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Applicable Analysis
Vol. 85, No. 1–3, January–March 2006, 249–276

Taylor & Francis
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On complex-valued solutions to a 2D Eikonal equation. Part three: analysis of a Bäcklund transformation

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Communicated by R. P. Gilbert

(*Received 26 July 2004; in final form 11 October 2004*)

Alla memoria del nostro maestro ed amico Carlo Pucci

The following equation $w_x^2 + w_y^2 + n^2(x, y) = 0$ and related partial differential equations and systems, which arise in a generalization of geometrical optics, are investigated from a theoretical point of view. Here x and y denote rectangular coordinates in the Euclidean plane, n is real-valued, strictly positive and smooth enough. Qualitative properties of smooth solutions were derived in Magnanini and Talenti (1999, On complex-valued solutions to a 2D Eikonal equation. Part One: qualitative properties. *Contemporary Mathematics*, **283**, 203–229). Partial differential equations governing Re(w) were treated in Magnanini and Talenti (2002, On Complex-Valued Solutions to a 2D Eikonal Equation. Part Two: Existence Theorems. *SIAM Journal on Mathematical Analysis*, **34**, 805–835). Here we put to use viscosity and variational methods, and a Bäcklund transformation relating Re(w) and Im(w).

Keywords: Partial differential equations; Bäcklund transformations; Critical points; Boundary value problems; Convex functionals; Minimizers; Free boundaries; Viscosity solutions

1991 Mathematics Subject Classifications: Primary 35J70, 35Q60; Secondary 49N60

1. Introduction

1.1. *Subject*

In the present article we let notations be as in appendix A; let n denote some *real-valued*, *strictly positive*, *sufficiently smooth* function of x and y; and take the following partial

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Applicable Analysis ISSN 0003-6811 print: ISSN 1563-504X online
http://www.tandf.co.uk/journals
DOI: 10.1080/00036810500277041

differential equation

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + n^2 = 0 \tag{1.1}$$

in hand.

1.2. *Motivations*

Equation (1.1) arises in acoustics, nuclear physics and optics. Suppose an isotropic, non-conducting, non-dissipative medium and a monochromatic electromagnetic field interact in absence of electric currents. Suppose the space dimension equals 2. Let n and ν denote the refractive index and the wave number, respectively. n is a real-valued field, whose reciprocal is proportional to the relevant velocity of propagation through the medium, and ν is a large constant parameter, whose reciprocal is proportional to the length of waves involved. The following Helmholtz equation

$$\Delta W + \nu^2 n^2 W = 0 \tag{1.2}$$

is an archetype of those partial differential equations that ensue from Maxwell's system and model the affairs mathematically. A distinctive feature of (1.2) is *stiffness* – the order of magnitude of ν is significantly greater than that of the other coefficients involved.

Expansions, which represent solutions asymptotically as $\nu \simeq \infty$, are a clue to (1.2). One of these expansions is provided by classical geometrical optics – see [1–3], for example. Though successful in describing both the propagation of light and the development of caustics via the mechanism of rays, geometrical optics is inherently unable to account for any optical process that takes place beyond a caustic. More comprehensive expansions are supplied by a theory, proposed by Felsen and coworkers and nicknamed *Evanescent Wave Tracking* (*EWT*) – see [4–12]. EWT does include geometrical optics; in addition, the former is credited to model certain effects that are excluded from the latter – for instance, the fast decaying waves that develop on the side of the caustic where the geometric optical rays fail to penetrate.

Basically, EWT calls for real-valued fields u, v, λ, μ – *all independent on* ν – such that (1.2) consists with the following expansion

$$W = \exp(\lambda + u\mu)[\cos(\mu + \nu v) + O(1/\nu)] \tag{1.3}$$

as ν approaches ∞. EWT may be built upon the ansatz consisting of the three items as below:

(i) W is given by either $W = \exp(\nu U)\cos(\nu V)$ or $W = \exp(\nu U)\sin(\nu V)$.
(ii) U and V obey the following partial differential system

$$(\Delta + \nu^2 n^2)\exp(\nu U)\begin{bmatrix}\cos(\nu V)\\ \sin(\nu V)\end{bmatrix} = 0.$$

(iii) The following expansion

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \nu^{-1} + O(\nu^{-2})$$

holds in a robust topology as ν approaches ∞.

Items (i) and (ii) obviously imply (1.2); items (i) and (iii) imply (1.3). Equations governing u, v, λ, μ are (1.4) and (1.5) below, and can be easily inferred from (ii) and (iii) as follows.

The differential system appearing in (ii) can be recast thus

$$(1/\nu)\Delta U + |\nabla U|^2 - |\nabla V|^2 + n^2 = 0, \quad (1/\nu)\Delta V + 2(\nabla U, \nabla V) = 0;$$

therefore (ii), (iii) and standard arguments cause u, v, λ, μ to obey both

$$|\nabla u|^2 - |\nabla v|^2 + n^2 = 0, \quad (\nabla u, \nabla v) = 0, \tag{1.4}$$

and

$$2\left\{\begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}\frac{\partial}{\partial x} + \begin{bmatrix} u_y & -v_y \\ v_y & u_y \end{bmatrix}\frac{\partial}{\partial y}\right\}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \Delta\begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{1.5}$$

Observe that (1.3), (1.4) and (1.5) become respectively

$$W = \exp(\lambda)[\cos(\mu + \nu v) + O(1/\nu)],$$
$$|\nabla v|^2 = n^2,$$

and

$$2(\nabla v, \nabla\lambda) + \Delta v = 0, \quad (\nabla v, \nabla\mu) = 0,$$

if u is zero. In other words, if u vanishes, the three formulas in question become the *geometric optical expansion*, the *Eikonal equation* and the *transport equation* of geometrical optics. Observe also that, if i denotes $\sqrt{-1}$, then (1.4) and (1.5) can be arranged in the following form

$$(u_x + iv_x)^2 + (u_y + iv_y)^2 + n^2 = 0, \tag{1.6}$$

$$2\left[(u_x + iv_x)\frac{\partial}{\partial x} + (u_y + iv_y)\frac{\partial}{\partial y}\right](\lambda + i\mu) + \Delta(u + iv) = 0, \tag{1.7}$$

respectively.

Equations (1.4) to (1.7) are the basis of EWT. Equation (1.6) is nothing but a copy of (1.1) – the theme of this article. It should be stressed here that the same equation also appears in a more exhaustive asymptotic analysis of (1.2), which leads to uniform expansions near caustics, and in modeling deeper diffraction processes – a propos information can be found in [13–16].

2. Framework

Our treatment of (1.1) relies upon an apparatus that is described next.

2.1. *Some transformations of Bäcklund type*

Let

$$f = \text{a } \textit{Young function},$$

and let g be the *conjugate* of f. Recall from convex analysis that f maps $[0,\infty[$ into $[0,\infty[$, vanishes at 0, increases and is convex; g is a Young function too, and

$$g(\rho) = \sup\{\rho\rho' - f(\rho') : 0 \le \rho' < \infty\}$$

for every nonnegative ρ. As a working hypothesis, assume that f is strictly increasing, strictly convex and smooth.

Bäcklund transformations attached to the following equations

$$\nabla v = \frac{n}{|\nabla u|}\, f'\left(\frac{|\nabla u|}{n}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\nabla u, \tag{2.1}$$

$$\nabla u = \frac{n}{|\nabla v|}\, g'\left(\frac{|\nabla v|}{n}\right)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\nabla v \tag{2.2}$$

play a basic role here. Information on Bäcklund transformations can be found in [17–20]. The Bäcklund transformations in hand operate on real-valued, tractable functions of two real variables, and return much the same items. They amount to suitably stretching or shrinking a gradient, hence rotating the same by 90° and (2.1) reads indeed

$$v_x = -nf'(\rho)\sin\omega, \quad v_y = nf'(\rho)\cos\omega,$$

if the polar coordinates defined in appendix B dress up the hodograph of u. Critical points of trial functions carry weight in the event that either f' or g' fails to vanish at 0, and can be circumvented by viscosity methods.

Under the hypotheses made *ab initio*, we have

$$g(f'(\rho)) = \rho f'(\rho) - f(\rho) \quad \text{and} \quad g'(f'(\rho)) = \rho$$

for every nonnegative ρ – in particular f' and the restriction of g' to the range of f' are *inverse* of one another (see e.g. [21], sections 12 and 26). Therefore (2.1) implies (2.2): that is to say, any Bäcklund transformation associated with (2.1) is *one-to-one*, and one Bäcklund transformation associated with (2.2) continues the *inverse* of the former.

The right-hand side of (2.1) is locally a gradient if and only if u obeys

$$\operatorname{div}\left\{\frac{n}{|\nabla u|}\, f'\left(\frac{|\nabla u|}{n}\right)\nabla u\right\} = 0 \tag{2.3}$$

– a *second-order differential equation in divergence form*, features of which are sketched in appendix C. The right-hand side of (2.2) is locally a gradient if and only if v obeys

$$\operatorname{div}\left\{\frac{n}{|\nabla v|}\, g'\left(\frac{|\nabla v|}{n}\right)\nabla v\right\} = 0. \tag{2.4}$$

Therefore a Bäcklund transformation attached to (2.2) acts on solutions of (2.4); a Bäcklund transformation attached to (2.1) maps solutions of (2.3) into solutions of (2.4).

Remarks (i) If

$$f(\rho) = \frac{\rho^2}{2}$$

for every nonnegative ρ, then $g = f$. Moreover, both (2.1) and (2.2) coincide with *Cauchy–Riemann equations* no matter how n is, and both (2.3) and (2.4) coincide with *Laplace equation*. (ii) Let p be any exponent larger then 1,

$$f(\rho) = \frac{\rho^p}{p}$$

for every nonnegative ρ, and $n \equiv 1$. Then (2.1) reads

$$v_x = -|\nabla u|^{p-2} u_y, \qquad v_y = |\nabla u|^{p-2} u_x$$

– a pair widely investigated in [22–24]. Equation (2.3) reads

$$\operatorname{div}\left\{|\nabla u|^{p-2}\nabla u\right\} = 0$$

– the so-called *p-Laplace equation*.

2.2. *A partial differential system*

The following system

$$|\nabla v| = nf'\left(\frac{|\nabla u|}{n}\right), \quad (\nabla u, \nabla v) = 0, \tag{2.5}$$

is central to our investigations. Observe the architecture of (2.5): the former equation relates the absolute values of the relevant gradients, the latter informs us that the same gradients are orthogonal. Observe also that (2.5) implies

$$|\nabla u| = ng'\left(\frac{|\nabla v|}{n}\right), \quad (\nabla u, \nabla v) = 0. \tag{2.6}$$

As propositions (i) and (ii) below show, a solution pair to (2.5) results from letting u obey (2.3), then casting v via (2.1); any solution pair to (2.5), whose Jacobian determinant is strictly positive, can be recovered via this process. In other words, a Bäcklund transformation underlying (2.1) serves the purpose of ruling out those pairs that satisfy system (2.5) and whose Jacobian determinant changes its sign.

Proposition (iii) informs us that decoupling system (2.5) results both in equation (2.3) and in equation (2.4). Proposition (iv) informs us that system (2.5) is elliptic.

(i) Equation (2.1) implies both (2.5) and the condition $\partial(u,v)/\partial(x,y) \geq 0$.

Proof Equation (2.1) implies (2.5) trivially. Since

$$\frac{\partial(u,v)}{\partial(x,y)} = \left(\nabla u, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla v\right),$$

(2.1) also gives

$$\frac{\partial(u,v)}{\partial(x,y)} = n\ |\nabla u|\ f'\left(\frac{|\nabla u|}{n}\right) \quad \text{and} \quad \frac{\partial(u,v)}{\partial(x,y)} = |\nabla u||\nabla v|. \qquad \blacksquare$$

(ii) System (2.5) and condition $\partial(u,v)/\partial(x,y) > 0$ imply (2.1).

Proof The following equations

$$|\nabla u|^2 \nabla v = (\nabla u, \nabla v)\nabla u + \frac{\partial(u,v)}{\partial(x,y)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u,$$

$$|\nabla u|^2 |\nabla v|^2 = (\nabla u, \nabla v)^2 + \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^2,$$

which result from either algebraic manipulations or a geometric argument, allow us to recast (2.5) this way

$$|\nabla u|^2 \nabla v = \frac{\partial(u,v)}{\partial(x,y)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u, \quad \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = n|\nabla u|\ f'\left(\frac{|\nabla u|}{n}\right). \qquad \blacksquare$$

(iii) System (2.5) implies both (2.3) and (2.4).

Proof Here we rely upon notations and formulas from appendix B. The latter equation in (2.5) results in the following pair:

$$Xv = 0, \quad \nabla v = (Yv) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\nabla u}{|\nabla u|}.$$

We have

$$\operatorname{div}\left\{|\nabla v| \frac{\nabla u}{|\nabla u|}\right\} = X|\nabla v| - h\ |\nabla v|,$$

because both sides of such an equation equal

$$\left(\nabla|\nabla v|, \frac{\nabla u}{|\nabla u|}\right) + |\nabla v| \text{ div } \frac{\nabla u}{|\nabla u|}.$$

We deduce successively $X(Yv) = h(Yv)$, $X|\nabla v| = h|\nabla v|$ and

$$\text{div}\left\{|\nabla v|\frac{\nabla u}{|\nabla u|}\right\} = 0.$$

Combining the last equation and the former in (2.5), gives (2.3).

Equation (2.4) follows from (2.6) via parallel arguments. ■

(iv) System (2.5) is either *elliptic* or *degenerate elliptic*. *Degeneracies* do occur at the *critical points* of u if f' fails to vanish at 0; *uniform ellipticity* prevails if f satisfies the following condition

$$0 < \text{Constant} \leq \frac{\rho f''(\rho)}{f'(\rho)} \leq \text{Constant} \tag{2.7}$$

for every nonnegative ρ.

Proof Differentiating the following map

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} nf'\left(\frac{|\nabla u|}{n}\right) - |\nabla v| \\ (\nabla u, \nabla v) \end{bmatrix}$$

results in the following linear partial differential operator

$$\begin{bmatrix} f''\left(\frac{|\nabla u|}{n}\right)\frac{u_x}{|\nabla u|} & -\frac{v_x}{|\nabla v|} \\ v_x & u_x \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} f''\left(\frac{|\nabla u|}{n}\right)\frac{u_y}{|\nabla u|} & -\frac{v_y}{|\nabla v|} \\ v_y & u_y \end{bmatrix} \frac{\partial}{\partial y},$$

whose characteristic determinant equals either side of the following equality

$$\det\left(-\begin{bmatrix} f''\left(\frac{|\nabla u|}{n}\right)\frac{u_x}{|\nabla u|} & -\frac{v_x}{|\nabla v|} \\ v_x & u_x \end{bmatrix} \mathrm{d}y + \begin{bmatrix} f''\left(\frac{|\nabla u|}{n}\right)\frac{u_y}{|\nabla u|} & -\frac{v_y}{|\nabla v|} \\ v_y & u_y \end{bmatrix} \mathrm{d}x\right)$$

$$= \left(\left(f''\left(\frac{|\nabla u|}{n}\right)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u)^T + \frac{\nabla v}{|\nabla v|} \cdot (\nabla v)^T\right)\begin{bmatrix} -\mathrm{d}y \\ \mathrm{d}x \end{bmatrix}, \begin{bmatrix} -\mathrm{d}y \\ \mathrm{d}x \end{bmatrix}\right).$$

If u and v obey (2.5), then formula (B.1) from appendix B and straightforward manipulations yield successively

$$\frac{\nabla v}{|\nabla v|} \cdot (\nabla v)^T = nf'(\rho)\begin{bmatrix} -\sin\omega \\ \cos\omega \end{bmatrix} \cdot [-\sin\omega \;\; \cos\omega]$$

and

$$f''\left(\frac{|\nabla u|}{n}\right)\frac{\nabla u}{|\nabla u|}\cdot(\nabla u)^T+\frac{\nabla v}{|\nabla v|}\cdot(\nabla v)^T$$
$$=n\begin{bmatrix}\cos\omega & -\sin\omega\\ \sin\omega & \cos\omega\end{bmatrix}\begin{bmatrix}\rho f''(\rho) & 0\\ 0 & f'(\rho)\end{bmatrix}\begin{bmatrix}\cos\omega & \sin\omega\\ -\sin\omega & \cos\omega\end{bmatrix}.$$

In other words, (2.5) causes

$$f''\left(\frac{|\nabla u|}{n}\right)\frac{\nabla u}{|\nabla u|}\cdot(\nabla u)^T+\frac{\nabla v}{|\nabla v|}\cdot(\nabla v)^T$$

to have the following eigenvalues

$$|\nabla u|\, f''\left(\frac{|\nabla u|}{n}\right) \quad\text{and}\quad nf'\left(\frac{|\nabla u|}{n}\right). \qquad \blacksquare$$

2.3. *Specifications*

The case where

$$f(\rho)=\frac{1}{2}\left[\rho\sqrt{\rho^2+1}+\log(\rho+\sqrt{\rho^2+1})\right] \tag{2.8}$$

for every nonnegative ρ is consequential from our point of view.

Equation (2.8) gives $f(0)=0$,

$$f'(\rho)=\sqrt{\rho^2+1},$$
$$f''(\rho)=\rho/f'(\rho), \quad 0\le \rho f''(\rho)/f'(\rho)=\rho^2/(1+\rho^2)<1,$$

for every nonnegative ρ; moreover, $f''=\sqrt{1-(f')^{-2}}$ and $f'''=(f')^{-3}$. The same equation causes the conjugate of f to obey

$$g(\rho)=0 \text{ if } 0\le\rho\le 1, \ =\frac{1}{2}\left[\rho\sqrt{\rho^2-1}-\log(\rho+\sqrt{\rho^2-1})\right] \text{ if } 1<\rho<\infty,$$
$$g'(\rho)=0 \text{ if } 0\le\rho\le 1, \ =\sqrt{\rho^2-1} \text{ if } 1<\rho<\infty; \tag{2.9}$$

moreover, to enjoy the following properties

$$g''(\rho)=\rho/g'(\rho)=\sqrt{1+(g'(\rho))^{-2}}, \quad 1<\rho g''(\rho)/g'(\rho)=\rho^2/(\rho^2-1),$$

and $g'''(\rho)=-(g'(\rho))^{-3}$ for every ρ larger than 1.

Peculiarities of f and g, which entail significant effects in the present context, are summarized thus. First,

$$f(\rho)=\frac{\rho^2}{2}+\frac{1}{2}\log(2\rho)+\frac{1}{4}+O(\rho^{-2}), \quad f(\rho)+g(\rho)=\rho^2+O(\rho^{-2}),$$

as $\rho \to \infty$ – both f and g *grow quadratically* at infinity. Second,

$$f(\rho) = \rho\left(1 + \frac{\rho^2}{6} - \frac{\rho^4}{40} + \cdots\right)$$

as $\rho \to 0$ – the surface generated by revolving the graph of f about the ordinate axis has a *conical point* at the origin. Third, g *fails* to increase strictly and to be strictly convex. Fourth, range of $f' =$ support of $g' = [1, \infty[$,

$$g'(f'(\rho)) = \rho \quad \text{and} \quad f'(g'(\rho)) = \max\{1, \rho\}$$

for every nonnegative ρ – g' is a *strict extension* of, and is different from, the inverse of f'.

Equations (2.8) and (2.9) embody a number of geometric features. First, f coincides with an *arclength* on the following parabola $\{(x, y) \in \mathbb{R}^2\colon x^2 = 2y\}$.

Second, f is related to a *pseudosphere*. The pseudosphere is a distinguished surface of revolution in Euclidean space $\mathbb{R}^3$, whose Gauss curvature equals a negative constant. The Riemannian structure, which the pseudosphere inherits from $\mathbb{R}^3$, is identical with the Riemannian structure of the Poincaré, or hyperbolic, half-plane: that is to say, the intrinsic geometry of the pseudosphere is a model of the hyperbolic geometry of Lobachevski. If A and B are constants and u is given by

$$r = \sqrt{x^2 + y^2}, \quad u(x, y) = A\left(\sqrt{1 - \frac{r^2}{A^2}} + \log\frac{r}{|A| + \sqrt{A^2 - r^2}}\right) + B,$$

then the graph of u is precisely a pseudosphere. In ([25], section 2.3) it is observed that, if n equals 1 and f obeys (2.8), then such a u is a *rotation invariant solution* to equation (2.3).

Third, the meridian curve of any pseudosphere is a *tractrix*. Recall that the tractrix, alias hundekurve, is the path of a dog pulled by an inextensible leash whose end runs along an axis. The evolute of a tractrix is a *catenary*. Equation (2.9) shows that the *evolute* of the following catenary $\{(x, y) \in \mathbb{R}^2\colon 2x = \cosh(2y),\ -\infty < y \le 0\}$ is precisely the *graph of the positive part of* g.

If (2.8) and (2.9) are in force, then (2.1) and (2.3) read

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \sqrt{1 + \frac{n^2}{u_x^2 + u_y^2}} \begin{bmatrix} -u_y \\ u_x \end{bmatrix}, \tag{2.10}$$

$$\frac{\partial}{\partial x}\left\{\sqrt{1 + \frac{n^2}{u_x^2 + u_y^2}}\, u_x\right\} + \frac{\partial}{\partial y}\left\{\sqrt{1 + \frac{n^2}{u_x^2 + u_y^2}}\, u_y\right\} = 0, \tag{2.11}$$

respectively; furthermore, (2.4) amounts to claiming that either the inequality $v_x^2 + v_y^2 \le n^2$ prevails or

$$v_x^2 + v_y^2 > n^2$$

and

$$\frac{\partial}{\partial x}\left\{\sqrt{1-\frac{n^2}{v_x^2+v_y^2}}\, v_x\right\}+\frac{\partial}{\partial y}\left\{\sqrt{1-\frac{n^2}{v_x^2+v_y^2}}\, v_y\right\}=0. \tag{2.12}$$

The following identity

$$\begin{aligned}&|\nabla u|^3\sqrt{n^2+|\nabla u|^2}\ \mathrm{div}\left\{\sqrt{n^2+|\nabla u|^2}\ \frac{\nabla u}{|\nabla u|}\right\}\\&\quad=\left(|\nabla u|^4+n^2u_y^2\right)u_{xx}-2n^2u_xu_yu_{xy}+\left(|\nabla u|^4+n^2u_x^2\right)u_{yy}+n|\nabla u|^2(\nabla u,\nabla n),\end{aligned}$$

which ensues from either the content of appendix C or an inspection, causes any sufficiently smooth solution of (2.11) to satisfy

$$\begin{aligned}&\left[(u_x^2+u_y^2)^2+n^2u_y^2\right]u_{xx}-2n^2u_xu_yu_{xy}+\left[(u_x^2+u_y^2)^2+n^2u_x^2\right]u_{yy}\\&\quad+n(u_x^2+u_y^2)(n_xu_x+n_yu_y)=0\end{aligned} \tag{2.13}$$

– a *semilinear* second-order partial differential equation with *polynomial nonlinearities*. On the other hand, Proposition 2.2.1 from [25] informs us that smooth solutions to (2.13) exist whose gradient vanishes exclusively in a set of measure 0, and that *do not* satisfy (2.11) in the sense of distributions – they make the left-hand side of (2.11) a well-defined distribution, which is supported by the set of the critical points, but is *not* zero.

As the content of appendix C shows, equation (2.13) is *degenerate elliptic*. A real-valued solution to (2.13) is *elliptic* if the gradient of u is nowhere equal to zero; a degeneracy occurs at any *critical point* of u. Theorem 3.1.2 from [25] basically says that smooth solutions to (2.13) *cannot* have *isolated* critical points; moreover, the critical points of smooth solutions to (2.13) spread along the *geometric optical rays*, whose definition is recalled in appendix C.

Manipulations show that any sufficiently smooth solution to (2.12) obeys

$$\begin{aligned}&\left[(v_x^2+v_y^2)^2-n^2v_y^2\right]v_{xx}+2n^2v_xv_yv_{xy}+\left[(v_x^2+v_y^2)^2-n^2v_x^2\right]v_{yy}\\&\quad-n(v_x^2+v_y^2)(n_xv_x+n_yv_y)=0\end{aligned} \tag{2.14}$$

– another semilinear second-order partial differential equation with polynomial nonlinearities. Solutions v to (2.14) such that $v_x^2+v_y^2>n^2$ are *elliptic*. The real-valued twice differentiable solutions to the *Eikonal equation* $v_x^2+v_y^2=n^2$ are *parabolic* solutions to (2.14). Solutions v to (2.14) such that $v_x^2+v_y^2<n^2$ are *hyperbolic*. A treatment of solutions to (2.14), which exhibit a mixed elliptic-hyperbolic character, is in [26].

Combining (2.5) and (2.8) results in the following 2×2 partial differential system

$$u_x^2 + u_y^2 - v_x^2 - v_y^2 + n^2 = 0, \quad u_x v_x + u_y v_y = 0. \tag{2.15}$$

Combining (2.6) and (2.9) results in the following system

$$u_x^2 + u_y^2 - \max\left\{v_x^2 + v_y^2, n^2\right\} + n^2 = 0, \qquad u_x v_x + u_y v_y = 0. \tag{2.16}$$

The analysis made in section 2.2 informs us that system (2.15) is *degenerate elliptic*. A solution pair is *elliptic* if the former component of such a pair is free from *critical points*; any critical point of the former component gives rise to a *degeneracy*. As is easy to see, quite the same statement applies to (2.16).

Observe that if w is a complex-valued function of x and y, moreover if

$$u = \mathrm{Re}(w) \quad \text{and} \quad v = \mathrm{Im}(w),$$

then

$$\mathrm{Re}(w_x^2 + w_y^2) = u_x^2 + u_y^2 - v_x^2 - v_y^2 \quad \text{and} \quad \mathrm{Im}(w_x^2 + w_y^2) = 2(u_x v_x + u_y v_y)$$

– therefore w obeys equation (1.1) if and only if its real and imaginary parts, u and v, obey system (2.15).

System (2.16) amounts to stating that either

$$u_x = u_y = 0 \quad \text{and} \quad v_x^2 + v_y^2 \leq n^2,$$

or $u_x^2 + u_y^2$ is different from 0, $v_x^2 + v_y^2$ is larger than n^2, and (2.15) holds. In particular, any pair u and v, which satisfies (2.16) in some domain either in a classical or in some generalized sense, automatically satisfies (2.15) in any open subdomain that is either contained or essentially contained in $\{(x, y) \colon |\nabla v(x, y)| > n\}$. Therefore, (2.16) can be viewed as a *free-boundary problem* for system (2.15) – the relevant *free boundary* being the inner part of $\partial\{(x, y) \colon |\nabla v(x, y)| > n\}$.

One might summarize the above remarks in the language of EWT by saying that: (i) the free boundary represents a caustic; (ii) the free-boundary problem models a non-geometric optical regime, which develops in the side of such a caustic where geometrical optics breaks down.

2.4. *Viscosity*

In the present article we treat the above equations and systems via a *vanishing viscosity method*, which is apt to circumvent the degeneracies in hand and mimics ideas from [27–30]. Our method consists of the following steps. (i) Let ε be a small positive parameter. (ii) Let f_ε and g_ε be a convenient pair of conjugate Young functions: let f_ε approach f effectively as $\varepsilon \to 0$, and simultaneously remove those features of f that cause uniform ellipticity to fail. (iii) Let f_ε and g_ε replace f and g in formulas (2.1) and (2.2), in equations (2.3) and (2.4), in systems (2.5) and (2.6), respectively. (iv) Shake well, and take limits as ε approaches 0.

The following lemmas put items (i) and (ii) in operation.

LEMMA 2.1 *Let* (2.8) *be in force*; *let* ε *obey*

$$0 < \varepsilon < 1, \tag{2.17}$$

and let f_ε *be given by*

$$f_\varepsilon(\rho) = \int_0^\rho t\left(\frac{1+t^2}{\varepsilon+t^2}\right)^{1/2(1-\varepsilon)} \mathrm{d}t \tag{2.18}$$

for every nonnegative ρ. *The following properties hold.*

(i) f_ε *vanishes at* 0, *strictly increases, is strictly convex and smooth – in other words,* f_ε *is a nice Young function.*

(ii)

$$f_\varepsilon(\rho) = \varepsilon^{-1/2(1-\varepsilon)} \cdot \frac{\rho^2}{2} \cdot \left[1 - \frac{\rho^2}{4\varepsilon} + O(\rho^4)\right],$$

$$f'_\varepsilon(\rho) = \varepsilon^{-1/2(1-\varepsilon)} \cdot \rho \cdot \left[1 - \frac{\rho^2}{2\varepsilon} + O(\rho^4)\right], \quad f''_\varepsilon(\rho) = \varepsilon^{-1/2(1-\varepsilon)}\left[1 - \frac{3\rho^2}{2\varepsilon} + O(\rho^4)\right]$$

as $\rho \to 0$. *In particular,* f'_ε *has a zero of multiplicity one at* 0.

(iii)

$$f_\varepsilon(\rho) = \frac{\rho^2}{2} + \frac{1}{2}\log(2\rho) + \frac{1}{4} - C_\varepsilon + O(\rho^{-2}),$$

$$f'_\varepsilon(\rho) = \rho + \frac{1}{2\rho} + O(\rho^{-3}), \quad f''_\varepsilon(\rho) = 1 - \frac{1}{2\rho^2} + O(\rho^{-4})$$

as $\rho \to \infty$. *Here* C_ε *is positive and obeys* $C_\varepsilon = O(\sqrt{\varepsilon})$ *as* $\varepsilon \to 0$.

(iv)

$$\sqrt{\varepsilon}\frac{2+\sqrt{\varepsilon}}{(1+\sqrt{\varepsilon})^2} \le \frac{\rho f''_\varepsilon(\rho)}{f'_\varepsilon(\rho)} = 1 - \frac{\rho^2}{(1+\rho^2)(\varepsilon+\rho^2)} < 1$$

for every positive ρ. *In particular,* $\rho f''_\varepsilon(\rho)/f'_\varepsilon(\rho)$ *is both bounded and bounded away from* 0 *as* ρ *ranges from* 0 *to* ∞.

(v) $(\partial/\partial\varepsilon)f_\varepsilon(\rho) < 0$ and $(\partial/\partial\varepsilon)f'_\varepsilon(\rho) < 0$ for every positive ρ.

(vi) f_ε *approaches* f *uniformly as* $\varepsilon \to 0$. *In fact,*

$$0 < f(\rho) - f_\varepsilon(\rho) < C_\varepsilon$$

for every positive ρ.

LEMMA 2.2 *Let* (2.9), (2.17) *and* (2.18) *be in force; let*

$$g_\varepsilon(\rho) = \text{the Young conjugate of } f_\varepsilon \tag{2.19}$$

The following properties hold.

(i) g_ε *strictly increases and is strictly convex,* g'_ε *equals the inverse of* f'_ε.

(ii)

$$g_\varepsilon(\rho) = \varepsilon^{1/2(1-\varepsilon)} \cdot \frac{\rho^2}{2} \cdot \left[1 + \varepsilon^{\varepsilon/(1-\varepsilon)} \cdot \frac{\rho^2}{4} + O(\rho^4)\right],$$

$$g'_\varepsilon(\rho) = \varepsilon^{1/2(1-\varepsilon)} \cdot \rho \cdot \left[1 + \varepsilon^{\varepsilon/(1-\varepsilon)} \cdot \frac{\rho^2}{2} + O(\rho^4)\right],$$

$$g''_\varepsilon(\rho) = \varepsilon^{1/2(1-\varepsilon)}\left[1 + \varepsilon^{\varepsilon/(1-\varepsilon)} \cdot \frac{3\rho^2}{2} + O(\rho^4)\right]$$

as $\rho \to 0$.

(iii)

$$g_\varepsilon(\rho) = \frac{\rho^2}{2} - \frac{1}{2}\log(2\rho) - \frac{1}{4} + C_\varepsilon + O(\rho^{-2}),$$

$$g'_\varepsilon(\rho) = \rho - \frac{1}{2\rho} + O(\rho^{-3}), \quad g''_\varepsilon(\rho) = 1 + \frac{1}{2\rho^2} + O(\rho^{-4})$$

(iv) *as* $\rho \to \infty$. *Here* C_ε *is the same quantity appearing in* (iii), *Lemma* 2.1.

$$\frac{g'_\varepsilon(\rho)}{\rho g''_\varepsilon(\rho)} = 1 - \left[1 + \varepsilon + (g'_\varepsilon(\rho))^2 + \varepsilon(g'_\varepsilon(\rho))^{-2}\right]^{-1}$$

for every positive ρ. *In particular,*

$$0 < 1 - \frac{g'_\varepsilon(\rho)}{\rho g''_\varepsilon(\rho)} \leq (1 + \sqrt{\varepsilon})^{-2}$$

for every positive ρ.

(v) $(\partial/\partial\varepsilon)g_\varepsilon(\rho) > 0$ *and* $(\partial/\partial\varepsilon)g'_\varepsilon(\rho) > 0$ *for every positive* ρ.

(vi) g_ε *approaches* g *uniformly as* $\varepsilon \to 0$.

(vii) g'_ε *approaches* g' *uniformly as* $\varepsilon \to 0$.

(viii) $(\partial/\partial\varepsilon)\big(\sqrt{\varepsilon}/g'_\varepsilon(\rho)\big) > 0$ *for every positive* ρ.

(ix) $\sqrt{\varepsilon}\rho/g'_\varepsilon(\rho)$ *approaches* $\sqrt{1 - \min\{1, \rho^2\}}$ *uniformly with respect to* ρ, *as* ρ *ranges between 0 and* ∞ *and* $\varepsilon \to 0$.

(x) $(\partial/\partial\varepsilon)\big(g'_\varepsilon(\rho)/g''_\varepsilon(\rho)\big) > 0$ *for every positive* ρ.

(xi) $\rho^{-2}\big[g'_\varepsilon(\rho)/\rho g''_\varepsilon(\rho)\big]$ *approaches* $\min\big\{1, \rho^{-4}\big\}$ *uniformly with respect to* ρ, *as* ρ *ranges between 0 and* ∞ *and* $\varepsilon \to 0$.

3. Main results

In the present section we consider system (2.16) and a related boundary value problem. Relevant ingredients can be specified as follows:

Ground domain. Let Ω be an *exterior domain* in $\mathbb{R}^2$, i.e. an open connected subset of the Euclidean plane whose complement is bounded.

Function spaces. Let $L^p(\Omega)$, $L^p_{\text{loc}}(\Omega)$, $C^\infty(\Omega)$ and $C_0^\infty(\Omega)$ have the usual significations. Define $W^{1,2}(\Omega)$ as the completion of $C^\infty(\Omega)$ under the norm given by

$$\| u \|^2_{W^{1,2}(\Omega)} = 16 \int_\Omega u^2 (x^2 + y^2 + 4)^{-2} \, dx \, dy + \int_\Omega |\nabla u|^2 \, dx \, dy. \tag{3.1}$$

Define $W_0^{1,2}(\Omega) =$ closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$ – the subset of $W^{1,2}(\Omega)$ consisting of those functions that vanish on $\partial\Omega$ in a generalized sense.

Observe that the measure $16(x^2 + y^2 + 4)^{-2} \, dx \, dy$, appearing in (3.1), can be thought of as the area element on the two-dimensional unit sphere $\mathbb{S}^2$, provided $\mathbb{S}^2$ is parametrized via a stereographic projection; recall also that $\int |\nabla u|^2 \, dx \, dy$ is invariant under conformal mappings. Hence $W^{1,2}(\Omega)$ can be identified with a space of standard Sobolev functions defined in some open subset of $\mathbb{S}^2$. More information on function spaces, which are involved throughout, can be found in [31].

THEOREM 3.1 *Assumptions*. (i) *Ω is essentially different from $\mathbb{R}^2$, i.e. area* $(\mathbb{R}^2 \setminus \Omega) > 0$. (ii) *$n$ is in $L^2(\Omega)$; $\partial n/\partial x$ and $\partial n/\partial y$ are in $L^2_{\text{loc}}(\Omega)$.* (iii) *j is any member of $W^{1,2}(\Omega)$.*

Assertion A solution pair to (2.16) *exists, enjoying the following properties.* (iv) *u and v belong to $W^{1,2}(\Omega)$.* (v) *u obeys the following boundary condition*

$$u \in j + W_0^{1,2}(\Omega), \tag{3.2}$$

i.e. equals j on $\partial\Omega$ in a generalized sense. (vi) *u and v are twice differentiable in a generalized sense and obey the following inequalities*

$$\begin{aligned} &\int_{\{(x,y):\, \text{dist}((x,y), \mathbb{R}^2 \setminus K) \geq r\}} \frac{|\nabla u|^2}{n^2 + |\nabla u|^2} \left| \nabla \nabla^T u \right|^2 \, dx \, dy \\ &\qquad \leq \text{Constant} \times \left[\int_K |\nabla n|^2 \, dx \, dy + r^{-2} \int_K \left(n^2 + |\nabla u|^2 \right) dx \, dy \right], \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\int_{\{(x,y):\, \text{dist}((x,y), \mathbb{R}^2 \setminus K) \geq r\}} \frac{|\nabla u|^4}{\left(n^2 + |\nabla u|^2 \right)^2} \left| \nabla \nabla^T v \right|^2 \, dx \, dy \\ &\qquad \leq \text{Constant} \times \left[\int_K |\nabla n|^2 \, dx \, dy + r^{-2} \int_K \left(n^2 + |\nabla u|^2 \right) dx \, dy \right] \end{aligned} \tag{3.4}$$

– here K is any nice compact subset of Ω, r is any positive number, and Constant stands for some absolute constant. (vii) *u obeys equation* (2.13) *almost everywhere in Ω.* (viii) *v is a viscosity solution to the following equation*

$$\begin{aligned} &\left[n^2 - R\left(\frac{|\nabla v|}{n} \right) v_y^2 \right] v_{xx} + 2R\left(\frac{|\nabla v|}{n} \right) v_x v_y v_{xy} + \left[n^2 - R\left(\frac{|\nabla v|}{n} \right) v_x^2 \right] v_{yy} \\ &\qquad - R\left(\frac{|\nabla v|}{n} \right) |\nabla v|^2 \, (\nabla \log n, \nabla v) = 0, \end{aligned} \tag{3.5}$$

$$R(\rho) = 1 \quad \textit{if } 0 \leq \rho \leq 1, \quad = \rho^{-4} \quad \textit{if } \rho > 1. \tag{3.6}$$

In particular, v is a viscosity solution to equation (2.14) *above in any open subset of* $\{(x, y) \in \Omega : |\nabla v(x, y)| > n(x, y)\}$, *and is a viscosity solution of the following equation*

$$\left(n^2 - v_y^2\right) v_{xx} + 2 v_x v_y v_{xy} + \left(n^2 - v_x^2\right) v_{yy} - |\nabla v|^2 (\nabla \log n, \nabla v) = 0 \tag{3.7}$$

in any open subset of $\{(x, y) \in \Omega : |\nabla v(x, y)| \leq n(x, y)\}$.

Remarks

(i) The *viscosity solutions*, mentioned in Theorem 3.1, can be defined as in [32] or [33, Chapter 10].

(ii) The following identity

$$\text{l.h.s. of (3.7)} = n^2 \left(\frac{1}{2}\ v_x \frac{\partial}{\partial x} + \frac{1}{2}\ v_y \frac{\partial}{\partial y} - \Delta v \right) \left[n^{-2} (v_x^2 + v_y^2) - 1 \right],$$

which ensues from either the content of appendices B and C or from an inspection, shows that any sufficiently smooth solution to the following *Eikonal equation*

$$v_x^2 + v_y^2 = n^2$$

obeys (3.7).

(iii) The function, which takes the value $\sqrt{1 + \rho^2} - 1$ at any nonnegative ρ, and the function, which takes the value $1 - \sqrt{1 - \rho^2}$ at any ρ such that $0 \leq \rho \leq 1$ and is $+\infty$ elsewhere, are *Young conjugate*. The *Euler–Lagrange equation* of the variational integral which takes the following values

$$\int \left[\sqrt{1 + n^{-2} (u_x^2 + u_y^2)} - 1 \right] n^2 \, \mathrm{d}x \, \mathrm{d}y$$

on suitable trial functions u, and the *Euler–Lagrange equation* of the variational integral which takes the following values

$$\int \left[1 - \sqrt{1 - n^{-2} (v_x^2 + v_y^2)} \right] n^2 \, \mathrm{d}x \, \mathrm{d}y$$

on suitable trial functions v, are a generalization of the *minimal surface equation* and *a special case of* (3.7), respectively.

(iv) In the case where $n \equiv 1$, the Legendre transformation given by

$$p = v_x(x, y), \ q = v_y(x, y), \ v(x, y) + V(p, q) = xp + yq,$$

turns equation (3.7) into the following one

$$(1 - p^2) \ \frac{\partial^2 V}{\partial p^2} - 2pq \ \frac{\partial^2 V}{\partial p \partial q} + (1 - q^2) \ \frac{\partial^2 V}{\partial q^2} = 0,$$

which is sometimes called *Buseman equation* – see [20], [34], [35].

4. Proofs of Lemmas 2.1 and 2.2

4.1. *Proof of Lemma 2.1*

Compare with ([37], Lemma A1). ■

4.2. *Proof of Lemma 2.2*

Items (i), (ii) and (iii) from Lemma 2.1 imply (i). They also ensure that the values of g_ε, g'_ε and g''_ε at any nonnegative σ result from the following set

$$\begin{aligned}&0 \le \rho < \infty, \quad f'_\varepsilon(\rho) = \sigma, \quad g_\varepsilon(\sigma) = \rho f'_\varepsilon(\rho) - f_\varepsilon(\rho),\\ &g'_\varepsilon(\sigma) = \rho, \quad f''_\varepsilon(\rho) g''_\varepsilon(\sigma) = 1.\end{aligned} \tag{4.1}$$

Items (ii), (iii), (iv) and (v) result from (4.1) and Lemma 2.1 via straightforward inspections. Note that (4.1) yields

$$\frac{g'_\varepsilon(\sigma)}{\sigma g''_\varepsilon(\sigma)} = \frac{\rho f''_\varepsilon(\rho)}{f'_\varepsilon(\rho)}.$$

Item (vi) is an obvious consequence of Lemma 2.1 and the very definition of Young conjugate. Note that $0 < g_\varepsilon(\sigma) - g(\sigma) < C_\varepsilon$ for every positive σ.

Since $[f'_\varepsilon(\rho)]^2(\varepsilon + \rho^2) \ge \rho^2(1+\rho^2)$ if $\rho \ge 0$, we have

$$g'_\varepsilon(\sigma) - g'(\sigma) \le \sqrt{\frac{\sigma^2-1}{2} + \sqrt{\left(\frac{\sigma^2-1}{2}\right)^2 + \varepsilon\sigma^2}} - \sqrt{\max\{1,\sigma^2\} - 1}$$

if $\sigma \ge 0$. Analysis shows that, as σ ranges from 0 to ∞, the right-hand side of the last inequality achieves its maximum if $\sigma = 1$. Therefore

$$g'_\varepsilon(\sigma) \le g'(\sigma) + \varepsilon^{1/4}$$

for every nonnegative σ. Since g'_ε is above g', item (vii) follows.

As $\varepsilon^{\varepsilon/2(1-\varepsilon)} f'_\varepsilon(\rho) f'_\varepsilon(\sqrt{\varepsilon}/\rho) = 1$ for every positive ρ, we have

$$\frac{\sqrt{\varepsilon}}{g'_\varepsilon(\sigma)} = g'_\varepsilon\left(\varepsilon^{\varepsilon/2(1-\varepsilon)}\sigma^{-1}\right) \tag{4.2}$$

for every positive σ. Equation (4.2) yields item (viii) easily. It also informs us that

$$\frac{\sqrt{\varepsilon}\sigma}{g'_\varepsilon(\sigma)} \to \varepsilon^{\varepsilon/2(1-\varepsilon)} \text{ as } \sigma \to 0, \qquad \frac{\sqrt{\varepsilon}\sigma}{g'_\varepsilon(\sigma)} \to \sqrt{\varepsilon} \quad \text{as} \quad \sigma \to \infty;$$

moreover,

$$\frac{\sqrt{\varepsilon}\sigma}{g'_\varepsilon(\sigma)} \to \sigma g'(1/\sigma) \quad \text{as} \quad \varepsilon \to 0$$

and σ is positive. Item (ix) is demonstrated.

Item (x) follows from (iv), (v) and (viii). Items (ii) and (iii) give

$$\sigma^{-2}\left[1-\frac{g'_\varepsilon(\sigma)}{\sigma g''_\varepsilon(\sigma)}\right]\to \varepsilon^{\varepsilon/(1-\varepsilon)} \quad \text{as} \quad \sigma\to 0, \quad \to 0 \quad \text{as} \quad \sigma\to\infty;$$

items (iv), (vii) and (ix) give

$$\sigma^{-2}\left[1-\frac{g'_\varepsilon(\sigma)}{\sigma g''_\varepsilon(\sigma)}\right]\to \left[1+(\sigma^2+1)(g'(\sigma))^2\right]^{-1} \quad \text{as} \quad \varepsilon\to 0$$

and σ is positive. Item (xi) ensues, owing to (x). (A calculus theorem, inferring uniform convergence from monotonicity and continuity, underlies the above arguments.) ■

5. Proof of Theorem 3.1

5.1. *Vanishing viscosity*

Let ε, f_ε, g_ε obey (2.17), (2.18), (2.19). Let u_ε and v_ε be in $W^{1,2}(\Omega)$ and such that

$$|\nabla v_\varepsilon| = nf'_\varepsilon\left(\frac{|\nabla u_\varepsilon|}{n}\right), \quad (\nabla u_\varepsilon, \nabla v_\varepsilon)=0. \tag{5.1}$$

According to the theory outlined in section 2, (5.1) is *uniformly elliptic*. We may propose u_ε be the solution to the following variational problem

$$\int_\Omega f_\varepsilon\left(\frac{|\nabla u|}{n}\right)n^2 \ \mathrm{d}x\,\mathrm{d}y = \text{minimum},$$
$$\text{under the condition: } u\in j+W_0^{1,2}(\Omega), \tag{5.2}$$

v_ε be given by the following Bäcklund transformation

$$\nabla v_\varepsilon = \frac{n}{|\nabla u_\varepsilon|}\, f'_\varepsilon\left(\frac{|\nabla u_\varepsilon|}{n}\right)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\nabla u_\varepsilon \tag{5.3}$$

and obey the following condition

$$\int_\Omega v_\varepsilon(x^2+y^2+4)^{-2}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{5.4}$$

Properties of f_ε and tools from the calculus of variations – see ([37], sections 3 and 4) – ensure that u_ε *exists*, *is unique*, *and is endowed with second-order derivatives in* $L^2_{\text{loc}}(\Omega)$;

the relevant *Euler–Lagrange equation* takes the following form

$$\varepsilon n^2\left(n^2+|\nabla u_\varepsilon|^2\right)\Delta u_\varepsilon+\left[|\nabla u_\varepsilon|^4+n^2\left(\frac{\partial u_\varepsilon}{\partial y}\right)^2\right]\frac{\partial^2 u_\varepsilon}{\partial x^2}-2n^2\frac{\partial u_\varepsilon}{\partial x}\frac{\partial u_\varepsilon}{\partial y}\frac{\partial^2 u_\varepsilon}{\partial x\partial y}$$
$$+\left[|\nabla u_\varepsilon|^4+n^2\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right]\frac{\partial^2 u_\varepsilon}{\partial y^2}+n|\nabla u_\varepsilon|^2(\nabla n,\nabla u_\varepsilon)=0. \tag{5.5}$$

The behavior of f_ε and properties of u_ε – which include both obeying the Euler–Lagrange equation and the membership to $W^{1,2}(\Omega)$ – ensure that the following differential form

$$\frac{n}{|\nabla u_\varepsilon|}\,f'\left(\frac{|\nabla u_\varepsilon|}{n}\right)\left(-\frac{\partial u_\varepsilon}{\partial y}\,\mathrm{d}x+\frac{\partial u_\varepsilon}{\partial x}\,\mathrm{d}y\right)$$

is closed and that the coefficients involved are square-integrable in Ω. On the other hand, we assumed that the complement of Ω is bounded and Ω is connected. We deduce successively that an appropriate winding number is 0, and the form in question is integrable. Therefore v_ε *exists in* $W^{1,2}(\Omega)$, *is unique and is endowed with second-order derivatives in* $L_{\mathrm{loc}}(\Omega)$.

Equation (5.3) implies successively

$$\begin{gathered}\nabla u_\varepsilon=\frac{n}{|\nabla v_\varepsilon|}\,g'_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\begin{bmatrix}0&1\\-1&0\end{bmatrix}\nabla v_\varepsilon,\\ |\nabla u_\varepsilon|=ng'_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right),\quad(\nabla u_\varepsilon,\nabla v_\varepsilon)=0,\\ \operatorname{div}\left\{\frac{n}{|\nabla v_\varepsilon|}\,g'_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\nabla v_\varepsilon\right\}=0.\end{gathered}\tag{5.6}$$

The latter equation can be recast as follows

$$\left[n^2-R_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\left(\frac{\partial v_\varepsilon}{\partial y}\right)^2\right]\frac{\partial^2 v_\varepsilon}{\partial x^2}+2\,R_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\frac{\partial v_\varepsilon}{\partial x}\frac{\partial v_\varepsilon}{\partial y}\frac{\partial^2 v_\varepsilon}{\partial x\partial y}$$
$$+\left[n^2-R_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\left(\frac{\partial v_\varepsilon}{\partial x}\right)^2\right]\frac{\partial^2 v_\varepsilon}{\partial y^2}-R_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)|\nabla v_\varepsilon|^2(\nabla\log n,\nabla v_\varepsilon)=0, \tag{5.7}$$

provided R_ε is given by

$$R_\varepsilon(\rho)=\rho^{-2}\left[1-\frac{\rho g''_\varepsilon(\rho)}{g'_\varepsilon(\rho)}\right]\quad\text{if }\rho>0,\quad R_\varepsilon(0)=\varepsilon^{\varepsilon/(1-\varepsilon)}. \tag{5.8}$$

Remark A suitable argument, which is omitted here for the sake of brevity, shows that v_ε is a solution to the following variational problem

$$\int_\Omega \left[g_\varepsilon\left(\frac{|\nabla v|}{n}\right) n^2 + \frac{\partial u_\varepsilon}{\partial y}\, v_x - \frac{\partial u_\varepsilon}{\partial x}\, v_y \right] \mathrm{d}x\,\mathrm{d}y = \text{minimum},$$

under the sole condition: $v \in W^{1,2}(\Omega)$.

5.2. *Bounds*

Inequalities (i) to (vi) below include the following ingredients: $K =$ a compact subset of Ω, whose interior is not empty and whose boundary is smooth enough; $r =$ any positive number; $\mathcal{K}(r) =$ the set of the interior points (x, y) of K such that the closed disk, center (x, y) and radius r, is contained in K; $p =$ any exponent larger than or equal to 1.

(i)

$$\int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x\,\mathrm{d}y \leq \text{(Constant independent of } \varepsilon).$$

(ii)

$$\int_{\mathcal{K}(r)} \frac{|\nabla u_\varepsilon|^2}{n^2 + |\nabla u_\varepsilon|^2} \left|\nabla\nabla^T u_\varepsilon\right|^2 \, \mathrm{d}x\,\mathrm{d}y \leq \int_K |\nabla n|^2 \, \mathrm{d}x\,\mathrm{d}y + r^{-2} \int_K \left(n^2 + |\nabla u_\varepsilon|^2\right) \mathrm{d}x\,\mathrm{d}y.$$

(iii)

$$|\nabla v_\varepsilon| \leq \sqrt{n^2 + |\nabla u_\varepsilon|^2}.$$

(iv)

$$\int_\Omega |\nabla v_\varepsilon|^2 \, \mathrm{d}x\,\mathrm{d}y \leq \int_\Omega \left(n^2 + |\nabla u_\varepsilon|^2\right) \mathrm{d}x\,\mathrm{d}y.$$

(v)

$$\left[\int_{\mathcal{K}(r)} |\nabla v_\varepsilon|^p \, \mathrm{d}x\,\mathrm{d}y\right]^{2/p} \leq \text{(Constant independent of } \varepsilon)$$

$$\times \left[\int_K |\nabla n|^2 \, \mathrm{d}x\,\mathrm{d}y + r^{-2} \int_K \left(n^2 + |\nabla u_\varepsilon|^2\right) \mathrm{d}x\,\mathrm{d}y\right].$$

(vi)

$$\int_{\mathcal{K}(r)} \left| \left(\nabla \frac{|\nabla u_\varepsilon|^2}{n^2 + |\nabla u_\varepsilon|^2}\, \nabla^T \right) v_\varepsilon \right|^2 \mathrm{d}x\,\mathrm{d}y \leq\ 20 \int_K |\nabla n|^2 \, \mathrm{d}x\,\mathrm{d}y + 18\, r^{-2} \int_K \left(n^2 + |\nabla u_\varepsilon|^2\right) \mathrm{d}x\,\mathrm{d}y.$$

Proof of (i) *and* (ii) Item (i) results from (5.2) and asymptotics of $vf\varepsilon$; (ii) results from (5.5). Details are in ([37], section 4.3). ■

Proof of (iii) *and* (iv) The former equation in (5.1), plus Lemma 2.1. ■

Proof of (v) *and* (vi) For notational convenience, we temporarily drop subscript ε's and denote u_ε and v_ε by u and v tout court.

Arguments from appendix B give

$$\nabla\sqrt{n^2+|\nabla u|^2}=\frac{1}{\sqrt{1+\rho^2}}\ \nabla n+\frac{\rho}{\sqrt{1+\rho^2}}(\nabla\nabla^T u)\begin{bmatrix}\cos\omega\\ \sin\omega\end{bmatrix},$$

hence we have

$$\left|\nabla\sqrt{n^2+|\nabla u|^2}\right|\le|\nabla n|+\frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}|\nabla\nabla^T u|.$$

Item (v) follows, via (ii) and (iii), and a Sobolev embedding theorem.

Equation (5.3), and formulas from appendix B give

$$\begin{aligned}&\left(\nabla\frac{|\nabla u|^2}{n^2+|\nabla u|^2}\ \nabla^T\right)v\cdot\begin{bmatrix}0&1\\-1&0\end{bmatrix}\\ &=[\varphi(\rho)-\rho\varphi'(\rho)]\ \nabla n\cdot[\cos\omega\ \sin\omega]+(\nabla\nabla^T u)\begin{bmatrix}\cos\omega&-\sin\omega\\ \sin\omega&\cos\omega\end{bmatrix}\\ &\times\begin{bmatrix}\varphi'(\rho)&0\\0&\varphi(\rho)/\rho\end{bmatrix}\begin{bmatrix}\cos\omega&\sin\omega\\-\sin\omega&\cos\omega\end{bmatrix},\end{aligned}$$

provided φ is defined by $\varphi(\rho)=\rho^2 f'_\varepsilon(\rho)/(1+\rho^2)$ for every nonnegative ρ. Items from Lemma 2.1 tell us that

$$0\le\rho\varphi'(\rho)-\varphi(\rho)\le 2\rho^2(1+\rho^2)^{-3/2},\quad \varphi'(\rho)\le 3\rho(1+\rho^2)^{-1/2}$$

for every nonnegative ρ. We infer

$$\left|\left(\nabla\frac{|\nabla u|^2}{n^2+|\nabla u|^2}\ \nabla^T\right)v\right|\le 4\cdot 3^{-3/2}|\nabla n|+3\frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}|\nabla\nabla^T u|.$$

Item (vi) follows, owing to (ii). ∎

5.3. *Convergence*

Let f and g be given by (2.8) and (2.9), respectively. Let u be the solution to the following variational problem

$$\int_\Omega f\left(\frac{|\nabla u|}{n}\right)n^2\ \mathrm{d}x\,\mathrm{d}y=\text{minimum},$$
$$\text{under the condition: } u\in j+W_0^{1,2}(\Omega). \tag{5.9}$$

As shown in ([36], section 2), u exists, is unique and enjoys the following properties. First, ∇u is in $(L^p_{\mathrm{loc}}(\Omega))^2$ for any p such that $1\le p<\infty$, and is differentiable in some generalized sense. Second, the following items hold as $\varepsilon\to 0$. (i) $u_\varepsilon\to u$

uniformly on every compact subset of Ω. (ii) $\nabla u_\varepsilon \to \nabla u$ strongly in $(L^p_{\mathrm{loc}}(\Omega))^2$, and weakly in $(L^2(\Omega))^2$. Third, u obeys (3.3), and satisfies equation (2.13) almost everywhere in Ω.

Condition (5.4), bounds from the previous subsection, and standard compactness theorems cause a function v to exist in $W^{1,2}(\Omega)$ and to enjoy the following properties. First, ∇v is in $(L^p_{\mathrm{loc}}(\Omega))^2$ for any p such that $1 \le p < \infty$; $|\nabla u|^2(n^2 + |\nabla u|^2)^{-1}\nabla^T v$ is endowed with a gradient in $(L^2_{\mathrm{loc}}(\Omega))^4$, and the latter obeys an appropriate analog of (3.4). Second, the following items hold as ε approaches 0 along a suitable sequence.

(iii) $v_\varepsilon \to v$ uniformly on every compact subset of Ω.
(iv) $\nabla v_\varepsilon \to \nabla v$ weakly, both in $(L^p_{\mathrm{loc}}(\Omega))^2$ and in $(L^2(\Omega))^2$.
(v) $|\nabla u_\varepsilon|^2(n^2 + |\nabla u_\varepsilon|^2)^{-1}\nabla^T v_\varepsilon \to |\nabla u|^2(n^2 + |\nabla u|^2)^{-1}\nabla^T v$ in $(L^p_{\mathrm{loc}}(\Omega))^2$.
We claim that
(vi) $|\nabla u|^2(n^2 + |\nabla u|^2)^{-1}\nabla v_\varepsilon \to |\nabla u|^2(n^2 + |\nabla u|^2)^{-1}\nabla v$ in $(L^p_{\mathrm{loc}}(\Omega))^2$ as ε approaches 0 along a suitable sequence.
(vii) $|\nabla v| \le \sqrt{n^2 + |\nabla u|^2}$.
(viii) v is endowed with generalized second-order derivatives, and obeys (3.4).

Proof of (vi) items (ii), (iv) and (v). ∎

Proof of (vii) Let K be any compact subset of Ω. The following inequalities

$$\int_K |\nabla v| \;\mathrm{d}x\,\mathrm{d}y$$
$$\le \limsup_{\varepsilon\to 0}\int_K |\nabla v_\varepsilon| \;\mathrm{d}x\,\mathrm{d}y \le \limsup_{\varepsilon\to 0}\int_K \sqrt{n^2 + |\nabla u_\varepsilon|^2}\;\mathrm{d}x\,\mathrm{d}y = \int_K \sqrt{n^2 + |\nabla u|^2}\;\mathrm{d}x\,\mathrm{d}y$$

result from the following arguments, respectively: either item (iii) or (iv), plus a classical semicontinuity theorem; bound (iii) from subsection 5.2; item (ii). Item (vii) ensues. ∎

Proof of (viii) The derivatives in question can be defined via the following formula

$$\frac{|\nabla u|^2}{n^2 + |\nabla u|^2}(\nabla\nabla^T v) = \left(\nabla \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\nabla^T\right) v - \left(\nabla \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\right)\cdot \nabla^T v. \tag{5.10}$$

The former term on the right-hand side of (5.10) is all right. Formulas from appendix B and (iii) from subsection 5.2 yield

$$n\nabla \frac{|\nabla u|^2}{n^2 + |\nabla u|^2} = \frac{2\rho}{(1+\rho^2)^2}\left\{(\nabla\nabla^T u)\begin{bmatrix}\cos\omega\\ \sin\omega\end{bmatrix} - \rho\nabla n\right\},$$
$$\frac{1}{2}\left|\left(\nabla \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\right)\cdot \nabla^T v\right| \le 2\cdot 3^{-3/2}|\nabla n| + \frac{|\nabla u|}{\sqrt{n^2 + |\nabla u|^2}}|\nabla\nabla^T u|.$$

Therefore, the latter term on the right-hand side of (5.10) is all right too. Item (viii) follows. ∎

Now we are in position to show that u and v obey (2.16). Let K be any compact subset in Ω. The former equation from (5.6) gives

$$\int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ |\nabla u_\varepsilon|\ \mathrm{d}x\,\mathrm{d}y = \int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ ng'_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\mathrm{d}x\,\mathrm{d}y. \tag{5.11}$$

Item (ii) implies

$$\int_K \frac{|\nabla u|^2}{\sqrt{n^2+|\nabla u|^2}}\ \mathrm{d}x\,\mathrm{d}y = \lim_{\varepsilon\to 0}\int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ |\nabla u_\varepsilon|\ \mathrm{d}x\,\mathrm{d}y. \tag{5.12}$$

We have

$$\int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ ng'_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\mathrm{d}x\,\mathrm{d}y \le \int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ ng'\left(\frac{|\nabla v_\varepsilon|}{n}\right)\mathrm{d}x\,\mathrm{d}y$$
$$+\sup\{g'_\varepsilon(\rho)-g'(\rho) : 0\le\rho<\infty\}\times\int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ n\ \mathrm{d}x\,\mathrm{d}y;$$

moreover,

$$\int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ ng'\left(\frac{|\nabla v_\varepsilon|}{n}\right)\mathrm{d}x\,\mathrm{d}y = \int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ ng'\left(\frac{|\nabla v|}{n}\right)\mathrm{d}x\,\mathrm{d}y + \text{a remainder},$$

$$|\text{remainder}|^2 \le \int_K \frac{|\nabla u|^2}{n^2+|\nabla u|^2}\ |\nabla v_\varepsilon-\nabla v|\ \mathrm{d}x\,\mathrm{d}y\times\int_K(2n+|\nabla v_\varepsilon|+|\nabla v|)\,\mathrm{d}x\,\mathrm{d}y.$$

The last line results from the following inequality

$$|g'(a)-g'(b)|\le\sqrt{|a-b|}\ \sqrt{2+a+b}\quad (a,b=\text{any positive numbers}),$$

and Cauchy–Schwarz inequality.

Consequently, Lemma 2.2, item (vi) and bound (v) from subsection 5.2 give

$$\liminf_{\varepsilon\to 0}\int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ ng'_\varepsilon\left(\frac{|\nabla v_\varepsilon|}{n}\right)\mathrm{d}x\,\mathrm{d}y \le \int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\ ng'\left(\frac{|\nabla v|}{n}\right)\mathrm{d}x\,\mathrm{d}y. \tag{5.13}$$

Formulas (5.11), (5.12) and (5.13) inform us that

$$\int_K \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\left[|\nabla u|-ng'\left(\frac{|\nabla v|}{n}\right)\right]\mathrm{d}x\,\mathrm{d}y\le 0.$$

On the other hand, (vii) implies

$$\frac{1}{\sqrt{n^2+|\nabla u|^2}}\left[|\nabla u| - ng'\left(\frac{|\nabla v|}{n}\right)\right]^2 \leq \frac{|\nabla u|}{\sqrt{n^2+|\nabla u|^2}}\left[|\nabla u| - ng'\left(\frac{|\nabla v|}{n}\right)\right].$$

We infer that

$$|\nabla u| - ng'\left(\frac{|\nabla v|}{n}\right) = 0.$$

Items (ii) and (iv), and the latter equation in (5.1) clearly imply $(\nabla u, \nabla v) = 0$.

System (2.16) is established. The last assertion of our theorem is a consequence of the following ingredients. First, equation (5.7). Second, formulas (3.6) and (5.8), and Lemma 2.2 – $R_\varepsilon(\rho)$ approaches $R(\rho)$ uniformly with respect to ρ, as ρ ranges in $[0,\infty[$ and $\varepsilon \to 0$. Third, the notion of viscosity solution and arguments from ([32], section 6). The proof is complete. ■

6. Additional remarks

Besides (2.17) and (2.18), other expressions of f_ε may serve the same purpose. For instance, one may retain (2.8), then let ε be any positive number and

$$f_\varepsilon(\rho) = f\left(\sqrt{\varepsilon^2+\rho^2}\right) - f(\varepsilon)$$

for every nonnegative ρ.

However, assertion (viii) from Theorem 3.1 is contingent upon the expression of f_ε that underlies the proof. For instance, if (2.17) is replaced by $\varepsilon =$ any positive number, and (2.18) is replaced by the following pair

$$f_\varepsilon(0) = 0,$$
$$f'_\varepsilon(\rho) = \frac{1}{2\sqrt{1+\varepsilon^2}}\left[(3+2\varepsilon^2)\frac{\rho}{\varepsilon} - \left(\frac{\rho}{\varepsilon}\right)^3\right] \quad \text{if } 0 \leq \rho < \varepsilon, \quad = \sqrt{1+\rho^2} \quad \text{if } \rho \geq \varepsilon,$$

then (3.6) must be replaced by

$$R(\rho) = 8\left[1 + 2\cos\left(\frac{2}{3}\arcsin\rho\right)\right]^{-3} \quad \text{if } 0 \leq \rho < 1, \quad = \rho^{-4} \quad \text{if } \rho \geq 1.$$

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Appendix A: notations

We let x and y denote rectangular coordinates in Euclidean plane $\mathbb{R}^2$, denote differentiations with respect to x and y either by $\partial/\partial x$ and $\partial/\partial y$ or by subscripts, and let

$$\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x$$

– the *Jacobian determinant* of u and v. Let the *gradient operator* be defined thus

$$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix},$$

and let div indicate the *divergence operator* – div acts on vector fields and stands for ∇^T. Let $\Delta = \operatorname{div} \nabla$ – the *Laplace operator* – and observe that

$$\nabla\nabla^T = \begin{bmatrix} \partial^2/\partial x^2 & \partial^2/\partial x\partial y \\ \partial^2/\partial x\partial y & \partial^2/\partial y^2 \end{bmatrix};$$

the *Hessian operator*.

We denote the scalar product of two vectors by parentheses and denote the absolute value of either a vector or a matrix by vertical bars – for instance, $(\nabla u, \nabla v)$ stands for $u_x v_x + u_y v_y$ whenever u and v are scalar fields; $|\nabla u|$ and $|\nabla\nabla^T u|$ stand for $\sqrt{u_x^2 + u_y^2}$ and $\sqrt{u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2}$, respectively.

Appendix B: subsidiary formulas

The *hodograph* of u is the range of ∇u. *Polar cordinates* ρ and ω in the hodograph of u may be defined either by

$$n\rho \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} = \nabla u, \tag{B.1}$$

or alternatively by

$$\rho = (1/n)\sqrt{u_x^2 + u_y^2}, \quad \cos\omega : u_x = \sin\omega : u_y.$$

Let X and Y be defined by either of the following equations

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}\nabla, \quad \nabla = \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} X + \begin{bmatrix} -\sin\omega \\ \cos\omega \end{bmatrix} Y. \tag{B.2}$$

Let h and k be defined by

$$-kX + hY = [X, Y]\mathrm{z} \tag{B.3}$$

– the *commutator* of X and Y.

Equations (B.1) and (B.3) show that

$$X = \frac{u_x}{|\nabla u|}\frac{\partial}{\partial x} + \frac{u_y}{|\nabla u|}\frac{\partial}{\partial y},$$

a differentiation along the *lines of steepest descent* of u; and that

$$Y = -\frac{u_y}{|\nabla u|}\frac{\partial}{\partial x} + \frac{u_x}{|\nabla u|}\frac{\partial}{\partial y},$$

a differentiation along the *level lines* of u. Equations (B.1), (B.2), (B.3), and (B.5) below inform us that

$$(1/h)\frac{\nabla u}{|\nabla u|}$$

is the *principal normal* to the *level lines* of u, and that

$$(1/k)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\nabla u}{|\nabla u|}$$

is the *principal normal* to the *lines of steepest descent* of u – in other words, h is a *signed curvature of the level lines* of u, and k is a *signed curvature of the lines of steepest descent* of u.

Equation (B.1) causes ρ and ω to satisfy

$$\nabla(n\rho)\cdot[\cos\omega \;\; \sin\omega] + n\rho\nabla\omega\cdot[-\sin\omega \;\; \cos\omega] = \nabla\nabla^T u,$$

therefore to satisfy

$$\nabla\rho = \frac{1}{n}(\nabla\nabla^T u)\begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} - \rho\nabla\log n, \quad \nabla\omega = \frac{1}{n\rho}(\nabla\nabla^T u)\begin{bmatrix} -\sin\omega \\ \cos\omega \end{bmatrix}. \tag{B.4}$$

Equations (B.1) to (B.4) give

$$\begin{aligned}
h &= -Y\omega, \quad k = X\omega, \\
h &= -|\nabla u|^{-3}\left\{u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2\right\}, \\
k &= |\nabla u|^{-3}\left\{(u_{yy} - u_{xx})u_xu_y + u_{xy}(u_x^2 - u_y^2)\right\}, \\
\nabla\omega &= -h\begin{bmatrix} -\sin\omega \\ \cos\omega \end{bmatrix} + k\begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}, \\
X\rho &= \Delta u/n + \rho(h - X\log n), \quad Y\rho = \rho(k - Y\log n), \\
\nabla\rho &= (\Delta u/n + k\rho)\begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix} + h\rho\begin{bmatrix} -\sin\omega \\ \cos\omega \end{bmatrix} - \rho\nabla\log n.
\end{aligned} \tag{B.5}$$

We also have

$$-k = \frac{\partial(u, 1/|\nabla u|)}{\partial(x, y)},$$

the *Jacobian determinant* of u and $1/|\nabla u|$; furthermore,

$$h = -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right), \quad k = \mathrm{div}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\frac{\nabla u}{|\nabla u|}\right).$$

Appendix C: features of equation (2.3)

Formulas (B.1) to (B.5) cause

$$\mathrm{div}\left\{\frac{n}{|\nabla u|}f'\left(\frac{|\nabla u|}{n}\right)\nabla u\right\}$$

to equal any of the following expressions:

$$(X - h)(nf'(\rho)),$$

$$f''(\rho)\Delta u + n[\rho f''(\rho) - f'(\rho)](h - X \log n),$$

$$\text{trace of } \left\{\begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}\begin{bmatrix} f''(\rho) & 0 \\ 0 & f'(\rho)/\rho \end{bmatrix}\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}(\nabla\nabla^T u)\right\}$$
$$- [\rho f''(\rho) - f'(\rho)]\left(\nabla n, \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}\right).$$

Therefore equation (2.3) can be recast in any of the following three forms

$$(X - h)(nf'(\rho)) = 0, \tag{C.1}$$

$$f''(\rho)\Delta u + n[\rho f''(\rho) - f'(\rho)](h - X \log n) = 0, \tag{C.2}$$

$$\begin{aligned}&\left[f''(\rho)(\cos)^2 + \frac{f'(\rho)}{\rho}(\sin)^2\right]u_{xx} + 2\left[f''(\rho) - \frac{f'(\rho)}{\rho}\right]\cos\ \sin\ u_{xy} \\ &\quad + \left[f''(\rho)(\sin)^2 + \frac{f'(\rho)}{\rho}(\cos)^2\right]u_{yy} = \left[f''(\rho) - \frac{f'(\rho)}{\rho}\right](\nabla \log n, \nabla u).\end{aligned} \tag{C.3}$$

Equations (C.1) and (C.2) bring geometric ingredients in evidence. Equation (C.1) shows that an ordinary differential equation governs

$$nf'\left(\frac{|\nabla u|}{n}\right)$$

along the lines of steepest descent of u. Equation (C.2) balances the curvature of the level lines of u and the curvature of certain geodesics. Recall that the geodesics in the following Riemannian metric

$$n\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$$

are called *rays* in geometical optics, and are characterized by the differential equation

$$(\nabla \log n, \text{ principal normal}) = 1.$$

Hence the value of

$$X \log n$$

at any point (x, y) equals the signed curvature at (x, y) of the ray which is tangent at (x, y) to a level line of u.

Equation (C.3) is a *quasi-linear* form of (2.3). Equation (C.3) is either *elliptic* or *degenerate elliptic*. *Degeneracies* do occur at critical points of u if $f'(0) \neq 0$; *uniform ellipticity* prevails if f satisfies (2.7).

Equations (2.3) and (C.3) result in

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla\omega = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \rho f''(\rho)/f'(\rho) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \nabla \log\rho + \nabla \log n \tag{C.4}$$

– a *Bäcklund transformation relating ρ and ω.* The ensuing equation

$$\operatorname{div}\left\{ \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \rho f''(\rho)/f'(\rho) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \nabla \log\rho \right\} + \Delta \log n = 0 \tag{C.5}$$

governs the gradient of any solution to equation (2.3).