

Generation of analytic semigroups and domain characterization for degenerate elliptic operators with unbounded coefficients arising in financial mathematics. Part II

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Abstract. This paper is devoted to study the generation of analytic semigroup for a family of degenerate elliptic operators (with unbounded coefficients) which includes well known operators arising in mathematical finance. The generation property is proved by assuming some compensation conditions among the coefficients and applying a suitable modification of the techniques developed in [16]. Using the results proved in [11] concerning the generation in the space $L^2(\mathbb{R}^d)$, we prove the generation results in $L^p(\mathbb{R}^d)$ for $p \in [1, +\infty]$. These results have several consequences in connection with the financial applications [3, 11].

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1. Introduction

In this paper we study the generation of analytic semigroups in $L^p(\mathbb{R}^d)$, with p in $[1, +\infty]$ for a family of degenerate elliptic operators with unbounded coefficients. These results can be employed to obtain existence, uniqueness and regularity estimates for the solutions of the associated (linear or semilinear) parabolic problems, through the well known theory of analytic semigroups (e.g. [12]). This has been done in [3] for the so-called “no-arbitrage” operators arising in pricing contingent

claims. We consider the following differential operator in \mathbb{R}^d

$$\mathcal{A}(x, D) = \sum_{i,j=1}^d \psi_i(x) \psi_j(x) a_{i,j}(x) D_{i,j} + \sum_{i=1}^d b_i(x) D_i - \gamma^2(x)$$

(denoted simply by \mathcal{A} in the following), where the weights $\psi_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 1, \dots, d$, are differentiable sublinear functions vanishing in not more than a negligible set Z , the matrix $\{a_{i,j}\}_{i,j=1,\dots,d}$ is bounded and uniformly elliptic, the coefficients $b_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 1, \dots, d$, are measurable functions and the function $\gamma : \mathbb{R}^d \mapsto \mathbb{R}$ is differentiable and locally square integrable with its first derivatives. The main difficulty to overcome here is the need of managing both the possible unboundedness of all the coefficients and the presence of zero's for those of the second order terms. In general these operators do not generate analytic nor strongly continuous semigroups (for instance the Ornstein-Uhlenbeck operator in one dimension, where $\psi = 1$, $b(x) = x$, $\gamma = 0$). However, we prove that choosing suitable compensation conditions on the coefficients this become possible.

In [11] we considered the operator \mathcal{A} defined in the whole space \mathbb{R}^d and we proved the generation of analytic semigroup in the space $L^2(\mathbb{R}^d)$, by an application of Hilbert space techniques. This was possible thanks to some preliminary a priori estimates, which are established by an appropriate choice of some compensation conditions among the coefficients of the operator. Then we obtained a characterization of the domain of the operator in $L^2(\mathbb{R}^d)$ by a localization procedure which was adapted to the growth rate of the weights ψ 's at infinity and close to the negligible set Z of all zeros of the ψ 's.

The aim of this paper is to pass from the $L^2(\mathbb{R}^d)$ case to the $L^p(\mathbb{R}^d)$ one, when $1 \leq p \leq +\infty$, by using a suitable modification of the Stewart's method [15, 16, 17]. Of course the fitting localization procedure become more complicated here, since it now depends also on the growth rate of the zero order coefficient γ .

A first result of our semigroup generation analysis is the existence of solutions of the no-arbitrage pricing problems, which is a central topic in the modern mathematical finance. However, a general existence result can be obtained via the probabilistic approach. So the main motivation to study these generation problems is based on the the question of regularity of solutions. This also in order to apply suitable numerical methods.

The paper is organized as follows. In Section 2 we introduce the notation and recall some results about the generation of analytic semigroup in the spaces $L^2(\mathbb{R}^d)$ proved in [11]. In Section 3 we prove the generation of analytic semigroup and we obtain the domain characterization in the spaces $L^p(\mathbb{R}^d)$, $2 < p < \infty$ and $L^\infty(\mathbb{R}^d)$. This result implies the generation of analytic semigroup in the spaces $L^p(\mathbb{R}^d)$ for $1 \leq p < 2$ using duality techniques.

2. Preliminary material and notation

Let Ω be an open subset of the d -dimensional Euclidean space \mathbb{R}^d . We denote by $C^\infty(\Omega)$ the linear space of all infinitely differentiable complex-valued functions on Ω , and we write $C_c^\infty(\Omega)$ for the linear submanifold of $C^\infty(\Omega)$ of all functions with compact support in Ω .

We denote by $W^{n,p}(\Omega)$ the usual *Sobolev space* (see, e.g., [1]), defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W^{n,p}(\Omega)} \equiv \sum_{|\alpha| \leq n} \left(\int_{\Omega} |D^\alpha u(x)|^p dx \right)^{1/p}$$

writing $L^p(\Omega)$ [resp. $H^n(\Omega)$] rather than $W^{0,p}(\Omega)$ [resp. $W^{n,2}(\Omega)$], and using the shorthands $W^{n,p}$ and L^p for $W^{n,p}(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$, respectively.

We denote by $W_{loc}^{n,p}$ [resp. L_{loc}^p , H_{loc}^n] the linear space of all measurable complex-valued functions on \mathbb{R}^d belonging to $W^{n,p}(\Omega)$ [resp. $L^p(\Omega)$, $H^n(\Omega)$] for every open subset Ω of \mathbb{R}^d having compact closure, and, for any fixed real-valued function $\xi \in W_{loc}^{n,p}$, we define the *weighted Sobolev space* $W_\xi^{n,p}$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the *weighted norm*

$$\|u\|_{W_\xi^{n,p}} \equiv \|\xi u\|_{W^{n,p}}.$$

It is well known that $W_\xi^{n,p}$ can also be defined as the space of all measurable functions u such that $\xi u \in W^{n,p}$. Similarly, for any choice of the functions α , β_i , $i = 1, \dots, d$, $\delta_{i,j}$, $i, j = 1, \dots, d$ belonging to L_{loc}^p , with $\text{essinf } |\alpha| > 0$, we introduce the weighted Sobolev spaces $W_{(\alpha,\beta)}^{1,p}$ and $W_{(\alpha,\beta,\delta)}^{2,p}$ defined as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the *weighted norm*

$$\|u\|_{W_{(\alpha,\beta)}^{1,p}} \equiv \|\alpha u\|_{L^p} + \sum_{i=1}^d \|\beta_i D_i u\|_{L^p}$$

and

$$\|u\|_{W_{(\alpha,\beta,\delta)}^{2,p}} \equiv \|\alpha u\|_{L^p} + \sum_{i=1}^d \|\beta_i D_i u\|_{L^p} + \sum_{i,j=1}^d \|\delta_{i,j} D_{i,j} u\|_{L^p}$$

respectively and we introduce also the spaces $W_{\xi,(\alpha,\beta)}^{1,p}$ [resp. $W_{\xi,(\alpha,\beta,\delta)}^{2,p}$] of all measurable functions u such that $\xi u \in W_{(\alpha,\beta)}^{1,p}$ [resp. $\xi u \in W_{(\alpha,\beta,\delta)}^{2,p}$], endowed with the norms

$$\|u\|_{W_{\xi,(\alpha,\beta)}^{1,p}} \equiv \|\xi u\|_{W_{(\alpha,\beta)}^{1,p}} \quad [\text{resp. } \|u\|_{W_{\xi,(\alpha,\beta,\delta)}^{2,p}} \equiv \|\xi u\|_{W_{(\alpha,\beta,\delta)}^{2,p}}].$$

Lastly we denote with L_ξ^p the space $W_\xi^{0,p}$.

Let us now consider the formal second-order differential operator

$$\mathcal{A}u \equiv \sum_{i,j=1}^d \psi_i(x) \psi_j(x) a_{i,j}(x) D_{i,j} u + \sum_{i=1}^d b_i(x) D_i u - \gamma^2(x) u. \quad (1)$$

Assumption 2.1. 1. For all $i, j = 1, \dots, d$, the coefficients $a_{i,j}(x)$ are bounded differentiable real-valued functions on \mathbb{R}^d such that $a_{i,j}(x) = a_{j,i}(x)$, and satisfying the strong ellipticity condition

$$\operatorname{Re} \sum_{i,j=1}^d a_{i,j}(x) z_i \bar{z}_j \geq E |z|^2 \quad \forall z \in \mathbb{C}^d,$$

for a suitable ellipticity modulus $E > 0$ independent of $x \in \mathbb{R}^d$;

2. for every $i = 1, \dots, d$, the coefficients $b_i(x)$ are measurable real-valued functions on \mathbb{R}^d , while $\gamma(x)$ is a real-valued function in L^2_{loc} with $\operatorname{ess\,inf}_1(\gamma) \geq 1$;
3. for all $i = 1, \dots, d$ the coefficients $\psi_i(x)$ are differentiable, and we have

$$|b_i(x)| \leq B_1 E^{1/2} \eta_{1,i}(x) |\psi_i(x)| \gamma(x) \quad \forall x \in \mathbb{R}^d, \quad (2)$$

$$|D_j(\psi_i(x) \psi_j(x) a_{i,j}(x))| \leq B_2 E^{1/2} \eta_{2,i,j}(x) |\psi_i(x)| \gamma(x) \quad \forall x \in \mathbb{R}^d,$$

for suitable constants B_1 and B_2 such that $B_1 + B_2 < 2$ and measurable positive functions $\eta_{1,i}(x)$ and $\eta_{2,i,j}(x)$ satisfying

$$\sum_{i=1}^d \eta_{1,i}^2(x) = d \sum_{i,j=1}^d \eta_{2,i,j}^2(x) = 1.$$

Assumption 2.1 allows us to reduce the analysis of the nonvariational case to the analysis of the variational one. Indeed, introducing the sesquilinear form $a(\cdot, \cdot)$ associated to the operator \mathcal{A} , given by

$$a(u, v) \equiv \hat{a}(u, v) - \int_{\mathbb{R}^d} \sum_{i,j=1}^d D_j(\psi_i(x) \psi_j(x) a_{i,j}(x)) D_i u(x) \bar{v}(x) dx,$$

for all $u \in H_{(\gamma, \psi)}^{1,p}$ $1 < p < \infty$ and $v \in H_{(\gamma, \psi)}^{1,q}$ where q is the conjugate of p , and writing

$$D(\mathcal{A}_p) \equiv \{u \in H_{(\gamma, \psi)}^{1,p} : \exists K(u) > 0 \text{ s.t. } |a(u, \varphi)| \leq K(u) \|\varphi\|_q \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d)\},$$

one can study the realization $\mathcal{A}_p : D(\mathcal{A}_p) \rightarrow L^p$ of \mathcal{A} by considering for each $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > 0$, the equation

$$(\lambda - \mathcal{A}_p)u = f. \quad (3)$$

In [11] the following results are proved:

Theorem 2.2. *Under Assumption 2.1, the operator $\mathcal{A}_2 : D(\mathcal{A}_2) \rightarrow L^2$ generates an analytic semigroup on L^2 .*

Moreover

¹This condition could be replaced by the seemingly more general $\operatorname{ess\,inf} \gamma > 0$, provided to employ a standard normalization procedure.

Corollary 2.3. *Under Assumption 2.1, for every solution $u \in D(\mathcal{A}_2)$ of (3), we have*

$$|\lambda|^{1/2} \|\gamma u\|_{L^2} \leq K' \|f\|_{L^2} \quad \text{and} \quad |\lambda|^{1/2} \|\psi_i D_i u\|_{L^2} \leq K'' \|f\|_{L^2},$$

for suitably chosen $K', K'' > 0$ independent of λ .

In order to obtain suitable estimates for the first-order derivatives we need the following assumption:

Assumption 2.4. Under 1. and 2. of Assumption 2.1, suppose in addition that γ is continuously differentiable and that, for all $i, j = 1, \dots, d$ and $x \in \mathbb{R}^d$, we have

$$|b_i(x)| \leq B_1 E^{1/2} \eta_{1,i}(x) |\psi_i(x)| \gamma(x),$$

$$|D_j(\psi_i(x) \psi_j(x) a_{i,j}(x))| \leq B_2 E^{1/2} \eta_{2,i,j}(x) |\psi_i(x)| \gamma(x),$$

$$2 |\psi_j(x) D_j \gamma(x) a_{i,j}(x)| \leq B_3 E^{1/2} \eta_{3,i,j}(x) \gamma^2(x),$$

for suitable constants B_1, B_2 and B_3 such that $B_1 + B_2 + B_3 < 2$ and suitable measurable functions $\eta_{1,i}(x), \eta_{2,i,j}(x)$ and $\eta_{3,i,j}(x)$ on \mathbb{R}^d satisfying $\sum_{i=1}^d \eta_{1,i}(x) = d \sum_{i,j=1}^d \eta_{2,i,j}^2(x) = d \sum_{i,j=1}^d \eta_{3,i,j}^2(x) = 1$.

We have

Theorem 2.5. *Under Assumption 2.4, both $\gamma^2 u$ and $\psi_i \gamma D_i u$ belong to L^2 , for every $i = 1, \dots, d$. More precisely, u belongs to $H_{(\gamma^2, \gamma \psi)}^1$, and*

$$\|u\|_{H_{(\gamma^2, \gamma \psi)}^1} \leq K \|f\|_{L^2}$$

holds true for a suitable $K > 0$. In particular, for every $i = 1, \dots, d$, also $b_i D_i u$ belongs to L^2 , and we have

$$\sum_{i=1}^d \|b_i D_i u\|_{L^2} \leq d^2 B_1 E^{1/2} \sum_{i=1}^d \|\psi_i \gamma D_i u\|_{L^2}.$$

Aiming to show that for all $i, j = 1, \dots, d$ the single summand $\psi_i(x) \psi_j(x) D_{i,j} u(x)$ belongs to L^2 , we need to strengthen our hypotheses on the coefficients $\psi_i(x)$'s. Therefore, having in mind our examples, we will assume then the negligibility of the set

$$Z \equiv \left\{ x \in \mathbb{R}^d : \psi_i(x) = 0, \text{ for some } i = 1, \dots, d \right\},$$

of all zeros of the $\psi_i(x)$'s, and the existence of a suitable countable covering of $\mathbb{R}^d - Z$ which allows us to perform a localization procedure. Such a covering will be made by rectangles of the type

$$R(x_0, r\psi) \equiv \left\{ x \in \mathbb{R}^d : |x_i - x_i^{(0)}| \leq r |\psi_i(x_0)|, \ i = 1, \dots, d \right\},$$

for $x_0 \equiv (x_1^{(0)}, \dots, x_d^{(0)}) \in \mathbb{R}^d - Z$ and $r > 0$.

Assumption 2.6. Under Assumption 2.4, suppose in addition that

- (i) for every $i = 1, \dots, d$, the differentiable function $\psi_i(x)$ belongs to H_{loc}^1 and the set Z is negligible;
- (ii) there exist real numbers $r_1 > 0$ and $L > 0$ such that for every $0 < r \leq r_1$ we can find a countable set $N_r \subset \mathbb{R}^d - Z$ such that
 - (a) the family $\mathcal{F}_1 \equiv \{R(x, r\psi)\}_{x \in N_r}$ is a covering of $\mathbb{R}^d - Z$;
 - (b) each rectangle of the family $\mathcal{F}_2 \equiv \{R(x, 2r\psi)\}_{x \in N_r}$ does not contain any element of Z and has a nonempty intersection with at most a fixed number n_0 of other rectangles of \mathcal{F}_2 itself;
 - (c) we have

$$\frac{1}{L} \leq \min_{i=1, \dots, d} \inf_{x \in R(x_0, 2r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \leq \max_{i=1, \dots, d} \sup_{x \in R(x_0, 2r\psi)} \frac{|\psi_i(x)|}{|\psi_i(x_0)|} \leq L,$$

for each $x_0 \in N_r$.

Remark 2.7. Assumption 2.6 is convenient for proving the characterization result. However, it is not easy to check it for given operators. In [11] it is proved that 2.6 is verified under a more treatable assumption, befitted with examples coming from financial mathematics. More precisely, it was shown that the Assumption 2.8 below implies 2.6.

Assumption 2.8. Under Assumption 2.4, suppose in addition that

- (i) Part (i) of Assumption 2.6 holds true;
- (ii) there exist $r_0 > 0$ (small), $R_0 > 0$ (large), and $\alpha > 0$ such that for every $x \in \left\{x \in \mathbb{R}^d : \text{dist}(x, Z) < r_0 \text{ or } \text{dist}(x, 0) > R_0\right\} \equiv \mathcal{D}(r_0, R_0)$ and every $i = 1, \dots, d$ we have

$$|D_j \psi_i(x)| \leq \alpha;$$

- (iii) for every $i = 1, \dots, d$ the function $\psi_i(x)$ depends only on the variable x_i .

Theorem 2.9. *Under Assumption 2.6, the functions*

$$\psi_i(x) \psi_j(x) D_{i,j} u(x)$$

belong to L^2 for all $i, j = 1, \dots, d$. More precisely, we have $u \in H_{(\gamma^2, \gamma\psi, \psi^2)}^2$ and the estimate

$$|\lambda| \|u\|_{L^2} + |\lambda|^{1/2} \|u\|_{H_{(\gamma, \psi)}^1} + \|u\|_{H_{(\gamma^2, \gamma\psi, \psi^2)}^2} \leq K \|f\|_{L^2}$$

holds true for a suitable $K > 0$.

Such results can be extended to the case of weighted Sobolev spaces:

Theorem 2.10. *Assume that Assumption 2.4 still holds true when replacing the first-order term of the operator \mathcal{A} with*

$$\sum_{i=1}^d b_i D_i + \sum_{i,j=1}^d \psi_i \psi_j a_{i,j} \left(\frac{D_i \xi}{\xi} D_j + \frac{D_j \xi}{\xi} D_i \right)$$

and the zero-order term with

$$-\gamma^2 + \sum_{i,j=1}^d \psi_i \psi_j a_{i,j} \left(\frac{D_{i,j} \xi}{\xi} + 2 \frac{D_i \xi D_j \xi}{\xi^2} \right) + \sum_{i=1}^d b_i \frac{D_i \xi}{\xi};$$

then the operator \mathcal{A} has a realization $\mathcal{A}_{2,\xi} : D(\mathcal{A}_{2,\xi}) \rightarrow L_\xi^2$ which generates an analytic semigroup on L_ξ^2 . Moreover, for each $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > 0$, the resolvent equation $\lambda u - \mathcal{A}_{2,\xi} u = f$ has, for every $f \in L^2$, a unique solution $u \in D(\mathcal{A}_{2,\xi})$, which satisfies the estimate

$$|\lambda| \|u\|_{L_\xi^2} + |\lambda|^{1/2} \|u\|_{H_{\xi,(\gamma,\psi)}^{1,2}} + \|u\|_{H_{\xi,(\gamma^2,\gamma\psi,\psi^2)}^{2,2}} \leq C \|f\|_{L_\xi^2},$$

for a suitable constant $C > 0$. In particular we have $D(\mathcal{A}_{2,\xi}) = H_{\xi,(\gamma^2,\gamma\psi,\psi^2)}^2$.

Remark 2.11. By using the Korn's argument it is possible to pass from generation results in the case of differentiable coefficients to similar result in the case of continuous coefficients. However, in such a general setting, it is impossible to find a general approach leading to this kind of results. This can be done in particular cases with different procedures.

3. Generation of analytic semigroups on $L^p(\mathbb{R}^d)$

In order to prove the L^p estimates we need an additional assumption. Actually we need the existence of a suitable countable covering of $\mathbb{R}^d - Z$ which allows us to perform a localization procedure. Let $r_{\gamma(x_0)}$ be the minimum between r and $\gamma(x_0)^{-1}$. Let

$$R(x_0, r_{\psi,\gamma}) \equiv \left\{ x \in \mathbb{R}^d : |x_i - x_i^{(0)}| \leq r_{\gamma(x_0)} |\psi_i(x_0)|, i = 1, \dots, d \right\},$$

for $x_0 \equiv (x_1^{(0)}, \dots, x_d^{(0)}) \in \mathbb{R}^d - Z$ and $r > 0$.

Assumption 3.1. Under Assumption 2.4, suppose in addition that

- (i) Part (i) of Assumption 2.6 holds true;
- (ii) There exist real numbers $r_1 > 0$ and $L > 0$ such that for every $0 < r \leq r_1$ we can find a countable set $N_r \subset \mathbb{R}^d - Z$ such that
 - (a) the family $\mathcal{F}_1 \equiv \{R(x, r_{\psi,\gamma})\}_{x \in N_r}$ is a covering of $\mathbb{R}^d - Z$;
 - (b) each rectangle of the family $\mathcal{F}_2 \equiv \{R(x, 2r_{\psi,\gamma})\}_{x \in N_r}$ does not contain any element of Z and has a nonempty intersection with at most a fixed number n_0 of other rectangles of \mathcal{F}_2 itself;
 - (c) we have

$$\frac{1}{L} \leq \inf_{x \in R(x_0, 2r_{\psi,\gamma})} \frac{|\gamma(x)|}{|\gamma(x_0)|} \leq \sup_{x \in R(x_0, 2r_{\psi,\gamma})} \frac{|\gamma(x)|}{|\gamma(x_0)|} \leq L,$$

for each $x_0 \in N_r$.

Remark 3.2. As done in Remark 2.7, it is possible to find stronger conditions that imply Assumption 3.1 and that are usually satisfied by the classical problems arising from financial mathematics.

Lemma 3.3. *Under Assumption 3.1, assume to have proved that for a $p \geq 2$ the solution u of the resolvent equation (3), related to some $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > 0$ belongs to $W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,p}$ and satisfies the estimate*

$$|\lambda| \|u\|_{L^p} + |\lambda|^{1/2} \|u\|_{W_{(\gamma, \psi)}^{1,p}} + \|u\|_{W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,p}} \leq C' \|f\|_{L^p} \quad (4)$$

for some $C' > 0$ and for each $f \in L^p$.

Let $q \in (p, p^*)$ and let $f \in L^q$. Assume that $u \in W_{(\gamma^2, \gamma\psi)}^{1,p}$ is a solution of (3). Then u satisfies the estimate

$$|\lambda| \|u\|_{L^q} + |\lambda|^{1/2} \|u\|_{W_{(\gamma, \psi)}^{1,q}} + \|u\|_{W_{(\gamma^2, \gamma\psi)}^{1,q}} \leq C \|f\|_{L^q}, \quad (5)$$

for every $\lambda \in \mathbb{C}$ whose real part is greater than a suitable fixed positive real number ω and for a suitable $C > 0$ independent of λ . Moreover if $p > d$ we may choose $q = \infty$.

Proof. For each $x_0 \equiv (x_1^0, \dots, x_d^0) \in \mathbb{R}^d - Z$, we also consider the change of variables $T_{x_0, \psi} : \mathbb{R}^d \mapsto \mathbb{R}^d$ defined by

$$T_{x_0, \psi}(x) \stackrel{\text{def}}{=} ((x_1 - x_1^0)/|\psi_1(x_0)|, \dots, (x_d - x_d^0)/|\psi_d(x_0)|),$$

of inverse

$$T_{x_0, \psi}^{-1}(x) = (x_1^0 + |\psi_1(x_0)| x_1, \dots, x_d^0 + |\psi_d(x_0)| x_d).$$

Furthermore, for every $r > 0$ we denote by $B(x_0, r)$ the d -dimensional ball centered at 0 with radius r and we write $B(x_0, r\psi)$ for the d -dimensional ellipsoid centered at x_0 with semiaxes $r|\psi_1(x_0)|, \dots, r|\psi_d(x_0)|$. Clearly

$$T_{x_0, \psi}(B(x_0, r\psi)) = B(0, r) \quad \text{and} \quad T_{x_0, \psi}^{-1}(B(0, r)) = B(x_0, r\psi).$$

Consider the change of variables

$$\tilde{u}(x) \stackrel{\text{def}}{=} (u \circ T_{x_0, \psi}^{-1})(x),$$

and let $\theta(x)$ be any smooth cut-off function such that

$$\begin{cases} \theta(t) = 1 & \text{if } t \in [0, 1] \\ \theta(t) = 0 & \text{if } t \in [2, +\infty[\end{cases},$$

for each $0 < r \leq r_0$ we can define a cut-off function on \mathbb{R}^d by setting

$$\theta_r(x) \stackrel{\text{def}}{=} \theta\left(\frac{|x|}{r}\right),$$

and we can consider the function

$$v(x) = \theta_r(x) \tilde{u}(x).$$

Clearly $v \in L^p$ satisfies the following equation

$$\begin{aligned} & \sum_{i,j=1}^d \frac{\tilde{\psi}_i(x)\tilde{\psi}_j(x)}{\psi_i(x_0)\psi_j(x_0)} \tilde{a}_{i,j}(x) D_{i,j}v(x) + \sum_{i=1}^d \frac{\tilde{b}_i(x)}{\psi_i(x_0)} D_i v(x) - \tilde{\gamma}^2 v(x) - \lambda v(x) \\ &= \theta_r(x) \tilde{f}(x) + \sum_{i=1}^d \frac{\tilde{b}_i(x)}{\psi_i(x_0)} D_i \theta_r(x) \tilde{u}(x) \\ &+ \sum_{i,j=1}^d \frac{\tilde{\psi}_i(x)\tilde{\psi}_j(x)}{\psi_i(x_0)\psi_j(x_0)} \tilde{a}_{i,j}(x) [\tilde{u}(x) D_{i,j} \theta_r(x) + D_i \theta_r(x) D_j \tilde{u}(x) + D_j \theta_r(x) D_i \tilde{u}(x)], \end{aligned}$$

whose right side \tilde{h} satisfies

$$\tilde{h} \in L^p(B(0, 2r)) \quad \text{and} \quad \tilde{h} = 0 \text{ on } \partial B(0, 2r).$$

On the other hand, by the assumption of the lemma the solution $v \in L^p$ satisfies the estimate

$$|\lambda| \|v\|_{L^p} + |\lambda|^{1/2} \|v\|_{W_{(\gamma, \psi)}^{1,p}} + \|v\|_{W_{(\gamma, \psi, \psi^2)}^{2,p}} \leq C_1 \|\tilde{h}\|_{L^p},$$

for a suitable $C > 0$ independent of λ .

Furthermore we can also prove that

$$\begin{aligned} \|\tilde{h}\|_{L^p} &\leq C_2 \left[\|\tilde{f}\|_{L^p(B(0, 2r))} + \frac{1}{r^2} \|\tilde{u}\|_{L^p(B(0, 2r))} + \frac{1}{r} \|D\tilde{u}\|_{L^p(B(0, 2r))} \right. \\ &\quad \left. + \frac{1}{r} \|\tilde{\gamma}\tilde{u}\|_{L^p(B(0, 2r))} \right], \end{aligned}$$

for a suitable $C_2 > 0$ where the last estimate comes from equation (2).

Combining the above estimates, we obtain

$$\begin{aligned} & |\lambda| \|\tilde{u}\|_{L^p(B(0, r))} + |\lambda|^{1/2} [\gamma(x_0) \|\tilde{u}\|_{L^p(B(0, r))} + \|D\tilde{u}\|_{L^p(B(0, r))}] \\ &+ \gamma^2(x_0) \|\tilde{u}\|_{L^p(B(0, r))} + \gamma(x_0) \|D\tilde{u}\|_{L^p(B(0, r))} + \|D^2\tilde{u}\|_{L^p(B(0, r))} \\ &\leq C \left[\|\tilde{f}\|_{L^p(B(0, 2r))} + \frac{1}{r^2} \|\tilde{u}\|_{L^p(B(0, 2r))} + \frac{1}{r} \|D\tilde{u}\|_{L^p(B(0, 2r))} \right. \\ &\quad \left. + \frac{1}{r} \gamma(x_0) \|\tilde{u}\|_{L^p(B(0, 2r))} \right], \end{aligned} \tag{6}$$

where we suppose r small enough in order that Assumption 3.1 holds. Now, if we take $q \in (p, p^*)$ and $\delta(q) = d/q - d/p + 1$, for every $\varepsilon > 0$ there exists (see [13, p. 66]) $C(\varepsilon) > 0$ such that

$$\|\tilde{u}\|_{L^q(B(0, r))} \leq \varepsilon r^{\delta(q)} \|D\tilde{u}\|_{L^p(B(0, r))} + C(\varepsilon) r^{\delta(q)-1} \|\tilde{u}\|_{L^p(B(0, r))}, \tag{7}$$

and

$$\|D\tilde{u}\|_{L^q(B(0, r))} \leq \varepsilon r^{\delta(q)} \|D^2\tilde{u}\|_{L^p(B(0, r))} + C(\varepsilon) r^{\delta(q)-2} \|\tilde{u}\|_{L^p(B(0, r))}. \tag{8}$$

Combining estimates (7) and (8) we have

$$\begin{aligned} & \frac{1}{r^2} \|\tilde{u}\|_{L^q(B(0,r))} + \frac{1}{r} \|D\tilde{u}\|_{L^q(B(0,r))} \\ & \leq \frac{1}{r^2} \left[\varepsilon r^{\delta(q)} \|D\tilde{u}\|_{L^p(B(0,r))} + C(\varepsilon) r^{\delta(q)-1} \|\tilde{u}\|_{L^p(B(0,r))} \right] \\ & \quad + \frac{1}{r} \left[\varepsilon r^{\delta(q)} \|D^2\tilde{u}\|_{L^p(B(0,r))} + C(\varepsilon) r^{\delta(q)-2} \|\tilde{u}\|_{L^p(B(0,r))} \right]. \end{aligned}$$

and, rearranging the terms,

$$\begin{aligned} & \frac{1}{r^2} \|\tilde{u}\|_{L^q(B(0,r))} + \frac{1}{r} \|D\tilde{u}\|_{L^q(B(0,r))} \\ & \leq C(\varepsilon) r^{\delta(q)-3} \|\tilde{u}\|_{L^p(B(0,r))} + \varepsilon r^{\delta(q)-2} \|D\tilde{u}\|_{L^p(B(0,r))} + \varepsilon r^{\delta(q)-1} \|D^2\tilde{u}\|_{L^p(B(0,r))}. \end{aligned}$$

Taking into account (6), the above estimate implies

$$\begin{aligned} & \frac{1}{r^2} \|\tilde{u}\|_{L^q(B(0,r))} + \frac{1}{r} \|D\tilde{u}\|_{L^q(B(0,r))} \\ & \leq \left[C(\varepsilon) r^{\delta(q)-3} (|\lambda| + \gamma(x_0)^2)^{-1} + \varepsilon r^{\delta(q)-2} (|\lambda|^{\frac{1}{2}} + \gamma(x_0))^{-1} + \varepsilon r^{\delta(q)-1} \right] \\ & \quad \left[\|\tilde{f}\|_{L^p(B(0,2r))} + \frac{1}{r^2} \|\tilde{u}\|_{L^p(B(0,2r))} + \frac{1}{r} \|D\tilde{u}\|_{L^p(B(0,2r))} + \frac{1}{r} \gamma(x_0) \|\tilde{u}\|_{L^p(B(0,2r))} \right]. \end{aligned}$$

By Hoelder inequality we get

$$\begin{aligned} & \frac{1}{r^2} \|\tilde{u}\|_{L^q(B(0,r))} + \frac{1}{r} \|D\tilde{u}\|_{L^q(B(0,r))} \\ & \leq \left[C(\varepsilon) r^{-2} (|\lambda| + \gamma(x_0)^2)^{-1} + \varepsilon r^{-1} (|\lambda|^{\frac{1}{2}} + \gamma(x_0))^{-1} + \varepsilon \right] \\ & \quad \left[\|\tilde{f}\|_{L^q(B(0,2r))} + \frac{1}{r^2} \|\tilde{u}\|_{L^q(B(0,2r))} + \frac{1}{r} \|D\tilde{u}\|_{L^q(B(0,2r))} + \frac{1}{r} \gamma(x_0) \|\tilde{u}\|_{L^q(B(0,2r))} \right]. \end{aligned}$$

Finally, if we take:

- $\varepsilon > 0$ a small number to be chosen later
- $r_0 = (|\lambda|^{\frac{1}{2}} + \gamma(x_0))^{-1}$
- $r = \alpha r_0$, where α is a number to be chosen later,

then we obtain

$$\begin{aligned} & \frac{1}{\alpha^2} (|\lambda|^{\frac{1}{2}} + \gamma(x_0))^2 \|\tilde{u}\|_{L^q(B(0,r_0))} + \frac{1}{\alpha} (|\lambda|^{\frac{1}{2}} + \gamma(x_0)) \|D\tilde{u}\|_{L^q(B(0,r_0))} \\ & \leq \left[C(\varepsilon) \frac{1}{\alpha^2} + \varepsilon \frac{1}{\alpha} + \varepsilon \right] \left[\|\tilde{f}\|_{L^q(B(0,2r_0))} + \frac{1}{\alpha^2} (|\lambda|^{\frac{1}{2}} + \gamma(x_0))^2 \|\tilde{u}\|_{L^q(B(0,2r_0))} \right. \\ & \quad \left. + \frac{1}{\alpha} (|\lambda|^{\frac{1}{2}} + \gamma(x_0)) \|D\tilde{u}\|_{L^q(B(0,2r_0))} + \frac{1}{\alpha} (|\lambda|^{\frac{1}{2}} + \gamma(x_0)) \gamma(x_0) \|\tilde{u}\|_{L^q(B(0,2r_0))} \right] \quad (9) \end{aligned}$$

So if $q < \infty$, by changing variable back, summing up to the covering and using assumption 3.1, we have

$$\begin{aligned} & \frac{1}{\alpha^2} (|\lambda| \|u\|_{L^q} + \|\gamma^2 u\|_{L^q} + \frac{1}{\alpha} (|\lambda|^{1/2} \|Du\|_{L^q} + \|\gamma Du\|_{L^q})) \\ & \leq 2L^4 n_0 \left[C(\varepsilon) \frac{1}{\alpha^2} + \varepsilon \frac{1}{\alpha} + \varepsilon \right] \left[\frac{1}{\alpha^2} (|\lambda| \|u\|_{L^q} + \|\gamma^2 u\|_{L^q}) \right. \\ & \quad \left. + \frac{1}{\alpha} (|\lambda|^{1/2} \|\gamma u\|_{L^q} + \|\gamma^2 u\|_{L^q}) + \frac{1}{\alpha} (|\lambda|^{1/2} \|Du\|_{L^q} + \|\gamma Du\|_{L^q}) + \|f\|_{L^q} \right]. \end{aligned}$$

The statement follows from the above estimate choosing $\varepsilon = \frac{1}{8L^4 n_0}$ and $\alpha = 4C(\varepsilon)L^4 n_0$. If $q = \infty$ the argument is easier. Actually by changing variable back and by localizing around the points where $u(x)$, $\gamma^2(x)u(x)$, $Du(x)$, $\psi(x)Du(x)$ and $\gamma(x)\psi(x)Du(x)$ attain the maximum, the result follows directly from (9) without using a covering argument. \square

Remark 3.4. Assume $f \in L^\infty$. Arguing as in [15] from the proof of the previous lemma it is possible to get an estimate for the second derivatives. Actually, starting from equation (6), by Hoelder inequality and choosing a suitable $r = \alpha(|\lambda|^{\frac{1}{2}} + \gamma(x_0))^{-1}$, one gets that for each $q > d$:

$$\sup_{x_0 \in \mathbb{R}^d} (|\lambda|^{\frac{1}{2}} + \gamma(x_0))^{\frac{d}{q}} \|\psi^2 D^2 u\|_{L^q(B(x_0, r_0 \psi))} \leq C \|f\|_{L^\infty}. \quad (10)$$

Remark 3.5. Estimate (4) implies the uniqueness of the solution of (3) in $W_{\gamma^2, \gamma\psi}^{1,q}$.

Lemma 3.6. *Under the assumptions of Lemma 3.3 we have that $u \in W_{\gamma^2, \gamma\psi, \psi^2}^{2,q}$ and*

$$\|\psi^2 D^2 u\|_{L^q} \leq C \|f\|_{L^q}.$$

Proof. Using the notation of the previous lemma we have that the function $\tilde{u}(x)$ satisfies the following equation in $B(x_0, 2r)$

$$\sum_{i,j=1}^d \frac{\tilde{\psi}_i(x) \tilde{\psi}_j(x)}{\psi_i(x_0) \psi_j(x_0)} \tilde{a}_{i,j}(x) D_{i,j} \tilde{u}(x) = - \sum_{i=1}^d \frac{\tilde{b}_i(x)}{\psi_i(x_0)} D_i \tilde{u}(x) - \tilde{\gamma}^2 \tilde{u}(x) + \tilde{f}(x) = \tilde{h}(x).$$

Noting that the second order differential operator

$$\tilde{u}(x) \mapsto \sum_{i,j=1}^d \frac{\tilde{\psi}_i(x) \tilde{\psi}_j(x)}{\psi_i(x_0) \psi_j(x_0)} \tilde{a}_{i,j}(x) D_{i,j} \tilde{u}(x)$$

is a strongly elliptic operator in $B(0, 2r)$ and, thanks to known regularity results (see [6] and also [9, Theor. 17.2, p.67], [10, 8.3, p.173]) it follows that $\tilde{u}(x) \in W^{2,q}(B(0, r))$ and

$$\|D^2 \tilde{u}\|_{L^q(B(x_0, 2r))} \leq C \|\tilde{h}\|_{L^q(B(x_0, 2r))}.$$

The statement follows changing variable back, summing up to the covering and applying the results of Lemma 3.3. \square

Theorem 3.7. *Assume that 3.1 holds. Then for each $2 \leq q \leq \infty$ the operator $\mathcal{A}_q : D(\mathcal{A}_q) \rightarrow L^q$ generates an analytic semigroup on L^p . Moreover if $q < \infty$ for every solution $u \in D(\mathcal{A}_q)$ of (3), we have*

$$|\lambda| \|u\|_{L^q} + |\lambda|^{\frac{1}{2}} \|u\|_{W_{(\gamma, \psi)}^{1,q}} + \|u\|_{W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,q}} \leq C \|f\|_{L^q} \quad (11)$$

for every $\lambda \in \mathbb{C}$ whose real part is greater than a suitable fixed positive real number ω and for a suitable $C > 0$ independent of λ .

If $q = \infty$ then for every solution $u \in D(\mathcal{A}_\infty)$ of (3), we have

$$|\lambda| \|u\|_{L^\infty} + |\lambda|^{\frac{1}{2}} \|u\|_{W_{(\gamma, \psi)}^{1,\infty}} + \|u\|_{W_{(\gamma^2, \gamma\psi)}^{1,\infty}} \leq C \|f\|_{L^\infty} \quad (12)$$

for every $\lambda \in \mathbb{C}$ whose real part is greater than a suitable fixed positive real number ω and for a suitable $C > 0$ independent of λ .

Proof. This Theorem is true for $p = 2$. Let $q \in (2, 2^*]$ where 2^* is the Sobolev exponent. Let u be a solution of (3) and assume that $f \in L^2 \cap L^q$. By applying Lemmata 3.3 and 3.6 we have that (11) holds.

If $f \in L^q$, we consider a sequence of functions $f_n \in L^2 \cap L^q$ converging to f in L^q . Then the sequence of associated solutions u_n is a Cauchy sequence in $W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,q}$ converging to a function u that is a solution of (3). Moreover this solution is unique by Remark 3.5. Therefore the result is proved for any $q \in (2, 2^*]$.

If $q \in (2^*, (2^*)^*)$ one can prove the result iterating the previous argument. After a finite number of steps the statement follows. \square

Using duality techniques one may prove a generation result in L^p with $1 \leq p < 2$.

Theorem 3.8. *Assume 3.1 holds. Let $1 \leq p < 2$, then the operator $\hat{\mathcal{A}}_p : D(\mathcal{A}_p) \rightarrow L^p$ generates an analytic semigroup on L^p .*

Proof. The statement follows if we show that a solution of (3) for $1 \leq p < 2$ exists and is unique, and moreover

$$|\lambda| \|u\|_{L^p} \leq C \|f\|_{L^p} \quad (13)$$

for every $\lambda \in \mathbb{C}$ whose real part is greater than a suitable fixed positive real number ω and for a suitable $C > 0$ independent of λ .

Assume first $f \in L^p \cap L^2$ so one has the existence of a solution of (3). Let $\hat{\mathcal{A}}$ the operator in variational form

$$\hat{\mathcal{A}}u \equiv \sum_{i,j=1}^d D_j(\psi_i(x)\psi_j(x)a_{i,j}(x)D_i u) + \sum_{i=1}^d b_i(x)D_i u - \gamma^2(x)u,$$

and let $\hat{\mathcal{A}}^*$ its adjoint.

Define the function space $H = \{g \in L^{p'} : \|g\|_{L^{p'}} = 1\}$ where p' is the conjugate of p . Then

$$\|u\|_{L^p} = \sup_{g \in H} \int u g dx.$$

For each $g \in H$, let v_g be the solution of the equation

$$(\hat{\mathcal{A}}^* - \lambda)v_g = g. \quad (14)$$

By Theorem 3.7

$$|\lambda| \|v_g\|_{L^{p'}} \leq C \|g\|_{L^{p'}} \leq C.$$

Therefore

$$\begin{aligned} \|u\|_{L^p} &= \sup_{g \in H} \int u(\hat{\mathcal{A}}^* - \lambda)v_g dx = \sup_{g \in H} \int (\hat{\mathcal{A}} - \lambda)uv_g dx = \sup_{g \in H} \int f v_g dx \\ &\leq C |\lambda|^{-1} \|f\|_{L^p}. \end{aligned}$$

Note that the duality argument implies directly the uniqueness of a variational solution of (3). We have only to prove the existence in the general case. If $f \in L^p$ we can find a sequence of function $f_n \in L^p \cap L^2$ converging to $f \in L^p$. By the previous estimate we have that the solutions u_n are a Cauchy sequence in L^p . Using the regularity of the coefficients, it is not difficult to prove that the functions u_n converge to the solution u of (3) that satisfies estimate (13). \square

Remark 3.9. If one assumes more regular coefficients, one may characterize the domain also in the case $p < 2$. Precisely for every solution $u \in D(\mathcal{A}_p)$ of (3) we have

$$|\lambda| \|u\|_{L^p} + |\lambda|^{\frac{1}{2}} \|u\|_{W_{(\gamma, \psi)}^{1,p}} + \|u\|_{W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,p}} \leq C \|f\|_{L^p} \quad (15)$$

or

$$|\lambda| \|u\|_{L^1} + |\lambda|^{\frac{1}{2}} \|u\|_{W_{(\gamma, \psi)}^{1,1}} + \|u\|_{W_{(\gamma^2, \gamma\psi)}^{1,1}} \leq C \|f\|_{L^1} \quad (16)$$

according to whether $1 < p < 2$ or $p = 1$, for every $\lambda \in \mathbb{C}$ whose real part is greater than a suitable fixed positive real number ω and for a suitable $C > 0$ independent of λ .

Briefly let $f \in L^p \cap L^2$ (the general case follows as before from standard density arguments). So for every g in H there exists a solution v_g of (14) and by Theorem 3.7 we have

$$|\lambda| \|v_g\|_{L^{p'}} + |\lambda|^{\frac{1}{2}} \|v_g\|_{W_{(\gamma, \psi)}^{1,p'}} + \|v_g\|_{W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,p'}} \leq C \quad (17)$$

if $1 < p < 2$ or

$$|\lambda| \|v_g\|_{L^\infty} + |\lambda|^{\frac{1}{2}} \|v_g\|_{W_{(\gamma, \psi)}^{1,\infty}} + \|v_g\|_{W_{(\gamma^2, \gamma\psi)}^{1,\infty}} \leq C$$

if $p = 1$.

Now (15) easily follows from (17) and estimates

$$\begin{aligned} \|u\|_{L^p} &\leq \|v_g\|_{L^{p'}} \|f\|_{L^p} \\ \|u\|_{W_{(\gamma, \psi)}^{1,p'}} &\leq \|v_g\|_{W_{(\gamma, \psi)}^{1,p'}} \|f\|_{L^p} \\ \|u\|_{W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,p}} &\leq \|v_g\|_{W_{(\gamma^2, \gamma\psi, \psi^2)}^{2,p'}} \|f\|_{L^p}. \end{aligned}$$

Similarly for (16).

Example. Consider the PDE for the price of a European contingent claim in the multifactor case, under the so-called no-arbitrage assumption

$$D_t v + \mathcal{A}v = 0,$$

where

$$\mathcal{A}v = \frac{1}{2} \text{Tr} ((\sigma \text{diag} x) (D_{i,j} v) (\sigma \text{diag} x)^*) + (r - \rho)(1 - \epsilon) \sum_{i=1}^d x_i D_i v - r v,$$

with terminal condition $v(x, T) = g(x)$ (see, e.g., [18]). Here $(\text{diag} x)$ is the diagonal matrix with the components of $x \equiv (x_1, \dots, x_d)$ on the main diagonal, r is the interest rate of a reference riskless asset in the market, σ is a given d -order matrix such that, writing σ^* for the transpose of σ , the matrix $\sigma^* \sigma$ is positive definite, $\rho \equiv \rho(x, t)$ is the dividend rate and $\epsilon \equiv \epsilon(x, t)$ is the tax rate on dividends. The solution $v \equiv v(x, t)$ represents the no-arbitrage price of a contingent claim having payoff $g \equiv g(x)$ at the expiration time T . In the case $d = 1$, $\rho = 0$, $\epsilon = 0$ and $g(x) = (x - E)^+$, where E is the maturity price of the option, we obtain the well-known Black and Scholes equation described in [4]. Also multifactor models, such as the ones appearing in [8], options on futures contracts, and swaps can be treated in our framework ([3, 18]), along with the example below.

Example. We consider here the structure model of interest rate derivatives. For the so-called affine single-term structure model the interest rate is modeled by the stochastic process $(X_t)_{t \geq 0}$ satisfying the differential equation

$$dX_t = (\alpha_1(t) + \alpha_2(t)X_t) dt + (\beta_1(t) + \beta_2(t)X_t) dW_t. \quad (18)$$

Suitably choosing the coefficients $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$ and $\beta_2(t)$, different term-structure models can be obtained. In particular two models fitting our framework can be obtained by choosing

1. $\alpha_1 = \alpha_2 = \beta_1 = 0$ [7]
2. $\beta_1 = 0$ [5].

The price of a zero-coupon bond maturing at date T is the solution of the Cauchy problem

$$D_t v + \mathcal{A}v = 0,$$

with the end terminal condition $v(x, T) = 1$, where

$$\mathcal{A}v = \frac{1}{2}(\beta_1 + \beta_2 x)^2 D_{x,x} v + (\alpha_1 + \alpha_2 x) D_x v.$$

We remark in addition that our results allow us to treat also multifactor models with time-dependent coefficients (see [8, 3]), and also semilinear perturbations of the above equations.

Example. The following equation, coming from nonlinear filtering, is considered in [2, 14]:

$$D_t = D_{x,x} + x D_x v - x^2 v, \quad t > 0, x \in \mathbb{R}$$

with initial condition $v(0, x) = g(x)$. It can be easily checked that the second order operator defined by the right-hand side of the above equation satisfies our assumptions.

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