ITERATION THEORY IN HYPERBOLIC DOMAINS

By

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Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of $\mathbb{C}$. In [5] Carl Cowen states that, under very general conditions, for any $f \in \mathrm{Hol}(\Delta, \Delta)$, there exists a linear transformation φ ($\varphi \in \mathrm{Aut}(\Delta)$ or $\varphi \in \mathrm{Aut}(\mathbb{C})$) and a function σ, analytic in Δ, such that $\sigma \circ f = \varphi \circ \sigma$. Moreover, under suitable normalizations, φ and σ are unique. This theorem can be considered as a "classification theorem", since it associates (uniquely) to any function $f \in \mathrm{Hol}(\Delta, \Delta)$ a linear fractional transformation φ. The investigation of the behaviour of the iterates $\{f^{\circ n}\}_{n \in \mathbb{N}}$ is then reduced to the description of the (known) behaviour of $\{\varphi^{\circ n}\}_{n \in \mathbb{N}}$. On the other hand, the solutions g of the functional equation $\sigma \circ g = \psi \circ \sigma$ (with σ, as above, such that $\sigma \circ f = \varphi \circ \sigma$ and ψ a linear transformation such that $\varphi \circ \psi = \psi \circ \varphi$) give rise to a class of analytic functions which can be called "generalized iterates" of f; and this class of functions "generated" by f (or, more precisely, by the functional equation of f) is closely related to the class of functions which commute under composition with the given function f (see, e.g., [6], [18]). This kind of study involves sophisticated techniques from iteration theory and invokes many geometric tools from the hyperbolic geometry of the disc Δ. It seems plausible that the techniques used to study iteration of analytic functions on the disc can be applied more widely; this paper makes a contribution in this direction. The theory developed by Cowen ([5], [6]) and others ([8], [13], [16], [18]) for functions holomorphic in Δ is extended to functions holomorphic on multiply connected hyperbolic domains of finite connectivity. A nice geometric characterization is provided for the semigroup of commuting locally-injective holomorphic maps of a hyperbolic domain D of regular type into itself, with a fixed point in D.

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1 Hyperbolic domains of regular type

A (noncompact) domain D of a compact Riemann surface $\hat{X}$ is *of regular type* if

(i) every connected component of the boundary of $D, \partial D$, is either a Jordan curve (that is, a simple closed continuous curve) or an isolated point; and

(ii) for every connected component Σ of ∂D, there exists a neighborhood V of Σ such that $V \cap \partial D = \Sigma$.

Hyperbolic domains of regular type form a large class of (hyperbolic) Riemann surfaces which have very good properties for our investigations.

Let Σ be a connected component of ∂D; we shall say that Σ is a *point component* if it is an isolated point, a *Jordan component* otherwise. Let us immediately note the following.

Lemma 1.1. *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type. Then ∂D has a finite number of connected components.*

Proof. Assume, by contradiction, that $\{\Sigma_n\}_{n\in\mathbb{N}}$ is an infinite sequence of distinct connected components of ∂D. Take $z_n \in \Sigma_n$ for any $n \in\mathbb{N}$. A subsequence of $\{z_n\}_{n\in\mathbb{N}}$ converges to a point $w_0 \in \partial D$; but then the connected component of ∂D containing w_0 cannot be separated from the other components of ∂D. □

Thus hyperbolic domains of regular type are finitely multiply connected hyperbolic domains. However, it is well-known that any multiply connected domain of finite connectivity can be mapped univalently onto a domain bounded by closed analytic Jordan curves and isolated points (see [11] and [9], p. 262). Furthermore the correspondence of the boundaries of the domains under such univalent mapping is completely understood. Thus it will be sufficient to consider hyperbolic domains of regular type for the study of all finitely multiply connected hyperbolic domains.

Since we are interested in hyperbolic domains and, in particular, in the behaviour of holomorphic maps at the boundary of such domains, we shall make heavy use of the properties of the universal covering Δ of hyperbolic domains. To fix terminology, let us recall some basic facts concerning covering spaces and maps of Riemann surfaces.

Let X and Y be two Riemann surfaces and

$$\pi_X : \tilde{X} \to X \qquad \text{and} \qquad \pi_Y : \tilde{Y} \to Y$$

their universal covering maps. Any $f \in \mathrm{Hol}(X,Y)$ admits a *lifting*, that is, a holomorphic function $\tilde{f} \in \mathrm{Hol}(\tilde{X},\tilde{Y})$ such that $f \circ \pi_X = \pi_Y \circ \tilde{f}$. The function $\tilde{f}$ is uniquely determined by its value at one point. In particular, since Δ is the universal covering of any hyperbolic Riemann surface X, if $f \in \mathrm{Hol}(X, X)$ has a fixed point in X (i.e., if there exists z_0 in X such that $f(z_0) = z_0$), we can always lift f to a map $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ with a fixed point w_0 in Δ, where $\pi_X(w_0) = z_0$. Suppose now that f has no fixed point in D; clearly no lifting $\tilde{f}$ can have a fixed point in Δ. But since $\tilde{f}$ is a holomorphic map of Δ into Δ, by Wolff's Lemma, [1], $\tilde{f}$ has a "fixed point $\tau(\tilde{f})$ on the boundary of Δ" in the sense of non-tangential limits, that is to say, according to the following definition.

Definition 1.2. Let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ and let $\sigma \in \partial\Delta$. We say that $c \in \mathbb{C} \cup \{\infty\}$ is the *non-tangential limit* (or *angular limit*) of f at $\sigma \in \partial\Delta$, or simply, that f tends to c as z tends to σ *non-tangentially*, if $f(z)$ approaches c as z tends to σ within the Stolz region $K(\sigma, M) = \{z \in \Delta \mid |\sigma - z|/(1 - |z|) < M\}$, for all $M > 1$. (See [1] for details on the shape of $K(\sigma, M)$.) We shall also write $K\text{-}\lim_{z \to \sigma} f(z) = c$.

Hence $\tilde{f}$ has a "fixed point $\tau(\tilde{f})$ on the boundary of Δ" if and only if $K\text{-}\lim_{z \to \tau(\tilde{f})} \tilde{f}(z) = \tau(\tilde{f})$.

Since for topological reasons the universal covering map π_D of the hyperbolic domain D gives a correspondence of the boundaries of Δ and of D, we have to study the boundary behaviour of π_D very accurately. Let Σ be a connected component of the boundary of a hyperbolic domain D of regular type and denote by C_Σ the largest open connected arc (possibly void) of points of the boundary of Δ corresponding to Σ; C_Σ is also called the *principal arc associated to* Σ. Having introduced the terminology we need, we recall the main results concerning the boundary behaviour of the universal covering map π_D to be used in the sequel.

Theorem 1.3 *([1]). Suppose that D is a multiply connected hyperbolic domain of regular type, and denote by $\pi : \Delta \to D$ the universal covering map of D. Let Σ be a connected component of the boundary of D. Then,*

(i) *if $\Sigma = \{a\}$ is a point component of ∂D, C_Σ is empty and if $\tau \in \partial\Delta$ is associated to Σ, then $\pi(z)$ tends to a as z tends to τ non-tangentially;*

(ii) *if Σ is a Jordan component of ∂D, C_Σ is not empty and if $C \subset \partial\Delta$ is an open arc associated to Σ, then $\pi(z)$ extends continuously to C and the image of C through this extension is exactly Σ.*

For later considerations, it is worth noticing that, by applying the Osgood–Taylor–Carathéodory Theorem [4], [9] we obtain that, if Σ is a Jordan component of ∂D and if $\tau \in C_\Sigma \subset \overline{\Delta}$, then π extends continuously to a neighborhood of τ (in $\overline{\Delta}$) and, furthermore, π is locally injective at τ.

2 Holomorphic extension of $f \in \mathrm{Hol}(D, D)$ on the point components of the boundary of a hyperbolic domain D of regular type

For our purpose, we recall the following version [1] of

Theorem 2.1 (*Big Picard Theorem*). *Let X be a hyperbolic Riemann surface contained in a compact Riemann surface $\hat{X}$ and let $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Then every $f \in \mathrm{Hol}(\Delta^*, X)$ extends holomorphically to a function $\hat{f} \in \mathrm{Hol}(\Delta, \hat{X})$.*

The first application of the Big Picard Theorem is the following lemma.

Lemma 2.2. *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type and let $f \in \mathrm{Hol}(D, D)$. Suppose first that ∂D, the boundary of D, has at least one Jordan component, so in particular $\overline{D} \neq \hat{X}$. Let P denote the set of point components of ∂D. Then any $f \in \mathrm{Hol}(D, D)$ extends to $\hat{f} \in \mathrm{Hol}(D \cup P, \overline{D})$.*

Proof. Assume first that D is bounded in $\hat{X}$; then Riemann's removable singularity theorem [7] allows us to extend $f \in \mathrm{Hol}(D, D)$ to $\hat{f} \in \mathrm{Hol}(D \cup P, \overline{D})$. We have only to prove that $\hat{f}(D \cup P) \subseteq \overline{D}$. Assume by contradiction that, if $p \in P$, $\hat{f}(p)$ belongs to an unbounded region T delimited by a Jordan component of ∂D; one can always find a neighborhood U of $\hat{f}(p)$ such that $U \subset\subset T$. Since $\hat{f}$ is continuous at p, there exists a neighborhood V of p in $D \cup \{p\}$ such that $\hat{f}(V) \subset U$. But, since $\hat{f}$ extends f and $f(D) \subset D$, we get a contradiction.

Assume now that D is unbounded in $\hat{X}$; then the Big Picard Theorem allows us to extend $f \in \mathrm{Hol}(D, D)$ to $\hat{f} \in \mathrm{Hol}(D \cup P, \hat{X})$. But, again, by continuity of $\hat{f}$ at each point of P, we may conclude that actually $\hat{f} \in \mathrm{Hol}(D \cup P, \overline{D})$. □

Suppose now that ∂D has no Jordan components so that $\partial D = \{x_1, ..., x_k\}$; in particular $\overline{D} = \hat{X}$. There are three cases:

(a) $\hat{X}$ *is hyperbolic*. In this case, since D is hyperbolic, ∂D may be empty. Since, moreover, D is of regular type and $\overline{D} = \hat{X}$, $\hat{X}$ itself is of regular type.

(b) $\hat{X}$ *is a torus*. In this case, since D is hyperbolic, $\partial D = \{x_1, ..., x_k\}$ contains at least one point ($k > 0$).

(c) $\hat{X}$ *is the Riemann sphere* $\hat{\mathbb{C}}$. In this case, since D is hyperbolic, $\partial D = \{x_1, ..., x_k\}$ contains at least three points ($k > 2$).

Let $\hat{f} \in \mathrm{Hol}(\hat{X}, \hat{X})$ be the extension of $f \in \mathrm{Hol}(D, D)$ by means of the Big Picard Theorem. Observe that, in general, if $\hat{X}$ is a torus, $\hat{f}(\hat{X}) = \hat{X}$; indeed, if $\hat{f}(\hat{X}) = \hat{X}$ minus at most one point, $\hat{f}$ would be constant, since $\hat{f}$ would be a holomorphic function of a non-hyperbolic Riemann surface onto a hyperbolic Riemann surface. For the same reasons, $\hat{f}(\hat{\mathbb{C}})$ must be $\hat{\mathbb{C}}$ minus at most two points. Summarizing, we can state the following.

Lemma 2.3. *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type. If $f \in$ $\mathrm{Hol}(D, D)$ does not have a fixed point in D and is non-constant, the extension $\hat{f}$ (by means of the Big Picard Theorem or Lemma 2.2) must have a fixed point in $\overline{D}$.*

This point which, as already observed, corresponds to the Wolff point on the boundary of Δ of the lifting $\tilde{f}$ of f must be an isolated point of ∂D when ∂D has no Jordan components; when ∂D has at least one Jordan component, it can be either an isolated point of ∂D or a point of a Jordan component of ∂D.

We now turn to obtaining an extension of holomorphic maps of a hyperbolic domain of regular type into itself in order to generalize our studies to the case of holomorphic maps of a hyperbolic domain of regular type into itself with a fixed point in the closure of this hyperbolic domain (which still has to be a hyperbolic domain). We proceed as follows: if $f \in \mathrm{Hol}(D, D)$ has a fixed point in D, there is nothing to do. Otherwise, there are two possibilities: either ∂D has at least one Jordan component or ∂D has no Jordan components. In the first case, if the fixed point of the extension $\hat{f} \in \mathrm{Hol}(\hat{X}, \hat{X})$ is an isolated point p of ∂D, we add this point p to D and consider $D' = D \cup \{p\}$ and $\dot{f} \in \mathrm{Hol}(D', D')$, the restriction of $\hat{f}$ to D', where $\hat{f}$ is as in Lemma 2.2. Observe that D' is still a hyperbolic domain of regular type. The case in which the fixed point of the extension $\hat{f} \in \mathrm{Hol}(\hat{X}, \hat{X})$ is not an isolated point of ∂D will be examined later. Suppose now that ∂D has no Jordan components; the extension $\hat{f} \in \mathrm{Hol}(\hat{X}, \hat{X})$ of $f \in \mathrm{Hol}(D, D)$ given in Lemma 2.2 necessarily has a fixed point p which is an isolated point p of ∂D. But if we now consider (as before) $D' = D \cup \{p\}$ and $\dot{f} \in \mathrm{Hol}(D', D')$, the restriction of $\hat{f}$ to D', we are not able to conclude that D' is still hyperbolic. To avoid this inconvenience, we shall assume that D is properly contained in the Riemann sphere $\hat{\mathbb{C}}$ minus three points, or in a torus minus two points. In this way, by adding a point to D, we still obtain a hyperbolic domain (of regular type).

With this procedure, we have generalized our considerations to hyperbolic domains of regular type D and to maps $f \in \mathrm{Hol}(D, D)$ with either a fixed point in D or a fixed point on a Jordan component of ∂D. Let us summarize by means of the following

Proposition 2.4. *Suppose that D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Assume, furthermore, that D is properly contained in the Riemann sphere $\hat{\mathbb{C}}$ minus three points, or in a torus minus two points. Then any $f \in \mathrm{Hol}(D, D)$ can be extended to $\dot{f} \in \mathrm{Hol}(D', D')$,* *where D' is a hyperbolic domain of regular type containing D and where $\dot{f}$ has either a fixed point in D' or a fixed point on a Jordan component of the boundary D'.*

3 The Jordan component case

Before considering the case of holomorphic maps with a fixed point on a Jordan component of the boundary of a hyperbolic domain of regular type D, let us transfer the Poincaré distance ω from Δ to D or, more generally, to any hyperbolic Riemann surface. Let X be a hyperbolic Riemann surface and denote by $\pi_X : \Delta \to X$ the universal covering map of X. Defining for $z, w \in X$

$$\omega_X(z,w) = \inf\{\omega(\tilde{z},\tilde{w}) : \tilde{z} \in {\pi_X}^{-1}(z), \tilde{w} \in {\pi_X}^{-1}(w)\},$$

we get a complete hyperbolic distance on X, which induces the standard topology [1]. The main property of this hyperbolic distance on an arbitrary hyperbolic Riemann surface is the analogue of the Schwarz–Pick Lemma for the Poincaré distance in Δ, namely [1]

Proposition 3.1. *Let X and Y be two hyperbolic Riemann surfaces and $f : X \to Y$ a holomorphic function. Let ω_X and ω_Y be the (induced) hyperbolic distances on X and on Y. Then for all $z, w \in X$ we have*

$$\omega_Y(f(z), f(w)) \leq \omega_X(z,w).$$

Take now a hyperbolic domain of regular type D endowed with the induced hyperbolic metric ω_D. Suppose that ∂D has at least one Jordan component and let $f \in \mathrm{Hol}(D, D)$ have a fixed point on one of these Jordan components which, as already observed, corresponds to the Wolff point $\tau(\tilde{f})$ on the boundary of Δ of the lifting $\tilde{f}$ of f. Since $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ has no fixed points in Δ, there exists [5] in a neighborhood of $\tau(\tilde{f})$ a *fundamental set for* $\tilde{f}$, i.e., an open, connected, simply connected subset $V_{\tilde{f}}$ of Δ such that

(i) $\tilde{f}(V_{\tilde{f}}) \subset V_{\tilde{f}}$ and

(ii) for any compact set K in Δ, there is a positive integer n so that $\tilde{f}^{\circ n}(K) \subset V_{\tilde{f}}$.

Moreover, it can be proved that $\tilde{f}$ is injective on $V_{\tilde{f}}$ [5].

Our aim is now to prove the following

Proposition 3.2. *Suppose that D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Let $f \in \mathrm{Hol}(D, D)$. Assume that ∂D, the boundary of D, has at least one Jordan component and that neither f nor any* *extension $\hat{f}$ of f has a fixed point in D. Then there exists, in a neighborhood of a fixed point of f on a Jordan component of ∂D, a fundamental set for f in D on which f is injective.*

Proof. For the construction of a fundamental set $V_{\tilde{f}}$ in Δ, where $\tilde{f}$ is injective, Cowen [5] uses a deep result of Pommerenke [14], which guarantees that for any compact set K in Δ, there exists an integer N such that $\tilde{f}$ is injective on $\bigcup_{n=N}^{\infty} \tilde{f}^{\circ n}(K)$. This fact is applied to the interior of each compact set of the family of exhaustive compact sets $K_r = \{z \in \Delta : |z| \leq 1 - 1/r\}$, and the construction of the desired fundamental set is carried out by gluing these unions of iterates of K_r; this set is, by construction, open and connected. To obtain a set which is also simply connected, one adds the eventual holes to the previous unions of iterates of K_r; the enlarged set is still fundamental, since $\tilde{f}$ is injective on the boundary of the holes (see [13] for details). The classical Wolff–Denjoy Theorem (see, e.g., [19], [1]) asserts that the iterates $\tilde{f}^{\circ n}(K)$ of a compact set K in Δ converge to the Wolff point $\tau(\tilde{f}) \in \partial\Delta$ of $\tilde{f}$; then, according to the observation which follows Theorem 1.3, one can always find an integer $N' > N$ in such a way that not only $\tilde{f}$ but also the covering map π_D is injective on $\bigcup_{n=N'}^{\infty} \tilde{f}^{\circ n}(K)$, and each step of the construction in [5] can be repeated in the very same way.

So let $V_{\tilde{f}}$ be the fundamental set for $\tilde{f}$ where $\tilde{f}$ and π_D are injective and let $V_f = \pi_D(V_{\tilde{f}})$. First of all, by definition, we have

$$f(V_f) = f(\pi_D(V_{\tilde{f}})) = \pi_D(\tilde{f}(V_{\tilde{f}})) \subset \pi_D(V_{\tilde{f}}) = V_f.$$

Let $z_0 = \pi_D(0)$ and let $K \subset D$ be any compact set in D. Consider $r = \sup_{z \in K} \omega_D(z, z_0)$; clearly $r < \infty$ since K is compact. We can always find a real number r^*, $0 < r^* < 1$, so that for $\overline{B(0,r^*)} = \{z \in \Delta : \omega_D(z,0) \leq r^*\}$ we have $\pi_D(\overline{B(0,r^*)}) = \{w \in D : \omega_D(w, z_0) \leq r\}$. Hence $K \subset \pi_D(\overline{B(0,r^*)})$. Now $\overline{B(0,r^*)}$ is a compact set in Δ; thus, since $V_{\tilde{f}}$ is a fundamental set for $\tilde{f}$ in Δ, there exists an integer n_0 such that for $n > n_0$, $\tilde{f}^{\circ n}(\overline{B(0,r^*)}) \subset V_{\tilde{f}}$. Then

$$f^{\circ n}(K) \subset f^{\circ n}(\pi_D(\overline{B(0,r^*)})) = \pi_D(\tilde{f}^{\circ n}(\overline{B(0,r^*)})) \subset \pi_D(V_{\tilde{f}}) = V_f \quad \text{for all } n > n_0.$$

So V_f is a fundamental set for f in a neighborhood of a fixed point of a Jordan component of ∂D; moreover, f is injective on V_f, since $\tilde{f}$ and π_D are and since $f \circ \pi_D = \pi_D \circ \tilde{f}$. □

Remark. The Theorem stated in [5] is actually proved for any $f \in \mathrm{Hol}(\Delta, \Delta)$ with the assumption that the value of the derivative of f at the Wolff point $\tau(f)$ is not 0; by this we mean that $f'(\tau(f)) \neq 0$ if $\tau(f)$ is a fixed point in Δ or, otherwise, $K\text{-}\lim_{z \to \tau(f)} f'(z) \neq 0$. If $\tau(f) \in \partial\Delta$, this assumption is always fulfilled since, by the Julia–Wolff–Carathéodory Theorem, if σ is a fixed point of f on the boundary of Δ, we have $K\text{-}\lim_{z\to\sigma} f'(z) = c \in \mathbb{R}^+$, so in particular, $K\text{-}\lim_{z\to\tau(f)} f'(z) \neq 0$; actually, by Wolff's Lemma [1], $0 < K\text{-}\lim_{z\to\tau(f)} f'(z) \leq 1$. The proof in the case of a fixed point in Δ, when the derivative of f is not zero, is an (obvious) application of the Local Inversion Theorem, and does not involve any other result. Suppose $f \in \mathrm{Hol}(D, D)$ has a fixed point z_0 in D. Assume that $\pi_D(w_0) = z_0$ and let $\tilde{f}$ be a lifting of f such that $\tilde{f}(w_0) = w_0$. By taking the derivative of $f(\pi_D(w)) = \pi_D(\tilde{f}(w))$ in w_0, one gets

$$\pi_D{}'(\tilde{f}(w_0)) \cdot \tilde{f}'(w_0) = \pi_D{}'(w_0) \cdot \tilde{f}'(w_0) = f'(\pi_D(w_0)) \cdot \pi_D{}'(w_0) = f'(z_0) \cdot \pi_D{}'(w_0)$$

and, since π_D is a local homeomorphism at w_0, $\pi_D{}'(w_0) \neq 0$, so that $\tilde{f}'(w_0) = f'(z_0)$.

Therefore, putting together the results of the previous section with Proposition 3.2, we get

Proposition 3.3. *Suppose that D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Let $f \in \mathrm{Hol}(D, D)$. Assume, in addition, that the value of the derivative of f at the fixed point (if any) in D is not* 0; *then there exists a fundamental set for f in D, on which f is injective.*

This proposition is the first step to extend the theorems proved by Cowen in [5] and in [6]. The proof of the main theorem of [5], viz. the classification theorem already recalled in the Introduction, starts from the existence of a fundamental set V_f for f in Δ, on which f is injective. All (minimal) conditions necessary for carrying out a construction similar to the one which appears in the proof of the main theorem of Cowen, [5], are also fulfilled in case of $f \in \mathrm{Hol}(D, D)$, where D is a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$. Hence, we can restate the main Theorem in [5] as follows:

Theorem 3.4. *Let D be a hyperbolic domain of regular type contained in a compact Riemann surface $\hat{X}$ and let $f \in \mathrm{Hol}(D, D)$ be neither a constant map nor an automorphism of D. Assume, in addition, that the value of the derivative of f at the fixed point in D (if any) is nonzero. Then there exist*

(0) *a fundamental set V_f for f in D, on which f is univalent;*

(1) *a domain Ω, which is either the complex plane $\mathbb{C}$ or the unit disc Δ;*

(2) *a linear fractional transformation φ mapping Ω onto Ω;*

(3) *an analytic map σ mapping D into Ω;*

such that

(i) *σ is univalent on V_f;*

(ii) *$\sigma(V_f)$ is a fundamental set for φ in Ω;*

(iii) $\sigma \circ f = \varphi \circ \sigma$.

Finally, φ is unique up to conjugation under a linear fractional transformation mapping Ω onto Ω, and the maps φ and σ depend only on f and not on the choice of the fundamental set V; that is, if φ_1 and σ_1 satisfy (i), (ii) *and* (iii) *then there exists an automorphism ρ of Ω such that $\varphi_1 = \rho^{-1} \circ \varphi \circ \rho$ and $\sigma_1 = \rho \circ \sigma$.*

The case of $f \in \mathrm{Aut}(D)$ (automorphism of D) is excluded in the classification of Theorem 3.4 since we have the following (strong) characterization result of the automorphisms of D [1].

Theorem 3.5. *Let $D \subset \hat{X}$ be a hyperbolic domain of regular type. If D is not doubly connected, then $\mathrm{Aut}(D)$ is finite. If D is doubly connected, then D is biholomorphic either to $\Delta^* = \Delta \setminus \{0\}$ or to an annulus $A(r,1) = \{z \in \mathbb{C} : r < |z| < 1\}$ for some real number r, $0 < r < 1$. Then every $\gamma \in \mathrm{Aut}(\Delta^*)$ is of the form $\gamma(z) = e^{i\vartheta} z$ ($\vartheta \in \mathbb{R}$) and every $\gamma \in \mathrm{Aut}(A(r,1))$ is either of the form $\gamma(z) = e^{i\vartheta} z$ or of the form $\gamma(z) = e^{i\vartheta} r z^{-1}$.*

By applying the same techniques used in [5], one can prove that, up to conjugation, only four cases may actually occur in the classification given by Theorem 3.4. In particular, we have

Proposition 3.6. *Assume that $f \in \mathrm{Hol}(D,D)$ has a fixed point z_0 in D; let $\pi_D(w_0) = z_0$ and $\tilde{f} \in \mathrm{Hol}(\Delta,\Delta)$ be the lifting of f such that $\tilde{f}(w_0) = w_0$. Then, with the notation of Proposition* 3.4, *$\Omega = \mathbb{C}$ and $\varphi(z) = sz$, with $0 < |s| < 1$ if and only if f has a fixed point z_0 in D and, moreover, $f'(z_0) = s$.*

Proof. As observed in the Remark to Proposition 3.2, $f'(z_0) = \tilde{f}'(w_0)$, so the proof can be carried out as in [5]. □

According to the results of the previous section, the other three possible cases may then occur only if f or any other extension $\hat{f}$ of f on the point component of ∂D has no fixed point in D.

For all the cases, however, it is possible to extend the definition of the pseudo-iteration semigroup, given in [6] for $f \in \mathrm{Hol}(\Delta,\Delta)$, to the case of $f \in \mathrm{Hol}(D,D)$, where D is a hyperbolic domain of regular type, since this definition essentially relies upon Theorem 3.4. Following the definition given in [6] for the case $D = \Delta$, we have

Definition 3.7. Assume that $f \in \mathrm{Hol}(D,D)$ is as in the hypothesis of Theorem 3.4 and let Ω, σ and φ be related to f as in Theorem 3.4. We say that $g \in \mathrm{Hol}(D,D)$ is in the pseudo-iteration semigroup of $f \in \mathrm{Hol}(D,D)$ if and only if there exists $\psi \in \mathrm{Aut}(\Omega)$ such that $\sigma \circ g = \psi \circ \sigma$ and $\psi \circ \varphi = \varphi \circ \psi$. We write $g \in \mathrm{SPI}(f)$.

Not every $g \in \mathrm{SPI}(f)$ commutes (under composition) with f, not even when the hyperbolic domain is the disc Δ (see [6], [18]); at the same time, however, if $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$, and if $\tilde{g} \in \mathrm{Hol}(\Delta, \Delta)$ belongs to $\mathrm{SPI}(\tilde{f})$, then $\tilde{f}$ and $\tilde{g}$ commute if and only if there is an open set U in Δ such that $\tilde{g}(U)$ and $\tilde{g}(\tilde{f}(U))$ are contained in the fundamental set $V_{\tilde{f}}$ of $\tilde{f}$.

Observe that this condition is always fulfilled when $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ has a fixed point w_0 in Δ. This is so since, on the one hand, $\tilde{g} \in \mathrm{Hol}(\Delta, \Delta)$ commutes with $\tilde{f}$ if and only if $\tilde{f}$ and $\tilde{g}$ have w_0 as a fixed point and, on the other, the condition is equivalent to the Schwarz–Pick Lemma for $\tilde{f}$ and for $\tilde{g}$ [1].

Suppose now that $f \in \mathrm{Hol}(D, D)$, where D is a hyperbolic domain of regular type, and that $g \in \mathrm{SPI}(f)$. Let $\tilde{f}$ and $\tilde{g}$ be their liftings. Since the Identity Principle holds for holomorphic functions, from the definition of lifting, one immediately has that $f \circ g = g \circ f$ if and only if $\tilde{f} \circ \tilde{g} = \tilde{g} \circ \tilde{f}$. Hence, keeping in mind the construction of the fundamental set V_f for $f \in \mathrm{Hol}(D, D)$, one can easily deduce the same condition for the case of a hyperbolic domain of regular type, namely

Proposition 3.8. *Let $f \in \mathrm{Hol}(D, D)$ and let $g \in \mathrm{Hol}(D, D)$ be in the pseudo-iteration semigroup of f. Then f and g commute if and only if there is an open set U in D such that $g(U)$ and $g(f(U))$ are contained in the fundamental set V_f of f.*

In particular, let $f \in \mathrm{Hol}(D, D)$ have a fixed point $z_0 \in D$ and $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ be the lifting of f such that $\tilde{f}(w_0) = w_0$ (where $\pi_D(w_0) = z_0$). In this case, as already remarked, the fundamental set V_f for f reduces to a neighborhood of z_0. Therefore, as in the case of the unit disc Δ, any $g \in \mathrm{SPI}(f)$ commutes with f, according to Proposition 3.1 and to the previous considerations. But even more can be said. From the classification of Theorem 3.4, one knows that, in the case examined, one has $\Omega = \mathbb{C}$ and $\varphi(z) = f'(z_0)z$. Now, from the definition, $g \in \mathrm{SPI}(f)$ if and only if there exists $\psi \in \mathrm{Aut}(\Omega)$ such that $\sigma \circ g = \psi \circ \sigma$ and $\psi \circ \varphi = \varphi \circ \psi$ (where σ is such that $\sigma \circ f = f'(z_0) \cdot \sigma$). These functional equations imply that $\psi(z) = \lambda z$ and $\lambda = g'(z_0)$.

Suppose now that f and g are non-constant holomorphic maps of D into D, not automorphisms of D, which commute under composition, i.e., $f \circ g = g \circ f$. In the case of $D = \Delta$ and $\tilde{f}, \tilde{g} \in \mathrm{Hol}(\Delta, \Delta)$ commute, Behan [3] proves that $\tilde{f}$ and $\tilde{g}$ have the same Wolff point. Moreover, it is possible to show (see [18]) that "almost" always $\tilde{g} \in \mathrm{SPI}(\tilde{f})$: the only exception arises when $\tilde{f}$ has its Wolff point $\tau_{\tilde{f}}$ on $\partial\Delta$ and for any $w_0 \in \Delta$ $\tilde{f}^{\circ n}(w_0)$ converges to $\tau_{\tilde{f}}$ tangentially. If this is the case, one can anyway prove that $\tilde{g} \in \mathrm{SPI}(\tilde{f} \circ \tilde{g})$ (see [6]). The proof of the above results essentially relies upon the uniqueness of the map σ, which appears in the definition of the pseudo-iteration semigroup of a map and whose existence and uniqueness are stated in the Main Theorem of [5] for the case $D = \Delta$. Some deeper results are obtained by considering specific geometric properties of the fundamental set (see [18]). In the case of a $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ with a fixed point w_0 in Δ, these conditions are easily fulfilled: one has only to consider, as a fundamental set for $\tilde{f}$, a neighborhood of w_0 in Δ on which $\tilde{f}$ and $\tilde{g}$ are injective. Assume now that D is a generic hyperbolic domain of regular type. Theorem 3.4, which provides the tools for the (generalized) definition of the pseudo-iteration semigroup for $f \in \mathrm{Hol}(D, D)$, asserts that the map σ is unique. Moreover, it has already been remarked that the case of a map $f \in \mathrm{Hol}(D, D)$ with a fixed point z_0 is very similar to the case of a $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ with a fixed point w_0 in Δ since, as in the case $D = \Delta$, one can take a small neighborhood of z_0 as a fundamental set of f. Hence, one can obtain (see [6] and [18])

Proposition 3.9. *Let D be a hyperbolic domain of regular type. Assume that f and g are non-constant holomorphic maps of D into D, not automorphisms of D, which commute under composition. Then, if f has a fixed point $z_0 \in D$, g also has z_0 as a fixed point, and, moreover, $g \in \mathrm{SPI}(f)$.*

Proposition 3.10. *Let D be a hyperbolic domain of regular type and $f \in \mathrm{Hol}(D, D)$ have a fixed point $z_0 \in D$. A function $g \in \mathrm{Hol}(D, D)$ commutes with f if and only if g is a solution of the functional equation*

$$\sigma \circ g = g'(z_0) \cdot \sigma,$$

where σ is the "unique" solution of the functional equation $\sigma \circ f = f'(z_0) \cdot \sigma$.

Remark. A functional equation of the form

$$S \circ f = \lambda \cdot S$$

is called *Schröder's functional equation.* We can summarize Proposition 3.10 by saying that, given a function $f \in \mathrm{Hol}(D, D)$ with a fixed point $z_0 \in D$ (D a hyperbolic domain of regular type), the set of functions holomorphic in D which commute with f coincides with the set of solutions of Schröder's functional equation in g, $\sigma \circ g = \lambda \cdot \sigma$, where σ is given by the functional equation $\sigma \circ f = f'(z_0) \cdot \sigma$.

Call this set of functions F. Take $g \in F$ and let $\tilde{g} \in \mathrm{Hol}(\Delta, \Delta)$ be the lifting of g such that $\tilde{g}(w_0) = w_0$ ($\pi_D(w_0) = z_0$). Since $g'(z_0) = \tilde{g}'(w_0)$, by the Schwarz–Pick Lemma we have $g'(z_0) \in \overline{\Delta}$. Let $\lambda : F \to \Delta$ be defined by $\lambda(g) = g'(z_0)$. Clearly, since z_0 is a fixed point, for any $g \in F$, λ is multiplicative, that is, $\lambda(g \circ h) = \lambda(g) \cdot \lambda(f)$.

In [16] Pranger shows that given $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ locally univalent, such that $\tilde{f}(0) = 0$ and $0 < |\tilde{f}'(0)| < 1$, then $\lambda(F)$ is a closed subset Γ of $\overline{\Delta}$, such that

(1) $0, 1 \in \Gamma$ and $\Gamma \cap \Delta \neq \{0\}$;

(2) if $t, s \in \Gamma$, then $t \cdot s \in \Gamma$;

(3) $\hat{\mathbb{C}} \setminus \Gamma$ is connected.

Conversely, given a closed subset Γ in $\overline{\Delta}$ with properties (1), (2) and (3) there exists a locally univalent $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ such that $\tilde{f}(0) = 0$, $0 < |\tilde{f}'(0)| < 1$ and $\lambda(\tilde{F}) = \Gamma$, where $\tilde{F}$ is the set of functions, holomorphic in Δ, which commute with f. (The choice of 0 as a fixed point is arbitrary, since Δ is homogeneous.)

Consider $f \in \mathrm{Hol}(D, D)$ with a fixed point z_0 and let $\tilde{f} \in \mathrm{Hol}(\Delta, \Delta)$ be the lifting of f such that $\tilde{f}(w_0) = w_0$ $(\pi_D(w_0) = z_0)$. Since $f'(z_0) = \tilde{f}'(w_0)$, by applying the results of Pranger to the lifting $\tilde{f}$ of f, one immediately gets that $\tilde{F}$ is a closed subset of $\overline{\Delta}$, which has properties (1), (2) and (3). On the other hand, consider a closed subset Γ in $\bar{\Delta}$ with properties (1), (2) and (3). Consider $D = \varphi(\hat{\mathbb{C}} \setminus \Gamma)$, where $\varphi(z) = 1/z$. Clearly $0 \in D$; furthermore the domain D is hyperbolic since Γ contains at least three points. Now, given any $t \in \Gamma$, if $w \in D$ then $t \cdot w \in D$. In fact, if $t \cdot w \notin D$, there exists $s \in \Gamma$ such that $t \cdot w = \varphi(s) = 1/s$. Then $w = 1/t \cdot s = \varphi(t \cdot s)$ and $t \cdot s \in \Gamma$, so that $w \notin D$, which is a contradiction.

Define $f_t(w) = t \cdot w \quad \forall t \in \Gamma$. Clearly, $\forall t \in \Gamma$, $f_t \in \mathrm{Hol}(D, D)$, $f_t(0) = 0$, $f_t'(0) = t \in \Gamma$ and $\{f_t\}_{t \in \Gamma}$ is a family of locally univalent commuting holomorphic maps.

So we have extended the results of Pranger, namely, we have proved

Theorem 3.11. *Let D be a hyperbolic domain of regular type, let $f \in \mathrm{Hol}(D, D)$ be a locally univalent map and let $z_0 \in D$ be such that $f(z_0) = z_0$ and that $0 < |f'(z_0)| < 1$. Let F be the set of holomorphic self-maps of D which commute with f and define $\lambda : F \to \bar{\Delta}$ by $\lambda(g) = g'(z_0)$. Then $\Gamma = \lambda(F)$ is a closed subset of $\bar{\Delta}$ such that*

(1) $0, 1 \in \Gamma$, $\Gamma \cap \Delta \neq \{0\}$,

(2) *if* $t, s \in \Gamma$, $\quad t \cdot s \in \Gamma$,

(3) $\hat{\mathbb{C}} \setminus \Gamma$ *is connected.*

Conversely, given a closed subset $\Gamma \subset \bar{\Delta}$ with properties (1), (2) *and* (3), *there exist a hyperbolic domain D which contains 0 and a family of locally univalent holomorphic maps $\{f_t : D \to D\}_{t \in \Gamma}$ such that $f_t(0) = 0$ and $f_t'(0) \in \Gamma \quad \forall t \in \Gamma$.*

Theorem 3.11 gives rise to a great number of examples and possibilities: Γ may be a closed segment, a finite number of closed segments, a spiral, a closed disc and a finite set of points, which all fulfill properties (1), (2) and (3). In particular, taking $\Gamma = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, 1\}$, Pranger shows that there exists a locally univalent map $f \in \mathrm{Hol}(\Delta, \Delta)$ such that the only maps which commute with f are its natural iterates.

An example of such functions is known in the set of entire functions, namely $f(z) = e^z - 1$. No explicit examples for holomorphic functions in hyperbolic domains D have been exhibited, even in the case $D = \Delta$. For instance, Cowen in [6] first shows that if $f \in \mathrm{Hol}(\Delta, \Delta)$ and $\sup\{|f(z)| \ : \ z \in \Delta\} = |||f||| < 1$, then there are infinitely many functions besides the natural iterates of f that commute with f and then gives an example (but not an explicit one) of a map $g \in \mathrm{Hol}(\Delta, \Delta)$ such that the only functions commuting with g are its natural iterates. Of course, one immediately deduces that $|||g||| = 1$; but since $g(z)$ is essentially defined by considering the action of the map $t \cdot w$ up to conjugation of the Riemann map of Δ onto a suitable domain D with countably many cuts, it is rather difficult to study g without the interwining Riemann mapping.

Although the techniques used by Pranger in Δ ([16]) and here, applied for any holomorphic map with a fixed point in a hyperbolic domain D of regular type, cannot be considered a direct method for obtaining explicit examples for each kind of map, one can give a precise definition of the map in some particular cases. Take, for instance, $\Gamma = [0, 1]$ as in Theorem 3.11 and let $\varphi(z) = 1/z$. Let $V = \hat{\mathbb{C}} \backslash \varphi(\Gamma)$ and $S = \{z \in \mathbb{C} \ : \ 0 < \mathrm{Im} z < 2\pi\}$. Then S is a one-sheeted covering of V by means of the (invertible) map $w \to e^w + 1$. Let ρ be the Riemann map of Δ onto S such that $\rho(0) = i\pi$. Take $c \in \Gamma$ and define

$$\tilde{f}_c(z) = \rho^{-1}(\log(c(e^{\rho(z)} + 1) - 1)).$$

One immediately verifies that for any $c \in \Gamma$, $\tilde{f}_c(0) = 0$ and $\tilde{f}_c'(0) = c$; moreover

$$\tilde{f}_c(\tilde{f}_b(z)) = \rho^{-1}(\log(c(e^{\rho(\rho^{-1}(\log(b(e^{\rho(z)}+1)-1))} + 1) - 1)) = \tilde{f}_{cb}(z).$$

Hence $\{\tilde{f}_c\}_{c \in \Gamma}$ is the set of locally univalent maps whose existence is proved in [16].

This example has an explicit formulation since the domain V can be easily covered. Slight modifications to this example give the following generalizations:

(1) $\Gamma = \{z \in \mathbb{C} \ : |z| < r, \ \ 0 < r < 1\} \cup \{t \in \mathbb{R} \ : \ 0 < t \leq 1\}$, $V = \hat{\mathbb{C}} \backslash \varphi(\Gamma)$, $S = \{z \in \mathbb{C} \ : \ 0 < \mathrm{Im} z < 2\pi, \ \ \mathrm{Re} w < \log(1/r - 1)\}$ and $\tilde{f}_c(z) = \rho^{-1}(\log(c(e^{\rho(z)} + 1) - 1))$, where ρ is the Riemann map of Δ onto S such that $\rho(0) = i\pi$;

(2) $\Gamma = \{t \cdot e^{i\frac{k}{s}\pi} \ : \ s \in \mathbb{N}, k = 0, 1, ..., 2s, t \in [0, 1]\}$, $V = \hat{\mathbb{C}} \backslash \varphi(\Gamma)$, $S = \{z \in \mathbb{C} \ : \ 0 < \mathrm{Im} z < 2\pi\} \backslash \{z \in \mathbb{C} \ : \mathrm{Im} w = \frac{k}{s}\pi\}$ and $\tilde{f}_c(z) = \rho^{-1}(\log(c(e^{\rho(z)} + 1) - 1))$, where ρ is the Riemann map of Δ onto S such that $\rho(0) = i\pi$.

Consider now the domain $D = \hat{\mathbb{C}} \backslash \{-1, 0, 1, \infty\}$; this domain is strictly contained in the thrice-punctured Riemann sphere $\hat{\mathbb{C}} \backslash \{-1, 1, \infty\}$ and in this sense it represents the "minimal"—even though the widest—hyperbolic domain of regular type for which the considerations of the previous sections may be applied. (Let us remark, before turning to details, that the thrice-punctured Riemann sphere is in some sense the optimal domain, because any meromorphic self-map on a bounded domain can be extended to the removable singularities by means of the Riemann Theorem. On the other hand, the Picard theorem asserts that a holomorphic map f maps a neighborhood of an essential singularity onto a domain which omits at most two points in $\hat{\mathbb{C}}$; moreover, any meromorphic map defined on $\hat{\mathbb{C}}$ is a rational map.)

The domain $D' = \hat{\mathbb{C}} \backslash \{0, 1, \infty\}$, which is hyperbolic, may be covered (see [2] and [12]) by H^+ by means of the *elliptic modular function* $z \mapsto J(z)$, defined by

$$J(z) = 16e^{\pi iz} \prod_{n=1}^{\infty} \left(\frac{1 + e^{2\pi inz}}{1 + e^{(2n-1)\pi iz}} \right)^8 .$$

This map is generally studied because it is invariant under the action of the modular group G, that is, the group of transformations of H^+

$$\mu(z) = \frac{az + b}{cz + d},$$

with a, d odd integers and b, c even integers such that $ad - bc = 1$.

In particular, it can be shown that the function J effects a one-to-one conformal mapping from the region R bounded by the imaginary axis, the line $\mathrm{Re}(z) = 1$ and the circle $|z - 1/2| = 1/2$ onto the upper half plane H^+. By reflection, the region R' symmetric to R with respect to the imaginary axis is mapped one-to-one onto the lower half plane H^-; moreover, each point of $D' = \hat{\mathbb{C}} \backslash \{0, 1, \infty\}$ is covered by H^+ infinitely many times, and J has no branch points. Finally, J cannot be holomorphically extended across the real axis; indeed, by invariance, $J \circ \mu = J$ for any μ in G, and, since the set $\{b/d \ : \ b \text{ even}, d \text{ odd}\}$ of zeros of transformations in G is dense in $\mathbb{R}$, if J were defined on the real axis it would be constant. Any transformation μ in G defines the identity map on D' by means of the relation $J \circ \mu = J$, but if one considers a map $\varphi : H^+ \to H^+$, such that for any $z_0 \in D$ there exists $w_0 \in D$ such that $f(J^{-1}(z_0)) \subseteq J^{-1}(w_0)$, it is possible to define a holomorphic map $f : D \to D$ such that $f \circ J = J \circ \varphi$. For instance, the map $\varphi(z) = z + 1$ defines $f(z) = z/(z - 1)$.

We can then define the functions $Q(z)$ which is related to $J(z)$ by the relation $Q(z) = J\left(\frac{1}{\pi i} \log z\right)$. The function Q is defined in $\Delta^* = \{z \in \mathbb{C} \ : \ 0 < |z| < 1\}$, but it is not difficult (see [12]) to see that actually $\lim_{z \to 0} Q(z) = 0$, so that Q can be considered defined and holomorphic in Δ. In accordance with that, one can find the product expansion

$$Q(z) = 16z \prod_{n=1}^{\infty} \left(\frac{1 + z^{2n}}{1 + z^{(2n-1)}} \right)^8 .$$

The same properties of J can be proven for Q; in particular, Q has no branch points and Δ covers $\hat{\mathbb{C}}\setminus\{0,1,\infty\}$—by means of Q—infinitely many times.

If we now define $Q_1(z) = \sqrt{Q(z^2)}$, we get a function which is holomorphic in Δ, with no branch points but such that it covers $D = \hat{\mathbb{C}}\setminus\{-1,0,1,\infty\}$, since ${Q_1}^2(z) = Q(z^2)$. Moreover, the function Q_1 is odd, as can be easily observed from the product expansion

$$Q_1(z) = 4z \prod_{n=1}^{\infty} \left(\frac{1+z^{4n}}{1+z^{(4n-2)}} \right)^4 .$$

Now, if φ is a map such that $f \circ Q_1 = Q_1 \circ \varphi$, according to the result of Section 2, f can be extended to a holomorphic map $\dot{f}$, defined in D and on a point component of ∂D, which coincides with f in D. In this particular case, it is very easy to see this when $\varphi(0) \subset Q_1^{-1}(0)$; indeed, if W_0 is any neighborhood of 0 in D, since Q_1 is continuous and $\lim_{z\to 0} Q_1(z) = 0$, there exists a neighborhood V_0 of 0 in Δ such that $Q_1(V_0) \subset W_0$. Then, since $f \circ Q_1 = Q_1 \circ \varphi$, necessarily the map f is continuous at 0 and then bounded. The Riemann theorem then allows us to extend f holomorphically to 0. The point 0 is moreover fixed for the map $\dot{f}$, and all the theorems for this case can be applied. The example given above, namely the map $\varphi(z) = z + 1$, perfectly fulfills all such properties; indeed, a map f, such that $f \circ Q_1 = Q_1 \circ \varphi$, is the map $f(z) = -z$, which has a fixed point at 0.

The map Q_1 gives rise to another covering; specifically, the map $z \in \Delta \mapsto P(z) = 2/\pi \arcsin Q_1(z)$ covers $\hat{\mathbb{C}}\setminus\mathbb{Z}\cup\{\infty\}$, which is hyperbolic but not of regular type, since the connected components of the boundary accumulate to ∞.

Finally, an explicit example of a family of self-maps, parameterized by $\mathbb{R}^+$, holomorphic in H^+, such that none of them commutes with any other of this family, has already been provided in [8]; this result makes use of the geometric properties of the fundamental set and gives a direct application of its construction.

References

[1] M. Abate, *Iteration Theory of Holomorphic Maps on Taut Manifolds*, Mediterranean Press, 1989.

[2] L. V. Ahlfors, *Complex Analysis*, Third Edition, McGraw-Hill, New York, 1979.

[3] D. F. Behan, *Commuting analytic functions without fixed points*, Proc. Amer. Math. Soc. **37** (1973), 114–120.

[4] R. B. Burckel, *An Introduction to Classical Complex Analysis*, Academic Press, New York, 1979.

[5] C. C. Cowen, *Iteration and the solution of functional equations for functions analytic in the unit disk*, Trans. Amer. Math. Soc. **265** (1981), 69–95.

[6] C. C. Cowen, *Commuting analytic functions*, Trans. Amer. Math. Soc. **283** (1984), 685–695.

[7] O. Forster, *Lectures on Riemann Surfaces*, Springer-Verlag, New York, 1981.

[8] G. Gentili and F. Vlacci, *Pseudo-iteration semigroups and commuting holomorphic maps*, Rend. Mat. Acc. Lincei (9) **5** (1994), 33–42.

[9] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, American Mathematical Society, Providence, 1969.

[10] J. Hadamard, *Two works on iteration and related questions*, Bull. Amer. Math. Soc. **50** (1944), 67–75.

[11] G. Julia, *Leçons sur la représentations conforme des aires multiplement connexes*, Gauthier-Villars, Paris, 1934.

[12] Z. Nehari, *Conformal Mapping*, Dover, New York, 1975.

[13] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[14] Ch. Pommerenke, *On the iteration of analytic functions in a half-plane, I*, J. London Math. Soc. **19** (1979), 439–447.

[15] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin–Heilderberg, 1992.

[16] W. A. Pranger, *Iteration of functions analytic on a disk*, Aequationes Math. **4** (1970), 201–204.

[17] E. Vesentini, *Capitoli scelti della teoria delle funzioni olomorfe*, Unione Matematica Italiana, 1984.

[18] F. Vlacci, *On commuting holomorphic maps in the unit disc of* $\mathbb{C}$, Complex Variables **30** (1996), 301–313.

[19] J. Wolff, *Sur une généralisation d'un théorème de Schwarz*, C. R. Acad. Sci. Paris **182** (1926), 918–920.

[20] J. Wolff, *Sur une généralisation d'un théorème de Schwarz*, C. R. Acad. Sci. Paris **183** (1926), 500–502.

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