SIAM J. CONTROL AND OPTIMIZATION
Vol. 30, No. 5, pp. 1092-1100, September 1992

005

A DIRICHLET BOUNDARY CONTROL PROBLEM FOR THE STRONGLY DAMPED WAVE EQUATION*

FRANCESCA BUCCI†

Abstract. A boundary control problem is considered for the strongly damped wave equation, and it is solved by dynamic programming arguments.

Key words. boundary control, Riccati equation, dynamic programming

AMS(MOS) subject classifications. 49, 49C20, 49B22

1. Introduction.

1.1. Statement of the problem and literature. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary $\partial\Omega$, and let $T > 0$ be fixed.

We are concerned with a boundary control problem for the strongly damped wave equation

$$
\begin{aligned}
&y_{tt}(t,x) = \Delta y(t,x) + c\Delta y_t(t,x) \qquad (t,x) \in\,]0,T[\, \times \Omega;\\
&y(0,x) = y_0(x), \quad y_t(0,x) = z_0(x) \qquad x \in \Omega;\\
&y(t,x) = u(t,x) \qquad (t,x) \in\,]0,T[\, \times \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where c is a positive constant; $y_0, z_0 \in L^2(\Omega)$; and we take u in $W^{1,2}(0, T; L^2(\partial\Omega))$.

Physical motivation for studying (1.1) arises from problems that may occur in the study of flexible structures in a bounded domain, controlled on the boundary through a Dirichlet boundary condition.

In recent years, boundary control problems have become of interest in optimal control theory. Flandoli [2], [3] and Lasiecka and Triggiani [4] study a general abstract class of dynamic that covers parabolic-like problems, namely, not only heat/diffusion equations, but also wave or plate equations with structural damping. In their works, they assume, as usual, that controls u belong to $L^2(0, T; L^2(\partial\Omega))$.

In [5] Lasiecka and Triggiani give several examples of partial differential equations, with boundary or point control, which can be reduced to that abstract model. Nevertheless, to the knowledge of the author, (1.1) has not been explicitly treated in relation to optimal contol problems.

In this paper, following the original idea of Balakrishnan for parabolic equations (see [1]), we derive a solution formula for (1.1) in the product space $H = L^2(\Omega) \times L^2(\Omega)$. This formula yields the couple (y, y_t) in terms of the time derivative of the control u_t.

Since we want to solve the control problem using dynamic programming techniques, we must work in the product space H, and we would expect that (y, y_t) belongs to $L^2(0, T; H)$. Therefore, due to the low regularity of the solutions to (1.1) under the assumption $u \in L^2(0, T; L^2(\partial\Omega))$, we take $u \in W^{1,2}(0, T; L^2(\partial\Omega))$.

* Received by the editors August 15, 1990; accepted for publication (in revised form) June 26, 1991.
† Dipartimento di Matematica, Via Buonarroti 2, 56127 Pisa, Italy.

Consistent with this choice, we consider the problem of minimizing the cost functional

$$
\begin{aligned}
J(u)= & \int_0^T dt \int_\Omega \{|(C_1 y(t,\cdot))(x)|^2+|(C_2 y_t(t,\cdot))(x)|^2\}\,dx \\
& +\int_0^T dt \int_{\partial\Omega} \{|u(t,x)|^2+|u_t(t,x)|^2\}\,d\sigma \\
& +\int_\Omega \{|(\Gamma_1 y(T,\cdot))(x)|^2+|(\Gamma_2 y_t(T,\cdot))(x)|^2\}\,dx
\end{aligned} \tag{1.2}
$$

overall u in $W^{1,2}(0, T; L^2(\partial\Omega))$, where $C_i,\ \Gamma_i \in \mathscr{L}(L^2(\Omega))$, $i=1,2, \Gamma_i$ are selfadjoint, and y is subject to the partial differential equation (1.1).

The purpose of § 2 of this paper is to show that it is possible to reformulate problem (1.1), (1.2) into a standard quadratic control problem. This goal is achieved by introducing suitable states and controls, namely, setting $W=(y\cdot y_t, u)$, $v=u'$.

Section 3 is devoted to showing that the theory developed in [3] can be applied to the new control problem, provided that Γ_2 belong to $\mathscr{L}(L^2(\Omega), H_0^{2\beta}(\Omega))$ for some $\beta\in(\frac14,\frac12)$.

1.2. Notation. Let X and Y be two Hilbert spaces. We denote norms and inner products with $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively.

We represent with $\mathscr{L}(X, Y)(\mathscr{L}(X)$ if $X=Y)$, $\Sigma(X)$, $\Sigma^+(X)$ the space of all bounded linear operators from X to Y, the space of all bounded selfadjoint operators in X, and the subset of $\Sigma(X)$ of nonnegative definite operators, respectively.

If T is a linear operator (generally unbounded) from X to Y, we denote its domain with $D(T)$ and its adjoint by T^*.

Moreover, we denote by $\rho(T)$ the resolvent set of T, by $\sigma(T)$ the spectrum of T, and by $R(\lambda, T)=(\lambda-T)^{-1}$ the resolvent operator, respectively. We set $\omega_T=\sup\{\operatorname{Re}\lambda\,|\,\lambda\in\sigma(T)\}$.

If T generates a C_0-semigroup $G(t)$ on X, we set $G(t)=e^{tT}$.

2. The abstract setting. Let $\Omega\subset\mathbb{R}^n$ be an open bounded set with smooth boundary $\partial\Omega$, and let $T>0$ be fixed. We study in $(0, T)\times\Omega$ the optimal control problem (1.1), (1.2).

We consider the Dirichlet realization of the Laplace operator in $L^2(\Omega)$, defined by $Ay=\Delta y$ for any $y\in D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, and we denote by D the Dirichlet mapping from $L^2(\partial\Omega)$ to $L^2(\Omega)$, defined by $Dv=w$, where

$$
\begin{aligned}
\Delta w &= 0 \quad \text{in } \Omega,\\
w(x) &= v(x) \qquad x\in\partial\Omega.
\end{aligned}
$$

As is proved in [6], $D\in\mathscr{L}(L^2(\partial\Omega), H^{1/2}(\Omega))$.

Moreover, we introduce the Hilbert spaces $H=L^2(\Omega)\times L^2(\Omega)$, $U=L^2(\partial\Omega)$ and define the linear operator $\mathscr{A}$ in the product space H as

$$
\begin{aligned}
&\mathscr{A}\begin{pmatrix} y\\ z\end{pmatrix} \doteq \begin{pmatrix} 0 & I\\ A & cA\end{pmatrix}\begin{pmatrix} y\\ z\end{pmatrix}=\begin{pmatrix} z\\ A(y+cz)\end{pmatrix},\\
&D(\mathscr{A})=\left\{\begin{pmatrix} y\\ z\end{pmatrix}\in H \;\middle|\; y+cz\in D(A)\right\}.
\end{aligned} \tag{2.1}
$$

It is well known that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $e^{t\mathscr{A}}$ on H of negative type.

For simplicity, we first assume that $u(0,\cdot)=0$ and consider the problem of minimizing (1.2) over the class of controls

$$W_0^{1,2}(0,T;L^2(\partial\Omega))=\{u\in W^{1,2}(0,T;L^2(\partial\Omega))\,|\,u(0,\cdot)=0\}.$$

Following a standard technique introduced by Balakrishnan (see [1]), we can reduce problem (1.1) to a homogeneous boundary problem. Then it is easy to check that the solution $Y=(y,y_t)$ to (1.1) satisfies

$$Y(t)=e^{t\mathcal{A}}\begin{pmatrix}y_0\\ z_0\end{pmatrix}-\mathcal{A}\int_0^t e^{(t-s)\mathcal{A}}Fu'(s)\,ds+Eu(t), \tag{2.2}$$

where $Y(t)=Y(t,\cdot)$, $u(t)=u(t,\cdot)$, and E, F are the linear operators in $\mathcal{L}(U,H)$, defined by

$$Eu=\begin{pmatrix}Du\\ 0\end{pmatrix},\qquad Fu=\begin{pmatrix}0\\ Du\end{pmatrix},$$

respectively.

Remark 2.1. If we apply

$$\mathcal{A}^{-1}=\begin{pmatrix}-cI & A^{-1}\\ I & 0\end{pmatrix}$$

to (2.2) and integrate by parts in t, we obtain that

$$\begin{pmatrix}-cy(t)+A^{-1}y'(t)\\ y(t)\end{pmatrix}=\mathcal{A}^{-1}\,e^{t\mathcal{A}}\begin{pmatrix}y_0\\ z_0\end{pmatrix}-\mathcal{A}\int_0^t e^{(t-s)\mathcal{A}}Fu(s)\,ds-cEu(t),$$

which easily yields regularity of solutions to (1.1) in terms of the regularity of the control u. We stress that, if $u\in L^2(0,T;L^2(\partial\Omega))$, then we only have $A^{-1}y'(t)\in L^2(0,T;H)$.

Therefore, because we want to use dynamic programming arguments, we cannot weaken the assumption $u\in W^{1,2}(0,T;L^2(\partial\Omega))$.

The cost functional can be written as

$$J(u)=\int_0^T\{|CY(s)|^2+|u(s)|^2+|u'(s)|^2\}\,ds+\langle P_0Y(T),Y(T)\rangle, \tag{2.3}$$

where

$$C=\begin{pmatrix}C_1 & 0\\ 0 & C_2\end{pmatrix},\qquad P_0=\begin{pmatrix}\Gamma_1^2 & 0\\ 0 & \Gamma_2^2\end{pmatrix},$$

and it is clear that $C\in\mathcal{L}(H)$, $P_0\in\Sigma^+(H)$.

Now note the control u as an auxiliary component of the state and define u' as a new control. More precisely, set

$$u'=v,\qquad W=\begin{pmatrix}Y\\ u\end{pmatrix} \tag{2.4}$$

and introduce a new states space $\bar{H}=H\times U$, while we set $\bar{U}=U$.

From (2.2), (2.4) it is rather easy to derive a semigroup formula to be satisfied by W in $\bar{H}$. To do that, we need the following lemma.

LEMMA 2.2. *Let $G:[0,+\infty)\to\mathcal{L}(\bar{H})$ be defined by*

$$t\to\begin{pmatrix}e^{t\mathcal{A}} & (I-e^{t\mathcal{A}})E\\ 0 & I\end{pmatrix}, \tag{2.5}$$

with $\mathcal{A}$ given by (2.1).

Then G is an analytic semigroup on $\bar{H}$ of type $=0$, *and its generator* $\mathcal{B}$ *is defined by*

$$D(\mathcal{B})=\left\{\begin{pmatrix} Y \\ u \end{pmatrix}\in \bar{H} \,\middle|\, Y-Eu\in D(\mathcal{A})\right\},$$

$$(2.6)\qquad \mathcal{B}\begin{pmatrix} Y \\ u \end{pmatrix}=\begin{pmatrix} \mathcal{A}(Y-Eu) \\ 0 \end{pmatrix}.$$

Moreover, $\rho(\mathcal{B})=\{\lambda\in\mathbb{C}\,|\,\lambda\in\rho(\mathcal{A}),\ \lambda\neq 0\}$, *and, for all* $\lambda\in\rho(\mathcal{B})$, *we have that*

$$(2.7)\qquad R(\lambda,\mathcal{B})=\begin{pmatrix} R(\lambda,\mathcal{A}) & -(1/\lambda)\mathcal{A}R(\lambda,\mathcal{A})E \\ 0 & (1/\lambda)I \end{pmatrix}.$$

Proof. We can easily check that G is a strongly continuous semigroup on $\bar{H}$. We now characterize the generator $\mathcal{B}$ of $G(t)$.

Let $\binom{Y}{u}\in\bar{H}$, $t>0$. We write

$$\frac{1}{t}(G(t)-I)\begin{pmatrix} Y \\ u \end{pmatrix}=\frac{1}{t}\begin{pmatrix} e^{t\mathcal{A}}-I & (I-e^{t\mathcal{A}})E \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y \\ u \end{pmatrix}$$
$$=\frac{1}{t}\begin{pmatrix} (e^{t\mathcal{A}}-I)(Y-Eu) \\ 0 \end{pmatrix}.$$

Therefore the limit $\lim_{t\to 0}(1/t)(G(t)-I)\binom{Y}{u}$ exists if and only if there exists $\lim_{t\to 0}(1/t)(e^{t\mathcal{A}}-I)(Y-Eu)$, that is, by definition, if $Y-Eu\in D(\mathcal{A})$.

In conclusion,

$$D(\mathcal{B})=\left\{\begin{pmatrix} Y \\ u \end{pmatrix}\in \bar{H} \,\middle|\, Y-Eu\in D(\mathcal{A})\right\}$$

and, for all $\binom{Y}{u}\in D(\mathcal{B})$,

$$\mathcal{B}\begin{pmatrix} Y \\ u \end{pmatrix}=\begin{pmatrix} \mathcal{A}(Y-Eu) \\ 0 \end{pmatrix},$$

and (2.6) holds true.

Also, formula (2.7) can be easily verified.

To show that $G(t)=e^{t\mathcal{B}}$ is an analytic semigroup on $\bar{H}$, we observe that, if $\omega>0$, we have that

$$\rho(\mathcal{B})\supset\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\,\lambda>\omega\},$$

and, since $e^{t\mathcal{A}}$ is an analytic semigroup of negative type, from (2.7) we easily deduce the bound

$$|R(\lambda,\mathcal{B})|\leqq\frac{M}{|\lambda-\omega|}\qquad \mathrm{Re}\,\lambda>\omega.\qquad\square$$

Remark 2.3. By using (2.6) and definition (2.1) of $\mathcal{A}$, we can write more explicitly

$$D(\mathcal{B})=\left\{\begin{pmatrix} y \\ z \\ u \end{pmatrix}\in \bar{H} \,\middle|\, y+cz-Du\in D(A)\right\},$$

$$\mathcal{B}\begin{pmatrix} y \\ z \\ u \end{pmatrix}=\begin{pmatrix} z \\ A(y+cz-Du) \\ 0 \end{pmatrix}.$$

By using the operators defined in Lemma 2.2, we can finally obtain the following theorem.

THEOREM 2.4. *Let Y be as in* (2.2), *W and v defined by* (2.4). *Then $W(t)$ satisfies*

$$W(t)=e^{t\mathcal{B}}W_0+(I-\mathcal{B})\int_0^t e^{(t-s)\mathcal{B}}\mathcal{G}v(s)\,ds, \tag{2.8}$$

where $e^{t\mathcal{B}}$, $\mathcal{B}$ are given by (2.5), (2.6), *respectively*; *$\mathcal{G}$ is the linear bounded operator from $\bar{U}$ to $\bar{H}$ defined by*

$$\mathcal{G}v=\begin{pmatrix} Ev-\mathcal{A}(I-\mathcal{A})^{-1}Fv \\ v \end{pmatrix}; \tag{2.9}$$

and $W_0=(y_0,z_0,0)^T$.

Proof. Formula (2.8) is proved by a short verification substituting (2.9) and $W_0=(y_0,z_0,0)^T$ into the second member of (2.8) and considering (2.5) and (2.6). □

In conclusion, the control problem (2.2), (2.3) can be reduced, in the abstract spaces $\bar{H}$, $\bar{U}$, to the problem of minimizing the quadratic functional

$$J(v)=\int_0^T (|\bar{C}W(s)|^2+|v(s)|^2)\,ds+\langle \bar{P}_0W(T),W(T)\rangle, \tag{2.10}$$

over all $v\in L^2(0,T;\bar{U})$, where

$$\bar{C}=\begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & I\end{pmatrix},\qquad \bar{P}_0=\begin{pmatrix} \Gamma_1^2 & 0 & 0 \\ 0 & \Gamma_2^2 & 0 \\ 0 & 0 & 0\end{pmatrix},$$

and W is subject to (2.8).

Suppose now that $\mathcal{B}$, $\mathcal{G}$, $\bar{C}$, $\bar{P}_0$ satisfy all conditions assumed by Flandoli in [3] to show the existence and uniqueness of the solutions to Riccati equation associated with problem (2.8)-(2.10). Obviously, once we have obtained the optimal control $v^*\in L^2(0,T;\bar{U})$ for problem (2.8)-(2.10), the optimal control u^* for the original problem (1.1), (1.2) is given by $u^*(t)=\int_0^t v^*(s)\,ds$.

Remark 2.5. Until now, we have supposed that $u(0)=0$. Otherwise, we can proceed as follows. We first assume that $u(0)=u_0\in U$ is fixed, and we derive the solution formula for (1.1) in H as follows:

$$Y(t)=e^{t\mathcal{A}}\begin{pmatrix} y_0-Du_0 \\ z_0\end{pmatrix}-\mathcal{A}\int_0^t e^{(t-s)\mathcal{A}}Fu'(s)\,ds+Eu(t). \tag{2.11}$$

By using the same method described in the case where $u_0=0$, we reduce the problem of minimizing (2.3) over the class of controls $u\in W^{1,2}(0,T;U)$ such that $u(0)=u_0$ (where Y is subject to (2.11)) to problem (2.8)-(2.10), with $W_0=(y_0,z_0,u_0)$.

If the theory developed in [3] applies to (2.8)-(2.10), then the Riccati feedback synthesis yields the optimal value $J(v^*)=\langle P(T)W_0,W_0\rangle$, where P is the solution to the Riccati equation associated with (2.8)-(2.10). This is a quadratic form with respect to u_0. Thus, to solve the original control problem in $W^{1,2}(0,T;U)$, it remains to minimize $J(v^*)$ with respect to u_0.

3. Solution of the control problem. We want to check hypotheses assumed in [3] to solve problem (2.8)-(2.10). We can immediately see that $\bar{C}\in\mathcal{L}(\bar{H})$, $\bar{P}_0\in\Sigma^+(\bar{H})$. As a consequence of Lemma 2.2, we also know that $\mathcal{B}$ generates an analytic semigroup of type less than 1.

Therefore it remains to show that

$$\exists 0<\alpha<1 \quad \text{such that } \mathscr{G}\in\mathscr{L}(\bar U, D((I-\mathscr{B})^\alpha)) \tag{3.1}$$

and that, under suitable assumptions on Γ_i,

$$\exists \beta\in(\tfrac12-\alpha,\tfrac12) \quad \text{such that } (I-\mathscr{B}^*)^\beta\sqrt{\bar P_0}\in\mathscr{L}(\bar H). \tag{3.2}$$

To prove the validity of (3.1), we give a characterization of the interpolation spaces $D_{\mathscr{B}}(\theta,2)$, for any $\theta\in(0,1)$. (As to relations between interpolation spaces and domains of fractional powers of linear operators, see [7, §§ 1.13–1.15], and the references contained therein.)

We start by showing the following lemma.

LEMMA 3.1. *For any* $\theta\in(0,1)$

$$D_{\mathscr{B}}(\theta,2)=\left\{\begin{pmatrix}Y\\u\end{pmatrix}\in\bar H\;\middle|\; Y-Eu\in D_{\mathscr{A}}(\theta,2)\right\}, \tag{3.3}$$

and the norm

$$\begin{pmatrix}Y\\u\end{pmatrix}\to\left\|\begin{pmatrix}Y\\u\end{pmatrix}\right\|_{\bar H}+\|Y-Eu\|_{D_{\mathscr{A}}(\theta,2)} \tag{3.4}$$

is equivalent to the norm of $D_{\mathscr{B}}(\theta,2)$.

Proof. We use the well-known characterization [7, § 1.14], below:

$$D_{\mathscr{B}}(\theta,p)=\{W\in\bar H: t\to\|t^\theta\mathscr{B}R(t,\mathscr{B})W\|\in L^p_*(a,+\infty)\}$$

with norm

$$\|W\|_{D_{\mathscr{B}}(\theta,p)}=\|W\|_{\bar H}+\|t^\theta\mathscr{B}R(t,\mathscr{B})W\|_{L^p_*(a,+\infty)}, \tag{3.5}$$

where $a\geqq\max(1,\omega_{\mathscr{B}})$, and $f\in L^p_*(a,+\infty)$ if

$$\int_a^{+\infty}|f(t)|^p\frac{dt}{t}<+\infty.$$

Let $\theta\in(0,1)$, $\binom{Y}{u}\in\bar H$, $t\geqq a$. By representation (2.7) of the resolvent $R(t,\mathscr{B})$ in terms of $R(t,\mathscr{A})$, it follows that

$$\mathscr{B}R(t,\mathscr{B})\begin{pmatrix}Y\\u\end{pmatrix}=\begin{pmatrix}\mathscr{A}R(t,\mathscr{A})(Y-Eu)\\0\end{pmatrix}. \tag{3.6}$$

Therefore

$$t\to\left\|t^\theta\mathscr{B}R(t,\mathscr{B})\begin{pmatrix}Y\\u\end{pmatrix}\right\|\in L^2_*(a,+\infty)$$

if and only if

$$t\to\|t^\theta\mathscr{A}R(t,\mathscr{A})(Y-Eu)\|\in L^2_*(a,+\infty),$$

and (3.3) holds true. The equivalence of the norms (3.4), (3.5) is again a consequence of (3.6). $\square$

Arguing as in Lemma 3.1, by means of the representation of the resolvent $R(t,\mathscr{A})$ in terms of $R(t^2/(ct+1);A)$, we can easily deduce the next lemma.

LEMMA 3.2. *For any* $\theta\in(0,1)$,

$$D_{\mathscr{A}}(\theta,2)=\left\{\begin{pmatrix}y\\z\end{pmatrix}\in H\;\middle|\; y+cz\in D_A(\theta,2)\right\},$$

and the norm

$$\begin{pmatrix} y \\ z \end{pmatrix} \to \left\| \begin{pmatrix} y \\ z \end{pmatrix} \right\|_H + \|y+cz\|_{D_A(\theta,2)}$$

is equivalent to the norm of $D_{\mathscr{A}}(\theta, 2)$.

COROLLARY 3.3. *For any* $\theta \in (0,1)$, *we have that*

$$D_{\mathscr{B}}(\theta, 2) = \left\{ \begin{pmatrix} y \\ z \\ u \end{pmatrix} \in \bar{H} \;\middle|\; y + cz - Du \in D_A(\theta, 2) \right\},$$

and the norm

$$\begin{pmatrix} y \\ z \\ u \end{pmatrix} \to \left\| \begin{pmatrix} y \\ z \\ u \end{pmatrix} \right\|_{\bar{H}} + \|y + cz - Du\|_{D_A(\theta,2)}$$

is equivalent to the norm of $D_{\mathscr{B}}(\theta, 2)$.

We are now able to verify condition (3.1).

PROPOSITION 3.4. *Let* $\mathscr{G}$ *be as in* (2.9). *Then there exists* $\theta \in (0, \frac{1}{4})$ *such that* $\mathscr{G} \in \mathscr{L}(\bar{U}, D((I-\mathscr{B})^\theta))$.

Proof. As a consequence of the inclusion

$$D_{\mathscr{B}}(\theta + \varepsilon, 2) \hookrightarrow D((I - \mathscr{B})^\theta), \tag{3.7}$$

which holds for any $\theta \in (0,1)$, $\varepsilon > 0$, it is sufficient to show that there exists $\theta \in (0, \frac{1}{4})$ such that $\mathscr{G} \in \mathscr{L}(\bar{U}, D_{\mathscr{B}}(\theta, 2))$.

Let $v \in \bar{U}$. From Lemma 3.1, $\mathscr{G}v \in D_{\mathscr{B}}(\theta, 2)$ for some $\theta \in (0,1)$ if and only if $R(1, \mathscr{A})Fv - Fv \in D_{\mathscr{A}}(\theta, 2)$ for the same θ.

Since $D \in \mathscr{L}(U, D_A(\theta, 2))$ for any $\theta \in (0, \frac{1}{4})$ [6], conclusion follows easily from Lemma 3.2. □

It remains to check (3.2).

Let θ be as in Proposition 3.4. It is sufficient to show the existence of $\beta \in (\frac{1}{2} - \theta, \frac{1}{2})$ such that

$$\sqrt{\bar{P}_0} \in \mathscr{L}(\bar{H}, D_{\mathscr{B}^*}(\beta, 2)), \tag{3.8}$$

where $\mathscr{B}^*$ is the adjoint of $\mathscr{B}$. After that, again as a consequence of (3.7)—which also holds true for $\mathscr{B}^*$—and by the closed graph theorem, we obtain that

$$\exists \beta \in (\tfrac{1}{2} - \theta, \tfrac{1}{2}) \quad \text{such that } (I - \mathscr{B}^*)^\beta \sqrt{\bar{P}_0} \in \mathscr{L}(\bar{H}).$$

By using the same arguments as in Lemmas 3.1 and 3.2, we can easily deduce the next lemma.

LEMMA 3.5. *For any* $\theta \in (0,1)$,

$$D_{\mathscr{B}^*}(\theta, 2) = \left\{ \begin{pmatrix} y \\ z \\ u \end{pmatrix} \in \bar{H} : z \in D_A(\theta, 2) \right\}, \tag{3.9}$$

and the norm

$$\begin{pmatrix} y \\ z \\ u \end{pmatrix} \to \left\| \begin{pmatrix} y \\ z \\ u \end{pmatrix} \right\|_{\bar{H}} + \|z\|_{D_A(\theta,2)}$$

is equivalent to the norm of $D_{\mathscr{B}^*}(\theta, 2)$.

Assume now that

$$\exists \beta \in (\tfrac{1}{2}-\theta, \tfrac{1}{2}) \quad \text{such that } \Gamma_2 \in \mathscr{L}(L^2(\Omega), D_A(\beta, 2)). \tag{3.10}$$

Then we have the following proposition.

PROPOSITION 3.6. *There exists* $\beta \in (\frac{1}{2}-\theta, \frac{1}{2})$ *such that*

$$(I-\mathscr{B}^*)^\beta \sqrt{\bar{P}_0} \in \mathscr{L}(\bar{H}).$$

Proof. Let

$$\begin{pmatrix} y \\ z \\ u \end{pmatrix} \in \bar{H}.$$

Then

$$\sqrt{\bar{P}_0}\begin{pmatrix} y \\ z \\ u \end{pmatrix} = \begin{pmatrix} \Gamma_1 y \\ \Gamma_2 z \\ 0 \end{pmatrix}.$$

By hypothesis (3.10) on Γ_2, and from (3.9), there exists $\beta \in (\frac{1}{2}-\theta, \frac{1}{2})$ such that

$$\sqrt{\bar{P}_0}\begin{pmatrix} y \\ z \\ u \end{pmatrix} \in D_{\mathscr{B}^*}(\beta, 2).$$

Moreover, as a consequence of Lemma 3.5, we can write

$$\left\| \sqrt{\bar{P}_0}\begin{pmatrix} y \\ z \\ u \end{pmatrix} \right\|_{D_{\mathscr{B}^*}(\beta,2)} = \left\| \begin{pmatrix} \Gamma_1 y \\ \Gamma_2 z \\ 0 \end{pmatrix} \right\|_{\bar{H}} + \|\Gamma_2 z\|_{D_A(\beta,2)},$$

and, again by (3.10), we deduce the bound

$$\left\| \sqrt{\bar{P}_0}\begin{pmatrix} y \\ z \\ u \end{pmatrix} \right\|_{D_{\mathscr{B}^*}(\beta,2)} \leqq \text{const} \left\| \begin{pmatrix} y \\ z \\ u \end{pmatrix} \right\|_{\bar{H}}.$$

Thus (3.8) holds true, and Proposition 3.6 is proved. □

Now, following [3], we can solve the Riccati equation associated with (2.8)–(2.10) and conclude by using dynamic programming that, for every $y_0, z_0 \in L^2(\Omega)$, there exists a unique feedback optimal control v^* for (2.8)–(2.10).

At this point, we can interpret the Riccati feedback synthesis of problem (2.8)–(2.10) in terms of the original control problem (1.1), (1.2).

Therefore we can finally state the following theorem.

THEOREM 3.7. *If* $C_i \in \mathscr{L}(L^2(\Omega))$, $\Gamma_i \in \Sigma(L^2(\Omega))$, $i=1,2$, $\Gamma_2 \in \mathscr{L}(L^2(\Omega), H_0^{2\beta}(\Omega))$ *for some* $\beta \in (\frac{1}{4}, \frac{1}{2})$, *then, for every* $y_0, z_0 \in L^2(\Omega)$, *there exists a unique feedback optimal control* u^* *for problem* (1.1), (1.2) *in* $W_0^{1,2}(0, T; L^2(\partial\Omega))$.

REFERENCES

[1] A. V. BALAKRISHNAN, *Applied Functional Analysis*, Springer-Verlag, New York, 1976.

[2] F. FLANDOLI, *Riccati equation arising in a boundary control problem with distributed parameters*, SIAM J. Control Optim., 22 (1984) pp. 76–86.

[3] F. FLANDOLI, *On the direct solutions of Riccati equations arising in boundary control theory*, Ann. Mat. Pura Appl., to appear.

[4] I. LASIECKA AND R. TRIGGIANI, *Dirichlet boundary control problem for parabolic equations with quadratic cost: Analyticity and Riccati's feedback synthesis*, SIAM J. Control Optim., 21 (1983), pp. 41–68.

[5] ———, *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Lecture Notes in Control and Inform. Sci., 164, Springer-Verlag, Berlin, New York, 1991.

[6] J. L. LIONS AND E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, New York, 1971.

[7] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutschen Verlag der Wissenschaften, Berlin, 1978.