mathrm{Spec} A, F) \simeq \mathbb{R}_\tau \mathrm{Hom}(\mathbb{R}\mathrm{Spec} A, F) \simeq F(A).$$

For an object $F \in \text{St}(\mathcal{C}, \tau)$, and $A \in \text{Comm}(\mathcal{C})$ we will use the following notation

$$\mathbb{R}F(A) := \mathbb{R}_\tau \underline{\text{Hom}}(\mathbb{R}\text{Spec } A, F).$$

Note that $\mathbb{R}F(A) \simeq RF(A)$, where R is a fibrant replacement functor on $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$. Note that there is always a natural morphism $F(A) \rightarrow \mathbb{R}F(A)$, which is an equivalence precisely when F is a stack.

Finally, another important consequence of assumption 1.3.2.2 is the local character of representable stacks.

PROPOSITION 1.3.2.8. *Let G be a representable stack and $F \rightarrow G$ be any morphism. Assume there exists a τ -covering family of representable stacks*

$$\{G_i \rightarrow G\},$$

such that each stack $F \times_G^h G_i$ is representable. Then F is a representable stack.

PROOF. Let $X \in \text{Aff}_{\mathcal{C}}$ be a fibrant object such that $G \simeq \underline{h}_X$. We can of course assume that $G = \underline{h}_X$. We can also assume that $F \rightarrow G$ is a fibration, and therefore that G and F are fibrant objects.

By choosing a refinement of the covering family $\{G_i \rightarrow G\}$, one can suppose that the covering family is finite and that each morphism $G_i \rightarrow G$ is the image by \underline{h} of a fibration $U_i \rightarrow X$ in $\text{Aff}_{\mathcal{C}}$. Finally, considering the coproduct $U = \coprod^l U_i \rightarrow X$ in $\text{Aff}_{\mathcal{C}}$ and using lemma 1.3.2.3 (1) one can suppose that the family $\{G_i \rightarrow G\}$ has only one element and is the image by \underline{h} of a fibration $U \rightarrow X$ in $\text{Aff}_{\mathcal{C}}$.

We consider the augmented simplicial object $U_* \rightarrow X$ in $\text{Aff}_{\mathcal{C}}$, which is the nerve of the morphism $U \rightarrow X$, and the corresponding augmented simplicial object $\underline{h}_{U_*} \rightarrow \underline{h}_X$ in $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$. We form the pullback in $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$

$$\begin{array}{ccc} F & \longrightarrow & \underline{h}_X \\ \uparrow & & \uparrow \\ F_* & \longrightarrow & \underline{h}_{U_*} \end{array}$$

which is a homotopy pullback because of our choices. In particular, for any n , F_n is a representable stack.

Clearly F_* is the nerve of the fibration $F \times_{\underline{h}_X} \underline{h}_U \rightarrow F$. As this last morphism is an epimorphism in $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ the natural morphism

$$|F_*| := \text{Hocolim}_n F_n \rightarrow F$$

is an isomorphism in $\text{St}(\mathcal{C}, \tau)$. Therefore, it remains to show that $|F_*|$ is a representable stack.

We will consider the category $s\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ of simplicial objects in $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$, endowed with its Reedy model structure (see [HAGI, §4.4] for details and notations). In the same way, we will consider the Reedy model structure on $s\text{Aff}_{\mathcal{C}}$, the category of simplicial objects in $\text{Aff}_{\mathcal{C}}$.

LEMMA 1.3.2.9. *There exists a simplicial object V_* in $\text{Aff}_{\mathcal{C}}$ and an isomorphism $\mathbb{R}h_{V_*} \simeq F_*$ in $\text{Ho}(s\text{Aff}_{\mathcal{C}}^{\sim, \tau})$.*

PROOF. First of all, our functor \underline{h} extends in the obvious way to a functor on the categories of simplicial objects

$$\underline{h} : s\text{Aff}_{\mathcal{C}} \rightarrow s\text{Aff}_{\mathcal{C}}^{\sim, \tau},$$

by the formula

$$(\underline{h}_{X_*})_n := \underline{h}_{X_n}$$

for any $X_* \in sAff_C$. This functor possesses a right derived functor

$$\mathbb{R}\underline{h} : \mathrm{Ho}(sAff_C) \longrightarrow \mathrm{Ho}(sAff_C^{\sim, \tau}),$$

which is easily seen to be fully faithful.

We claim that the essential image of $\mathbb{R}\underline{h}$ consists of all simplicial objects $F_* \in \mathrm{Ho}(sAff_C^{\sim, \tau})$ such that for any n , F_n is a representable stack. This will obviously imply the lemma. Indeed, as $\mathbb{R}\underline{h}$ commutes with homotopy limits, one sees that this essential image is stable by (U-small) homotopy limits. Also, any object $F_* \in sAff_C^{\sim, \tau}$ can be written as a homotopy limit

$$F_* \simeq \mathrm{Holim}_n \mathbb{R}Cosk_n(F_*)$$

of its derived coskeleta (see [HAGI, §4.4]). Recall that for a fibrant object F_* in $sAff_C^{\sim, \tau}$ one has

$$Cosk_n(F_*)_p \simeq \mathbb{R}Cosk_n(F_*)_p \simeq (F_*)^{Sk_n \Delta^p},$$

where, for a simplicial set K , $(F_*)^K$ is defined to be the equalizer of the two natural maps

$$\prod_{[n]} (F_n)^{K_n} \rightrightarrows \prod_{[n] \rightarrow [m]} (F_m)^{K_n}.$$

Now, for any simplicial set K , and any integer $n \geq 0$, there is a homotopy push out square

$$\begin{array}{ccc} Sk_n K & \longrightarrow & Sk_{n+1} K \\ \uparrow & & \uparrow \\ \prod_{K \partial \Delta^{n+1}} \partial \Delta^{n+1} & \longrightarrow & \prod_{K \Delta^{n+1}} \Delta^{n+1}. \end{array}$$

Using this, and the fact that $K \mapsto F_*^K$ sends homotopy push outs to homotopy pullbacks when F_* is fibrant, we see that for any fibrant object $F_* \in sAff_C^{\sim, \tau}$, and any n , we have a homotopy pullback diagram in $sAff_C^{\sim, \tau}$

$$\begin{array}{ccc} \mathbb{R}Cosk_n(F_*) & \longrightarrow & \mathbb{R}Cosk_{n-1}(F_*) \\ \uparrow & & \uparrow \\ A_*^n & \longrightarrow & B_*^n. \end{array}$$

Here, A_*^n and B_*^n are defined by the following formulas

$$\begin{aligned} A_*^n : \Delta^{op} &\longrightarrow sAff_C^{\sim, \tau} \\ [p] &\mapsto \prod_{(\Delta^p) \Delta^{n+1}} F_*^{\Delta^{n+1}} \\ B_*^n : \Delta^{op} &\longrightarrow sAff_C^{\sim, \tau} \\ [p] &\mapsto \prod_{(\Delta^p) \partial \Delta^{n+1}} F_*^{\partial \Delta^{n+1}}. \end{aligned}$$

Therefore, by induction on n , it is enough to see that if F_* is fibrant then $Cosk_0(F_*)$, A_*^n and B_*^n all belongs to the essential image of $\mathbb{R}\underline{h}$.

The simplicial object $Cosk_0(F_*)$ is isomorphic to the nerve of $F_0 \longrightarrow *$, and therefore is the image by $\mathbb{R}\underline{h}$ of the nerve of a fibration $X \longrightarrow *$ in $sAff_C$ representing F_0 . This shows that $Cosk_0(F_*)$ is in the image of $\mathbb{R}\underline{h}$.

We have $F_*^{\Delta^{n+1}} = F_{n+1}$, which is a representable stack. Let $X_{n+1} \in sAff_C$ be a fibrant object such that F_{n+1} is equivalent to $\underline{h}_{X_{n+1}}$. Then, as \underline{h} commutes with limits, we see that A_*^n is equivalent the image by \underline{h} of the simplicial object

$$[p] \mapsto \prod_{(\Delta^p) \Delta^{n+1}} X_{n+1}.$$

In the same way, $F_*^{\partial\Delta^{n+1}}$ can be written as a finite homotopy limit of F_i 's, and therefore is a representable stack. Let Y_{n+1} be a fibrant object in $AffC$ such that $F_*^{\partial\Delta^{n+1}}$ is equivalent to $\underline{h}_{Y_{n+1}}$. Then, B_*^n is equivalent to the image by \underline{h} of the simplicial object

$$[p] \mapsto \prod_{(\Delta^p)^{\partial\Delta^{n+1}}} Y_{n+1}.$$

This proves the lemma. \square

We now finish the proof of Proposition 1.3.2.8. Let V_* be a simplicial object in $sAffC$ such that $F_* \simeq \mathbb{R}\underline{h}_{V_*}$. The augmentation $F_* \rightarrow \mathbb{R}\underline{h}_{U_*}$ gives rise to a well defined morphism in $\text{Ho}(sAffC^{\sim,\tau})$

$$q : \mathbb{R}\underline{h}_{V_*} \rightarrow \mathbb{R}\underline{h}_{U_*}.$$

We can of course suppose that V_* is a cofibrant object in $sAffC$. As U_* is the nerve of a fibration between fibrant objects it is fibrant in $sAffC$. Therefore, as $\mathbb{R}\underline{h}$ is fully faithful, we can represent q , up to an isomorphism, as the image by \underline{h} of a morphism in $sAffC$

$$r : V_* \rightarrow U_*.$$

On the level of commutative monoids, the morphism r is given by a morphism $B_* \rightarrow C_*$ of co-simplicial objects in $Comm(C)$. By construction, for any morphism $[n] \rightarrow [m]$ in Δ , the natural morphism

$$F_m \rightarrow F_n \times_{\mathbb{R}\underline{h}_{U_n}} \mathbb{R}\underline{h}_{U_m}$$

is an isomorphism in $\text{St}(C, \tau)$. This implies that the underlying co-simplicial B_* -module of C_* is homotopy cartesian in the sense of Def. 1.2.12.1. Our assumption 1.3.2.2 (3) implies that if $Y := \text{Hocolim}_n V_* \in AffC$, then the natural morphism

$$V_* \rightarrow U_* \times_X^h Y$$

is an isomorphism in $\text{Ho}(sAffC)$. As homotopy colimits in $AffC^{\sim,\tau}$ commute with homotopy pullbacks, this implies that

$$|F_*| \simeq |\mathbb{R}\underline{h}_{V_*}| \simeq |\mathbb{R}\underline{h}_{U_*}| \times_{\mathbb{R}\underline{h}_X}^h \mathbb{R}\underline{h}_Y.$$

But, as $\mathbb{R}\underline{h}_{U_*} \rightarrow \mathbb{R}\underline{h}_X$ is the homotopy nerve of an epimorphism we have

$$|\mathbb{R}\underline{h}_{U_*}| \simeq \mathbb{R}\underline{h}_X,$$

showing finally that $|F_*|$ is isomorphic to $\mathbb{R}\underline{h}_Y$ and therefore is a representable stack. \square

Finally, we finish this first section by the following description of the comma model category $AffC^{\sim,\tau}/\underline{h}_X$, for some fibrant object $X \in AffC$. This is not a completely trivial task as the model category $AffC^{\sim,\tau}$ is not right proper.

For this, let $A \in Comm(C)$ such that $X = \text{Spec } A$, so that A is a cofibrant object in $Comm(C)$. We consider the comma model category $A - Comm(C) = (AffC/X)^{op}$, which is also the model category $Aff_{A-Mod}^{op} = Comm(A - Mod)$. The model pre-topology τ on $AffC$ induces in a natural way a model pre-topology τ on $AffC/X = Aff_{A-Mod}$. Note that there exists a natural equivalence of categories, compatible with the model structures, between $(AffC/X)^{\sim,\tau}$ and $AffC^{\sim,\tau}/\underline{h}_X$. We consider the natural morphism $h_X \rightarrow \underline{h}_X$. It gives rise to a Quillen adjunction

$$AffC^{\sim,\tau}/\underline{h}_X \rightarrow AffC^{\sim,\tau}/\underline{h}_X \quad AffC^{\sim,\tau}/\underline{h}_X \leftarrow AffC^{\sim,\tau}/\underline{h}_X$$

where the right adjoint sends $F \rightarrow \underline{h}_X$ to $F \times_{\underline{h}_X} h_X \rightarrow h_X$.

PROPOSITION 1.3.2.10. *The above Quillen adjunction induces a Quillen equivalence between $Aff_C^{\sim, \tau}/h_X \simeq (Aff_C/X)^{\sim, \tau}$ and $Aff_C^{\sim, \tau}/\underline{h}_X$.*

PROOF. For $F \rightarrow \underline{h}_X$ a fibrant object, and $Y \in Aff_C/X$, there is a homotopy cartesian square

$$\begin{array}{ccc} (F \times_{\underline{h}_X} h_X)(Y) & \longrightarrow & h_X(Y) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & \underline{h}_X(Y). \end{array}$$

As the morphism $h_X(Y) \rightarrow \underline{h}_X(Y)$ is always surjective up to homotopy when Y is cofibrant, this implies easily that the derived pullback functor

$$\mathrm{Ho}(Aff_C^{\sim, \tau}/\underline{h}_X) \rightarrow \mathrm{Ho}(Aff_C^{\sim, \tau}/h_X)$$

is conservative. Therefore, it is enough to show that the forgetful functor

$$\mathrm{Ho}(Aff_C/X)^{\sim, \tau} \rightarrow \mathrm{Ho}(Aff_C^{\sim, \tau}/\underline{h}_X)$$

is fully faithful.

Using the Yoneda lemma for Aff_C/X , we have

$$\mathrm{Map}_{(Aff_C/X)^{\sim, \tau}}(h_Y, h_Z) \simeq \mathrm{Map}_{Aff_C/X}(Y, Z)$$

for any two objects Y and Z in Aff_C/X . Therefore, there exists a natural fibration sequence

$$\mathrm{Map}_{(Aff_C/X)^{\sim, \tau}}(h_Y, h_Z) \longrightarrow \mathrm{Map}_{Aff_C}(Y, Z) \longrightarrow \mathrm{Map}_{Aff_C}(Z, X).$$

In the same way, the Yoneda lemma for Aff_C implies that there exists a fibration sequence

$$\mathrm{Map}_{Aff_C^{\sim, \tau}/\underline{h}_X}(h_Y, h_Z) \longrightarrow \mathrm{Map}_{Aff_C}(Y, Z) \longrightarrow \mathrm{Map}_{Aff_C}(Z, X).$$

This two fibration sequences implies that the forgetful functor induces equivalences of simplicial sets

$$\mathrm{Map}_{Aff_C/X)^{\sim, \tau}}(h_Y, h_Z) \simeq \mathrm{Map}_{Aff_C^{\sim, \tau}/\underline{h}_X}(Y, Z).$$

In other words, the functor

$$\mathrm{Ho}(Aff_C/X)^{\sim, \tau} \rightarrow \mathrm{Ho}(Aff_C^{\sim, \tau}/\underline{h}_X)$$

is fully faithful when restricted to the full subcategory of representable stacks. But, any object in $\mathrm{Ho}(Aff_C/X)^{\sim, \tau}$ is a homotopy colimit of representable stacks. Furthermore, as the derived pullback

$$\mathrm{Ho}(Aff_C^{\sim, \tau}/\underline{h}_X) \rightarrow \mathrm{Ho}(Aff_C^{\sim, \tau}/h_X)$$

commutes with homotopy colimits (as homotopy pullbacks of simplicial sets do), this implies that the functor

$$\mathrm{Ho}(Aff_C/X)^{\sim, \tau} \rightarrow \mathrm{Ho}(Aff_C^{\sim, \tau}/\underline{h}_X)$$

is fully faithful on the whole category. \square

The important consequence of Prop. 1.3.2.10 comes from the fact that it allows to see objects in $\mathrm{Ho}(Aff_C^{\sim, \tau}/\underline{h}_X)$ as functors

$$A - \mathrm{Comm}(\mathcal{C}) \rightarrow \mathcal{SSet}.$$

This last fact will be used implicitly in the sequel of this work.

As explained in the introduction to this chapter, we will need to fix a class \mathbf{P} of morphisms in $\mathcal{A}ff_{\mathcal{C}}$. Such a class will be then used to glue representable stacks to get a *geometric stack*. In other words, geometric stacks will be the objects obtained by taking some quotient of representable stacks by equivalence relations whose structural morphisms are in \mathbf{P} . Of course, different choices of \mathbf{P} will lead to different notions of geometric stacks. To fix his intuition the reader may think of \mathbf{P} as being the class of smooth morphisms (though in some applications \mathbf{P} can be something different).

From now, and all along this section, we fix a class \mathbf{P} of morphism in $\mathcal{A}ff_{\mathcal{C}}$, that is stable by equivalences. As the Yoneda functor

$$\mathbb{R}h : \mathrm{Ho}(\mathcal{A}ff_{\mathcal{C}})^{op} \longrightarrow \mathrm{St}(\mathcal{C}, \tau)$$

is fully faithful we can extend the notion of morphisms belonging to \mathbf{P} to its essential image. So, a morphism of representable objects in $\mathrm{Ho}(SPr(\mathcal{A}ff_{\mathcal{C}}^{\sim, \tau}))$ is in \mathbf{P} if by definition it correspond to a morphism in $\mathrm{Ho}(\mathcal{A}ff_{\mathcal{C}})$ which is in \mathbf{P} . We will make the following assumptions on morphisms of \mathbf{P} with respect to the topology τ , making "being in \mathbf{P} " into a τ -local property.

ASSUMPTION 1.3.2.11. (1) *Covering families consist of morphisms in \mathbf{P} i.e. for any τ -covering family $\{U_i \rightarrow X\}_{i \in I}$ in $\mathcal{A}ff_{\mathcal{C}}$, the morphism $U_i \rightarrow X$ is in \mathbf{P} for all $i \in I$.*

(2) *Morphisms in \mathbf{P} are stable by compositions, equivalences and homotopy pull-backs.*

(3) *Let $f : X \rightarrow Y$ be a morphism in $\mathcal{A}ff_{\mathcal{C}}$. If there exists a τ -covering family*

$$\{U_i \rightarrow X\}$$

such that each composite morphism $U_i \rightarrow Y$ lies in \mathbf{P} , then f belongs to \mathbf{P} .

(4) *For any two objects X and Y in $\mathcal{A}ff_{\mathcal{C}}$, the two natural morphisms*

$$X \rightarrow X \coprod^L Y \quad Y \rightarrow X \coprod^L Y$$

are in \mathbf{P} .

The reader will notice that assumptions 1.3.2.2 and 1.3.2.11 together imply the following useful fact.

LEMMA 1.3.2.12. *Let $\{X_i \rightarrow X\}$ be a finite family of morphisms in \mathbf{P} . The total morphism*

$$\coprod_i^L X_i \rightarrow X$$

is also in \mathbf{P} .

PROOF. We consider the family of natural morphisms

$$\{X_j \rightarrow \coprod_i^L X_i\}_j.$$

According to our assumption 1.3.2.2 it is a τ -covering family in $\mathcal{A}ff_{\mathcal{C}}$. Moreover, each morphism $X_j \rightarrow \coprod_i^L X_i$ and $X_j \rightarrow X$ is in \mathbf{P} , so assumption 1.3.2.11 (3) implies that so is $\coprod_i^L X_i \rightarrow X$. \square

We finish this section by the following definition.

DEFINITION 1.3.2.13. A Homotopical Algebraic Geometry context (or simply HAG context) is a 5-tuple $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P})$, where $(\mathcal{C}, \mathcal{C}_0, \mathcal{A})$ is a HA context in the sense of Def. 1.1.0.11, τ is a model pre-topology on $\text{Aff}_{\mathcal{C}}$, \mathbf{P} is a class of morphism in $\text{Aff}_{\mathcal{C}}$, and such that assumptions 1.3.2.2 and 1.3.2.11 are satisfied.

1.3.3. Main definitions and standard properties

From now on, we fix a HAG context $(\mathcal{C}, \mathcal{C}_0, \mathcal{A}, \tau, \mathbf{P})$ in the sense of Def. 1.3.2.13. We will consider the model category of stacks $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ as described in the previous section, and introduce the notion of *geometric* and *n-geometric stacks*, which are objects in $\text{Aff}_{\mathcal{C}}^{\sim, \tau}$ satisfying certain properties.

The basic geometric idea is that a stack is geometric if it is obtained by taking the quotient of a representable stack X (or more generally of a disjoint union of representable stacks) by the action of a groupoid object X_1 acting on X , such that X_1 is itself representable, and such that the source morphism $X_1 \rightarrow X$ is a morphism in the chosen class \mathbf{P} . If one thinks of \mathbf{P} as being the class of certain smooth morphisms, being geometric is thus equivalent of being a quotient by a smooth groupoid action.

It turns out that this notion is not enough for certain applications, as some naturally arising stacks are obtained as quotients by an action of a groupoid in geometric stacks rather than in representable stacks (e.g. the quotients by a group-stack action). We will therefore also introduce the notion of *n-geometric stack*, which is defined inductively as a stack obtained as a quotient by an action of a groupoid object in $(n-1)$ -geometric stacks whose source morphism is in \mathbf{P} . Of course, for this definition to make sense one must also explain, inductively on n , what are the morphisms in \mathbf{P} between n -geometric stacks.

The inductive definition we give below uses a different (though equivalent) point of view, closer to the original definition of algebraic stacks due to Deligne-Mumford and M. Artin. It says that a stack F is *n-geometric* if for any pair of points of F , the stack of equivalences between them is $(n-1)$ -geometric, and if moreover it receives a morphism, which is surjective and in \mathbf{P} , from a representable stack (or from a disjoint union of representable stacks). The equivalence of this definition with the previously mentioned quotient-by-groupoids point of view will be established in the next section (see Prop. 1.3.4.2).

- DEFINITION 1.3.3.1.**
- (1) A stack is (-1) -geometric if it is representable.
 - (2) A morphism of stacks $f : F \rightarrow G$ is (-1) -representable if for any representable stack X and any morphism $X \rightarrow G$, the homotopy pullback $F \times_G^h X$ is a representable stack.
 - (3) A morphism of stacks $f : F \rightarrow G$ is in $(-1)\text{-}\mathbf{P}$ if it is (-1) -representable, and if for any representable stack X and any morphism $X \rightarrow G$, the induced morphism

$$F \times_G^h X \rightarrow X$$

is a \mathbf{P} -morphism between representable stacks.

Now let $n \geq 0$.

- (1) Let F be any stack. An n -atlas for F is a \mathbf{U} -small family of morphisms $\{U_i \rightarrow F\}_{i \in I}$ such that
 - (a) Each U_i is representable.
 - (b) Each morphism $U_i \rightarrow F$ is in $(n-1)\text{-}\mathbf{P}$.

(c) *The total morphism*

$$\coprod_{i \in I} U_i \longrightarrow F$$

is an epimorphism.

- (2) A stack F is n -geometric if it satisfies the following two conditions.
 - (a) The diagonal morphism $F \longrightarrow F \times^h F$ is $(n-1)$ -representable.
 - (b) The stack F admits an n -atlas.
- (3) A morphism of stacks $F \longrightarrow G$ is n -representable if for any representable stack X and any morphism $X \longrightarrow G$, the homotopy pullback $F \times_G^h X$ is n -geometric.
- (4) A morphism of stacks $F \longrightarrow G$ is in $n\text{-}\mathbf{P}$ (or has the property $n\text{-}\mathbf{P}$, or is a $n\text{-}\mathbf{P}$ -morphism) if it is n -representable and if for any representable stack X , any morphism $X \longrightarrow G$, there exists an n -atlas $\{U_i\}$ of $F \times_G^h X$, such that each composite morphism $U_i \longrightarrow X$ is in \mathbf{P} .

REMARK 1.3.3.2. In the above definition, condition (2a) follows from condition (2b). This is not immediate now but will be an easy consequence of the description of geometric stacks as quotients by groupoids given in the next section. We prefer to keep the definition of n -geometric stacks with the two conditions (2a) and (2b) as it is very similar to the usual definition of algebraic stacks found in the literature (e.g. in [La-Mo]).

The next Proposition gives the fundamental properties of geometric n -stacks.

PROPOSITION 1.3.3.3.

- (1) Any $(n-1)$ -representable morphism is n -representable.
- (2) Any $(n-1)\text{-}\mathbf{P}$ -morphism is a $n\text{-}\mathbf{P}$ -morphism.
- (3) n -representable morphisms are stable by isomorphisms, homotopy pullbacks and compositions.
- (4) $n\text{-}\mathbf{P}$ -morphisms are stable by isomorphisms, homotopy pullbacks and compositions.

PROOF. We use a big induction on n . All the assertions are easily verified for $n = -1$ using our assumptions 1.3.2.11 on the morphisms in \mathbf{P} . So, we fix an integer $n \geq 0$ and suppose that all the assertions are true for any $m < n$; let's prove that they all remain true at the level n .

(1) By definition 1.3.3.1 it is enough to check that a $(n-1)$ -geometric stack F is n -geometric. But an $(n-1)$ -atlas for F is a n -atlas by induction hypothesis (which tells us in particular that a $(n-2)\text{-}\mathbf{P}$ -morphism is a $(n-1)\text{-}\mathbf{P}$ -morphism), and moreover the diagonal of F is $(n-2)$ -representable thus $(n-1)$ -representable, again by induction hypothesis. Therefore F is n -geometric.

(2) Let $f : F \longrightarrow G$ be an $(n-1)\text{-}\mathbf{P}$ -morphism. By definition it is $(n-1)$ -representable, hence n -representable by (1). Let $X \longrightarrow G$ be a morphism with X representable. Since f is in $(n-1)\text{-}\mathbf{P}$, there exists an $(n-1)$ -atlas $\{U_i\}$ of $F \times_G^h X$ such that each $U_i \longrightarrow X$ is in \mathbf{P} . But, as already observed in (1), our inductive hypothesis, shows that any $(n-1)$ -atlas is an n -atlas, and we conclude that f is also in $n\text{-}\mathbf{P}$.

(3) Stability by isomorphisms and homotopy pullbacks is clear by definition. To prove the stability by composition, it is enough to prove that if $f : F \longrightarrow G$ is an n -representable morphism and G is n -geometric then so is F .

Let $\{U_i\}$ be an n -atlas of G , and let $F_i := F \times_G^h U_i$. The stacks F_i are n -geometric, so we can find an n -atlas $\{V_{i,j}\}_j$ for F_i , for any i . By induction hypothesis (telling us in particular that $(n-1)$ - \mathbf{P} -morphisms are closed under composition) we see that the family of morphisms $\{V_{i,j} \rightarrow F\}$ is an n -atlas for F . It remains to show that the diagonal of F is $(n-1)$ -representable.

There is a homotopy cartesian square

$$\begin{array}{ccc} F \times_G^h F & \longrightarrow & F \times^h F \\ \downarrow & & \downarrow \\ G & \longrightarrow & G \times^h G. \end{array}$$

As G is n -geometric, the stability of $(n-1)$ -representable morphisms under homotopy pullbacks (true by induction hypothesis) implies that $F \times_G^h F \rightarrow F \times^h F$ is $(n-1)$ -representable. Now, the diagonal of F factors as $F \rightarrow F \times_G^h F \rightarrow F \times^h F$, and therefore by stability of $(n-1)$ -representable morphisms by composition (true by induction hypothesis), we see that it is enough to show that $F \rightarrow F \times_G^h F$ is $(n-1)$ -representable. Let X be a representable stack and $X \rightarrow F \times_G^h F$ be any morphism. Then, we have

$$F \times_{F \times_G^h F}^h X \simeq X \times_{(F \times_G^h X) \times^h (F \times_G^h X)}^h (F \times_G^h X).$$

As by hypothesis $F \times_G^h X$ is n -geometric this shows that $F \times_{F \times_G^h F}^h X$ is $(n-1)$ -geometric, showing that $F \rightarrow F \times_G^h F$ is $(n-1)$ -representable.

(4) Stability by isomorphisms and homotopy pullbacks is clear by definition. Let $F \rightarrow G \rightarrow H$ be two n - \mathbf{P} -morphisms of stacks. By (3) we already know the composite morphism to be n -representable. By definition of being in n - \mathbf{P} one can assume that H is representable. Then, there exists an n -atlas $\{U_i\}$ for G such that each morphism $U_i \rightarrow H$ is in \mathbf{P} . Let $F_i := F \times_G^h U_i$, and let $\{V_{i,j}\}_j$ be an n -atlas for F_i such that each $V_{i,j} \rightarrow U_i$ is in \mathbf{P} . Since by induction hypothesis $(n-1)$ - \mathbf{P} -morphisms are closed under composition, we see that $\{V_{i,j}\}$ is indeed an n -atlas for F such that each $V_{i,j} \rightarrow H$ is in \mathbf{P} . \square

An important consequence of our descent assumption 1.3.2.2 and Prop. 1.3.2.8 is the following useful proposition.

PROPOSITION 1.3.3.4. *Let $f : F \rightarrow G$ be any morphism such that G is an n -geometric stack. We suppose that there exists a n -atlas $\{U_i\}$ of G such that each stack $F \times_G^h U_i$ is n -geometric. Then F is n -geometric.*

If furthermore each projection $F \times_X^h U_i \rightarrow U_i$ is in n - \mathbf{P} , then so is f .

PROOF. Using the stability of n -representable morphisms by composition (see Prop. 1.3.3.3 (3)) we see that it is enough to show that f is n -representable. The proof goes by induction on n . For $n = -1$ this is our corollary Prop. 1.3.2.8. Let us assume $n \geq 0$ and the proposition proved for rank less than n . Using Prop. 1.3.3.3 (3) it is enough to suppose that G is a representable stack X .

Let $\{U_i\}$ be an n -atlas of X as in the statement, and let $\{V_{i,j}\}$ be an n -atlas for $F_i := F \times_X^h U_i$. Then, the composite family $V_{i,j} \rightarrow F$ is clearly an n -atlas for F . It remains to prove that F has an $(n-1)$ -representable diagonal.

The diagonal of F factors as

$$F \rightarrow F \times_X^h F \rightarrow F \times^h F.$$

The last morphism being the homotopy pullback

$$\begin{array}{ccc} F \times_X^h F & \longrightarrow & F \times_X^h F \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times_X^h X \end{array}$$

is representable and therefore $(n-1)$ -representable. Finally, let Z be any representable stack and $Z \rightarrow F \times_X^h F$ be a morphism. Then, the morphism

$$Z \times_{F \times_X^h F}^h F \rightarrow X$$

satisfies the conditions of the proposition 1.3.3.4 for the rank $(n-1)$. Indeed, for any i we have

$$(Z \times_{F \times_X^h F}^h F) \times_X^h U_i \simeq (Z \times_X^h U_i) \times_{F_i \times_{U_i}^h F_i}^h F_i.$$

Therefore, using the induction hypothesis we deduce that the stack $Z \times_{F \times_X^h F}^h F$ is $(n-1)$ -geometric, proving that $F \rightarrow F \times_X^h F$ is $(n-1)$ -representable.

The last part of the proposition follows from the fact that any n -atlas $\{V_{i,j}\}$ of F_i is such that each morphism $V_{i,j} \rightarrow X$ is in $n\text{-P}$ by construction. \square

COROLLARY 1.3.3.5. *The full subcategory of n -geometric stacks in $\text{St}(\mathcal{C}, \tau)$ is stable by homotopy pullbacks, and by U -small disjoint union if $n \geq 0$.*

PROOF. Let $F \rightarrow H \leftarrow G$ be a diagram of stacks. There are two homotopy cartesian squares

$$\begin{array}{ccc} F \times^h G & \longrightarrow & G \\ \downarrow & & \downarrow \\ F & \longrightarrow & \bullet \end{array} \qquad \begin{array}{ccc} F \times_H^h G & \longrightarrow & H \\ \downarrow & & \downarrow \\ F \times^h G & \longrightarrow & H \times^h H, \end{array}$$

showing that the stability under homotopy pullbacks follows from the stability of n -representable morphisms under compositions and homotopy pullbacks.

Let us prove the second part of the corollary, concerning U -small disjoint union. Suppose now that $n \geq 0$ and let F be $\coprod_i F_i$ with each F_i an n -geometric stack. Then, we have

$$F \times^h F \simeq \coprod_{i,j} F_i \times^h F_j.$$

For any representable stack X , and any morphism $X \rightarrow F \times^h F$, there exists a 0 -atlas $\{U_k\}$ of X , and commutative diagrams of stacks

$$\begin{array}{ccc} U_k & \longrightarrow & F_{i(k)} \times^h F_{j(k)} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \coprod_{i,j} F_i \times^h F_j. \end{array}$$

We apply Prop. 1.3.3.4 to the morphism

$$G := F \times_{F \times^h F}^h X \rightarrow X$$

and for the covering $\{U_k\}$. We have

$$G \times_X^h U_k \simeq \emptyset \text{ if } i(k) \neq j(k)$$

and

$$G \times_X^h U_k \simeq F_{i(k)} \times_{F_{i(k)} \times^h F_{i(k)}}^h U_k$$

otherwise. Prop. 1.3.3.4 implies that G is $(n-1)$ -geometric, and therefore that the diagonal of F is $(n-1)$ -representable. Finally, the same argument and assumption 1.3.2.11 show that the disjoint union of n -atlases of the F_i 's will form an n -atlas for F . \square

Finally, let us mention the following important additional property.

PROPOSITION 1.3.3.6. *Let $f : F \rightarrow G$ be an n -representable morphism. If f is in $m\text{-}\mathbf{P}$ for $m > n$ then it is in $n\text{-}\mathbf{P}$.*

PROOF. By induction on m it is enough to treat the case $m = n + 1$. The proof goes then by induction on n . For $n = -1$ this is our assumption 1.3.2.11 (3). For $n \geq 0$ we can by definition assume G is a representable stack and therefore that F is n -geometric. Then, there exists an $(n+1)$ -atlas $\{U_i\}$ for F such that each $U_i \rightarrow G$ is in \mathbf{P} . By induction, $\{U_i\}$ is also an n -atlas for F , which implies that the morphism f is in fact in \mathbf{P} . \square

The last proposition implies in particular that for an n -representable morphism of stacks the property of being in $n\text{-}\mathbf{P}$ does not depend on n . We will therefore give the following definition.

DEFINITION 1.3.3.7. *A morphism in $\text{St}(\mathcal{C}, \tau)$ is in \mathbf{P} if it is in $n\text{-}\mathbf{P}$ for some integer n .*

1.3.4. Quotient stacks

We will now present a characterization of geometric stacks in terms of quotient by groupoid actions. This point of view is very much similar to the presentation of manifolds by charts, and much less intrinsic than definition Def. 1.3.3.1. However, it is sometimes more easy to handle as several stacks have natural presentations as quotients by groupoids.

Let X_* be a Segal groupoid object in a model category M (Def. 1.3.1.6). Inverting the equivalence

$$X_2 \rightarrow X_1 \times_{X_0}^h X_1$$

and composing with $d_1 : X_2 \rightarrow X_1$ gives a well defined morphism in $\text{Ho}(M)$

$$\mu : X_1 \times_{X_0}^h X_1 \rightarrow X_1,$$

that is called *composition*. In the same way, inverting the equivalence

$$X_2 \rightarrow X_1 \times_{d_0, X_0, d_0}^h X_1,$$

and composing with

$$\begin{aligned} Id \times^h s_0 : X_1 &\rightarrow X_1 \times_{d_0, X_0, d_0}^h X_1 \\ d_2 : X_2 &\rightarrow X_1 \end{aligned}$$

gives a well defined isomorphism in $\text{Ho}(M)$

$$i : X_1 \rightarrow X_1,$$

called *inverse*. It is easy to check that $d_1 \circ i = d_0$ as morphisms in $\text{Ho}(M)$, showing that the two morphisms d_0 and d_1 are always isomorphic in $\text{Ho}(M)$. Finally, using condition (1) of Def. 1.3.1.6 we see that for any $i > 0$, all the face morphisms

$$X_i \rightarrow X_{i-1}$$

of a Segal groupoid object are isomorphic in $\text{Ho}(M)$.

DEFINITION 1.3.4.1. A Segal groupoid object X_* in $\text{Ho}(SPr(Aff_{\mathcal{C}}^{\sim, \tau}))$ is an n - \mathbf{P} Segal groupoid if it satisfies the following two conditions.

- (1) The stacks X_0 and X_1 are disjoint unions of n -geometric stacks.
- (2) The morphism $d_0 : X_1 \rightarrow X_0$ is in n - \mathbf{P} .

As n -geometric stacks are stable by homotopy pullbacks, X_i is a disjoint union of n -geometric stacks for any i and any n - \mathbf{P} Segal groupoid X_* . Furthermore, the condition (2) of Def. 1.3.1.6 implies that the two morphisms

$$d_0, d_1 : X_1 \rightarrow X_0$$

are isomorphic as morphisms in $\text{St}(\mathcal{C}, \tau)$. Therefore, for any n - \mathbf{P} Segal groupoid X_* , all the faces $X_i \rightarrow X_{i-1}$ are in \mathbf{P} .

PROPOSITION 1.3.4.2. Let $F \in \text{Ho}(SPr(Aff_{\mathcal{C}}^{\sim, \tau}))$ be a stack and $n \geq 0$. The following conditions are equivalent.

- (1) The stack F is n -geometric.
- (2) There exists an $(n-1)$ - \mathbf{P} Segal groupoid object X_* in $SPr(Aff_{\mathcal{C}}^{\sim, \tau})$, such that X_0 is a disjoint union of representable stacks, and an isomorphism in $\text{Ho}(SPr(Aff_{\mathcal{C}}^{\sim, \tau}))$

$$F \simeq |X_*| := \text{Hocolim}_{[n] \in \Delta} X_n.$$

- (3) There exists an $(n-1)$ - \mathbf{P} Segal groupoid object X_* in $SPr(Aff_{\mathcal{C}}^{\sim, \tau})$, and an isomorphism in $\text{Ho}(SPr(Aff_{\mathcal{C}}^{\sim, \tau}))$

$$F \simeq |X_*| := \text{Hocolim}_{[n] \in \Delta} X_n.$$

If these conditions are satisfied we say that F is the quotient stack of the $(n-1)$ - \mathbf{P} Segal groupoid X_* .

PROOF. We have already seen that a 0-geometric stack is n -geometric for any n . Therefore, (2) \Rightarrow (3). It remains to show that (1) implies (2) and (3) implies (1).

- (1) \Rightarrow (2) Let F be an n -geometric stack, and $\{U_i\}$ be an n -atlas for F . We let

$$p : X_0 := \coprod_i U_i \rightarrow F$$

be the natural projection. Up to an equivalence, we can represent p by a fibration $X_0 \rightarrow F$ between fibrant objects in $SPr(Aff_{\mathcal{C}}^{\sim, \tau})$. We define a simplicial object X_* to be the nerve of p

$$X_n := \underbrace{X_0 \times_F X_0 \times_F \cdots \times_F X_0}_{n \text{ times}}.$$

Clearly, X_* is a groupoid object in $SPr(Aff_{\mathcal{C}}^{\sim, \tau})$ in the usual sense, and as p is a fibration between fibrant objects it follows that it is also a Segal groupoid object in the sense of Def. 1.3.1.6. Finally, as F is n -geometric, one has $X_1 \simeq \coprod_{i,j} U_i \times_F^h U_j$ which is therefore an $(n-1)$ -geometric stack by Cor. 1.3.3.5. The morphism $d_0 : X_1 \rightarrow X_0 \simeq \coprod_i U_i$ is then given by the projections $U_i \times_F^h U_j \rightarrow U_i$ which are in $(n-1)$ - \mathbf{P} as $\{U_i\}$ is an n -atlas. This implies that X_* is an $(n-1)$ - \mathbf{P} Segal groupoid such that X_0 is a disjoint union of representable.

LEMMA 1.3.4.3. The natural morphism

$$|X_*| \rightarrow F$$

is an isomorphism in $\text{Ho}(SPr(Aff_{\mathcal{C}}^{\sim, \tau}))$.

PROOF. For any fibration of simplicial sets $f : X \rightarrow Y$, we know that the natural morphism from the geometric realization of the nerve of f to Y is equivalent to an inclusion of connected components. This implies that the morphism $|X_*| \rightarrow F$ induces an isomorphism on the sheaves π_i for $i > 0$ and an injection on π_0 . As the morphism $p : X_0 \rightarrow F$ is an epimorphism and factors as $X_0 \rightarrow |X_*| \rightarrow F$, the morphism $|X_*| \rightarrow F$ is also an epimorphism. This shows that it is surjective on the sheaves π_0 and therefore is an isomorphism in $\text{Ho}(SPr(Aff_C^{\sim, \tau}))$. \square

The previous lemma finishes the proof of (1) \Rightarrow (2).

(3) \Rightarrow (1) Let X_* be an $(n-1)$ -P Segal groupoid and $F = |X_*|$. First of all, we recall the following important fact.

LEMMA 1.3.4.4. *Let M be a \mathbf{U} -model topos, and X_* be a Segal groupoid object in M with homotopy colimit $|X_*|$. Then, for any $n > 0$, the natural morphism*

$$X_n \rightarrow X_0 \times_{|X_*|}^h X_0 \times_{|X_*|}^h \cdots \times_{|X_*|}^h X_0$$

$n \text{ times}$

is an isomorphism in $\text{Ho}(M)$.

PROOF. This is one of the standard properties of model topoi. See Thm. 1.3.1.7. \square

Let $\{U_i\}$ be an $(n-1)$ -atlas for X_0 , and let us consider the composed morphisms

$$f_i : U_i \rightarrow X_0 \rightarrow F.$$

Clearly, $\coprod_i U_i \rightarrow F$ is a composition of epimorphism, and is therefore a epimorphism. In order to prove that $\{U_i\}$ form an n -atlas for F it is enough to prove that the morphism $X_0 \rightarrow F$ is in $(n-1)$ -P.

Let X be any representable stack, $X \rightarrow F$ be a morphism, and let G be $X_0 \times_F^h X$. As the morphism $X_0 \rightarrow F$ is an epimorphism, we can find a covering family $\{Z_j \rightarrow X\}$, such that each Z_j is representable, and such that there exists a commutative diagram in $\text{St}(\mathcal{C}, \tau)$

$$\begin{array}{ccc} X_0 & \xrightarrow{\quad} & F \\ & \searrow & \uparrow \\ & & X \\ & \nearrow & \uparrow \\ & & \coprod_j Z_j \end{array}$$

By assumption 1.3.2.11 (1) we can also assume that each morphism $Z_j \rightarrow X$ is in \mathbf{P} , and therefore that $\{Z_j \rightarrow X\}$ is a 0-atlas of X .

In order to prove that $G \rightarrow X$ is in $(n-1)$ -P it is enough by Prop. 1.3.3.4 to prove that each stack $G_j := G \times_X^h Z_j \simeq X_0 \times_F^h Z_j$ is $(n-1)$ -geometric, and furthermore that each projection $G_j \rightarrow Z_j$ is in \mathbf{P} . We have

$$G_j \simeq (X_0 \times_F^h X_0) \times_{X_0}^h Z_j.$$

Therefore, lemma 1.3.4.4 implies that $G_j \simeq X_1 \times_{X_0}^h Z_j$, showing that each G_j is $(n-1)$ -geometric and finally that G is $(n-1)$ -geometric. Furthermore, this also shows that each projection $G_j \rightarrow Z_j$ is of the form $X_1 \times_{X_0}^h Z_j \rightarrow Z_j$ which is in \mathbf{P} by hypothesis on the Segal groupoid object X_* . \square

COROLLARY 1.3.4.5. *Let $f : F \rightarrow G$ be an epimorphism of stacks and $n \geq 0$. If F is n -geometric and f is $(n-1)$ -representable and in \mathcal{P} , then G is n -geometric.*

PROOF. Indeed, let $U \rightarrow F$ be an n -atlas, and let $g : U \rightarrow G$ be the composition. The morphism g is still an epimorphism and $(n-1)$ -representable, and thus we can assume that F is 0-representable. The morphism f being an epimorphism, G is equivalent to the quotient stack of the Segal groupoid X_* which is the homotopy nerve of f . By assumption this Segal groupoid is an $(n-1)$ -Segal groupoid and thus G is n -geometric by Prop. 1.3.4.2 (1). \square

1.3.5. Quotient stacks and torsors

Writing a stack as a quotient stack of a Segal groupoid is also useful in order to describe associated stacks to certain pre-stacks. Indeed, it often happens that a pre-stack is defined as the quotient of a Segal groupoid, and we are going to show in this section that the associated stack of such quotients can be described using an adequate notion of torsor over a Segal groupoid. This section is in fact completely independent of the notion of geometricity and concern pure stacky statements that are valid in any general model topos. We have decided to include this section as it helps understanding the quotient stack construction presented previously. However, it is not needed for the rest of this work and proofs will be more sketchy than usual.

We let M be a general \mathcal{U} -model topos in the sense of Def. 1.3.1.5. The main case of application will be $M = \text{Aff}_{\mathcal{C}}^{\sim, \tau}$ but we rather prefer to state the results in the most general setting (in particular we do not even assume that M is t -complete).

Let X_* be a Segal groupoid object in M in the sense of Def. 1.3.1.6, and we assume that each X_n is a fibrant object in M . We will consider sM , the category of simplicial objects in M , which will be endowed with its levelwise projective model structure, for which fibrations and equivalences are defined levelwise. We consider sM/X_* , the model category of simplicial objects over X_* . Finally, we let Z be a fibrant object in M , and $X_* \rightarrow Z$ be a morphism in sM (where Z is considered as a constant simplicial object), and we assume that the induced morphism

$$|X_*| \rightarrow Z$$

is an isomorphism in $\text{Ho}(M)$.

We define a Quillen adjunction

$$\phi : sM/X_* \rightarrow M/Z \quad sM/X_* \leftarrow M/Z : \psi$$

in the following way. For any $Y \rightarrow Z$ in M , we set

$$\psi(Y) := Y \times_Z X_* \in sM,$$

or in other words $\psi(Y)_n = Y \times_Z X_n$ and with the obvious transitions morphisms. The left adjoint to ψ sends a simplicial object $Y_* \rightarrow X_*$ to its colimit in M

$$\phi(Y_*) := \text{Colim}_{n \in \Delta^{op}} Y_n \rightarrow \text{Colim}_n X_n \in \Delta^{op} \rightarrow Z.$$

It is easy to check that (ϕ, ψ) is a Quillen adjunction.

PROPOSITION 1.3.5.1. *The functor*

$$\mathbb{R}\psi : \text{Ho}(M/Z) \rightarrow \text{Ho}(sM/X_*)$$

is fully faithful. Its essential image consists of all $Y_* \rightarrow X_*$ such that for any morphism $[n] \rightarrow [m]$ in Δ the square

$$\begin{array}{ccc} Y_m & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & X_n \end{array}$$

is homotopy cartesian.

PROOF. Let $Y \rightarrow Z$ in M/Z . Proving that $\mathbb{R}\psi$ is fully faithful is equivalent to prove that the natural morphism

$$|Y \times_Z^h X_*| \rightarrow Y$$

is an isomorphism in $\mathrm{Ho}(M)$. But, using the standard properties of model topoi (see Thm. 1.3.1.7), we have

$$|Y \times_Z^h X_*| \simeq Y \times_Z^h |X_*| \simeq Y$$

as $|X_*| \simeq Z$. This shows that $\mathbb{R}\psi$ is fully faithful.

By definition of the functors ϕ and ψ , it is clear that $\mathbb{R}\psi$ takes its values in the subcategory described in the statement of the proposition. Conversely, let $Y_* \rightarrow X_*$ be an object in $\mathrm{Ho}(M)$ satisfying the condition of the proposition. As X_* is a Segal groupoid object, we know that X_* is naturally equivalent to the homotopy nerve of the augmentation morphism $X_0 \rightarrow Z$ (see Thm. 1.3.1.7). Therefore, the object $\mathbb{R}\psi\mathbb{L}\phi(Y_*)$ is by definition the homotopy nerve of the morphism

$$|Y_*| \times_Z^h X_0 \rightarrow |Y_*|.$$

But, we have

$$|Y_*| \times_Z^h X_0 \simeq |Y_* \times_Z^h X_0| \simeq Y_0$$

by hypothesis on Y_* . Therefore, the object $\mathbb{R}\psi\mathbb{L}\phi(Y_*)$ is naturally isomorphic in $\mathrm{Ho}(sM/X_*)$ to the homotopy nerve of the natural

$$Y_0 \rightarrow |Y_*|.$$

Finally, as X_* is a Segal groupoid object, so is Y_* by assumption. The standard properties of model topoi (see Thm. 1.3.1.7) then tell us that Y_* is naturally equivalent to the homotopy nerve of $Y_0 \rightarrow |Y_*|$, and thus to $\mathbb{R}\psi\mathbb{L}\phi(Y_*)$ by what we have just done. \square

The model category sM/X_* , or rather its full subcategory of objects satisfying the conditions of Prop. 1.3.5.1, can be seen as the category of objects in M together with an action of the Segal groupoid X_* . Proposition 1.3.5.1 therefore says that the homotopy theory of stacks over $|X_*|$ is equivalent to the homotopy theory of stacks together with an action of X_* . This point of view will now help us to describe the stack associated to $|X_*|$.

For this, let F be a fixed fibrant object in M . We define a new model category $sM/(X_*, F)$ in the following way. Its objects are pairs (Y_*, f) , where $Y_* \rightarrow X_*$ is an object in sM/X_* and $f : \mathrm{Colim}_n Y_n \rightarrow F$ is a morphism in M . Morphisms $(Y_*, f) \rightarrow (Y'_*, f')$ are given by morphisms $Y_* \rightarrow Y'_*$ in sM/X_* , such that

$$\begin{array}{ccc} \mathrm{Colim}_n Y_n & \longrightarrow & \mathrm{Colim}_n Y'_n \\ & \searrow f & \swarrow f' \\ & F & \end{array}$$

commutes in M . The model structure on $sM/(X_*, F)$ is defined in such a way that fibrations and equivalences are defined on the underlying objects in sM . The model category $sM/(X_*, F)$ is also the comma model category $sM/(X_* \times F)$ where F is considered as a constant simplicial object in M .

DEFINITION 1.3.5.2. *An object $Y_* \in sM/(X_*, F)$ is a X_* -torsor on F if it satisfies the following two conditions.*

- (1) *For all morphism $[n] \rightarrow [m]$ in Δ , the square*

$$\begin{array}{ccc} Y_m & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & X_n \end{array}$$

is homotopy cartesian.

- (2) *The natural morphism*

$$|Y_*| \longrightarrow F$$

is an isomorphism in $\text{Ho}(M)$.

The space of X_* -torsors over F , denoted by $\text{Tors}_{X_*}(F)$, is the nerve of the sub category of fibrant objects $sM/(X_*, F)^f$, consisting of equivalences between X_* -torsors on F .

Suppose that $f : F \rightarrow F'$ is a morphism between fibrant objects in M . We get a pullback functor

$$sM/(X_*, F') \rightarrow sM/(X_*, F),$$

which is right Quillen, and such that the induced functor on fibrant objects

$$sM/(X_*, F')^f \rightarrow sM/(X_*, F)^f$$

sends X_* -torsors over F to X_* -torsors over F' (this uses the commutation of homotopy colimits with homotopy pullbacks). Therefore, restricting to the sub categories of equivalences, we get a well defined morphism between spaces of torsors

$$f^* : \text{Tors}_{X_*}(F') \rightarrow \text{Tors}_{X_*}(F).$$

By applying the standard strictification procedure, we can always suppose that $(f \circ g)^* = g^* \circ f^*$. This clearly defines a functor from $(M^f)^{op}$, the opposite full subcategory of fibrant objects in M^f , to SSet

$$\text{Tors}_{X_*} : (M^f)^{op} \rightarrow \text{SSet}.$$

This functor sends equivalences in M^f to equivalences of simplicial sets, and therefore induces a $\text{Ho}(\text{SSet})$ -enriched functor (using for example [D-K1])

$$\text{Tors}_{X_*} : \text{Ho}(M^f)^{op} \simeq \text{Ho}(M)^{op} \rightarrow \text{Ho}(\text{SSet}).$$

In other words, there are natural morphisms in $\text{Ho}(\text{SSet})$

$$\text{Map}_M(F, F') \rightarrow \text{Map}_{\text{SSet}}(\text{Tors}_{X_*}(F'), \text{Tors}_{X_*}(F)),$$

compatible with compositions.

The main classification result is the following. It gives a way to describe the stack associated to $|X_*|$ for some Segal groupoid object X_* in M .

PROPOSITION 1.3.5.3. *Let X_* be a Segal groupoid object in M and Z be a fibrant model for $|X_*|$ in M . Then, there exists an element $\alpha \in \pi_0(\text{Tors}_{X_*}(Z))$, such that for any fibrant object $F \in M$, the evaluation at α*

$$\alpha^* : \text{Map}_M(F, Z) \rightarrow \text{Tors}_{X_*}(F)$$

is an isomorphism in $\text{Ho}(\text{SSet})$.

PROOF. We can of course assume that X_* is cofibrant in sM , as everything is invariant by changing X_* with something equivalent. Let $\text{Colim}_n X_n \rightarrow Z$ be a morphism in M such that Z is fibrant and such that the induced morphism

$$|X_*| \rightarrow Z$$

is an equivalence. Such a morphism exists as $\text{Colim}_n X_n$ is cofibrant in M and computes the homotopy colimit $|X_*|$. The element $\alpha \in \pi_0(\text{Tors}_{X_*}(Z))$ is defined to be the pair $(X_*, p) \in \text{Ho}(sM/(X_*, Z))$, consisting of the identity of X_* and the natural augmentation $p: \text{Colim}_n X_n \rightarrow Z$. Clearly, α is a X_* -torsor over Z .

Applying Cor. A.0.5 of Appendix A, we see that there exists a homotopy fiber sequence

$$\text{Map}_M(F, Z) \longrightarrow N((M/Z)_W^f) \longrightarrow N(M_W^f)$$

where the $(M/Z)_W^f$ (resp. M_W^f) is the subcategory of equivalences between fibrant objects in M/Z (resp. in M), and the morphism on the right is the forgetful functor and the fiber is taken at F . In the same way, there exists a homotopy fiber sequence

$$\text{Tors}_{X_*}(F) \longrightarrow N(((sM/X_*)_W^f)^{\text{cart}}) \longrightarrow N(M_W^f)$$

where $(sM/X_*)_W^f$ is the subcategory of sM/X_* consisting of equivalence between fibrant objects satisfying condition (1) of Def. 1.3.5.2, and where the morphism on the right is induced by the homotopy colimit functor of underlying simplicial objects and the fiber is taken at F . There exists a morphism of homotopy fiber sequences of simplicial sets

$$\begin{array}{ccccc} \text{Map}_M(F, Z) & \longrightarrow & N((M/Z)_W^f) & \longrightarrow & N(M_W^f) \\ \alpha^* \downarrow & & \downarrow & & \downarrow \text{Id} \\ \text{Tors}_{X_*}(F) & \longrightarrow & N(((sM/X_*)_W^f)^{\text{cart}}) & \longrightarrow & N(M_W^f) \end{array}$$

and the arrow in the middle is an equivalence because of our proposition 1.3.5.1. \square

1.3.6. Properties of morphisms

We fix another class \mathbf{Q} of morphisms in Affc , which is stable by equivalences, compositions and homotopy pullbacks. Using the Yoneda embedding the notion of morphisms in \mathbf{Q} (or simply \mathbf{Q} -morphisms) is extended to a notion of morphisms between representable stacks.

DEFINITION 1.3.6.1. We say that morphisms in \mathbf{Q} are compatible with τ and \mathbf{P} (or equivalently that morphisms in \mathbf{Q} are local with respect to τ and \mathbf{P}) if the following two conditions are satisfied:

- (1) If $f: X \rightarrow Y$ is a morphism in Affc such that there exists a covering family

$$\{U_i \rightarrow X\}$$

with each composite morphism $U_i \rightarrow Y$ in \mathbf{Q} , then f belongs to \mathbf{Q} .

- (2) If $f: X \rightarrow Y$ is a morphism in Affc and there exists a covering family

$$\{U_i \rightarrow Y\}$$

such that each homotopy pullback morphism

$$X \times_Y^h U_i \rightarrow U_i$$

is in \mathbf{Q} , then f belongs to \mathbf{Q} .

For a class of morphism \mathbf{Q} , compatible with τ and \mathbf{P} in the sense above we can make the following definition.

DEFINITION 1.3.6.2. Let \mathbf{Q} be a class of morphisms in Aff_C , stable by equivalences, homotopy pullbacks and compositions, and which is compatible with τ and \mathbf{P} in the sense above. A morphism of stacks $f : F \rightarrow G$ is in \mathbf{Q} (or equivalently is a \mathbf{Q} -morphism) if it is n -representable for some n , and if for any representable stack X and any morphism $X \rightarrow G$ there exists an n -atlas $\{U_i\}$ of $F \times_G^h X$ such that each morphism $U_i \rightarrow X$ between representable stacks is in \mathbf{Q} .

Clearly, because of our definition 1.3.6.1, the notion of morphism in \mathbf{Q} of definition 1.3.6.2 is compatible with the original notion. Furthermore, it is easy to check, as it was done for \mathbf{P} -morphisms, the following proposition.

- PROPOSITION 1.3.6.3.** (1) Morphisms in \mathbf{Q} are stable by equivalences, compositions and homotopy pullbacks.
 (2) Let $f : F \rightarrow G$ be any morphism between n -geometric stacks. We suppose that there exists a n -atlas $\{U_i\}$ of G such that each projection $F \times_X^h U_i \rightarrow U_i$ is in \mathbf{Q} . Then f is in \mathbf{Q} .

PROOF. Exercise. □

We can also make the following two general definitions of morphisms of stacks.

DEFINITION 1.3.6.4. Let $f : F \rightarrow G$ be a morphism of stacks.

- (1) The morphism is categorically locally finitely presented if for any representable stack $X = \mathbb{R}\text{Spec } A$, any morphism $X \rightarrow G$, and any \mathbf{U} -small filtered system of commutative A -algebras $\{B_i\}$, the natural morphism

$$\text{Hocolim}_i \text{Map}_{\text{Aff}_C^{\tau}/X}(\mathbb{R}\text{Spec } B_i, F \times_G^h X) \rightarrow \text{Map}_{\text{Aff}_C^{\tau}/X}(\mathbb{R}\text{Spec}(\text{Hocolim}_i B_i), F \times_G^h X)$$

is an isomorphism in $\text{Ho}(\text{SSet})$.

- (2) The morphism f is quasi-compact if for any representable stack X and any morphism $X \rightarrow G$ there exists a finite family of representable stacks $\{X_i\}$ and an epimorphism

$$\coprod_i X_i \rightarrow F \times_G^h X.$$

- (3) The morphism f is categorically finitely presented if it is categorically locally finitely presented and quasi-compact.
 (4) The morphism f is a monomorphism if the natural morphism

$$F \rightarrow F \times_G^h F$$

is an isomorphism in $\text{St}(C, \tau)$.

- (5) Assume that the class \mathbf{Q} of finitely presented morphism in Aff_C is compatible with the model topology τ in the sense of Def. 1.3.6.1. The morphism f is locally finitely presented if it is a \mathbf{Q} -morphism in the sense of Def. 1.3.6.2. It is a finitely presented morphism if it is quasi-compact and locally finitely presented.

Clearly, the above notions are compatible with the one of definition 1.2.6.1, in the sense that a morphism of commutative monoids $A \rightarrow B$ has a certain property in the sense of Def. 1.2.6.1 if and only if the corresponding morphism of stacks $\mathbb{R}\text{Spec } B \rightarrow \mathbb{R}\text{Spec } A$ has the same property in the sense of Def. 1.3.6.4.

PROPOSITION 1.3.6.5. *Quasi-compact morphisms, categorically (locally) finitely presented morphisms, (locally) finitely presented morphisms and monomorphisms are stable by equivalences, composition and homotopy pullbacks.*

PROOF. Exercise. □

REMARK 1.3.6.6. When the class of finitely presented morphisms in $\mathcal{A}ff\mathcal{C}$ is compatible with the topology τ , Def. 1.3.6.4 gives us two different notions of locally finitely presented morphism which are a priori rather difficult to compare. Giving precise conditions under which they coincide is however not so important as much probably they are already different in some of our main examples (e.g. for each example for which the representable stacks are not truncated, as in the complicial, or brave new algebraic geometry presented in §2.3 and §2.4). In this work we have chosen to use only the non categorical version of locally finitely presented morphisms, the price to pay being of course that they do not have easy functorial characterization.

1.3.7. Quasi-coherent modules, perfect modules and vector bundles

For a commutative monoid A in \mathcal{C} , we define a category $A - QCoh$, of quasi-coherent modules on A (or equivalently on $Spec A$) in the following way. Its objects are the data of a B -module M_B for any commutative A -algebra $B \in A - Comm(\mathcal{C})$, together with an isomorphism

$$\alpha_u : M_B \otimes_B B' \longrightarrow M_{B'}$$

for any morphism $u : B \longrightarrow B'$ in $A - Comm(\mathcal{C})$, such that one has $\alpha_u \circ (\alpha_u \otimes_{B'} B'') = \alpha_{u \circ u} \otimes_{B'} B''$ for any pair of morphisms

$$B \xrightarrow{u} B' \xrightarrow{v} B''$$

in $A - Comm(\mathcal{C})$. Such data will be denoted by (M, α) . A morphism in $A - QCoh$, from (M, α) to (M', α') is given by a family of morphisms of B -modules $f_B : M_B \longrightarrow M'_B$, for any $B \in A - Comm(\mathcal{C})$, such that for any $u : B \rightarrow B'$ in $A - Comm(\mathcal{C})$ the diagram

$$\begin{array}{ccc} M_B \otimes_B B' & \xrightarrow{\alpha_u} & M_{B'} \\ \downarrow f_B \otimes_{B'} B' & & \downarrow f_{B'} \\ (M'_B) \otimes_B B' & \xrightarrow{\alpha'_u} & M'_{B'} \end{array}$$

commutes. As the categories $A - Mod$ and $Comm(\mathcal{C})$ are all \mathbf{V} -small, so are the categories $A - QCoh$.

There exists a natural projection $A - QCoh \longrightarrow A - Mod$, sending (M, α) to M_A , and it is straightforward to check that it is an equivalence of categories. In particular, the model structure on $A - Mod$ will be transported naturally on $A - QCoh$ through this equivalence. Fibrations (resp. equivalences) in $A - QCoh$ are simply the morphisms $f : (M, \alpha) \longrightarrow (M', \alpha')$ such that $f_A : M_A \longrightarrow M'_A$ is a fibration (resp. an equivalence).

Let now $f : A \longrightarrow A'$ be a morphism of commutative monoids in \mathcal{C} . There exists a pullback functor

$$f^* : A - QCoh \longrightarrow A' - QCoh$$

defined by $f^*(M, \alpha)_B := M_B$ for any $B \in A - Comm(\mathcal{C})$, and for $u : B \longrightarrow B'$ in $A - Comm(\mathcal{C})$ the transition morphism

$$f^*(M, \alpha)_B \otimes_B B' = M_B \otimes_B B' \longrightarrow f^*(M, \alpha)_{B'} = M_{B'}$$