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Amalgams of finite inverse semigroups

Alessandra Cherubini [a,*], John Meakin [b], Brunetto Piochi [c]

[a] *Dipartimento di Matematica, Politecnico di Milano, Piazza L. da Vinci 32, I 20133 Milano, Italy*
[b] *Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0323, USA*
[c] *Dipartimento di Matematica "U. Dini", Università degli Studi di Firenze, viale Morgagni 67/a, I 50134 Firenze, Italy*

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Abstract

We show that the word problem is decidable for an amalgamated free product of finite inverse semigroups (in the category of inverse semigroups). This is in contrast to a recent result of M. Sapir that shows that the word problem for amalgamated free products of finite semigroups (in the category of semigroups) is in general undecidable.

Keywords: Word problem; Semigroups amalgams; Inverse semigroups; Schützenberger graph

1. Introduction

If S_1 and S_2 are semigroups (groups) such that $S_1 \cap S_2 = U$ is a nonempty subsemigroup (subgroup) of both S_1 and S_2 then $[S_1, S_2; U]$ is called an amalgam of semigroups (groups). The amalgamated free product $S_1 *_U S_2$ associated with this amalgam in the category of semigroups (groups) is defined by the usual universal diagram.

* Corresponding author.
E-mail addresses: alessandra.cherubini@polimi.it (A. Cherubini), jmeakin@math.unl.edu (J. Meakin), piochi@math.unifi.it (B. Piochi).

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The amalgam $[S_1, S_2; U]$ is said to be strongly embeddable in a semigroup (group) S if there are injective homomorphisms $\phi_i : S_i \to S$ such that $\phi_1|_U = \phi_2|_U$ and $S_1\phi_1 \cap S_2\phi_2 = U\phi_1 = U\phi_2$. It is well known that every amalgam of groups embeds in a group (and hence in the amalgamated free product of the group amalgam). However, an early example of Kimura [9] shows that semigroup amalgams do not necessarily embed in any semigroup. On the other hand, T.E. Hall [6] showed that every amalgam of inverse semigroups (in the category of inverse semigroups) embeds in an inverse semigroup, and hence in the corresponding amalgamated free product in the category of inverse semigroups.

An inverse semigroup is a semigroup S with the property that for each element $a \in S$ there is a unique element $a^{-1} \in S$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$. A consequence of the definition is that idempotents commute in any inverse semigroup. One may also define a natural partial order on such a semigroup S by $a \leqslant b$ iff $a = eb$ for some idempotent e of S.

Inverse semigroups arise very naturally in mathematics as semigroups of partial one-one maps on a set (or partial isometries of a metric space, or homeomorphisms between open subsets of a topological space, or local diffeomorphisms of a differentiable manifold etc). We refer the reader to the book of Petrich [15] for basic results and notation about inverse semigroups and to the more recent books of Lawson [10] and Paterson [14] for many references to the connections between inverse semigroups and other branches of mathematics.

Recently Birget, Margolis, and Meakin [3] showed that even under very nice conditions on a semigroup amalgam $[S_1, S_2; U]$, the corresponding amalgamated free product $S_1 *_U S_2$ in the category of semigroups may have undecidable word problem, quite in contrast to the situation for amalgamated free products of groups. This result was further strengthened by Sapir [16] who showed that an amalgamated free product of finite semigroups may have undecidable word problem. However, in the present paper we show that the word problem is decidable for any amalgamated free product of finite inverse semigroups in the category of inverse semigroups.

We refer the reader to [15] for information about free inverse semigroups, and in particular for a description of Munn's solution [13] to the word problem for the free inverse semigroup on a set in terms of Munn trees. Munn's work was greatly extended by Stephen [17] who introduced the notion of Schützenberger graphs associated with presentations of inverse semigroups and their role in the study of the word problem. We refer to the papers by Jones [7] and Jones, Margolis, Meakin, and Stephen [8] for information about the structure of free products of inverse semigroups in the category of inverse semigroups, and to the papers by Haataja, Margolis, and Meakin [5], Bennett [1,2], Stephen [18], and Cherubini, Meakin, and Piochi [4] for detailed information about various classes of amalgamated free products of inverse semigroups. We will make heavy use of Bennett's ideas in our study of amalgamated free products of finite inverse semigroups in this paper. Our strategy for solving the word problem for an amalgamated free product of finite inverse semigroups is to provide a construction of their Schützenberger graphs, very much along the lines of Bennett's construction of the Schützenberger graphs of a lower bounded amalgam [1]. We briefly recall some relevant notation and refer to [1,4,17] and [15] for any undefined notation and terminology.

We will denote the free inverse semigroup on a set X by $\mathrm{FIS}(X)$. It is the quotient of the free semigroup $(X \cup X^{-1})^+$ by the least congruence ρ that makes the resulting quotient semigroup inverse (see [15]). We denote the inverse semigroup S presented by a set X of generators and a set T of relations by $S = \mathrm{Inv}\langle X \mid T \rangle$. This is the quotient of the free semigroup $(X \cup X^{-1})^+$ by the least congruence τ that contains ρ and the relations in T. We refer to [11,17] or the survey paper [12] for much information about presentations of inverse semigroups. Crucial to the study of presentations of inverse semigroups is the notion of the Schützenberger automaton $\mathcal{A}(X, T, w)$ for a word $w \in (X \cup X^{-1})^+$. This automaton has underlying graph $S\Gamma(X, T, w)$ whose set of vertices is the $\mathcal{R}$-class containing $w\tau$ and whose edges consist of all triples (s, x, t) where s and t are $\mathcal{R}$-related to $w\tau$ in S, $x \in X \cup X^{-1}$, and $s.x\tau = t$: we view this edge as being directed from s to t. The graph $S\Gamma(X, T, w)$ is an *inverse word graph* over X (i.e., a connected graph whose edges are labeled over $X \cup X^{-1}$ in such a way that each edge e labeled by x has a unique inverse edge labeled by x^{-1}) and is also deterministic. The automaton $\mathcal{A}(X, T, w)$ is then defined as the (inverse) automaton on this underlying graph that has as initial state the vertex $ww^{-1}\tau$ and as terminal state the vertex $w\tau$. The importance of these automata stems from the fact that for any two words $w, w' \in (X \cup X^{-1})^+$, $w\tau = w'\tau$ if and only if $\mathcal{A}(X, T, w) = \mathcal{A}(X, T, w')$, or equivalently if these two automata accept the same language [17]. In view of this we will occasionally abuse notation slightly and denote $\mathcal{A}(X, T, w)$ by $\mathcal{A}(X, T, w\tau)$ when it is convenient.

We refer the reader to Stephen's original paper [17] for a description of an iterative procedure for constructing the Schützenberger automaton $\mathcal{A}(X, T, w)$ from the linear automaton of w by repeated applications of the process of expansions and determinations (edge foldings). The essential idea is that one constructs iteratively a sequence of automata that "approximate" the Schützenberger automaton of w in the sense that the languages that they accept become successively better approximates of the language of the Schützenberger automaton. An inverse automaton $\mathcal{B}$ is called an approximate automaton for $\mathcal{A}(X, T, w)$ if there is a word $w' \in L(\mathcal{B})$ such that $w'\tau = w\tau$ and every word in $L(\mathcal{B})$ is greater than or equal to w in the natural partial order on the inverse semigroup S, i.e., $L(\mathcal{B}) \subseteq L(\mathcal{A})$. One solves the word problem for a presentation of an inverse semigroup S by effectively constructing the associated Schützenberger automata or an approximation to the Schützenberger automaton that enables to solve the word problem. It is evident that these automata are finite if S is finite.

In his paper [1], Bennett constructs the Schützenberger automata for amalgamated free products of a class of amalgams that he refers to as "lower bounded" amalgams of inverse semigroups. Our construction of the Schützenberger automata corresponding to an amalgamated free product of finite inverse semigroups closely follows the construction of Bennett, but differs from Bennett's construction in some technical ways, as amalgams of finite inverse semigroups are not necessarily lower bounded.

2. V-quotients

We denote by $i(p)$ (respectively $t(p)$) the initial (respectively terminal) vertex of a path p and by $l(p)$ the word labeling the path p in an inverse word graph. We say that p is a path from α to β if $i(p) = \alpha$ and $t(p) = \beta$. In this case we also say that the word $l(p)$ labels a path from α to β. If p is a path in Λ with $i(p) = t(p) = \alpha$ then p is called a loop based at α. If Λ is a deterministic inverse word graph, and if w labels a path from α to β in Λ, it is convenient to write $\beta = \alpha w$. We also say that αw *exists* for the word $w \in (X \cup X^{-1})^+$ in this case.

There is an evident notion of a *morphism* between inverse word graphs. This is just a graph morphism that preserves labeling of edges. Morphisms between inverse word graphs are referred to as *V-homomorphisms* in [17]. A surjective morphism is an edge surjective V-epimorphism in the sense of [17]. If Λ is an inverse word graph over X and η is an equivalence relation on the set of vertices of Γ, the corresponding quotient graph Λ/η is called a *V-quotient* of Λ (see [17] for details). This notion extends to the concept of a V-quotient of an inverse automaton in the obvious way. There is a least equivalence relation on the vertices of an inverse automaton Λ such that the corresponding V-quotient is deterministic. A deterministic V-quotient of Λ is called a *DV-quotient* in this paper. There is a natural V-homomorphism from Λ onto a V-quotient of Λ.

It is convenient to record the following lemma for later use in this paper.

Lemma 1. *Let Λ be a deterministic inverse word graph over X, let Γ be the V-quotient of Λ obtained by identifying vertices $\alpha_1, \alpha_2, \ldots, \alpha_n$ of Λ, and let Δ be the determinized form of Γ. Let $\equiv$ be the smallest equivalence relation on the set of vertices of Λ such that*

(1) *$\alpha_i \equiv \alpha_j$ for all i and j, and*
(2) *if $\beta_1 \equiv \beta_2$ and $\beta_1 w$ and $\beta_2 w$ both exist for some word $w \in (X \cup X^{-1})^+$, and some vertices β_i of Λ, then $\beta_1 w \equiv \beta_2 w$.*

Then two vertices γ_1 and γ_2 of Λ are identified in the DV-quotient Δ if and only if $\gamma_1 \equiv \gamma_2$.

Proof. It is clear that two vertices γ_1 and γ_2 of Λ get identified in Δ if $\gamma_1 \equiv \gamma_2$, since if β_1 gets identified with β_2 and there is some word w such that $\beta_1 w$ and $\beta_2 w$ both exist, then $\beta_1 w$ gets identified with $\beta_2 w$. To prove the converse, note that by Theorem 4.4 of Stephen [17], γ_1 gets identified with γ_2 if and only if there is some Dyck word d (i.e., a word that freely reduces to 1) in $(X \cup X^{-1})^+$ such that d labels a path from γ_1 to γ_2 in Γ. We prove that $\gamma_1 \equiv \gamma_2$ by induction on the length of d.

If $|d| = 2$, then $d = xx^{-1}$ for some letter $x \in X \cup X^{-1}$. Since Λ is deterministic, if $\gamma_1 \neq \gamma_2$, then we must have that x labels an edge from γ_1 to v_i and x^{-1} labels an edge from v_j to γ_2 for some $i \neq j$. But since v_i and v_j were identified in Γ, then clearly $\gamma_1 \equiv \gamma_2$. This gives a base for the induction.

Suppose that $d = d_1 d_2 \ldots d_k$ for some Dyck words d_i and that $k > 1$. Then d_1 labels a path in Γ from γ_1 to some vertex δ_2, d_2 labels a path in Γ from δ_2 to $\delta_3, \ldots,$ and d_k labels a path in Γ from δ_k to γ_2. By induction, $\gamma_1 \equiv \delta_2 \equiv \delta_3 \equiv \cdots \equiv \gamma_2$.

So assume that d cannot be written as a product of smaller Dyck words, and that $|d| > 2$. Then we have $d = xcx^{-1}$ for some Dyck word c with $|c| < |d|$. Now c labels a path from some vertex β_1 of Γ to some other vertex β_2. By induction, $\beta_1 \equiv \beta_2$, and $\beta_1 x^{-1} = \gamma_1$ and $\beta_2 x^{-1} = \gamma_2$, so by part (2) of the definition of $\equiv$, we have $\gamma_1 \equiv \gamma_2$, as required. □

Now let $S = \mathrm{Inv}\langle X \mid T\rangle$ be a finite inverse semigroup and let Λ be an inverse word graph over X. We will always assume X and T to be finite in this paper. Recall from [17] that Λ is called *closed* (relative to the presentation) if Λ is deterministic and whenever $u = v$ is a relation in T and u (respectively v) labels a path in Λ from a vertex α to a vertex β, then v (respectively u) also labels a path in Λ from α to β.

Lemma 2. *Let $(\lambda, \Lambda, \lambda)$ be a nontrivial closed inverse automaton relative to a presentation $S = \mathrm{Inv}\langle X \mid T\rangle = (X \cup X^{-1})^+/\tau$ of a finite inverse semigroup S. Then there exists a unique minimum idempotent $e = u\tau \in S$ such that $(\lambda, \Lambda, \lambda)$ is a DV-quotient of the Schützenberger automaton $\mathcal{A}(e) = \mathcal{A}(X, T, u) = (\lambda^*, \Lambda^*, \lambda^*)$. In particular, Λ is finite. If a word $y \in (X \cup X^{-1})^+$ labels a λ–θ path in $(\lambda, \Lambda, \lambda)$, then y also labels a λ^*–θ^* path in Λ^* for some vertex θ^*. If y labels a path starting at λ in Λ and $y\tau$ is an idempotent of S, then y labels a loop at λ in Λ.*

Proof. Consider the automaton $(\lambda, \Lambda, \lambda)$. Since this automaton is nontrivial, there is some word u such that $u\tau$ is an idempotent of S and u labels a loop in Λ based at λ. Let $e = u\tau$ be the minimum idempotent of S such that u labels a loop in Λ based at λ. Denote by $\mathcal{A}(e) = (\lambda^*, \Lambda^*, \lambda^*)$ the Schützenberger automaton of u relative to $\langle X \mid T\rangle$. (In fact $\lambda^* = e$ and Λ^* is the Schützenberger graph of u relative to $\langle X \mid T\rangle$.) If v is a word in $(X \cup X^{-1})^+$ which labels a loop in $\mathcal{A}(e)$ based at λ^* then there is a finite sequence of automata $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ such that $\mathcal{A}_1$ is the linear automaton of u, each $\mathcal{A}_{i+1}$ is obtained from $\mathcal{A}_i$ by a full expansion relative to T or an edge folding, and $v \in L(\mathcal{A}_n)$. The fact that $(\lambda, \Lambda, \lambda)$ is closed with respect to T implies that if this same sequence of expansions and edge foldings is performed in $(\lambda, \Lambda, \lambda)$, then induction on the number of steps shows that v labels a loop in $(\lambda, \Lambda, \lambda)$ based at λ.

By Theorem 2.5 of [17] there exists a homomorphism ϕ from Λ^* to Λ that maps λ^* onto λ. If y labels a λ–θ path in Λ it follows that uyy^{-1} labels a loop in Λ based at λ. By minimality of e this implies that $e = (uyy^{-1})\tau$, so uyy^{-1} labels a loop in $\mathcal{A}(e)$ based at λ^*. Again, since u labels a loop in $\mathcal{A}(e)$ based at λ^*, it follows that y labels a λ^*–θ^* path in Λ^* for some vertex θ^*. In particular, this implies that ϕ is surjective.

To prove the last statement of the lemma, note that if $y\tau$ is an idempotent of S, then $yy^{-1}\tau = y\tau$, but yy^{-1} labels a loop based at λ, so $y\tau = yy^{-1}\tau \geqslant e$. But this means that y labels a loop based at λ^* in $\mathcal{A}(e)$, and so the image of this loop must be a loop labeled by y and based at λ in Λ. □

Remark. Note that the Schützenberger automaton $\mathcal{A}(e)$ of Lemma 2 contains every Schützenberger automaton $\mathcal{A}(f)$ which has $(\lambda, \Lambda, \lambda)$ as a DV-quotient, for f idempotent. In fact suppose that there exists an idempotent $f = v\tau$ of S such that $(\lambda, \Lambda, \lambda)$ is a DV-quotient of the corresponding Schützenberger automaton $\mathcal{A}(f)$. Now v labels a loop in Λ based at λ, whence $f \geqslant e$, so that $\mathcal{A}(f)$ is embedded into $\mathcal{A}(e)$. In the sequel we will refer to $\mathcal{A}(e)$ as the *maximum determinizing Schützenberger automaton* of $(\lambda, \Lambda, \lambda)$. Clearly the automaton $(\lambda, \Lambda, \lambda)$ accepts a larger language than its maximum determinizing Schützenberger automaton in general.

Recall from [17] that if $(\alpha, \Lambda_1, \beta)$ and $(\gamma, \Lambda_2, \delta)$ are two birooted inverse word graphs, then $(\alpha, \Lambda_1, \beta) \times (\gamma, \Lambda_2, \delta)$ is the birooted inverse word graph obtained as the V-quotient of the union of these two birooted graphs by identifying β and γ. The next result is an immediate consequence of Lemma 5.2 in [17].

Lemma 3. *Let e and f be idempotents of some inverse semigroup $S = \mathrm{Inv}\langle X \mid T\rangle$, with corresponding Schützenberger automata $\mathcal{A}(e)$ and $\mathcal{A}(f)$. Then $\mathcal{A}(e) \times \mathcal{A}(f)$ approximates the Schützenberger automaton $\mathcal{A}(ef)$. Furthermore, if $(\alpha, \Lambda, \alpha)$ is a V-quotient of $\mathcal{A}(e)$ and (β, Γ, β) is a V-quotient of $\mathcal{A}(f)$, then $(\alpha, \Lambda, \alpha) \times (\beta, \Gamma, \beta)$ is a V-quotient of $\mathcal{A}(e) \times \mathcal{A}(f)$.*

We remark that automata that are closed with respect to T are not necessarily Schützenberger automata relative to the presentation. For example, the inverse monoid

$$S = \mathrm{Inv}\langle a, b \mid a^2 = b^3 = 1,\ ba = ab^2\rangle$$

is clearly the symmetric group on three letters, so it has only one $\mathcal{D}$-class and hence only one Schützenberger graph, so the DV-quotient of this graph obtained by identifying the vertices corresponding to the group elements 1 and b is a graph with two vertices, so it is not a Schützenberger graph, but it is closed with respect to these defining relations. Note also that b labels a loop at 1 in this graph, but b does not label a loop in the Cayley graph (Schützenberger graph) of S. So loops in a DV-quotient of a Schützenberger graph do not all lift to loops in the Schützenberger graph.

3. Finite amalgams

If $[S_1, S_2; U]$ is an amalgam of finite inverse semigroups and $u \in U$, we denote by $w_i(u)$ the natural image of u in S_i under the embedding of U into S_i. If S_i is presented as $S_i = \mathrm{Inv}\langle X_i \mid R_i\rangle = (X_i \cup X_i^{-1})^+/\eta_i$, where the X_i are disjoint alphabets, then the words $w_i(u)$ are viewed as words in the alphabet X_i and $S_1 *_U S_2 = \mathrm{Inv}\langle X \mid R \cup W\rangle = (X \cup X^{-1})^+/\tau$, where $X = X_1 \cup X_2$, $R = R_1 \cup R_2$ and W is the set of all pairs $(w_1(u), w_2(u))$ for $u \in U$. Furthermore, if $v_i \in (X_i \cup X_i^{-1})^+$ and $v \in (X \cup X^{-1})^+$, then $\mathcal{A}(X, R_i, v_i)$ will denote the Schützenberger automaton of the word v_i relative to $\langle X_i \mid R_i\rangle$ and $\mathcal{A}(X, R \cup W, v)$ will denote the Schützenberger automaton of the word v relative to $\langle X \mid R \cup W\rangle$. We shall adhere to this notation throughout the remainder of the paper.

We recall some notation from [1,8]. Suppose that Γ is an inverse word graph labeled over $X = X_1 \cup X_2$: then an edge of Γ that is labeled from $X_i \cup X_i^{-1}$ (for some $i \in \{1, 2\}$) is said to be *colored* by i. A subgraph of Γ is called *monochromatic* if all of its edges have the same color. A *lobe* of Γ is defined to be a maximal monochromatic connected subgraph of Γ. The coloring of edges extends to coloring of lobes. Two lobes are said to be *adjacent* if they share common vertices, called *intersections*. If $v \in V(\Gamma)$ is an intersection vertex, then it is common to two unique lobes, which we denote by $\Delta_1(v)$ and $\Delta_2(v)$, colored respectively by 1 and 2. We define the *lobe graph* $T(\Gamma)$ to be the graph whose vertices are the lobes of Γ and whose edges correspond to adjacency of lobes. We say that Γ is *cactoid* if its lobe graph is a finite tree and adjacent lobes have precisely one common intersection.

Theorem 1 [8, Theorem 4.1]. *The Schützenberger automata of the free product $S_1 * S_2$ relative to $\langle X \mid R\rangle$ are, up to isomorphism, precisely (a transversal of) the cactoid inverse automata over X whose lobes are Schützenberger graphs relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$.*

We refer the reader to [8] for details of the iterative procedure used to construct the Schützenberger automata for free products of inverse semigroups and to [1] for the iterative construction of the Schützenberger automata of a "lower bounded" amalgam of inverse semigroups. Our construction below of the Schützenberger automata corresponding to an amalgam of finite inverse semigroups very closely follows Bennett's construction [1], the major difference being that the lobes of the automata under construction are closed DV-quotients of Schützenberger automata relative to $\langle X_i \mid R_i\rangle$ (for $i \in \{1,2\}$), rather than Schützenberger automata, as in [1]. While we will attempt to refer as much as possible to Bennett's construction, the fact that our lobes are not Schützenberger automata does cause some technical difficulties, and several of Bennett's constructions need to be modified. We first need to modify the central construction used in [8]. The idea of this construction is to start with a cactoid automaton over X, close one of its lobes relative to the appropriate presentation $\langle X_i \mid R_i\rangle$, and then make the resulting automaton deterministic.

Let $\mathcal{A} = (\alpha, \Delta, \beta)$ be a finite inverse automaton over X. We define the *closure* of $\mathcal{A}$ with respect to a presentation $\langle X \mid T\rangle$ to be the automaton $\mathrm{cl}(\mathcal{A})$ such that $\mathrm{cl}(\mathcal{A})$ is closed with respect to the presentation, $L(\mathcal{A}) \subseteq L(\mathrm{cl}(\mathcal{A}))$, and if Γ is any other closed automaton with respect to the presentation such that $L(\mathcal{A}) \subseteq L(\Gamma)$, then $L(\mathrm{cl}(\mathcal{A})) \subseteq L(\Gamma)$. The existence of a unique automaton with these properties follows from the work of Stephen [17,18], in particular from Theorem 2.5 of [17] and Lemma 3.4 of [18]. If $\mathcal{A}$ is the linear automaton of some word w then $\mathrm{cl}(\mathcal{A})$ is the Schützenberger automaton $\mathcal{A}(X, T, w)$.

Construction 1. Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton over X. Let Δ be a lobe of Γ, colored by i, that is not closed relative to $\langle X_i \mid R_i\rangle$. Let λ be any vertex of Δ, let $\mathrm{cl}(\Delta)$ be a disjoint copy of the closure of Δ relative to $\langle X_i \mid R_i\rangle$, and let λ^* denote the natural image of λ in $\mathrm{cl}(\Delta)$. Construct the V-quotient $\mathcal{A}^* = (\alpha^*, (\Gamma \cup \mathrm{cl}(\Delta))/\kappa, \beta^*)$, where κ is the least V-equivalence that identifies λ with λ^* and makes the image deterministic, and let α^*, β^* denote the respective images of α and β.

Lemma 4. *Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton over X.*

- (i) *The automaton $\mathcal{A}^*$ constructed from $\mathcal{A}$ by an application of Construction 1 is also a finite cactoid inverse automaton. Moreover, if $\mathcal{A}$ approximates $\mathcal{A}(X, R, w)$ (respectively $\mathcal{A}(X, R \cup W, w)$) for some word $w \in (X \cup X^{-1})^+$, then so does $\mathcal{A}^*$.*
- (ii) *After iteratively applying Construction 1 finitely many times, starting from $\mathcal{A}$, we eventually arrive at a cactoid automaton $\mathcal{A}'$ with the property that each lobe of $\mathcal{A}'$ is a DV-quotient of some Schützenberger graph relative to either $\langle X_1 \mid R_1\rangle$*

or $\langle X_2 \mid R_2\rangle$, $\mathcal{A}'$ is closed with respect to R, and $\mathcal{A}'$ approximates $\mathcal{A}(X, R, w)$ (respectively $\mathcal{A}(X, R \cup W, w)$) if $\mathcal{A}$ does.

(iii) *In addition, if this construction is applied iteratively starting from the linear automaton of a word $w \in (X \cup X^{-1})^+$, then the resulting automaton $\mathcal{A}'$ is the Schützenberger automaton $\mathcal{A}(X, R, w)$ (respectively $\mathcal{A}(X, R \cup W, w)$) of w in the free product $S_1 * S_2$.*

Proof. The proof of this is essentially a slight modification of the proofs of Propositions 3.1, 3.2, and 3.3 and Theorem 3.4 of [8], so we just outline the proof here and refer the reader to [8] for additional details.

Without loss of generality let us assume that Δ is colored by the color 1. The closure $(\lambda^*, \mathrm{cl}(\Delta), \lambda^*)$ of $(\lambda, \Delta, \lambda)$ is a finite inverse automaton obtained by applying finitely many elementary expansions and edge foldings [17] and is a DV-quotient of some Schützenberger automaton relative to $\langle X_1 \mid R_1\rangle$ by Lemma 2. The automaton $\mathcal{A}^*$ is still a cactoid automaton, by essentially the same argument as is used in the proof of Proposition 3.2 of [8] and is an approximate automaton of $\mathcal{A}(w)$ if $\mathcal{A}$ is an approximate automaton of $\mathcal{A}(w)$, by Lemmas 1.3 and 1.5 of [8].

The graph Γ^* has at most as many lobes as Γ. Each of its lobes is either a lobe of Γ or was obtained from lobes $\Delta_1, \Delta_2, \ldots, \Delta_k$ of $\Gamma \cup \mathrm{cl}(\Delta)$ by identifying intersection vertices $v_1, v_2, \ldots, v_k$, forming products $\Delta_i \times \Delta_j$ and folding edges (see [8] for details). From Lemmas 2 and 3, it follows that the lobes of $\mathcal{A}^*$ are DV-quotients of approximate automata of Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$.

The second statement in the Lemma follows easily from the fact that the automata constructed after an application of Construction 1 have finitely many lobes. The final statement (iii) is Theorem 3.4 of [8]. □

Let v be an intersection vertex of an inverse automaton over X, with corresponding lobes $\Delta_1(v)$ and $\Delta_2(v)$. Let $e_i(v)$ denote the minimum idempotent of S_i labeling a loop based at v in Δ_i (for $i = 1, 2$) and let $U_i(e_i(v)) = \{u \in U\colon u$ labels a loop in Δ_i based at $v\}$. If $U_i(e_i(v))$ is nonempty, it is a finite subsemigroup of U, so it has a minimum idempotent which we denote by $f(e_i(v))$. It is clear that if the automaton is deterministic and its lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$, then $\mathcal{A}(X_i, R_i, e_i(v))$ is the maximum determinizing Schützenberger automaton of $(v, \Delta_i(v), v)$.

Remark. If Δ_i is a Schützenberger graph, then $U_i(e_i(v)) = \{u \in U \mid e_i(v) \leqslant_i u\}$, where $\leqslant_i$ denotes the natural partial order in S_i. (This was the definition used by Bennett in [1].) However, these definitions do not in general coincide if Δ_i is not a Schützenberger graph, as one readily sees by examining the example after Lemma 2.

We say that an inverse automaton Γ over X *has property* (L) if for every intersection vertex v of Γ we have

$$\text{either} \quad U_1\big(e_1(v)\big) = U_2\big(e_2(v)\big) = \emptyset \quad \text{or} \quad f\big(e_1(v)\big) = f\big(e_2(v)\big).$$

Construction 2 (a). Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton over X whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and suppose that $\mathcal{A}$ does not satisfy property (L). Without loss of generality, by the last statement in Lemma 2, there exists an intersection vertex v of $\mathcal{A}$ such that $U_1(e_1(v)) \neq \emptyset$ and $w_2(f(e_1(v))) \notin U_2(e_2(v))$. (The other case is dual.) Let $f = w_2(f(e_1(v)))$ and form the product $\mathcal{B} = (v, \Gamma, v) \times \mathcal{A}(X_2, R_2, f)$. The union of the images of $\Delta_2(v)$ and $\mathcal{A}(X_2, R_2, f)$ is a lobe of $\mathcal{B}$ that is a V-quotient of a Schützenberger automaton relative to R_2 by Lemma 3. By repeated applications of Construction 1 we obtain a rooted cactoid automaton $\mathcal{B}' = (v', \Gamma', v')$ which is closed relative to $\langle X \mid R \rangle$ and whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$. The automaton $\mathcal{A}' = (\alpha', \Gamma', \beta')$ (where α' and β' are the respective images of α and β) is the automaton obtained from $\mathcal{A}$ by an application of Construction 2(a) at the vertex v. (It is intended that this construction encompasses the dual case to the one described here as well.)

Lemma 5. *Let $w \in (X \cup X^{-1})^+$ and let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and suppose that $\mathcal{A}$ is an approximate automaton for $\mathcal{A}(X, R \cup W, w)$. If $\mathcal{A}'$ is the automaton obtained from $\mathcal{A}$ by an application of Construction* 2(a), *then $\mathcal{A}'$ is also a finite cactoid inverse automaton whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and $\mathcal{A}'$ approximates $\mathcal{A}(X, R \cup W, w)$. Repeated applications of Construction* 2(a) *to such an automaton $\mathcal{A}$ terminate in a finite number of steps in a finite deterministic cactoid inverse automaton $\mathcal{A}^*$ that satisfies property (L) (and whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$).*

Proof. The proof is really just an adaptation of the proofs of Lemmas 2.2 and 2.3 of [1], the essential difference being that the lobes of the automata under consideration are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ as opposed to Schützenberger automata. The proof of Lemma 2.2 of [1] carries through with almost no change in this setting. This, combined with Lemma 4 yields the first claim in our lemma. The last claim (the fact that repeated applications of Construction 2(a) must terminate after finitely many steps in an automaton that satisfies property (L)) follows again by adapting the proof of Lemma 2.3 of [1] to the current setting, but is actually easier than Bennett's proof of that lemma since the semigroups S_1 and S_2 are both finite, so there are only finitely many possible graphs that can arise as closed DV-quotients of Schützenberger automata relative to $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$. Any application of Construction 2(a) at a vertex v replaces a closed DV-quotient of a Schützenberger graph in either S_1 or S_2 by another closed DV-quotient of a Schützenberger graph, and the new graph has either more edges or more loops (i.e., has higher rank fundamental group) than the original graph. Finiteness of each S_i puts an upper bound on the number of edges and the rank of the fundamental group of these graphs. We refer the reader to Bennett's proof of Lemmas 2.2 and 2.3 in his paper [1] for full details. $\square$

We say that an inverse automaton $\mathcal{A}$ over X has the *loop equality property* if $U_1(e_1(v)) = U_2(e_2(v))$ for each intersection vertex v of $\mathcal{A}$. If all lobes are in fact Schützenberger graphs, then this concept coincides with the concept of the *lower bound equality property* of Bennett [1]. It is clear that if $\mathcal{A}$ satisfies the loop equality property, then it must also satisfy property (L), but the converse is false in general.

Lemma 6. *Let $\mathcal{A}$ be a finite cactoid inverse automaton over X whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$ and suppose that v is an intersection vertex of two lobes $\Delta_1(v)$ and $\Delta_2(v)$ such that $f(e_1(v)) = f(e_2(v))$. If there is a path in $\Delta_1(v)$, starting at v and labeled by a word $w_1(u)$ for some $u \in U$, then there is a path in $\Delta_2(v)$, starting at v and labeled by the word $w_2(u)$.*

Proof. Clearly $e_1(v) \leqslant w_1(u)w_1(u)^{-1}$ and $f(e_2(v)) = f(e_1(v)) \leqslant w_1(u)w_1(u)^{-1}$. Now $f(e_2(v)) \leqslant_2 w_2(u)w_2(u)^{-1}$ since both of these elements are in the image of U in S_2, so $e_2(v) \leqslant_2 w_2(u)w_2(u)^{-1}$. By the remark after Lemma 2, $(v, \Delta_2(v), v)$ is a DV-quotient of $(v, S\Gamma(e_2(v)), v)$. Since $w_2(u)w_2(u)^{-1}$ labels a loop based at v in $S\Gamma(e_2(v))$, it follows that $w_2(u)w_2(u)^{-1}$ labels a loop in $\Delta_2(v)$ based at v. But this means that $w_2(u)$ labels a path in $\Delta_2(v)$ based at v. □

Construction 2(b). Let $\mathcal{A}$ be a finite cactoid inverse automaton over X whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$ and suppose that $\mathcal{A}$ satisfies property (L) but does not satisfy the loop equality property. Then either there exists some intersection vertex v of $\mathcal{A}$ and a nonidempotent element $u \in U$ such that $w_1(u) \in U_1(e_1(v))$ and $w_2(u) \notin U_2(e_2(v))$ or there exists an intersection vertex v with the dual property (with subscripts interchanged). Without loss of generality assume that the first case occurs. In Δ_1 there is a loop based at v labeled by $w_1(u)$, while in $\Delta_2(v)$ there is a v–v' path labeled by $w_2(v)$ for some v', by Lemma 6. Form the V-quotient $\mathcal{B}$ of $\mathcal{A}$ obtained by identifying v and v' in $\Delta_2(v)$. Then apply Constructions 1 and 2(a) to the resulting automaton $\mathcal{B}$, obtaining an automaton $\mathcal{A}'$. We say that $\mathcal{A}'$ is obtained from $\mathcal{A}$ by an application of Construction 2(b).

Lemma 7. *Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$, suppose that $\mathcal{A}$ satisfies property (L) and that $\mathcal{A}$ approximates $\mathcal{A}(X, R \cup W, w)$ for some word w. Then the automaton $\mathcal{A}'$ obtained from $\mathcal{A}$ by an application of Construction* 2(b) *also has lobes that are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$, $\mathcal{A}'$ approximates $\mathcal{A}(X, R \cup W, w)$ and $\mathcal{A}'$ satisfies property (L). Successive applications of Construction* 2(b) *lead after finitely many steps to a finite cactoid inverse automaton $\mathcal{A}^*$ whose lobes are closed DV-quotients of Schützenberger automata relative to $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$ such that $\mathcal{A}^*$ approximates $\mathcal{A}(X, R \cup W, w)$ and $\mathcal{A}^*$ has the loop equality property.*

Proof. It is clear by Lemmas 2 and 5 that the lobes of $\mathcal{A}'$ are closed DV-quotients of appropriate Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$ and that $\mathcal{A}'$ satisfies property (L). Let $\mathcal{C}$ be the automaton obtained from $\mathcal{A}$ by sewing on to $\mathcal{A}$ a loop labeled by $w_2(u)$ based at the vertex v. This operation is an "elementary expansion" in the sense of Stephen [17], since $w_1(u) = w_2(u)$ is a relation in W and $w_1(u)$ labels a loop in $\Delta_1(v)$ by assumption. It is clear that the determinized form of the intermediate automaton $\mathcal{B}$ in the description of Construction 2(b) is obtained from $\mathcal{C}$ by a finite sequence edge foldings, so this automaton is an approximate automaton of $\mathcal{A}(X, R \cup W, w)$ by Lemma 5.6 and Theorem 5.7 of [17]. Hence by Lemmas 4 and 5 above, $\mathcal{A}'$ is an approximate automaton of $\mathcal{A}(X, R \cup W, w)$.

The proof that a finite sequence of applications of Construction 2(b) terminates in an automaton that satisfies the loop equality property is again a modification of Bennett's proof of his Lemma 2.3 in [1]. Each application of Construction 2(b) effectively introduces an additional relation of the form $w_1(u) = w_2(u)$ for some $u \in U$ at some intersection vertex v. The construction may also decrease the number of lobes and the number of intersection vertices of the resulting automaton, but each intersection vertex has an image that is also an intersection vertex in the resulting automaton, and loops labeled by $w_i(u)$ in a lobe $\Delta_i(v)$ are transformed into loops with the same label in the new automaton. Finiteness of the automata and of the semigroup U forces this process to stop after finitely many steps in an automaton that satisfies the loop equality property. □

Remark. Construction 2(b) provides one of the essential differences between the argument presented in this paper and Bennett's argument [1]. It is a consequence of this construction that the lobes of the automata under construction are DV-quotients of Schützenberger automata (as opposed to Schützenberger automata) relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$.

We next consider the *related pair separation property* of Bennett. Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite inverse automaton over X whose lobes are closed DV-quotients of Schützenberger automata relative to $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and let v be an intersection vertex of $\mathcal{A}$ for which $U_1(e_1(v)) = U_2(e_2(v))$. Consider a word $u \in U$ such that $w_1(u)$ labels a v–v_1 path in $\Delta_1(v)$ for some vertex v_1. Then $w_1(u)$ labels a v^*–v_1^* path in the maximum determinizing Schützenberger automaton $\mathcal{A}(e_1(v))$ by Lemma 2. Hence $w_1(u)w_1(u)^{-1} \geqslant e_1(v)$, whence this element belongs to $U_1(e_1(v)) = U_2(e_2(v))$. It follows that $w_2(v)$ also labels a v–v_2 path in Δ_2 for some vertex v_2. Following Bennett [1], we say that (v_1, v_2) is a *related pair* of the intersection vertex v. By a very minor modification of Bennett's argument in the first part of Section 3 of his paper [1], we see that the relation $R(v)$ consisting of all pairs (v_1, v_2) such that (v_1, v_2) is a related pair of v defines a partial one–one map from $V(\Delta_1(v))$ to $V(\Delta_2(v))$. The equivalence relation on Γ generated by $R(v)$ thus identifies the two coordinates of each related pair without identifying any two vertices from the same lobe.

Let $\mathcal{A}$ be a finite inverse automaton over X whose lobes are closed DV-quotients of Schützenberger automata relative to $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and which satisfies the loop equality property. We say that $\mathcal{A}$ has the *related pair separation property* if for any lobe Δ of $\mathcal{A}$ (with color 1 without loss of generality) and for any two intersection vertices v and v' of Δ that are vertices of Δ but are not common to the same *pair* of lobes of $\mathcal{A}$, there is no word $u \in U$ such that $w_1(u)$ labels a path in Δ from v to v'.

Construction 3. Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton whose lobes are closed DV-quotients of some Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and suppose that $\mathcal{A}$ has the loop equality property. Let v_0 and v_1 be two different intersection vertices of a lobe Δ_2 that is (without loss of generality) colored by the color 2 and suppose that there is a path labeled by $w_2(u)$ from v_0 to v_1 for some $u \in U$. Let Δ_0 and Δ_1 be the two lobes (colored by 1) adjacent to Δ_2 and intersecting Δ_2 in v_0 and v_1, respectively. Since $\mathcal{A}$ has the loop equality property, there is a path in Δ_0 from v_0 to v_0' labeled by $w_1(u)$ for some vertex v_0'. Form the graph $\tilde{\Gamma}$ by disconnecting Γ at v_0 and replacing v_0 with $v_0(0)$ and $v_0(2)$ in Δ_0 and Δ_2, respectively. Denote by T_0 the component of $\tilde{\Gamma}$ that contains $v_0(0)$ and by T_2 the component that contains $v_0(2)$. Now put $\mathcal{B} = (v_0', T_0, v_0') \times (v_1, T_2, v_1)$. Clearly all lobes of $\mathcal{B}$ except at most $\Delta_0 \times \Delta_1$ are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$. By Lemma 3, $\Delta_0 \times \Delta_2$ is a V-quotient of an approximate automaton relative to $\langle X_1 \mid R_1 \rangle$, so we can apply Constructions 1, 2(a), and 2(b) to the automaton $\mathcal{B}$. Denote the natural images in $\mathcal{B}$ of α by α' and of β by β' and let $\mathcal{A}' = (\alpha', \Gamma', \beta')$ be the resulting automaton. We say that $\mathcal{A}'$ is obtained from $\mathcal{A}$ by an application of Construction 3.

Lemma 8. *Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$, that approximates $\mathcal{A}(X, R \cup W, w)$ for some word w and has the loop equality property. If $\mathcal{A}'$ is the automaton obtained from $\mathcal{A}$ by an application of Construction* 3, *then $\mathcal{A}'$ also is a finite cactoid inverse automaton whose lobes are closed DV-quotients of Schützenberger automata relative to $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$, that has the loop equality property and approximates $\mathcal{A}(X, R \cup W, w)$. Furthermore, repeated applications of this construction will terminate in a finite number of steps in an automaton that has the related pair separation property.*

Proof. The only thing that needs to be proved is that the automaton $\mathcal{A}'$ approximates $\mathcal{A}(X, R \cup W, w)$: all other statements in the lemma are immediate. Thus we have to show that $L(\mathcal{A}') \subseteq L(\mathcal{A}(X, R \cup W, w))$ and that in $L(\mathcal{A}')$ there exists a word τ-equivalent to w. Let $\mathcal{B}$ be the automaton constructed in the description of Construction 3 and denote the natural image of α (respectively β) in $\mathcal{B}$ by α (respectively β) again. Let $\mathcal{A}'' = (\alpha, \Sigma, \beta)$ be the resulting automaton, where Σ is the underlying graph of $\mathcal{B}$. By Lemmas 4, 5, and 7 it suffices to check that $\mathcal{A}''$ is an approximate automaton for $\mathcal{A}(X, R \cup W, w)$.

Let $v_0' = v_1$ be denoted by v in $\mathcal{A}''$. Now let $z \in L(\mathcal{A}'')$. Then there exists in Σ an α–β path labeled by z. Every α–β path which belongs entirely to the same component T_0 or T_2 (if any) was already an α–β path in Γ, whence its label belongs to $L(\mathcal{A}(X, R \cup W, w))$. So consider an α–β path λ in $\mathcal{A}''$ containing v and which can be split into parts which belong to different components. Consider $\lambda = \gamma\delta$, where $t(\delta) = \beta$, $i(\delta) = v$ and if $\beta = v$ then $\lambda = \gamma$, else δ belongs entirely to the same component as the vertex β, and factor $\gamma = \gamma_1\gamma_2 \cdots \gamma_n$ where

- $i(\gamma_1) = \alpha$, $t(\gamma_n) = v$, $i(\gamma_{i+1}) = t(\gamma_i) = v$, $i = 1, \ldots, n-1$,
- each of the paths γ_i belongs entirely to one of the components T_0 or T_2, $i = 1, \ldots, n$,

- γ_{i+1} and γ_i belong to different components T_0 and T_2, for $i = 1, \ldots, n-1$; the same holds for δ and γ_n if $\gamma \neq \lambda$.

Let us prove by induction on n that if γ_n is a path in T_0 (respectively T_2), then there exists an α–v_0' (respectively α–v_1) path γ^* in the graph Γ which is labeled by a word which is less than or equal to $l(\gamma)$ in $S = (X \cup X^{-1})^+/\tau$. This will prove that $L(\mathcal{A}'') \subseteq L(\mathcal{A}(X, R \cup W, w))$, since this language consists of all words that are greater than or equal to w in S.

Let $n = 1$. Then $\gamma = \gamma_1$ is an α–v_0' path belonging to the component T_0, whence also it is a path in Γ (similarly if γ_1 belongs to T_2).

Now let $\gamma = \gamma_1\gamma_2 \cdots \gamma_n$ and suppose that γ_n belongs to T_0, so that γ_n labels a loop based at v_0'. Thus γ_{n-1} belongs to T_2 and there exists an α–v_1 path γ' in the graph Γ such that $l(\gamma') \leqslant l(\gamma_1\gamma_2 \cdots \gamma_{n-1})$ in S. In addition, there exists a v_1–v_0 path labeled by $w_2(u)$, a v_0–v_0' path labeled by $w_1(u^{-1})$, and thus an α–v_0' path labeled by $l(\gamma_1\gamma_2 \cdots \gamma_{n-1})w_2(u)w_1(u)^{-1}l(\gamma_n)$ in Γ. Also $l(\gamma)\tau \geqslant (l(\gamma_1\gamma_2 \cdots \gamma_{n-1})w_1(u)w_1(u)^{-1} \cdot l(\gamma_n))\tau = (l(\gamma_1\gamma_2 \cdots \gamma_{n-1})w_2(u)w_1(u)^{-1}l(\gamma_n))\tau$.

The case when γ_n is in T_2 is symmetric.

Finally, note that in $\mathcal{A} = (\alpha, \Gamma, \beta)$ there exists an α–β path labeled by a word w' such that $w'\tau = w\tau$. If this path has no vertex equal to v_0 then its label also labels a path in $\mathcal{A}''$. So, consider $w' = l(\gamma_1)l(\gamma_2) \cdots l(\gamma_n)$, where

– $i(\gamma_1) = \alpha$, $t(\gamma_n) = \beta$, $i(\gamma_{i+1}) = t(\gamma_i) = v_0$, $i = 1, \ldots, n-1$,
– each of the paths γ_i belongs entirely to the same of the components T_0 and T_2, $i = 1, \ldots, n$ and γ_{i+1} and γ_i belong to different components T_0 and T_2 for $i = 1, \ldots, n-1$.

Suppose, without loss of generality, that γ_i belongs to T_0 for i even.

Certainly in the automaton $\mathcal{A}''$ there is an α–β path labeled by the word $w'' = l(\gamma_1)w_2(u^{-1})w_1(u)l(\gamma_2)w_1(u^{-1})w_2(u)l(\gamma_3) \cdots l(\gamma_n)$ and $w''\tau = w'''\tau$ where $w''' = l(\gamma_1)w_2(u)^{-1}w_2(u)l(\gamma_2)w_1(u)^{-1}w_1(u) \cdots l(\gamma_n)$.

Clearly $w'''\tau \leqslant w'\tau$ in S. But w''' labels a path from α to β in Γ, so $w''' \in L(\mathcal{A}) \subseteq L(\mathcal{A}(X, R \cup W, w))$. Hence $w'''\tau \geqslant w'\tau$ in S. Thus $w'''\tau = w''\tau = w'\tau$ in S. This shows that $\mathcal{A}''$ is an approximate automaton for $\mathcal{A}(X, W \cup R, w)$, as required. □

We now consider the *adjacent lobe assimilation* property of Bennett [1].

Construction 4. Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite inverse word graph, whose lobes are closed DV-quotients of some Schützenberger automata relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$ and which has the loop equality property and the related pair separation property. Then for each intersection vertex v and for every v–v_1 path in $\Delta_1(v)$ labeled by $w_1(u)$ for some $u \in U$ there exists a unique v–v_2 path in $\Delta_2(v)$ labeled by $w_2(u)$, and conversely; v_1 and v_2 cannot be intersection vertices of our graph by the related pair separation property. Identify v_1 and v_2, i.e., consider the V-quotient of the graph Γ with respect to the equivalence relation $v_1 = v_2$ and repeat this construction with respect to all related pairs in $\Delta_1(v)$ and $\Delta_2(v)$. Since all lobes are finite, then we end after finitely many identifications: we say that the two lobes $\Delta_1(v)$ and $\Delta_2(v)$ were *assimilated*.

Lemma 9. *Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a finite cactoid inverse automaton, whose lobes are closed DV-quotients of some Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and which has the loop equality property and the related pair separation property and suppose that $\mathcal{A}$ approximates $\mathcal{A}(X, R \cup W, w)$. After finitely many applications of Construction* 4*, we get a finite inverse automaton whose lobes are closed DV-quotients of some Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$, which approximates $\mathcal{A}(X, R \cup W, w)$, has the loop equality property and the related pair separation property and where all adjacent lobes are assimilated.*

Proof. Denote by v' the vertex of the graph Γ' obtained from Γ by identifying v_1 with v_2 in Construction 4. We first show that $U_1(e_1(v')) = U_2(e_2(v'))$ in Γ'. Recall that there exists a v–v_1 path in $\Delta_1(v)$ labeled by $w_1(u)$ for some $u \in U$ and a v–v_2 path in $\Delta_2(v)$ labeled by $w_2(u)$.

If $u' \in U_1(e_1(v'))$, then u' labels a loop based at v_1 in $\Delta_1(v)$ and so $uu'u^{-1} \in U_1(e_1(v))$, as it labels a loop based in v in $\Delta_1(v)$; hence by the loop equality property $uu'u^{-1} \in U_2(e_2(v))$ and labels a loop based at v in $\Delta_2(v)$. But u labels a v–v_2 path in $\Delta_2(v)$ so that u' also labels a loop based at v_2 whence $u' \in U_2(e_2(v'))$.

This enables us to repeat Construction 4 as many times as we need on each pair of lobes, obtaining an automaton that satisfies the loop equality property after each step. The related pair separation property still holds after every application of the construction, since all the vertices we are working on are connected by paths whose labels belong to U. By the finiteness of U and of the number of lobes of the automaton we finish after finitely many applications of this construction.

Note that an application of Construction 4 may also be accomplished by sewing on to $\mathcal{A}$ a path labeled by $w_2(u)$ from v to v_1 (in the notation of the construction) and then folding edges in the resulting automaton. It follows from Lemma 5.6 and Theorem 5.7 of [17] that the resulting automaton is also an approximate automaton of $\mathcal{A}(X, R \cup W, w)$. □

Since assimilation does not affect adjacency of lobes, a lobe path is reduced in Γ if and only if it is reduced in the assimilated form of Γ. Following Bennett [1], we say that an inverse automaton $\mathcal{A}$ whose lobes are closed DV-quotients of Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ is *opuntoid* if:

(i) it has the loop equality property;
(ii) it has the adjacent lobe assimilation property;
(iii) it has no nontrivial reduced lobe loops (i.e., its lobe graph is a tree).

From the discussions above, it is clear that the automaton that we obtain from the linear automaton of a word $w \in (X \cup X^{-1})^+$ by closing under repeated applications of Constructions 1–4 above is a finite inverse opuntoid automaton that approximates $\mathcal{A}(X, R \cup W, w)$. We refer to this automaton as the *core automaton* of w and denote it by $\mathrm{Core}(w)$: this is not the Schützenberger automaton of w and it is also not the case that $\mathrm{Core}(w) = \mathrm{Core}(w')$ if $w\tau = w'\tau$, but as we shall see below, the Schützenberger automaton $\mathcal{A}(X, R \cup W, w)$ is readily obtained from $\mathrm{Core}(w)$ by successive applications of Construction 5 below, and carries all of the essential information of $\mathcal{A}(X, R \cup W, w)$.

Let Γ be an opuntoid graph and let $v \in V(\Gamma)$ be a vertex belonging to a lobe Δ_i colored by $i \in \{1, 2\}$. Then (again analogously to Bennett [1]), we say that v is a *bud* of Γ if it is not an intersection vertex and $U_i(e_i(v))$ is not empty. This is equivalent to saying that there is some path in Δ_i starting at v and labeled by an element $u \in U$, because in that case, $uu^{-1} \in U_i(e_i(v))$. (Clearly, no such path can end in an intersection vertex, by the adjacent lobe assimilation property.) The graph Γ is *complete* if it has no buds: an opuntoid automaton is complete if its underlying graph is complete.

Construction 5. Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be an opuntoid automaton, suppose that $\mathcal{A}$ is not complete and let v be a bud, so v is not an intersection, belonging to a lobe Δ_2 colored say by 2, with $U_2(e_2(v)) \neq \emptyset$. Form the automaton $\mathcal{B} = (v^*, \Gamma^*, v^*) = (v, \Gamma, v) \times \mathcal{A}(X_1, R_1, f)$, with $f = w_1(f(e_2(v)))$. By Lemma 6, if $u \in U_2(e_2(v))$, then $w_1(u)$ labels a path starting at v and ending at v', say, in the new adjoined lobe $\mathcal{A}(X_1, R_1, f)$, but this path is not necessarily a loop. Form a lobe Δ_1 by first identifying all such vertices v' with v in $\mathcal{A}(X_1, R_1, f)$, then determinizing, and then closing with respect to R_1. Finally, apply Construction 4 at the vertex v to assimilate Δ_2 and the new lobe Δ_1, and denote the resulting automaton by $\mathcal{A}^*$.

Lemma 10. *Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be an opuntoid automaton whose lobes are closed DV-quotients of some Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and suppose that $\mathcal{A}$ approximates $\mathcal{A}(X, R \cup W, w)$. Then each application of Construction 5 leads to an opuntoid automaton $\mathcal{A}^*$ whose lobes are closed DV-quotients of some Schützenberger automata relative to either $\langle X_1 \mid R_1 \rangle$ or $\langle X_2 \mid R_2 \rangle$ and which approximates $\mathcal{A}(X, R \cup W, w)$. In particular, the new automaton $\mathcal{A}^*$ has one more lobe than $\mathcal{A}$, and the automaton $\mathcal{A}$ is unchanged by this process.*

It is convenient to split the proof of this lemma into several parts.

Lemma 11. *Fix the notation as in the statement of Construction 5. Let u be an element of U such that $w_2(u)$ labels a loop based at v in Δ_2 but u is not an idempotent of U. Let n be the smallest integer such that u^n is an idempotent of U. Then $w_1(u)^n$ labels a loop based at v in $\mathcal{A}(X_1, R_1, f)$. Denote by v_i the vertex in $\mathcal{A}(X_1, R_1, f)$ at the end of the path starting at v and labeled by $w_1(u)^i$, for $i = 1, \dots, n-1$. Then for all i, a word $s \in (X_1 \cup X_1^{-1})^+$ labels a path in $\mathcal{A}(X_1, R_1, f)$ starting at v_i if and only if s labels a word in $\mathcal{A}(X_1, R_1, f)$ starting at v. Furthermore, s labels a loop based at v in $\mathcal{A}(X_1, R_1, f)$ if and only if s labels a loop based at v_i in $\mathcal{A}(X_1, R_1, f)$.*

Proof. Suppose first that s labels a path in $\mathcal{A}(X_1, R_1, f)$ starting at v. Then ss^{-1} labels a loop in $\mathcal{A}(X_1, R_1, f)$ based at v, so $ss^{-1} \geqslant f$. Hence $uss^{-1}u^{-1} \geqslant ufu^{-1}$ in U. Since ufu^{-1} is an idempotent of U that labels a path based at v in $\mathcal{A}(X_1, R_1, f)$, we have $ufu^{-1} \geqslant f$, and hence $uss^{-1}u^{-1} \geqslant f$, whence s labels a path starting at v_1 in $\mathcal{A}(X_1, R_1, f)$. If s labels a loop at v, since $\mathcal{A}(X_1, R_1, f)$ is a Schützenberger automaton, then usu^{-1} must label a loop at v also, since u labels a path from v to v_1, so s labels a loop at v_1. A similar argument applies to the vertices v_i for $i > 1$.

Conversely, if s labels a path starting at v_1 in $\mathcal{A}(X_1, R_1, f)$, then $uss^{-1}u^{-1} \geqslant f$, so $u^n ss^{-1}u^{-n} \geqslant u^{n-1} f u^{-(n-1)}$ in U. This latter idempotent is greater than or equal to f, again by minimality of f. Also, u^n is an idempotent of U. Hence $u^n ss^{-1} \geqslant f$, and it follows that $ss^{-1} \geqslant f$, whence s labels a path in $\mathcal{A}(X_1, R_1, f)$, starting at v. If s labels a loop at v_1, then as above, $u^n su^{-n}$ labels a loop at v, and since u^n labels a loop at v this means that s labels a loop at v. A similar argument applies if we start at a vertex v_i for $i > 1$. This verifies the claim above. □

Lemma 12. *Fix the notation as in the statement of Lemma* 11*. Let Δ be the DV-quotient of $\mathcal{A}(X_1, R_1, f)$ obtained by identifying all of the vertices $v_1, v_2, \dots, v_{n-1}$ with v and then determinizing. Then two vertices γ_1 and γ_2 of $\mathcal{A}(X_1, R_1, f)$ are identified in Δ if and only if there is some word w that labels a path in $\mathcal{A}(X_1, R_1, f)$ from v_i to γ_1 and a path in $\mathcal{A}(X_1, R_1, f)$ from v_j to γ_2 for some i, j. Furthermore, a word $s \in (X_1 \cup X_1^{-1})^+$ labels a path in $\mathcal{A}(X_1, R_1, f)$ starting at γ_1 if and only if s labels a path in $\mathcal{A}(X_1, R_1, f)$ starting at γ_2.*

Proof. Define $\gamma_1 \sim \gamma_2$ if and only if there exists a word w labeling a path from v_i to γ_1 and from v_j to γ_2 for some i, j. We claim that $\sim$ coincides with the equivalence relation $\equiv$ of Lemma 1. Clearly $\sim$ is included in $\equiv$. We show that $\sim$ is an equivalence relation.

Suppose that $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. Then there exist words w and s and vertices v_i, v_j, v_k, v_l such that w labels a path from v_i to γ_1 and a path from v_j to γ_2 and s labels a path from v_k to γ_2 and a path from v_l to γ_3. There is a path labeled by u^t from v_j to v_k for some t. Hence $u^t sw^{-1}$ labels a loop in $\mathcal{A}(X_1, R_1, f)$ based at v_j. By Lemma 11, $u^t sw^{-1}$ also labels a loop in $\mathcal{A}(X_1, R_1, f)$ based at v_i. This loop must go from v_i to some vertex v_h (via the path labeled by u^t), then from v_h to some vertex β (via a path labeled by s), and then back to v_i (via a path labeled by w^{-1}). But since w labels a path from v_i to γ_1, we must have w^{-1} labels a path from γ_1 to v_i, and so $\beta = \gamma_1$. Hence there is a path labeled by s from v_h to γ_1, and also a path labeled by s from v_l to γ_3, so $\gamma_1 \sim \gamma_3$.

It is also clear from Lemma 11 that if $\gamma_1 \sim \gamma_2$ and s is a word in $(X \cup X^{-1})^+$, then s labels a path in $\mathcal{A}(X_1, R_1, f)$ starting from γ_1 if and only if s labels a path in $\mathcal{A}(X_1, R_1, f)$ starting from γ_2 (just extend the path labeled by some word w from v_i to γ_1 and from v_j to γ_2: ws labels a path starting at v_i if and only if it also labels a path starting at v_j, by Lemma 11). Hence $\sim$ satisfies the two properties defining the equivalence relation $\equiv$, and so $\sim$ is equal to $\equiv$. The last statement in the lemma also follows from the above argument. □

Lemma 13. *In the notation of Lemma* 12*, the lobe Δ is closed with respect to the relations R_1.*

Proof. Suppose that γ_1 and γ_2 are two vertices of Δ and that there is a path in Δ labeled by a word s from γ_1 to γ_2 and that $s = t$ is a relation in R_1. We must show that t also labels a path in Δ from γ_1 to γ_2. Assume first that neither γ_1 nor γ_2 is equal to the image of v in the natural morphism from $\mathcal{A}(X_1, R_1, f)$ to Δ. Thus we may regard γ_1 and γ_2 as vertices of $\mathcal{A}(X_1, R_1, f)$.

By construction of Δ from $\mathcal{A}(X_1, R_1, f)$, and by Lemma 12, there must be a factorization of the word s as a product $s = s_1 s_2 \ldots s_k$ and vertices $\delta_i, \beta_i, i = 1, \ldots, k$, where $\delta_1 = \gamma_1, \beta_k = \gamma_2$, s_i labels a path from δ_i to β_i, for $i = 1, \ldots, k$ and $\beta_i \sim \delta_{i+1}$ for $i = 1, \ldots, k-1$.

By Lemma 12, s_2 labels a path from β_1 to some vertex μ_2 such that $\mu_2 \sim \beta_2 \sim \delta_3$, and then s_3 labels a path from μ_2 to some vertex μ_3 with $\mu_3 \sim \beta_3 \sim \delta_4$, and so on. Thus we eventually produce a path labeled by $s = s_1 s_2 \ldots s_k$ in $\mathcal{A}(X_1, R_1, f)$ starting at $\delta_1 = \gamma_1$ and ending at a vertex $\mu_k \sim \gamma_2$. Since $\mathcal{A}(X_1, R_1, f)$ is closed with respect to the presentation R_1, it follows that there is a path in $\mathcal{A}(X_1, R_1, f)$ from γ_1 to μ_k labeled by the word t. Since $\mu_k \sim \gamma_2$, we see that there is a path from γ_1 to γ_2 labeled by t in Δ. Hence Δ is closed with respect to the relations R_1, as required. A similar argument applies if one or both of the vertices γ_i is equal to the image of v in the natural map from $\mathcal{A}(X_1, R_1, f)$ to Δ. $\square$

Proof of Lemma 10. We first need to prove that when Construction 5 is applied, we have $U_1(e_1(v)) = U_2(e_2(v))$ at the new intersection point v (in the notation of Construction 5). By construction, we clearly have $U_2(e_2(v)) \subseteq U_1(e_1(v))$. To prove the converse, we need to show that every loop based at v in Δ_1 labeled by an element of U, also labels a loop based at v in Δ_2. Now the lobe Δ_1 of Construction 5 is obtained by identifying all vertices $v_i(u)$ of $\mathcal{A}(X_1, R_1, f)$, as described above, for all words u that label loops at v in Δ_2, then determinizing, and then closing with respect to R_1. But by Lemma 13, the DV-quotient of $\mathcal{A}(X_1, R_1, f)$ obtained by performing the identifications and the determinizing is already closed with respect to R_1. Thus we need only to consider loops in this DV-quotient Δ_1 of $\mathcal{A}(X_1, R_1, f)$.

So let u' be an element of U such that $w_1(u')$ labels a loop based at v in Δ_1. By factoring the word u' as a product $u' = u_1 u_2 \ldots u_k$ where each u_i labels an appropriate path in $\mathcal{A}(X_1, R_1, f)$ and by applying an argument very similar to the argument used in the proof of Lemma 13, we see that u' labels a path in $\mathcal{A}(X_1, R_1, f)$ from $v_i(u)$ to $v_j(\bar{u})$ for some $u, \bar{u} \in U$ such that $w_2(u)$ and $w_2(\bar{u})$ label loops in Δ_2 based at v, and some i, j. Then $w_1(u)^i w_1(u') w_1(\bar{u})^{-j}$ labels a loop based at v in $\mathcal{A}(X_1, R_1, f)$.

It follows that $u^i u' \bar{u}^{-j} \geqslant_1 f$. Also, $u^i u' \bar{u}^{-j} \in U$ of course. Now by the remark after Lemma 2, Δ_2 is a DV-quotient of $\mathcal{A}(X_2, R_2, e_2(v))$ and $u^i u' \bar{u}^{-j} \geqslant_2 f \geqslant_2 e_2(v)$. It follows that $w_2(u^i) w_2(u') w_2(\bar{u}^{-j})$ labels a loop based at v in $\mathcal{A}(X_2, R_2, e_2(v))$, and hence in Δ_2. Since $w_2(u)$ and $w_2(\bar{u})$ label loops in Δ_2 based at v, we see that $w_2(u')$ also labels a loop in Δ_2 based at v. Hence $U_1(e_1(v)) = U_2(e_2(v))$ at the new intersection point v (in the notation of Construction 5).

It is now clear that after we apply Construction 4, the resulting automaton $\mathcal{A}^*$ is opuntoid. We need only verify that $\mathcal{A}^*$ is an approximate automaton for $\mathcal{A}(w) = (\alpha, S\Gamma(w\tau), \beta)$. It suffices to prove that the automaton $\mathcal{A}'$ obtained from $\mathcal{A}$ by adding the lobe Δ_1 at v is an approximate automaton for $\mathcal{A}(w) = (\alpha, S\Gamma(w\tau), \beta)$.

In fact $L(\mathcal{A}) \subseteq L(\mathcal{A}')$, so that in $L(\mathcal{A}')$ there is a word w' such that $w'\tau = w\tau$. Consider a word s which labels in $\mathcal{A}'$ an α–β path in $\mathcal{A}'$. If this path does not contain any edges in Δ_1, then clearly $s \in L = L(\mathcal{A}(w)) = \{z \in (X \cup X^{-1})^+ \mid z\tau \geqslant w\tau\}$. In addition, if $s = s_1 s_2$ where s_1 labels a path from α to v in $\mathcal{A}$ and s_2 labels a path from v to β in $\mathcal{A}$, and if u is an element of U such that $w_2(u)$ labels a loop in Δ_2 based at v, then we also see that

$s_1w_2(f)w_2(u)s_2 \in L$. So assume that s factors as $s = s_1t_1s_2t_2 \dots t_ks_{k+1}$ where s_1 labels a α–v path in $\mathcal{A}$, t_i labels a loop in Δ_1 based at v, s_i labels a loop in $\mathcal{A}$ based at v for $i = 1, \dots, k$, and s_{k+1} labels a path from v to β in $\mathcal{A}$.

Proceed by induction on k, the above case $k = 0$, where no edges of the α–β path labeled by s are in Δ_1, being the basis for the induction. So we may assume that $s_1t_1 \dots s_{k-1}t_{k-1}s_kw_2(f)w_2(u)^{-1}s_{k+1} \in L$ for each element $u \in U$ such that $w_2(u)$ labels a loop in Δ_2 based at v. By Lemma 2, t_k labels a path from v to some vertex β in $\mathcal{A}(X_1, R_1, f)$, where β is identified with v in the DV-quotient Δ_1 of $\mathcal{A}(X_1, R_1, f)$, as constructed in Lemma 12. By the construction of this DV-quotient, there is some word $u \in U$ such that $w_2(u)$ labels a loop in Δ_2 and $w_1(u)$ labels a path from β to v in Δ_1. Thus $t_kw_1(u)$ labels a loop based at v in $\mathcal{A}(X_1, R_1, f)$, whence $t_kw_1(u) \geqslant_1 w_1(f)$.

Now by induction hypotheses, $s_1t_1 \dots s_{k-1}t_{k-1}(s_ks_{k+1}) \geqslant_S w$, and by $t_kw_1(u) \geqslant_S w_1(f)$, we get $s_1t_1 \dots s_kt_kw_1(u)s_{k+1} \geqslant_S s_1t_1 \dots s_kw_1(f)s_{k+1} \geqslant_S w$, so $s = s_1t_1 \dots s_kt_ks_{k+1} \geqslant_S s_1t_1 \dots s_kt_kw_1(u)w_1(u)^{-1}s_{k+1} \geqslant_S s_1t_1 \dots s_kw_1(f)w_1(u)^{-1}s_{k+1} =_S s_1t_1 \dots s_kw_2(f) \cdot w_2(u)^{-1}s_{k+1} \geqslant_S w$. Hence $s \in L$, as required. □

We are now in a position to prove the main theorem of the paper.

Theorem 2. *Let $S = S_1 *_U S_2$ be an amalgamated free product of finite inverse semigroups S_1 and S_2 amalgamating a common inverse subsemigroup U, where $S_i = \mathrm{Inv}\langle X_i \mid R_i\rangle$ are given finite presentations of S_i for $i = 1, 2$. Then the word problem for S is decidable.*

Proof. Let w_1 and w_2 be two words in $(X \cup X^{-1})^+$. We need a decision procedure to show whether $w_2 \in L(\mathcal{A}(X, R \cup W, w_1))$ or not. Suppose that $|w_2| = n$. Iteratively apply Constructions 1, 2(a), 2(b), 3, and 4 to the word w_1 to obtain an automaton $\mathcal{A}$ that is an approximate automaton for $\mathcal{A}(X, R \cup W, w_1)$. Applications of Construction 5 to this and subsequent automata leave $\mathcal{A}$ unchanged. By Lemma 10, the opuntoid nature of all subsequent automata means that the lobe graph of each of these automata is obtained from the previous lobe graph (tree) by adding one more vertex and edge, and that the only change that results by applying Construction 5 is to add one more lobe to the original automaton. Apply Construction 5 to $\mathcal{A}$ and subsequent automata enough times so that either no further application of Construction 5 is possible, or we build all automata whose lobe graphs contain all possible paths of length n starting from the initial lobe (the lobe containing the initial vertex) of the automaton $\mathcal{A}$. The word w_2 is accepted by the automaton $\mathcal{A}(X, R \cup W, w_1)$ if and only if it is accepted by one of the automata iteratively obtained from $\mathcal{A}$ by application of Construction 5. Thus we have a finite decision procedure to test whether $w_2 \in L(\mathcal{A}(X, R \cup W, w_1))$. By the results of Stephen [17], this provides a solution to the word problem for S. □

Recalling our definition of opuntoid automaton (slightly different from Bennett's), one can use arguments very similar to [1] Lemma 5.4 to show that:

Theorem 3. *Let $S = S_1 *_U S_2$ be an amalgamated free product of finite inverse semigroups S_1 and S_2 amalgamating a common inverse subsemigroup U, where $S_i = \mathrm{Inv}\langle X_i \mid R_i\rangle$ are given finite presentations of S_i for $i = 1, 2$. Let $X = X_1 \cup X_2$, $R = R_1 \cup R_2$ and W be* *the set of all pairs* $(w_1(u), w_2(u))$ *for* $u \in U$. *Then Schützenberger automata relative to* $\langle X \mid R \cup W \rangle$ *are complete opuntoid automata.*

Proof. Note first that a complete opuntoid automaton which approximates the Schützenberger automaton $\mathcal{A}(X, R \cup W, w)$ for some word $w \in X^+$ is isomorphic to the Schützenberger automaton. In fact its lobes are closed with respect to the presentation $\langle X_i \mid R_i \rangle$, whence it is closed with respect to $\langle X \mid R \rangle$. But it is complete, whence it is also closed with respect to $\langle X \mid W \rangle$.

Now, let us start from a core automaton $\mathrm{Core}(w)$. If it is complete, it is the Schützenberger automaton of w relative to $\langle X \mid R \cup W \rangle$. Otherwise repeated applications of Construction 5 give a sequence of opuntoid automata $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{A}'' \subset \cdots$ which approximate the Schützenberger automaton of w. This sequence forms a direct system A in the category of inverse automata over X, whose direct limit

$$\lim A = \bigcup_{k=1,\dots,\infty} \mathcal{A}^k$$

also approximates the Schützenberger automaton whence, being complete, it is the Schützenberger automaton $\mathcal{A}(X, R \cup W, w)$. $\square$

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