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CNN model for studying dynamics and travelling wave solutions of FitzHugh–Nagumo equation

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Abstract

In this paper, a cellular neural network (CNN) model of FitzHugh–Nagumo equation is introduced. Dynamical behavior of this model is investigated using harmonic balance method. For the CNN model of FitzHugh–Nagumo equation, propagation of solitary waves have been proved.
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1. Introduction

The most widely used mathematical model of excitation and propagation of impulse (action potential) in nerve membranes is the FitzHugh–Nagumo equation. In [6], it has been shown that this equation or the original Nagumo active pulse transmission can be unified under the umbrella of a one-dimensional reaction–diffusion cellular neural network (CNN) where the cells are of a degenerate case of Chua's oscillator. CNNs are dynamic nonlinear circuits having mainly locally recurrent circuit topology, in other words, a local interconnection of simple circuit units called cells. Each CNN is defined mathematically by its cell dynamics and synaptic law, which specifies each cell's interaction with its neighbors. In this paper, we shall focus on reaction–diffusion CNNs [4,11] having a linear synaptic law that approximates a spatial Laplacian operator.

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An autonomous CNN made of universal cells [5] and coupled to each other by only one layer of linear resistors provides a unified active medium for generating wave phenomena. In our case, since the FitzHugh–Nagumo equation is only a simplification of the classic Hodgkin–Huxley equations [8] for nerve conduction, we shall prove that with appropriate choice of circuit parameters a CNN represents a more general and versatile model of nerve conduction.

Consider now FitzHugh–Nagumo equation in the form

$$u_t = u(u-a)(1-u) - w + u_{xx},$$
$$w_t = \varepsilon(u - bw), \tag{1}$$

where u_t is the first partial derivative of $u(t,x)$ with respect to t, u_{xx} is the second derivative of u with respect to x, w_t is the first derivative of $w(t,x)$ with respect to t, $b \geqslant 0$, $0 < a < \frac{1}{2}$, $0 < \varepsilon \ll 1$, u is a membrane potential in a nerve axon, w is an auxiliary variable. In this equation the steady state $u = w = 0$ represents the resting state of the nerve. Since ε is a small parameter, w is a slow variable compared to u, and in an initial time period we may assume that w does not change appreciably, i.e., $w = 0$. Thus,

$$u_t = u(u-a)(1-u) + u_{xx},$$

which is the well-known Nagumo's equation. Here $u = 0$ corresponds to the resting state and $u = 1$ to the excited state of the nerve. For this equation, it has been proved that both $u = 0$ and $u = 1$ are stable, whereas $u = a$ is unstable. Therefore, there is a threshold phenomenon. The model with $w = 0$ thus predicts that a finite localized stimulus can be sufficient to trigger a wave front such that the nerve is in its resting state before its passage and its excited state afterwards. Thus the information is passed along the nerve. For a model of nerve conduction to be realistic, there must be a mechanism for returning to the rest state, so that the nerve may again be excited by a stimulus. This is the role of the slow variable w.

In Section 2, we shall present an autonomous CNN model for FitzHugh–Nagumo equation (1). In Section 3, a special spectral technique related to the Harmonic Balance method will be applied for studying the dynamic behavior of the CNN model and the existence of periodic solutions will be predicted. In Section 4, we shall prove the existence of solitary waves for our CNN model.

2. CNN model for FitzHugh–Nagumo equation

2.1. Basic definition of a CNN

Since its invention in 1988 [5], the investigation of CNNs has evolved to cover a very broad class of problems and frameworks. Many researchers have made significant contributions to the study of CNN phenomena using different mathematical tools [12]. CNN is simply an analog dynamic processor array, made of cells, which contain linear capacitors, linear resistors, linear and nonlinear controlled sources. Let us consider a two-dimensional grid with 3×3 neighborhood system as is shown in Fig. 1.

The squares are the circuit units—cells, and the links between the cells indicate that there are interactions between linked cells. One of the key features of a CNN is that the individual cells are nonlinear dynamical systems, but that the coupling between them is linear. Roughly speaking, one

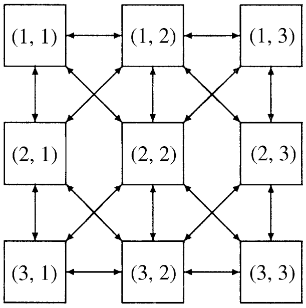

Fig. 1.

could say that these arrays are nonlinear but they have a linear spatial structure, which makes the use of techniques for their investigation common in engineering or physics attractive.

We shall give the general definitions of a CNN which follows the original one [5].

Definition 1. A CNN is a

(a) 2-, 3-, or n-dimensional array of
(b) mainly identical dynamical systems, called cells, which satisfies two properties:
(c) most interactions are local within a finite radius r, and
(d) all state variables are continuous valued signals.

Definition 2. A CNN is defined mathematically by four specifications:

(1) CNN cell dynamics, which presents the state equation,
(2) CNN synaptic law, which gives the interactions between cells,
(3) Boundary conditions,
(4) Initial conditions.

Now in terms of Definition 2, for a general CNN whose cells are made of time-invariant circuit elements, each cell $C(i,j)$ is characterized by its CNN cell dynamics:

$$\dot{x}_{ij} = -g(x_{ij}, u_{ij}, I^{s}_{ij}), \tag{2}$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usually a scalar. In most cases, the interactions (spatial coupling) with the neighbor cell $C(i+k, j+l)$ are specified by a CNN synaptic law:

$$I^{s}_{ij} = A_{ij,kl}x_{i+k,j+l} + \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \tilde{B}_{ij,kl} * u_{i+k,j+l}(t), \tag{3}$$

where f is any sigmoid function: $|f(x_{ij})| \leqslant c = \text{const}$, and $(\mathrm{d}f(x_{ij})/\mathrm{d}x_{ij}) \geqslant 0$. The first term $A_{ij,kl}x_{i+k,j+l}$ of (3) is simply a linear feedback of the states of the neighborhood nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood. In this paper, we assume that CNN has no inputs, i.e., $u \equiv 0$ [4] and we shall henceforth refer to this zero-input CNN as an autonomous CNN. For analytical investigations, it is often necessary to assume an autonomous CNN of infinite size, i.e., $N \to \infty$. In this case, the boundary conditions are replaced by the prescribed behavior of the solution at infinity. For CNN with nearest-neighbor coupling, the following three boundary conditions are typical.

(1) Fixed (Dirichlet) boundary condition:

$$x_0(t) \equiv x_{N+1}(t) \equiv 0;$$

(2) Zero-flux (Neumann) boundary condition:

$$x_0(t) \equiv x_1(t),$$

$$x_{N+1}(t) \equiv x_N(t).$$

The boundaries act like mirrors reflecting the two extreme cells of the linear array.

(3) Periodic boundary condition:

$$x_0(t) \equiv x_N(t),$$

$$x_{N+1} \equiv x_1(t)$$

making the array circular. Patterns and waves are usually observed with boundary conditions (2) and (3). In our case we shall consider our autonomous CNN model for FitzHugh–Nagumo equation with periodic boundary conditions (3), i.e., we shall have a circular array.

2.2. Modelling FitzHugh–Nagumo equation via CNN

As we mentioned in the introduction, FitzHugh–Nagumo equation can be presented by a reaction–diffusion autonomous CNN where the cells are a degenerate special case of Chua's oscillator [4,6]. CNN is called a reaction–diffusion CNN because it is described by a discretized version of the well-known system of nonlinear PDEs, called in the literature as the reaction–diffusion equations [2]: $\partial u/\partial t = f(u) + D\nabla^2 u$, where $u \in R^n$, $f \in R^n$, D is an $(n \times n)$ diagonal matrix whose diagonal elements, D_i are called the diffusion coefficients, and

$$\nabla^2 u_i = \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2}, \quad i = 1, 2, \ldots, n$$

is the Laplacian operator in R^2. There are several ways to approximate the Laplacian operator $\nabla^2 u_i$ in discrete space by a CNN synaptic law with an appropriate A-template [11].

We map $u(x,t)$ into a CNN layer such that the state voltage of a CNN cell $v_{xkl}(t)$ [5] at a grid point (k,l) is associated with $u(kh,t)$, $h = \Delta x$, and such that the second spatial partial derivative can be written as

$$u_{xx} \sim \frac{1}{h^2}\,[u(x+h,t) - u(x,t) - (u(x,t) - u(x-h,t))]$$

$$= \frac{1}{h^2}\,[u_{k+1,l} - 2u_{k,l} + u_{k-1,l}],$$

where h is the uniform grid size.

Using this approximation and by the similarity indicated in [4], it is easy to design the CNN model of Eq. (1).

(1) CNN cell dynamics:

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = u_j(u_j - a)(1 - u_j) - w_j + I_j^{\mathrm{s}},$$

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = \varepsilon(u_j - bw_j), \quad 1 \leqslant j \leqslant N. \tag{4}$$

(2) CNN synaptic law:

$$I_j^{\mathrm{s}} = \frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1}). \tag{5}$$

Let us assume for simplicity that the grid size of our CNN model is $h = 1$, and let us denote the nonlinearity $f(u_j) = u_j^2(1 + a) - u_j^3$. Substituting (5) into (4) we obtain

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = u_{j-1} - (2 + a)u_j + u_{j+1} + f(u_j) - w_j,$$

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = \varepsilon(u_j - bw_j), \quad 1 \leqslant j \leqslant N. \tag{6}$$

System (6) is actually a system of ODEs which is identified as the state equation of an autonomous CNN made of N cells.

(3) Boundary conditions: The boundary conditions affect the steady-state solutions of CNN considerably here. We take periodic boundary conditions (3), because they yield the most regular topology of the array (all cells are identical), which will be conveniently exploited for the analysis.

3. Dynamic behavior of the CNN model

In this section, we shall introduce an approximative method for studying the dynamics of the CNN model (6), based on a special Fourier transform. The idea of using Fourier expansion for finding the solutions of PDEs is well known in physics. It is used to predict what spatial frequencies or modes will dominate in nonlinear PDEs. In CNN literature, this approach has been developed for analyzing the dynamics of CNNs with symmetric templates [7]. When all state trajectories of a CNN converge toward equilibrium points, an idea borrowed from physics and signal processing is to analyze them in spatial frequency domain. The array is viewed as a spatial digital filter, the coupling template parameters being the coefficients of a spatial infinite impulse response filter. Until the nonlinearity begins to modify significantly the states, this analysis is very accurate. Afterwards, it will depend on the type of template parameters. In the case where steady-state trajectories oscillate, it is convenient to use both a space-frequency, as before, and a time-frequency transform.

In this paper, we investigate the dynamic behavior of a CNN model (6) by the use of special spectral technique related to harmonic balance method well known in control theory and in the study of electronic oscillators as describing the function method [9,10]. This method is based on the fact that all cells in CNN are identical [5], and therefore by introducing a suitable double transform, the network can be reduced to a Lur'e system (Fig. 2) to which the describing function technique [9] is applied for discovering the existence and characteristics of periodic solutions.

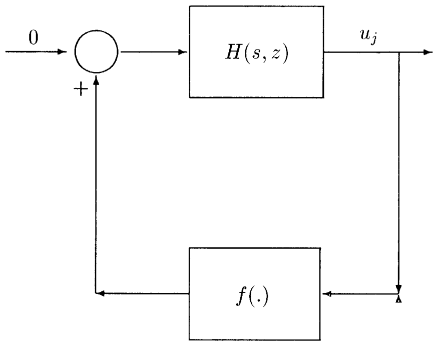

Fig. 2.

Let us introduce the double Fourier transform $F(s,z)$ of functions $f_k(t)$ discrete in space and continuous in time:

$$F(s,z) = \sum_{k=-\infty}^{k=\infty} z^{-k} \int_{-\infty}^{\infty} f_k(t) \exp(-st)\,dt. \tag{7}$$

In fact, this is continuous-time discrete-space Fourier transform (CTDSFT) from continuous time t and discrete space k to continuous temporal frequency ω, and continuous spatial frequency Ω, such that $z = \exp(i\Omega)$, $s = i\omega$.

Applying the above transform (7) to system (6), we obtain

$$sU(s,z) = z^{-1}U(s,z) - (2+a)U(s,z) + zU(s,z) + F(s,z) - W(s,z),$$

$$sW(s,z) = \varepsilon(U(s,z) - bW(s,z)). \tag{8}$$

From (8) the double transform of u_j, $U(s,z)$ can be expressed as a function of the double Fourier transform of $f(u_j)$, $F(s,z)$:

$$U(s,z) = \frac{s+\varepsilon b}{s^2 + sA + \varepsilon B} N(s,z), \tag{9}$$

$A = a + 2 - z^{-1} - z + \varepsilon b$, $B = 1 + b(a+2) - bz^{-1} - bz$.

Therefore, dynamical system (8) can be represented in the Lur'e form shown in Fig. 2, where the linear part is the transfer function

$$H(s,z) = \frac{s+\varepsilon b}{s^2 + sA + \varepsilon B}$$

and the nonlinear one is the function $f(.)$.

According to the above Lur'e diagram (Fig. 2), the transfer function $H(s,z)$ can be presented in terms of ω_0 and Ω_0, i.e., $H(s,z) = H_{\Omega_0}(\omega_0)$:

$$H_{\Omega_0}(\omega_0) = \frac{U_{\Omega_0}(\omega_0)}{W_{\Omega_0}(\omega_0)}. \tag{10}$$

We are looking for possible periodic solutions of our CNN model (6) of the form

$$u_j(t) = \xi(\Omega_0 j + \omega_0 t), \tag{11}$$

for some function $\xi : \mathbf{R} \to \mathbf{R}$ and for some $0 \leqslant \Omega_0 \leqslant 2\pi$, $\omega_0 = 2\pi/T_0$, where $T_0 > 0$ is the minimal period of (11). For the circular array the possible values for Ω_0 can be easily obtained. As $u_j(t)$ is assumed to be periodic, with minimal period T_0, one has

$$\xi(\Omega_0 j + \omega_0 t) = \xi(\Omega_0 j + \omega_0 t + k\omega_0 T_0) \tag{12}$$

for any $k \in \mathbf{N}$. On the other hand, the periodic boundary conditions (3) impose that

$$\xi(\omega_0 t) = \xi(\Omega_0 N + \omega_0 t). \tag{13}$$

Combining (12) with $j = 0$ and (13), we get

$$\Omega_0 = \frac{k}{N}\,\omega_0 T_0 = \frac{2\pi k}{N}, \quad 0 \leqslant k \leqslant N-1, \tag{14}$$

where the range of k is determined by the condition $0 \leqslant \Omega_0 \leqslant 2\pi$.

Now according to (11) we shall suppose that the state variable has the form

$$u_j(t) = U_{m_0} \sin(\omega_0 t + \mathrm{j}\Omega_0), \tag{15}$$

which amounts to specify as ansatz for (11), which is $\xi(\psi) = U_{m_0} \sin\psi$.

Then we shall approximate the output by the fundamental component of its Fourier expansion:

$$w_j(t) \simeq W_{m_0} \sin(\omega_0 t + \mathrm{j}\Omega_0), \tag{16}$$

with

$$W_{m_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} N(U_{m_0}\sin\psi)\sin\psi\,\mathrm{d}\psi = U_{m_0}^3(-\tfrac{3}{4}). \tag{17}$$

Thus, the ratio of the CTDSFTs of these periodic solutions is

$$H_{\Omega_0}(\omega_0) = \frac{U_{\Omega_0}(\omega_0)}{W_{\Omega_0}(\omega_0)} = \frac{U_{m_0}}{W_{m_0}}. \tag{18}$$

On the other hand, if we substitute $s = \mathrm{i}\omega_0$ and $z = \exp(\mathrm{i}\Omega_0)$ in (9) we obtain

$$H_{\Omega_0}(\omega_0) = \frac{\mathrm{i}\omega_0 + \varepsilon b}{-\omega_0^2 + \mathrm{i}\omega_0 \hat{A} + \varepsilon \hat{B}}, \tag{19}$$

where

$$\hat{A} = a + 2 - 2\cos\Omega_0 + \varepsilon b, \quad \hat{B} = 1 - 2b\cos\Omega_0 + b(a+2).$$

According to (18) and (19) the following constraints hold:

$$\mathrm{Re}(H_{\Omega_0}(\omega_0)) = \frac{U_{m_0}}{W_{m_0}},$$

$$\mathrm{Im}(H_{\Omega_0}(\omega_0)) = 0. \tag{20}$$

Thus (14), (17) and (20) give us the necessary set of equations for finding the unknowns U_{m_0}, ω_0, Ω_0. As we mentioned above, we are looking for a periodic wave solution of (6); therefore, U_{m_0} will determine the approximate amplitude of the wave, and $T_0 = 2\pi/\omega_0$ will determine the wave speed.

Combining (19) and (20) we get

$$\frac{\omega_0^2(\hat{A}-\varepsilon b)+\varepsilon^2 b\hat{B}}{\omega_0^4+\omega_0^2(\hat{A}^2-2\varepsilon\hat{B})+\varepsilon^2\hat{B}^2}=\frac{U_{m_0}}{W_{m_0}},$$

$$\frac{\omega_0(\varepsilon\hat{B}-\varepsilon b\hat{A})-\omega_0^3}{\omega_0^4+\omega_0^2(\hat{A}^2-2\varepsilon\hat{B})+\varepsilon^2\hat{B}^2}=0. \tag{21}$$

After solving (21) we obtain the solutions

$$\omega_0=\sqrt{\varepsilon\hat{B}-\varepsilon b\hat{A}},$$

$$U_{m_0}=\sqrt{\frac{4}{3}\frac{\varepsilon b^2\hat{A}-b\hat{A}^2+\hat{A}\hat{B}}{b\hat{A}-\varepsilon b^2-\hat{B}}}. \tag{22}$$

Now according to the describing function method, if for a given value of Ω_0 (20) we can find a solution (ω_0, U_{m_0}) of (21), then we can predict the existence of a periodic solution with an amplitude U_{m_0} and a period of approximately $T_0=2\pi/\omega_0$. Therefore, the following theorem has been proved.

Theorem 1. *CNN model (6) of the FitzHugh–Nagumo equation (1) with circular array of N cells has periodic state solutions $u_j(t)$ with a finite set of spatial frequencies $\Omega_0=2\pi k/N$, $0\leqslant k\leqslant N-1$ and a period $T_0=2\pi/\omega_0$.*

Remark 1. By applying the above Harmonic balance method we have been able to obtain a characterization of the periodic steady-state solutions of our CNN model. In order to validate the accuracy of the achieved results it would be useful to have a possible initial condition from which the network will reach, at steady state, a steady-state solution characterized by the desired value of Ω_0. One such possibility is to take an initial condition $x_j(0)=\sin(\Omega_0 j)$, $1\leqslant j\leqslant p$.

Remark 2. In the above analysis the higher-order harmonics occurring at the output of the nonlinear block are neglected, i.e. the nonlinear block is replaced by a constant gain having the same input, which minimizes the mean squared error between the output from the nonlinearity and that from the gain itself [2]. This assumption is often referred to us as the filtering hypothesis. This is a very important condition for the accuracy of the predictions.

4. Propagation of solitary waves in CNN model of FitzHugh–Nagumo equation

In literature concerning FitzHugh–Nagumo equation (1) [2], it is shown that under required conditions there exists a solitary travelling wave solution of (1) for ε small corresponding to the triggered response of a nerve. The threshold behavior ensures that only stimuli above a certain level will trigger a nerve impulse. This impulse consists of a wave of excitation, the leading wave front, followed by a retractory period, when the nerve is not amenable to the triggering of another impulse, followed by relaxation to the steady state. Another nerve impulse may then be triggered.

Let us consider CNN model (6). For system (6), the travelling wave solutions are

$$u_j(t) = u(\eta),$$
$$w_j(t) = w(\eta), \quad 1 \leqslant j \leqslant N, \tag{23}$$

where $\eta = t - \mathrm{j}h$, $h > 0$ is a parameter.

If we compare the travelling wave solution (23) with the predicted periodic solutions (15) and (16), we can see that they are similar $u(t-\mathrm{j}h) \to U_{m_0}\sin(\omega_0 t + \mathrm{j}\Omega_0)$, $w(t-\mathrm{j}h) \to W_{m_0}\sin(\omega_0 t + \mathrm{j}\Omega_0)$. Therefore, we are able to make a conclusion that our CNN model (6) of FitzHugh–Nagumo equation has a solitary wave solution with period of the wave of approximately $T_0 = 2\pi/\omega_0$ and with amplitude of the wave U_{m_0}.

Substituting (23) in (6) we obtain

$$\dot{u} = u(\eta - h) - 2u(\eta) - u(\eta + h) + f(u) + w(\eta),$$
$$\dot{w} = \varepsilon(u - bw), \tag{24}$$

where the dot denotes differentiation with respect to η, $f(u) = u(u-a)(1-u)$. Note that η is the coordinate moving along the array with a velocity equal to $c = 1/h$. Then the two difference terms $[u(\eta - h) - u(\eta)] - [u(\eta) - u(\eta + h)]$ can be replaced approximately by the first derivatives $-\dot{u}/h$ and $+\dot{u}/h$, respectively. Hence, from (24) we obtain

$$\dot{u} = \frac{1}{1+2c} f(u) + \frac{1}{1+2c} w,$$
$$\dot{w} = \varepsilon(u - bw). \tag{25}$$

Let us assume that the nonlinear function $f(u) = u(u-a)(1-u)$ satisfies for some $a \in (0,1)$ the following conditions:

$$f(0) = f(a) = f(1) = 0, \quad f < 0 \in (0,a), \; f > 0 \in (a,1), \; f'(0) < 0, \; f'(1) < 0,$$
$$F(1) = \int_0^1 f(z)\,\mathrm{d}z > 0. \tag{26}$$

Then the following theorem is known [2] for FitzHugh–Nagumo equation (1) with $f(u)$ satisfying (26):

Theorem 2. *The stationary problem* (25) *for FitzHugh–Nagumo equation with zero flux boundary conditions, where* $f(u)$ *satisfies* (26), *has three or more solutions with* $0 \leqslant u \leqslant 1$. *The two stationary solutions* $u \equiv 0$ *and* $u \equiv 1$ *are both asymptotically stable as solutions of the corresponding initial-boundary-value problem, whereas* $u \equiv a$ *is unstable.*

The phase plane for the stationary solutions of FitzHugh–Nagumo equation are given in Fig. 3. By definition, the equilibrium points of (25) should satisfy

$$\frac{1}{1+2c} f(u^*) - w^* = 0,$$
$$\varepsilon u^* - \varepsilon b w^* = 0. \tag{27}$$

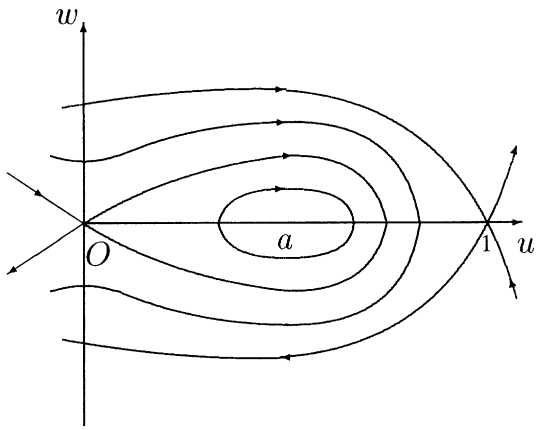

Fig. 3.

Clearly, $u \equiv 0$, $u \equiv a$, and $u \equiv 1$ are solutions of the stationary problem. To prove the asymptotic stability or instability, we shall linearize (25) about $(0,0)$, $(1,0)$, $(a,0)$. The Jacobian matrix for (27) is

$$J = \begin{vmatrix} \frac{1}{1+2c} f'(u) & \frac{1}{1+2c} \\ 1 & -b \end{vmatrix}.$$

Let us consider first the equilibrium point $(0,0)$. In this case we have the following eigenvalue equation:

$$\begin{vmatrix} -\frac{a}{1+2c} - \lambda & \frac{1}{1+2c} \\ 1 & -b-\lambda \end{vmatrix} = 0$$

or

$$\lambda^2 + \lambda\left(b + \frac{a}{1+2c}\right) + \frac{1}{1+2c}(ab-1) = 0.$$

If $ab - 1 < 0$, then the origin is a saddle point. Analogously it can be shown that $(1,0)$ is a saddle point too.

We require a trajectory from $(0,0)$ to $(1,0)$ in the phase plane remaining in the strip $0 \leqslant u \leqslant 1$. Any such wave front must be monotonic. This is easily seen from phase plane (Fig. 3), noting that a trajectory is directed towards the right if w is positive and to the left if it is negative. It follows that the equilibrium points $(0,0)$ and $(1,0)$ cannot be centers or foci, since solutions close to such points must oscillate. Trajectories which pass from one equilibrium point to another are known as heteroclinic orbits.

Let us write system (25) in the following way:

$$\dot{z} = X(z;\mu), \tag{28}$$

$$z = (u,w), \quad \mu = (b,c,\varepsilon), \quad a_1 := (0,0), \quad a_2 := (1,0).$$

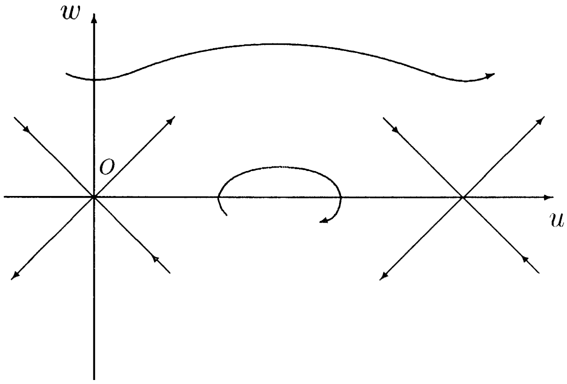

Fig. 4.

Then we can show that (25) has a heteroclinic solution $z_1^*(\eta)$ from a_1 to a_2 ($z_2^*(\eta)$ from a_2 to a_1) for certain parameter values. This solution corresponds to a travelling wave of FitzHugh–Nagumo equation which satisfies

$$\lim_{\eta\to-\infty} z_1^*(\eta) = a_1, \quad \lim_{\eta\to+\infty} z_1^*(\eta) = a_2, \tag{29}$$

$$\lim_{\eta\to-\infty} z_2^*(\eta) = a_2, \quad \lim_{\eta\to+\infty} z_2^*(\eta) = a_1, \tag{30}$$

Therefore the following theorem holds:

Theorem 3. *For CNN model of FitzHugh–Nagumo equation* (6) *where* $f(u)$ *satisfies* (26), *there is* $c > 0$ *such that there exists a wave front from* 0 *to* 1.

The phase plane for travelling wave solutions of FitzHugh–Nagumo equation where $f(u)$ satisfies (26) is given in Fig. 4.

5. Conclusion

In this paper, we present the derivation of the CNN implementations through spatial discretization, which suggests a methodology for converting PDEs to CNN templates. The CNN solution of a PDE has four basic properties—they are

(i) continuous in time;
(ii) continuous and bounded in value;
(iii) continuous in interaction parameters; and
(iv) discrete in space.

As it was stated in [11], some autonomous CNNs represent an excellent approximation to the nonlinear partial differential equations (PDEs). Although the CNN equations describing reaction–diffusion systems are with the large number of cells, they can exhibit new phenomena that cannot be obtained from their limiting PDEs. This demonstrates that an autonomous CNN is in some sense more general than its associated nonlinear PDE.

The results presented in this paper show that with appropriate choice of circuit parameters, the generic third-order CNN model (6) represents a more general and versatile model of nerve conduction than the FitzHugh–Nagumo equation. It is well known that FitzHugh–Nagumo equation is currently the model of choice in simulating the mathematical neurophysiology of nerve conduction because it is much simpler than the Hodgkin–Huxley equations. Therefore, we have proved once again that an autonomous CNN can serve as a unifying paradigm for active wave propagation. For researchers outside of engineering, the CNN paradigm will find increasing applications in view of the large and rapidly expanding body of knowledge being generated in the CNN research community. By simply translating any such nonengineering but CNN-based phenomenon into a corresponding CNN paradigm, many tools, results, and concepts developed for CNNs [12] can be used to understand, explain and control such a phenomenon.

6. Uncited references

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