Some global in time results for integro-differential parabolic inverse problems.

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Abstract We discuss a global in time existence and uniqueness result for an inverse problem arising in the theory of heat conduction for materials with memory. The novelty lies in the fact this is a global in time well posed problem in the sense of Hadamard, for semilinear parabolic inverse problems of integrodifferential type.

1 Introduction

In this paper we discuss some strategies we can use in the study of parabolic integrodifferential inverse problems. The choice of the strategy depends on what type of nonlinearities are involved. We consider the heat equation for materials with memory since it is one of the most important physical examples to which our methods apply. Other models, for instance in the theory of population dynamics, can also be considered within our framework. We recall, for the sake of completeness, the heat equation for materials with memory. Let Ω be an open and bounded set in $\mathbb{R}^3$ and T be a positive real number. The evolution equation for the temperature u is given, for $(t, x) \in [0, T] \times \Omega$, by

$$D_t u(t,x) = k\Delta u(t,x) + \int_0^t h(t-s)\Delta u(s,x)\,ds + F(u(t,x)), \tag{1.1}$$

where k is the diffusivity coefficient, h accounts for the memory effects and F is the heat source. In the inverse problem we consider, besides the temperature u, also h as a further unknown, and to determine it we add an additional measurement on u represented in integral form by

$$\int_\Omega \phi(x)u(t,x)\,dx = G(t), \qquad \forall t \in [0,T], \tag{1.2}$$

where ϕ and G are given functions representing the type of device used to measure u (on a suitable part of the body Ω) and the result of the measurement, respectively. We associate with (1.1)–(1.2) the initial-boundary conditions, for example of Neumann type:

$$\begin{cases} u(0,x) = u_0(x), & x \in \overline{\Omega}, \\ D_\nu u(t,x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \end{cases} \tag{1.3}$$

ν denoting the outward normal unit vector.
So one of the problems we are going to investigate is the following.

PROBLEM 1.1 (The Inverse Problem with two types of nonlinearities): *determine the temperature* $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$ *and the convolution kernel* $h : [0,T] \times \Omega \longrightarrow \mathbb{R}$ *satisfying* (1.1)–(1.3).

In the case when F is independent of u, but depends only on x and on t, we assume that the heat source is placed in a given position, but its time dependence is unknown, so we can suppose that

$$F(t,x) = f(t)g(x),$$

where f has to be determined and g is a given datum. Then we also assume that the diffusion coefficient k is unknown. The second inverse problem we will study is as follows.

PROBLEM 1.2 (An inverse problem with a nonlinearity of convolution type): *determine the temperature* $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$, *the diffusion coefficient* k *and the functions* $h : [0,T] \longrightarrow \mathbb{R}$, $f : [0,T] \longrightarrow \mathbb{R}$ *satisfying the system*

$$\begin{cases} D_t u(t,x) = k\Delta u(t,x) + \int_0^t h(t-s)\Delta u(s,x)\,ds + f(t)g(x), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \end{cases} \tag{1.4}$$

with the additional conditions

$$\int_{\overline{\Omega}} u(t,x)\mu_j(dx) = G_j(t), \quad \forall t \in [0,T], \quad j = 1,2, \tag{1.5}$$

where g, u_0, G_1, G_2 are given data and μ_1 and μ_2 are finite Borel measures in $C(\overline{\Omega})$.

REMARK 1.1 The additional conditions considered for Problem 1.2 (cf. (1.5)) is more general than the one considered for Problem 1.1 (cf. (1.2)). This is due to the fact that in Problem 1.2 we will choose the space of continuous functions as reference space. Such a setting has the advantage to allow additional measurements of the temperature also on the boundary of Ω, while in the L^p-setting (if we consider additional measurements of type (1.5)), one is compelled to make further measurements inside the body. In fact, for Problem 1.2 the measure μ_j $(1 \leq j \leq 2)$ is Borel measure in $\overline{\Omega}$, e.g., concentrated on the surface $\partial\Omega$, while, in the other case, ϕ (cf. (1.2)) is an element of $L^{p'}(\Omega)$ with $1/p + 1/p' = 1$.

Several identification problems involving the heat equation with memory have been faced and solved in the recent years; see for example [4, 5, 6, 8, 9, 12, 14] and the literature therein. The type of results we find are theorems of *local in time* existence and uniqueness for the solution of the inverse problem considered. What is still an open problem is to find *global in time* existence and uniqueness theorems for a sufficiently large class of nonlinearities that involve the function $F(u(t,x))$.

Since in this paper we want to show what kind of difficulties we have to overcome to solve Problem 1.1 (Problem 1.2, even though it has more unknowns, from a technical view point is a particular case) we make the following classification of the difficulties one has to face when dealing with this kind of inverse problems.

The main difficulty arises because there are two types of nonlinearities: one is in the convolution term $\int_0^t h(t-s)\Delta u(s,x)\,ds$, while the second one is obviously due to the nonlinear function $F(u(t,x))$.

There are several papers in which nonlinearities of convolution type have been studied. In particular, in [10, 11] the authors prove global in time results, in suitable weighted spaces, for convolution kernels that depend also on one space variable.

Such spaces are the natural tool to face inverse problems in which there are only nonlinearities of convolution type.

The presence of the nonlinear function $F(u(t,x))$ of the unknown u leads us to look for *a priori* estimates for the unknowns u and h, so that from a local in time result we obtain a global in time one.

The problem that arises with both nonlinearities is that the weighted spaces are not suitable in treating the nonlinear term $F(u(t,x))$. What has been done in the recent paper [7] is to find methods that allow us to treat both nonlinearities simultaneously.

In the case where we are looking for a local in time solution there is a wide class of function spaces in which it is possible to set our problem; see for example [4, 5, 6, 12, 14, 20, 22], but in the case we have to find *a priori* estimates just a few spaces are useful to this aim.

In this paper we present global in time results in the space of bounded functions with values in an interpolation space for a problem that involves only the nonlinearity of convolution type and then we show the very recent results in the Sobolev setting in the case there are both type of nonlinearities.

In the literature we can find the recent paper [21], in which the authors prove a *conditioned* global in time result for a phase-field model using a different strategy with respect to ours that suits well for the particular coupling of the equations of the phase-field system they consider. This is due to the fact that the two types of nonlinearities belong to two different equations.

Our final target is to generalize the technique developed in [7] to the different phase-field models that we can find in the literature; see for example [2, 3, 16, 20, 24, 25].

Let us explain what are the main differences in dealing with one or two types of nonlinearities showing the strategies we use.

In the case when the term F is a given datum that does not depend on the temperature u, or as in Problem 1.2 where $F = fg$ with f unknown, we use the weighted spaces, to be introduced in the sequel, and we proceed as follows.

(1) In the case it is possible to formulate our problem in at least two function spaces, we consider an abstract formulation of the inverse problem relating it to a Banach space X. This is not strictly necessary if the results hold just in the case when X can be uniquely chosen.

(2) We choose a functional setting. For example we can take the space of bounded functions on $[0, T]$ or the Sobolev spaces on $[0, T]$ with values in the Banach space X and we select the related optimal regularity theorem for the linearized version of the problem.

(3) We prove that the abstract version of the problem is equivalent to a suitable fixed point system.

(4) Since the fixed point system contains integral operators, we have to estimate them in the weighted spaces we are considering (exponential weight $e^{\sigma t}$, $\sigma \in \mathbb{R}^+$, $t \in [0, T]$ is usually used). The estimates for the integral operators must be such that suitable constants depending on σ approaches zero as $\sigma \to \infty$.

(5) By the Contraction Principle we prove that the equivalent problem has a unique solution, so we get existence and uniqueness of a solution to our inverse problem.

(6) We apply the abstract results to the concrete problem.

Let us come to the doubly nonlinear case in which F depends on u. The main idea to solve the problem in this case is to prove that there exists a local in time solution of the inverse problem in Sobolev spaces without weights, then we linearize the convolution term and we find a priori estimates for u and for the convolution kernel h. More precisely we proceed as follows.

(a) In this case we do not give an abstract formulation since at the moment we are able to prove our results only in the Sobolev setting.

(b). We use the Sobolev spaces $W^{k,p}(0,T;L^p(\Omega))$.

(c) Analogue to (3), but the concrete system is considered instead of the abstract one.

(d) The fixed point system contains integral operators; we have to estimate them in the Sobolev spaces we have chosen.

(e) We apply the Contraction Principle to prove that there exists a unique *local in time* solution. Thanks to the equivalence theorem previously obtained we get *local in time* existence and uniqueness of the solution to our inverse problem. We prove a *global in time* uniqueness result without the condition that F_u be bounded.

(f) We linearize the convolution term thanks to the *local in time* existence and uniqueness theorem. We observe that a unique solution $(\hat{u}, \hat{h})$ exists in $[0,\tau]$ for some $\tau > 0$. We set $v_\tau(t) = v(\tau + t)$ and $h_\tau(t) = h(\tau + t)$ and consider, for $0 < t < \tau$, the splitting

$$\int_0^{\tau+t} h(\tau + t - s)\Delta v(s,x)ds = h_\tau * \Delta\hat{v}(t,x) + \hat{h} * \Delta v_\tau(t,x) + \tilde{F}(t,x),$$

where the symbol $*$ stands for the convolution (see below) and $\tilde{F}(t,x)$ is a given data depending on the known functions $(\hat{u}, \hat{h})$. This way of rewriting the convolution term allows us to avoid the weighted spaces that have a bad behavior when we deal with the nonlinearity $F(u)$.

(g) We deduce the a priori estimates for $v_\tau(t)$ and $h_\tau(t)$ for $0 < t < \tau$ with the condition F_u be bounded. In a finite number of steps we extend the solution to the interval $[0,T]$.

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2 Functional settings and preliminary material

Let X be a Banach space with norm $\|\cdot\|$ and let $T > 0$. We denote by $C([0,T];X)$ the usual space of continuous functions with values in X, while we denote by $\mathcal{B}([0,T];X)$ the space of bounded functions with values in X. $\mathcal{B}([0,T];X)$ will be endowed with the sup-norm

$$\|u\|_{\mathcal{B}([0,T];X)} := \sup_{0\le t\le T} \|u(t)\| \tag{2.1}$$

and $C([0,T];X)$ will be considered a closed subspace of $\mathcal{B}([0,T];X)$. We will use the notations $C([0,T];\mathbb{R}) = C([0,T])$ and $\mathcal{B}([0,T];\mathbb{R}) = \mathcal{B}([0,T])$. By $\mathcal{L}(X)$ we denote the space of all bounded linear operators from X into itself equipped with the sup–norm, while $\mathcal{L}(X;\mathbb{R}) = X'$ is the space of all bounded linear functionals on X considered with the natural norm. We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $s \in \mathbb{Z}$, $s \geq 2$ we set $W^{s,p}_B(\Omega) := \{f \in W^{s,p}(\Omega) : D_\nu f \equiv 0\}$. We denote by $B^s_{p,q}(\Omega)$ $(s > 0,\ 1 \leq p, q \leq +\infty)$ the Besov spaces. The symbol $(\cdot,\cdot)_{\theta,p}$ stands for the real interpolation functor $(0 < \theta < 1,\ 1 \leq p \leq +\infty)$. For all $h \in L^1(0,T)$ and $f : (0,T) \to X$ we define the convolution

$$h * f(t) := \int_0^t h(t-s)f(s)ds,$$

whenever the integral has a meaning. Let $p \in [1,+\infty)$, $m \in \mathbb{N}_0$; if $f \in W^{m,p}(0,T;X)$ (see [1]), we set

$$\|f\|_{W^{m,p}(0,T;X)} := \sum_{j=0}^{m-1} \|f^{(j)}(0)\|_X + \|f^{(m)}\|_{L^p(0,T;X)}.$$

For the sake of brevity we define the Banach space

$$X(T,p) = W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)),$$

where $T \in \mathbb{R}^+$, $p \in [1,+\infty]$. If $u \in X(T,p)$ we set

$$\|u\|_{X(T,p)} = \|u\|_{W^{1,p}(0,T;L^p(\Omega))} + \|u\|_{L^p(0,T;W^{2,p}(\Omega))}.$$

We now give the definition:

DEFINITION 2.1 *Let $A : \mathcal{D}(A) \subseteq X \to X$ be a linear operator, possibly with $\overline{\mathcal{D}(A)} \neq X$. Operator A is said to be sectorial if it satisfies the following assumptions:*

- *there exist $\theta \in (\pi/2, \pi)$ and $\omega \in \mathbb{R}$, such that any $\lambda \in \mathbb{C} \setminus \{\omega\}$ with $|\arg(\lambda - \omega)| \leq \theta$ belongs to the resolvent set of A.*
- *there exists $M > 0$ such that $\|(\lambda - \omega)(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M$ for any $\lambda \in \mathbb{C} \setminus \{\omega\}$ with $|\arg(\lambda - \omega)| \leq \theta$.*

The above definition of sectorial operator is important to define the semigroup of bounded linear operators $\{e^{tA}\}_{t\geq 0}$, in $\mathcal{L}(X)$, so that $t \to e^{tA}$ is an analytic function from $(0,+\infty)$ to $\mathcal{L}(X)$.
Let us define the family of interpolation spaces (see [23] or [29]) $\mathcal{D}_A(\theta,\infty)$, $\theta \in (0,1)$, between $\mathcal{D}(A)$ and X by

$$\mathcal{D}_A(\theta,\infty) = \left\{x \in X : \ |x|_{\mathcal{D}_A(\theta,\infty)} := \sup_{0<t<1} t^{1-\theta}\|Ae^{tA}x\| < \infty\right\} \tag{2.2}$$

with the norm

$$\|x\|_{\mathcal{D}_A(\theta,\infty)} = \|x\| + |x|_{\mathcal{D}_A(\theta,\infty)}. \tag{2.3}$$

We also set

$$\mathcal{D}_A(1+\theta,\infty) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}_A(\theta,\infty)\}. \tag{2.4}$$

$\mathcal{D}_A(1+\theta,\infty)$ turns out to be a Banach space when equipped with the norm

$$\|x\|_{\mathcal{D}_A(1+\theta,\infty)} = \|x\| + \|Ax\|_{\mathcal{D}_A(\theta,\infty)}. \tag{2.5}$$

For Problem 1.2 we will use the following optimal regularity result:

THEOREM 2.1 (Optimal regularity in spaces $\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$) *Let X be a Banach space. Consider the Cauchy Problem:*

$$(CP) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.6}$$

where $A : \mathcal{D}(A) \to X$ is a sectorial operator and $\theta \in (0,1)$. For any $f \in C([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$, $u_0 \in \mathcal{D}_A(\theta+1,\infty)$ the Cauchy problem (CP) admits a unique solution $u \in C^1([0,T];X) \cap C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta+1,\infty))$.

Proof. See the book [23] or the original paper [27].

As we have discussed in the introduction for Problem 1.1 at the moment we have the only possibility to choose Sobolev spaces, if we want a global in time result. For this reason we do not formulate the inverse problem in an abstract setting. As a consequence the optimal regularity result we are in need of is formulated just for the concrete case.

THEOREM 2.2 (Optimal regularity in spaces $X(T,p)$) *Let Δ be the Laplace operator and $k_0 \in \mathbb{R}^+$. Consider the problem*

$$\begin{cases} D_t u(t,x) = k_0 \Delta u(t,x) + F(t,x), & (t,x) \in [0,T] \times \Omega, \\ D_\nu u(t,x') = 0, & (t,x') \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{2.7}$$

Then, if $p \in (1,+\infty)$, $F \in L^p(0,T;L^p(\Omega))$ and $u_0 \in (L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p}$, (2.7) has a unique solution $u \in X(T,p)$. Moreover, for all $T_0 \in \mathbb{R}^+$, there exists $C(T_0) \in \mathbb{R}^+$, such that, if $0 < T \le T_0$,

$$\|u\|_{X(T,p)} \le C(T_0)(\|F\|_{L^p(0,T;L^p(\Omega))} + \|u_0\|_{(L^p(\Omega),W^{2,p}_B(\Omega))_{1-1/p,p}}). \tag{2.8}$$

Proof. It is that of Theorem 8.1 in [15].

From Theorem 3.5 in [17], we have that, for $p \in (1,+\infty)$:

$$(L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p} = \begin{cases} B^{2(1-1/p)}_{p,p}(\Omega) & \text{if } 1 < p < 3, \\ \{f \in B^{2(1-1/p)}_{p,p}(\Omega) : D_\nu f \equiv 0\} & \text{if } 3 < p < +\infty. \end{cases} \tag{2.9}$$

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3 The main results

We present in the following subsections the results we have obtained in the case we deal only with the nonlinearity of convolution type and the case in which both nonlinearities are involved. The space of bounded functions is used in the first case, while the Sobolev setting is used in the second case.

3.1 The case of one nonlinearity of convolution type

We give the Inverse Problem 1.2 an abstract formulation and then we apply the abstract Theorem 3.1 to the concrete case.

PROBLEM 3.1 (Inverse Abstract Problem (IAP)) *Let A be a sectorial operator in X. Determine a positive number k and three functions u, h, f, such that*

$$(\alpha) \begin{cases} u \in C^2([0,T];X) \cap C^1([0,T];\mathcal{D}(A)), \\ D_t u \in \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty)), \quad D_t^2 u \in \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty)), \end{cases}$$

(β) $h \in C([0,T])$,

(γ) $f \in C^1([0,T])$,

satisfying the system

$$\begin{cases} u'(t) = kAu(t) + \int_0^t h(t-s)Au(s)ds + f(t)g, \quad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{3.1}$$

and the additional conditions:

$$\langle u(t), \phi_j \rangle = G_j(t), \quad t \in [0,T], \quad j = 1,2, \tag{3.2}$$

where ϕ_j ($j = 1,2$) are given bounded linear functionals on X, and G_j, u_0, g are given data.

We study the (IAP) under the following assumptions:

(H1) $\theta \in (0,1)$, X is a Banach space and A is a sectorial operator in X.

(H2) $u_0 \in \mathcal{D}_A(1+\theta+\varepsilon, \infty)$, for some $\varepsilon \in (0, 1-\theta)$.

(H3) $g \in \mathcal{D}_A(\theta+\varepsilon, \infty)$, for some $\varepsilon \in (0, 1-\theta)$.

(H4) $\phi_j \in X'$, for $j = 1, 2$.

(H5) $G_j \in C^2([0,T])$, for $j = 1, 2$.

(H6) The matrix

$$M := \begin{pmatrix} < Au_0, \phi_1 > & < g, \phi_1 > \\ < Au_0, \phi_2 > & < g, \phi_2 > \end{pmatrix} \tag{3.3}$$

is invertible and its inverse is defined by

$$M^{-1} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3.4}$$

(H7) We require that the linear system

$$\begin{cases} k_0 < Au_0, \phi_1 > + f_0 < g, \phi_1 >= G_1'(0), \\ k_0 < Au_0, \phi_2 > + f_0 < g, \phi_2 >= G_2'(0), \end{cases} \tag{3.5}$$

has a unique solution (k_0, f_0) with $k_0 > 0$.

(H8) $v_0 := k_0 Au_0 + f_0 g \in \mathcal{D}_A(1+\theta, \infty)$.

(H9) $< u_0, \phi_j >= G_j(0), \quad < v_0, \phi_j >= G_j'(0), \quad j = 1, 2.$

REMARK 3.1 Owing to (H6) the first component k_0 of the solution (k_0, f_0) is positive if and only if

$$\frac{1}{\det M}[G_1'(0) < g, \phi_2 > - G_2'(0) < g, \phi_1 >] > 0.$$

The main abstract result is the following:

THEOREM 3.1 *Assume that conditions (H1)-(H9) are fulfilled. Then Problem 3.1 has a unique (global in time) solution (k, u, h, f), with $k \in \mathbb{R}^+$, and u, h, f satisfying conditions (α), (β) and (γ).*

Proof. See Section 4 for the main steps of the proof or Section 5 in [13] for all the details.

An application to the concrete case. We choose as reference space

$$X = C(\overline{\Omega}), \tag{3.6}$$

where Ω is an open bounded set in $\mathbb{R}^n$ (in the introduction we have considered the physical case $n = 3$, but the result holds for any $n \in \mathbb{N}$) with boundary of class $C^{2(1+\theta+\varepsilon)}$, for some $\theta \in (0, 1/2)$, $\varepsilon \in (0, (1/2) - \theta)$. We define

$$\begin{cases} \mathcal{D}(A) = \Big\{u \in \bigcap_{1\le p<+\infty} W^{2,p}(\Omega) \ : \Delta u \in C(\overline{\Omega}), \quad D_\nu u|_{\partial\Omega} = 0\Big\}, \\ Au := \Delta u, \quad \forall u \in \mathcal{D}(A). \end{cases} \tag{3.7}$$

It was proved by Stewart (see [28]) that A is a sectorial operator in X. Then we recall the following characterizations concerning the interpolation spaces related to A (see [23]):

$$\mathcal{D}_A(\xi,\infty) = \begin{cases} C^{2\xi}(\overline{\Omega}), \quad \text{if} \quad 0 < \xi < 1/2, \\ \{u \in C^{2\xi}(\overline{\Omega}) : \ D_\nu u|_{\partial\Omega} = 0\}, \quad \text{if} \quad 1/2 < \xi < 1. \end{cases} \tag{3.8}$$

Consequently, if $0 < \xi \le \theta + \varepsilon$, we have

$$\mathcal{D}_A(1+\xi;\infty) = \{u \in C^{2(1+\xi)}(\overline{\Omega}) : D_\nu u|_{\partial\Omega} = 0\}. \tag{3.9}$$

So we consider the Inverse Problem 1.2 under the following assumptions:

(K1) Ω is an open bounded set in $\mathbb{R}^n$ with boundary of class $C^{2(1+\theta+\varepsilon)}$, for some $\theta \in (0, 1/2)$, $\varepsilon \in (0, (1/2) - \theta)$.

(K2) $u_0 \in C^{2(1+\theta+\varepsilon)}(\overline{\Omega})$, $\quad D_\nu u_0|_{\partial\Omega} = 0$.

(K3) $g \in C^{2(\theta+\varepsilon)}(\overline{\Omega})$.

(K4) For $j = 1, 2$, μ_j is a bounded Borel measure in $\overline{\Omega}$. We set, for $\psi \in X$,

$$<\psi, \phi_j> := \int_{\overline{\Omega}} \psi(x)\mu_j(dx). \tag{3.10}$$

(K5) Suppose that (H5) holds.

(K6) Suppose that (H6) holds with ϕ_j ($j = 1, 2$) defined in (3.10) and A defined in (3.7).

(K7) Suppose that (H7) holds.

(K8) $v_0 := k_0\Delta u_0 + f_0 g \in C^{2(1+\theta)}(\overline{\Omega})$, $\quad D_\nu v_0|_{\partial\Omega} = 0$.

(K9) Suppose that (H9) holds.

Applying Theorem 3.1 we immediately deduce:

THEOREM 3.2 *Assume that conditions (K1)-(K9) are satisfied. Then the Inverse Problem 1.2 has a unique (global in time) solution (k, u, h, f), such* *that*

$$u \in C^2([0,T]; C(\overline{\Omega})) \cap C^1([0,T]; \mathcal{D}(A)),$$
$$D_t u \in \mathcal{B}([0,T]; C^{2(1+\theta)}(\overline{\Omega})), \quad D_t^2 u \in \mathcal{B}([0,T]; C^{2\theta}(\overline{\Omega})),$$
$$h \in C([0,T]), \qquad f \in C^1([0,T]), \qquad k \in \mathbb{R}^+,$$

A being defined in (3.7).

3.2 The case of two nonlinearities

We solve the Inverse Problem 1.1 under the following conditions on the data:

(I1) Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2.

(I2) $p \in (1, +\infty)$, $n \in \mathbb{N}$, with $n < 2p$.

(I3) $u_0 \in W_B^{2,p}(\Omega)$.

(I4) $\phi \in W^{2,p'}(\Omega)$. We set $\psi := \Delta\phi$.

(I5) $F \in C^\infty(\mathbb{R})$.

(I6) $v_0 := \Delta u_0 + F(u_0) \in (L^p(\Omega), W_B^{2,p}(\Omega))_{1-1/p,p}$.

(I7) $g \in W^{2,p}(0,T)$.

(I8) $\Phi(u_0) = g(0)$ and $\Phi(v_0) = g'(0)$.

(I9) $\Phi(\Delta u_0) := \int_\Omega \phi(x)\Delta u_0(x)dx \neq 0$.

(I10) F_u is bounded.

THEOREM 3.3 (Global in time). *Let the assumptions (I1)–(I10) hold. Let $T > 0$ and $p \geq 2$. Then Problem 1.1 has a unique solution*

$$(u,h) \in [W^{2,p}(0,T; L^p(\Omega)) \cap W^{1,p}(0,T; W^{2,p}(\Omega))] \times L^p(0,T).$$

Proof. See Section 5 for the main steps of the proof or Section 7 in [7] for all the details.

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4 The strategy for nonlinearity of convolution type in weighted spaces

We now want to show in a more explicit way how we obtain our results sketching some proofs. We follow the list in the introduction.

Step (1). The abstract formulation is Problem 3.1.

Step (2). The functional setting is the space of bounded function with values in an interpolation space, see (α) for the function u.

Step (3). We reformulate Problem 3.1 in terms of the equivalent nonlinear fixed point system (4.10). In proving Theorem 4.1 we find out a set of regularity and compatibility conditions on the data that makes the inverse problem well posed. To this aim, we start by introducing some notations. We set

$$A_0 := k_0 A. \tag{4.1}$$

As $k_0 > 0$, see (H7), A_0 is a sectorial operator in X. Next, we set, for $t \in [0,T]$,

$$\overline{h}_1(t) := a_{11} G_1''(t) + a_{12} G_2''(t), \tag{4.2}$$

$$\overline{w}_1(t) := a_{21} G_1''(t) + a_{22} G_2''(t), \tag{4.3}$$

$$\overline{v}(t) := e^{tA_0} v_0. \tag{4.4}$$

We immediately observe that $\overline{h}_1$ and $\overline{w}_1$ belong to $C([0,T])$ and, owing to Theorem 2.1, $\overline{v} \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$. Next, we define, again for $t \in [0,T]$,

$$\overline{h}(t) := \overline{h}_1(t) - k_0 a_{11} < A\overline{v}(t), \phi_1 > - k_0 a_{12} < A\overline{v}(t), \phi_2 >, \tag{4.5}$$

$$\overline{w}(t) := \overline{w}_1(t) - k_0 a_{21} < A\overline{v}(t), \phi_1 > - k_0 a_{22} < A\overline{v}(t), \phi_2 > . \tag{4.6}$$

Of course, $\overline{h}$ and $\overline{w}$ belong to $C([0,T])$. Finally, we introduce the following (nonlinear) operators, defined for every $(v,h,w) \in [C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))] \times C([0,T]) \times C([0,T])$:

$$\mathcal{R}_1(v,h,w)(t) := e^{tA_0} * (hAu_0 + wg + h * Av)(t), \tag{4.7}$$

$$\begin{aligned} \mathcal{R}_2(v,h,w)(t) := & -k_0[a_{11} < A\mathcal{R}_1(v,h,w)(t), \phi_1 > \\ & + a_{12} < A\mathcal{R}_1(v,h,w)(t), \phi_2 >] \\ & - a_{11} < h * A\overline{v}(t), \phi_1 > \\ & - a_{12} < h * A\overline{v}(t), \phi_2 > \\ & - a_{11} < h * A\mathcal{R}_1(v,h,w)(t), \phi_1 > \\ & - a_{12} < h * A\mathcal{R}_1(v,h,w)(t), \phi_2 >, \end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathcal{R}_3(v,h,w)(t) := -k_0[a_{21} &< A\mathcal{R}_1(v,h,w)(t),\phi_1 > \\ &+a_{22} < A\mathcal{R}_1(v,h,w)(t),\phi_2 >] \\ &-a_{21} < h * A\overline{v}(t),\phi_1 > \\ &-a_{22} < h * A\overline{v}(t),\phi_2 > \\ &-a_{21} < h * A\mathcal{R}_1(v,h,w)(t),\phi_1 > \\ &-a_{22} < h * A\mathcal{R}_1(v,h,w)(t),\phi_2 > . \end{aligned} \tag{4.9}$$

We observe that $\mathcal{R}_2$ and $\mathcal{R}_3$ are well defined, because

$$\mathcal{R}_1(v,h,w) \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty)).$$

Moreover, $\mathcal{R}_2(v,h,w)$ and $\mathcal{R}_3(v,h,w)$ both belong to $C([0,T])$.
Now we can introduce the following problem:

PROBLEM 4.1 *Determine three functions v, h, w, such that*

(α') $v \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}(1+\theta,\infty))$,

(β') $h \in C([0,T])$,

(γ') $w \in C([0,T])$,

satisfying the system

$$\begin{cases} v &= \overline{v} + \mathcal{R}_1(v,h,w), \\ h &= \overline{h} + \mathcal{R}_2(v,h,w), \\ w &= \overline{v} + \mathcal{R}_3(v,h,w). \end{cases} \tag{4.10}$$

THEOREM 4.1 (Equivalence) *Let A be a sectorial operator in the Banach space X and $\theta \in (0,1)$. Let us assume that the data g, u_0, ϕ_j and G_j $(j = 1,2)$ satisfy the conditions (H1)-(H9). Suppose that k, u, f, h satisfy Problem 3.1, with $k \in \mathbb{R}_+$ and u, f, h fulfilling the regularity conditions (α), (β), (γ). Then*

(I) $k = k_0$, the triplet (v,h,w), where $v = u'$, $w = f'$ satisfies the conditions (α'), (β'), (γ') and solves Problem 4.1;

(II) conversely, if (v,h,w), with the above regularity, is a solution of the Problem 4.1, then the triplet (u,h,f), where $u = u_0 + 1*v$, $f = f_0 + 1*w$, satisfies the regularity conditions (α), (β), (γ) and solves Problem 3.1 with $k = k_0$.

Proof. It is Theorem 4.2 in [13] and it is based on Theorem 2.1.

Step (4). *Fundamental estimates for the integral operators $\mathcal{R}_j$, $j = 1, 2, 3$, in the weighted space.*

We introduce some crucial estimates that will be essential to obtain global in time existence and uniqueness of a solution to our inverse problem. Let $\lambda > 0$, $T > 0$, $\theta \in (0,1)$. If $f \in \mathcal{B}([0,T]; X)$, we set

$$\|u\|_{\mathcal{B}_\lambda([0,T];X)} := \sup_{0 \le t \le T} e^{-\lambda t} \|u(t)\|. \tag{4.11}$$

We denote by $C_\lambda([0,T]; X)$ the space $C([0,T]; X)$ equipped with the norm $\|\cdot\|_{\mathcal{B}_\lambda([0,T];X)}$.
We will use the notations $C_\lambda([0,T]; \mathbb{R}) = C_\lambda([0,T])$ and $\mathcal{B}_\lambda([0,T]; \mathbb{R}) = \mathcal{B}_\lambda([0,T])$. We now state some useful estimates in these weighted spaces for the solution of the Cauchy problem given by Theorem 2.1. We will list in the following theorems what kind of estimates we are in need of.

THEOREM 4.2 *Let $A : \mathcal{D}(A) \to X$ be a sectorial operator, $\theta \in (0,1)$. Let us suppose that $f \in C([0,T]; X) \cap \mathcal{B}([0,T]; \mathcal{D}_A(\theta, \infty))$. Then the following estimates hold:*

$$\|e^{tA} * f\|_{C_\lambda([0,T];X)} \le \frac{C_0}{1+\lambda} \|f\|_{C_\lambda([0,T];X)}; \tag{4.12}$$

if $\theta \le \xi \le 1 + \theta$,

$$\|e^{tA} * f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\xi,\infty))} \le \frac{C(\theta,\xi)}{(1+\lambda)^{1+\theta-\xi}} \|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \tag{4.13}$$

with C_0 and $C(\theta, \xi)$ independent of f and λ.

Proof. It is that of Theorem 4.3 in [13].

THEOREM 4.3 *Let $\lambda \ge 0$, $h \in C([0,T])$ and $u \in C([0,T]; X) \cap \mathcal{B}([0,T]; \mathcal{D}_A(\theta, \infty))$. Then there exists $C > 0$, independent of T, λ, h, u, such that, if*

$$h * u(t) := \int_0^t h(t-s) u(s) ds, \tag{4.14}$$

$h * u \in C([0,T]; X) \cap \mathcal{B}([0,T]; \mathcal{D}_A(\theta, \infty))$ and satisfies the following estimate:

$$\begin{aligned} \|h * u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \le C \min\Big\{ & T \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \\ & (1+\lambda)^{-1} \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))}, \\ & (1+\lambda)^{-1} \|h\|_{\mathcal{B}([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \Big\}. \end{aligned} \tag{4.15}$$

Proof. It is that of Theorem 4.4 in [13].

The above theorems give us the possibility to estimate the operators $\mathcal{R}_j$, $j = 1, 2, 3$.

LEMMA 4.1 *Assume that conditions (H1)-(H9) are satisfied. Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ be the operators defined in (4.7), (4.7), (4.8), respectively. Then there exists $C > 0$ such that, for all $\lambda \geq 0$, $v, v_1, v_2 \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$, $h, h_1, h_2 \in C([0,T])$ and $w, w_1, w_2 \in C([0,T])$ we have*

$$\begin{aligned}\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \leq C[&\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &+(1+\lambda)^{-\varepsilon}(\|h\|_{\mathcal{B}_\lambda([0,T])} + \|w\|_{\mathcal{B}_\lambda([0,T])}) \\ &+(1+\lambda)^{-1}(\|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])} \\ &+\|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) \\ &+\|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}];\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad \leq C[\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])}\|v_1-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad + \|h_2-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad + (1+\lambda)^{-\varepsilon}(\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} + \|w_1-w_2\|_{\mathcal{B}_\lambda([0,T])}) \\ &\qquad + (1+\lambda)^{-1}\|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}];\end{aligned} \tag{4.17}$$

and, if $i \in \{2,3\}$,

$$\begin{aligned}&\|\mathcal{R}_i(v,h,w)\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad \leq C[\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} + (1+\lambda)^{-1}\|h\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad + \|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}];\end{aligned} \tag{4.18}$$

$$\begin{aligned}&\|\mathcal{R}_i(v_1,h_1,w_1) - \mathcal{R}_i(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad \leq C[\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad + (1+\lambda)^{-1}\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad + \|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v_1,h_1,w_1)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &+\|h_2-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}].\end{aligned} \tag{4.19}$$

Proof. It is that of Lemma 5.1 in [13] and it is based on Theorems 4.2 and 4.3.

Step 5. By the Contraction Principle we prove Theorem 3.1. Let $\lambda \geq 0$. We set

$$Y(\lambda) := (C_\lambda([0,T];\mathcal{D}(A)) \cap \mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty)) \times C_\lambda([0,T])^2, \quad (4.20)$$

and we endow it with the norm

$$\|(v,h,w)\|_{Y(\lambda)} := \max\{\|v\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}, \|h\|_{\mathcal{B}_\lambda([0,T])}, \|w\|_{\mathcal{B}_\lambda([0,T])}\}, \quad (4.21)$$

with $v \in C_\lambda([0,T];\mathcal{D}(A)) \cap \mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))$, $h \in C_\lambda([0,T])$ and $w \in C_\lambda([0,T])$. With the norm (4.21) $Y(\lambda)$ becomes a Banach space.
Let $\lambda \geq 0$, $\rho > 0$ and set

$$Y(\lambda,\rho) := \{(v,h,w) \in Y(\lambda) : \|(v,h,w) - (\overline{v},\overline{h},\overline{w})\|_{Y(\lambda)} \leq \rho\}. \quad (4.22)$$

Then, for every $\rho > 0$, $Y(\lambda,\rho)$ is a closed subset of $Y(\lambda)$.
Now we introduce the following operator N: if $(v,h,w) \in Y(\lambda)$, we set

$$N(v,h,w) := (\overline{v} + \mathcal{R}_1(v,h,w), \overline{h} + \mathcal{R}_2(v,h,w), \overline{w} + \mathcal{R}_3(v,h,w)). \quad (4.23)$$

Clearly N is a nonlinear operator in $Y(\lambda)$.
Now we show that Problem 3.1 has a solution (k,u,h,f), with the regularity properties (α), (β), (γ).
Applying Theorem 4.1, we are reduced to looking for a solution (v,h,w) of system (4.10), satisfying the conditions (α'), (β'), (γ'). This is equivalent to looking for a fixed point of N in $Y(\lambda)$, for some $\lambda \geq 0$. For the details see Section 5 in [13].

Step (6). An application of the abstract result is Theorem 3.2.

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5 The strategy for the case of two nonlinearities in Sobolev spaces

We show the main ideas on which is based the global in time result for the doubly nonlinear problem.

Step (a)–(b). We consider in this case the concrete formulation of the problem since the correct functional setting is the Sobolev space

$$X(T,p) = W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)).$$

Step (c). An equivalent reformulation of the problem is the following:

THEOREM 5.1 *Let the assumptions (I1)–(I9) hold. Let u and h verify the conditions*

$$u \in W^{2,p}([0,T];L^p(\Omega)) \cap W^{1,p}([0,T];W^{2,p}(\Omega)), \quad h \in L^p([0,T]), \quad (5.1)$$

and solve the system (1.1)–(1.3). We set $v := D_t u$, so v and h satisfy the conditions

$$v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)), \quad h \in L^p([0,T]). \tag{5.2}$$

Then v and h solve the system (5.4) and (5.6). On the other hand, let v, h satisfy the conditions (5.2) and solve the system (5.4) and (5.6). If we set $u := u_0 + 1 * v$, then u, h verify the conditions (5.1) and solve the system (1.1)–(1.3).

Proof. We split the proof into two steps.

Step 1. Suppose that the problem (1.1)–(1.3) has a solution

$$u \in W^{2,p}(0,T;L^p(\Omega)) \cap W^{1,p}(0,T;W^{2,p}(\Omega)), \quad h \in L^p(0,T).$$

We set

$$D_t u(t,x) := v(t,x), \tag{5.3}$$

and we differentiate the first equation in (1.1) to get

$$\begin{cases} D_t v(t,x) = \Delta v(t,x) + h(t)\Delta u_0 + h * \Delta v(t,x) \\ \qquad\qquad + F_u(u_0(x) + 1 * v(t,x))v(t,x), \\ v(0,x) := v_0 = Au_0(x) + F(u_0(x)), \ x \in \Omega, \\ D_\nu v(t,x) = 0, \quad t \in [0,T], \ x \in \partial\Omega. \end{cases} \tag{5.4}$$

If (I1)–(I9) hold, obviously we have $v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ and $h \in L^p(0,T)$. Apply now functional Φ (cf. (I9)) to the first equation and, keeping in mind that $\Phi(D_t u)(t) = g'(t)$ and $\Phi(D_t^2 u)(t) = g''(t)$, we get

$$\begin{aligned} g''(t) &= \Phi[\Delta v(t,\cdot)] + h(t)\Phi[\Delta u_0(\cdot)] \\ &\quad + h * \Phi[\Delta v(t,\cdot)] + \Phi[F_u(u_0(\cdot) + 1 * v(t,\cdot))v(t,\cdot)]. \end{aligned} \tag{5.5}$$

We can write, setting $\chi^{-1} := \Phi[\Delta u_0(\cdot)] \neq 0$,

$$\begin{aligned} h(t) &= \chi g''(t) - \chi\Phi[F_u(u_0(\cdot) + 1 * v(t,\cdot))v(t,\cdot)] \\ &\quad - \chi\Phi[\Delta v(t,\cdot)] - \chi h * \Phi[\Delta v(t,\cdot)]. \end{aligned} \tag{5.6}$$

Step 2. Suppose now that $v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ and $h \in L^p(0,T)$ satisfy system (5.4) and (5.6). Since $D_t u(t,x) := v(t,x)$ we observe that the first equation in (5.4) can be rewritten as

$$D_t[D_t u(t,x) - \Delta u(t,x) - h * \Delta u(t,x) - F(u(t,x))] = 0,$$

which gives

$$D_t u(t,x) - \Delta u(t,x) - h * \Delta u(t,x) - F(u(t,x)) = C(x). \tag{5.7}$$

Setting $t = 0$, we have

$$v_0(x) - \Delta u_0(x) - f(u_0(x)) = C(x),$$

so we get $C = 0$. So, from equation (5.7) we deduce the first equation in (1.1). Consider the equation for h in (5.5), it can be written as:

$$D_t g'(t) = D_t[\Phi[\Delta u(t,\cdot)] + h * \Phi[\Delta u(t,\cdot)] + \Phi[F(u(t,\cdot))]]. \tag{5.8}$$

This gives

$$c + g'(t) = \Phi[\Delta u(t,\cdot)] + h * \Phi[\Delta u(t,\cdot)] + \Phi[F(u(t,\cdot))].$$

At $t = 0$ we have

$$c + g'(0) = \Phi[\Delta u_0(\cdot) + F(u_0(\cdot))].$$

Since $\Delta u_0(x) + F(u_0(x)) = v_0$ and $g'(0) = \Phi[v_0]$ we get $c = 0$. Then the equation

$$D_t \Phi[u(t,\cdot)] = g'(t)$$

becomes

$$c' + \Phi[u(t,\cdot)] = g(t).$$

Setting $t = 0$ and recalling the compatibility condition $\Phi[u(0)] = g(0)$, we get $c' = 0$ so that

$$\Phi[u(t,\cdot)] = g(t).$$

Step (d). We get the preliminary lemmas that are necessary to estimate the operators entering in the equivalent reformulation of the problem.

THEOREM 5.2 *Let X be a Banach space, $p \in (1, +\infty)$, $\tau \in \mathbb{R}^+$, $h \in L^p(0,\tau)$, $f \in L^p(0,\tau; X)$. Then $h * f \in L^p(0,\tau;X)$ and*

$$\|h * f\|_{L^p(0,\tau;X)} \leq \tau^{1-1/p} \|h\|_{L^p(0,\tau)} \|f\|_{L^p(0,\tau;X)}.$$

Proof. It is that of Theorem 3.2 in [7].

THEOREM 5.3 *Let X be a Banach space, $\tau \in \mathbb{R}^+$, $p \in (1, +\infty)$, $z \in W^{1,p}(0,\tau;X)$, with $z(0) = 0$. Then*

$$\|z\|_{L^\infty(0,\tau;X)} \leq \tau^{1-1/p} \|z\|_{W^{1,p}(0,\tau;X)}, \tag{5.9}$$

$$\|z\|_{L^p(0,\tau;X)} \leq p^{-1/p} \tau \|z\|_{W^{1,p}(0,\tau;X)}. \tag{5.10}$$

Proof. It is that of Theorem 3.3 in [7].

THEOREM 5.4 *Under the conditions (I1) and (I2), $W^{2,p}(\Omega)$ is continuously embedded in $C(\overline{\Omega})$ and is a space of pointwise multipliers for $W^{s,p}(\Omega)$, for $s = 0, 1, 2$.*

Proof. It is that of Theorem 3.4 in [7].

THEOREM 5.5 *Under the assumptions (I1),(I2) and (I3), if $S \in C^\infty(\mathbb{R})$, then the map $v \to S \circ v$ is of class C^∞ from $W^{2,p}(\Omega)$ into itself. Moreover, for all $k \in \mathbb{N}_0$, $(S \circ \cdot)^{(k)}$ is bounded with values in $\mathcal{L}^k(W^{2,p}(\Omega), W^{2,p}(\Omega))$ in every bounded subset of $W^{2,p}(\Omega)$.*

Proof. It is that of Theorem 3.5 in [7].

THEOREM 5.6 *Assume that (I1) and (I2) are satisfied, $S \in C^\infty(\mathbb{R})$, $u_0 \in W^{2,p}(\Omega)$. Let $R \in \mathbb{R}^+$, $0 < \tau \leq T$ and let V_1 and V_2 be elements of $X(\tau, p)$ such that*

$$\max_{j \in \{1,2\}} \|V_j\|_{L^p(0,\tau;W^{2,p}(\Omega))} \leq R.$$

Then

$$\begin{aligned} &\|S(u_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{L^p(0,\tau;L^p(\Omega))} \\ &\leq C(R,T)\tau^{(p-1)/(2p)}\|V_1 - V_2\|_{X(\tau,p)}. \end{aligned}$$

Proof. It is that of Theorem 3.6 in [7].

THEOREM 5.7 *Let $p \in (1, +\infty)$, Ω satisfying (H1), $\phi \in L^{p'}(\Omega)$, $\tau \in \mathbb{R}^+$. We define in $L^p(0,\tau; L^p(\Omega))$ the operator*

$$\Phi[f](t) := \int_\Omega \phi(x) f(t,x)dx.$$

If $u \in X(\tau,p)$, we consider the map $u \to \Phi[\Delta u]$. Then $\Phi[\Delta u] \in L^p(0,\tau)$ and

$$\|\Phi[\Delta u]\|_{L^p(0,\tau)} \leq \omega(\tau)\|u\|_{X(\tau,p)}, \tag{5.11}$$

with $\omega(\tau) > 0$, independent of u, and $\lim_{\tau \to 0} \omega(\tau) = 0$.

Proof. It is that of Theorem 3.7 in [7].

Step (e). *The local in time existence theorem and the global in time uniqueness theorem without the condition F_u bounded.*

THEOREM 5.8 (Local in time existence and uniqueness). *Let the assumptions (I1)–(I9) hold. Then there exists $\tau \in (0, T]$, depending on the data, such that the inverse problem (1.1)–(1.3) has a unique solution*

$$(u, h) \in [W^{2,p}(0, \tau; L^p(\Omega)) \cap W^{1,p}(0, \tau; W^{2,p}(\Omega))] \times L^p(0, \tau).$$

Proof. It is that of Theorem 2.1 in [7] and its proof is based on Theorems 2.2 and 5.2–5.7.

THEOREM 5.9 (Global in time uniqueness). *Let the assumptions (I1)–(I9) hold. If $\tau \in (0, T]$, and if the inverse problem in Definition 1.2 has two solutions $(u_j, h_j) \in W^{2,p}(0, \tau; L^p(\Omega)) \cap W^{1,p}(0, \tau; W^{2,p}(\Omega)) \times L^p(0, \tau)$, $j \in \{1, 2\}$, then $u_1 = u_2$ and $h_1 = h_2$.*

Proof. It is that of Theorem 2.2 in [7] and its proof is based on Theorems 2.2 and 5.2–5.7.

Step (f). *We linearize the convolution term.* From the local in time existence and uniqueness theorem we observe that a unique solution $\widehat{u}$, $\widehat{h}$ exists in $[0, \tau]$ for some $\tau > 0$. So we can consider the equations:

$$\begin{cases} D_t v(\tau + t, x) = \Delta v(\tau + t, x) + h(\tau + t)\Delta u_0 \\ + \int_0^{\tau+t} h(\tau + t - s)\Delta v(s, x)ds + F_u(u_0(x) + 1 * v(\tau + t, x))v(\tau + t, x), \\ v(\tau, x) = u_\tau(x), \quad x \in \Omega, \\ D_\nu v(\tau + t, x) = 0, \quad t \in [0, T], \ x \in \partial\Omega, \\ \Phi[v(\tau + t, \cdot)] := \int_\Omega \phi(x) v(\tau + t, x)dx = g'(\tau + t), \quad t \in [0, T]. \end{cases} \tag{5.12}$$

We define the new unknowns

$$v_\tau(t) = v(\tau + t), \quad h_\tau(t) = h(\tau + t), \quad g_\tau(t) = g(\tau + t), \tag{5.13}$$

and we observe that

$$1 * v(\tau + t, x) = \int_0^{\tau+t} v(s)ds = \int_0^\tau \widehat{v}(s)ds + \int_\tau^{\tau+t} v(s)ds$$

$$= \int_0^\tau \widehat{v}(s)ds + \int_0^t v_\tau(s')ds'$$

where we have set $s - \tau = s'$ and defined

$$\widetilde{u}_0(x) := u_0(x) + 1 * \widehat{v}(t, x).$$

So we can rewrite

$$F_u(u_0(x) + 1 * v(\tau + t, x))\, v(\tau + t, x) = F_u(\widetilde{u}_0(x) + 1 * v_\tau(t, x))\, v_\tau(t, x).$$

Let now $0 < t < \tau$. Thanks to the splitting

$$\int_0^{\tau+t} h(\tau + t - s)\Delta v(s, x)ds$$

$$= \int_0^{\tau} \widehat{h}(\tau + t - s)\Delta\widehat{v}(s, x)ds + \int_\tau^{t+\tau} \widehat{h}(\tau + t - s)\Delta v(s, x)ds$$

$$= \int_0^{t} h_\tau(t - s)\Delta\widehat{v}(s, x)ds + \int_t^{\tau} \widehat{h}(\tau + t - s)\Delta\widehat{v}(s, x)ds$$

$$+ \int_\tau^{t+\tau} \widehat{h}(\tau + t - s)\Delta v(s, x)ds \tag{5.14}$$

and setting $s - \tau = s'$, we have

$$\int_\tau^{t+\tau} \widehat{h}(\tau + t - s)\Delta v(s, x)ds = \int_0^{t} \widehat{h}(t - s')\Delta v_\tau(s', x)ds'.$$

Consequently, the convolution term becomes linear in the unknowns involved in the convolution so that the system becomes:

$$\begin{cases} D_t v_\tau(t, x) = \Delta v_\tau(t, x) + h_\tau(t)\Delta u_0 + h_\tau * \Delta\widehat{v}(t, x) \\ +\widehat{h} * \Delta v_\tau(t, x) + F_u(\widetilde{u}_0(x) + 1 * v_\tau(t, x))v_\tau(t, x) + \widetilde{F}(t, x) \\ v_\tau(0, x) = u_\tau(x), \quad x \in \Omega, \\ D_\nu v_\tau(t, x) = 0, \quad t \in [0, T],\ x \in \partial\Omega, \\ \Phi[v_\tau(t, \cdot)] := \int_\Omega \phi(x)v_\tau(t, x)dx = g'_\tau(t), \quad t \in [0, T], \end{cases} \tag{5.15}$$

where we have set

$$\widetilde{F}(t, x) := \int_t^{\tau} \widehat{h}(\tau + t - s)\Delta\widehat{v}(s, x)ds.$$

Steps (g) and (h). *We deduce the a priori estimates for v_τ and h_τ.*
The idea is to get – thanks to the fact that F_u is bounded – the *a priori* estimates for the unknowns v_τ and h_τ. The proof is based on the following lemma.

LEMMA 5.1 *Assume that the assumptions (I1)–(I10) are fulfilled, $p \geq 2$. Let $(\widehat{v}, \widehat{h}) \in X(\tau, p) \times L^p(0, \tau)$ be a solution of (5.4)–(5.6) in $[0, \tau] \times \Omega$, with $0 < \tau < T$. Then, there exists $C > 0$, such that, for all $\delta \in (0, \tau \wedge (T - \tau)]$, if $(v, h) \in X(\tau + \delta, p) \times L^p(0, \tau + \delta)$ is a solution of (5.4)–(5.6) in $[0, \tau + \delta] \times \Omega$, then*

$$\|v_\tau\|_{X(\tau+\delta,p)} + \|h_\tau\|_{L^p(0,\tau+\delta)} \leq C.$$

Proof. It is that of Lemma 7.3 in [7].

Now, in a finite number of steps we can extend the solution to the interval $[0, T]$. For all the details see Section 7 in [7].

References

[1] R. Adams: *Sobolev spaces*, Pure and Applied Mathematics, vol. 65, Plenum Press (1975).

[2] P. Colli, G. Gilardi and M. Grasselli: Global smooth solution to the standard phase field models with memory, Adv. Differential Equations, **2** (1997), 453–486.

[3] P. Colli, G. Gilardi and M. Grasselli: Well-posedness of the weak formulation for the phase field models with memory, Adv. Differential Equations, **2** (1997), 487–508.

[4] F. Colombo: Direct and inverse problems for a phase field model with memory, J. Math. Anal. Appl., **260** (2001), 517–545.

[5] F. Colombo: An inverse problem for a generalized Kermack-McKendrick model, J. Inv. Ill Posed Prob., **10** (2002), 221–241.

[6] F. Colombo and D. Guidetti: A unified approach to non linear integrodifferential inverse problems of parabolic type, Zeit. Anal. Anwend., **21** (2002), 431–464.

[7] F. Colombo and D. Guidetti: A global in time result for a nonlinear parabolic integrodifferential inverse problem, Quad. Dip. Mat. Politecnico di Milano (2005).

[8] F. Colombo and A. Lorenzi: Identification of time and space dependent relaxation kernels in the theory of materials with memory I, J. Math. Anal. Appl., **213** (1997), 32–62.

[9] F. Colombo and A. Lorenzi: Identification of time and space dependent relaxation kernels in the theory of materials with memory II, J. Math. Anal. Appl., **213** (1997), 63–90.

[10] F. Colombo, D. Guidetti and A. Lorenzi: Integrodifferential identification problems for the heat equation in cylindrical domains, Adv. Math. Sci. Appl., **13** (2003), 639–662.

[11] F. Colombo, D. Guidetti and A. Lorenzi: Integrodifferential identification problems for thermal materials with memory in nonsmooth plane domains, Dynamic Systems and Applications, **12** (2003), 533–560.

[12] F. Colombo, D. Guidetti and V. Vespri: Identification of two memory kernels and the time dependence of the heat source for a parabolic conserved phase-field model, to appear in: Math. Meth. Appl. Sci. (2005).

[13] F. Colombo, D. Guidetti and V. Vespri: A global in time result for an integrodifferential parabolic inverse problem in the space of bounded functions, submitted.

[14] F. Colombo and V. Vespri: A semilinear integrodifferential inverse problem, Evolution equations, J. Goldstein, R. Nagel and S. Romanelli editors, Marcel Dekker, Inc., Cap 6, **234** (2003), 91–104.

[15] R. Denk, M. Hieber and J. Prüss: R-boundedness, Fourier multipliers and problems of elliptic and parabolic type, preprint, Technische Universität Darmstadt, n. 2156 (2001).

[16] C. Giorgi, M. Grasselli and V. Pata: Uniform attractors for a phase-field model with memory and quadratic nonlinearity, Indiana Univ. Math. J., **48** (1999), 1395–1445.

[17] D. Guidetti: On interpolation with boundary conditions, Math. Zeit., **207** (1991), 439,460.

[18] D. Guidetti and A. Lorenzi: A mixed type identification problem related to a phase-field model with memory, preprint.

[19] M. Grasselli: An inverse problem in population dynamics, Num. Funct. Anal. Optim., **18** (1997), 311–323.

[20] M. Grasselli, V. Pata and F.M. Vegni: Longterm dynamics of a conserved phase-field system with memory, Asymp. Analysis, **33** (2003), 261–320.

[21] A. Lorenzi, E. Rocca and G. Schimperna: Direct and inverse problems for a parabolic integrodifferential system of Caginalp type, to appear in Adv. Math. Sci. Appl., (2005).

[22] A. Lorenzi and E. Sinestrari: An inverse problem in the theory of materials with memory, Nonlinear Anal. T. M. A., **12** (1988), 1317–1335.

[23] A. Lunardi: *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and Their Application, Birkhäuser, Basel, 1995.

[24] A. Novick and Cohen: The Cahn-Hilliard equation: mathematical and

modeling perspectives, Adv. Math. Sci. Appl., **8** (1998), 965–985.

[25] A. Novick and Cohen: Conserved phase field equations with memory, in *Curvature Flows and Related Topics*, 1994, GAKUTO Internat. Ser. Math. Sci. Appl., Vol. 5, Gakkōtosho, Tokyo, (1995), 179–197.

[26] A. Pazy: *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[27] E. Sinestrari: On the Cauchy problem in spaces of continuous functions, J. Math. Anal. Appl., **107** (1985), 16–66.

[28] H.B. Stewart: Generation of analytic semigroups by strongly elliptic operators, Trans. Amer. Math. Soc., **199** (1974), 141–162.

[29] H. Triebel: *Interpolation theory, function spaces, differential operators,* North-Holland, Amsterdam, 1978.

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Volume 251

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