A global in time result for an integro-differential parabolic inverse problem in the space of bounded functions

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Abstract

The problem we deal with arises in the theory of heat propagation in materials with memory. We consider the identification of both the relaxation kernel and the time dependence of the heat source for an integro–differential equation of parabolic type. We prove an existence and uniqueness theorem, *global in time*, for the abstract version of the problem and we give an application to the concrete case. The novelties of this work are: the choice of the functional setting (spaces of bounded functions with values in an interpolation space) and *existence and uniqueness results, global in time,* which are not easy to obtain for inverse problems.

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approach, heat equation with memory, identification of the memory kernel and of the heat source, global in time results.

1 Introduction

Integro–differential parabolic inverse problems constitute an important class since one of the classical models is represented by the heat equation for materials with memory. In several problems in physics and chemistry the heat equation with memory is coupled with one or more parabolic or hyperbolic equations, ruling further state variables, in particular in the theory of phase transition for materials with memory we find several models that have been recently studied from several view points: for instance, as direct problems in Hilbert spaces and as dynamical systems (long time behavior). We list here in the following, without claim of completeness, some papers and books in which one can find some models and results involving the heat equation: [1, 2, 3, 4, 5, 6, 7, 8, 18, 19, 20, 22, 26, 27, 28]. The list of references related to inverse integro-differential parabolic problems is long and we recall just some of the papers that share our strategies and methods, based on analytic semigroup theory, fixed point arguments and optimal regularity theorems, such techniques where first used in [24].

In the papers [12, 13, 16, 17] the authors study inverse problems for the heat equation with memory in non smooth domains and with memory kernels depending on time and on one space variable, in Hölder and Sobolev (of fractional order) spaces. In [15] has been proposed a unified approach for fully non linear problems which contains as particular cases the theory of combustion of materials with memory and some models of spread of disease, again in the setting of Hölder and Sobolev spaces. In [9, 10, 11, 21] the authors treat inverse problems for a model in population dynamics and for phase-field models in the Hölder setting.
In [14] the authors have used for the first time the same spaces used here but with no weights, so they obtain local in time results for a population dynamic model. In the recent and interesting paper [23], the authors consider an inverse problem for a phase field model and they prove an existence and uniqueness (under conditions) result global in time in a Hilbert space.

The novelties of this work are: the choice of the functional setting (spaces of bounded functions with values in an interpolation space) and *existence and uniqueness results, global in time,* which are not easy to obtain for inverse problems. We point out that the functional setting we have assumed leads to solutions of the problem which are less regular in time and more regular in space, compared with the existent literature on this topic. Our global in time result is proved for an abstract version of the problem and then we give an application to the concrete case.

For sake of simplicity we limit ourself to the heat equation with memory but the estimates we have obtained in Section 3 can be applied to get global in time results for more general linear models with convolution kernels.

We now formulate the problem we are going to investigate. Let Ω be an open bounded set in $\mathbb{R}^3$ with a suitably regular boundary (to be specified in the sequel) and $T > 0$. We can easily deduce the evolution equation for the temperature u by the continuity equation (for $(t,x) \in [0,T] \times \Omega$)

$$D_t u(t,x) + \mathrm{div}\, J(t,x) - f_0(t,x) = 0, \tag{1.1}$$

where the vector J denotes the density of heat flow per unit surface area per unit time and f_0 is the heat source per unit volume per unit time in Ω. We recall that the well known Fourier's law for materials with memory is given by

$$J(t,x) = -k\nabla u(t,x) - k' \int_0^t h(t-s)\nabla u(s,x)\, ds, \tag{1.2}$$

where we suppose that the diffusion coefficient k is a positive real number and for sake of simplicity the coefficient k' is assumed to be equal to 1. The convolution kernel h, which accounts for the thermal memory, is supposed to depend on time, only. To obtain the equation ruling the evolution of the temperature we replace (1.2) into the continuity equation (1.1) and we get

$$D_t u(t,x) = \mathrm{div}\, [k\nabla u(t,x) + \int_0^t h(t-s)\nabla u(s,x)\, ds] + f_0(t,x). \tag{1.3}$$

We now have to focus our attention on the fact that the memory kernel h *is not* a physical observable. This is the fact that motivates the inverse problem because h has to be considered unknown. A further difficulty arises when the heat source is placed in a fixed position of the material but the time dependence is not known. We suppose that

$$f_0(t,x) = f(t)g(x) \tag{1.4}$$

where $g(x)$ is a given datum, but $f(t)$ has to be considered a further unknown of the problem.
To determine, simultaneously, both the unknown functions h, f and the temperature u we need, for example, additional measurements on the temperature on suitable parts of the material, that can be represented in integral form (see (1.6)). We are now in position to state our problem.

Problem 1.1. (The Inverse Problem (IP)) *Determine the temperature* $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$*, the diffusion coefficient* k *and the functions* $h : [0,T] \longrightarrow \mathbb{R}$*,* $f : [0,T] \longrightarrow \mathbb{R}$ *satisfying system*

$$\begin{cases} D_t u(t,x) = k\Delta u(t,x) + \int_0^t h(t-s)\Delta u(s,x)\, ds + f(t)g(x), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \end{cases} \tag{1.5}$$

with the additional conditions

$$\int_\Omega u(t,x)\mu_j(dx) = G_j(t), \quad \forall t \in [0,T], \quad j = 1,2, \tag{1.6}$$

where g*,* u_0*,* G_1*,* G_2 *are given data and* μ_1 *and* μ_2 *are finite Borel measures in* $C(\overline{\Omega})$*.*

The plan of the paper is the following.

- Section 2 contains the functional setting of our problem and the statements of the main abstract and concrete results.

- In Section 3 we prove some fundamental estimates in weighted spaces which are of crucial importance to get global in time results.

- Section 4 contains a suitably equivalent reformulation of the Inverse Abstract Problem in terms of Volterra integral equations of the second kind.

- In Section 5 we prove, via fixed point argument, the main result of this paper, i.e. Theorem 2.5, which is a consequence of the preliminary results obtained in the previous sections.

2 Definitions and main results

The results that we are going to recall in this section hold in the case X is a Banach space with norm $\|\cdot\|$. Let $T > 0$, we denote by $C([0,T];X)$ the usual space of continuous functions with values in X, while we denote by $\mathcal{B}([0,T];X)$ the space of bounded functions with values in X. $\mathcal{B}([0,T];X)$ will be endowed with the sup–norm

$$\|u\|_{\mathcal{B}([0,T];X)} := \sup_{0\le t\le T} \|u(t)\| \tag{2.1}$$

and $C([0,T];X)$ will be considered a closed subspace of $\mathcal{B}([0,T];X)$.

We will use the notations $C([0,T];\mathbb{R}) = C([0,T])$ and $\mathcal{B}([0,T];\mathbb{R}) = \mathcal{B}([0,T])$.

By $\mathcal{L}(X)$ we denote the space of all bounded linear operators from X into itself equipped with the sup–norm, while $\mathcal{L}(X;\mathbb{R}) = X'$ is the space of all bounded linear functionals on X considered with the natural norm. We set $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

Definition 2.1. *Let $A : \mathcal{D}(A) \subseteq X \to X$, be a linear operator, possibly with $\overline{\mathcal{D}(A)} \neq X$. Operator A is said to be sectorial if it satisfies the following assumptions:*

- *there exist $\theta \in (\pi/2, \pi)$ and $\omega \in \mathbb{R}$, such that any $\lambda \in \mathbb{C}\setminus\{\omega\}$ with $|\arg(\lambda-\omega)| \le \theta$ belongs to the resolvent set of A.*

- *there exists $M > 0$ such that $\|(\lambda-\omega)(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \le M$ for any $\lambda \in \mathbb{C}\setminus\{\omega\}$ with $|\arg(\lambda-\omega)| \le \theta$.*

The fact that the resolvent set of A is not void implies that A is closed, so that $\mathcal{D}(A)$ endowed with the graph norm becomes a Banach space.

According to the definition of sectorial operator it is possible to define the semigroup $\{e^{tA}\}_{t\ge 0}$, of bounded linear operators in $\mathcal{L}(X)$, so that $t \to \mathrm{e}^{tA}$ is an analytic function from $(0,\infty)$ to $\mathcal{L}(X)$ satisfying for $k \in \mathbb{N}$ the relations

$$\frac{d^k}{dt^k} e^{tA} = A^k \mathrm{e}^{tA}, \quad t > 0, \tag{2.2}$$

and

$$Ae^{tA}x = e^{tA}Ax, \quad x \in \mathcal{D}(A), \quad t > 0. \tag{2.3}$$

Moreover, for $k \in \mathbb{N}_0$, $\varepsilon > 0$, there exist positive constants $M_{k,\epsilon}$, such that

$$\|t^k A^k e^{tA}\|_{\mathcal{L}(X)} \leq M_{k,\epsilon} e^{(\omega+\varepsilon)t}, \qquad t > 0. \tag{2.4}$$

For more details see for example [25, 29, 30]. Let us define the family of interpolation spaces (see [25] or [32]) $\mathcal{D}_A(\theta,\infty)$, $\theta \in (0,1)$, between $\mathcal{D}(A)$ and X by

$$\mathcal{D}_A(\theta,\infty) = \Big\{x \in X : \ |x|_{\mathcal{D}_A(\theta,\infty)} := \sup_{0<t<1} t^{1-\theta}\|Ae^{tA}x\| < \infty\Big\} \tag{2.5}$$

with the norm

$$\|x\|_{\mathcal{D}_A(\theta,\infty)} = \|x\| + |x|_{\mathcal{D}_A(\theta,\infty)}. \tag{2.6}$$

It is straightforward to verift that, if $T > 0$, $k \in \mathbb{N}$, $f \in \mathcal{D}_A(\theta,\infty)$, $t \in (0,T]$, then

$$\|A^k e^{tA} f\| \leq C(T,k)\|f\|_{\mathcal{D}_A(\theta,\infty)} t^{\theta-k}. \tag{2.7}$$

We also set

$$\mathcal{D}_A(1+\theta,\infty) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}_A(\theta,\infty)\}, \tag{2.8}$$

$\mathcal{D}_A(1+\theta,\infty)$ turns out to be a Banach space when equipped with the norm

$$\|x\|_{\mathcal{D}_A(1+\theta,\infty)} = \|x\| + \|Ax\|_{\mathcal{D}_A(\theta,\infty)}. \tag{2.9}$$

If $\theta \leq \xi \leq 1+\theta$ and $x \in \mathcal{D}_A(1+\theta,\infty)$,

$$\|x\|_{\mathcal{D}_A(\xi,\infty)} \leq C(\theta,\xi)\|x\|^{1+\theta-\xi}_{\mathcal{D}_A(\theta,\infty)}\|x\|^{\xi-\theta}_{\mathcal{D}_A(\theta+1,\infty)}, \tag{2.10}$$

with $C(\theta,\xi)$ independent of x. Optimal regularity results and the analytic semigroup theory are fundamental tools in the study of direct and inverse parabolic problems. Our strategy is to formulate the abstract version of the inverse problem in terms of a system of equivalent fixed point equations. Several optimal regularity results are at our disposal for such equivalent formulation. Consider the following optimal regularity result for the Cauchy Problem (CP):

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{2.11}$$

Let $A : \mathcal{D}(A) \to X$ be a sectorial operator and $\theta \in (0,1)$. In [25], [30] we can find the proofs of the following results:

Theorem 2.2. (Strict solution in spaces $\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$) *For any* $f \in C([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$, $u_0 \in \mathcal{D}_A(\theta+1,\infty)$ *the Cauchy problem (CP) admits a unique solution* $u \in C^1([0,T];X) \cap C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta+1,\infty))$.

Remark 2.3. We point out that the fundamental fact that leads us to prove global in time results is the idea of introducing the weighted spaces $\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))$. In the next section we give all the necessary estimates to prove them.

2.1 The main abstract result

Problem 2.4. (Inverse Abstract Problem (IAP)) *Let A be a sectorial operator in X. Determine a real number k and three functions u, h, f, such that*

$$
\begin{aligned}
&(\alpha) \quad \begin{cases} u \in C^2([0,T];X) \cap C^1([0,T];\mathcal{D}(A)), \\ D_t u \in \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty)), \qquad D_t^2 u \in \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty)), \end{cases} \\
&(\beta) \quad h \in C([0,T]), \\
&(\gamma) \quad f \in C^1([0,T]),
\end{aligned}
$$

satisfying system

$$
\begin{cases} u'(t) = kAu(t) + \int_0^t h(t-s)Au(s)ds + f(t)g, \qquad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.12}
$$

and the additional conditions:

$$
\langle u(t), \phi_j \rangle = G_j(t), \quad t \in [0,T], \quad j = 1,2, \tag{2.13}
$$

where ϕ_j ($j = 1,2$) are given bounded linear functionals on X, and G_j, u_0, g are given data.

We study the (IAP) under the following assumptions:

(H1) $\theta \in (0,1)$, X is a Banach space and A is a sectorial operator in X.

(H2) $u_0 \in \mathcal{D}_A(1+\theta+\varepsilon,\infty)$, for some $\varepsilon \in (0, 1-\theta)$.

(H3) $g \in \mathcal{D}_A(\theta+\varepsilon,\infty)$, for some $\varepsilon \in (0, 1-\theta)$.

(H4) $\phi_j \in X'$, for $j = 1,2$.

(H5) $G_j \in C^2([0,T])$, for $j = 1,2$.

(H6) We set

$$
M := \begin{pmatrix} < Au_0, \phi_1 > & < g, \phi_1 > \\ < Au_0, \phi_2 > & < g, \phi_2 > \end{pmatrix} \tag{2.14}
$$

and we suppose that the matrix M is invertible. We define

$$
M^{-1} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{2.15}
$$

Observe that owing to (H6) the system

$$
\begin{cases} k_0 < Au_0, \phi_1 > + f_0 < g, \phi_1 >= G_1'(0), \\ k_0 < Au_0, \phi_2 > + f_0 < g, \phi_2 >= G_2'(0), \end{cases} \tag{2.16}
$$

has a unique solution (k_0, f_0).

(H7) We require that

$$
k_0 > 0.
$$

(H8) $v_0 := k_0 A u_0 + f_0 g \in \mathcal{D}_A(1+\theta, \infty)$.

(H9) $< u_0, \phi_j >= G_j(0), \quad < v_0, \phi_j >= G_j'(0), \quad j = 1, 2.$

The main abstract result is the following:

Theorem 2.5. *Assume that conditions (H1)-(H9) are fulfilled. Then Problem 2.4 has a unique (global in time) solution* (k, u, h, f)*, with* $k \in \mathbb{R}^+$*, and* u*,* h*,* f *satisfying conditions* (α)*,* (β) *and* (γ)*.*

Proof. See Section 5. □

2.2 An application

We choose as reference space

$$X = C(\overline{\Omega}), \tag{2.17}$$

where Ω is an open bounded set in $\mathbb{R}^3$ with boundary of class $C^{2(1+\theta+\varepsilon)}$, for some $\theta \in (0, 1/2)$, $\varepsilon \in (0, (1/2) - \theta)$. We define

$$\begin{cases} \mathcal{D}(A) = \{u \in \bigcap_{1 \le p < +\infty} W^{2,p}(\Omega) \ : \Delta u \in C(\overline{\Omega}), \quad D_\nu u|_{\partial\Omega} = 0\}, \\ Au := \Delta u, \quad \forall u \in \mathcal{D}(A). \end{cases} \tag{2.18}$$

It was proved by Stewart (see [31]) that A is a sectorial operator in X. Then we recall the following characterizations concerning the interpolation spaces related to A (see [25]):

$$\mathcal{D}_A(\xi, \infty) = \begin{cases} C^{2\xi}(\overline{\Omega}), \quad \text{if} \quad 0 < \xi < 1/2, \\ \{u \in C^{2\xi}(\overline{\Omega}) : \ D_\nu u|_{\partial\Omega} = 0\}, \quad \text{if} \quad 1/2 < \xi < 1. \end{cases} \tag{2.19}$$

It is known that, if $0 < \xi \le \theta + \varepsilon$, we have

$$\mathcal{D}_A(1+\xi, \infty) = \{u \in C^{2(1+\xi)}(\overline{\Omega}) : D_\nu u|_{\partial\Omega} = 0\}. \tag{2.20}$$

So we consider the Inverse Problem 1.1 under the following assumptions:

(K1) Ω is an open bounded set in $\mathbb{R}^3$ with boundary of class $C^{2(1+\theta+\varepsilon)}$, for some $\theta \in (0, 1/2)$, $\varepsilon \in (0, (1/2) - \theta)$.

(K2) $u_0 \in C^{2(1+\theta+\varepsilon)}(\overline{\Omega}), \quad D_\nu u_0|_{\partial\Omega} = 0.$

(K3) $g \in C^{2(\theta+\varepsilon)}(\overline{\Omega})$.

(K4) For $j = 1, 2$, μ_j is a bounded Borel measure in $\overline{\Omega}$. We set, for $\psi \in X$,

$$< \psi, \phi_j >:= \int_{\Omega} \psi(x) \mu_j(dx). \tag{2.21}$$

(K5) Suppose that (H5) holds.

(K6) Suppose that (H6) holds with ϕ_j $(j = 1, 2)$ defined in (2.21) and A defined in (2.18).

(K7) Suppose that (H7) holds.

(K8) $v_0 := k_0\Delta u_0 + f_0 g \in C^{2(1+\theta)}(\overline{\Omega}), \quad D_\nu v_0|_{\partial\Omega} = 0.$

(K9) Suppose that (H9) holds.

Applying the main result of the paper (Theorem 2.5), we deduce the following:

Theorem 2.6. *Assume that conditions (K1)-(K9) are satisfied. Then the Inverse Problem 1.1 has a unique (global in time) solution* (k, u, h, f)*, such that*

$$\begin{aligned}
&u \in C^2([0,T]; C(\overline{\Omega})) \cap C^1([0,T]; \mathcal{D}(A)),\\
&D_t u \in \mathcal{B}([0,T]; C^{2(1+\theta)}(\overline{\Omega})), \quad D_t^2 u \in \mathcal{B}([0,T]; C^{2\theta}(\overline{\Omega})),\\
&h \in C([0,T]),\\
&f \in C^1([0,T]),\\
&k \in \mathbb{R}^+,
\end{aligned}$$

with A *defined in (2.18).*

3 The weighted spaces

In this section we introduce some crucial estimates that will be essential to obtain existence and uniqueness global in time of a solution for our inverse problem. Let $\lambda > 0$, $T > 0$, $\theta \in (0,1)$. If $f \in \mathcal{B}([0,T]; X)$, we set

$$\|u\|_{\mathcal{B}_\lambda([0,T];X)} := \sup_{0 \le t \le T} e^{-\lambda t}\|u(t)\|. \tag{3.1}$$

We denote by $C_\lambda([0,T]; X)$ the space $C([0,T]; X)$ equipped with the norm $\|\cdot\|_{\mathcal{B}_\lambda([0,T];X)}$. We will use the notations $C_\lambda([0,T]; \mathbb{R}) = C_\lambda([0,T])$ and $\mathcal{B}_\lambda([0,T]; \mathbb{R}) = \mathcal{B}_\lambda([0,T])$. We now prove some useful estimates in these weighted spaces for the solution of the Cauchy problem given by Theorem 2.2.

Theorem 3.1. *Let* $A : \mathcal{D}(A) \to X$ *be a sectorial operator,* $\theta \in (0,1)$*. Let us suppose that* $f \in C([0,T]; X) \cap \mathcal{B}([0,T]; \mathcal{D}_A(\theta, \infty))$*. Then the following estimates hold:*

$$\|e^{tA} * f\|_{C_\lambda([0,T];X)} \le \frac{C_0}{1+\lambda}\|f\|_{C_\lambda([0,T];X)}; \tag{3.2}$$

if $\theta \le \xi \le 1+\theta$*,*

$$\|e^{tA} * f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\xi,\infty))} \le \frac{C(\theta,\xi)}{(1+\lambda)^{1+\theta-\xi}}\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \tag{3.3}$$

with C_0 *and* $C(\theta,\xi)$ *independent of* f *and* λ*.*

Proof. In the following section the symbol C, sometimes with an index, will denote different positive constants.

We show (3.2). Owing to (2.4), we have

$$\begin{aligned} e^{-\lambda t}\|e^{tA} * f(t)\| &= e^{-\lambda t}\| \int_0^t e^{(t-s)A} f(s)\, ds\| \\ &\le M_0\|f\|_{C_\lambda([0,T];X)} \int_0^t e^{(\omega+1-\lambda)s}\, ds \\ &\le \tfrac{C_0}{1+\lambda}\|f\|_{C_\lambda([0,T];X)}. \end{aligned} \tag{3.4}$$

To get (3.3), owing to (2.10), it suffices to consider the cases $\xi = \theta$ and $\xi = 1+\theta$. If $\xi = \theta$, we have:

$$\begin{aligned} & e^{-\lambda t}|e^{tA} * f(t)|_{\mathcal{D}_A(\theta,\infty)} \\ &= e^{-\lambda t} \sup_{0<\tau<1} \tau^{1-\theta}\| \int_0^t Ae^{(t+\tau-s)A} f(s) ds\|. \end{aligned}$$

If $t > s > 0$ and $\tau \in (0,1)$, we have

$$\begin{aligned} e^{-\lambda t}\tau^{1-\theta}\|Ae^{(t+\tau-s)A}f(s)\| &\le e^{-\lambda t}|e^{(t-s)A}f(s)|_{\mathcal{D}_A(\theta,\infty)} \\ &\le C_1 e^{-\lambda t}|f(s)|_{\mathcal{D}_A(\theta,\infty)} \\ &\le C_1 e^{-\lambda(t-s)}\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \end{aligned}$$

so we get

$$\begin{aligned} e^{-\lambda t}|e^{tA} * f(t)|_{\mathcal{D}_A(\theta,\infty)} &\le C_1\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \int_0^t e^{-\lambda(t-s)} ds \\ &\le \frac{C_2}{\lambda}\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}. \end{aligned}$$

Next, we consider the case $\xi = 1+\theta$. We start by estimating $A(e^{tA} * f)$ in $C_\lambda([0,T];X)$:

$$\begin{aligned} e^{-\lambda t}\|A(e^{tA} * f)(t)\| &\le e^{-\lambda t}\int_0^t \|Ae^{(t-s)A}f(s)\|\, ds \\ &\le C_1\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \int_0^t \tfrac{e^{-\lambda(t-s)}}{(t-s)^{1-\theta}}\, ds. \end{aligned} \tag{3.5}$$

If $\lambda > 0$, we observe that

$$\begin{aligned} \int_0^t \tfrac{e^{-\lambda(t-s)}}{(t-s)^{1-\theta}} ds &= \int_0^t \tfrac{e^{-\lambda s}}{s^{1-\theta}} ds \\ &\le \int_0^{+\infty} e^{-\lambda s} s^{\theta-1} ds \\ &= \lambda^{-\theta}\, \Gamma(\theta). \end{aligned}$$

So, we have obtained

$$\|A(e^{tA} * f)\|_{\mathcal{B}_\lambda([0,T];X)} \le \frac{C}{(1+\lambda)^\theta}\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}. \tag{3.6}$$

It remains to estimate $A(e^{tA} * f)$ in $\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))$; if $\lambda > 0$, $\tau \in (0,1)$, $0 < s < t \le T$, we have

$$\begin{aligned} e^{-\lambda t}\tau^{1-\theta}\|A^2 e^{(\tau+t-s)A}f(s)\| &\le e^{-\lambda t}\tau^{1-\theta}\|A^2 e^{(\tau+t-s)A}\|_{\mathcal{L}(\mathcal{D}_A(\theta,\infty);X)}\|f(s)\|_{\mathcal{D}_A(\theta,\infty)} \\ &\le e^{-\lambda(t-s)}\tau^{1-\theta}\|A^2 e^{(\tau+t-s)A}\|_{\mathcal{L}(\mathcal{D}_A(\theta,\infty);X)}\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &\le Ce^{-\lambda(t-s)}\tfrac{\tau^{1-\theta}}{(\tau+t-s)^{2-\theta}}\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}. \end{aligned}$$

So we get the estimates

$$\begin{aligned} & e^{-\lambda t}|A(e^{tA} * f)(t)|_{\mathcal{D}_A(\theta,\infty)} \\ & = e^{-\lambda t} \sup_{0<\tau<1} \tau^{1-\theta} \|Ae^{\tau A} A \int_0^t e^{(t-s)A} f(s) ds\| \\ & \leq C\|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \sup_{0<\tau<1} \tau^{1-\theta} \int_0^t \frac{e^{-\lambda s}}{(\tau+s)^{2-\theta}} ds \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \tau^{1-\theta} \int_0^t \frac{e^{-\lambda s}}{(\tau+s)^{2-\theta}} ds &= \int_0^{t/\tau} \frac{e^{-\lambda \tau s}}{(1+s)^{2-\theta}} ds \\ &\leq \int_0^{+\infty} \frac{1}{(1+s)^{2-\theta}} ds. \end{aligned}$$

This concludes the proof. □

Theorem 3.2. *Let $\lambda \geq 0$, $h \in C([0,T])$ and $u \in C([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$. Then there exists $C > 0$, independent of T, λ, h, u, such that, if*

$$h * u(t) := \int_0^t h(t-s)u(s)ds, \tag{3.8}$$

$h * u \in C([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$ and satisfies the following estimate:

$$\|h * u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \leq C \min\left\{ \begin{array}{l} T\|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \\ (1+\lambda)^{-1} \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))}, \\ (1+\lambda)^{-1} \|h\|_{\mathcal{B}([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \end{array} \right\}. \tag{3.9}$$

Proof. If $0 \leq t \leq T$, we have

$$\begin{aligned} \|e^{-\lambda t}(h * u)(t)\| &\leq \int_0^t e^{-\lambda(t-s)} |h(t-s)| e^{-\lambda s} \|u(s)\| ds \\ &\leq \min\Big\{ T\|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];X)}, \\ &\int_0^t e^{-\lambda s} ds \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}([0,T];X)}, \\ &\int_0^t e^{-\lambda s} ds \|h\|_{\mathcal{B}([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];X)} \Big\} \\ &\leq C \min\Big\{ T\|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];X)}, \\ &(1+\lambda)^{-1} \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}([0,T];X)}, \\ &(1+\lambda)^{-1} \|h\|_{\mathcal{B}([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];X)} \Big\}. \end{aligned}$$

Moreover, if $\tau \in (0,1)$, we have

$$\begin{aligned} e^{-\lambda t} \|\tau^{1-\theta} A e^{\tau A}(h * u)(t)\| &\leq \int_0^t e^{-\lambda(t-s)} |h(t-s)| e^{-\lambda s} \|\tau^{1-\theta} A e^{\tau A} u(s)\| ds \\ &\leq \min\{ T\|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \\ &\int_0^t e^{-\lambda s} ds \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))}, \\ &\int_0^t e^{-\lambda s} ds \|h\|_{\mathcal{B}([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \} \\ &\leq C \min\{ T\|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty)}, \\ &(1+\lambda)^{-1} \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))}, \\ &(1+\lambda)^{-1} \|h\|_{\mathcal{B}([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \}. \end{aligned}$$

These estimates imply the conclusions immediately. □

Theorem 3.3. *Let $f \in \mathcal{B}([0,T];X)$. Assume that $f(0) = 0$ and suppose that f is continuous in 0. Then*

$$\lim_{\lambda\to+\infty} \|f\|_{\mathcal{B}_\lambda([0,T];X)} = 0.$$

In particular, if h and u satisfy the assumptions of Theorem 3.2, then

$$\lim_{\lambda\to+\infty} \|h * u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} = 0.$$

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be such that, if $0 \le t \le \delta$, then $\|f(t)\| \le \varepsilon$. So, if $0 \le t \le \delta$, we have $e^{-\lambda t}\|f(t)\| \le \varepsilon$. If $\delta \le t \le T$,

$$e^{-\lambda t}\|f(t)\| \le e^{-\delta\lambda}\|f\|_{\mathcal{B}([0,T];X)} \le \varepsilon$$

if λ is sufficiently large. If h and u satisfy the conditions of Theorem 3.2, we have, from Theorem 3.2, for $0 < \delta \le T$,

$$\|h * u\|_{\mathcal{B}([0,\delta];\mathcal{D}_\theta(A))} \le C\delta\|h\|_{\mathcal{B}([0,T])}\|u\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))},$$

so that the first part of the statement is applicable. □

4 An equivalent fixed point system

In this section we reformulate Problem 2.4 in terms of the equivalent nonlinear fixed point system (4.10). In proving Theorem 4.2 we find out a set of regularity and compatibility conditions on the data that makes the inverse problem well posed. Moreover, starting from the equivalent fixed point system we obtain existence and uniqueness results for system (IAP) via the Contraction Principle.

To this aim, we start by introducing some notations. We set

$$A_0 := k_0 A. \tag{4.1}$$

As $k_0 > 0$, see (H7), A_0 is a sectorial operator in X. Next, we set, for $t \in [0,T]$,

$$\overline{h}_1(t) := a_{11}G_1''(t) + a_{12}G_2''(t), \tag{4.2}$$

$$\overline{w}_1(t) := a_{21}G_1''(t) + a_{22}G_2''(t), \tag{4.3}$$

$$\overline{v}(t) := e^{tA_0}v_0. \tag{4.4}$$

We immediately observe that $\overline{h}_1$ and $\overline{w}_1$ belong to $C([0,T])$ and, owing to Theorem 2.2, $\overline{v} \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$. Next, we define, again for $t \in [0,T]$,

$$\overline{h}(t) := \overline{h}_1(t) - k_0 a_{11} < A\overline{v}(t), \phi_1 > -k_0 a_{12} < A\overline{v}(t), \phi_2 >, \tag{4.5}$$

$$\overline{w}(t) := \overline{w}_1(t) - k_0 a_{21} < A\overline{v}(t), \phi_1 > -k_0 a_{22} < A\overline{v}(t), \phi_2 > . \tag{4.6}$$

Of course, $\overline{h}$ and $\overline{w}$ belong to $C([0,T])$. Finally, we introduce the following (nonlinear) operators, defined for every $(v,h,w) \in [C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))] \times C([0,T]) \times C([0,T])$:

$$\mathcal{R}_1(v,h,w)(t) := e^{tA_0} * (hAu_0 + wg + h * Av)(t), \tag{4.7}$$

$$\begin{aligned} \mathcal{R}_2(v,h,w)(t) \quad := \quad & -k_0[a_{11} < A\mathcal{R}_1(v,h,w)(t), \phi_1 > + a_{12} < A\mathcal{R}_1(v,h,w)(t), \phi_2 >] \\ & -a_{11} < h * A\overline{v}(t), \phi_1 > -a_{12} < h * A\overline{v}(t), \phi_2 > \\ & -a_{11} < h * A\mathcal{R}_1(v,h,w)(t), \phi_1 > -a_{12} < h * A\mathcal{R}_1(v,h,w)(t), \phi_2 >, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \mathcal{R}_3(v,h,w)(t) \quad := \quad & -k_0[a_{21} < A\mathcal{R}_1(v,h,w)(t), \phi_1 > + a_{22} < A\mathcal{R}_1(v,h,w)(t), \phi_2 >] \\ & -a_{21} < h * A\overline{v}(t), \phi_1 > -a_{22} < h * A\overline{v}(t), \phi_2 > \\ & -a_{21} < h * A\mathcal{R}_1(v,h,w)(t), \phi_1 > -a_{22} < h * A\mathcal{R}_1(v,h,w)(t), \phi_2 > . \end{aligned} \tag{4.9}$$

We observe that $\mathcal{R}_2$ and $\mathcal{R}_3$ are well defined, because

$$\mathcal{R}_1(v,h,w) \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty)).$$

Moreover, $\mathcal{R}_2(v,h,w)$ and $\mathcal{R}_3(v,h,w)$ both belong to $C([0,T])$.
Now we can introduce the following problem:

Problem 4.1. *Determine three functions v, h, w, such that*

$$\begin{array}{ll} (\alpha') & v \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}(1+\theta,\infty)), \\ (\beta') & h \in C([0,T]), \\ (\gamma') & w \in C([0,T]), \end{array}$$

satisfying system

$$\left\{ \begin{array}{lcl} v & = & \overline{v} + \mathcal{R}_1(v,h,w), \\ h & = & \overline{h} + \mathcal{R}_2(v,h,w), \\ w & = & \overline{v} + \mathcal{R}_3(v,h,w). \end{array} \right. \tag{4.10}$$

Theorem 4.2. (Equivalence) *Let A be a sectorial operator in the Banach space X and $\theta \in (0,1)$. Let us assume that the data g, u_0, ϕ_j and G_j $(j = 1,2)$ satisfy the conditions (H1)-(H9) . Suppose that k, u, f, h satisfy Problem 2.4, with $k \in \mathbb{R}$ and u, f, h fulfilling the regularity conditions (α), (β), (γ). Then*

(I) $k = k_0$;

(II) the triplet (v,h,w), where $v = u'$, $w = f'$ satisfies the conditions (α'), (β'), (γ') and solves Problem 4.1;

(III) conversely, if (v,h,w), with the above regularity, is a solution of the Problem 4.1, then the triplet (u,h,f), where $u = u_0 + 1 * v$, $f = f_0 + 1 * w$, satisfies the regularity conditions (α), (β), (γ) and solves Problem 2.4 with $k = k_0$.

Proof. Let (u,h,f) satisfy the regularity conditions (α), (β), (γ) and solve Problem 2.4, with $k \in \mathbb{R}$. Applying ϕ_j $(j = 1,2)$ to the first equation in (2.12) and, taking $t = 0$, we get

$$k < Au_0, \phi_j > + f(0) < g, \phi_j > = G_j'(0).$$

Employing (H6), we immediately obtain

$$k = k_0, \qquad f(0) = f_0. \tag{4.11}$$

So (I) is proved. Now we set

$$v := u', \qquad w := f'.$$

Then $v \in C^1([0,T];X) \cap C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$, $v' \in \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$, $w \in C([0,T])$. In particular, (v,h,w) satisfies (α'), (β'), (γ'). Differentiating the first equation in (2.12) and, using (4.11) and (H8), we obtain

$$\begin{cases} v'(t) = k_0 Av(t) + h(t)Au_0 + h * Av(t) + w(t)g, \quad t \in [0,T], \\ v(0) = v_0. \end{cases} \tag{4.12}$$

Applying ϕ_j $(j = 1,2)$ to the first equation in (4.12), we get, for $t \in [0,T]$,

$$h(t) < Au_0, \phi_j > + w(t) < g, \phi_j >= G_j''(t) - k_0 < Av(t), \phi_j > - < h * Av(t), \phi_j > . \tag{4.13}$$

It follows that

$$\begin{aligned} h(t) &= \overline{h}_1(t) - k_0[a_{11} < Av(t), \phi_1 > + a_{12} < Av(t), \phi_2 >] \\ &\quad - a_{11} < h * Av(t), \phi_1 > - a_{12} < h * Av(t), \phi_2 >, \end{aligned} \tag{4.14}$$

$$\begin{aligned} w(t) &= \overline{w}_1(t) - k_0[a_{21} < Av(t), \phi_1 > + a_{22} < Av(t), \phi_2 >] \\ &\quad - a_{21} < h * Av(t), \phi_1 > - a_{22} < h * Av(t), \phi_2 > . \end{aligned} \tag{4.15}$$

Now we apply the variation of parameter formula to (4.12), to get

$$\begin{aligned} v(t) &= \overline{v}(t) + e^{tA_0} * (hAu_0 + h * Av + wg)(t) \\ &= \overline{v}(t) + \mathcal{R}_1(v,h,w)(t), \end{aligned} \tag{4.16}$$

hence, from (4.14)-(4.15), we obtain

$$h = \overline{h} + \mathcal{R}_2(v,h,w),$$

$$w = \overline{w} + \mathcal{R}_3(v,h,w).$$

So (v,h,w) satisfies Problem 4.1. On the other hand, assume that v, h, w satisfy conditions (α'), (β'), (γ') and system (4.10). As $v_0 \in \mathcal{D}_A(1+\theta,\infty)$ and $hAu_0 + h * Av + wg \in C([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$, we have that $v \in C^1([0,T];X) \cap C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$ and solves system (4.12). From (4.10), we obtain also that (4.14), (4.15) and (4.13) are satisfied. Applying ϕ_j $(j = 1,2)$ to the first equation in (4.12), we get

$$< v'(t), \phi_j >= G_j''(t), \qquad t \in [0,T]. \tag{4.17}$$

Now we set $u := u_0 + 1 * v$ and $f := f_0 + 1 * w$. Then, owing to (H2), u satisfies (α), while f satisfies (γ). Observe also that the first equation in (4.12) can be written in the form

$$u'' = D_t(k_0 Au + h * Au + fg). \tag{4.18}$$

As $u'(0) = v_0$ and $f(0) = f_0$, from (H8) we obtain that system (2.12) is satisfied, with $k = k_0$. Finally, applying (H9) and (4.17), we can conclude that even (2.13) is satisfied. □

5 Proof of Theorem 2.5

To prove Theorem 2.5, we shall employ the following:

Lemma 5.1. *Assume that conditions (H1)-(H9) are satisfied.*

Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ be the operators defined in (4.7), (4.8), (4.9), respectively. Then there exists $C > 0$ such that, $\forall \lambda \geq 0$, $\forall v, v_1, v_2 \in C([0,T]; \mathcal{D}(A)) \cap \mathcal{B}([0,T]; \mathcal{D}_A(1+\theta, \infty))$, $\forall\, h, h_1, h_2 \in C([0,T])$, $\forall w, w_1, w_2 \in C([0,T])$,

$$\begin{aligned} \|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} &\leq C[\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &+ (1+\lambda)^{-\varepsilon}(\|h\|_{\mathcal{B}_\lambda([0,T])} + \|w\|_{\mathcal{B}_\lambda([0,T])}) \\ &+ (1+\lambda)^{-1}(\|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])} + \|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) \\ &+ \|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}]; \end{aligned} \tag{5.1}$$

$$\begin{aligned} \|\mathcal{R}_1(v_1,h_1,w_1) - {} & \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & \leq C[\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])}\|v_1-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & +\|h_2-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & +(1+\lambda)^{-\varepsilon}(\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} + \|w_1-w_2\|_{\mathcal{B}_\lambda([0,T])}) \\ & +(1+\lambda)^{-1}\|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}]; \end{aligned} \tag{5.2}$$

and, if $i \in \{2,3\}$,

$$\begin{aligned} \|\mathcal{R}_i(v,h,w)\|_{\mathcal{B}_\lambda([0,T])} \leq {} & C(\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & +(1+\lambda)^{-1}\|h\|_{\mathcal{B}_\lambda([0,T])} \\ & +\|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}); \end{aligned} \tag{5.3}$$

$$\begin{aligned} & \|\mathcal{R}_i(v_1,h_1,w_1) - \mathcal{R}_i(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T])} \\ & \leq C(\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & +(1+\lambda)^{-1}\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} + \|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v_1,h_1,w_1)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & +\|h_2-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}). \end{aligned} \tag{5.4}$$

Proof. Owing to (H2)-(H3) and Theorem 3.1 (applied replacing θ with $\theta+\varepsilon$ and ξ with $1+\theta$), we have, for some $C_1 > 0$, independent of λ, h, w,

$$\|e^{tA_0} * (hAu_0 + wg)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \leq C_1(1+\lambda)^{-\varepsilon}(\|h\|_{\mathcal{B}_\lambda([0,T])} + \|w\|_{\mathcal{B}_\lambda([0,T])}). \tag{5.5}$$

Moreover, employing again Theorem 3.1 and Theorem 3.2, we have

$$\begin{aligned} & \|e^{tA_0} * (h * Av)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & \leq C_2\|h * Av\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & \leq C_2(\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} + \|(h-\overline{h}) * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & +\|\overline{h} * A(v-\overline{v})\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} + \|(h-\overline{h}) * A(v-\overline{v})\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & \leq C_3[\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} + (1+\lambda)^{-1}(\|A\overline{v}\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))}\|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])} \\ & +\|\overline{h}\|_{\mathcal{B}([0,T])}\|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) + \|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}], \end{aligned} \tag{5.6}$$

with C_2 and C_3 independent of λ, v, h. So (5.1) follows from (5.5) and (5.6). Concerning (5.2), we have

$$\begin{aligned} \|e^{tA_0} * ((h_1 - h_2)Au_0 + & (w_1 - w_2)g)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & \leq C_1(1+\lambda)^{-\varepsilon}(\|h_1 - h_2\|_{\mathcal{B}_\lambda([0,T])} + \|w_1 - w_2\|_{\mathcal{B}_\lambda([0,T])}). \end{aligned} \tag{5.7}$$

Next, always using Theorems 3.1 and 3.2, we have

$$\begin{aligned} \|e^{tA_0} * (h_1 * Av_1 - & h_2 * Av_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & \leq C_2\|h_1 * Av_1 - h_2 * Av_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & \leq C_2(\|(h_1 - h_2) * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & +\|\overline{h} * A(v_1 - v_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & +\|(h_1 - h_2) * A(v_1 - \overline{v})\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & +\|(h_2 - \overline{h}) * A(v_1 - v_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}) \\ & \leq C_4[(1+\lambda)^{-1}(\|A\overline{v}\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))} \\ & \|h_1 - h_2\|_{\mathcal{B}_\lambda([0,T])} + \|\overline{h}\|_{\mathcal{B}([0,T])}\|v_1 - v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) \\ & +\|h_1 - h_2\|_{\mathcal{B}_\lambda([0,T])}\|v_1 - \overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) \\ & +\|h_2 - \overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v_1 - v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))})], \end{aligned} \tag{5.8}$$

with C_4 independent of λ, v_1,v_2, h_1, h_2. So (5.2) follows from (5.7) and (5.8). The formulas (5.3) and (5.4) can be shown analogously. Concerning (5.3), it is convenient to observe that

$$h * A\mathcal{R}_1(v,h,w) = (h - \overline{h}) * A\mathcal{R}_1(v,h,w) + \overline{h} * A\mathcal{R}_1(v,h,w),$$

while (5.4) follows from the identity

$$\begin{aligned} h_1 * A\mathcal{R}_1(v_1,h_1,w_1) - & h_2 * A\mathcal{R}_1(v_2,h_2,w_2) \\ & = (h_1 - h_2) * A\mathcal{R}_1(v_1,h_1,w_1) \\ & +(h_2 - \overline{h}) * A[\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)] \\ & +\overline{h} * A[\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)]. \end{aligned} \tag{5.9}$$

□

Proof of Theorem 2.5. Let $\lambda \geq 0$. We set

$$Y(\lambda) := (C_\lambda([0,T];\mathcal{D}(A)) \cap \mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty)) \times C_\lambda([0,T])^2, \tag{5.10}$$

endowed with the norm

$$\|(v,h,w)\|_{Y(\lambda)} := \max\{\|v\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}, \|h\|_{\mathcal{B}_\lambda([0,T])}, \|w\|_{\mathcal{B}_\lambda([0,T])}\}, \tag{5.11}$$

with $v \in C_\lambda([0,T];\mathcal{D}(A)) \cap \mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))$, $h \in C_\lambda([0,T])$, $w \in B_\lambda([0,T])$, with the norm (5.11) we have that $Y(\lambda)$ becomes a Banach space. Let $\lambda \geq 0$, $\rho > 0$ and set

$$Y(\lambda,\rho) := \{(v,h,w) \in Y(\lambda) : \|(v,h,w) - (\overline{v},\overline{h},\overline{w})\|_{Y(\lambda)} \leq \rho\}. \tag{5.12}$$

Then, for every $\rho > 0$, $Y(\lambda,\rho)$ is a closed subset of $Y(\lambda)$.

Now we introduce the following operator N: if $(v,h,w) \in Y(\lambda)$, we set

$$N(v,h,w) := (\overline{v} + \mathcal{R}_1(v,h,w), \overline{h} + \mathcal{R}_2(v,h,w), \overline{w} + \mathcal{R}_3(v,h,w)), \tag{5.13}$$

clearly N is a nonlinear operator in $Y(\lambda)$.
Now we show that Problem 2.4 has a solution (k,u,h,f), with the regularity properties (α), (β), (γ).
Applying Theorem 4.2, we are reduced to look for a solution (v,h,w) of system (4.10), satisfying the conditions (α'), (β'), (γ'). This is equivalent to look for a fixed point of N in $Y(\lambda)$, for some $\lambda \geq 0$.
First step : We show that there exists $\overline{\rho}_1 > 0$, such that, if $0 < \rho \leq \overline{\rho}_1$, there exists $\lambda_1(\rho) \geq 0$, so that, for $\lambda \geq \lambda_1(\rho)$, $N(Y(\lambda,\rho)) \subseteq Y(\lambda,\rho)$.
In fact, if $(v,h,w) \in Y(\lambda,\rho)$, we have, applying (5.1) and (5.3),

$$\begin{aligned} \|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} &\leq C[\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &+(1+\lambda)^{-\varepsilon}(2\rho + \|\overline{h}\|_{C([0,T])} + \|\overline{w}\|_{C([0,T])}) \\ &+2(1+\lambda)^{-1}\rho + \rho^2] \\ &\leq C_1[\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &+(\rho+1)(1+\lambda)^{-\varepsilon} + \rho^2], \end{aligned}$$

for $i \in \{2,3\}$,

$$\begin{aligned} \|\mathcal{R}_i(v,h,w)\|_{\mathcal{B}_\lambda([0,T])} &\leq C_2\{(1+\rho)[\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &+(\rho+1)(1+\lambda)^{-\varepsilon} + \rho^2] \\ &+(1+\lambda)^{-1}(\rho + \|\overline{h}\|_{C([0,T])})\} \\ &\leq C_3(1+\rho)[\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &+(\rho+1)(1+\lambda)^{-\varepsilon} + \rho^2], \end{aligned}$$

with C_1, C_2, C_3 independent of λ and ρ. Now we choose $\overline{\rho}_1 > 0$, such that, if $0 < \rho \leq \overline{\rho}_1$,

$$\max\{\ C_1\rho^2,\ C_3(1+\rho)\rho^2\ \} < \rho. \tag{5.14}$$

Then, owing to Theorem 3.3, for any $\rho \in (0,\overline{\rho}_1]$, there exists $\lambda(\rho) \geq 0$, such that, if $\lambda \geq \lambda(\rho)$ and $(v,h,w) \in Y(\lambda,\rho)$

$$\begin{aligned} \|N(v,h,w) - (\overline{v},\overline{h},\overline{w})\|_{Y(\lambda)} = \max\{ \quad &\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}, \\ &\|\mathcal{R}_2(v,h,w)\|_{\mathcal{B}_\lambda([0,T])}, \\ &\|\mathcal{R}_3(v,h,w)\|_{\mathcal{B}_\lambda([0,T])} \quad \} \\ &\leq \rho. \end{aligned}$$

Second step : We show that there exists $\overline{\rho}_2 > 0$, such that, if $0 < \rho \leq \overline{\rho}_2$, there exists $\lambda_2(\rho) \geq 0$, so that, if $\lambda \geq \lambda_2(\rho)$, and (v_1,h_1,w_1) and (v_2,h_2,w_2) are elements of $Y(\lambda,\rho)$,

$$\|N(v_1,h_1,w_1) - N(v_2,h_2,w_2)\|_{Y(\lambda)} \leq \frac{1}{2}\|(v_1,h_1,w_1) - (v_2,h_2,w_2)\|_{Y(\lambda)}. \tag{5.15}$$

Indeed, let $\rho \in (0,\overline{\rho}_1]$ and $\lambda \geq \lambda_1(\rho)$. Then, by (5.2) and (5.4), we have

$$\begin{aligned} \|\mathcal{R}_1(v_1,h_1,w_1) - \ &\mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\leq C[\rho(\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} + \|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) \\ &+(1+\lambda)^{-\varepsilon}(\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} + \|w_1-w_2\|_{\mathcal{B}_\lambda([0,T])}) \\ &+(1+\lambda)^{-1}\|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}] \\ &\leq 2C[\rho + (1+\lambda)^{-\varepsilon} + (1+\lambda)^{-1}]\|(v_1-v_2,h_1-h_2,w_1-w_2)\|_{Y(\lambda)}. \end{aligned}$$

and , for $i \in \{2,3\}$,

$$\begin{aligned}\|\mathcal{R}_i(v_1,h_1,w_1)- & \mathcal{R}_i(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T])} \\ & \leq C_4(1+\rho)[\rho+(1+\lambda)^{-\varepsilon}+(1+\lambda)^{-1}] \\ & \times\|(v_1-v_2,h_1-h_2,w_1-w_2)\|_{Y(\lambda)},\end{aligned}$$

with C_4 independent of λ and ρ. Now, we choose $\overline{\rho}_2 \in (0,\overline{\rho}_1]$, in such a way that

$$\max\{\ 2C\rho,\ C_4\rho(1+\rho)\ \} < 1/2. \tag{5.16}$$

Then,for any $\rho \in (0,\overline{\rho}_2]$, there exists $\lambda_2(\rho) \geq 0$, such that, if $\lambda \geq \lambda_2(\rho)$ and (v_1,h_1,w_1) and (v_2,h_2,w_2) in $Y(\lambda,\rho)$, we have (5.15).

Third step: existence of a solution. We apply the results of the two previous steps: we fix $\rho \in (0,\overline{\rho}_2\})$ and $\lambda \geq \max\{\lambda_1(\rho),\lambda_2(\rho)\}$. Then N maps $Y(\lambda,\rho)$ into itself and, restricted to this subset, is a contraction. Then Banach's fixed point theorem allows to conclude that there exists a unique $(v,h,w) \in Y(\lambda,\rho)$, which is a fixed point for N. From the equivalence Theorem 4.2, we conclude that a solution exists.

Fourth step: uniqueness of a solution . Let (k_j,u_j,h_j,f_j) $(j = 1,2)$ be solutions of Problem 2.4. We want to show that they coincide. We already know that $k_1 = k_2 = k_0$. We set

$$v_j := u_j', \qquad w_j := f_j', \qquad j = 1,2. \tag{5.17}$$

Then, owing to Theorem 4.2, v_1 and v_2 belong to $C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$, w_1 and w_2 are in $C([0,T])$, and (v_1,h_1,w_1) and (v_2,h_2,w_2) are both fixed points of N. For $j = 1,2$, we have, owing to (5.1),

$$\begin{aligned}\|v_j-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} & = \|\mathcal{R}_1(v_j,h_j,w_j)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & \leq C[\|\overline{h}*A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ & +(1+\lambda)^{-\varepsilon}(\|h_j\|_{\mathcal{B}_\lambda([0,T])}+\|w_j\|_{\mathcal{B}_\lambda([0,T])}) \\ & +(1+\lambda)^{-1}(\|h_j-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}+\|v_j-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) \\ & +\|h_j-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v_j-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}].\end{aligned}$$

Applying Theorem 3.3 and observing that $\mathcal{R}_2(v_j,h_j,w_j)(0) = 0$, we have

$$\lim_{\lambda\to+\infty}\|h_j-\overline{h}\|_{\mathcal{B}_\lambda([0,T])} = 0,$$

so we conclude that

$$\lim_{\lambda\to+\infty}\|v_j-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} = \lim_{\lambda\to+\infty}\|\mathcal{R}_1(v_j,h_j,w_j)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} = 0.$$

From (5.3), we obtain, for $i \in \{2,3\}$,

$$\begin{aligned}\|\mathcal{R}_i(v_j,h_j,w_j)\|_{\mathcal{B}_\lambda([0,T])} \leq \ & C(\|\mathcal{R}_1(v_j,h_j,w_j)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ & +(1+\lambda)^{-1}\|h_j\|_{\mathcal{B}_\lambda([0,T])} \\ & +\|h_j-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v_j,h_j,w_j)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}),\end{aligned}$$

so that we conclude that, for $j = 1,2$,

$$\lim_{\lambda\to+\infty}\|(v_j,h_j,w_j)-(\overline{v},\overline{h},\overline{w})\|_{Y(\lambda)} = 0. \tag{5.18}$$

Now we choose λ sufficiently large, so that

$$\max\{\|(v_j, h_j, w_j) - (\overline{v}, \overline{h}, \overline{w})\|_{Y(\lambda)} : j = 1, 2\} \leq \rho \leq \overline{\rho}_2, \tag{5.19}$$

and, if necessary, we increase λ in such a way that $\lambda \geq \max\{\lambda_1(\rho), \lambda_2(\rho)\}$ and (5.19) continues to hold. Then (v_1, h_1, w_1) and (v_2, h_2, w_2) are both fixed points of N in $Y(\lambda, \rho)$. We conclude that they coincide.

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