Annals of Mathematics, **156** (2002), 931–946

Matzoh ball soup: Heat conductors with a stationary isothermic surface

By Rolando Magnanini and Shigeru Sakaguchi*

Abstract

We consider a bounded heat conductor that satisfies the exterior sphere condition. Suppose that, initially, the conductor has temperature 0 and, at all times, its boundary is kept at temperature 1. We show that if the conductor contains a proper sub-domain, satisfying the interior cone condition and having constant boundary temperature at each given time, then the conductor must be a ball.

1. Introduction

A *matzoh ball* is a dumpling, made of special unleavened crackers, that one takes from the refrigerator and drops into boiling stock (see [R-R] for a recipe). The physical situation at hand can be modeled in the general Euclidean space $\mathbb{R}^N$ as an initial-boundary value problem for the heat equation: in a bounded domain Ω — the *matzoh ball* — the normalized temperature $u = u(x,t)$ at a point $x \in \Omega$ and time $t > 0$ satisfies the heat equation:

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, +\infty), \tag{1.1}$$

and the two conditions:

$$u = 1 \quad \text{on} \quad \partial\Omega \times (0, +\infty), \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \Omega \times \{0\}. \tag{1.3}$$

A conjecture, posed in [Kl] by M. S. Klamkin and referred to by L. Zalcman in [Z] as the *Matzoh Ball Soup*, was settled affirmatively by G. Alessandrini in [A1]–[A2]. In [A2], under the assumption that every point of $\partial\Omega$ is regular

*This research was partially supported by a Grant-in-Aid for Scientific Research (C) (♯10640175) and (B) (♯12440042) of Japan Society for the Promotion of Science, by a 1999-2000 Grant of the Italian MURST and by the Visiting Professors program of INDAM-GNAFA-CNR.

Key words: Heat equation, overdetermined problems, stationary surfaces.

AMS subject classifications: Primary 35K05, 35K20, 35J05; Secondary 35J25, 35B38, 35B40.

with respect to the Laplacian, it was proved that if all the spatial isothermic surfaces of u are *invariant with time* then Ω must be a *ball*. (Of course, the values of u vary with time on its spatial isothermic surfaces.)

The case where the homogeneous initial data in (1.3) is replaced by a function in the space $L^2(\Omega)$ was also considered in [A1]–[A2] and, with the help of J. Serrin's celebrated symmetry theorem for elliptic equations [Ser], was settled in the following terms: if all the spatial isothermic surfaces of the solution u to (1.1) with homogeneous Dirichlet boundary condition and initial data $\varphi \in L^2(\Omega)$ are invariant with time, then either φ is an eigenfunction of the Laplacian or Ω is a ball.

The analogous question where condition (1.2) is replaced by the homogeneous Neumann boundary condition was examined and answered positively (see [Sa1, Theorem 1]) with the aid of the classification theorem for *isoparametric hypersurfaces in Euclidean space* due to T. Levi-Civita and B. Segre (see [LC], [Seg]). The method used in [Sa1] can be applied to give an alternative proof of Alessandrini's results.

An important observation is that, in order to prove Klamkin's conjecture [Kl], both methods employed in [A1]–[A2] and [Sa1] need to assume that *infinitely many* isothermic surfaces of u are invariant with time. As a natural consequence of this remark, one may wonder if the requirement that a finite number (possibly only one) of level surfaces of u are invariant with time implies that Ω is a ball.

Our main result in this direction is the following.

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is a domain, with boundary ∂D, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$.*

Assume that the solution u to problem (1.1)–(1.3) *satisfies the following condition*:

$$u(x,t) = a(t), \quad (x,t) \in \partial D \times (0,+\infty), \tag{1.4}$$

for some function $a : (0,+\infty) \to (0,+\infty)$.

Then Ω must be a ball.

We recall that Ω satisfies the *exterior sphere condition* if for every $y \in \partial\Omega$ there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, where $B_r(z)$ denotes an open ball centered at $z \in \mathbb{R}^N$ and with radius $r > 0$. Also, D satisfies the *interior cone condition* if for every $x \in \partial D$ there exists a finite right spherical cone K_x with vertex x such that $K_x \subset \overline{D}$ and $\overline{K_x} \cap \partial D = \{x\}$.

When Ω is convex, we observe that there is no need to require that D satisfies the interior cone condition. Indeed, a classical result shows that the function $x \mapsto \log(1 - u(x,t))$ is concave for each given time $t > 0$ (see [B-L], [Ko]). This fact and the analyticity of u in x, imply that, for each $t > 0$, there exists a point $x(t) \in \Omega$ — the *cold spot* — such that

$$\{x \in \Omega : \nabla u(x,t) = 0\} = \{x \in \Omega : u(x,t) = \min_{y \in \Omega} u(y,t)\} = \{x(t)\}.$$

Thus, we can conclude that, with the exception of the cold spot and the boundary $\partial\Omega$, the isothermic surfaces in a convex conductor are always smooth closed convex hypersurfaces. The following result is then an easy consequence of Theorem 1.1.

Corollary 1.2. *Let Ω be a bounded convex domain in $\mathbb{R}^N$, $N \geq 2$, and suppose that D is a domain such that $\overline{D} \subset \Omega$. Assume that the solution u to problem* (1.1)–(1.3) *satisfies condition* (1.4).

Then Ω must be a ball.

The proof of Theorem 1.1 exploits arguments different from the ones used in [A1]–[A2] and [Sa1]. Our technique is essentially based on two ingredients.

One ingredient is a careful study of the asymptotic behavior of $u(x,t)$ as $t \to 0^+$ or, more conveniently, the asymptotic behavior as $s \to +\infty$ of the function $W = W(x,s)$ defined by

$$W(x,s) = s \int_0^{+\infty} u(x,t)\ e^{-s\ t} dt,\ \ s > 0. \tag{1.5}$$

Notice that W is the solution of the following elliptic boundary value problem:

$$\Delta W - s\ W = 0 \qquad \text{in } \Omega, \tag{1.6}$$

$$W = 1 \qquad \text{on } \partial\Omega. \tag{1.7}$$

A result in [Va] (see also [F-W] and [E-I]) shows that, as $s \to +\infty$, the function $-\frac{1}{\sqrt{s}} \log W(x,s)$ converges uniformly on $\overline{\Omega}$ to the function $d = d(x)$ defined by

$$d(x) = \text{dist }(x, \partial\Omega), \quad x \in \Omega. \tag{1.8}$$

Moreover, if u satisfies (1.4), then for any fixed $s > 0$, W is constant on ∂D: indeed,

$$W(x,s) = s \int_0^{+\infty} a(t)\ e^{-s\ t} dt := A(s),\ \ x \in \partial D. \tag{1.9}$$

In Section 3, by using these observations, we will show two facts:

(i) $\Omega = D + B_R(0)$, where $B_R(0)$ is the ball centered at the origin and with radius

$$R = \lim_{s \to +\infty} \{-\frac{1}{\sqrt{s}} \log A(s)\}; \tag{1.10}$$

in other words $\partial\Omega$ and ∂D are *parallel* surfaces;

(ii) ∂D is analytic and, since ∂D is a level surface of d, also $\partial\Omega$ must be real analytic.

The second ingredient of our proof is the *balance law* proved in Theorem 2.1. Let G be a domain in $\mathbb{R}^N$; a solution $v = v(x,t)$ to the heat equation in $G \times (0, +\infty)$ is such that $v(x_0, t) = 0$, for some $x_0 \in G$ and for every $t > 0$, if and only if

$$\int_{\partial B_r(x_0)} v(x,t)\, dS_x = 0, \text{ for every } r \in [0, d_*) \text{ and every } t > 0, \tag{1.11}$$

where $d_* = \text{dist}\,(x_0, \partial G)$. If v is bounded, we introduce a function $V = V(x,s)$ defined as in (1.5) by replacing u with v and we derive from (1.11) that

$$\int_{\partial B_r(x_0)} V(x,s)\, dS_x = 0, \text{ for every } r \in [0, d_*) \text{ and every } s > 0. \tag{1.12}$$

By (1.12) and the study of the asymptotic behavior of the integral in (1.12) as $s \to +\infty$ (see Theorem 2.3) we will show in Theorem 3.2 that if the solution u to (1.1)–(1.3) satisfies (1.4) then

$$\prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(x)\right] = \text{constant}, \quad \text{for every } x \in \partial\Omega. \tag{1.13}$$

Here, $\kappa_j(x), j = 1, \ldots, N-1$, denotes the j^{th} principal curvature of the surface $\partial\Omega$ at the point $x \in \partial\Omega$ (we refer to §2 for a definition of κ_j).

If $N = 2$, condition (1.13) directly implies that Ω is a ball. When $N \geq 3$, we derive the same conclusion with the help of A. D. Aleksandrov's uniqueness theorem [Alek].

Theorem 2.1 was stated without proof in [M-S2]. To make the present paper self-contained, we present a short proof of Theorem 2.1 together with a new proof of a result (Corollary 2.2) proved in [M-S1]. A more general version of Theorem 2.3 will appear in a forthcoming paper ([M-S3]).

2. A balance law and an asymptotic estimate

In this section, we shall construct the two main tools for proving our symmetry results. One of them is the following *balance law.*

THEOREM 2.1. *Let G be a domain in $\mathbb{R}^N$, $N \geq 2$, let x_0 be a point in G and set $d_* = \text{dist}\,(x_0, \partial G)$. Suppose that $v = v(x,t)$ is a solution of the heat equation in $G \times (0, +\infty)$.*

Then the following assertions are equivalent:

(i) $v(x_0, t) = 0$ *for every* $t \in (0, +\infty)$;

(ii) *for every* $(r,t) \in [0,d_*) \times (0,+\infty)$

$$\int_{\partial B_r(x_0)} v(x,t)\ dS_x = 0. \tag{2.1}$$

Proof. By a translation, we can suppose that $x_0 = 0$, the origin of $\mathbb{R}^N$. If (2.1) holds, then $v(0,t) = 0$ for every $t \in (0,+\infty)$ clearly. If $v(0,t) = 0$ for every $t \in (0,+\infty)$, we will show that the function $p = p(r,t)$ defined in $[0,d_*) \times (0,+\infty)$ by

$$p(r,t) = \int_{\partial B_1(0)} v(rx,t)\ dS_x, \tag{2.2}$$

which is analytic with respect to r in $[0,d_*)$, is a solution of the initial value problem:

$$p_t = p_{rr} + \frac{N-1}{r}\ p_r \ \text{ in } (0,d_*) \times (0,+\infty), \tag{2.3}$$

$$p(0,t) = p_r(0,t) = 0, \quad t \in (0,+\infty). \tag{2.4}$$

Hence, (2.1) follows from the fact that

$$\frac{\partial^k p}{\partial r^k}(0,t) = 0,\ t \in (0,+\infty),\ k = 0,1,2,\dots,$$

by induction on the integer k (see [Sa2] for a similar argument).

It is evident that $p(0,t) = 0$ for every $t \in (0,+\infty)$. As in [M-S1], by using the heat equation in radial coordinates, we write

$$0 = \int_{\partial B_1(0)} \left(\partial_t - \partial_r^2 - \frac{N-1}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}}\right) v(rx,t)\ dS_x,$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{N-1} \equiv \partial B_1(0)$. Then (2.3) follows from the fact that $\int_{\partial B_1(0)} \Delta_{\mathbb{S}^{N-1}} v(rx,t)\ dS_x = 0$.

Finally, from (2.3), we have

$$p_r(0,t) = \frac{1}{N-1}\ \lim_{r\to 0}\ r\ (p_t - p_{rr}) = 0$$

for every $t \in (0,+\infty)$. □

The following corollary provides another proof of a result first demonstrated in [M-S1].

Corollary 2.2. *Assume* G, x_0, d_* *and* v *as in Theorem* 2.1. *Then the following assertions are equivalent*:

(i) $\nabla v(x_0,t) = 0$ *for every* $t \in (0,+\infty)$;

(ii) *for every* $(r,t) \in [0, d_*) \times (0, +\infty)$

$$\int\limits_{\partial B_r(x_0)} (x - x_0)\ v(x,t)\ dS_x = 0. \tag{2.5}$$

Proof. Since each component of $\nabla v(x,t)$ satisfies the heat equation, by Theorem 2.1 we have that (i) is equivalent to

$$\int\limits_{\partial B_r(x_0)} \nabla v(x,t)\ dS_x = 0 \text{ for every } (r,t) \in [0, d_*) \times (0, +\infty).$$

Integrating the latter formula with respect to r yields

$$\int\limits_{B_r(x_0)} \nabla v(x,t)\ dx = 0 \text{ for every } (r,t) \in [0, d_*) \times (0, +\infty),$$

which, by the divergence theorem, is equivalent to (2.5). □

Theorem 2.3 below provides our second tool for the proofs of our symmetry results. In order to state it, we need to introduce some notation and definitions.

Take a point $x \in \partial\Omega$ and a unit vector $\omega \in T_x(\partial\Omega)$ — the *tangent space* to $\partial\Omega$ at x — and let $\sigma \mapsto \gamma(\sigma)$ be a smooth curve on $\partial\Omega$, parametrized according to its arclength $\sigma \in [0, L]$, such that $\gamma(0) = x$ and $\gamma'(0) = \omega$.

Define a function $\mathcal{S}_x : \{\omega \in T_x(\partial\Omega) : |\omega| = 1\} \to \mathbb{R}$ by

$$\mathcal{S}_x(\omega) = \gamma''(0) \cdot \nu(x),$$

where $\nu(x)$ is the interior unit normal vector to $\partial\Omega$ at x and the dot denotes scalar product. Notice that $\mathcal{S}_x(\omega)$ is the curvature of the curve γ at x, by the Frenet-Serret formulae.

Let $d(x)$ be defined by (1.8); since $\nu(x) = \nabla d(x)$ and $\gamma'(\sigma) \cdot \nu(\gamma(\sigma)) = 0$ for every $\sigma \in [0, L]$, by differentiating with respect to σ this latter equation, we obtain:

$$\gamma''(\sigma) \cdot \nu(\gamma(\sigma)) = -\gamma'(\sigma) \cdot [\nabla^2 d(\gamma(\sigma))\ \gamma'(\sigma)],$$

where $\nabla^2 d$ denotes the Hessian matrix of d, and hence

$$\mathcal{S}_x(\omega) = -\omega \cdot [\nabla^2 d(x)\ \omega], \quad \omega \in T_x(\partial\Omega) \text{ with } |\omega| = 1. \tag{2.6}$$

We can extend $\mathcal{S}_x$ to a bilinear form — the *shape operator at* x — on $\mathbb{R}^N = T_x(\mathbb{R}^N)$ by observing that $\omega \cdot [\nabla^2 d(x)\ \omega] = 0$ for every ω proportional to $\nu(x) = \nabla d(x)$; in fact, $\nabla^2 d(x)\ \nabla d(x) = 0$, since $|\nabla d|^2 = 1$ on Ω (see [G-H-L]). The critical values of $\mathcal{S}_x(\omega)$ on the unit sphere $\mathbb{S}^{N-1}$ — the eigenvalues of $-\nabla^2 d(x)$ – are 0 and the *principal curvatures* $\kappa_1(x), \dots, \kappa_{N-1}(x)$ of $\partial\Omega$ at x (see [G-T, Lemma 14.17]).

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a domain with C^2 boundary $\partial\Omega$ and let $\kappa_1, \dots, \kappa_{N-1}$ denote the principal curvatures of $\partial\Omega$.*

Let $B_R(x_0) \subset \Omega$ be an open ball with radius $R > 0$ centered at x_0 and suppose that the set $\partial\Omega \cap \partial B_R(x_0)$ is made of a finite number of points $p_1, \dots, p_K$ such that $\kappa_j(p_k) < \frac{1}{R}$ for every $j = 1, \dots, N-1$ and every $k = 1, \dots, K$.

Let $W = W(x,s)$ be the solution to problem (1.6)–(1.7). *Then, the following formula holds for every function φ continuous on $\mathbb{R}^N$:*

$$\lim_{s\to+\infty} s^{\frac{N-1}{4}} \int\limits_{\partial B_R(x_0)} \varphi(x)\ W(x,s)\ dS_x = (2\pi)^{\frac{N-1}{2}} \sum_{k=1}^{K} \varphi(p_k) \left\{\prod_{j=1}^{N-1} [\frac{1}{R} - \kappa_j(p_k)]\right\}^{-\frac{1}{2}}. \tag{2.7}$$

The proof of Theorem 2.3 is based on Lemma 2.4 below, where we show that the two functions

$$W_\varepsilon^{\pm}(x,s) = \exp\{-\sqrt{s(1\mp\varepsilon)}\ d(x)\}, \tag{2.8}$$

where $d(x)$ is defined by (1.8), provide respectively an upper and a lower barrier for W in Ω for large values of s.

LEMMA 2.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$. Let $W(x,s)$ be the solution to* (1.6)–(1.7).

Then, for every $\varepsilon > 0$, there exists a positive number s_ε such that

$$W_\varepsilon^-(x,s) \le W(x,s) \le W_\varepsilon^+(x,s) \tag{2.9}$$

for every $x \in \overline{\Omega}$ and every $s \ge s_\varepsilon$, where $W_\varepsilon^-(x,s)$ and $W_\varepsilon^+(x,s)$ are defined in (2.8).

Proof. Choose a number $\delta > 0$ such that the function $d = d(x)$ defined in (1.8) is of class C^2 in the set $\overline{\Omega_\delta}$ where

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}. \tag{2.10}$$

Let $W_\varepsilon^{\pm}(x,s)$ be given by (2.8). A straightforward computation gives

$$\Delta W_\varepsilon^{\pm} - s\ W_\varepsilon^{\pm} = \mp\varepsilon\sqrt{s}\ \{\sqrt{s} \pm \frac{\sqrt{(1\mp\varepsilon)}}{\varepsilon}\ \Delta d\}\ W_\varepsilon^{\pm} \quad \text{in } \ \Omega_\delta.$$

Set $M_\delta = \max\limits_{\overline{\Omega}_\delta} |\Delta d|$; if $s \ge \frac{1+\varepsilon}{\varepsilon^2}\ M_\delta^2$, then

$$\begin{array}{l} \Delta W_\varepsilon^{+} - s\ W_\varepsilon^{+} \le 0 \\ \Delta W_\varepsilon^{-} - s\ W_\varepsilon^{-} \ge 0 \end{array} \quad \text{in } \ \Omega_\delta. \tag{2.11}$$

Since the function $-\frac{1}{\sqrt{s}}\log W(x,s)$ converges uniformly on $\overline{\Omega}$ to $d(x)$ as $s \to +\infty$, (see [Va], [E-I]), there exists a number $s^* > 0$ such that

$$-\delta(1-\sqrt{1-\varepsilon}) \le -\frac{1}{\sqrt{s}}\log W(x,s) - d(x) \le \delta(\sqrt{1+\varepsilon}-1),\ x \in \overline{\Omega},$$

for every $s \ge s^*$. Hence, since $d(x) \ge \delta$ for every $x \in \Omega \setminus \Omega_\delta$, we obtain

$$W_\varepsilon^-(x,s) \le W(x,s) \le W_\varepsilon^+(x,s), \quad x \in \Omega \setminus \Omega_\delta, \tag{2.12}$$

for every $s \ge s^*$. Moreover,

$$W_\varepsilon^-(x,s) = W(x,s) = W_\varepsilon^+(x,s) = 1, \quad x \in \partial\Omega, \tag{2.13}$$

for every $s > 0$, clearly.

Choose $s_\varepsilon = \max(s^*, \frac{1+\varepsilon}{\varepsilon^2}\ M_\delta^2)$. Then by the comparison principle, from (2.11), (2.12) and (2.13), we have

$$W_\varepsilon^-(x,s) \le W(x,s) \le W_\varepsilon^+(x,s), \quad x \in \Omega_\delta, \tag{2.14}$$

for every $s \ge s_\varepsilon$. Combining (2.14) with (2.12) yields (2.9). □

Proof of Theorem 2.3. We will show preliminarily that

$$\lim_{s\to+\infty} s^{\frac{N-1}{4}} \int\limits_{\partial B_R(x_0)} \varphi(x)\ e^{-\sqrt{s}\ d(x)}\ dS_x = (2\pi)^{\frac{N-1}{2}} \sum_{k=1}^{K} \varphi(p_k) \left\{\prod_{j=1}^{N-1}[\frac{1}{R} - \kappa_j(p_k)]\right\}^{-\frac{1}{2}}. \tag{2.15}$$

Let $p_h \in \{p_1, \dots, p_K\}$; by using a partition of unity, we can suppose that supp φ does not contain the point $2x_0 - p_h$ and any p_k different from p_h.

Let $\mathbb{R}^{N-1} \ni \sigma = (\sigma_1, \dots, \sigma_{N-1}) \mapsto x(\sigma) \in \partial B_R(x_0)$ be a parametrization of $\partial B_R(x_0)$ such that $x(0) = p_h$; a convenient choice of $x(\sigma)$ is the stereographic projection from the point $2x_0 - p_h$ onto the tangent space to $\partial B_R(x_0)$ at p_h. Precisely, take an orthonormal basis $\xi^1, \dots, \xi^N$ of $\mathbb{R}^N$ with $\xi^N = (x_0 - p_h)/R$, and put:

$$x(\sigma) = \frac{2R|\sigma|^2}{4R^2 + |\sigma|^2}\ \xi^N + \frac{4R^2}{4R^2+|\sigma|^2}\sum_{j=1}^{N-1}\sigma_j\xi^j + p_h.$$

By this change of coordinates, the integral in (2.15) becomes

$$\int\limits_{\partial B_R(x_0)} \varphi(x)\ e^{-\sqrt{s}\ d(x)}\ dS_x = \int\limits_{\mathbb{R}^{N-1}} \varphi(x(\sigma))\ e^{-\sqrt{s}\ d(x(\sigma))} J(\sigma)\ d\sigma,$$

where

$$J(\sigma) \equiv \sqrt{\det\left(\frac{\partial x(\sigma)}{\partial\sigma_i}\cdot\frac{\partial x(\sigma)}{\partial\sigma_j}\right)} = \left(\frac{4R^2}{4R^2+|\sigma|^2}\right)^{N-1},$$

since

$$\frac{\partial x(\sigma)}{\partial \sigma_i} \cdot \frac{\partial x(\sigma)}{\partial \sigma_j} = \left(\frac{4R^2}{4R^2 + |\sigma|^2} \right)^2 \delta_{ij}, \ i, j = 1, \dots, N-1. \tag{2.16}$$

Here δ_{ij} is Kronecker's symbol.

Let $d^*(\sigma) = d(x(\sigma))$. Then $d^*(0) = 0$. We will later observe that $\nabla d^*(0) = 0$ and $\nabla^2 d^*(0)$ is positive definite. Moreover, since supp φ does not contain any p_k different from p_h, we may assume that $d^*(\sigma) > 0$ if $\sigma \neq 0$. Hence, by Laplace's method (see [deB, p. 71] for example), or by the stationary phase method (see [Ev, pp. 208–217] for example), we infer that

$$\begin{aligned} \lim_{s \to +\infty} s^{\frac{N-1}{4}} \int_{\mathbb{R}^{N-1}} \varphi(x(\sigma)) e^{-\sqrt{s}\ d(x(\sigma))} J(\sigma)\ d\sigma \\ = (2\pi)^{\frac{N-1}{2}}\ \varphi(p_h)\ J(0)\ (\det \nabla^2 d^*(0))^{-\frac{1}{2}}. \end{aligned} \tag{2.17}$$

Formula (2.15) will result from (2.17) by observing that $J(0) = 1$ and that

$$\det \nabla^2 d^*(0) = \prod_{j=1}^{N-1} [\frac{1}{R} - \kappa_j(p_h)], \tag{2.18}$$

as it will be clear from the following argument.

Differentiating $d^*(\sigma)$ twice yields:

$$\begin{aligned} \frac{\partial^2 d^*}{\partial \sigma_i \partial \sigma_j}(\sigma) \quad = \quad & \frac{\partial x}{\partial \sigma_i}(\sigma) \cdot \left(\nabla^2 d(x(\sigma))\ \frac{\partial x}{\partial \sigma_j}(\sigma) \right) \\ & + \nabla d(x(\sigma)) \cdot \frac{\partial^2 x}{\partial \sigma_i \partial \sigma_j}(\sigma), \quad i, j = 1, \dots, N-1, \end{aligned} \tag{2.19}$$

for every $\sigma \in \mathbb{R}^{N-1}$, where the dot stands for scalar product of vectors in $\mathbb{R}^N$.

Since $x(\sigma) \in \partial B_R(x_0)$ for every $\sigma \in \mathbb{R}^{N-1}$, we obtain:

$$\frac{\partial x}{\partial \sigma_i}(\sigma) \cdot (x(\sigma) - x_0) = 0, \quad i = 1, \dots, N-1,$$

$$\frac{\partial^2 x}{\partial \sigma_i \partial \sigma_j}(\sigma) \cdot (x(\sigma) - x_0) + \frac{\partial x}{\partial \sigma_i}(\sigma) \cdot \frac{\partial x}{\partial \sigma_j}(\sigma) = 0, \quad i, j = 1, \dots, N-1,$$

for every $\sigma \in \mathbb{R}^{N-1}$. The fact that $-\nabla d(p_h) = (x(0) - x_0)/R$ then yields that

$$\nabla d(p_h) \cdot \frac{\partial x}{\partial \sigma_i}(0) = 0, \quad i = 1, \dots, N-1,$$

$$\nabla d(p_h) \cdot \frac{\partial^2 x}{\partial \sigma_i \partial \sigma_j}(0) = \frac{1}{R}\ \frac{\partial x}{\partial \sigma_i}(0) \cdot \frac{\partial x}{\partial \sigma_j}(0), \quad i, j = 1, \dots, N-1,$$

and hence

$$\nabla d^*(0) = 0,$$

$$\frac{\partial^2 d^*}{\partial\sigma_i\partial\sigma_j}(0) = \frac{\partial x}{\partial\sigma_i}(0)\cdot\left\{\left[\nabla^2 d(p_h)+\frac{1}{R}\ I\right]\ \frac{\partial x}{\partial\sigma_j}(0)\right\},\ i,j=1,\dots,N-1,$$

by (2.19), where I is the $N\times N$ identity matrix.

By (2.16), the vectors $\frac{\partial x}{\partial\sigma_i}(0), i=1,\dots,N-1$, make an orthonormal basis of the tangent space $T_{p_h}(\partial\Omega)=T_{p_h}(\partial B_R(x_0))$; therefore, we conclude that the eigenvalues of $\nabla^2 d^*(0)$ are $\frac{1}{R}-\kappa_j(p_h)$ $(j=1,\dots,N-1)$ and hence (2.18) holds.

We now prove formula (2.7). It suffices to prove it for any nonnegative φ, since any φ can be written as $\varphi=\varphi^+-\varphi^-$ where $\varphi^+=\max\{\varphi,0\}$ and $\varphi^-=\max\{-\varphi,0\}$.

By Lemma 2.4, we have for every $s\ge s_\varepsilon$ and for every nonnegative φ:

$$\int\limits_{\partial B_R(x_0)}\varphi(x)\ W_\varepsilon^-(x,s)\ dS_x\le\int\limits_{\partial B_R(x_0)}\varphi(x)\ W(x,s)\ dS_x\le\int\limits_{\partial B_R(x_0)}\varphi(x)\ W_\varepsilon^+(x,s)\ dS_x;$$

then, (2.15) and the definition (2.8) of $W_\varepsilon^\pm(x,s)$ give

$$\limsup_{s\to+\infty}\ s^{\frac{N-1}{4}}\int\limits_{\partial B_R(x_0)}\varphi(x)\ W(x,s)\ dS_x$$

$$\le\left(\frac{2\pi}{\sqrt{1-\varepsilon}}\right)^{\frac{N-1}{2}}\ \sum_{k=1}^K\varphi(p_k)\ \left\{\prod_{j=1}^{N-1}\left[\frac{1}{R}-\kappa_j(p_k)\right]\right\}^{-\frac{1}{2}}$$

and

$$\liminf_{s\to+\infty}\ s^{\frac{N-1}{4}}\int\limits_{\partial B_R(x_0)}\varphi(x)\ W(x,s)\ dS_x$$

$$\ge\left(\frac{2\pi}{\sqrt{1+\varepsilon}}\right)^{\frac{N-1}{2}}\ \sum_{k=1}^K\varphi(p_k)\ \left\{\prod_{j=1}^{N-1}\left[\frac{1}{R}-\kappa_j(p_k)\right]\right\}^{-\frac{1}{2}},$$

for every $\varepsilon>0$. By letting ε tend to 0, we obtain (2.7) and the proof is concluded. □

3. Symmetry results

In Lemma 3.1 below, we prove analyticity of ∂D and $\partial\Omega$ by using our balance law.

LEMMA 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is a domain satisfying the interior cone condition and such that $\overline{D} \subset \Omega$.*

Assume that the solution $u = u(x,t)$ to problem (1.1)–(1.3) *satisfies condition* (1.4). *Let R be the positive constant given by* (1.10).

Then the following assertions hold:

(i) *for every $x \in \partial D$, $d(x) = R$, where d is defined by* (1.8);

(ii) *∂D is analytic*;

(iii) *$\partial\Omega$ is analytic and $\partial\Omega = \{x \in \mathbb{R}^N : \mathrm{dist}(x, D) = R\}$*;

(iv) *the mapping: $\partial D \ni x \mapsto y(x) \equiv x - R\,\nu^*(x) \in \partial\Omega$ is a diffeomorphism, where $\nu^*(x)$ denotes the interior unit normal vector to ∂D at $x \in \partial D$*;

(v) *for every $x \in \partial D$, $\nabla d(y(x)) = \nu^*(x)$ and $\overline{B_R(x)} \cap \partial\Omega = \{y(x)\}$*;

(vi) *let $\kappa_j(y)$, $j = 1, \dots, N-1$ denote the j^{th} principal curvature at $y \in \partial\Omega$ of the analytic surface $\partial\Omega$; then $\kappa_j(y) < \frac{1}{R}$, $j = 1, \dots, N-1$, for every $y \in \partial\Omega$.*

Proof. (i) As already observed, under our assumptions, for each fixed $s > 0$, the function $W = W(x,s)$, defined by (1.5), is the solution to problem (1.6)–(1.7) and satisfies (1.9). Since Ω enjoys the exterior sphere condition, we can apply a result in [Va] (see also [E-I] and [F-W]): as $s \to +\infty$, the function $-\frac{1}{\sqrt{s}} \log W(x,s)$ converges uniformly on $\overline{\Omega}$ to the function $d(x)$ defined by (1.8), and hence we get (i).

(ii) It suffices to show that, for every point $x \in \partial D$, there exists a time $t^* > 0$ such that $\nabla u(x, t^*) \neq 0$; then, analyticity of ∂D will follow from analyticity of u with respect to the space variable.

Assume by contradiction that there exists a point $x_0 \in \partial D$ such that $\nabla u(x_0, t) = 0$ for every $t > 0$. Since u is continuous up to $\partial\Omega \times (0, +\infty)$, by Corollary 2.2 (ii), we can infer that

$$\int_{\partial B_R(x_0)} (x - x_0)\; u(x,t)\; dS_x = 0 \quad \text{for every } t > 0;$$

hence

$$\int_{\partial B_R(x_0)} (x - x_0)\; W(x,s)\; dS_x = 0 \quad \text{for every } s > 0, \tag{3.1}$$

in view of (1.5).

On the other hand, since D satisfies the interior cone condition, there exists a finite right spherical cone K with vertex at x_0 such that $K \subset \overline{D}$ and

$\overline{K} \cap \partial D = \{x_0\}$. By translating and rotating if needed, we can suppose that $x_0 = 0$ and that K is the set $\{x \in B_\rho(0) : x_N < -|x|\cos\theta\}$, where $\rho \in (0, R)$ and $\theta \in (0, \frac{\pi}{2})$.

Since $K \subset \overline{D}$ and $\overline{K} \cap \partial D = \{0\}$, assertion (i) implies that

$$d(x) > R \quad \text{for every } x \in K. \tag{3.2}$$

The set defined by

$$V = \{x \in \partial B_R(0) : x_N \geq R\sin\theta\} \tag{3.3}$$

is such that

$$\partial\Omega \cap \partial B_R(0) \subset V, \tag{3.4}$$

because, otherwise, there would be a point in K contradicting (3.2).

Thus, from (3.4) it follows that we can choose a number $\delta > 0$ such that

$$d(x) \geq 5\delta \quad \text{for every } x \in \partial B_R(0) \cap \{x_N \leq 0\}. \tag{3.5}$$

Since we know that $-\frac{1}{\sqrt{s}} \log W(x,s)$ converges uniformly on $\overline{\Omega}$ to $d(x)$ as $s \to +\infty$, we can choose $s^* > 0$ such that

$$\left| -\frac{1}{\sqrt{s}} \log W(x,s) - d(x) \right| < \delta,$$

for every $x \in \overline{\Omega}$ and every $s \geq s^*$. This latter inequality, together with (3.3), (3.4), and (3.5), gives, for every $s \geq s^*$, the following two estimates:

$$\int\limits_{\partial B_R(0)\cap\{x_N\leq 0\}} x_N\, W(x,s)\, dS_x \quad \geq \quad -\frac{1}{2} R\, e^{-4\delta\sqrt{s}}\, \mathcal{H}^{N-1}\left(\partial B_R(0)\right), \tag{3.6}$$

$$\int\limits_{V\cap\overline{\Omega}_{2\delta}} x_N\, W(x,s)\, dS_x \quad \geq \quad R\sin\theta\, e^{-3\delta\sqrt{s}}\, \mathcal{H}^{N-1}\left(V \cap \overline{\Omega}_{2\delta}\right).$$

Here $\mathcal{H}^{N-1}(\cdot)$ denotes the $(N-1)$-dimensional Hausdorff measure and $\Omega_{2\delta}$ is defined by (2.10).

A consequence of (3.6) is that, for every $s \geq s^*$,

$$\begin{aligned} &\int_{\partial B_R(0)} x_N\, W(x,s)\, dS_x \\ &\quad \geq \int_{V\cap\overline{\Omega}_{2\delta}} x_N\, W(x,s)\, dS_x + \int_{\partial B_R(0)\cap\{x_N\leq 0\}} x_N\, W(x,s)\, dS_x \\ &\quad \geq R\, e^{-3\delta\sqrt{s}} \left[\sin\theta\, \mathcal{H}^{N-1}\left(V\cap\overline{\Omega}_{2\delta}\right) - \frac{1}{2} e^{-\delta\sqrt{s}}\, \mathcal{H}^{N-1}\left(\partial B_R(0)\right)\right]. \end{aligned}$$

Therefore, we obtain a contradiction by observing that the first term of this chain of inequalities equals zero, by (3.1), while the last term can be made positive by choosing $s > 0$ sufficiently large.

(iii), (iv), and (v). Let

$$\Gamma = \{x \in \mathbb{R}^N : \text{dist}\,(x, D) = R\};$$

it is clear that $\Gamma \subset \partial\Omega$. Take any point $x \in \partial D$. Then, there exists a unique point $y \in \partial\Omega$ such that $\overline{B_R(x)} \cap \partial\Omega = \{y\}$. Indeed, since ∂D is analytic by (ii), if $\tilde{y} \in \overline{B_R(x)} \cap \partial\Omega$ and $\tilde{y} \neq y$, then

$$\frac{y-x}{|y-x|} = -\nu^*(x) = \frac{\tilde{y}-x}{|\tilde{y}-x|},$$

where $\nu^*(x)$ is the interior unit normal vector to ∂D at x, which is a contradiction. Since Ω enjoys the exterior sphere property, there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, and hence $\overline{B_r(z)} \cap \overline{B_R(x)} = \{y\}$. Therefore,

$$\text{dist}\,(z, D) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \overline{D} = \{x\}. \tag{3.7}$$

Let $\kappa_j^*,\ j = 1, \dots, N-1$, denote the principal curvatures of the surface ∂D; (3.7) implies that

$$\kappa_j^*(x) \geq -\frac{1}{r+R},\ j = 1, \dots, N-1.$$

Since $\kappa_j^* > -\frac{1}{R}$ on ∂D, for every $j = 1, \dots, N-1$, Γ is an analytic hypersurface diffeomorphic to ∂D (see [G-T, Lemma 14.16]), and hence Γ equals $\partial\Omega$. Assertions (iii), (iv), and (v) then follow at once.

(vi) Take any point $y \in \partial\Omega$. Assertions (iii) and (iv) imply that there exists a unique $x \in \partial D$ such that $\overline{B_R(y)} \cap \overline{D} = \{x\}$. Since ∂D is analytic, D satisfies the interior sphere condition, that is there exists a ball $B_r(z) \subset D$ such that $\overline{B_r(z)} \cap \partial D = \{x\}$. Therefore,

$$d(z) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \partial\Omega = \{y\}, \tag{3.8}$$

and consequently

$$\kappa_j(y) \leq \frac{1}{r+R},\ j = 1, \dots, N-1.$$

Assertion (vi) is proved. □

THEOREM 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is a domain satisfying the interior cone condition with boundary ∂D and such that $\overline{D} \subset \Omega$.*

Assume that the solution $u = u(x,t)$ to problem (1.1)–(1.3) *satisfies condition* (1.4).

Then, $\partial\Omega$ is analytic and (1.13) *holds with R given by* (1.10). *In particular, if $N = 2$, Ω must be a ball.*

Proof. First of all, by Lemma 3.1, both $\partial\Omega$ and ∂D are analytic. Let p and q be two distinct points in $\partial\Omega$ and let

$$P = p + R\,\nabla d(p),\ Q = q + R\,\nabla d(q). \tag{3.9}$$

Assertions (iv) and (v) from Lemma 3.1 guarantee that $P, Q \in \partial D$ and $P \neq Q$. (In fact, $p = y(P)$ and $q = y(Q)$ in (iv).)

For $x \in B_R(0)$, consider the function

$$v(x,t) = u(x+P,t) - u(x+Q,t); \tag{3.10}$$

$v = v(x,t)$ satisfies the heat equation in $B_R(0) \times (0,+\infty)$ and by (1.4)

$$v(0,t) = u(P,t) - u(Q,t) = 0,$$

for every $t > 0$. Since v is continuous up to $\partial B_R(0) \times (0,+\infty)$, by Theorem 2.1 we obtain

$$\int\limits_{\partial B_R(0)} v(x,t)\ dS_x = 0$$

for every $t > 0$, and hence

$$\int\limits_{\partial B_R(P)} u(x,t)\ dS_x = \int\limits_{\partial B_R(Q)} u(x,t)\ dS_x$$

for every $t > 0$. Therefore, in view of (1.5), we have

$$\int\limits_{\partial B_R(P)} W(x,s)\ dS_x = \int\limits_{\partial B_R(Q)} W(x,s)\ dS_x \tag{3.11}$$

for every $s > 0$. Assertions (v) and (vi) from Lemma 3.1 make sure that we can apply Theorem 2.3 (with $\varphi = 1$) to (3.11). We multiply both sides of (3.11) by $s^{\frac{N-1}{4}}$ and take the limits as $s \to +\infty$. Since $\partial B_R(P) \cap \partial\Omega = \{p\}$ and $\partial B_R(Q) \cap \partial\Omega = \{q\}$, after some manipulation, we obtain:

$$\prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(p)\right] = \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(q)\right],$$

that is (1.13) holds. □

We quote A. D. Aleksandrov's uniqueness theorem from [Alek, p. 412], adjusted to our notations. A special case of this theorem is the well-known *Soap-Bubble Theorem* (see also [R]).

Theorem 3.3 (Aleksandrov). *Let $\Phi = \Phi(\kappa_1, \cdots, \kappa_{N-1})$ be a continuously differentiable function, defined for $\kappa_1 \geq \cdots \geq \kappa_{N-1}$, and subject to the condition $\frac{\partial \Phi}{\partial \kappa_i} > 0$ $(i = 1, \cdots, N-1)$.*

Suppose that in $\mathbb{R}^N$ we have a twice-differentiable closed surface S without self-intersections and with bounded principal curvatures.

If on the surface S the function Φ of its principal curvatures $\kappa_1, \cdots, \kappa_{N-1}$ has at all points one and the same value, then S is a sphere.

Proof of Theorem 1.1. By Theorem 3.2, it suffices to consider the case where $N \geq 3$.

We set

$$\Phi = \Phi(\kappa_1, \cdots, \kappa_{N-1}) = -\prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j\right] \tag{3.12}$$

and observe that

$$\frac{\partial \Phi}{\partial \kappa_i} > 0 \quad (i = 1, \cdots, N-1), \text{ if } \max_{1 \le j \le N-1} \kappa_j < \frac{1}{R}.$$

Since condition (1.13) holds by Theorem 3.2, we infer that the function Φ is constant on $\partial\Omega$.

Therefore, by applying Theorem 3.3 to each connected component of $\partial\Omega$, we conclude that $\partial\Omega$ must be a sphere. □

Remark. The method of proof of Theorem 3.3 is called *Aleksandrov's reflection principle* or *the method of moving planes*, which is based on the maximum principle for elliptic partial differential equations of second order.

In fact, by using local coordinates, the condition $\Phi(\kappa_1, \ldots, \kappa_{N-1}) =$ constant on the surface S can be converted into a second order partial differential equation which is of elliptic type, since $\frac{\partial \Phi}{\partial \kappa_i} > 0$ $(i = 1, \cdots, N-1)$. In the case the function Φ is given by (3.12), we obtain an equation of Monge-Ampère type.

Acknowledgement. The authors would like to thank Professor Toshio Mikami for the reference [F-W] on the asymptotic behavior as $t \to 0$ of the solution of problem (1.1)–(1.3) and Professor M. Bardi for the references [E-I] and [Va].

Università di Firenze, Firenze, Italy
E-mail address: magnanin@math.unifi.it

Ehime University, Ehime, Japan
E-mail address: sakaguch@dpc.ehime-u.ac.jp

References

[Alek] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large V, *Vestnik Leningrad Univ*. **13** (1958), 5–8 (English translation: *Trans. Amer. Math. Soc.* **21** (1962), 412–415).

[A1] G. Alessandrini, Matzoh ball soup: a symmetry result for the heat equation, *J. Analyse Math*. **54** (1990), 229–236.

[A2] ———, Characterizing spheres by functional relations on solutions of elliptic and parabolic equations, *Applicable Anal*. **40** (1991), 251–261.

[B-L] H. L. Brascamp and E. H. Lieb, On extensions of Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation, *J. Funct. Anal*. **22** (1976), 366–389.

[deB] N. G. de Bruijn, *Asymptotic Methods in Analysis*, *Bibliotheca Mathematica* **4**, North-Holland Publ. Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publ. Inc., New York, 1958.

[Ev] L. C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1998.

[E-I] L. C. Evans and H. Ishii, A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities, *Ann. Inst. Henri Poincaré* **2** (1985), 1–20.

[F-W] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York, 1984.

[G-H-L] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian Geometry* (Second Edition), Springer-Verlag, New York, 1993.

[G-T] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Second Edition), Springer-Verlag, New York, 1983.

[Kl] M. S. Klamkin, A physical characterization of a sphere (Problem 64-5*), *SIAM Review* **6** (1964), 61.

[Ko] N. J. Korevaar, Convex solutions to nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J*. **32** (1983), 603–614.

[LC] T. Levi-Civita, Famiglie di superficie isoparametriche nell'ordinario spazio euclideo, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur*. **26** (1937), 355–362.

[M-S1] R. Magnanini and S. Sakaguchi, The spatial critical points not moving along the heat flow, *J. Analyse Math*. **71** (1997), 237–261.

[M-S2] ———, The spatial critical points not moving along the heat flow II: The centrosymmetric case, *Math. Z*. **230** (1999), 695–712, Corrigendum, *ibid*. **232** (1999), 389.

[M-S3] ———, On heat conductors with a stationary hot spot, *Ann. Mat. Pura Appl.*, to appear.

[R] R. C. Reilly, Mean curvature, the Laplacian, and soap bubbles, *Amer. Math. Monthly* **89** (1982), 180–188, 197–198.

[R-R] I. S. Rombauer and M. Rombauer Becker, *Joy of Cooking*, Plume, New American Library, New York, 1973.

[Sa1] S. Sakaguchi, When are the spatial level surfaces of solutions of diffusion equations invariant with respect to the time variable?, *J. Analyse Math*. **78** (1999), 219–243.

[Sa2] ———, Stationary critical points of the heat flow in spaces of constant curvature, *J. London Math. Soc*. **63** (2001), 400–412.

[Seg] B. Segre, Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur*. **27** (1938), 203–207.

[Ser] J. Serrin, A symmetry problem in potential theory, *Arch. Rational Mech. Anal*. **43** (1971), 304–318.

[Va] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, *Comm. Pure Appl. Math*. **20** (1967), 431–455.

[Z] L. Zalcman, Some inverse problems of potential theory, *Contemp. Math*. **63** (1987), 337–350.

(Received April 11, 2001)