



UNIVERSITÀ
DEGLI STUDI
FIRENZE

FLORE

Repository istituzionale dell'Università degli Studi di Firenze

Diffusive corrections to asymptotics of a strong-field quantum transport equation

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

Original Citation:

Diffusive corrections to asymptotics of a strong-field quantum transport equation / C. MANZINI; G. FROSALI. - In: PHYSICA D-NONLINEAR PHENOMENA. - ISSN 0167-2789. - STAMPA. - 239:(2010), pp. 1402-1415. [10.1016/j.physd.2009.10.016]

Availability:

The webpage <https://hdl.handle.net/2158/243397> of the repository was last updated on

Published version:

DOI: 10.1016/j.physd.2009.10.016

Terms of use:

Open Access

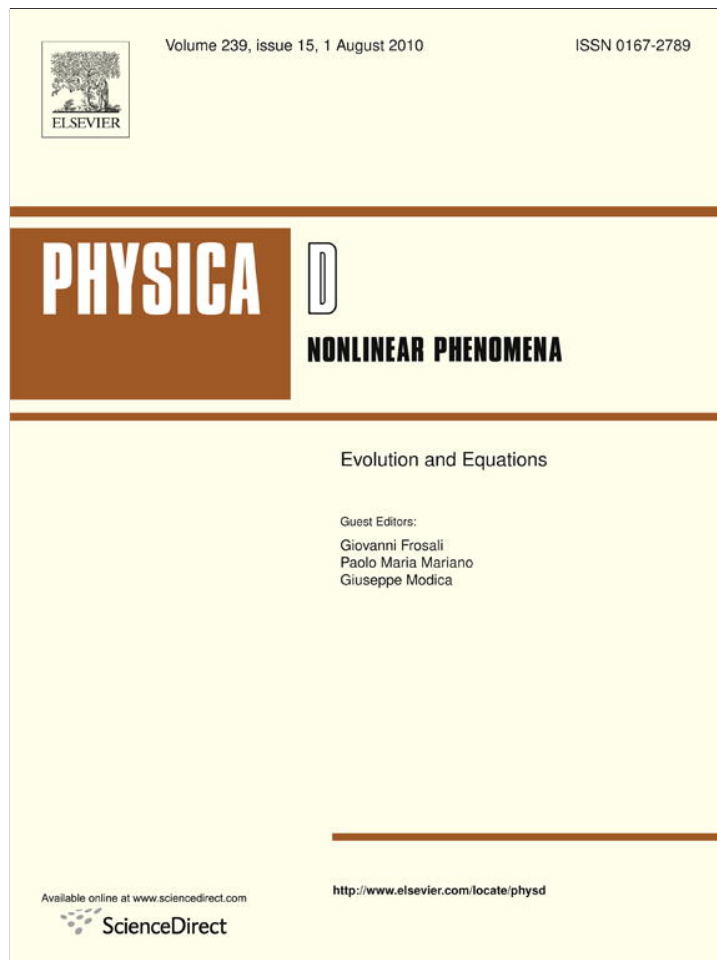
La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

Publisher copyright claim:

La data sopra indicata si riferisce all'ultimo aggiornamento della scheda del Repository FloRe - The above-mentioned date refers to the last update of the record in the Institutional Repository FloRe

(Article begins on next page)

Provided for non-commercial research and education use.
Not for reproduction, distribution or commercial use.



This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

<http://www.elsevier.com/copyright>



Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Diffusive corrections to asymptotics of a strong-field quantum transport equation

Chiara Manzini, Giovanni Frosali*

Dipartimento di Matematica "G.Sansone", Università di Firenze - Via S.Marta 3, I-50139 Firenze, Italy

ARTICLE INFO

Article history:

Available online 1 November 2009

Keywords:

Asymptotic analysis
 Quantum drift-diffusion model
 Wigner equation
 Open quantum systems
 Singularly perturbed parabolic equations

ABSTRACT

The asymptotic analysis of a linear high-field Wigner-BGK equation is developed by a modified Chapman-Enskog procedure. By an expansion of the unknown Wigner function in powers of the Knudsen number ϵ , evolution equations are derived for the terms of zeroth and first order in ϵ . In particular, a quantum drift-diffusion equation for the position density of electrons, with an ϵ -order correction on the field terms, is obtained. Well-posedness and regularity of the approximate problems are established, and a rigorous proof that the difference between exact and asymptotic solutions is of order ϵ^2 , uniformly in time and for arbitrary initial data is given.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

Quantum mechanics has recently proved an essential tool for modeling the new generation of nanodevices [1]. However, the adoption of quantum models requires a delicate compromise with quantum statistics principles. Hamiltonian dynamics is described at the quantum level, either in terms of wave-functions (via Schrödinger–Poisson–systems), or of density-matrix operators (via von Neumann equation). For different reasons, both formulations are not suitable for simulations: precisely, the wave-function approach cannot be extended to picture dissipative dynamics of open quantum systems, while the density matrix approach is not appropriate to describe finite position domains, due to its non-local character. For the same reasons, it is instead convenient to employ (Wigner) quasi-distribution functions [2,3]. Nevertheless, a phase-space description of a multi-dimensional dynamics presents well-known computational drawbacks. On the other hand, quantum hydrodynamic models seem to be a promising tool both from the numerical and the analytical point of view [4,5]. Similarly, in semi-classical semiconductor theory, the interest of modeler has shifted from the Boltzmann equation to hydrodynamic systems, and they have been widely studied both for a physical validation and from an analytical and numerical point of view (cf. [6] and the references therein). A rigorous derivation of quantum hydrodynamic models from more fundamental ones, in either Schrödinger or Wigner formulation, is an open and analytically demanding problem [7–9]. This is the motivation of the present paper.

The preliminary step for switching from the kinetic picture to a macroscopic one consists of including dissipative mechanisms in the evolution model, for example, the interaction of the quantum

system with the environment. In the weak coupling limit, a Markovian dynamics can be still adopted, and the description via an (operator) evolution equation in Lindblad form is considered quantum-physically correct [10]. From this class of evolution equations, kinetic models of open systems can be derived via a Wigner transform. In Section 2 we shall briefly review the most popular Wigner models of irreversible dynamics.

In this paper, we consider the case of an open quantum system in a high-field regime, more precisely, of an electron ensemble subject to an external potential, whose effect is comparable with the interaction with the ion crystal. Much interest was devoted in the recent past to high-field models in the classical charged particle transport, defined by the regime where drift velocity dominates thermal velocity. Some mathematical results on the analysis of runaway particles for the Boltzmann equation are available [11–13]. Moreover, the literature on high-field asymptotics is rather rich, see [14–17] and the papers cited there in.

Including high-field effects has great relevance in semiconductor simulation. A macroscopic model of this evolution is expected to contain field-dependent transport parameters, that are typically deduced via fitting procedures. We refer the reader to [18] for an updated review of derivations of semi-classical high-field drift-diffusion models by diverse limit procedures: in particular, in [18], explicit field-dependent mobilities are obtained from an energy-transport model. On the contrary, in [19], a high-field drift-diffusion model with non-explicit field-dependent coefficients is derived, as the limit of a Spherical Harmonics Expansion of the semi-classical Boltzmann equation.

We present a rigorous derivation of a Quantum drift-diffusion (QDD) equation with an explicit field-dependent mobility and diffusion coefficient. We shall start from the Wigner equation with an additional linear Bhatnager–Gross–Krook (BGK) term, modeling the interaction with the environment, and then adapt the equation to the high-field case, by rescaling it in terms of the Knudsen

* Corresponding author.

E-mail address: giovanni.frosali@unifi.it (G. Frosali).

number ϵ . Thus, our contribution is the quantum counterpart of [11]. We recall that, in [20], the starting point is the Wigner-BGK equation as well, but collisions are considered to be the strongest mechanism during the evolution (moderately high-field regime), and the relaxation term is derived via a Chapman–Enskog procedure. In our case, the additional relaxation term is instead an $\mathcal{O}(\hbar^2)$ -approximation of the Wigner-transformed relaxation term in operator formulation (cf. Section 2). Moreover, we perform an asymptotic expansion of the unknown Wigner function in terms of ϵ , according to a modified Chapman–Enskog procedure introduced in [21]. This method has been applied to many kinetic models and constitutes a valuable tool for a rigorous asymptotic derivation of macroscopic models (cf. Section 5). We substitute the Wigner unknown in the ordinary evolution problem, with the expansion of order ϵ^2 , and we get an approximated problem: in particular, an equation with unknown the electron position-density. This equation is precisely the QDD equation corrected by the $\mathcal{O}(\hbar^2)$ -Bohmian term of order ϵ , and by field-dependent terms, of order ϵ as well. These terms contain the same field-dependent coefficients obtained in the semi-classical case [18,11].

The well-posedness of the $\mathcal{O}(\epsilon^2)$ -approximated problem is discussed in Sections 7 and 8, and finally, in Section 9, we prove that the difference between the solutions of the ordinary and of the approximated evolution problems is also of order ϵ^2 . In conclusion, with the present analysis we obtain a QDD equation with field-dependent mobility and diffusion coefficients and we prove rigorously that, up to a certain degree of accuracy, it constitutes a model of quantum transport in the high-field case. From the analytical point of view, this equation is a second-order parabolic PDE with non-homogeneous coefficients. In particular, it belongs to the class of singularly perturbed equations; accordingly, the well-posedness result, together with the regularity estimates derived in Section 8, are complementary to the discussion in [22] about the same class of equations with constant coefficients. A counterpart of our analysis is the well-posedness study of the quantum drift-diffusion equation, in the fourth-order formulation obtained via a “classical-equilibrium” approximation [5]. We remark that the asymptotic procedure used here presents analogies with the Chapman–Enskog one in kinetic theory; nevertheless, generally the latter does not deal with the “initial layer” problem, namely, the instants close to the initial one are excluded from the analysis, due to the rapid changes of the solution [11]. The only exception we know is [23]. In contrast, in the presented approach the initial layer problem is solved simultaneously with the derivation of the drift-diffusion equation.

2. Wigner-BGK equations

Let us consider a quantum system with d degrees of freedom, evolving under the effect of an external potential $V = V(x)$, $x \in \mathbb{R}^d$. The Wigner equation with unknown the quasi-distribution function $w = w(x, v, t)$, $(x, v) \in \mathbb{R}^{2d}$, $t > 0$, provides a kinetic description of the evolution of the system. It reads

$$\frac{\partial w}{\partial t} + v \cdot \nabla_x w - \mathcal{O}[V]w = 0, \quad (x, v) \in \mathbb{R}^{2d}, t > 0, \quad (2.1)$$

with the pseudo-differential operator $\mathcal{O}[V]$ defined by

$$\begin{aligned} (\mathcal{O}[V]w)(x, v, t) &= \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta V(x, \eta) w(x, v', t) e^{i(v-v') \cdot \eta} dv' d\eta \\ &= \frac{i}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \delta V(x, \eta) \mathcal{F} w(x, \eta, t) e^{iv \cdot \eta} d\eta, \end{aligned} \quad (2.2)$$

where

$$\delta V(x, \eta) := \frac{1}{\hbar} \left[V \left(x + \frac{\hbar \eta}{2m} \right) - V \left(x - \frac{\hbar \eta}{2m} \right) \right]$$

and $\mathcal{F}f(\eta) \equiv [\mathcal{F}_{v \rightarrow \eta} f](\eta)$ denotes the partial Fourier transform of w with respect to the variable v ; the corresponding new variable is denoted by η . In the Fourier-transformed space $\mathbb{R}_x^d \times \mathbb{R}_\eta^d$ the operator $\mathcal{O}[V]$ is the multiplication operator by the function $i \delta V$; in symbols,

$$\mathcal{F}(\mathcal{O}[V]w)(x, \eta) = i \delta V(x, \eta) \mathcal{F} w(x, \eta). \quad (2.3)$$

Eq. (2.1) corresponds via a Wigner-transform to the von Neumann equation describing the conservative dynamics of an isolated quantum system [3]. Successive modifications of the Wigner model have been proposed to picture an irreversible interaction of the system with the environment. In [24] a scattering term is derived by a weak-coupling limit; however, due to its non-locality, it is not suitable for simulations and for mathematical analysis. A second possibility is an additional diffusive term, as in the quantum counterpart of Fokker–Planck (FP) equation of classical kinetic theory [25] (cf. [26] for the latest derivation and [27,28] for the latest well-posedness results). Unlike the Wigner equation with the scattering term, the quantum FP equation is the Wigner-transformed version of a Markovian master equation in Lindblad form, namely, it is the kinetic version of a quantum-physically correct model [29]. The shape of the drift-diffusion equations corresponding to the low-field, respectively high-field, scaling of the classical, respectively quantum, FP equations are presented in [30]. Another possibility is to insert a BGK operator, either linear or non-linear, like in [31], meaning that after a time $1/\nu$ the system relaxes to a prescribed state w_{eq} ; namely,

$$\begin{aligned} \frac{\partial w}{\partial t} + v \cdot \nabla_x w - \mathcal{O}[V]w &= -\nu(w - w_{\text{eq}}), \\ (x, v) \in \mathbb{R}^{2d}, t > 0. \end{aligned} \quad (2.4)$$

In the recent literature [7,32,20,9], diverse relaxation-time states w_{eq} have been proposed.

The standard picture is that the system converges to a state of thermodynamical equilibrium with the surrounding environment at temperature T . The operator $e^{-\beta H}$ gives the statistical equilibrium state at (constant) temperature $T = 1/k\beta$ (k is the Boltzmann constant), H being the energy operator associated to the system. The von Neumann equation modified by a relaxation-time term containing $e^{-\beta H}$ is in Lindblad form [33]. Accordingly, a Wigner-BGK model, being the Wigner-transformed version of that equation, formally belongs to the class of quantum-physically correct kinetic models. The same Wigner-BGK model shall contain the Wigner-transformed $e^{-\beta H}$ as the relaxation-time state. In his pioneering article [3], Wigner applies an expansion in terms of \hbar to the Wigner function corresponding to the operator $e^{-\beta H}$ and obtains the classical equilibrium distribution function on the phase space with correction of non-odd order in \hbar :

$$\begin{aligned} w_{\text{W}}(x, v) &:= \left(\frac{m}{2\pi\hbar} \right)^d e^{-\beta \mathcal{E}} \left\{ 1 + \hbar^2 \frac{\beta^2}{24} \left[-\frac{3}{m} \Delta V + \frac{\beta}{m} |\nabla V|^2 \right. \right. \\ &\quad \left. \left. + \beta \sum_{r,s=1}^d v_r v_s \frac{\partial^2 V}{\partial x_r \partial x_s} \right] + \mathcal{O}(\hbar^4) \right\}, \end{aligned} \quad (2.5)$$

where $\mathcal{E}(x, v) := mv^2/2 + V(x)$ is the total energy of the system. Let us call w_{eq} its local (in time and space) version, defined by

$$w_{\text{eq}}(x, v, t) := C(x, t) w_{\text{W}}(x, v),$$

with C to be chosen. By assuming, like in [32],

$$\int w_{\text{eq}}(x, v, t) dv = \int w(x, v, t) dv =: n[w](x, t) \equiv n(x, t), \quad (2.6)$$

and since, by direct computation,

$$\int w_W(x, v) dv = \left(\frac{m}{2\pi \hbar^2 \beta} \right)^{d/2} e^{-\beta V} \times \left\{ 1 + \hbar^2 \frac{\beta^2}{12m} \left[-\Delta V + \frac{\beta}{2} |\nabla V|^2 \right] + \mathcal{O}(\hbar^4) \right\},$$

the local Wigner thermal equilibrium function w_{eq} equals

$$w_{eq}(x, v, t) = n(x, t) \left(\frac{\beta m}{2\pi} \right)^{d/2} e^{-\beta m v^2/2} \times \left\{ 1 + \hbar^2 \frac{\beta^2}{24} \left[-\frac{1}{m} \Delta V + \beta \sum_{r,s=1}^d v_r v_s \frac{\partial^2 V}{\partial x_r \partial x_s} \right] + \mathcal{O}(\hbar^4) \right\}. \quad (2.7)$$

In (2.7) one can recognize the classical (normalized) Maxwellian

$$F(v) := \left(\frac{\beta m}{2\pi} \right)^{d/2} e^{-\beta m v^2/2}, \quad (2.8)$$

the particle density n as parameter and the constant temperature $1/k\beta$, with an additional correction term of order \hbar^2 . We shall consider the expression (2.7) as the $\mathcal{O}(\hbar^2)$ -approximation of the Wigner function associated to the state to which the quantum system shall approach.

An alternative strategy to determine the relaxation-time state is presented in [34]; it is an extension to the quantum case of Levermore's one for classical kinetic equations ([35], cf. [6] for semi-classical equations). It consists of tackling a constrained minimization problem for the relative entropy of the quantum system under consideration, with respect to the environment. In the quantum case the procedure is performed at the operator level in terms of the operators describing the states of the quantum system, due to the non-local definition of the entropy. However, the constraints for the minimization procedure are considered at the kinetic level. Thus, the Wigner transform \mathcal{W} is used intensively to switch from the operator formulation to the kinetic one, any time it is required by the procedure. Due to that, the expression of the minimizer of the entropy formally derived in [7] is non-explicit. Nevertheless, in [7], is formally proved that $\mathcal{W}\{\exp \mathcal{W}^{-1} f\} = \exp f + \mathcal{O}(\hbar^2)$ with f defined on the phase-space. Accordingly, the (formal) minimizer reads

$$w_A(x, v, t) := e^{(A-\beta m v^2/2)} \left\{ 1 + \hbar^2 \frac{\beta^2}{8} \left[+\frac{1}{m} \Delta A + \frac{\beta}{3m} |\nabla A|^2 + \frac{\beta}{3} \sum_{r,s=1}^d v_r v_s \frac{\partial^2 A}{\partial x_r \partial x_s} \right] + \mathcal{O}(\hbar^4) \right\} \quad (2.9)$$

with $A = A(x, t)$ Lagrange multiplier used for the constrained minimization procedure, i.e.

$$\int w_A(x, v, t) dv = n(x, t).$$

By comparison of the expression (2.9) with (2.7), it can be easily seen that they coincide if one identifies the Lagrange multiplier A with $-\beta V$. In [9] it is indeed proved that $A = -\beta V + \mathcal{O}(\hbar^2)$ holds.

Remark 2.1. It is crucial to recall that the correspondence via Wigner-transform of the operator and the kinetic formulations is merely formal, unless certain assumptions are posed both on the Wigner functions and on the operators [36]. This point creates analytical difficulties in stating rigorously the well-posedness of the derivation in [7]. For the same reason, the analysis of Wigner equations is set in the Hilbert space L^2 , since the necessary condition for the rigorous correspondence is satisfied [36] (cf. [28], e.g.). \square

As a consequence of the previous discussion, in the present article we shall adopt the Wigner-BGK equation (2.4) containing (2.7) on the right-hand side as the model of the open quantum system evolution. In particular, we remark that we shall consider the operator on the right-hand side as an $\mathcal{O}(\hbar^2)$ -approximation, in the kinetic framework, of the dissipative dynamics induced by the interaction with the environment.

3. The high-field Wigner-BGK equation

Our aim is describing an open quantum system subject to a strong external potential; in particular, the action of the potential is to be considered comparable with the interaction with the environment. In order to adapt the Wigner-BGK equation (2.4) to this specific case, we rewrite it by using dimensionless variables and, for this purpose, we introduce the time-scales of the action of the external potential and of the interaction with the environment. Let us call t_V the potential characteristic time and t_C the mean free time between interactions of the system with the background. Then we introduce $x' = x/x_0$, $v' = v/v_0$, $t' = t/t_0$, with x_0 , v_0 , t_0 characteristic quantities, and we call $w' = w(x', v', t')$ the rescaled Wigner function (observe that we can indeed neglect to rescale the Wigner function). Thus, we obtain

$$\frac{x_0}{v_0 t_0} \frac{\partial}{\partial t} w + v \cdot \nabla_x w - \frac{x_0}{v_0 t_V} \Theta[V] w = -\frac{x_0}{v_0 t_C} v (w - w_{eq}),$$

$$t > 0, (x, v) \in \mathbb{R}^{2d},$$

where we have omitted the prime everywhere. If we introduce the relation $x_0 = v_0 t_0$, we obtain

$$\frac{\partial}{\partial t} w + v \cdot \nabla_x w - \frac{t_0}{t_V} \Theta[V] w = -\frac{t_0}{t_C} v (w - w_{eq}),$$

$$t > 0, (x, v) \in \mathbb{R}^{2d}.$$

In the following we assume that the times t_V and t_C are comparable, in the sense

$$\frac{t_V}{t_0} \approx \frac{t_C}{t_0} \approx \epsilon, \quad (3.1)$$

where $\epsilon := l/x_0$ is the Knudsen number, since $l := v_0 t_C$ is the characteristic length corresponding to the classical mean free path. This corresponds to saying that the external potential and the interactions coexist during the evolution. In particular, $\epsilon \approx 0$ corresponds to an evolution in which the effect of the interactions is dominant on the transport ($t_C \ll t_0$ or equivalently $l \ll x_0$). However, at this time the action of the external potential has the same strength, due to the assumption (3.1) ($t_V \ll t_0$). In fact, the resulting equation is

$$\epsilon \frac{\partial w}{\partial t} + \epsilon v \cdot \nabla_x w - \Theta[V] w = -v (w - w_{eq}),$$

$$t > 0, (x, v) \in \mathbb{R}^{2d}. \quad (3.2)$$

We recall that it is the quantum counterpart of the one studied by Poupaud in [11].

Now we put (3.2) in abstract form. As motivated in Remark 2.1, a suitable setting for problems in Wigner formulation is the Hilbert space $L^2(\mathbb{R}^{2d})$. However, in order to give a rigorous sense to the expression

$$n(x) := \int w(x, v) dv, \quad \forall x \in \mathbb{R}^d, \quad (3.3)$$

which enters the equation via the definition (2.7) of w_{eq} , we introduce the subspace $X_k := L^2(\mathbb{R}^{2d}, (1 + |v|^{2k}) dx dv; \mathbb{R})$, with $k \in \mathbb{N}$, endowed with the norm

$$\|w\|_{X_k}^2 = \int_{\mathbb{R}^{2d}} |w(x, v)|^2 (1 + |v|^{2k}) dx dv.$$

Let us call X_k^v the Hilbert space $L^2(\mathbb{R}^d, (1 + |v|^{2k})dv; \mathbb{R})$ and H_k^m the Sobolev space $H_x^m \otimes X_k^v$. The weight k has to be chosen according to the space dimension d : we call d -admissible

$$k \in \mathbb{N} \quad \text{such that } 2k > d.$$

The function $n(x)$ in (3.3) is well-posed for all $w \in X_k$ with d -admissible k , since

$$\left| \int_{\mathbb{R}^d} w(x, v) dv \right| \leq C(d, k) \left(\int_{\mathbb{R}^d} |w(x, v)|^2 (1 + |v|^k)^2 dv \right)^{1/2}, \quad \forall x \in \mathbb{R}^d$$

by the Hölder inequality (cf. [37]). We define the streaming operator S by

$$Sw = -v \cdot \nabla_x w, \quad D(S) = \{w \in X_k \mid Sw \in X_k\},$$

and the operators

$$\mathcal{A}w := \Theta[V]w, \quad \mathcal{C}w := -(vw - \Omega w), \quad \forall w \in X_k \quad (3.4)$$

with the operator Ω defined by

$$\Omega w(x, v) := vF(v) \left\{ 1 + \hbar^2 \frac{\beta^2}{24} \left[-\frac{1}{m} \Delta V + \beta \sum_{r,s=1}^d v_r v_s \frac{\partial^2 V}{\partial x_r \partial x_s} \right] \right\} \times \int w(x, v') dv'.$$

The function $F(v)$ is the normalized Maxwellian, given by (2.8). Observe that we substitute the function w_{eq} defined in (2.7) with the operator Ωw , that differs from w_{eq} by terms of order \hbar^4 . Let us call $F^{(2)}$ the $O(\hbar^2)$ -coefficient in the above definition of Ω

$$F^{(2)}(x, v) \equiv F^{(2)}[V](x, v) = \frac{\beta^2}{24} \left[-\frac{1}{m} \Delta V + \beta \sum_{r,s=1}^d v_r v_s \frac{\partial^2 V}{\partial x_r \partial x_s} \right] F(v),$$

such that

$$\Omega w(x, v) \equiv v n[w](x) [F(v) + \hbar^2 F^{(2)}(x, v)].$$

Observe that such an expression for Ωw can be seen as an $O(\hbar^2)$ -correction to the classical product $n(x)F(v)$.

In conclusion, we write Eq. (3.2) in the abstract form

$$\begin{cases} \epsilon \frac{dw}{dt} = \epsilon Sw + \mathcal{A}w + \mathcal{C}w, \\ \lim_{t \rightarrow 0^+} \|w(t) - w_0\|_{X_k} = 0 \end{cases} \quad (3.5)$$

where w_0 is the initial condition.

In next Lemma we specify under which assumptions the abstract definition (3.4) of the operator $\mathcal{A} + \mathcal{C}$ is well-posed.

Lemma 3.1. *If $V \in H_x^k$ with d -admissible k and $\Delta V \in L_x^\infty$, then the operator $\mathcal{A} + \mathcal{C}$ is well-defined from X_k into itself, and is bounded by*

$$\|\mathcal{A} + \mathcal{C}\|_{\mathcal{B}(X_k)} \leq C(d, k) \left[\|V\|_{H_x^k} + v \|\Delta V\|_{L_x^\infty} \|F\|_{X_{k+2}^v} + v \|F\|_{X_k^v} + v \right].$$

Moreover, $\mathcal{A} + \mathcal{C}$ is well-defined from X_k^v into itself, and is bounded by

$$\|\mathcal{A} + \mathcal{C}\|_{\mathcal{B}(X_k^v)} \leq C(d, k) \left(\|V\|_{H_x^k} + v \|\Delta V(x)\| \|F\|_{X_{k+2}^v} + v \|F\|_{X_k^v} + v \right). \quad (3.6)$$

Proof. Here and in the following we denote by C non necessarily equal constants.

The arguments are similar to those in [38], so we just give a sketch of the proof. First of all, by (cf. (2.3)), for all $w \in X_k$ it holds that

$$\begin{aligned} \|\Theta[V]w\|_{X_k}^2 &= C \|\delta V \mathcal{F} w\|_{L_{x,\eta}^2}^2 + C \left\| \sum_{i=1}^d \frac{\partial^k}{\partial \eta_i^k} (\delta V \mathcal{F} w) \right\|_{L_{x,\eta}^2}^2 \\ &\leq 2C \|V\|_{L_x^\infty}^2 \|w\|_{L_{x,v}^2}^2 + C \left\| \sum_{i=1}^d \frac{\partial^k}{\partial \eta_i^k} (\delta V \mathcal{F} w) \right\|_{L_{x,\eta}^2}^2, \end{aligned}$$

since $2k > d$ guarantees that $H_x^k \hookrightarrow L_x^\infty$. Here, the constant C is due to the Fourier transform. Then, by applying the product-formula rule and using Sobolev embeddings for the functions V and $\mathcal{F} w \in L_x^2 \otimes H_\eta^k$, it follows that $\|\Theta[V]\|_{\mathcal{B}(X_k)} \leq C \|V\|_{H_x^k}$. Moreover, for all $w \in X_k$ with $2k > d$,

$$\begin{aligned} \|\Omega w\|_{X_k}^2 &\leq v \int_{\mathbb{R}^{2d}} (1 + |v|^{2k}) \left(1 + \frac{\beta^4 \hbar^4}{24^2 m^2} |\Delta V|^2(x) \right) |F(v)|^2 \\ &\quad \times \left| \int_{\mathbb{R}^d} w(x, v') dv' \right|^2 dx dv \\ &\quad + \int_{\mathbb{R}^{2d}} (1 + |v|^{2k}) \frac{\beta^6 \hbar^4}{24^2 m^4} \left| \sum_{r,s=1}^d v_r v_s \frac{\partial^2 V(x)}{\partial x_r \partial x_s} F(v) \right|^2 \\ &\quad \times \left| \int_{\mathbb{R}^d} w(x, v') dv' \right|^2 dx dv \\ &\leq v C (1 + \|\Delta V\|_{L_x^\infty}^2) \|F\|_{X_k^v}^2 \|w\|_{X_k}^2 \\ &\quad + v C \|\Delta V\|_{L_x^\infty}^2 \|F\|_{X_{k+2}^v}^2 \|w\|_{X_k}^2, \end{aligned} \quad (3.7)$$

since $F \in X_k$, $\forall k$. Then the estimate of $\|\mathcal{C}\|_{\mathcal{B}(X_k)}$ is straightforward. The estimate (3.6) in X_k^v can be proved analogously. ■

We remark that the existence and uniqueness of a solution in X_k of the initial value system (3.5) for any $\epsilon > 0$ can be stated under the assumptions of Lemma 3.1 by using arguments of semigroup theory, analogously to [37].

4. Well-posedness of the problem with $\epsilon = 0$

The aim of this paper is the asymptotic analysis of the system (3.5) with a Chapman–Enskog type procedure. The first step of the analysis is solving Eq. (3.5) with $\epsilon = 0$. This corresponds to individuating the Wigner function describing the state of the system in case the interaction of the environment and the action of the potential are dominant with respect to the transport. We remark that the function w_{eq} defined by (2.7) describes the state to which the system relaxes under the sole interaction with the environment.

We consider the equation $(\mathcal{A} + \mathcal{C})w = 0$ in the space X_k : the variable x can be considered as a parameter in the analysis, thus we shall study $(\mathcal{A} + \mathcal{C})w = 0$ in the space X_k^v for any fixed $x \in \mathbb{R}^d$. However, with an abuse of language, we shall also denote the operators by the same letters when x is fixed. We can state the following proposition.

Proposition 4.1. *If $V \in H_x^{\tilde{k}}$ with $\tilde{k} = \max\{2, k\}$, and d -admissible k , then for a fixed $x \in \mathbb{R}^d$*

$$\ker(\mathcal{A} + \mathcal{C}) = \{cM(v), c \in \mathbb{R}\} \subset X_k^v, \quad (4.1)$$

with

$$M(x, v) := v \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}F(\eta)}{v - i\delta V(x, \eta)} \left(1 - \frac{\beta \hbar^2}{24m^2} \times \sum_{r,s=1}^d \eta_r \eta_s \frac{\partial^2 V(x)}{\partial x_r \partial x_s} \right) \right\} (x, v), \quad v \in \mathbb{R}^d \quad (4.2)$$

for any fixed x . Moreover, for all $h \in X_k^v$, $(\mathcal{A} + \mathcal{C})w = h$ has a solution if and only if

$$\int_{\mathbb{R}^d} h(v)dv = 0. \quad (4.3)$$

Remark 4.1. It can be immediately deduced by the characterization (4.1) that the solution of the equation $(\mathcal{A} + \mathcal{C})w = 0$ in X_k is unique, except for a factor depending only on x . \square

Proof. By definition,

$$\ker(\mathcal{A} + \mathcal{C}) := \{w \in X_k^v \mid (v - \Theta[V])w = \Omega w\}.$$

For all $h \in X_k^v$, the Fourier-transformed version of $(v - \Theta[V])w = h$ reads $(v - i\delta V)\mathcal{F}w = \mathcal{F}h$. Thus,

$$w(v) = (v - \Theta[V])^{-1}h(v) := \mathcal{F}^{-1} \left(\frac{\mathcal{F}h(\eta)}{v - i\delta V(\eta)} \right) (v) \quad (4.4)$$

is the unique solution; equivalently, the operator $(v - \Theta[V])$ is invertible in X_k^v with bounded inverse, defined by (4.4). Precisely,

$$\begin{aligned} \|w\|_{X_k^v}^2 &= \|(v - \Theta[V])^{-1}h\|_{X_k^v}^2 \\ &= C \int \frac{|\mathcal{F}h(\eta)|^2}{v^2 + |\delta V(\eta)|^2} d\eta + C \sum_{r=1}^d \int \left| \frac{\partial^k}{\partial \eta_r^k} \frac{\mathcal{F}h(\eta)}{v - i\delta V(\eta)} \right|^2 d\eta \\ &\leq \frac{C}{v^2} \|h\|_{L_v^2}^2 + C \sum_{r=1}^d \int \left| \frac{\partial^k}{\partial \eta_r^k} \frac{\mathcal{F}h(\eta)(v + i\delta V(\eta))}{v^2 + (\delta V)^2(\eta)} \right|^2 d\eta, \end{aligned}$$

then, by applying product formula, it can be checked that, if $2k > d$,

$$\|(v - \Theta[V])^{-1}\|_{\mathcal{B}(X_k^v)} \leq C(1 + \|V\|_{H_x^k}). \quad (4.5)$$

Then,

$$\begin{aligned} w \in \ker(\mathcal{A} + \mathcal{C}) &\Leftrightarrow w = (v - \Theta[V])^{-1}\Omega w \\ \Leftrightarrow w &= v n[w](v - \Theta[V])^{-1}(F + \hbar^2 F^{(2)}[V]) \Leftrightarrow w = n[w]M(v), \end{aligned} \quad (4.6)$$

by definition of the operators $(v - \Theta[V])^{-1}$ and Ω . From this follows the characterization (4.1) of $\ker(\mathcal{A} + \mathcal{C})$, with the function M defined by

$$M(x, v) := v(v - \Theta[V])^{-1}(F(v) + \hbar^2 F^{(2)}[V](x, v)) \quad \forall v \in \mathbb{R}^d,$$

with the fixed $x \in \mathbb{R}^d$. Since $F + \hbar^2 F^{(2)}[V] \in X_k$ for all $k \in \mathbb{N}$, provided $\Delta V \in L_x^2$; then, due to the assumption on V and to (4.5), $M \in X_k^v$ if $2k > d$. For all $h \in X_k$, solving $(\mathcal{A} + \mathcal{C})w = h$ is equivalent to $(\mathcal{I} - (v - \Theta[V])^{-1}\Omega)w = -(v - \Theta[V])^{-1}h$. Moreover, by the equivalence (4.5), $\ker(\mathcal{A} + \mathcal{C}) = \ker(\mathcal{I} - (v - \Theta[V])^{-1}\Omega)$. Since $\ker(\mathcal{A} + \mathcal{C}) \neq \{0\}$, the operator $\mathcal{I} - (v - \Theta[V])^{-1}\Omega$ is not injective. If the operator $(v - \Theta[V])^{-1}\Omega$ is compact, by the Fredholm alternative, this is equivalent to $R((v - \Theta[V])^{-1}\Omega) \neq X_k^v$. The equation $(\mathcal{I} - (v - \Theta[V])^{-1}\Omega)w = M$ has indeed no solution, since

$$\int_{\mathbb{R}^d} (\mathcal{I} - (v - \Theta[V])^{-1}\Omega)w(v)dv = 0, \quad \forall w \in X_k^v,$$

(by the definition of the operator $(v - \Theta[V])^{-1}\Omega$), while, instead, $\int M(v)dv = \int F(v)dv = 1$. Analogously, for all $u \in \ker(\mathcal{A} + \mathcal{C})$, $(\mathcal{I} - (v - \Theta[V])^{-1}\Omega)w = u$ has no solution.

In conclusion, if we show that $(v - \Theta[V])^{-1}\Omega$ is a compact operator, then we can conclude by the Fredholm alternative that $(\mathcal{I} - (v - \Theta[V])^{-1}\Omega)w = h$ has a solution iff $\int h(v)dv = 0$. Analogously to Lemma 1 in [11], one can construct, by the Rellich–Kondrachov theorem, a sequence of bounded finite rank operators converging to $(v - \Theta[V])^{-1}\Omega$. Thus the thesis follows. \blacksquare

Finally, let us compute the first and the second moments of the function M :

Lemma 4.1. Let $V \in H_x^{k+2}$ with d -admissible k . Then, the function M defined by (4.2) satisfies

$$(\mathcal{A} + \mathcal{C})w = 0 \Leftrightarrow w = n[w]M \quad \text{with} \quad \int M(x, v)dv = 1, \quad (4.7)$$

$$\int vM(x, v)dv = -\frac{1}{vm} \nabla V(x), \quad (4.8)$$

$$\begin{aligned} \int v \otimes vM(x, v)dv \\ = \frac{\mathcal{I}}{\beta m} + \frac{2}{v^2 m^2} \nabla V \otimes \nabla V + \frac{\beta \hbar^2}{12m^2} \nabla \otimes \nabla V. \end{aligned} \quad (4.9)$$

Proof. (4.7) follows by (4.5). Moreover, since $V \in H_x^{k+2}$ with $2k > d$, then the function M

$$M(x, v) = v \mathcal{F}^{-1} \left(\frac{\mathcal{F}(F + \hbar^2 F^{(2)})}{v - i\delta V(x, \eta)} \right) (x, v),$$

belongs to X_{k+2} . By calculus rules in the Fourier space, since F is smooth, it holds

$$\int vM(x, v)dv = iv \left[\nabla_\eta \left(\frac{\mathcal{F}(F + \hbar^2 F^{(2)})}{v - i\delta V(x, \eta)} \right) \right] (x, 0).$$

By taking the derivative and using

$$\mathcal{F}(F + \hbar^2 F^{(2)})(x, 0) = \mathcal{F}F(0) = 1,$$

$$\nabla_\eta \mathcal{F}(F + \hbar^2 F^{(2)})(x, 0) = 0,$$

and $(\nabla_\eta \delta V)(x, 0) = \nabla_x V(x)/m$, one gets (4.8).

Analogously, the second moments of M are well defined, and, by the calculus rules, it holds that

$$\int v_i v_j M(x, v)dv = -v \left[\frac{\partial^2}{\partial \eta_i \partial \eta_j} \frac{\mathcal{F}(F + \hbar^2 F^{(2)})}{v - i\delta V} \right] (x, 0),$$

$$\forall i, j = 1, \dots, d,$$

and

$$\begin{aligned} &-\frac{\partial^2}{\partial \eta_i \partial \eta_j} \left(\frac{\mathcal{F}F}{v - i\delta V} \right) (x, 0) \\ &= -\frac{1}{v} \left(\frac{\partial^2 \mathcal{F}F}{\partial \eta_i \partial \eta_j} \right) (x, 0) + \frac{2}{v^3} \left(\frac{\partial \delta V}{\partial \eta_i} \frac{\partial \delta V}{\partial \eta_j} \right) (x, 0) \\ &= \frac{1}{v\beta m} + \frac{2}{v^3 m^2} \frac{\partial V(x)}{\partial x_i} \frac{\partial V(x)}{\partial x_j}, \\ &-\frac{\partial^2}{\partial \eta_i \partial \eta_j} \left(\frac{\mathcal{F}F^{(2)}}{v - i\delta V} \right) (x, 0) = -\frac{1}{v} \left(\frac{\partial^2 \mathcal{F}F^{(2)}}{\partial \eta_i \partial \eta_j} \right) (x, 0) \\ &= \frac{\beta}{12m^2 v} \frac{\partial^2 V(x)}{\partial x_i \partial x_j}. \end{aligned}$$

Thus the thesis follows. \blacksquare

Remark 4.2. The state that describes the system under the effect of the interaction with the environment and of the strong potential is described by the function nM , with M defined by (4.2). The fluid velocity relative to such state is non-zero and given by (4.8). In contrast, the velocity of the system in the state w_{eq} defined by (2.7) (i.e., when it is only subject to the influence of the environment), is $\int v w_{eq} dv = 0$, as expected since it is an equilibrium state. Moreover, the expression of the second moment tensor (4.9) has to be compared with

$$\int v \otimes v w_{eq} dv = n \left(\frac{\mathcal{I}}{\beta m} + \frac{\beta \hbar^2}{12m^2} \nabla \otimes \nabla V \right).$$

They differ by the second summand in (4.9) that is to be referred to the strong-field assumption (cf. [11]). \square

As a consequence of Proposition 4.1, the following subspace is well-defined

$$(X_k)_M := \{ \alpha(x)M(x, v), \alpha \in L^2_x \} \subset X_k,$$

which coincides with $\ker(\mathcal{A} + \mathcal{C})$ when $\mathcal{A} + \mathcal{C}$ is considered as an operator on X_k . Accordingly, we can decompose the space X_k as

$$X_k = (X_k)_M \oplus (X_k)^0 \tag{4.10}$$

with

$$(X_k)^0 := \left\{ w \in X_k \mid \int w(x, v) dv = 0 \right\},$$

and define the corresponding spectral projection \mathcal{P} from X_k into $(X_k)_M$, by

$$\mathcal{P}w := M \int_{\mathbb{R}^d_v} w(x, v) dv,$$

and $\mathcal{Q} := \mathcal{I} - \mathcal{P}$. The following corollary is still a preliminary result for our asymptotic procedure.

Corollary 4.1. *Let $V \in H^k_x$ with d -admissible k . Then, the operator $\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}$ is an isomorphism of $(X_k)^0$ onto itself, with*

$$\|\mathcal{A} + \mathcal{C}\|_{\mathcal{B}(X_k)} \leq C(d, k) \left(\|V\|_{H^k_x} + \nu \right). \tag{4.11}$$

If, in addition, $V \in H^{k+j}_x$ with $j > 0$, then $\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}$ is an isomorphism of $(H^j_k)^0$ onto itself, with

$$\|\mathcal{A} + \mathcal{C}\|_{\mathcal{B}(H^j_k)} \leq C(d, k, j) \left(\|V\|_{H^{k+j}_x} + \nu \right). \tag{4.12}$$

Proof. The operator $\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}$, when considered as an operator acting on $(X_k)^0$, reduces to

$$\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}u = \mathcal{O}[V]u - \nu u, \quad \forall u \in (X_k)^0. \tag{4.13}$$

Then the thesis follows from Lemma 3.1, Proposition 4.1, and by the skew-symmetry of the pseudo-differential operator. The second statement and estimate (4.12) can be proved analogously. \blacksquare

5. The asymptotic expansion

According to the decomposition of the space X_k , every function $w \in X_k$ can be written as $w = \mathcal{P}w + \mathcal{Q}w$, with $\mathcal{P}w \in (X_k)_M$ and $\mathcal{Q}w \in (X_k)^0$. Let us call $\varphi := \mathcal{P}w$ and $\psi := \mathcal{Q}w$. Observe that, for all $w \in X_k$, $\int \mathcal{P}w(x, v) dv = n[w](x)$, while $\int \mathcal{Q}w(x, v) dv = 0$, that is, we separate the part of w that contributes to the density $n[w]$ from the other one. Precisely, it holds $\mathcal{P}w = n[w]M$, by definition.

Applying formally the projection \mathcal{P} , respectively \mathcal{Q} , to the Wigner-BGK equation (3.5) with unknown w , we obtain the following system of equations with unknown φ and ψ

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \mathcal{P}S\mathcal{P}\varphi + \mathcal{P}S\mathcal{Q}\psi \\ \frac{\partial \psi}{\partial t} = \mathcal{Q}S\mathcal{P}\varphi + \mathcal{Q}S\mathcal{Q}\psi + \frac{1}{\epsilon}\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\psi \end{cases} \tag{5.1}$$

where we used $(\mathcal{A} + \mathcal{C})\mathcal{P}\varphi = 0$ and $\mathcal{P}(\mathcal{A} + \mathcal{C})\mathcal{Q}\psi = 0$, together with the initial conditions

$$\varphi(0) = \varphi_0 = \mathcal{P}w_0, \quad \psi(0) = \psi_0 = \mathcal{Q}w_0. \tag{5.2}$$

System (5.1) consists of an evolution problem with unknown functions $\varphi = n[w]M$ and ψ , and it is supplemented by the initial conditions (5.2). It is a reformulation of (3.5).

Since we expect the solution w to be subject to rapid changes for small times, we split the functions φ and ψ into the sums of the “bulk” parts $\bar{\varphi}$ and $\bar{\psi}$ and of the “initial layer” parts $\tilde{\varphi}$ and $\tilde{\psi}$,

$$\varphi(t) = \bar{\varphi}(t) + \tilde{\varphi}\left(\frac{t}{\epsilon}\right), \quad \psi(t) = \bar{\psi}(t) + \tilde{\psi}\left(\frac{t}{\epsilon}\right).$$

The bulk part $\bar{\varphi}$ is left *unexpanded* and the other parts are expanded in terms of ϵ as follows

$$\begin{aligned} \bar{\varphi}(\tau) &= \bar{\varphi}_0(\tau) + \epsilon \bar{\varphi}_1(\tau) + \epsilon^2 \bar{\varphi}_2(\tau) + \dots \\ \bar{\psi}(t) &= \bar{\psi}_0(t) + \epsilon \bar{\psi}_1(t) + \epsilon^2 \bar{\psi}_2(t) + \dots \\ \tilde{\psi}(\tau) &= \tilde{\psi}_0(\tau) + \epsilon \tilde{\psi}_1(\tau) + \epsilon^2 \tilde{\psi}_2(\tau) + \dots, \end{aligned} \tag{5.3}$$

with $\tau = t/\epsilon$. Accordingly, Eq. (5.1) for the bulk part terms of the expansion up to the order ϵ^2 become

$$\begin{cases} \frac{\partial \bar{\varphi}}{\partial t} = \mathcal{P}S\mathcal{P}\bar{\varphi} + \mathcal{P}S\mathcal{Q}\bar{\psi}_0 + \epsilon \mathcal{P}S\mathcal{Q}\bar{\psi}_1 \\ 0 = \mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\bar{\psi}_0 \\ 0 = \mathcal{Q}S\mathcal{P}\bar{\varphi} + \mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\bar{\psi}_1 \end{cases} \tag{5.4}$$

while the equations for the initial layer parts read

$$\begin{cases} \frac{\partial \tilde{\varphi}_0}{\partial \tau} = 0, \\ \frac{\partial \tilde{\varphi}_1}{\partial \tau} = \mathcal{P}S\mathcal{Q}\tilde{\psi}_0(\tau) \\ \frac{\partial \tilde{\psi}_0}{\partial \tau} = \mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\tilde{\psi}_0(\tau) \\ \frac{\partial \tilde{\psi}_1}{\partial \tau} = \mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\tilde{\psi}_1(\tau) + \mathcal{Q}S\mathcal{Q}\tilde{\psi}_0(\tau) \end{cases} \tag{5.5}$$

and the initial conditions (5.2) yield

$$\begin{cases} \bar{\varphi}(0) + \tilde{\varphi}_0(0) + \epsilon \tilde{\varphi}_1(0) = \varphi_0 \\ \bar{\psi}_0(0) + \tilde{\psi}_0(0) = \psi_0 \\ \tilde{\psi}_1(0) + \tilde{\psi}_1(0) = 0. \end{cases} \tag{5.6}$$

System (5.4), together with (5.5)–(5.6), is an $\mathcal{O}(\epsilon^2)$ -approximated version of (5.1) with (5.2), once the expansion (5.3) has been introduced. In fact, the equations in (5.4) can be decoupled: by Corollary 4.1, the operator $\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}$ is invertible in $(X_k)^0$, thus

$$\bar{\psi}_0 \equiv 0 \tag{5.7}$$

$$\bar{\psi}_1 = -(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})^{-1}\mathcal{Q}S\mathcal{P}\bar{\varphi}, \tag{5.8}$$

which implies

$$\frac{\partial \bar{\varphi}}{\partial t} = \mathcal{P}S\mathcal{P}\bar{\varphi} - \epsilon \mathcal{P}S\mathcal{Q}(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})^{-1}\mathcal{Q}S\mathcal{P}\bar{\varphi}. \tag{5.9}$$

Thus, system (5.4) reduces to the system (5.8)–(5.9), with unknown functions $\bar{\varphi}(x, v, t) = n(x, t)M(x, v)$ and $\tilde{\psi}_1$. The next section

shall be dedicated to reformulate Eq. (5.9) as an equation with unknown n . The analysis of system (5.5), with unknown $\tilde{\varphi}$ and $\tilde{\psi}$ and initial conditions (5.6), is postponed to Section 7: it shall provide an appropriate initial condition for Eq. (5.9). Finally, in Sections 7 and 8 we shall establish a well-posedness result for the approximated problem. In our main theorem (cf. Theorem 9.1), we shall prove that the solution $\varphi + \psi$ of equations (5.1) indeed differs from $[\tilde{\varphi}(t) + \tilde{\varphi}_0(\tau) + \epsilon \tilde{\varphi}_1(\tau)] + [\tilde{\psi}_0(t) + \epsilon \tilde{\psi}_1(t) + \tilde{\psi}_0(\tau) + \epsilon \tilde{\psi}_1(\tau)]$, satisfying the approximated problem (5.4)–(5.6), by a term of order ϵ^2 .

6. The high-field quantum drift-diffusion equation

The aim of the present section is the reformulation of the abstract equation (5.9) as an equation with unknown n .

Lemma 6.1. *Let $V \in H_x^{k+2}$ with d -admissible k . Eq. (5.9) with unknown $\tilde{\varphi}(x, v, t) = n(x, t)M(x, v)$ can be rewritten as an evolution equation with unknown $n(x, t)$ of the form*

$$\begin{aligned} \frac{\partial n}{\partial t} - \frac{1}{vm} \nabla \cdot (n \nabla V) - \frac{\epsilon}{v\beta m} \nabla \cdot \nabla n \\ - \frac{\epsilon}{v^3 m^2} [\nabla \cdot (n(\nabla \otimes \nabla)V \nabla V) + \nabla \cdot \nabla \cdot (n \nabla V \otimes \nabla V)] \\ - \frac{\epsilon \beta \hbar^2}{12vm^2} \nabla \cdot \nabla \cdot (n \nabla \otimes \nabla V) = 0. \end{aligned} \quad (6.1)$$

Remark 6.1. The first line of (6.1) consists of the terms of the classical DD equation. The second line is peculiar to the strong-field assumption, being a correction of order ϵ , and consists of the additional term

$$\frac{1}{v} \frac{\nabla V \otimes \nabla V}{v^2 m^2}$$

in the pressure tensor, and of the term

$$\frac{1}{v} \left(\frac{(\nabla \otimes \nabla)V \nabla V}{v^2 m^2} \right)$$

in the drift term. Both terms are quadratic in the potential V . The second line can also be written as

$$-\frac{\epsilon}{v^3 m^2} \nabla \cdot [\nabla V \otimes \nabla V \nabla n + n(2\nabla \otimes \nabla V \nabla V + \Delta V \nabla V)].$$

This expression is the same obtained in [11] from the semi-classical Boltzmann equation with high-field scaling. The last line is the quantum pressure term (cf. [32,8]). \square

The proof requires the following preliminary lemmata.

Lemma 6.2. *Let $V \in H_x^{k+2}$ with d -admissible k . Then the equation*

$$(\mathcal{A} + \mathcal{C})w_i = M \left(-v_i + \int v_i M dv \right), \quad (6.2)$$

admits a unique solution $w = D_2 = (D_{21}, \dots, D_{2d})$, with $D_{2i} \in (X_{k+1})^0, \forall i = 1, \dots, d$. Moreover, let D be the matrix defined by

$$D_{ij}(x) := \int v_i D_{2j}(x, v) dv,$$

then

$$D(x) = \frac{1}{v} \left(\frac{\mathcal{I}}{\beta m} + \frac{1}{v^2 m^2} \nabla V \otimes \nabla V + \frac{\beta \hbar^2}{12m^2} \nabla \otimes \nabla V \right) (x). \quad (6.3)$$

Proof. Since the right-hand side of Eq. (6.2) belongs to $(X_{k+1})^0$, it satisfies the compatibility condition (4.3) and there exist $D_{2i} \in (X_{k+1})^0, \forall i = 1, \dots, d$ satisfying (6.2). More explicitly, D_2 solves

$$(\Theta[V] - v)D_2(x, v) = -M(x, v) \left[v + \frac{\nabla V(x)}{vm} \right], \quad (6.4)$$

since $\int D_2(v) dv = 0$ and by (4.8). Multiplying the left-hand side of Eq. (6.4) by $v + \nabla V/(vm)$ and integrating over \mathbb{R}^d we obtain

$$\begin{aligned} \int \left(v + \frac{\nabla V(x)}{vm} \right) \otimes (\Theta[V] - v) D_2(x, v) dv \\ = -v \int v_i D_{2j}(x, v) dv, \end{aligned}$$

by using the skew-symmetry and $D_{2i} \in (X_{k+1})^0$. Thus Eq. (6.4) gives

$$\begin{aligned} v D_{ij}(x) &= \int \left(v + \frac{\nabla V(x)}{vm} \right) \otimes \left(v + \frac{\nabla V(x)}{vm} \right) M(x, v) dv \\ &= \int v \otimes v M(x, v) dv - \frac{1}{v^2 m^2} \nabla V(x) \otimes \nabla V(x). \end{aligned} \quad (6.5)$$

From (4.9), the thesis follows. \blacksquare

Remark 6.2. By considering the expression (4.8) for the fluid velocity, we can recognize in (6.5) the classical definition of the pressure tensor in terms of M . This is to be expected, since the function M is the solution of the evolution problem with $\epsilon = 0$. Thus, the term with diffusion tensor D is what we expected to find as correction of first order in ϵ . By (6.3), it consists of the standard temperature and quantum pressure tensors, and of the additional tensor $1/(v^3 m^2) \nabla V(x) \otimes \nabla V(x)$, to be related to the strong-field assumption (cf. Remark 4.2). \square

Lemma 6.3. *Let $V \in H_x^{k+2}$ with d -admissible k . The following equation*

$$(\mathcal{A} + \mathcal{C})w = -v \cdot \nabla_x M + M \int v \cdot \nabla_x M dv, \quad (6.6)$$

admits a unique solution $D_1 \in (X_{k+1})^0$. Moreover, the vector W defined by

$$W(x) := \int v D_1(x, v) dv$$

can be calculated explicitly

$$\begin{aligned} W(x) &= \frac{1}{v} \left(2 \frac{\nabla \otimes \nabla V}{v^2 m^2} \nabla V(x) + \frac{\Delta V \nabla V}{v^2 m^2} \right. \\ &\quad \left. + \frac{\beta \hbar^2}{12m^2} \nabla \cdot \nabla \otimes \nabla V \right) (x). \end{aligned} \quad (6.7)$$

Proof. Under the regularity assumptions on $V, M \in H_{k+1}^1 := H_x^1 \otimes X_{k+1}^v$ and the right-hand side of Eq. (6.6) belongs to $(X_{k+1})^0$, thus there exists $D_1 \in (X_{k+1})^0$ solving

$$(\Theta[V] - v)D_1(x, v) = -v \cdot \nabla_x M(x, v) - M(x, v) \nabla \cdot \frac{\nabla V(x)}{vm},$$

which is equivalent to (6.6), since $\int D_1(v) dv = 0$. Multiplying by $v + \nabla V/vm$ and integrating over \mathbb{R}^d we obtain

$$\begin{aligned} v \int v D_1(x, v) dv \\ = \int \left(v + \frac{\nabla V(x)}{vm} \right) \nabla_x \cdot \left(v + \frac{\nabla V(x)}{vm} \right) M(x, v) dv \\ = \nabla_x \cdot \int v \otimes v M(x, v) dv - \frac{1}{v^2 m^2} \Delta V(x) \nabla V(x). \end{aligned} \quad (6.8)$$

The thesis follows directly from (4.9) and

$$\nabla \cdot (\nabla V \otimes \nabla V) = \Delta V \nabla V + (\nabla \otimes \nabla) V \nabla V. \quad \blacksquare$$

Proof of Lemma 6.1. First of all, let us write the explicit expression of the operators appearing in Eq. (5.9). Observe that $V \in H_x^{k+2}$ implies $M \in H_{k+1}^1$. By definition,

$$\mathcal{P}S\mathcal{P}\bar{\varphi} = -M \int v \cdot \nabla_x(nM)dv = -M \nabla_x \cdot \left(n \int v M dv \right), \quad (6.9)$$

and $(\mathcal{P}S\mathcal{Q}g) = (\mathcal{P}Sg) - (\mathcal{P}S\mathcal{P}g)$, i.e., explicitly,

$$\begin{aligned} (\mathcal{P}S\mathcal{Q}g) &= -M \left(\int v \cdot \nabla_x g dv - \int \nabla_x g dv \cdot \int v M dv \right. \\ &\quad \left. - \int g dv \nabla_x \cdot \int v M dv \right). \end{aligned} \quad (6.10)$$

Moreover $\mathcal{Q}S\mathcal{P}\bar{\varphi} = (S\mathcal{P} - \mathcal{P}S\mathcal{P})\bar{\varphi}$, i.e., explicitly,

$$\begin{aligned} \mathcal{Q}S\mathcal{P}\bar{\varphi} &= n \left[-v \cdot \nabla_x M + M \int v \cdot \nabla_x M dv \right] \\ &\quad + \nabla_x n \cdot \left[M \left(-v + \int v M dv \right) \right]. \end{aligned} \quad (6.11)$$

By Lemmata 6.2 and 6.3, $D_2(x, v) \equiv D_{2,i}(x, v)$ and $D_1(x, v)$ are solutions with $D_{2,i}, D_1 \in (X_{k+1})^0$ of Eqs. (6.2) and (6.6), respectively. Then, by some manipulations,

$$\mathcal{P}S\mathcal{Q}(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})^{-1}\mathcal{Q}S\mathcal{P}\bar{\varphi} = \mathcal{P}S(D_2 \cdot \nabla n) + \mathcal{P}S(D_1 n), \quad (6.12)$$

where the right-hand side can be written explicitly as

$$\begin{aligned} \mathcal{P}S(D_2 \cdot \nabla n) &= -M \int v \cdot \nabla_x (D_2 \cdot \nabla n) dv \\ &= -M \nabla_x \cdot \left[\left(\int v \otimes D_2 dv \right) \cdot \nabla n \right], \\ \mathcal{P}S(D_1 n) &= -M \int v \cdot \nabla_x (D_1 n) dv = -M \nabla_x \cdot \left[n \int v D_1 dv \right]. \end{aligned}$$

Hence, by simplifying the common factor M , Eq. (5.9) reads

$$\frac{\partial n}{\partial t} = -\nabla_x \cdot \left(n \int v M dv \right) + \epsilon \nabla_x \cdot (D \cdot \nabla n + nW),$$

and the thesis follows by using (6.3) and (6.7). \blacksquare

As a consequence of (6.12), Eq. (5.8) defining the other non-zero term of the bulk part expansion, $\tilde{\psi}_1$, can be rewritten as

$$\tilde{\psi}_1(x, t) = -\{D_2 \cdot \nabla n + D_1 n\}. \quad (6.13)$$

The explicit version of this expression shall be given in Eq. (8.20).

7. Rigorous results: The initial layer part

The aim of the present section is to prove the existence and regularity of the solutions of Eq. (5.5), together with the initial conditions (5.6). The first equation in (5.5) yields

$$\tilde{\varphi}_0(\tau) \equiv 0,$$

since we expect that $\lim_{\tau \rightarrow \infty} \tilde{\varphi}_0(\tau) = 0$. The equation for $\tilde{\psi}_0$ with the appropriate initial condition coming from (5.6)–(5.7), is

$$\begin{cases} \frac{\partial \tilde{\psi}_0}{\partial \tau} = \mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\tilde{\psi}_0 \\ \tilde{\psi}_0(0) = \psi_0. \end{cases} \quad (7.1)$$

We recall that the operator $\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}$ on $(X_k)^0$ reduces to

$$\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}w = \Theta[V]w - \nu w, \quad \forall w \in (X_k)^0,$$

(cf. (4.13)). By the product shape in Fourier-variables of the pseudo-differential operator (cf. (2.3)), it is more convenient to consider the equation for $\mathcal{F}\tilde{\psi}_0$, that looks like

$$\frac{\partial}{\partial \tau} \mathcal{F}\tilde{\psi}_0(x, \eta, \tau) = (i\delta V(x, \eta) - \nu) \mathcal{F}\tilde{\psi}_0(x, \eta, \tau).$$

Thus, we define, for all $w \in L^2(\mathbb{R}^6; \mathbb{R})$, the semigroup $G(\tau)$

$$\begin{aligned} G(\tau)w(x, v) &:= \mathcal{F}^{-1} \left(e^{i\delta V(x, \eta) - \nu} \tau \mathcal{F}w(x, \eta) \right) \\ &= e^{-\nu\tau} \mathcal{F}^{-1} \left(e^{i\delta V(x, \eta)\tau} \mathcal{F}w(x, \eta) \right), \quad \forall \tau \geq 0. \end{aligned} \quad (7.2)$$

The function $\tilde{\psi}_0(\tau) \equiv G(\tau)\psi_0$ formally satisfies system (7.1). Moreover,

Lemma 7.1. *If $w \in (X_k)^0$ and $V \in H_x^k$ with d -admissible k , then there exist $0 < \nu_k < \nu$, and a constant $C(\|V\|_{H_x^k}) > 0$, such that*

$$\|G(\tau)w\|_{X_k} \leq C(\|V\|_{H_x^k}) e^{-\nu_k \tau} \|w\|_{X_k}. \quad (7.3)$$

If, in addition, $w \in (H_k^j)^0$ and $V \in H_x^{k+j}$, then

$$\|G(\tau)w\|_{H_k^j} \leq C(\|V\|_{H_x^{k+j}}) e^{-\nu_{k+j} \tau} \|w\|_{H_k^j}, \quad (7.4)$$

with appropriate $C(\|V\|_{H_x^{k+j}}) > 0$, and $0 < \nu_{k+j} < 1$. Eq. (7.2) defines a strongly continuous semigroup on $(X_k)^0$ (respectively on $(H_k^j)^0$).

Proof. By definition we have

$$\begin{aligned} \|G(\tau)w\|_{X_k} &\leq C e^{-\nu\tau} \left(\|e^{i\delta V(x, \eta)\tau} \mathcal{F}w(x, \eta)\|_{L_{x, \eta}^2} \right. \\ &\quad \left. + \|\nabla_\eta^k e^{i\delta V(x, \eta)\tau} \mathcal{F}w(x, \eta)\|_{L_{x, \eta}^2} \right) \\ &\leq C e^{-\nu\tau} \left(\|w\|_{L_{x, v}^2} + P_k(\tau \|V\|_{H_x^k}) \|w\|_{X_k} \right) \\ &\leq e^{-\nu_k \tau} \max_{\tau \geq 0} \{ e^{-(\nu - \nu_k)\tau} P_k(\tau \|V\|_{H_x^k}) \} \|w\|_{X_k}, \end{aligned}$$

where $0 < \nu_k < \nu$ and P_k is a polynomial of degree k . The estimate (7.4) can be proved analogously. The last assertion follows immediately by applying the Hille–Yosida Thm., thanks to (7.3) (respectively (7.4)). \blacksquare

With Lemmata 4.1 and 7.1 we can prove the following proposition.

Proposition 7.1. *If $w_0 \in H_{k+1}^1$ and $V \in H_x^{k+2}$, with d -admissible k , then all terms of the initial layer expansion are well-defined and satisfy the following estimates:*

$$\|\tilde{\psi}_0(\tau)\|_{X_k} \leq M_1 e^{-\nu_k \tau} \|w_0\|_{X_k}, \quad (7.5)$$

$$\|\tilde{\varphi}_1(\tau)\|_{X_k} \leq M_2 e^{-\nu_{k+2} \tau} \|w_0\|_{H_{k+1}^1}, \quad (7.6)$$

$$\|\tilde{\psi}_1(\tau)\|_{X_k} \leq M_3 e^{-\nu_{k+2} \tau} \|w_0\|_{H_{k+1}^1}, \quad (7.7)$$

for some constants M_1, M_2 and M_3 (depending on the norms of V).

Proof. The unique solution of system (7.1) is

$$\tilde{\psi}_0(\tau) = G(\tau)\psi_0, \quad (7.8)$$

and (7.5) follows immediately from (7.3) since $\psi_0 = \mathcal{P}w_0 \in X_k$. Now we consider the second equation in Eq. (5.5), the following one:

$$\frac{\partial \tilde{\varphi}_1}{\partial \tau}(\tau) = \mathcal{P}S\mathcal{Q}\tilde{\psi}_0(\tau).$$

The right-hand side is well-defined by considering the definition of the operator $\mathcal{P}S\mathcal{Q}$ (cf. (6.10)), together with Lemma 7.1, since $\psi_0 \in (H_{k+1}^1)^0$ and $V \in H_x^{k+2}$. By integrating with respect to τ and taking into account that initial layer should satisfy $\lim_{\tau \rightarrow \infty} \tilde{\varphi}_1(\tau) = 0$, we obtain

$$\begin{aligned} \tilde{\varphi}_1(\tau) &= - \int_{\tau}^{\infty} \mathcal{P}S\mathcal{Q}\tilde{\psi}_0(s)ds \\ &= - \int_{\tau}^{\infty} \mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]G(s)\psi_0ds \\ &= -\mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1} \int_{\tau}^{\infty} \mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}G(s)\psi_0ds. \end{aligned}$$

The last integral is well-defined since the integrand $\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}G(s)\psi_0$ is equal to $G(s)\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\psi_0$, which is continuous in H_{k+1}^1 (by Lemma 7.1). Moreover $\mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1} \in \mathcal{L}(H_{k+1}^1, X_k)$ (by Corollary 4.1), then it can be taken outside the integral. Since

$$\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}G(s)\psi_0 = \frac{\partial G(s)\psi_0}{\partial s},$$

thanks to the exponential decay of G in H_{k+1}^1 and the continuity of the operator $\mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}$, we obtain

$$\tilde{\varphi}_1(\tau) = \mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}G(\tau)\psi_0, \quad (7.9)$$

and, in particular,

$$\tilde{\varphi}_1(0) = \mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\psi_0,$$

which provides the initial datum. Then, (7.6) follows from the estimate

$$\begin{aligned} \|\tilde{\varphi}_1(\tau)\|_{X_k} &\leq \|\mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\| \|G(\tau)\psi_0\|_{H_{k+1}^1} \\ &\leq M_2 e^{-\nu_k+2\tau} \|\psi_0\|_{H_{k+1}^1}, \end{aligned} \quad (7.10)$$

where $\|\cdot\|$ denotes the norm in $\mathcal{L}(H_{k+1}^1, X_k)$. Finally, we prove that the equation

$$\frac{\partial \tilde{\psi}_1}{\partial \tau}(\tau) = \mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}\tilde{\psi}_1(\tau) + \mathcal{Q}S\mathcal{Q}\tilde{\psi}_0(\tau)$$

is classically solvable. The initial condition for $\tilde{\psi}_1$ can be obtained from Eq. (5.6), together with Eq. (5.8) for $\tilde{\psi}_1$,

$$\tilde{\psi}_1(0) = -\tilde{\psi}_1(0) = [\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\mathcal{Q}S\mathcal{P}\bar{\varphi}(0),$$

and by considering

$$\begin{aligned} \bar{\varphi}(0) &= \varphi_0 - \tilde{\varphi}_0(0) - \epsilon\tilde{\varphi}_1(0) \\ &= \varphi_0 - \epsilon\mathcal{P}S\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\psi_0. \end{aligned} \quad (7.11)$$

Since $\tilde{\psi}_1(0)$ is by itself a correction of order ϵ , we neglect the term of order ϵ in the expression for $\bar{\varphi}(0)$, which yields

$$\tilde{\psi}_1(0) = [\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\mathcal{Q}S\mathcal{P}\varphi_0.$$

By Lemma 7.1, G is a semigroup on $(H_{k+1}^1)^0$ and, thanks to the assumption on w_0 , ψ_0 is in $\mathcal{D}(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})$ when the operator $\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}$ is defined in $D(S) \cap (H_{k+1}^1)^0$. Therefore $\tilde{\psi}_0(\tau) = G(\tau)\psi_0$ is differentiable on $[0, \infty[$ in X_{k+1} so that the inhomogeneous term $\mathcal{Q}S\mathcal{Q}\tilde{\psi}_0(\tau)$ is differentiable on $[0, \infty[$ in X_k . This, together with $\tilde{\psi}_1(0) = (\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})^{-1}\mathcal{Q}S\mathcal{P}\varphi_0 \in D(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})$, shows that

$$\tilde{\psi}_1(\tau) = G(\tau)\tilde{\psi}_1(0) + \int_0^{\tau} G(\tau - \sigma)\mathcal{Q}S\mathcal{Q}G(\sigma)\psi_0d\sigma \quad (7.12)$$

is a classical solution. The estimate (7.7) follows from $[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\mathcal{Q}S\mathcal{P} \in \mathcal{L}(H_k^1, X_k)$ and from (7.5):

$$\begin{aligned} \|\tilde{\psi}_1(\tau)\|_{X_k} &\leq K_1 e^{-\nu_k\tau} \|\varphi_0\|_{X_k} + K_2 e^{-\nu_k\tau} \int_0^{\tau} e^{(\nu_k - \nu_{k+2})\sigma} \|\psi_0\|_{H_{k+1}^1} d\sigma \\ &\leq K_1 e^{-\nu_k\tau} \|\varphi_0\|_{X_k} + K_3 e^{-\nu_{k+2}\tau} \|\psi_0\|_{H_{k+1}^1} \\ &\leq M_3 e^{-\nu_{k+2}\tau} \|w_0\|_{H_{k+1}^1}. \quad \blacksquare \end{aligned}$$

In order to obtain an initial value for Eq. (6.1) with unknown $\bar{\varphi} = nM$, we consider again (7.11). Let us call $n_0(x) = \int w_0(x, v)dv$, such that $\varphi_0 = n_0M$ and, by dividing both sides of the expression (7.11) by M , it yields

$$n(x, 0) = n_0(x) + \epsilon \int v \cdot \nabla_x \mathcal{F}^{-1} \left(\frac{\mathcal{F}\psi_0}{i\delta V - \nu} \right) (x, v)dv, \quad (7.13)$$

by using the explicit expression the operator $(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})^{-1}$ (cf. (4.13)). In the following we shall call

$$\begin{aligned} n(x, 0) &= n_0(x) + \epsilon n_1(x) \quad \text{with} \\ n_1(x) &:= \int v \cdot \nabla_x \mathcal{F}^{-1} \left(\frac{\mathcal{F}\psi_0}{i\delta V - \nu} \right) (x, v)dv. \end{aligned} \quad (7.14)$$

The explicit expression for (7.9) can be obtained analogously and reads

$$\tilde{\varphi}_1(\tau) = -M \int v \cdot \nabla_x \mathcal{F}^{-1} \left(\frac{\mathcal{F}G(\tau)\psi_0}{i\delta V - \nu} \right) dv.$$

8. Well-posedness of the high-field QDD equation

In this section, we establish a well-posedness and regularity result for Eq. (6.1), with a given external potential V . The equation can be rewritten in divergence form as

$$\frac{\partial n}{\partial t} - \mathcal{D}n - \mathcal{G}n - \mathcal{E}n = 0, \quad (8.1)$$

where we denote

$$\mathcal{D}n = \epsilon \nabla \cdot (\mathcal{D}\nabla n), \quad \mathcal{G}n = \epsilon \nabla \cdot (\mathcal{W}n), \quad \mathcal{E}n = \nabla \cdot (\mathcal{E}n)$$

with

$$\begin{aligned} \mathcal{D} &\equiv \mathcal{D}(x) \\ &:= \frac{1}{v} \left(\frac{\mathcal{I}}{\beta m} + \frac{1}{v^2 m^2} \nabla V \otimes \nabla V + \frac{\beta \hbar^2}{12m^2} \nabla \otimes \nabla V \right) (x), \\ \mathcal{W} &\equiv \mathcal{W}(x) \\ &:= \frac{1}{v} \left(2 \frac{\nabla \otimes \nabla V}{v^2 m^2} \nabla V + \frac{\Delta V \nabla V}{v^2 m^2} + \frac{\beta \hbar^2}{12m^2} \nabla \cdot \nabla \otimes \nabla V \right) (x), \\ \mathcal{E} &\equiv \mathcal{E}(x) := \frac{\nabla V(x)}{\nu m}, \quad \forall x \in \mathbb{R}^d. \end{aligned}$$

Assumption 8.1. V belongs to H_x^{k+2} with d -admissible k and is such that

$$\exists c > 0 \quad \text{s.t.} \quad \mathcal{D}(x)y \otimes y \geq c|y|^2, \quad \forall x, y \in \mathbb{R}^d.$$

This is a technical assumption implying that \mathcal{D} is a uniformly elliptic differential operator and thus ensuring the well-posedness of the drift-diffusion equation. A modified version of the well-posedness theorem will be the subject of future work.

Proposition 8.1. *Let V satisfy Assumption 8.1 and, in addition, $\nabla \Delta V \in W_x^{j-1,\infty}$ with $j \in \mathbb{N}$. Then the unique global solution $n = n(t)$ of Eq. (8.1) with $n(0) \in L_x^2$ satisfies $n(t) \in H_x^j$ for $t > 0$, and the following estimate*

$$\|n(t)\|_{H_x^j} \leq M_j(\epsilon t)^{-j/2} \|n(0)\|_{L_x^2} \quad (8.2)$$

holds with $M_j > 0$, for $\epsilon, t \rightarrow 0^+$.

In the following, by $\nabla F \in L^2(\mathbb{R}_x^d)$ we mean $\nabla F \in (L_x^2)^d$. Moreover, we consider $0 < \epsilon < 1$ and the constants are independent of ϵ , unless specified.

Proof. By Assumption 8.1 on the potential V , the operator \mathcal{D} defined on $D(\mathcal{D}) = H_x^2$ generates an analytic contraction semigroup $(T(t))_{t \geq 0}$ on L_x^2 (cf. Thm.VI.5.22 of [39]).

Let us derive here a basic estimate we shall use intensively in the following. By Assumption 8.1, for all $u \in H_x^2$

$$\|\nabla \cdot \nabla u\|_{L_x^2} \leq C \|\mathcal{D} \nabla u \otimes \nabla u\|_{L_x^2} \leq \frac{C}{\epsilon} (\|\mathcal{D} u\|_{L_x^2} + \|\text{div} \mathcal{D} \cdot \nabla u\|_{L_x^2}),$$

where the second inequality follows from the definition of the operator \mathcal{D} . Moreover, by using that for all $u \in H_x^2$,

$$\|\nabla u\|_{L_x^2} \leq C_\delta \|u\|_{L_x^2} + \delta \|\nabla \cdot \nabla u\|_{L_x^2}, \quad \forall \delta > 0, \quad (8.3)$$

with $C_\delta > 0$, it holds

$$\|\text{div} \mathcal{D} \cdot \nabla u\|_{L_x^2} \leq \|\text{div} \mathcal{D}\|_{L_x^\infty} (C_\delta \|u\|_{L_x^2} + \delta \|\nabla \cdot \nabla u\|_{L_x^2}), \quad \forall \delta > 0,$$

with $C_\delta > 0$. Thus,

$$\|\nabla \cdot \nabla u\|_{L_x^2} \leq \frac{c}{\epsilon(1 - \delta \|\text{div} \mathcal{D}\|_{L_x^\infty} / \epsilon)} (\|\text{div} \mathcal{D}\|_{L_x^\infty} C_\delta \|u\|_{L_x^2} + \|\mathcal{D} u\|_{L_x^2}),$$

and, in conclusion, for an appropriate choice of $\delta > 0$, some constant $C > 0$ exists such that

$$\|\nabla \cdot \nabla u\|_{L_x^2} \leq \frac{C}{\epsilon} (\|u\|_{L_x^2} + \|\mathcal{D} u\|_{L_x^2}). \quad (8.4)$$

The operator \mathcal{G} can be written as $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$ with $\mathcal{G}_1 f := \epsilon W \cdot \nabla f$ defined on H_x^1 , and $\mathcal{G}_2 f := \epsilon \nabla \cdot W f$, defined on L_x^2 . The operator \mathcal{G}_1 is \mathcal{D} -bounded, i.e., for all $n \in D(\mathcal{D})$,

$$\|\mathcal{G}_1 n\|_{L_x^2} \leq \epsilon \|W\|_{L_x^\infty} C (\|n\|_{L_x^2} + \|\nabla \cdot \nabla n\|_{L_x^2}) \quad (8.5)$$

$$\begin{aligned} &\leq \|W\|_{L_x^\infty} (C \|n\|_{L_x^2} + C \|\mathcal{D} n\|_{L_x^2}) \\ &\leq b \|n\|_{L_x^2} + a \|\mathcal{D} n\|_{L_x^2}, \end{aligned} \quad (8.6)$$

by using (8.4) and $\epsilon < 1$. Moreover, the \mathcal{D} -bound a_1 defined by

$$a_1 := \inf \{a \geq 0 \mid \exists b > 0 \text{ s.t. (8.6) holds}\}$$

is zero, by substituting (8.5) with (8.3). The operator $\mathcal{G}_2 f := \epsilon \nabla \cdot W f$, defined on L_x^2 , is bounded. The operator \mathcal{E} can be written as $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ with $\mathcal{E}_1 f := E \cdot \nabla f$, defined on H_x^1 , \mathcal{D} -bounded with \mathcal{D} -bound $a_2 = 0$, since, for all $n \in D(\mathcal{D})$,

$$\begin{aligned} \|\mathcal{E}_1 n\|_{L_x^2} &\leq \|E\|_{L_x^\infty} (C_\delta \|n\|_{L_x^2} + \delta \|\nabla \cdot \nabla n\|_{L_x^2}) \\ &\leq C \|E\|_{L_x^\infty} \left(C_\delta \|n\|_{L_x^2} + \frac{\delta}{\epsilon} (\|n\|_{L_x^2} + \|\mathcal{D} n\|_{L_x^2}) \right), \quad \forall \delta > 0, \end{aligned} \quad (8.7)$$

by using (8.3) and (8.4). The operator $\mathcal{E}_2 f := \nabla \cdot E f$ is defined on L_x^2 and bounded.

Thus, by Thm. III.2.10 of [39], $(\mathcal{D} + \mathcal{G} + \mathcal{E}, D(\mathcal{D}))$ generates an analytic semigroup on L_x^2 that we shall denote by $(S(t))_{t \geq 0}$. More precisely

$$\|(\mathcal{D} + \mathcal{G} + \mathcal{E})^\alpha S(t) u\|_{L_x^2} \leq M_\alpha t^{-\alpha} \|u\|_{L_x^2}, \quad t \rightarrow 0^+, \forall \alpha \geq 0 \quad (8.8)$$

with M_α independent of ϵ , by employing Lemma III.2.6 of [39].

In order to derive estimate (8.2), let us start from the following inequality

$$\epsilon^{m/2} \|u\|_{H_x^m} \leq C \|(\mathcal{D} + \mathcal{G} + \mathcal{E})^{m/2} u\|_{L_x^2} \quad (8.9)$$

that is yielded by similar arguments to (8.4). By combining (8.9) with (8.8), we get

$$\begin{aligned} \|S(t) u\|_{H_x^j} &\leq C \epsilon^{-j/2} \|(\mathcal{D} + \mathcal{G} + \mathcal{E})^{j/2} S(t) u\|_{L_x^2} \\ &\leq C_j (\epsilon t)^{-j/2} \|u\|_{L_x^2}, \end{aligned} \quad (8.10)$$

which holds for all $u \in L_x^2$ and for small t . ■

The singularity with respect to t in (8.2) can be easily removed:

Corollary 8.1. *Let the assumptions of the Proposition 8.1 hold. In addition, let $n(0)$ belong to H_x^j . Then the solution $n(t)$ belongs to H_x^j for all $t > 0$ and satisfies*

$$\|n(t)\|_{H_x^j} \leq C \epsilon^{-j/2} \|n(0)\|_{H_x^j}, \quad (8.11)$$

for $\epsilon, t \rightarrow 0^+$.

Proof. The following inequality holds

$$\epsilon^{m/2} \|u\|_{H_x^m} \leq C \|(\mathcal{D} + \mathcal{G} + \mathcal{E})^{m/2} u\|_{L_x^2} \leq C \|u\|_{H_x^m} \quad (8.12)$$

for all $u \in D((\mathcal{D} + \mathcal{G} + \mathcal{E})^{m/2})$, with $m \leq j$, and for $\epsilon \rightarrow 0^+$ (cf. (8.9)). Then, in particular,

$$\begin{aligned} \|S(t) u\|_{H_x^m} &\leq C \epsilon^{-m/2} \|(\mathcal{D} + \mathcal{G} + \mathcal{E})^{m/2} S(t) u\|_{L_x^2} \\ &\leq C \epsilon^{-m/2} \|(\mathcal{D} + \mathcal{G} + \mathcal{E})^{m/2} u\|_{L_x^2} \\ &\leq C \epsilon^{-m/2} \|u\|_{H_x^m}, \end{aligned} \quad (8.13)$$

which holds for all $u \in D((\mathcal{D} + \mathcal{G} + \mathcal{E})^{m/2})$, for small t and ϵ : the first inequality sign corresponds to the first one in Eq. (8.12), the second inequality follows by exchanging $(\mathcal{D} + \mathcal{G} + \mathcal{E})^{m/2}$ with $S(t)$ and the third one comes from Eq. (8.12). ■

Remark 8.1. Observe that in the low-field case, the QDD equation looks like Eq. (8.1) with $W \equiv 0$ and

$$D \equiv D(x) := \frac{1}{\nu} \left(\frac{\mathcal{I}}{\beta m} + \frac{\beta \hbar^2}{12 m^2} \nabla \otimes \nabla V \right) (x).$$

The corresponding well-posedness result requires the sole Assumption 8.1 on the potential V . Additionally, in order to establish the estimate (8.2) for all $j \in \mathbb{N}$, the assumption $\Delta V \in W_x^{j-1,\infty}$ is to be added. This result is to be compared with the analysis in [5], where the fourth order non-linear equation, obtained by the approximation $\nabla \log n = -\beta \nabla V + \mathcal{O}(\hbar^2)$, is tackled, cf. [32]. □

By increasing regularity of the initial datum, we can remove the singular behavior of the estimate (8.2) with respect to t and ϵ .

Corollary 8.2. *Let V satisfy Assumption 8.1 and $\nabla \Delta V \in W_x^{2j-1,\infty}$. Then the solution $n(t)$ of Eq. (8.1) with $n(0) \in D(\mathcal{D}^j)$ satisfies, for $\epsilon, t \rightarrow 0^+$*

$$\|n(t)\|_{H_x^j} \leq C \|n(0)\|_{H_x^{2j}}. \quad (8.14)$$

Moreover, the following refinement holds

$$\|n(t)\|_{H_x^j} \leq C \|n(0)\|_{H_x^j}. \quad (8.15)$$

Proof. We prove the thesis in the case $j = 1$. For $j > 1$ the thesis follows by an induction procedure similarly to [22]. Due to the regularity with respect to the variable x of the solution $n(t)$, for $t > 0$, we can find the evolution equation for ∇n by differentiating

$$\begin{aligned} \nabla \left(\frac{\partial}{\partial t} n \right) &= \frac{\partial}{\partial t} (\nabla n) = \nabla (\mathcal{D} + \mathcal{G} + \mathcal{E}) n \\ &= (\mathcal{D} + \mathcal{G} + \mathcal{E}) \nabla n - [(\mathcal{D} + \mathcal{G} + \mathcal{E}), \nabla] n, \end{aligned}$$

where we denote by $[(\mathcal{D} + \mathcal{G} + \mathcal{E}), \nabla]$ the commutator among the two operators. Since

$$\begin{aligned} & -[(\mathcal{D} + \mathcal{G} + \mathcal{E}), \partial_k] \\ &= \epsilon \sum_{i,j} \partial_i (\partial_k D_{ij} \partial_j n) + \sum_i \partial_i (\partial_k (\epsilon W_i + E_i) n) \\ &=: (\mathcal{D}' + \mathcal{G}' + \mathcal{E}') n, \end{aligned}$$

∇n satisfies

$$\frac{\partial}{\partial t} (\nabla n) = (\mathcal{D} + \mathcal{G} + \mathcal{E}) \nabla n + (\mathcal{D}' + \mathcal{G}' + \mathcal{E}') n. \quad (8.16)$$

The solution of the previous equation can be expressed by the Duhamel formula via the analytic semigroup $S(t)$ generated by $(\mathcal{D} + \mathcal{G} + \mathcal{E})$, as

$$\nabla n(t) = S(t) \nabla n(0) + \int_0^t S(t-s) (\mathcal{D}' + \mathcal{G}' + \mathcal{E}') n(s) ds. \quad (8.17)$$

Moreover we can estimate

$$\begin{aligned} \|\nabla n(t)\|_{L_x^2} &\leq C \|\nabla n(0)\|_{L_x^2} + C \int_0^t \|(\mathcal{D}' + \mathcal{G}' + \mathcal{E}') n(s)\|_{L_x^2} ds \\ &\leq C \|\nabla n(0)\|_{L_x^2} + C \int_0^t (\epsilon \|n(s)\|_{H_x^2} + \|n(s)\|_{H_x^1}) ds \\ &\leq C \|n(0)\|_{H_x^2} + C \int_0^t \|\nabla n(s)\|_{L_x^2} ds. \end{aligned} \quad (8.18)$$

by using (8.11) with $j = 2$, provided $n(0) \in H_x^2$. Therefore, by Gronwall lemma, we derive (8.14). In order to prove (8.15), we apply the first inequality in (8.10) with $j = 2$ to the function $n(t) = S(t)n(0)$ and we obtain

$$\|n(t)\|_{H_x^2} \leq \frac{C}{\epsilon} \|(\mathcal{D} + \mathcal{G} + \mathcal{E}) S(t) n(0)\|_{L_x^2}.$$

Then we use (8.8) for the term $(\mathcal{D} + \mathcal{G} + \mathcal{E})^{1/2} S(t) [(\mathcal{D} + \mathcal{G} + \mathcal{E})^{1/2} n(0)]$ with $j = 1$, and we get

$$\begin{aligned} \|n(t)\|_{H_x^2} &\leq \frac{C}{\epsilon} \|(\mathcal{D} + \mathcal{G} + \mathcal{E})^{1/2} S(t) (\mathcal{D} + \mathcal{G} + \mathcal{E})^{1/2} n(0)\|_{L_x^2} \\ &\leq \frac{C}{\epsilon} t^{-1/2} \|(\mathcal{D} + \mathcal{G} + \mathcal{E})^{1/2} n(0)\|_{L_x^2} \\ &\leq \frac{C}{\epsilon} t^{-1/2} \|n(0)\|_{H_x^1}, \end{aligned} \quad (8.19)$$

where for the last inequality the estimate (8.12) with $m = 1$ is used. Hence it holds for all $n(0) \in \mathcal{D}((\mathcal{D} + \mathcal{G} + \mathcal{E})^{1/2}) \equiv H_x^1$. By using (8.19) in the second line of (8.18) and using integrability of $t^{-1/2}$, we get (8.15) in the case $j = 1$. ■

The other (non-null) term of the bulk part is $\bar{\psi}_1$, which is of first order in ϵ . Since it satisfies

$$\bar{\psi}_1 = -(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})^{-1} \mathcal{Q} S \mathcal{P}(nM)$$

(cf. Eq. (5.8)), by using the definitions (6.11) and (4.13), it can be written explicitly as

$$\begin{aligned} \bar{\psi}_1(x, v, t) &= \nabla n(x, t) \cdot \mathcal{F}^{-1} \left\{ \frac{1}{i\delta V - v} \mathcal{F} \left(vM + \frac{M\nabla V}{vm} \right) \right\} (x, v) \\ &\quad + n(x, t) \mathcal{F}^{-1} \left\{ \frac{1}{i\delta V - v} \mathcal{F} \left(v \cdot \nabla_x M + \frac{M\Delta V}{vm} \right) \right\} (x, v), \\ \forall (x, v, t) &\in \mathbb{R}^{2d} \times \mathbb{R}^+. \end{aligned} \quad (8.20)$$

Thus, the estimates for the solution n in the previous corollary are the crucial ingredient to establish well-posedness of the definition (8.20) and the behavior with respect to time of the function $\bar{\psi}_1$ and of its derivatives.

Another fundamental aspect is the shape of the initial datum $n(0)$ for Eq. (8.1): by (7.14), it is given by $n(0) = n_0 + \epsilon n_1$, and the following estimate holds

$$\begin{aligned} \|n(0)\|_{H_x^j} &\leq \|n_0\|_{H_x^j} + \epsilon \|n_1\|_{H_x^j} \\ &\leq \|w_0\|_{H_x^j} + \epsilon C (\|V\|_{H_x^{k+j+2}}) \|w_0\|_{H_x^{j+1}}, \end{aligned} \quad (8.21)$$

for all d -admissible k , by using the estimate (4.12).

Proposition 8.2. *Let n be a solution of the drift-diffusion (8.1) with initial value $n(0)$ given by (7.14), with $w_0 \in H_{k+1}^4$, and with V satisfying Assumption 8.1 and $\nabla \Delta V \in W_x^{5,\infty}$. Then $\bar{\psi}_1$ is strongly differentiable with respect to $t > 0$, and for every $t > 0$ it satisfies*

$$\bar{\psi}_1(t) \in \mathcal{D}(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}) \cap \mathcal{D}(\mathcal{Q}\delta\mathcal{Q}).$$

Moreover there exists a constant $M > 0$ such that, for $\epsilon, t \rightarrow 0^+$,

$$\|\partial_t \bar{\psi}_1(t)\|_{X_k} \leq M \|w_0\|_{H_{k+1}^4}, \quad (8.22)$$

$$\|\partial_t \bar{\psi}_1(t)\|_{H_k^1} \leq M(1 + 1/t) \|w_0\|_{H_{k+1}^4}, \quad (8.23)$$

$$\|SQ \bar{\psi}_1(t)\|_{H_k^1} \leq M \|w_0\|_{H_{k+1}^4}. \quad (8.24)$$

Proof. If we differentiate with respect to t the expression (8.20), the only t -dependent functions are ∇n and n , explicitly

$$\begin{aligned} \partial_t \bar{\psi}_1(x, v, t) &= \partial_t (\nabla n)(x, t) \cdot A(x, v) + \partial_t n(x, t) B(x, v), \\ \forall (x, v, t) &\in \mathbb{R}^{2d} \times \mathbb{R}^+ \end{aligned} \quad (8.25)$$

where the functions A_i, B , defined in (8.20), are sufficiently regular, because of the assumptions on V . The differentiability of n with respect to t follows from analyticity of the semigroup $(S(t))_{t \geq 0}$. The differentiability of ∇n , instead, follows from the expression (8.17): since each term is continuously differentiable in time, so is ∇n .

Moreover, by using $\partial_t \nabla n = \nabla \partial_t n$ and the evolution equation for n ,

$$\begin{aligned} \|\partial_t (\nabla n)\|_{L_x^2} &= \|\nabla (\mathcal{D} + \mathcal{G} + \mathcal{E}) S(t) n(0)\|_{L_x^2} \\ &\leq \|(\mathcal{D} + \mathcal{G} + \mathcal{E}) S(t) n_0\|_{H_x^1} + \epsilon \|(\mathcal{D} + \mathcal{G} + \mathcal{E}) S(t) n_1\|_{H_x^1} \\ &\leq \|S(t) n_0\|_{H_x^3} + \epsilon \|S(t) n_1\|_{H_x^3} \leq C (\|w_0\|_{H_k^3} + \epsilon \|w_0\|_{H_{k+1}^4}) \\ &\leq C \|w_0\|_{H_{k+1}^4} \end{aligned} \quad (8.26)$$

where we split $n(0) = n_0 + \epsilon n_1$ and we use (8.15), together with (8.21). Similarly,

$$\begin{aligned} \|\partial_t (\nabla n)\|_{H_x^1} &\leq C \left(\|\nabla (\mathcal{D} + \mathcal{G} + \mathcal{E}) S(t) n(0)\|_{L_x^2} \right. \\ &\quad \left. + \|\nabla \cdot \nabla (\mathcal{D} + \mathcal{G} + \mathcal{E}) S(t) n(0)\|_{L_x^2} \right) \\ &\leq C \left(\|w_0\|_{H_{k+1}^4} + \|(\mathcal{D} + \mathcal{G} + \mathcal{E}) n_0\|_{H_x^2} \right. \\ &\quad \left. + \frac{\epsilon}{\epsilon t} \|(\mathcal{D} + \mathcal{G} + \mathcal{E}) n_1\|_{L_x^2} \right) \\ &\leq C \left(\|w_0\|_{H_{k+1}^4} + \|w_0\|_{H_k^4} + \frac{1}{t} \|w_0\|_{H_{k+1}^3} \right), \end{aligned} \quad (8.27)$$

where the first term in the inequality comes from (8.26). To obtain the second and the third terms, first commute $S(t)$ with $(\mathcal{D} + \mathcal{G} + \mathcal{E})$ and use $n(0) = n_0 + \epsilon n_1$, then apply estimate (8.15) to get the second term and (8.2) to get the third one.

Finally, inequality (8.27) follows from estimates (8.12) and (8.21). In order to prove (8.24), let us consider again the abstract definition of $\tilde{\psi}_1$ (see (5.8)):

$$\tilde{\psi}_1(t) = -(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q})^{-1}(\mathcal{Q}S\mathcal{P})(n(t)M).$$

Since $\mathcal{Q}S\mathcal{P}\tilde{\varphi}$ reads (see (6.11))

$$\begin{aligned} \mathcal{Q}S\mathcal{P}nM &= n \left[-v \cdot \nabla_x M + M \int v \cdot \nabla_x M dv \right] \\ &\quad + \nabla_x n \cdot \left[M \left(-v + \int v M dv \right) \right], \end{aligned}$$

under the present hypotheses, $\mathcal{Q}S\mathcal{P}(nM)$ belongs to H_k^2 , thus $\tilde{\psi}_1(t) \in \mathcal{D}(\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}) \cap \mathcal{D}(\mathcal{Q}S\mathcal{Q})$. By (8.20),

$$\begin{aligned} \mathcal{Q}\tilde{\psi}_1(x, v, t) &= \nabla \cdot \nabla n(x, t) v \cdot A(x, v) \\ &\quad + \nabla n(x, t) \cdot (v \cdot \nabla \cdot A + vB)(x, v) \\ &\quad + n(x, t) v \cdot \nabla B(x, v). \end{aligned}$$

Thus, in order to estimate $\|\mathcal{Q}\tilde{\psi}_1(t)\|_{X_k}$ and $\|\mathcal{Q}\tilde{\psi}_1(t)\|_{H_k^1}$, it is necessary to evaluate $\|\nabla \cdot \nabla n(t)\|_{L_x^2}$ and $\|\nabla \cdot \nabla n(t)\|_{H_x^1}$, $\|\nabla n(t)\|_{L_x^2}$ and $\|\nabla n(t)\|_{H_x^1}$, respectively. In particular,

$$\begin{aligned} \|\nabla \cdot \nabla n(t)\|_{H_x^1} &\leq C\|S(t)n(0)\|_{H_x^3} \leq C\|n(0)\|_{H_x^3} \\ &\leq C\|w_0\|_{H_k^3} + \epsilon C\|w_0\|_{H_{k+1}^4}, \end{aligned}$$

again by (8.15). Thus, we can conclude by using the regularity properties of A_i, B . ■

Remark 8.2. Observe that it is possible to remove the singularity for $t \rightarrow 0^+$ in the estimate (8.23), by assuming $w_0 \in H_{k+1}^5$ and modifying last two lines of (8.27) as follows

$$\begin{aligned} \|\partial_t(\nabla n)\|_{H_x^1} &\leq C \left(\|w_0\|_{H_{k+1}^4} + \|(\mathcal{D} + \mathcal{G} + \mathcal{E})n_0\|_{H_x^2} \right. \\ &\quad \left. + \epsilon \|(\mathcal{D} + \mathcal{G} + \mathcal{E})n_1\|_{H_x^2} \right) \\ &\leq C \left(\|w_0\|_{H_{k+1}^4} + \|w_0\|_{H_k^4} + \epsilon \|w_0\|_{H_{k+1}^5} \right). \quad \square \end{aligned} \quad (8.28)$$

9. Estimate of the error

In this section we prove rigorously that the high-field QDD equation, derived from the asymptotic expansion up to the first order in ϵ , is an approximation of order ϵ^2 of the high-field Wigner-BGK system (3.5). To this aim, we consider the errors obtained by replacing the functions $\mathcal{P}w = \varphi$ and $\mathcal{Q}w = \psi$ by the terms of their expansion up to first order in ϵ . We shall prove the following

Theorem 9.1. *If the initial value w_0 belongs to H_{k+1}^4 and V satisfies Assumption 8.1 and $\nabla \Delta V \in W_x^{5,\infty}$, then for any $T, 0 < T < \infty$, there is a constant C independent of ϵ such that*

$$\begin{aligned} \left\| \varphi(t) + \psi(t) - [\tilde{\varphi}(t) + \epsilon \tilde{\psi}_1(t) + \tilde{\psi}_0(t/\epsilon) + \epsilon \tilde{\varphi}_1(t/\epsilon) \right. \\ \left. + \epsilon \tilde{\psi}_1(t/\epsilon)] \right\|_{X_k} \leq C\epsilon^2, \end{aligned} \quad (9.1)$$

uniformly for $0 \leq t \leq T$.

This result relies on the estimates established in Propositions 7.1 and 8.2, about the behavior with respect to time of the initial layer functions $\tilde{\varphi}_1, \tilde{\psi}_1$ and the bulk functions. Let us split the error in two contributions

$$\begin{aligned} y(t) &= \varphi(t) - [\tilde{\varphi}(t) + \epsilon \tilde{\varphi}_1(\tau)], \\ z(t) &= \psi(t) - [\tilde{\psi}_0(\tau) + \epsilon \tilde{\psi}_1(t) + \epsilon \tilde{\psi}_1(\tau)] \end{aligned} \quad (9.2)$$

where $\tau = \frac{t}{\epsilon}$. The evolution equations for the errors y and z can be deduced from those satisfied by their components (cf. systems (5.1), (5.4), (5.5)). Hence, we have

$$\begin{cases} \frac{\partial y}{\partial t} = \mathcal{P}S\mathcal{P}y + \mathcal{P}S\mathcal{Q}z + f \\ \frac{\partial z}{\partial t} = \mathcal{Q}S\mathcal{P}y + \mathcal{Q}S\mathcal{Q}z + \frac{1}{\epsilon}\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}z + g \end{cases} \quad (9.3)$$

with initial conditions

$$y(0) = 0, \quad z(0) = O(\epsilon^2),$$

and inhomogeneous terms f and g defined by

$$\begin{aligned} f(t) &= \epsilon \left[\mathcal{P}S\mathcal{P}\tilde{\varphi}_1(\tau) + \mathcal{P}S\mathcal{Q}\tilde{\psi}_1(\tau) \right] \\ g(t) &= \epsilon \left[-\frac{\partial \tilde{\psi}_1}{\partial t} + \mathcal{Q}S\mathcal{Q}\tilde{\psi}_1(t) + \mathcal{Q}S\mathcal{P}\tilde{\varphi}_1(\tau) + \mathcal{Q}S\mathcal{Q}\tilde{\psi}_1(\tau) \right]. \end{aligned}$$

It is convenient to separate the evolution of the error due to the initial layer from that related to the bulk part. Let us define

$$r = y + z = r_i + r_b$$

with

$$r_i = -\epsilon \tilde{\varphi}_1 - \tilde{\psi}_0 - \epsilon \tilde{\psi}_1, \quad r_b = \varphi + \psi - \tilde{\varphi} - \epsilon \tilde{\psi}_1. \quad (9.4)$$

The derivation of estimate (9.1) is split according to (9.4) in the next two Lemmata.

Lemma 9.1. *Under the assumptions $V \in H_x^{k+4}$ and $w_0 \in H_{k+2}^2$, for any $T, 0 < T < \infty$, there is a constant C independent of ϵ such that*

$$\|r_i(t)\|_{X_k} \leq C\epsilon^2, \quad (9.5)$$

uniformly for $0 \leq t \leq T$.

Proof. The initial layer error $r_i = r_i(t)$ satisfies the equation

$$\begin{aligned} \frac{\partial r_i}{\partial t}(t) &= S r_i(t) + \frac{1}{\epsilon}(\mathcal{A} + \mathcal{C})r_i(t) + \epsilon S(\tilde{\varphi}_1 + \tilde{\psi}_1) \left(\frac{t}{\epsilon} \right), \\ r_i(0) &= 0. \end{aligned} \quad (9.6)$$

The operator $S + (\mathcal{A} + \mathcal{C})/\epsilon$ generates a uniformly bounded semigroup in $X_k, Z(t)$, cf. [40]. Thus, the mild solution of (9.6) is given by

$$r_i(t) = Z(t)r_i(0) + \epsilon \int_0^t Z(t-s)S(\tilde{\varphi}_1 + \tilde{\psi}_1)(s/\epsilon)ds,$$

with

$$\|r_i(t)\|_{X_k} \leq C\epsilon \int_0^t \|S(\tilde{\varphi}_1 + \tilde{\psi}_1)(s/\epsilon)\|_{X_k} ds. \quad (9.7)$$

The estimate of $\|S(\tilde{\varphi}_1 + \tilde{\psi}_1)(s/\epsilon)\|_{X_k}$ is a bit tedious, thus we simply sketch it. It is convenient to use the projections \mathcal{P}, \mathcal{Q} and evaluate $\mathcal{P}S\mathcal{P}\tilde{\varphi}_1, \mathcal{Q}S\mathcal{P}\tilde{\varphi}_1, \mathcal{P}S\mathcal{Q}\tilde{\psi}_1$, and $\mathcal{Q}S\mathcal{Q}\tilde{\psi}_1$ separately. By their definitions (cf. (6.9), (6.11)) $\mathcal{P}S\mathcal{P}, \mathcal{Q}S\mathcal{P} \in \mathcal{L}(H_k^1, X_k)$,

provided $V \in H_x^{k+2}$, thus the following modification of the estimate (7.10) for $\tilde{\varphi}_1$ holds,

$$\begin{aligned} & \|\mathcal{P}\mathcal{S}\mathcal{P}\tilde{\varphi}_1(\tau)\|_{X_k} \\ & \leq \|\mathcal{P}\mathcal{S}\mathcal{P}\| \|\mathcal{P}\mathcal{S}\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\| \|G(\tau)\psi_0\|_{H_{k+1}^2} \\ & \leq \text{Me}^{-\nu_{k+3}\tau} \|\psi_0\|_{H_{k+1}^2}, \end{aligned} \tag{9.8}$$

since $\mathcal{P}\mathcal{S}\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1} \in \mathcal{L}(H_{k+1}^2, H_k^1)$, provided $V \in H_x^{k+4}$. Analogously,

$$\begin{aligned} \|\mathcal{Q}\mathcal{S}\mathcal{P}\tilde{\varphi}_1(\tau)\|_{X_k} & \leq \|\mathcal{Q}\mathcal{S}\mathcal{P}\| \|\mathcal{P}\mathcal{S}\mathcal{Q}[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\| \|G(\tau)\psi_0\|_{H_{k+1}^2} \\ & \leq \text{Me}^{-\nu_{k+3}\tau} \|\psi_0\|_{H_{k+1}^2}. \end{aligned} \tag{9.9}$$

Let us recall the expression for $\tilde{\psi}_1$ (cf. (7.12))

$$\tilde{\psi}_1(\tau) = G(\tau)\tilde{\psi}_1(0) + \int_0^\tau G(\tau - \sigma)\mathcal{Q}\mathcal{S}\mathcal{Q}G(\sigma)\psi_0 d\sigma.$$

We evaluate $\|\mathcal{P}\mathcal{S}\mathcal{Q}\tilde{\psi}_1(\tau)\|_{X_k}$ and $\|\mathcal{Q}\mathcal{S}\mathcal{Q}\tilde{\psi}_1(\tau)\|_{X_k}$. Both $\mathcal{P}\mathcal{S}\mathcal{Q}$ and $\mathcal{Q}\mathcal{S}\mathcal{Q}$ belong to $\mathcal{L}(H_{k+1}^1, X_k)$. Moreover $[\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\mathcal{Q}\mathcal{S}\mathcal{P} \in \mathcal{L}(H_k^1, X_k)$ by definition, provided $V \in H_x^k$; thus

$$\begin{aligned} & \|\mathcal{Q}\mathcal{S}\mathcal{Q}G(\tau)\tilde{\psi}_1(0)\|_{X_k} \\ & \leq \|\mathcal{Q}\mathcal{S}\mathcal{Q}G(\tau)\| \|\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}]^{-1}\mathcal{Q}\mathcal{S}\mathcal{P}\varphi_0\|_{H_{k+1}^1} \\ & \leq \text{Ke}^{-\nu_k\tau} \|\varphi_0\|_{H_{k+1}^2}, \end{aligned}$$

provided $V \in H_x^{k+2}$. Concerning the second term, we obtain

$$\begin{aligned} & \left\| \mathcal{Q}\mathcal{S}\mathcal{Q} \int_0^\tau G(\tau - \sigma)\mathcal{Q}\mathcal{S}\mathcal{Q}G(\sigma)\psi_0 d\sigma \right\|_{X_k} \\ & \leq \|\mathcal{Q}\mathcal{S}\mathcal{Q}\| \left\| \int_0^\tau G(\tau - \sigma)\mathcal{Q}\mathcal{S}\mathcal{Q}G(\sigma)\psi_0 d\sigma \right\|_{H_{k+1}^1} \\ & \leq \text{Ke}^{-\nu_{k+2}\tau} \int_0^\tau e^{(\nu_{k+2} - \nu_{k+4})\sigma} \|\varphi_0\|_{H_{k+2}^2} d\sigma \\ & \leq \text{Ke}^{-\nu_{k+4}\tau} \|\varphi_0\|_{H_{k+2}^2}, \end{aligned}$$

provided $V \in H_x^{k+4}$. In conclusion,

$$\|\mathcal{Q}\mathcal{S}\mathcal{Q}\tilde{\psi}_1(\tau)\|_{X_k} \leq \text{Ke}^{-\nu_{k+4}\tau} \|\varphi_0\|_{H_{k+2}^2},$$

and analogously,

$$\|\mathcal{P}\mathcal{S}\mathcal{Q}\tilde{\psi}_1(\tau)\|_{X_k} \leq \text{Le}^{-\nu_{k+4}\tau} \|\varphi_0\|_{H_{k+2}^2},$$

for some constant $L > 0$. Finally, it is possible to find constants $\bar{\nu} > 0$ and $\bar{M}(\|V\|_{H_x^{k+4}}) > 0$ such that

$$\|S(\tilde{\varphi}_1 + \tilde{\psi}_1)(\tau)\|_{X_k} \leq \bar{M}(\|V\|_{H_x^{k+4}}) e^{-\bar{\nu}\tau} \|w_0\|_{H_{k+2}^2}.$$

Coming back to the estimate (9.7) of r_i , for any time t we have

$$\|r_i(t)\|_{X_k} = C\bar{M}\epsilon \int_0^t e^{-\bar{\nu}s/\epsilon} \|w_0\|_{H_{k+2}^2} ds \leq C\|f_0\|_{H_{k+2}^2} \epsilon^2. \quad \blacksquare$$

Lemma 9.2. Under the assumptions of Proposition 8.2, for any $T, 0 < T < \infty$, there is a constant C independent of ϵ such that

$$\|r_b(t)\|_{X_k} \leq C\epsilon^2, \tag{9.10}$$

uniformly for $0 \leq t \leq T$.

Proof. The error of the bulk part of the asymptotic expansion satisfies (9.3) with $f = 0$ and, instead of g ,

$$g_b(t) = \epsilon \left[-\frac{\partial \tilde{\psi}_1}{\partial t} + \mathcal{Q}\mathcal{S}\mathcal{Q}\tilde{\psi}_1(t) \right].$$

Since the inhomogeneous term $g_b(t)$ has a non-uniform behavior with respect to ϵ for small times, we split the inhomogeneous term $g(t)$ into the sum of two functions, say g_{b0} and g_{b1} , as follows

$$g_{b0}(t) = \eta_\epsilon g_b(t), \quad g_{b1} = g_b(t) - g_{b0}(t),$$

where η_ϵ is a non-increasing C^∞ -function such that

$$\eta_\epsilon(t) = \begin{cases} 1 & \text{for } t < \epsilon/2, \\ 0 & \text{for } t > 3\epsilon/2. \end{cases}$$

We write the error r_b as the sum of two parts $r_b = r_{b0} + r_{b1}$, solving the equation

$$\frac{\partial r_{b0}}{\partial t} = Sr_{b0} + \frac{1}{\epsilon}(\mathcal{A} + \mathcal{C})r_{b0} + \epsilon g_{b0}, \quad r_{b0}(0) = 0,$$

and an analogous one with the inhomogeneous term g_{b1} . Concerning the error r_{b0} , Proposition 8.2 yields the following estimate

$$\begin{aligned} \|r_{b0}(t)\|_{X_k} & \leq K\epsilon \int_0^{3\epsilon/2} \|g_{b0}(s)\|_{X_k} ds \\ & \leq K\epsilon \int_0^{3\epsilon/2} \left(\left\| \frac{\partial \tilde{\psi}_1}{\partial s}(s) \right\|_{X_k} + \|\mathcal{S}\mathcal{Q}\tilde{\psi}_1(s)\|_{X_k} \right) ds \\ & \leq K\epsilon \int_0^{3\epsilon/2} \|w_0\|_{H_{k+1}^4} ds \leq K\|w_0\|_{H_{k+1}^4} \epsilon^2. \end{aligned}$$

Finally, we consider the evolution equation for r_{b1} and represent it as

$$r_{b1} = \hat{r}_{b1} + h(t),$$

where the auxiliary function h solves the problem

$$\frac{\partial h}{\partial t} = \frac{1}{\epsilon}\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}h + \epsilon g_{b1}, \quad h(0) = 0.$$

Consequently, the function \hat{r}_{b1} satisfies the initial value problem

$$\frac{\partial \hat{r}_{b1}}{\partial t} = S\hat{r}_{b1} + \frac{1}{\epsilon}(\mathcal{A} + \mathcal{C})\hat{r}_{b1} + \mathcal{S}\mathcal{Q}h, \quad \hat{r}_{b1}(0) = 0,$$

thus it can be easily estimated in terms of the auxiliary function h as

$$\|\hat{r}_{b1}(t)\|_{X_k} \leq \int_{\epsilon/2}^t \|\mathcal{S}\mathcal{Q}h(s)\|_{X_k} ds.$$

Again by the properties of the operator $(\mathcal{A} + \mathcal{C})$, the solution h is given by

$$h(t) = \begin{cases} 0 & \text{for } t < \epsilon/2, \\ \epsilon \int_{\epsilon/2}^t G_\epsilon(t-s)g_{b1}(s) ds & \text{for } t > \epsilon/2, \end{cases}$$

where $G_\epsilon(\tau)$ is the bounded semigroup generated by $(1/\epsilon)\mathcal{Q}(\mathcal{A} + \mathcal{C})\mathcal{Q}$. Finally, applying again Proposition 8.2

$$\begin{aligned} \|\hat{r}_{b1}(t)\|_{X_k} & \leq \epsilon \int_{\epsilon/2}^t \|\mathcal{S}\mathcal{Q}h(s)\|_{X_k} ds \\ & \leq \epsilon K \int_{\epsilon/2}^t \int_{\epsilon/2}^s e^{-\nu_{k+1}\frac{s-s'}{\epsilon}} \|g_{b1}(s')\|_{H_k^1} ds' ds \\ & \leq \epsilon K \int_{\epsilon/2}^t \int_{\epsilon/2}^s e^{-\nu_{k+1}\frac{s-s'}{\epsilon}} \left(1 + \frac{1}{s} \right) \|w_0\|_{H_{k+1}^4} ds' ds \\ & \leq K\|w_0\|_{H_{k+1}^4} \epsilon^2, \end{aligned}$$

for any $t \in [0, T]$, where the constant K depends on T . In conclusion,

$$\|r_b(t)\|_{X_k} \leq K \|w_0\|_{H_{k+1}^4} \epsilon^2.$$

These two Lemmata prove [Theorem 9.1](#) and consequently demonstrate rigorously that the high-field QDD equation (6.1) is an approximation of order ϵ^2 of the high-field Wigner-BGK equation (3.2).

Acknowledgements

The authors are grateful to Luigi Barletti and Jacek Banasiak, for many helpful discussions on the position of the problem. The authors wish to thank the referees for their comments and suggestions. This work was performed under the auspices of the *National Group for Mathematical Physics* of the *Istituto Nazionale di Alta Matematica* and was partly supported by the *Italian Ministry of University (MIUR National Project “Mathematical Problems of Kinetic Theories”*, Cofin2004).

References

- [1] P.A. Markowich, C. Ringhofer, C. Schmeiser, *Semiconductor Equations*, Springer, Wien, 1990.
- [2] A. Arnold, A. Jüngel, Multi-scale modeling of quantum semiconductor devices, in: A. Mielke (Ed.), *Analysis, Modeling and Simulation of Multiscale Problems*, Springer, Berlin, 2006, pp. 331–363.
- [3] E. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.* 40 (1932) 749–759.
- [4] A. Jüngel, *Quasi-hydrodynamic Semiconductor Equations*, Birkhäuser, Basel, 2001.
- [5] A. Jüngel, R. Pinnau, Global non-negative solutions of a nonlinear fourth-order parabolic equation for quantum systems, *SIAM J. Math. Anal.* 32 (2000) 760–777.
- [6] A.M. Anile, G. Mascali, V. Romano, Recent developments in hydrodynamical modeling of semiconductors, in: A.M. Anile (Ed.), *Mathematical Problems in Semiconductor Physics*, in: *Lecture Notes in Math.*, vol. 1823, Springer, Berlin, 2003, pp. 1–56.
- [7] P. Degond, F. Méhats, C. Ringhofer, Quantum energy-transport and drift-diffusion models, *J. Stat. Phys.* 118 (2005) 625–665.
- [8] I. Gasser, P. Markowich, Quantum hydrodynamics, Wigner transforms and the classical limit, *Asymptot. Anal.* 14 (1997) 97–116.
- [9] A. Jüngel, D. Matthes, A derivation of the isothermal quantum hydrodynamic equations using entropy minimization, *Z. Angew. Math. Mech.* 85 (2005) 806–814.
- [10] G. Lindblad, On the generators of quantum dynamical semigroups, *Comm. Math. Phys.* 48 (1976) 119–130.
- [11] F. Poupaud, Runaway phenomena and fluid approximation under high fields in semiconductor kinetic theory, *Z. Angew. Math. Mech.* 72 (1992) 359–372.
- [12] L. Arlotti, G. Frosali, Runaway particles for a Boltzmann-like transport equation, *Math. Models Methods Appl. Sci.* 2 (2) (1992) 203–221.
- [13] N. Ben Abdallah, H. Chaker, The high field asymptotics for degenerate semiconductors, *Math. Models Methods Appl. Sci.* 11 (7) (2001) 1253–1272.
- [14] C. Cercignani, I.M. Gamba, C.D. Levermore, High field approximations to a Boltzmann–Poisson system and boundary conditions in a semiconductor, *Appl. Math. Lett.* 10 (4) (1997) 111–117.
- [15] C. Cercignani, I.M. Gamba, J.W. Jerome, C.-W. Shu, Device benchmark comparisons via kinetic, hydrodynamic, and high-field models, *Comput. Methods Appl. Mech. Eng.* 181 (4) (2000) 381–392.
- [16] N. Ben Abdallah, H. Chaker, C. Schmeiser, The high field asymptotics for a fermionic Boltzmann equation: Entropy solutions and kinetic shock profiles, *J. Hyperbolic Differ. Equ.* 4 (2007) 679–704.
- [17] F. Berthelin, N.J. Mauser, F. Poupaud, High-field limit from a kinetic equation to multidimensional scalar conservation laws, *J. Hyperbolic Differ. Equ.* 4 (2007) 123–145.
- [18] P. Degond, A. Jüngel, High-field approximations of the energy-transport model for semiconductors with non-parabolic band structure, *Z. Angew. Math. Phys.* 52 (2001) 1053–1070.
- [19] N. Ben Abdallah, P. Degond, P. Markowich, C. Schmeiser, High field approximations of the spherical harmonics expansion model for semiconductors, *Z. Angew. Math. Phys.* 52 (2001) 201–230.
- [20] C. Gardner, C. Ringhofer, The Chapman–Enskog expansion and the quantum hydrodynamic model for semiconductor devices, *VLSI Des.* 10 (2000) 415–435.
- [21] J.R. Mika, J. Banasiak, Singularly Perturbed Evolution Equations with Applications to Kinetic Theory, World Scientific, Singapore, 1995.
- [22] J. Banasiak, Singularly perturbed linear and semilinear hyperbolic systems: Kinetic theory approach to some folk’s theorems, *Acta Appl. Math.* 49 (1997) 199–228.
- [23] J. Banasiak, Diffusion approximation of the linear Boltzmann equation with and without an external field: Analysis of initial layer, *J. Math. Anal. Appl.* 205 (1) (1997) 216–238.
- [24] F. Fromlet, P.A. Markowich, C. Ringhofer, A Wigner function approach to phonon scattering, *VLSI Des.* 9 (1999) 339–350.
- [25] A.O. Caldeira, A.J. Leggett, Path integral approach to quantum Brownian motion, *Physica A* 121 (1983) 587–616.
- [26] F. Castella, L. Erdős, F. Fromlet, P.A. Markowich, Fokker–Planck equations as scaling limits of reversible quantum systems, *J. Stat. Phys.* 100 (2000) 543–601.
- [27] A. Arnold, E. Dhamo, C. Manzini, The Wigner–Poisson–Fokker–Planck system: Global-in-time solution and dispersive effects, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (4) (2007) 645–676.
- [28] A. Arnold, E. Dhamo, C. Manzini, Dispersive effects in quantum kinetic equations, *Indiana Univ. Math. J.* 56 (3) (2007) 1299–1331.
- [29] A. Arnold, C. Sparber, Quantum dynamical semigroups for diffusion models with Hartree interaction, *Comm. Math. Phys.* 251 (1) (2004) 179–207.
- [30] A. Arnold, J.A. Carrillo, I. Gamba, C.W. Shu, Low and high field scaling limits for the Vlasov and the Wigner–Poisson–Fokker–Planck systems, *Transport Theory Statist. Phys.* 30 (2–3) (2001) 43–100.
- [31] L.L. Bonilla, R. Escobedo, Wigner–Poisson and non-local drift-diffusion equation for semiconductor superlattices, *Math. Models Methods Appl. Sci.* 15 (8) (2005) 1253–1272.
- [32] C. Gardner, The quantum hydrodynamic model for semiconductor devices, *SIAM J. App. Math.* 54 (2) (1994) 409–427.
- [33] A. Arnold, Self-consistent relaxation-time models in quantum mechanics, *Comm. Partial Differential Equations* 21 (3–4) (1996) 473–506.
- [34] P. Degond, C. Ringhofer, Quantum moment hydrodynamics and the entropy principle, *J. Stat. Phys.* 112 (2003) 587–628.
- [35] C.D. Levermore, Moment closure hierarchies for kinetic theories, *J. Stat. Phys.* 83 (1996) 1021–1065.
- [36] P.L. Lions, T. Paul, Sur le mesures de Wigner, *Rev. Mat. Iberoam.* 9 (3) (1993) 553–618.
- [37] C. Manzini, L. Barletti, An analysis of the Wigner–Poisson problem with time-dependent, inflow boundary conditions, *Nonlinear Anal.* 60 (1) (2004) 77–100.
- [38] C. Manzini, The three dimensional Wigner–Poisson problem with inflow boundary conditions, *J. Math. Anal. Appl.* 313 (1) (2006) 184–196.
- [39] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 1999.
- [40] G. Frosali, C. van der Mee, S. Pavveri-Fontana, Conditions for runaway phenomena in the kinetic theory of particle swarms, *J. Math. Phys.* 30 (5) (1989) 1177–1186.