Moebius transformations and the Poincare' distance in the quaternionic setting

Möbius Transformations and the Poincaré Distance in the Quaternionic Setting

Cinzia Bisi & Graziano Gentili

ABSTRACT. In the space $\mathbb{H}$ of quaternions, we investigate the natural, invariant geometry of the open, unit disc $\Delta_{\mathbb{H}}$ and of the open half-space $\mathbb{H}^+$. These two domains are diffeomorphic via a Cayley-type transformation. We first study the geometrical structure of the groups of Möbius transformations of $\Delta_{\mathbb{H}}$ and $\mathbb{H}^+$ and identify original ways of representing them in terms of two (isomorphic) groups of matrices with quaternionic entries. We then define the cross-ratio of four quaternions, prove that, when real, it is invariant under the action of the Möbius transformations, and use it to define the analog of the Poincaré distances and differential metrics on $\Delta_{\mathbb{H}}$ and $\mathbb{H}^+$.

1. Introduction

The study of the intrinsic geometry of the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane, bi-holomorphic via the Cayley transformation to the upper half-plane Π^+ of $\mathbb{C}$, is very rich and of great, classical interest. The main tool for the study of this geometry is the Poincaré distance, which turns out to be the integrated distance of the Poincaré differential metric. In fact the holomorphic self-maps of $\mathbb{D}$ turn out to be contractions, and hence the group of all holomorphic automorphisms of $\mathbb{D}$ are isometries, for the Poincaré distance (and differential metric). As a consequence, an approach typical of differential geometry can be adopted to study the geometric theory of holomorphic self-maps of any simply connected domain strictly contained in $\mathbb{C}$. In fact, by the Riemann representation theorem, any such domain is bi-holomorphic to $\mathbb{D}$ (and to the upper half plane Π^+). In this setting the classical groups $SU(1,1)$ and $SL(2,\mathbb{R})$ come into the scenary: when quotiented by their centers, they represent the group of

all holomorphic automorphisms (the so called Möbius transformations) of $\mathbb{D}$ and Π^+, respectively.

When endowed with the Poincaré differential metric, the open, unit disc $\mathbb{D}$ acquires a structure of Riemannian surface of constant negative curvature, whose geodesics are the arcs of circles or straight lines which intersect the boundary $\partial\mathbb{D}$ orthogonally.

It is interesting to notice that the Poincaré distance can be defined on $\mathbb{D}$ by means of the family of all geodesics mentioned above. Following the approach of Siegel, [17], given two points $z_1, z_2 \in \mathbb{D}$ one can define the two ends z_3, z_4 of the (unique) geodesic passing through z_1 and z_2 as the intersections of this geodesic with $\partial\mathbb{D}$. One then orders the four points "cyclically" and defines the Poincaré distance $\delta_\mathbb{D}(z_1, z_2)$ as half the logarithm of the cross-ratio of the four points z_1, z_2, z_3, z_4.

With this in mind, in the present paper we consider the space $\mathbb{H}$ of quaternions and study the geometry of the open, unit disc $\Delta_\mathbb{H} = \{q \in \mathbb{H} : |q| < 1\}$ and of the half-space $\mathbb{H}^+ = \{q \in \mathbb{H} : \operatorname{Re}(q) > 0\}$, which turn out to be diffeomorphic via a Cayley-type transformation. More precisely, we give this paper a double aim. The first one is to study the groups of *Möbius transformations* of $\Delta_\mathbb{H}$ and of $\mathbb{H}^+$ (i.e., the groups of all quaternionic, fractional, linear transformations which leave $\Delta_\mathbb{H}$ and $\mathbb{H}^+$ invariant, respectively). The second aim is to give a direct, geometric definition of the analogue of the Poincaré distance (i.e., the real, hyperbolic distance) and differential metric in the quaternionic setting, to investigate their most interesting properties, and to explicitly describe the invariant geometry of the classical hyperbolic domains $\Delta_\mathbb{H}$ and $\mathbb{H}^+$ of $\mathbb{H}$.

In what follows, the elements of the skew field $\mathbb{H}$ of real quaternions will be denoted by $q = x_0 + ix_1 + jx_2 + kx_3$ where the x_ℓ are real, and i, j, k are imaginary units (i.e., their square equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. We will denote by $\mathbb{S}_\mathbb{H}^3$ the sphere of quaternions of unitary modulus $\{q \in \mathbb{H} : |q| = 1\}$ and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e., $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Notice that if $I \in \mathbb{S}$, then $I^2 = -1$; for this reason the elements of $\mathbb{S}$ are called imaginary units. We will also often use the fact that for any non-real quaternion $q \in \mathbb{H} \setminus \mathbb{R}$, there exist, and are unique, $x, y \in \mathbb{R}$ with $y > 0$, and $I \in \mathbb{S}$ such that $q = x + yI$.

In Section 2, in order to identify the group of all quaternionic, fractional, linear transformations, we start by studying the problem of finding the inverse of a quaternionic 2×2 matrix. This problem corresponds to solving, when possible, a linear system of four quaternionic equations, and leads us to define, in a direct and very natural way, the so called Dieudonné determinant of a quaternionic 2×2 matrix:

Definition 1.1. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix with quaternionic entries, then the (*Dieudonné*) *determinant* of A is defined to be the non negative real number

$$(1.1) \qquad \det_\mathbb{H}(A) = \sqrt{|a|^2\,|d|^2 + |c|^2\,|b|^2 - 2\operatorname{Re}(c\bar{a}b\bar{d})}.$$

The notion of quaternionic determinant appears in the literature in a much more general setting and uses at that level the tool of quasideterminants, [9, 16]. Here we study the principal properties of this determinant—giving simple, direct proofs of our assertions—also for the sake of completeness (see also [4, 8]). We then exploit these properties to investigate the structure of the group of all quaternionic, fractional, linear transformations of $\mathbb{H}$ with an intrinsic approach. In fact in Section 3 we set

$$\mathbb{G} = \{g(q) = (aq + b)(cq + d)^{-1} : a, b, c, d \in \mathbb{H},\ g \text{ invertible}\},$$
$$\mathrm{GL}(2, \mathbb{H}) = \{A,\ 2 \times 2 \text{ matrix with quaternionic entries} : \det_\mathbb{H}(A) \neq 0\},$$
$$\mathrm{SL}(2, \mathbb{H}) = \{A \in \mathrm{GL}(2, \mathbb{H}) : \det_\mathbb{H}(A) = 1\},$$

and we give a direct proof of the following result already established in [20]:

Theorem 1.2. *The set $\mathbb{G}$ of all quaternionic, fractional, linear transformations is a group with respect to composition. The map*

$$(1.2) \qquad \Phi : A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is a group homomorphism of $\mathrm{GL}(2, \mathbb{H})$ *onto* $\mathbb{G}$ *whose kernel is the center of* $\mathrm{GL}(2, \mathbb{H})$, *that is the subgroup*

$$\left\{ \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}.$$

In Section 4 we extend the structure-theorem of the complex, fractional, linear transformations to the quaternionic environment and prove that the group $\mathbb{G}$ is generated by all the similarities, $L(q) = aq + b$ ($a,\ b \in \mathbb{H},\ a \neq 0$) and the inversion $R(q) = q^{-1}$. Moreover, all the elements of $\mathbb{G}$ turn out to be conformal.

If the role of the complex cross-ratio is crucial in complex, projective geometry, its (real) generalizations to higher dimensions in $\mathbb{R}^n$ seem not to have a minor role in conformal geometry. In fact L. Ahlfors, while studying the conformal structure of $\mathbb{R}^n$, has given in [3] three different definitions of the cross-ratio of 4 points of $\mathbb{R}^n$. The one that we give here is the specialization to the quaternionic case of the definition given by C. Cao and P.L. Waterman in [5], and is new with respect to the ones given by Ahlfors. In fact the definition that we adopt of cross-ratio has the peculiar feature that the quaternionic, fractional, linear transformations act on it transforming its value by (quaternionic) conjugation (see Corollary 4.6). We prove, in particular, the following result.

Proposition 1.3. *Let*

$$C\mathcal{R}(q_1, q_2, q_3, q_4) := (q_1 - q_3)(q_1 - q_4)^{-1}(q_2 - q_4)(q_2 - q_3)^{-1}$$

be the cross-ratio of the four quaternions q_1, q_2, q_3, q_4. When the cross ratio of four quaternions is real, then it is invariant under the action of all quaternionic, fractional, linear transformations.

The above result has a great deal of interest in view of the following theorem (already proven in [5] in the Clifford setting).

Proposition 1.4. *Four pairwise distinct points q_1, q_2, q_3, $q_4 \in \mathbb{H}$ lie on a same (one-dimensional) circle or straight line if, and only if, their cross-ratio is real.*

When A.F. Möbius introduced the notion of what we call nowadays a fractional, linear transformation, what he had in mind was only a homeomorphism of the extended, complex plane $\mathbb{C} \cup \{\infty\}$ onto itself which maps circles onto circles. Adopting this point of view, still in Section 4 we define the families $\mathcal{F}_i$, for $i = 3$, 2, 1, respectively as $\mathcal{F}_i = S_i \cup P_i$ where S_i is the family of all i-(real) dimensional spheres and P_i is the family of all i-(real) dimensional affine subspaces of $\mathbb{H}$. Then, with an approach different from the one used by Wilker in [20], we give an original, direct proof of the following result.

Theorem 1.5. *The group $\mathbb{G}$ of all quaternionic, fractional, linear transformations maps elements of $\mathcal{F}_i$ onto elements of $\mathcal{F}_i$, for $i = 3, 2, 1$.*

The aim of Section 5 is to find a geometric approach to the definition of the quaternionic Poincaré distance on $\Delta_\mathbb{H}$ (often simply called Poincaré distance when no confusion can arise). To this aim we adopt the point of view used by C.L. Siegel, [17], for the homologous problem in the complex case and use the terminology introduced by Ahlfors in [3]. In fact, to start with, we define the non-Euclidean line through two points q_1, q_2 as the unique circle, or diameter, containing the two points and intersecting $\partial \Delta_\mathbb{H}$ orthogonally in the two ends q_3, q_4. The Poincaré distance of $\Delta = \Delta_\mathbb{H}$ is then defined by

$$(1.3) \qquad \delta_\Delta(q_1, q_2) = \frac{1}{2} \log(CR(q_1, q_2, q_3, q_4))$$

where the four points are arranged cyclically on the non-Euclidean line through q_1 and q_2. Notice that on each complex plane $L_I = \mathbb{R} + I\mathbb{R}$ (for any imaginary unit I) the quaternionic Poincaré distance coincides with the classical Poincaré distance of $\Delta_I = \Delta_\mathbb{H} \cap L_I$.

The structure of the group $\mathbb{M}$ of Möbius transformations of $\Delta_\mathbb{H}$ is studied, for example, in [6], in terms of the (classical) group $\mathrm{Sp}(1, 1)$. If $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the group $\mathrm{Sp}(1, 1)$ is defined (see, e.g., [11]) as

$$(1.4) \qquad \mathrm{Sp}(1, 1) = \{A \in \mathrm{GL}(2, \mathbb{H}) : {}^t\bar{A}HA = H\}$$

and it can be written equivalently as (see, e.g., [6])

$$\mathrm{Sp}(1, 1) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : |a| = |d|, \ |b| = |c|, \ |a|^2 - |c|^2 = 1, \ \bar{a}b = \bar{c}d, \ a\bar{c} = b\bar{d} \right\}.$$

It allows to rephrase and complete a result of [6] as follows:

Theorem 1.6. *The quaternionic, fractional, linear transformation defined by formula* $g(q) = (aq + b)(cq + d)^{-1}$ *is a Möbius transformation of* $\Delta_\mathbb{H}$ *if and only if* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}(1, 1)$. *Moreover, the map*

(1.5)
$$\varphi : \mathrm{Sp}(1, 1) \to \mathbb{M}$$
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is a group homomorphism whose kernel is the center of $\mathrm{Sp}(1, 1)$, *that is the subgroup*

$$\left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

By means of the statement of Theorem 1.6 we are able to obtain, for the quaternionic Möbius transformations, a characterization which closely resembles the classical representation of the complex Möbius transformations. A similar result is stated without proof in [13].

Theorem 1.7. *Each quaternionic Möbius transformation of the form*

$$g(q) = (aq + b) \cdot (cq + d)^{-1} \in \mathbb{M}$$

can be written uniquely as:

(1.6)
$$g(q) = \alpha(q - q_0)(1 - \overline{q_0}q)^{-1}\beta^{-1}$$

where $q_0 = -a^{-1}b \in \Delta_\mathbb{H}$ *and where* $\alpha = a/|a| \in \partial\Delta_\mathbb{H}$, $\beta = d/|d| \in \partial\Delta_\mathbb{H}$.

The description of the group of all Möbius transformations of $\Delta_\mathbb{H}$ given in Theorem 1.7 is different from the one given in a more general setting in [3].

Using Propositions 1.3 and 1.4 we sum up by proving the following result, whose statement is implicit in the work of Wilker [20] (see also [14]).

Proposition 1.8. *The Poincaré distance of* $\Delta_\mathbb{H}$ *is invariant under the action of the group of all Möbius transformations* $\mathbb{M}$ *and of the map* $q \mapsto \bar{q}$.

It is now possible to mimic the definition of the classical, complex Poincaré differential metric of $\mathbb{D} \subset \mathbb{C}$ to set the length of the vector $\tau \in \mathbb{H}$ for the Poincaré metric at $q \in \Delta_\mathbb{H}$ to be the number:

(1.7)
$$\langle \tau \rangle_q = \frac{|\tau|}{1 - |q|^2}.$$

Formula (1.7) leads now to the definition of the (square of the) Poincaré length element at $q \in \Delta_{\mathbb{H}}$:

$$ds^2 = \frac{|d_I q|^2}{(1 - |q|^2)^2}$$

where $q = x + yI$ and $d_I q = dx + I\,dy$ (for $I \in \mathbb{S}$). The quaternionic Poincaré differential metric given above can also be obtained by specializing to the case of quaternions the definition given in the more general setting of the study of conformal geometry of $\mathbb{R}^n$ by Ahlfors, [3]. At the end of Section 5, the following results are proved:

Theorem 1.9. *All the elements of the group $\mathbb{M}$ of Möbius transformations of $\Delta_{\mathbb{H}}$, as well as the map $q \mapsto \bar{q}$, leave the Poincaré differential metric invariant.*

Proposition 1.10. *The Poincaré distance δ_Δ of the unit disc $\Delta_{\mathbb{H}}$ is the integrated distance of the Poincaré differential metric of $\Delta_{\mathbb{H}}$.*

Section 6 is dedicated to transfer the Poincaré distance and differential metric of $\Delta_{\mathbb{H}}$ to $\mathbb{H}^+$ via a Cayley-type transformation. The results obtained in $\mathbb{H}^+$ are homologous to those which hold in $\Delta_{\mathbb{H}}$. Nevertheless, in this setting, we are able to give an original, nice description of the group of all Möbius transformations $\mathbb{M}(\mathbb{H}^+)$ of $\mathbb{H}^+$, in terms of a group of matrices $\mathrm{SL}(\mathbb{H}^+)$ which plays the role played by the group $\mathrm{SL}(2, \mathbb{R})$ in the complex case.

Theorem 1.11. *If $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then the set of matrices defined by*

$$\mathrm{SL}(\mathbb{H}^+) = \{A \in \mathrm{GL}(2, \mathbb{H}) : {}^t\bar{A} K A = K\}$$

is a subgroup of $\mathrm{SL}(2, \mathbb{H})$ of real dimension 10. This subgroup can be equivalently characterized as

$$\mathrm{SL}(\mathbb{H}^+) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{H},\ \mathrm{Re}(a\bar{c}) = \mathrm{Re}(b\bar{d}) = 0,\ \bar{b}c + \bar{d}a = 1 \right\}$$

and as

$$\mathrm{SL}(\mathbb{H}^+) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{H},\ \mathrm{Re}(c\bar{d}) = \mathrm{Re}(a\bar{b}) = 0,\ a\bar{d} + b\bar{c} = 1 \right\}.$$

Moreover, the map

$$\Psi : \mathrm{SL}(\mathbb{H}^+) \to \mathbb{M}(\mathbb{H}^+)$$

(1.8)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is a group homomorphism whose kernel is the center of $\mathrm{SL}(\mathbb{H}^+)$, that is the subgroup

$$\left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The last result of this paper states the following.

Theorem 1.12. *The two subgroups* $\mathrm{SL}(\mathbb{H}^+)$ *and* $\mathrm{Sp}(1,1)$ *of* $\mathrm{SL}(2,\mathbb{H})$ *are isomorphic.*

2. THE DETERMINANT OF 2×2 MATRICES WITH QUATERNIONIC ENTRIES

As it is well known, the determinant of a matrix with quaternionic entries cannot be defined as in the case of matrices with real or complex entries. Nevertheless, the study of the quaternionic analogue of the fractional, linear and Möbius, complex transformations leads us to an interesting generalization of the notion of determinant, in the case of 2×2 quaternionic matrices. The notion of quaternionic determinant appears in the literature in a much more general setting and exploits at that level the tool of quasideterminants, [9, 16]. Here we will present the main features of the determinant of 2×2 quaternionic matrices—giving simple, direct proofs of our assertions—for the sake of completeness (see also [4, 8]).

We will denote by $M(2,\mathbb{H})$ the $\mathbb{H}$-vector space (right or left, depending on the setting) of 2×2 matrices with quaternionic entries. A matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M(2,\mathbb{H})$ is said to be right-invertible if and only if there exists $\left[\begin{smallmatrix} x & y \\ t & z \end{smallmatrix}\right] \in M(2,\mathbb{H})$ such that

$$(2.1) \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ t & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

i.e., if and only if the following system of linear equations

$$(2.2) \qquad \begin{cases} ax + bt & = 1 \\ cx + dt & = 0 \\ ay + bz & = 0 \\ cy + dz & = 1 \end{cases}$$

has a (unique) solution $(x,y,t,z) \in \mathbb{H}^4$. We can now present the following proposition, already established in [20]. We give a detailed proof here.

Proposition 2.1. *The following three statements are equivalent:*

(1) *the matrix* $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M(2,\mathbb{H})$ *is right-invertible;*

(2) $b(c - db^{-1}a) \neq 0$ *or* $a(d - ca^{-1}b) \neq 0$;

(3) $c(b - ac^{-1}d) \neq 0$ *or* $d(a - bd^{-1}c) \neq 0$.

Proof. We will begin by proving that (1) implies (2). The first equation of (2.2) implies that $a \neq 0$ or $b \neq 0$. If $a \neq 0$ then, using the third equation in (2.2), we obtain $y = -a^{-1}bz$ and substituting in the fourth equation of the same

system we get $(d - ca^{-1}b)z = 1$. Therefore we obtain $(d - ca^{-1}b) \neq 0$ and $a(d - ca^{-1}b) \neq 0$. At this point an easy computation shows that in this case

$$
(2.3) \quad \begin{bmatrix} x & y \\ t & z \end{bmatrix} = \begin{bmatrix} a^{-1} + a^{-1}b(d - ca^{-1}b)^{-1}ca^{-1} & -a^{-1}b(d - ca^{-1}b)^{-1} \\ -(d - ca^{-1}b)^{-1}ca^{-1} & (d - ca^{-1}b)^{-1} \end{bmatrix}.
$$

If we are in the case $b \neq 0$ then, using as above system (2.2), we obtain $z = -b^{-1}ay$ and $(c - db^{-1}a)y = 1$, yielding $(c - db^{-1}a) \neq 0$ and hence $b(c - db^{-1}a) \neq 0$. As before, an easy computation shows now that

$$
(2.4) \quad \begin{bmatrix} x & y \\ t & z \end{bmatrix} = \begin{bmatrix} -(c - db^{-1}a)^{-1}db^{-1} & (c - db^{-1}a)^{-1} \\ b^{-1} + b^{-1}a(c - db^{-1}a)^{-1}db^{-1} & -b^{-1}a(c - db^{-1}a)^{-1} \end{bmatrix}.
$$

To prove that (2) implies (1), it is enough to notice that when (2) is assumed true, matrix (2.3) or (2.4) is well defined and that it is (by construction) the inverse of A. The proof of the equivalence of (1) and (3) is completely analogous to the one given above.   □

Remark 2.2. As one may expect, when $ab \neq 0$, then the two forms (2.3) and (2.4) of the inverse of A do coincide. If $abcd \neq 0$, then the inverse matrix of A assumes an even nicer form,

$$
\begin{bmatrix} x & y \\ t & z \end{bmatrix} = \begin{bmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{bmatrix}
$$

which allows a Cramer-type rule to solve 2×2 linear systems with quaternionic coefficients (see also [16]).

Proposition 2.3. *All the right-invertible elements of $M(2, \mathbb{H})$ are also left-invertible. The set $\mathrm{GL}(2, \mathbb{H})$ of all invertible elements of $M(2, \mathbb{H})$ is a group.*

Proof. Since the matrix multiplication in $M(2, \mathbb{H})$ is associative and has a neutral element, the assertion follows from general results of group theory (see, e.g. [12]).   □

Let us now compute

$$
\begin{aligned}
|a(d - ca^{-1}b)|^2 &= a(d - ca^{-1}b)(\bar{d} - \bar{b}\bar{a}^{-1}\bar{c})\bar{a} \\
&= a(|d|^2 - d\bar{b}\bar{a}^{-1}\bar{c} - ca^{-1}b\bar{d} + |c|^2 |a|^{-2} |b|^2)\bar{a} \\
&= |a|^2 |d|^2 - a(2\,\mathrm{Re}(d\bar{b}\bar{a}^{-1}\bar{c}))\bar{a} + |c|^2 |b|^2 \\
&= |a|^2 |d|^2 - |a|^2(2\,\mathrm{Re}(d\bar{b}\bar{a}^{-1}\bar{c})) + |c|^2 |b|^2 \\
&= |a|^2 |d|^2 + |c|^2 |b|^2 - 2\,\mathrm{Re}(d\bar{b}a\bar{c}).
\end{aligned}
$$

Similarly we obtain

$$|b(c - db^{-1}a)|^2 = |a|^2\,|d|^2 + |c|^2\,|b|^2 - 2\,\mathrm{Re}(c\bar{a}b\bar{d}\,).$$

We analogously get

$$
\begin{aligned}
(2.5) \qquad |c(b - ac^{-1}d)|^2 &= |d(a - bd^{-1}c)|^2 \\
&= |a|^2\,|d|^2 + |c|^2\,|b|^2 - 2\,\mathrm{Re}(a\bar{c}d\bar{b}).
\end{aligned}
$$

Since $\mathrm{Re}(uv) = \mathrm{Re}(\bar{u}\bar{v})$ for any $u, v \in \mathbb{H}$, we also have $\mathrm{Re}(c\bar{a}b\bar{d}\,) = \mathrm{Re}(a\bar{c}d\bar{b})$, and therefore we obtain the result below.

Lemma 2.4. *The following equalities hold*

$$
\begin{aligned}
|a(d - ca^{-1}b)|^2 &= |b(c - db^{-1}a)|^2 = |c(b - ac^{-1}d)|^2 \\
&= |d(a - bd^{-1}c)|^2 = |a|^2\,|d|^2 + |c|^2\,|b|^2 - 2\,\mathrm{Re}(c\bar{a}b\bar{d}\,).
\end{aligned}
$$

Remark 2.5. For all $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M(2, \mathbb{H})$, it turns out that

$$
\begin{aligned}
(2.6) \qquad |a|^2\,|d|^2 + |c|^2\,|b|^2 &- 2\,\mathrm{Re}(c\bar{a}b\bar{d}\,) \\
&\geq |a|^2\,|d|^2 + |c|^2\,|b|^2 - 2|a|\,|d|\,|b|\,|c| \\
&= (|a|\,|d| - |b|\,|c|)^2 \geq 0.
\end{aligned}
$$

Proposition 2.1, Lemma 2.4 and Remark 2.5 naturally lead to the following definition, which can also be found in [9, 16].

Definition 2.6. If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M(2, \mathbb{H})$, then the (*Dieudonné*) *determinant* of A is defined to be the non negative real number

$$(2.7) \qquad \det_{\mathbb{H}}(A) = \sqrt{|a|^2\,|d|^2 + |c|^2\,|b|^2 - 2\,\mathrm{Re}(c\bar{a}b\bar{d}\,)}.$$

Remark 2.7. It is worthwhile noticing that when $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has complex (or real) entries, then $\det_{\mathbb{H}}(A) = |ad - bc| = |\det(A)|$, i.e., the new notion of determinant coincides with the modulus of the classical determinant.

The interest of the preceding definition is made clear by the following result.

Proposition 2.8. *A matrix $A \in M(2, \mathbb{H})$ is invertible if, and only if,* $\det_{\mathbb{H}}(A) \neq 0$.

Proof. The proof is a direct consequence of Proposition 2.1 and Lemma 2.4. $\qquad\qquad\square$

We end this section by proving that the analogue of the Binet-Cauchy formula holds for $\det_{\mathbb{H}}$. This fact is established in a more general setting in [9, 16], where the proof is based on the properties of quasideterminants and does not contain all the details. It is also implicitly stated in [20]. In any case we give here a simple proof.

Lemma 2.9. *For any* λ, $\mu \in \mathbb{H}$ *and any matrix* $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in M(2, \mathbb{H})$ *we have:*

(i) $\det_{\mathbb{H}} \begin{bmatrix} x & y\lambda \\ z & t\lambda \end{bmatrix} = \det_{\mathbb{H}} \begin{bmatrix} x\lambda & y \\ z\lambda & t \end{bmatrix} = |\lambda| \det_{\mathbb{H}} \begin{bmatrix} x & y \\ z & t \end{bmatrix};$

(ii) $\det_{\mathbb{H}} \begin{bmatrix} \mu x & \mu y \\ z & t \end{bmatrix} = \det_{\mathbb{H}} \begin{bmatrix} x & y \\ \mu z & \mu t \end{bmatrix} = |\mu| \det_{\mathbb{H}} \begin{bmatrix} x & y \\ z & t \end{bmatrix};$

(iii) *If the matrix* Y *is obtained from the matrix* X *by:*

    (iii.a) *substituting to a row the sum of the two rows, or*

    (iii.b) *substituting to a column the sum of the two columns,*

    then $\det_{\mathbb{H}}(X) = \det_{\mathbb{H}}(Y)$.

Proof. A direct substitution and computation show the assertions. □

Proposition 2.10 (Binet property). *For all* $A, B \in M(2, \mathbb{H})$ *we have that*

$$\det_{\mathbb{H}}(AB) = \det_{\mathbb{H}}(A) \det_{\mathbb{H}}(B).$$

Proof. We can suppose that the matrices A, B are invertible (otherwise the proof is immediate). If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $AB = \begin{bmatrix} ae+bg & bh+af \\ ce+dg & dh+cf \end{bmatrix}$.

We will operate now on the matrix AB step by step, and use Lemma 2.9 at each step, to compute its determinant. If $h \neq 0$ then, by Lemma 2.9 (i), (iii), we have:

$$\det_{\mathbb{H}}(AB) = \det_{\mathbb{H}} \begin{bmatrix} ae + bg & bh + af \\ ce + dg & dh + cf \end{bmatrix}$$

$$= \det_{\mathbb{H}} \begin{bmatrix} ae + bg - (bh + af)h^{-1}g & bh + af \\ ce + dg - (dh + cf)h^{-1}g & dh + cf \end{bmatrix}.$$

Now

$$\begin{bmatrix} ae + bg - (bh + af)h^{-1}g & bh + af \\ ce + dg - (dh + cf)h^{-1}g & dh + cf \end{bmatrix} = \begin{bmatrix} a(e - fh^{-1}g) & bh + af \\ c(e - fh^{-1}g) & dh + cf \end{bmatrix}$$

and again by Lemma 2.9 (i), (iii) (since B is invertible, see Proposition 2.1),

$$\det{}_{\mathbb{H}} \begin{bmatrix} a(e - fh^{-1}g) & bh + af \\ c(e - fh^{-1}g) & dh + cf \end{bmatrix}$$

$$= \det{}_{\mathbb{H}} \begin{bmatrix} a(e - fh^{-1}g) & bh + af - a(e - fh^{-1}g)(e - fh^{-1}g)^{-1}f \\ c(e - fh^{-1}g) & dh + cf - c(e - fh^{-1}g)(e - fh^{-1}g)^{-1}f \end{bmatrix}.$$

We have

$$\begin{bmatrix} a(e - fh^{-1}g) & bh + af - a(e - fh^{-1}g)(e - fh^{-1}g)^{-1}f \\ c(e - fh^{-1}g) & dh + cf - c(e - fh^{-1}g)(e - fh^{-1}g)^{-1}f \end{bmatrix}$$

$$= \begin{bmatrix} a(e - fh^{-1}g) & bh \\ c(e - fh^{-1}g) & dh \end{bmatrix}$$

and, by the same argument,

$$\det{}_{\mathbb{H}} \begin{bmatrix} a(e - fh^{-1}g) & bh \\ c(e - fh^{-1}g) & dh \end{bmatrix} = \det{}_{\mathbb{H}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} |c(e-fh^{-1}g)h| = \det{}_{\mathbb{H}}(A) \det{}_{\mathbb{H}}(B).$$

In the remaining case in which $h = 0$, the coefficient f does not vanish and the matrix AB becomes

$$\begin{bmatrix} ae + bg & af \\ ce + dg & cf \end{bmatrix}.$$

Again, by Lemma 2.9 (i), (iii), we conclude

$$\det{}_{\mathbb{H}}(AB) = \det{}_{\mathbb{H}} \begin{bmatrix} ae + bg & af \\ ce + dg & cf \end{bmatrix} = \det{}_{\mathbb{H}} \begin{bmatrix} af(f^{-1}e) + bg & af \\ cf(f^{-1}e) + dg & cf \end{bmatrix}$$

$$= \det{}_{\mathbb{H}} \begin{bmatrix} bg & af \\ dg & cf \end{bmatrix} = |g| \, |f| \det{}_{\mathbb{H}}(A) = \det{}_{\mathbb{H}}(A) \det{}_{\mathbb{H}}(B). \qquad \square$$

3. Fractional Linear Transformations and their Properties

For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbb{H})$, with $c \neq 0$ or $d \neq 0$, the map

$$L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is called a (*quaternionic*) *fractional linear map*. To identify constant maps, we will give the following characterization:

Proposition 3.1. *The fractional linear map* $L_A(q) = (aq + b) \cdot (cq + d)^{-1}$ *is constant if, and only if,* $\det_\mathbb{H}(A) = 0$.

Proof. If the fractional linear transformation L_A is constant, i.e., if $L_A(q) = k$, for all $q \in \mathbb{H}$, then

$$(aq + b) \cdot (cq + d)^{-1} = k,$$
$$aq + b = kcq + kd,$$
$$(a - kc)q = kd - b,$$

for all $q \in \mathbb{H}$. Thus

$$(3.1) \qquad \begin{cases} a - kc = 0, \\ kd - b = 0, \end{cases}$$

yielding

$$A = \begin{bmatrix} kc & kd \\ c & d \end{bmatrix}$$

and $\det_\mathbb{H}(A) = 0$.

Conversely, if $\det_\mathbb{H}^2(A) = |a|^2 |d|^2 + |c|^2|b|^2 - 2\,\mathrm{Re}(c\bar{a}b\bar{d}) = 0$ and $abcd = 0$, then $cb = 0$ or $ad = 0$. If $cb = 0$, then $\det_\mathbb{H}^2(A) = |a|^2 |d|^2 = 0$ and so $ad = 0$. Similarly if $ad = 0$, then $\det_\mathbb{H}^2(A) = |c|^2 |b|^2 = 0$ and so $cb = 0$. Thus $ad = 0$ and $cb = 0$. If $c = 0$ then, by definition, $d \neq 0$ and hence $a = 0$, yielding $L_A(q) = bd^{-1}$ for all $q \in \mathbb{H}$. On the other hand, if $b = 0$ then, either $a = 0$ and $L_A \equiv 0$, or $d = 0$ implying $L_A(q) = ac^{-1}$ for all $q \in \mathbb{H}$. To conclude the proof, we notice that when $abcd \neq 0$, then by Proposition 2.4 we obtain for example $c = db^{-1}a$, which leads to

$$L_A(q) = (aq + b)(db^{-1}aq + d)^{-1}$$
$$= (aq + b)[db^{-1}(aq + bd^{-1}d)]^{-1}$$
$$= (aq + b)(aq + b)^{-1}bd^{-1} = bd^{-1}$$

for all $q \in \mathbb{H}$. $\square$

In analogy with the case of the complex plane $\mathbb{C}$, we give the following definition.

Definition 3.2. For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbb{H})$, the map

$$L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is called a (*quaternionic*) *fractional linear transformation* if $\det_{\mathbb{H}}(A) \neq 0$ i.e., if $A \in \mathrm{GL}(2, \mathbb{H})$.

In order to give a coherent, self-contained treatment of the subject, we give here an intrinsic, direct proof of the following result, already established in [20].

Theorem 3.3. *The set $\mathbb{G}$ of all quaternionic fractional linear transformations is a group with respect to composition. The map*

$$(3.2) \qquad \Phi : A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is a group homomorphism of $\mathrm{GL}(2, \mathbb{H})$ *onto* $\mathbb{G}$ *whose kernel is the center of* $\mathrm{GL}(2, \mathbb{H})$, *that is the subgroup*

$$\left\{ \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} : t \in \mathbb{R} \setminus \{0\} \right\}.$$

Proof. It is a straightforward computation to prove that, if $L_1, L_2 \in \mathbb{G}$ are such that $\Phi(A_1) = L_1$ and $\Phi(A_2) = L_2$ for some $A_1, A_2 \in \mathrm{GL}(2, \mathbb{H})$, then $\Phi(A_1 \cdot A_2) = L_1 \circ L_2$. Moreover, $\Phi(I_2) = Id$ is the identity map. As a consequence, Φ is a surjective homomorphism, and hence $\mathbb{G}$ is a group.

Now $L_A(q) = (aq + b) \cdot (cq + d)^{-1} = q$ for all $q \in \mathbb{H}$, if, and only if, $qcq + qd - aq - b = 0$ for all $q \in \mathbb{H}$ and hence $c = 0 = b$ and $a = d \in \mathbb{R}$. The last assertion follows immediately. □

If we set $\mathrm{SL}(2, \mathbb{H}) = \{A \in \mathrm{GL}(2, \mathbb{H}) : \det_{\mathbb{H}}(A) = 1\}$ then, as an application of the Binet formula (see Proposition 2.10), we obtain that $\mathrm{SL}(2, \mathbb{H})$ is a subgroup of $\mathrm{GL}(2, \mathbb{H})$ and that the following corollary holds.

Corollary 3.4. *The map*

$$(3.3) \qquad \Phi : A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is a group homomorphism of $\mathrm{SL}(2, \mathbb{H})$ *onto* $\mathbb{G}$ *whose kernel is the center of* $\mathrm{SL}(2, \mathbb{H})$, *that is the subgroup*

$$\left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Proof. The proof relies upon the fact that, for all $t \in \mathbb{R} \setminus \{0\}$ and all $A \in \mathrm{GL}(2, \mathbb{H})$, we have $\det_{\mathbb{H}}(tA) = t^2 \det_{\mathbb{H}}(A) > 0$. □

4. The Quaternionic Cross-ratio

We will generalize the classical definition of complex cross-ratio to the non commutative case of the Hamilton numbers, and study its peculiar properties.

Proposition 4.1. *Given three distinct* α, β, $\gamma \in \mathbb{H}$, *the fractional linear trans-formation defined by*

$$(\gamma - \beta)(\gamma - \alpha)^{-1}(q - \alpha)(q - \beta)^{-1}$$

maps α *to* 0, β *to* ∞ *and* γ *to* 1. *Moreover, all fractional linear transformations with the same property are of the form:*

$$k(\gamma - \beta)(\gamma - \alpha)^{-1}(q - \alpha)(q - \beta)^{-1}k^{-1}$$

with k *any element of* $\mathbb{H} \setminus \{0\}$.

Proof. Let us consider a generic element of $\mathbb{G}$ defined by

$$L_A(q) = (aq + b)(cq + d)^{-1},$$

and require that

$$L_A(\alpha) = (a\alpha + b)(c\alpha + d)^{-1} = 0,$$
$$L_A(\beta) = (a\beta + b)(c\beta + d)^{-1} = \infty,$$
$$L_A(\gamma) = (a\gamma + b)(c\gamma + d)^{-1} = 1.$$

It follows that

$$\begin{cases} a\alpha + b = 0, \\ c\beta + d = 0, \\ (a\gamma + b) = (c\gamma + d), \end{cases}$$

$$\begin{cases} b = -a\alpha, \\ d = -c\beta, \\ a(\gamma - \alpha) = c(\gamma - \beta), \end{cases}$$

$$\begin{cases} a = c(\gamma - \beta)(\gamma - \alpha)^{-1}, \\ b = -c(\gamma - \beta)(\gamma - \alpha)^{-1}\alpha, \\ d = -c\beta, \end{cases}$$

and therefore

$$L_A(q) = [c(\gamma - \beta)(\gamma - \alpha)^{-1}q - c(\gamma - \beta)(\gamma - \alpha)^{-1}\alpha](cq - c\beta)^{-1}$$
$$= c(\gamma - \beta)(\gamma - \alpha)^{-1}(q - \alpha)(q - \beta)^{-1}c^{-1}. \qquad \square$$

Following the approach due to C. Cao and P.L. Waterman, [5], we will now define the cross-ratio of 4-tuples of quaternions.

Definition 4.2. The quaternionic *cross-ratio* of four points q_1, q_2, q_3, q_4 in $\mathbb{H} \cup \{\infty\}$ is defined as:

$$CR(q_1, q_2, q_3, q_4) := (q_1 - q_3)(q_1 - q_4)^{-1}(q_2 - q_4)(q_2 - q_3)^{-1}.$$

To investigate the behaviour of the quaternionic cross-ratio under the action of the group of fractional, linear transformations, we will make use of the following decomposition lemma:

Lemma 4.3. *The group* $\mathbb{G}$ *is generated by the following four types of fractional linear transformations:*
  (i) $L_1(q) = q + b$, $b \in \mathbb{H}$;
  (ii) $L_2(q) = aq$, $a \in \mathbb{H}$, $|a| = 1$;
  (iii) $L_3(q) = rq$, $r \in \mathbb{R}^+ \setminus \{0\}$;
  (iv) $L_4(q) = q^{-1}$.
Moreover, all the elements of $\mathbb{G}$ *are conformal.*

Proof. Consider the fractional linear transformation

$$L_A(q) = (aq + b)(cq + d)^{-1}.$$

If $c = 0$, then $L_A(q) = (aq + b)d^{-1} = [d(aq + b)^{-1}]^{-1}$. If instead $c \neq 0$, simply notice that

$$L_A(q) = (aq + b)(cq + d)^{-1} = ac^{-1} + (b - ac^{-1}d)(cq + d)^{-1}$$

where $(b - ac^{-1}d) \neq 0$ since $\det_{\mathbb{H}}(A) \neq 0$ (see Proposition 2.4). This concludes the proof of the first part of the statement. The proof of the conformality of all the elements of $\mathbb{G}$ can be accomplished by observing that L_1, L_2, L_3 are obviously conformal, and by proving that L_4 is conformal as well. In fact the conjugation $q \mapsto \bar{q}$ is conformal and the $\mathbb{R}$-differential of the map $\overline{L_4(q)} = q/|q|^2$ at the point $q = x_0 + x_1 i + x_2 j + x_3 k \equiv (x_0, x_1, x_2, x_3)$ is represented (up to multiplying by $1/(x_0^2 + x_1^2 + x_2^2 + x_3^2)^2$) by the conformal matrix

$$\begin{bmatrix} -x_0^2 + x_1^2 + x_2^2 + x_3^2 & -2x_1x_0 & -2x_2x_0 & -2x_3x_0 \\ -2x_0x_1 & x_0^2 - x_1^2 + x_2^2 + x_3^2 & -2x_2x_1 & -2x_3x_1 \\ -2x_0x_2 & -2x_1x_2 & x_0^2 + x_1^2 - x_2^2 + x_3^2 & -2x_3x_2 \\ -2x_0x_3 & -2x_1x_3 & -2x_2x_3 & x_0^2 + x_1^2 + x_2^2 - x_3^2 \end{bmatrix}$$

$\square$

A set of generators analogous to the one given in the proposition above is presented in [14], in the more general Clifford setting.

Proposition 4.4. *With reference to Lemma 4.3, if $L \in \mathbb{G}$ is of type* (i) *or* (iii), *then*

$$CR(L(q_1), L(q_2), L(q_3), L(q_4)) = CR(q_1, q_2, q_3, q_4).$$

If instead $L \in \mathbb{G}$ is of type (iv), *then*

$$CR(L(q_1), L(q_2), L(q_3), L(q_4)) = q_3 CR(q_1, q_2, q_3, q_4) q_3^{-1}.$$

Finally, if $L(q) = aq$ is of type (ii), *then*

$$CR(L(q_1), L(q_2), L(q_3), L(q_4)) = a CR(q_1, q_2, q_3, q_4) a^{-1}$$

with $a \in \mathbb{S}_{\mathbb{H}}^3$.

This last statement, whose proof in our setting is a straightforward computation, is established in the general Clifford environment and in a more compact form, in [5]. Our statement has interesting consequences which will lead us to find out peculiar geometric properties of the quaternionic fractional linear transformations. Denote, as already established, by $\mathbb{S}$ the 2-sphere of pure imaginary units $\{x_1 i + x_2 j + x_3 k \in \mathbb{H} : x_1^2 + x_2^2 + x_3^2 = 1\}$ of $\mathbb{H}$ and consider, for $x, y \in \mathbb{R}$, the 2-sphere $x + y\mathbb{S}$ with center x and radius $|y|$. Then we have the following result.

Lemma 4.5. *For any x and $y \in \mathbb{R}$, the 2-sphere $x + y\mathbb{S}$ is such that $q(x + y\mathbb{S})q^{-1} = x + y\mathbb{S}$, for any $q \in \mathbb{H} \setminus \{0\}$.*

Proof. For any $x + yI \in x + y\mathbb{S}$, we have $q(x + yI)q^{-1} = qxq^{-1} + qyIq^{-1} = x + yqIq^{-1}$. Now $|qIq^{-1}| = 1$ and $\mathrm{Re}(qIq^{-1}) = \mathrm{Re}(Iq^{-1}q) = \mathrm{Re}(I) = 0$. Therefore $qIq^{-1} \in \mathbb{S}$, which concludes the proof. □

Proposition 4.4 and Lemma 4.5 directly imply that the orbits of the cross-ratio of four quaternions (under the action of the group of fractional linear transformations) are 2-spheres of type $x + y\mathbb{S}$, as established in the following corollary.

Corollary 4.6. *Let $CR(q_1, q_2, q_3, q_4) = x + yI \in x + y\mathbb{S}$ be the cross-ratio of the four quaternions q_1, q_2, q_3, q_4. Then*

$$\left\{ CR(L(q_1), L(q_2), L(q_3), L(q_4)) : L \in \mathbb{G} \right\} = x + y\mathbb{S}.$$

In particular, when the cross ratio of four quaternions is real, then it is invariant under the action of all fractional, linear transformations.

Let us set $S_3 = \{q + r\mathbb{S}^3_{\mathbb{H}} : q \in \mathbb{H},\ r \in \mathbb{R}^+ \setminus \{0\}\}$ to be the family of all 3-(real)-dimensional spheres of $\mathbb{H}$, and denote by $\mathcal{P}_3$ the family of all 3-(real)-dimensional affine spaces of $\mathbb{H}$. If $\mathcal{F}_3 = S_3 \cup \mathcal{P}_3$, then we can state and prove the following result, which closely resembles the classical statement that holds for all fractional linear transformations of $\mathbb{C}$. This result can also be viewed as a consequence of Wilker's identification of quaternionic linear fractional transformations with the classical Möbius group of the extended four dimensional space (see [20]); nevertheless, our approach is simpler and intrinsic to the quaternions.

Proposition 4.7. *The group* $\mathbb{G}$ *of all fractional, linear transformations maps elements of* $\mathcal{F}_3$ *onto elements of* $\mathcal{F}_3$, *i.e., it transforms the family of all 3-spheres and 3-dimensional, affine planes of* $\mathbb{H}$ *onto itself.*

Proof. Indeed the family of sets $\mathcal{F}_3$ is the family of zero-sets of the quadratic equations

$$(4.1) \qquad \alpha(q\bar{q}) + \beta q + \bar{q}\bar{\beta} + \gamma = 0$$

where $\alpha,\ \gamma \in \mathbb{R}$ and $\beta \in \mathbb{H}$. In fact, if we set $q = x_0 + x_1 i + x_2 j + x_3 k$ and $\beta = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$, equation (4.1) becomes

$$\alpha(x_0^2 + x_1^2 + x_2^2 + x_3^2) + 2\,\mathrm{Re}(\beta q) + \gamma = 0,$$

i.e.,

$$(4.2) \qquad \alpha(x_0^2 + x_1^2 + x_2^2 + x_3^2) + 2(\beta_0 x_0 - \beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3) + \gamma = 0.$$

By varying $\alpha,\ \gamma$ in $\mathbb{R}$ and β in $\mathbb{H}$, we obtain the entire family $\mathcal{P}_3$ as the family of zero-sets of (4.2) when $\alpha = 0$, and the entire family S_3 when $\alpha \neq 0$. At this point, it is enough to prove that the elements of $\mathbb{G}$ transform an equation of type (4.1) in an equation of the same type. If $L_1(q) = q + b$, with $b \in \mathbb{H}$, then equation (4.1) becomes

$$(4.3a) \qquad \alpha((q+b)\overline{(q+b)}) + \beta(q+b) + \overline{(q+b)}\bar{\beta} + \gamma = 0,$$

$$(4.3b) \qquad \alpha(q\bar{q}) + \alpha(2\,\mathrm{Re}(q\bar{b})) + 2\,\mathrm{Re}(\beta q) + \alpha|b|^2 + 2\,\mathrm{Re}(\beta b) + \gamma = 0,$$

$$(4.3c) \qquad \alpha(q\bar{q}) + \alpha(2\,\mathrm{Re}(\bar{b}q)) + 2\,\mathrm{Re}(\beta q) + \alpha|b|^2 + 2\,\mathrm{Re}(\beta b) + \gamma = 0,$$

$$(4.3d) \qquad \alpha(q\bar{q}) + 2\,\mathrm{Re}((\alpha\bar{b} + \beta)q) + \alpha|b|^2 + 2\,\mathrm{Re}(\beta b) + \gamma = 0,$$

$$(4.3e) \qquad \alpha(q\bar{q}) + (\alpha\bar{b} + \beta)q + \bar{q}(\overline{\alpha\bar{b} + \beta}) + \alpha|b|^2 + 2\,\mathrm{Re}(\beta b) + \gamma = 0,$$

which is still an equation of the same type. If $L_2(q) = aq$, with $a \in \mathbb{S}^3_{\mathbb{H}}$, then (4.1) becomes

(4.4a) $$\alpha(aq)(\overline{aq}) + \beta(aq) + (\overline{aq})\bar{\beta} + \gamma = 0,$$

(4.4b) $$\alpha(q\bar{q}) + (\beta a)q + \bar{q}(\bar{a}\bar{\beta}) + \gamma = 0,$$

(4.4c) $$\alpha(q\bar{q}) + (\beta a)q + \bar{q}(\overline{\beta a}) + \gamma = 0.$$

Under the action of $L_3(q) = rq$, with $r \in \mathbb{R}^+ \setminus \{0\}$, equation (4.1) transforms into

(4.5) $$r^2\alpha(q\bar{q}) + (r\beta)q + \bar{q}(\overline{r\beta}) + \gamma = 0$$

while, for $L_4(q) = q^{-1}$, it becomes

(4.6a) $$\alpha + \beta\bar{q} + q\bar{\beta} + \gamma(q\bar{q}) = 0,$$

(4.6b) $$|\beta|^2\alpha + |\beta|^2\bar{q}\beta + \bar{\beta}q|\beta|^2 + |\beta|^2\gamma(q\bar{q}) = 0,$$

(4.6c) $$\gamma(q\bar{q}) + \bar{\beta}q + \bar{q}\beta + \alpha = 0.$$

What is established in Lemma 4.3 leads to the conclusion of the proof. ◻

The geometrical properties of the elements of the group $\mathbb{G}$ are quite interesting: they are a generalization, and an extension to higher dimensions, of the geometrical properties of the classical group of complex fractional linear transformations. To give a clear idea of what we mean by this, we will denote by $\mathcal{F}_i$, for $i = 1, 2$, the family consisting of all i-(real)-dimensional spheres and all i-(real)-dimensional affine spaces of $\mathbb{H}$ and state the following result:

Corollary 4.8. *The group $\mathbb{G}$ of all fractional, linear transformations maps elements of $\mathcal{F}_2$ onto elements of $\mathcal{F}_2$ and elements of $\mathcal{F}_1$ onto elements of $\mathcal{F}_1$, i.e., it transforms the family of all 2-spheres and 2-dimensional, affine planes of $\mathbb{H}$ onto itself and the family of all circles and affine lines of $\mathbb{H}$ onto itself.*

Proof. Since all the elements of $\mathcal{F}_2$ and $\mathcal{F}_1$ are obtained as finite intersections of element of $\mathcal{F}_3$, the proof is a consequence of Proposition 4.7. ◻

The above corollary will play a key role while, in the sequel, we will define the Poincaré distance on the open unit disc $\Delta_{\mathbb{H}}$ of $\mathbb{H}$. To prepare the tools to be able to give such a definition, we will study the characterizing properties of the quaternionic cross-ratio. We point out that the first part of the following statement is proven in [5] in the more general Clifford setting. We present here a direct proof to maintain the paper self-contained.

Theorem 4.9. *Four pairwise distinct points* q_1, q_2, q_3, $q_4 \in \mathbb{H}$ *lie on the same* (*one-dimensional*) *circle if, and only if, their cross-ratio is real. The two pairs of points* q_1, q_2 *and* q_3, q_4 *lying on the same circle separate each other if, and only if,* $CR(q_1, q_2, q_3, q_4) < 0$.

Proof. The three pairwise distinct points q_2, q_3, q_4 determine a unique circle (or line) $C \subset \mathbb{H}$. In view of Proposition 4.1, take $L \in \mathbb{G}$ that maps q_2, q_3, q_4 respectively to 1, 0, ∞ and let $q_0 = L(q_1)$.

Then, by Corollary 4.8, L carries C onto the real axis $\mathbb{R}$ of $\mathbb{H}$ and, by Proposition 4.4, it is such that $CR(L(q_1), L(q_2), L(q_3), L(q_4)) = CR(q_0, 1, 0, \infty) = q_0$ is conjugated to $CR(q_1, q_2, q_3, q_4)$. We conclude that $q_0 \in \mathbb{R}$ if, and only if, $q_1 \in C$. Equivalently $CR(L(q_1), L(q_2), L(q_3), L(q_4)) = q_0 \in \mathbb{R}$ if, and only if, $CR(q_1, q_2, q_3, q_4) \in \mathbb{R}$ if, and only if, q_1, q_2, q_3, $q_4 \in C$. This proves the first part of our assertion.

To complete the proof notice that $CR(q_0, 1, 0, \infty) = q_0 < 0$ if, and only if, the two pairs of points q_0, 1 and 0, ∞ separate each other on the real axis. Since L maps the circular arc A from q_3 to q_4 through q_2 onto the positive real half axis, then (by the continuity of L) the pre-image q_1 of q_0 cannot belong to A. Therefore q_1, q_2 and q_3, q_4 separate each other. □

When defining the Poincaré distance on the open, unit disc of $\mathbb{H}$, we will be interested in the case in which the two pairs of points q_1, q_2 and q_3, q_4 lie on a same circle and do not separate each other. In this case, if we keep q_2, q_3, q_4 fixed and move q_1 from q_3 to q_4, by way of q_2, then, within the environment established in the proof of Theorem 4.9, the point q_0 moves from 0 to ∞ by way of 1. In view of

$$(4.7) \qquad\qquad CR(q_1, q_2, q_3, q_4) = CR(q_0, 1, 0, \infty) = q_0$$

during this procedure the cross ratio $CR(q_1, q_2, q_3, q_4)$ takes on all the positive values; in particular the value 1 comes up when $q_1 = q_2$. Moreover, we have the following result.

Proposition 4.10. *Let* q_1, q_2, q_3, $q_4 \in \mathbb{H}$ *be pairwise distinct points, arranged cyclically on a circle. Then* $CR(q_1, q_2, q_3, q_4) > 1$. *Moreover,* $q_1 = q_2$ *if, and only if,* $CR(q_1, q_2, q_3, q_4) = 1$.

Proof. If q_1, q_2, q_3, q_4 are arranged cyclically, so are q_0, 1, 0, ∞. Therefore, in view of (4.7), we have $q_0 = CR(q_0, 1, 0, \infty) > 1$. □

5. Möbius Transformations and the Poincaré Distance on $\Delta_{\mathbb{H}}$

We are now ready to construct a Poincaré-type distance, which we will simply call Poincaré distance, on the open unit disc $\Delta_{\mathbb{H}}$ of $\mathbb{H}$. We will do this by developing, in the quaternionic case, a variation of an approach adopted by Ahlfors in a different algebraic situation placed in the n-dimensional real vector space $\mathbb{R}^n$, [3].

We will start by defining the non-euclidean line through any two points q_1, $q_2 \in \Delta_{\mathbb{H}}$. We will use a "slicewise" approach.

Definition 5.1. If $q_1 \neq q_2 \in \Delta_{\mathbb{H}}$ are $\mathbb{R}$-linearly dependent, i.e., if they lie on a same diameter of the disc $\Delta_{\mathbb{H}} \subset \mathbb{H} \cong \mathbb{R}^4$, then we define the *non-Euclidean line* through q_1 and q_2 to be this diameter. When q_1, q_2 are $\mathbb{R}$-linearly independent, then they belong to a unique circle that intersects $\mathbb{S}_{\mathbb{H}}^3 = \partial \Delta_{\mathbb{H}}$ orthogonally and that will be defined to be the *non-Euclidean line* through q_1 and q_2.

To clarify the geometrical significance of the above definition, let us remark that any circle C which intersects $\mathbb{S}_{\mathbb{H}}^3$ orthogonally belongs to the 2-dimensional, real vector space $\Pi(C)$ spanned by the two vectors obtained as $\mathbb{S}_{\mathbb{H}}^3 \cap C$. Now, when q_1, $q_2 \in C$ are $\mathbb{R}$-linearly independent, they span a 2-dimensional, real vector space $\Pi(q_1, q_2) \subset \mathbb{H}$, which obviously must coincide with $\Pi(C)$. Therefore C is the classical non-Euclidean line of the 2-(real)-dimensional, open, unit disc $\Delta_{\mathbb{H}} \cap \Pi(q_1, q_2)$ passing through q_1, q_2.

Theorem 5.2. *For any given q_1, $q_2 \in \Delta_{\mathbb{H}}$, with $q_1 \neq q_2$, the unique non-Euclidean line ℓ containing q_1 and q_2 is the circle or the straight line determined by the four points q_1, q_2, $\overline{q_1}^{-1}$, $\overline{q_2}^{-1}$.*

Proof. Suppose q_1, q_2 are $\mathbb{R}$-linearly independent. Since $\overline{q_1}^{-1} = q_1 |q_1|^{-2}$ and $\overline{q_2}^{-1} = q_2 |q_2|^{-2}$, the four given points determine the 2-dimensional real subspace $\Pi(q_1, q_2)$ spanned by q_1, q_2. An easy computation shows that

$$\mathcal{CR}(q_1, q_2, \overline{q_1}^{-1}, \overline{q_2}^{-1}) = |q_1|^2 \, |q_2|^2 - q_2 \overline{q_1} - q_1 \overline{q_2} + 1 \in \mathbb{R}.$$

Thus q_1, q_2, $\overline{q_1}^{-1}$, $\overline{q_2}^{-1}$ lie on a same circle $\ell \subset \Pi(q_1, q_2)$. To prove that the circle ℓ is orthogonal to $\mathbb{S}_{\mathbb{H}}^3$, notice that the points q_1, q_2, $q_1 |q_1|^{-2}$, $q_2 |q_2|^{-2} \in \Pi(q_1, q_2) \equiv \mathbb{R}^2$, when placed on the complex plane $\mathbb{C}$ via the identification $\mathbb{R}^2 \cong \mathbb{C}$, can still be written as q_1, q_2, $\overline{q_1}^{-1}$, $\overline{q_2}^{-1}$. Therefore the proof reduces to the classical proof for the complex plane (see, e.g., [17]). The remaining case in which q_1, q_2 are $\mathbb{R}$-linearly dependent is straightforward. $\square$

We now turn our attention to investigate the structure of the group $\mathbb{M}$ of *Möbius transformations*, i.e., of the subgroup $\mathbb{M}$ of $\mathbb{G}$ consisting of all fractional linear transformations mapping the quaternionic, open, unit disc $\Delta_{\mathbb{H}}$ onto itself. First of all we recall that, once named $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the (classical) group of matrices (with quaternionic entries) $\mathrm{Sp}(1,1)$ is defined as (see, e.g., [11])

$$(5.1) \qquad \mathrm{Sp}(1,1) = \{A \in M(2, \mathbb{H}) : {}^t \bar{A} H A = H\}$$

and it can be written equivalently as (see, e.g., [6])

$$\mathrm{Sp}(1,1) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : |a| = |d|, \ |b| = |c|, \ |a|^2 - |c|^2 = 1, \ \bar{a}b = \bar{c}d, \ a\bar{c} = b\bar{d} \right\}.$$

In terms of $\mathrm{Sp}(1,1)$, we can rephrase and complete a result of [6] as follows:

Theorem 5.3. *The quaternionic, fractional linear transformation defined by formula* $g(q) = (aq + b)(cq + d)^{-1}$ *is a Möbius transformation of* $\Delta_{\mathbb{H}}$ *if and only if* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}(1,1)$. *Moreover, the map*

$$\varphi : \mathrm{Sp}(1,1) \to \mathbb{M}$$

(5.2)
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq + b) \cdot (cq + d)^{-1}$$

is a group homomorphism whose kernel is the center of $\mathrm{Sp}(1,1)$, *that is the subgroup*

$$\left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Notice, in particular, that for all $A \in \mathrm{Sp}(1,1)$, we have

(5.3) $$\det_{\mathbb{H}}(A) = \sqrt{|a|^4 + |c|^4 - 2|a|^2 |c|^2} = (|a|^2 - |c|^2) = 1$$

and hence $\mathrm{Sp}(1,1) \subset \mathrm{SL}(2, \mathbb{H})$.

By means of the statement of Theorem 5.3 we are able to obtain, for the quaternionic Möbius transformations, a characterization which closely resembles the classical representation of the complex Möbius transformations. A similar result is stated without proof in [13].

Theorem 5.4. *Each quaternionic Möbius transformation*

$$g(q) = (aq + b) \cdot (cq + d)^{-1} \in \mathbb{M}$$

can be written uniquely as:

(5.4) $$g(q) = \alpha(q - q_0)(1 - \overline{q_0}q)^{-1}\beta^{-1}$$

where $q_0 = -u \tanh(t) = -a^{-1}b \in \Delta_{\mathbb{H}}$ *and where* $\alpha = a/|a| \in \mathbb{S}^3_{\mathbb{H}}$, $\beta = d/|d| \in \mathbb{S}^3_{\mathbb{H}}$.

Proof. Since Theorem 5.3 implies that $|a|^2 - |c|^2 = 1$ and that $|d|^2 - |b|^2 = 1$, we obtain

$$a = \alpha \cosh(t), \quad b = \gamma \sinh(t),$$
$$d = \beta \cosh(t), \quad c = \delta \sinh(t),$$

with α, β, γ, $\delta \in \mathbb{S}_{\mathbb{H}}^3$. Moreover, $\bar{a}b = \bar{c}d$ yields $\bar{\alpha}\gamma = \bar{\delta}\beta$. We now recall that $\bar{\alpha} = \alpha^{-1}$ because $|\alpha| = 1$, and hence $\alpha^{-1}\gamma = \delta^{-1}\beta$. If $u := \alpha^{-1}\gamma = \delta^{-1}\beta$, then $\alpha = \gamma u^{-1}$, $\beta = \delta u$. Finally

$$G = \begin{bmatrix} \gamma u^{-1} \cosh(t) & \gamma \sinh(t) \\ \delta \sinh(t) & \delta u \cosh(t) \end{bmatrix}$$

and the fractional linear map associated to G becomes

$$g(q) = ((\gamma u^{-1} \cosh(t))q + \gamma \sinh(t))((\delta \sinh(t))q + \delta u \cosh(t))^{-1}.$$

We extract $\gamma u^{-1} \cosh(t)$ from the first factor and $\delta u \cosh(t)$ from the second factor and we obtain

$$\begin{aligned} g(q) &= (\gamma u^{-1} \cosh(t))(q + u \tanh(t))(u^{-1} \tanh(t)q + 1)^{-1}(\delta u \cosh(t))^{-1} \\ &= (\gamma u^{-1})(q + u \tanh(t))(u^{-1} \tanh(t)q + 1)^{-1}u^{-1}\delta^{-1} \end{aligned}$$

where γ, u, $\delta \in \mathbb{S}_{\mathbb{H}}^3$, and $u \tanh(t) = \overline{u^{-1} \tanh(t)}$. Therefore the Möbius transformations of the unit disc are of the form

$$g(q) = \alpha(q - q_0)(1 - \overline{q_0}q)^{-1}\beta^{-1}$$

where α, $\beta \in \mathbb{S}_{\mathbb{H}}^3$ and where $q_0 = -u \tanh(t) = -a^{-1}b \in \Delta_{\mathbb{H}}$. It is now an easy exercise to verify that the maps of the form (5.4) transform the unit disc of $\mathbb{H}$ onto itself. Indeed:

$$\begin{aligned} 1 &- \alpha(q - q_0)(1 - (\overline{q_0})q)^{-1}\beta^{-1}\overline{\beta^{-1}}(1 - \bar{q}q_0)^{-1}(\bar{q} - \overline{q_0})\bar{\alpha} \\ &= 1 - |(q - q_0)|^2 \, |(1 - \overline{q_0}q)^{-1}|^2 \\ &= (|1 - \overline{q_0}q|^2 - |q - q_0|^2)|(1 - \overline{q_0}q)^{-1}|^2 \\ &= (1 - \bar{q}q_0 - \overline{q_0}q + |q_0|^2 \, |q|^2 - |q|^2 + q\overline{q_0} + q_0\bar{q} - |q_0|^2)|(1 - \overline{q_0}q)^{-1}|^2 \\ &= (1 - q_0\overline{q_0})(1 - q\bar{q})|(1 - \overline{q_0}q)^{-1}|^2 \end{aligned}$$

because $2\,\mathrm{Re}(\bar{q}q_0) - 2\,\mathrm{Re}(q\overline{q_0}) = 0$.

An alternative proof can be obtained as follows.

If $g(q) = (aq + b)(cq + d)^{-1} \in \mathbb{G}$ belongs to the group of the Möbius transformations $\mathbb{M}$ and fixes 0, then $b = 0$ and, by the given characterization of $Sp(1,1)$, $c = 0$ and $|a| = |d| = 1$. Therefore each Möbius transformation which fixes 0 is of type $g(q) = aqd^{-1}$. Now let g be a Möbius transformation such that $g(0) = -p_0 = bd^{-1}$. If we compose g with $h(q) = (q + p_0)(1 + \overline{p_0}q)^{-1}$, then

$(h \circ g)$ fixes 0, and hence $(h \circ g)(q) = aqd^{-1}$. Finally

$$\begin{aligned}
g(q) &= h^{-1}(aqd^{-1}) = (aqd^{-1} - p_0)(1 - \overline{p_0}aqd^{-1})^{-1}\\
&= a(q - \bar{a}p_0 d)\bar{d}d(1 - \bar{d}\overline{p_0}aq)^{-1}d^{-1}\\
&= a(q - q_0)(1 - \overline{q_0}q)^{-1}d^{-1}
\end{aligned}$$

where $q_0 = \bar{a}p_0 d = a(-bd^{-1})d = -a^{-1}b$. $\qquad\square$

As we already mentioned, the Möbius transformations form a subgroup of the group $\mathbb{G}$ of all fractional linear transformations of $\mathbb{H}$. It is of interest to consider two Möbius transformations in their form (5.4) and find the form (5.4) of their composition. Indeed, given the two transformations

$$(5.5) \qquad\qquad g_1(q) = a(q - q_0)(1 - \overline{q_0}q)^{-1}b^{-1},$$

$$(5.6) \qquad\qquad g_2(q) = c(q - p_0)(1 - \overline{p_0}q)^{-1}d^{-1}$$

(with $|a| = |b| = |c| = |d| = 1$ and $|q_0| < 1$, $|p_0| < 1$) it is easy to verify that the composition $g = g_1 \circ g_2$ is the transformation associated to the matrix

$$\begin{bmatrix} a & -aq_0 \\ -b\overline{q_0} & b \end{bmatrix}\begin{bmatrix} c & -cp_0 \\ -d\overline{p_0} & d \end{bmatrix} = \begin{bmatrix} ac + aq_0 d\overline{p_0} & -(acp_0 + aq_0 d) \\ -(b\overline{q_0}c + bd\overline{p_0}) & b\overline{q_0}cp_0 + bd \end{bmatrix}.$$

Since

$$\begin{aligned}
|ac + aq_0 d\,\overline{p_0}| &= |ac|\,|1 + c^{-1}q_0 d\,\overline{p_0}|\\
&= [1 + |p_0|^2\,|q_0|^2 + 2\,\mathrm{Re}(p_0\bar{d}\,\overline{q_0}c)]^{1/2},\\
|b\overline{q_0}cp_0 + bd| &= |bd|\,|1 + d^{-1}\overline{q_0}cp_0|\\
&= [1 + |p_0|^2\,|q_0|^2 + 2\,\mathrm{Re}(\overline{p_0}\bar{c}q_0 d)]^{1/2},
\end{aligned}$$

and since

$$\mathrm{Re}(p_0\bar{d}\,\overline{q_0}c) = \mathrm{Re}(cp_0\bar{d}\,\overline{q_0}) = \mathrm{Re}(\overline{(cp_0)}\,\overline{(d\,\overline{q_0})}) = \mathrm{Re}(\overline{p_0}\bar{c}q_0 d),$$

then

$$|ac + aq_0 d\,\overline{p_0}| = |b\overline{q_0}cp_0 + bd|.$$

Moreover,

$$\begin{aligned}
\overline{(ac + aq_0 d\,\overline{p_0})}(acp_0 + aq_0 d) &= (p_0\bar{d}\,\overline{q_0}\bar{a} + \bar{c}\bar{a})(acp_0 + aq_0 d)\\
&= p_0 + p_0|q_0|^2 + \bar{c}q_0 d + p_0\bar{d}\,\overline{q_0}cp_0
\end{aligned}$$

and

$$\overline{(b\overline{q_0}cp_0 + bd)}(b\overline{q_0}c + bd\,\overline{p_0}) = (\bar{d}\bar{b} + \overline{p_0}\bar{c}q_0\bar{b})(b\overline{q_0}c + bd\,\overline{p_0})$$
$$= \overline{p_0} + \overline{p_0}|q_0|^2 + \bar{d}\,\overline{q_0}c + \overline{p_0}\bar{c}q_0 d\,\overline{p_0}.$$

In conclusion we can write

(5.7) $$g(q) = \alpha(q - w_0)(1 - \overline{w_0}q)^{-1}\beta^{-1}$$

where

$$\alpha = \frac{(ac + aq_0 d\,\overline{p_0})}{|ac + aq_0 d\,\overline{p_0}|}, \quad \beta = \frac{bd + b\overline{q_0}cp_0}{|bd + b\overline{q_0}cp_0|},$$

$$w_0 = (ac + aq_0 d\,\overline{p_0})^{-1}(acp_0 + aq_0 d)$$

$$= \frac{p_0 + p_0|q_0|^2 + \bar{c}q_0 d + p_0\bar{d}\,\overline{q_0}cp_0}{|ac + aq_0 d\,\overline{p_0}|^2}.$$

Notice that the inverse of the Möbius transformation

$$g(q) = \alpha(q - q_0)(1 - \overline{q_0}q)^{-1}\beta^{-1}$$

is given by $g^{-1}(q) = \alpha^{-1}(q + \alpha q_0\bar{\beta})(1 + \beta\overline{q_0}\bar{\alpha}q)^{-1}\beta$.

Remark 5.5. The determinant $\det_{\mathbb{H}}(M)$ of the matrix M associated to the Möbius transformation $g(q) = \alpha(q - q_0)(1 - \overline{q_0}q)^{-1}\beta^{-1}$ is equal to $(1 - |q_0|^2)$, in accordance with what happens in the complex case.

We are now ready to develop the announced geometric approach to the definition of the quaternionic Poincaré distance on $\Delta_{\mathbb{H}}$. To this end, we consider the non-euclidean line ℓ determined by $q_1, q_2 \in \Delta_{\mathbb{H}}$ as given in Definition 5.1 and call *ends* of ℓ the two intersection points $\ell \cap \mathbb{S}^3_{\mathbb{H}}$. We name q_3 and q_4 such ends, so that q_1, q_2, q_3, q_4 are arranged cyclically on ℓ. Then we define:

(5.8) $$\delta_\Delta(q_1, q_2) = \frac{1}{2}\log(CR(q_1, q_2, q_3, q_4))$$

to be the *Poincaré distance* between q_1 and q_2.

Proposition 5.6. *The map* $\delta_\Delta(q_1, q_2) = \frac{1}{2}\log(CR(q_1, q_2, q_3, q_4))$ *defined in* (5.8) *is a distance.*

Before proving that δ_Δ is actually a distance, we need to state properties of invariance for it. The following geometrical feature of the elements of $\mathbb{M}$ is important in the sequel.

Lemma 5.7. *The Möbius transformations map non-Euclidean lines of* $\Delta_{\mathbb{H}}$ *onto non-Euclidean lines of* $\Delta_{\mathbb{H}}$.

Proof. Thanks to Lemma 4.3, the Möbius transformations are conformal. Since any Möbius transformation maps $\mathbb{S}^3_{\mathbb{H}}$ onto itself, the proof is concluded in view of Corollary 4.8. □

We will now make some remarks on the geometrical properties of the Möbius transformations. For a non real q_0, let us consider

$$g(q) = (q - q_0)(1 - \overline{q_0}q)^{-1}.$$

Remark 5.8. The map g transforms any 2-dimensional real plane containing the line $\ell_{q_0} = \{tq_0 : t \in \mathbb{R}\}$ onto a 2-dimensional real plane containing the same line ℓ_{q_0}. Any other 2-dimensional real vector subspace not containing the point q_0 is mapped onto a 2-dimensional sphere orthogonal to $\mathbb{S}^3_{\mathbb{H}}$.

This last remark can be explained as follows: since $g(q_0) = 0$ and $g(0) = -q_0$, by Corollary 4.8 and in view of the conformality of g (see Lemma 4.3), we get that $g(\ell_{q_0}) = \ell_{q_0}$. Therefore Corollary 4.8 leads to the proof of the first part of the remark. If a 2-dimensional real plane π does not contain the line ℓ_{q_0}, then $0 \notin g(\pi)$ and, again by conformality and by Corollary 4.8, we obtain that $g(\pi)$ is a 2-dimensional sphere orthogonal to $\mathbb{S}^3_{\mathbb{H}}$.

Let us now consider the group $\mathbb{M}^*$ of *extended Möbius transformations* defined as the union of all the Möbius transformations $g \in \mathbb{M}$ and all maps h obtained as $h(q) = g(\bar{q})$ for $g \in \mathbb{M}$ (see [20]).

Proposition 5.9. *The Poincaré distance of* $\Delta_{\mathbb{H}}$ *is invariant under the action of the group of all extended Möbius transformations* $\mathbb{M}^*$.

Proof. Let us start the proof by recalling that the map $q \mapsto \bar{q}$ is conformal and transforms $\Delta_{\mathbb{H}}$ onto itself. Therefore, as all Möbius transformations, it maps non-Euclidean lines onto non-Euclidean lines transforming ends in ends. To conclude the proof, let us observe that the cross-ratio, when real, is invariant with respect to the action of all elements of $\mathbb{M}$ (see Corollary 4.6) and with respect to $q \mapsto \bar{q}$. □

Proof of Proposition 5.6. By Proposition 4.10, the cross ratio $CR(q_1, q_2, q_3, q_4)$ of the four points is a real positive number greater than 1 and therefore its real logarithm is well defined and positive. We want to prove now that δ_Δ is symmetric. Interchanging q_1 and q_2 requires interchanging q_3 and q_4 to maintain the cyclical order. After simple computations, we find:

$$CR(q_1, q_2, q_3, q_4) = k_1 k_2 \in \mathbb{R}$$

where $k_1 = (q_1 - q_3)(q_1 - q_4)^{-1} \in \mathbb{H}$ and $k_2 = (q_2 - q_4)(q_2 - q_3)^{-1} \in \mathbb{H}$. Similarly:

$$CR(q_2, q_1, q_4, q_3) = k_2 k_1 \in \mathbb{R}.$$

Since $k_1 k_2 \in \mathbb{R}$ it follows that $k_1 k_2 = k_2 k_1$. Hence

$$CR(q_1, q_2, q_3, q_4) = CR(q_2, q_1, q_4, q_3)$$

and hence
$$\delta_\Delta(q_1, q_2) = \delta_\Delta(q_2, q_1),$$

i.e., δ_Δ is symmetric. We also have that $q_1 = q_2$ if, and only if, $\delta_\Delta(q_1, q_2) = 0$ because $CR(q_1, q_1, q_3, q_4) = 1$ if, and only if, $q_1 = q_2$ (see Proposition 4.10). The last thing to prove is the triangle inequality. To this purpose, for any $q_0, q_1,$ $q_2 \in \Delta_{\mathbb{H}}$ we have to show that

$$(5.9) \qquad \delta_\Delta(q_1, q_2) \le \delta_\Delta(q_1, q_0) + \delta_\Delta(q_0, q_2).$$

In view of Theorem 5.4, consider the following Möbius transformation of $\Delta_{\mathbb{H}}$ onto itself,

$$(5.10) \qquad L(q) = \lambda_1 (q - a)(1 - \bar{a}q)^{-1} \lambda_2$$

where $a = q_1$, $\lambda_1 = |q_2 - q_1|(q_2 - q_1)^{-1}$, $\lambda_2 = (1 - \overline{q_1}q_2)|1 - \overline{q_1}q_2|^{-1}$ and $t = |q_2 - q_1| \, |1 - \overline{q_1}q_2| \in \mathbb{R}$. The transformation L maps q_1 to 0 and q_2 to $t \in \mathbb{R}^+$. By Proposition 5.9, the map δ_Δ is invariant under the action of L, and hence (5.9) is equivalent to

$$(5.11) \qquad \delta_\Delta(0, t) \le \delta_\Delta(0, L(q_0)) + \delta_\Delta(L(q_0), t).$$

Let $L(q_0) = x + yI$, for some $x, y \in \mathbb{R}$, $I \in \mathbb{S}$. Clearly $L(q_0)$, 0, $t \in L_I \cap \Delta_{\mathbb{H}} = \Delta_I$. Since, by construction, δ_Δ restricted to Δ_I coincides with the Poincaré distance of Δ_I, we prove (5.11) and conclude the proof. $\square$

At this point we are ready to exhibit a formula for the Poincaré distance of $\Delta_{\mathbb{H}}$. In fact, by the definition of δ_Δ (see (5.8)) we have, for any $t \in \mathbb{R}^+$:

$$(5.12) \qquad \delta_\Delta(0, t) = \frac{1}{2} \log(CR(0, t, 1, -1)) = \frac{1}{2} \log \frac{1 + t}{1 - t}.$$

Since the Poincaré distance is invariant by rotations we get

$$(5.13) \qquad \delta_\Delta(0, q) = \delta_\Delta(0, |q|) = \frac{1}{2} \log \frac{1 + |q|}{1 - |q|}.$$

In general, if we consider the isometry $q \mapsto (q - q_1)(1 - \overline{q_1}q)^{-1}$, for arbitrary $q_1,$ $q_2 \in \Delta_{\mathbb{H}}$, we obtain that:

$$(5.14) \qquad \delta_\Delta(q_1, q_2) = \delta_\Delta(0, \, |q_2 - q_1| \, |1 - \overline{q_1}q_2|^{-1})$$

$$= \frac{1}{2} \log \left(\frac{1 + |q_1 - q_2| \, |1 - \overline{q_1}q_2|^{-1}}{1 - |q_1 - q_2| \, |1 - \overline{q_1}q_2|^{-1}} \right).$$

We will end this section by defining and studying the main properties of the analogous of the Poincaré differential metric — which we will often simply call

Poincaré metric — in the case of the unit disc $\Delta_\mathbb{H}$ of $\mathbb{H}$. To this aim we mimic the definition of the classical complex Poincaré differential metric of $\mathbb{D} \subset \mathbb{C}$ to set the length of the vector $\tau \in \mathbb{H}$ for the Poincaré metric at $q \in \Delta_\mathbb{H}$ to be the number:

$$(5.15) \qquad \langle \tau \rangle_q = \frac{|\tau|}{1 - |q|^2}.$$

Formula (5.15) leads now to the definition of the (square of the) Poincaré length element at $q \in \Delta_\mathbb{H}$:

$$ds^2 = \frac{|d_I q|^2}{(1 - |q|^2)^2}$$

where $q = x + yI$ and $d_I q = dx + I\, dy$ (for $I \in \mathbb{S}$).

The following result has its own independent interest, as we will see later.

Theorem 5.10. *All the elements of the group* $\mathbb{M}^*$ *of extended Möbius transformations of* $\Delta_\mathbb{H}$ *leave the Poincaré differential metric (5.15) invariant.*

Proof. To begin with we recall that, by Theorem 5.4, all the elements of $\mathbb{M}$ can be written uniquely as:

$$(5.16) \qquad h(q) = \alpha(q - q_0)(1 - \overline{q_0} q)^{-1} \beta^{-1}$$

where $q_0 \in \Delta_\mathbb{H}$ and where $\alpha, \beta \in \mathbb{S}^3_\mathbb{H}$. Since the (right and left) multiplication by elements of $\mathbb{S}^3_\mathbb{H}$ obviously leaves the Poincaré differential metric invariant, we are left to prove the invariance of the differential metric under the action of the Möbius transformations of type

$$(5.17) \qquad g(q) = (q - q_0)(1 - \overline{q_0} q)^{-1}.$$

By Lemma 4.3, the map g can be decomposed as follows

$$(5.18) \qquad g(q) = -\overline{q_0}^{-1} + (-q_0 + \overline{q_0}^{-1})(1 - \overline{q_0} q)^{-1}.$$

Since (again by Lemma 4.3) all the elements of $\mathbb{M}$ are conformal, we will compute the dilation coefficients of the differentials of the single components of g. The dilation coefficient of the map $q \mapsto (-q_0 + \overline{q_0}^{-1})q$ is

$$|(-q_0 + \overline{q_0}^{-1})| = \frac{1 - |q_0^2|}{|q_0|}.$$

What is written in the proof of Lemma 4.3 yields that the dilation coefficient of $(1 - \overline{q_0} q)^{-1}$ is $|q_0|/|1 - \overline{q_0} q|^2$. Therefore the total dilation coefficient of g is $(1 - |q_0|^2)/|1 - \overline{q_0} q|^2$. Since $1 - |g(q)|^2 = (1 - |q_0|^2)/|1 - \overline{q_0} q|^2$, we have proved the assertion for all the Möbius transformations. To conclude the proof it is enough to notice that the dilation coefficient of (the differential of) the map $q \mapsto \bar{q}$ is equal to 1. $\qquad \square$

The map L defined in (5.10) sends two (arbitrary) points q_1 and q_2 of $\Delta_{\mathbb{H}}$, to 0 and $t \in \mathbb{R}^+$ (respectively), which belong to each Δ_I. Then, as in the case of the complex disc, we find that the Poincaré distance δ_Δ is such that

$$(5.19) \qquad \delta_\Delta(q_1, q_2) = \delta_\Delta(0, t) = \inf \int_\ell ds = \inf \int_\ell \frac{|d_I q|}{1 - |q|^2}$$

where the infimum has been taken on all the arcs ℓ which are piece-wise differentiable and which join 0 and t. Therefore we have the following result:

Proposition 5.11. *The Poincaré distance δ_Δ of the unit disc $\Delta_{\mathbb{H}}$ is the integrated distance of the Poincaré differential metric of $\Delta_{\mathbb{H}}$.*

6. MÖBIUS TRANSFORMATIONS AND THE POINCARÉ DISTANCE ON $\mathbb{H}^+$

Similarly to what happens in the case of the complex plane, the quaternionic half-space $\mathbb{H}^+ = \{q \in \mathbb{H} : \operatorname{Re}(q) > 0\}$ is diffeomorphic to the open, unit disc $\Delta_{\mathbb{H}}$ via the (biregular) Cayley transformation $\psi(q) = (1 + q)(1 - q)^{-1} \in \mathbb{G}$, (see [10]). We can state here the following result.

Lemma 6.1. *The Cayley transformation $\psi(q) = (1 + q)(1 - q)^{-1}$ maps non-Euclidean lines of $\Delta_{\mathbb{H}}$ onto real, affine, half-lines or arcs of circles, which are orthogonal to $\partial \mathbb{H}^+$.*

Proof. By Corollary 4.8, the conformality of ψ (see Lemma 4.3), and the fact that it transforms $\mathbb{S}^3_{\mathbb{H}}$ onto $\partial \mathbb{H}^+$, lead to the conclusion. □

It becomes now easy to define the Poincaré-type distance on $\mathbb{H}^+$. Given any two points $q_1 \neq q_2 \in \mathbb{H}^+$, we can in fact consider the unique, affine half-line or arc of circle, ℓ, of $\mathbb{H}^+$ which contains q_1, q_2 and intersects $\partial \mathbb{H}^+$ orthogonally. We call such an ℓ the *non-Euclidean line* (of $\mathbb{H}^+$) containing q_1 and q_2.

We then define the two intersections q_3, q_4 of ℓ with $\partial \mathbb{H}^+$ to be the *ends* of ℓ (one of them might be ∞) in such a way that q_1, q_2, q_3, q_4 are arranged cyclically on ℓ. Then we set

$$(6.1) \qquad \omega(q_1, q_2) = \frac{1}{2} \log(\mathcal{CR}(q_1, q_2, q_3, q_4))$$

to be the *Poincaré distance* (of $\mathbb{H}^+$) between q_1 and q_2.

Theorem 6.2. *The map ω defined by (6.1) is a distance and the Cayley transformation $\psi : \Delta_{\mathbb{H}} \to \mathbb{H}^+$ is an isometry with respect to the Poincaré distances of $\Delta_{\mathbb{H}}$ and $\mathbb{H}^+$.*

Proof. Corollary 4.6 implies that ψ leaves invariant the cross ratio of four points belonging to a same non-Euclidean line of $\Delta_{\mathbb{H}}$. The assertion follows. □

We end the paper with the description of all the isometries of the Poincaré distance ω of $\mathbb{H}^+$.

Proposition 6.3. *The Poincaré distance of* $\mathbb{H}^+$ *is invariant under the action of the group* $\mathbb{M}^*(\mathbb{H}^+) := \psi \mathbb{M}^* \psi^{-1}$*, where* $\mathbb{M}^*$ *is the group of extended Möbius transformations and* ψ *is the Cayley transformation.*

Definition 6.4. The group of all linear fractional transformations of $\mathbb{H}^+$ will be denoted by $\mathbb{M}(\mathbb{H}^+)$ and called the group of *Möbius transformations* of $\mathbb{H}^+$.

The study of an explicit description of the group $\mathbb{M}(\mathbb{H}^+) = \psi \mathbb{M} \psi^{-1}$ of all fractional linear transformations of $\mathbb{H}^+$ has a natural independent interest. We will perform it here, starting from the identification of its isotropy subgroup $\mathbb{M}(\mathbb{H}^+)_\infty$ at the point ∞.

Proposition 6.5. *If* $g \in \mathbb{M}(\mathbb{H}^+)_\infty$*, then there exist* $b, d \in \mathbb{H}$ *with* $d \neq 0$ *and* $\mathrm{Re}(bd^{-1}) = 0$*, such that* g *is the fractional linear transformation associated to the matrix*

$$(6.2) \qquad \begin{bmatrix} |d|^{-2}d & b \\ 0 & d \end{bmatrix}$$

that is,

$$(6.3) \qquad g(q) = |d|^{-2}dqd^{-1} + bd^{-1}.$$

Proof. Let $g(q) = (aq + b)(cq + d)^{-1}$. Condition $g(\infty) = \infty$ implies that $c = 0$. Since $g \in \mathbb{M}(\mathbb{H}^+)$, if $\mathrm{Re}(q) = 0$, then $\mathrm{Re}(g(q)) = 0$. Therefore, if we set $q = yI$, with $y \in \mathbb{R}$ and $I \in \mathbb{S}$, we have $\mathrm{Re}(ayId^{-1}) + \mathrm{Re}(bd^{-1}) = 0$ for all $y \in \mathbb{R}$ and all $I \in \mathbb{S}$. This is equivalent to require $\mathrm{Re}(aId^{-1}) = 0$ for all $I \in \mathbb{S}$ and $\mathrm{Re}(bd^{-1}) = 0$. Let us set $a = a_0 + a_1 L$ and $d^{-1} = d_0 + d_1 M$, with $a_0, a_1, d_0, d_1 \in \mathbb{R}$ and $L, M \in \mathbb{S}$. The equality $\mathrm{Re}(aId^{-1}) = 0$ becomes

$$(6.4) \qquad \mathrm{Re}(a_0 d_0 I + a_0 d_1 IM + a_1 d_0 LI + a_1 d_1 LIM) = 0.$$

Now, if I is orthogonal to both L and M, then (6.4) becomes (see [10] for notations)

$$(6.5) \qquad \mathrm{Re}(a_1 d_1 LIM) = a_1 d_1 \langle L \times I, M \rangle = 0.$$

If L and M are $\mathbb{R}$-linearly dependent, equation (6.5) gives no conditions. If, otherwise, L and M are $\mathbb{R}$-linearly independent it implies $a_1 d_1 = 0$ which directly yields $a, d \in \mathbb{R}$, and the assertion follows. We can therefore suppose from now on that L and M are $\mathbb{R}$-linearly dependent. If we choose $I = L = M$, equation (6.4) reduces to

$$(6.6) \qquad a_0 d_1 + a_1 d_0 = 0.$$

Since $a_0 d_1 + a_1 d_0 = \mathrm{Im}(ad^{-1})$, this leads to $ad^{-1} \in \mathbb{R}$, i.e., $a = rd$ for some $r \in \mathbb{R}$. Taking into account that $\mathrm{Re}(q) > 0$ implies $\mathrm{Re}(g(q)) > 0$, the real number r has to be strictly positive, and can be chosen to be equal to $|d|^{-2}$ without loss of generality. The assertion is proved. $\qquad \square$

Remark 6.6. In the proof of the last theorem, the choice of r is such that the Dieudonné determinant of the matrices of type (6.2) representing the elements of $\mathbb{M}(\mathbb{H}^+)_\infty$ is 1.

Let us consider the element $f_\gamma \in \mathbb{M}(\mathbb{H}^+)$ defined by

$$(6.7) \qquad\qquad\qquad f_\gamma(q) = (q - \gamma)^{-1}$$

and associated to the matrix

$$(6.8) \qquad\qquad\qquad \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix}$$

with $\mathrm{Re}(\gamma) = 0$. We have that $f_\gamma(\gamma) = \infty$.

Theorem 6.7. *Let $g \in \mathbb{M}(\mathbb{H}^+) \setminus \mathbb{M}(\mathbb{H}^+)_\infty$. Then there exist $\alpha, \beta, \gamma \in \mathbb{H}$ with $\alpha \neq 0$ and $\mathrm{Re}(\gamma) = 0 = \mathrm{Re}(\beta\alpha^{-1})$ such that $g(q) = (|\alpha|^{-2}\gamma\alpha q + \gamma\beta + \alpha)(|\alpha|^{-2}\alpha q + \beta)^{-1}$ is associated to the matrix*

$$(6.9) \qquad\qquad\qquad \begin{bmatrix} |\alpha|^{-2}\gamma\alpha & \gamma\beta + \alpha \\ |\alpha|^{-2}\alpha & \beta \end{bmatrix}.$$

Proof. Let $g(q) = (aq + b)(cq + d)^{-1}$ be a fractional linear transformation of $\mathbb{H}^+$ such that $g(\infty) = ac^{-1} = \gamma \in \partial\mathbb{H}^+ = \{q \in \mathbb{H} : \mathrm{Re}(q) = 0\}$. Then $(f_\gamma \circ g)$ fixes ∞ and by Proposition 6.5 there exist $\alpha, \beta \in \mathbb{H}$ with $\mathrm{Re}(\beta\alpha^{-1}) = 0$ such that:

$$(6.10) \qquad\qquad \begin{bmatrix} 0 & 1 \\ 1 & -ac^{-1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} |\alpha|^{-2}\alpha & \beta \\ 0 & \alpha \end{bmatrix}.$$

Therefore:

$$(6.11) \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \gamma & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} |\alpha|^{-2}\alpha & \beta \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} |\alpha|^{-2}\gamma\alpha & \gamma\beta + \alpha \\ |\alpha|^{-2}\alpha & \beta \end{bmatrix}. \qquad\square$$

The above results give us an idea on how to describe the group of all Möbius transformations of $\mathbb{H}^+$ in a more direct form. In fact we will prove now the following result.

Theorem 6.8. *If $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then the set of matrices defined by*

$$SL(\mathbb{H}^+) = \{A \in M(2, \mathbb{H}) : {}^t\bar{A}KA = K\}$$

is a subgroup of $\mathrm{SL}(2, \mathbb{H})$ *of real dimension 10. Moreover, this subgroup can be equivalently characterized as*

$$\mathrm{SL}(\mathbb{H}^+) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbb{H}) : \mathrm{Re}(a\bar{c}) = 0, \ \mathrm{Re}(b\bar{d}) = 0, \ \bar{b}c + \bar{d}a = 1 \right\}$$

and as

$$\mathrm{SL}(\mathbb{H}^+) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbb{H}) : \mathrm{Re}(c\bar{d}) = 0, \ \mathrm{Re}(a\bar{b}) = 0, \ a\bar{d} + b\bar{c} = 1 \right\}.$$

Proof. A direct computation, based on the definition of Dieudonné determinant (2.6), shows that $\det_{\mathbb{H}}(\,^t\bar{A}) = \det_{\mathbb{H}}(A)$. Now, for any $A \in M(2, \mathbb{H})$, the relation $^t\bar{A}KA = K$ implies, via the Binet property (Proposition 2.10) that $\mathrm{SL}(\mathbb{H}^+) \subset \mathrm{SL}(2, \mathbb{H})$. In what follows, we will use the fact that, by Proposition 2.3, the right inverse of any matrix $A \in \mathrm{SL}(2, \mathbb{H})$ is also the left inverse of A, and will be denoted by A^{-1}. A direct computation shows

$$(6.12) \qquad\qquad\qquad ^t\overline{AB} = {}^t\bar{B}\,^t\bar{A}$$

for all $A, B \in \mathrm{SL}(2, \mathbb{H})$. When $B = A^{-1}$, equation (6.12) becomes

$$(6.13) \qquad\qquad\qquad I = {}^t\overline{(A^{-1})}\,^t\bar{A}$$

and hence we obtain that

$$(6.14) \qquad\qquad\qquad (^t\bar{A})^{-1} = {}^t\overline{(A^{-1})}$$

for all $A \in \mathrm{SL}(2, \mathbb{H})$. We will now prove that $\mathrm{SL}(\mathbb{H}^+)$ is a group. In fact, for all $A, B \in \mathrm{SL}(\mathbb{H}^+)$, we have by (6.12)

$$(6.15) \qquad\qquad ^t\overline{(AB)}K(AB) = {}^t\bar{B}(^t\bar{A}KA)B = {}^t\bar{B}KB = K$$

and therefore $AB \in \mathrm{SL}(\mathbb{H}^+)$, which then turns out to be closed with respect to matrix multiplication. Since, by definition, for all $A \in \mathrm{SL}(\mathbb{H}^+)$ we have

$$(6.16) \qquad\qquad\qquad ^t\bar{A}KA = K$$

then, by multiplying both members of equation (6.16) on the right by A^{-1}, and on the left by $(^t\bar{A})^{-1}$, we obtain

$$(^t\bar{A})^{-1}[\,^t\bar{A}KA]A^{-1} = (^t\bar{A})^{-1}KA^{-1},$$
$$[(^t\bar{A})^{-1}\,^t\bar{A}]K[AA^{-1}] = (^t\bar{A})^{-1}KA^{-1},$$

whence

$$K = ({}^{t}\bar{A})^{-1}KA^{-1}.$$

Now, by using (6.14), the last equation gives

$$K = {}^{t}\overline{(A^{-1})}KA^{-1},$$

i.e., $A^{-1} \in \mathrm{SL}(\mathbb{H}^{+})$. Therefore we have proved that $\mathrm{SL}(\mathbb{H}^{+})$ is a group.

Take now any

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbb{H}).$$

Since

$${}^{t}\bar{A}KA = \begin{bmatrix} 2\,\mathrm{Re}(a\bar{c}) & \bar{c}b + \bar{a}d \\ \bar{b}c + \bar{d}a & 2\,\mathrm{Re}(b\bar{d}) \end{bmatrix}$$

we obtain that $A \in \mathrm{SL}(\mathbb{H}^{+})$ if and only if $\mathrm{Re}(a\bar{c}) = 0$, $\mathrm{Re}(b\bar{d}) = 0$, $\bar{b}c + \bar{d}a = 1$. As a consequence, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(\mathbb{H}^{+})$, we have that

$$\begin{bmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{d}a + \bar{b}c & \bar{d}b + \bar{b}d \\ \bar{c}a + \bar{a}c & \bar{c}b + \bar{a}d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, again by Proposition 2.3,

$$A^{-1} = \begin{bmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{bmatrix}$$

and since we have proved that $A^{-1} \in \mathrm{SL}(\mathbb{H}^{+})$, we obtain

$${}^{t}\overline{A^{-1}}KA^{-1} = \begin{bmatrix} 2\,\mathrm{Re}(c\bar{d}) & c\bar{b} + d\bar{a} \\ a\bar{d} + b\bar{c} & 2\,\mathrm{Re}(a\bar{b}) \end{bmatrix}.$$

Summing up, we have obtained that $\mathrm{Re}(a\bar{c}) = 0$, $\mathrm{Re}(b\bar{d}) = 0$, $\bar{b}c + \bar{d}a = 1$ if and only if $A \in \mathrm{SL}(\mathbb{H}^{+})$, if and only if $A^{-1} \in \mathrm{SL}(\mathbb{H}^{+})$, if and only if $\mathrm{Re}(c\bar{d}) = 0$, $\mathrm{Re}(a\bar{b}) = 0$, $a\bar{d} + b\bar{c} = 1$. Finally, a direct computation shows that the real dimension of $\mathrm{SL}(\mathbb{H}^{+})$ is 10.   □

The following group isomorphism assumes an interesting geometrical meaning. Let $C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ be the matrix associated to the Cayley transform (see Lemma 6.1), and define $\Phi : \mathrm{SL}(\mathbb{H}^{+}) \to \mathrm{Sp}(1,1)$ as the conjugation map $\Phi(A) = C^{-1}AC$. We have the following result.

Theorem 6.9. *The group* $\mathrm{SL}(\mathbb{H}^+)$ *is isomorphic to* $\mathrm{Sp}(1,1)$ *via the application* $\Phi : A \mapsto C^{-1}AC$.

Proof. Recall that $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is easy to verify that $C^{-1} = \frac{1}{2}{}^t C$, ${}^t C^{-1}HC^{-1} = -\frac{1}{2}K$, and ${}^t CKC = -2H$. As a consequence, by applying Theorem 6.8, we obtain

$$
{}^t\overline{[\Phi(A)]}H[\Phi(A)] = {}^t\overline{(C^{-1}AC)}HC^{-1}AC = {}^t C\, {}^t\bar{A}({}^t C^{-1}HC^{-1})AC
$$
$$
= -\frac{1}{2}{}^t C\, {}^t\bar{A}KAC = -\frac{1}{2}{}^t CKC = 2\frac{1}{2}H = H
$$

for all $A \in \mathrm{SL}(\mathbb{H}^+)$. This last equality proves the inclusion $\Phi(\mathrm{SL}(\mathbb{H}^+)) \subseteq \mathrm{Sp}(1,1)$. Moreover,

$$
{}^t\overline{[\Phi^{-1}(M)]}K[\Phi^{-1}(M)] = {}^t(\overline{CMC^{-1}})K(CMC^{-1})
$$
$$
= {}^t C^{-1}\, {}^t\bar{M}({}^t CKC)MC^{-1}
$$
$$
= -2\,{}^t C^{-1}({}^t\bar{M}HM)C^{-1}
$$
$$
= -2({}^t C^{-1}HC^{-1}) = 2\frac{1}{2}K = K
$$

for all $M \in \mathrm{Sp}(1,1)$, which implies $\mathrm{Sp}(1,1) \subseteq \Phi(\mathrm{SL}(\mathbb{H}^+))$ and ends the proof, since Φ is obviously an injective homomorphism. $\qquad\square$

We will conclude the paper with the following result, which urges a comparison with the complex case.

Theorem 6.10. *The map*

$$
\Psi : \mathrm{SL}(\mathbb{H}^+) \to \mathbb{M}(\mathbb{H}^+)
$$

(6.17)

$$
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto L_A(q) = (aq + b) \cdot (cq + d)^{-1}
$$

is a group homomorphism whose kernel is the center of $\mathrm{SL}(\mathbb{H}^+)$, *that is the subgroup*

$$
\left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.
$$

Proof. Since the subsets of matrices $A, N \subset \mathrm{GL}(2, \mathbb{H})$ defined by

$$
N = \left\{ \begin{bmatrix} |\alpha|^{-2}\gamma\alpha & \gamma\beta + \alpha \\ |\alpha|^{-2}\alpha & \beta \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{H},\ \mathrm{Re}(\gamma) = 0,\ \mathrm{Re}(\beta\bar{\alpha}) = 0 \right\},
$$

$$
A = \left\{ \begin{bmatrix} |d|^{-2}d & b \\ 0 & d \end{bmatrix} : b, d \in \mathbb{H},\ \mathrm{Re}(b\bar{d}) = 0 \right\},
$$

are contained in $\mathrm{SL}(\mathbb{H}^+)$ then, by Proposition 6.2 and Theorem 6.7, we have that $\mathbb{M}(\mathbb{H}^+) \subseteq \Psi(\mathrm{SL}(\mathbb{H}^+))$. Now, if $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(\mathbb{H}^+)$, then $\bar{d}a + \bar{b}c = 1$, $\mathrm{Re}(a\bar{c}) = 0$ and $\mathrm{Re}(b\bar{d}) = 0$. If $q \in \mathbb{H} \setminus \mathbb{R}$, we have

$$
\begin{aligned}
\mathrm{Re}((aq + b)(cq + d)^{-1}) = \mathrm{Re}((aq + b)\overline{(cq + d)}) &= \mathrm{Re}((aq + b)(\bar{q}\bar{c} + \bar{d})) \\
&= \mathrm{Re}(a|q|^2\bar{c} + aq\bar{d} + b\bar{q}\bar{c} + b\bar{d}) = \mathrm{Re}(aq\bar{d} + b\bar{q}\bar{c}) \\
&= \mathrm{Re}(q\bar{d}a - q\bar{c}b) = \mathrm{Re}(q(\bar{d}a - \bar{c}b)) \\
&= \mathrm{Re}(q(1 - (\bar{b}c + \bar{c}b))) = 0.
\end{aligned}
$$

The last equalities lead to $\Psi(A)(\partial\mathbb{H}^+) = \partial\mathbb{H}^+$. Since $\mathrm{Re}(\Psi(A)(1)) = 1$, we have $\Psi(\mathrm{SL}(\mathbb{H}^+)) \subseteq \mathbb{M}(\mathbb{H}^+)$. The same argument used in Theorem 3.3 leads to the identification of the kernel of Ψ and allows at this point the conclusion of the proof. □

The new description of the group of quaternionic, Möbius transformations $\mathbb{M}(\mathbb{H}^+)$ of $\mathbb{H}^+$ in terms of the group of matrices $\mathrm{SL}(\mathbb{H}^+)$ is interesting and promises developments in several directions. We plan to investigate, for example, the analog of the Fuchsian subgroups and their possible role in the construction of Riemann-type $\mathbb{H}$-surfaces.

The Cayley transformation $\psi : \Delta_\mathbb{H} \to \mathbb{H}^+$, defined by

$$
\psi(q) = (1 + q)(1 - q)^{-1},
$$

has real coefficients, and therefore it maps every $L_I = \mathbb{R} + I\mathbb{R} \cong \mathbb{C}$ $(I \in \mathbb{S})$ onto itself. The argument used in the complex case leads to the definition of the Poincaré differential metric on $\mathbb{H}^+$: for any $q \in \mathbb{H}^+$ and any $\tau \in \mathbb{H}$ the length of the vector τ for the Poincaré metric at q is expressed by

$$
(6.18) \qquad \langle\tau\rangle_q = \frac{|\tau|}{2|\mathrm{Re}(q)|}.
$$

Formula (6.18) leads as before to the definition of the (square of the) Poincaré length element in $\mathbb{H}^+$:

$$
ds^2_{\mathbb{H}^+} = \frac{|d_I q|^2}{4|\mathrm{Re}(q)|^2}.
$$

We end the paper by stating the following result, whose proof is straightforward:

Theorem 6.11. *The Poincaré differential metric* (6.18) *is invariant under the action of the group* $\mathbb{M}^*(\mathbb{H}^+)$ *of all extended Möbius transformations of* $\mathbb{H}^+$. *Moreover, the Poincaré distance* ω *of* $\mathbb{H}^+$ *defined by*

$$
\omega(q_1, q_2) = \frac{1}{2}\log(C\mathcal{R}(q_1, q_2, q_3, q_4))
$$

is the integrated distance of the Poincaré differential metric.

References

[1] L.V. Ahlfors, *Möbius transformations and Clifford numbers*, Differential Geometry and Complex Analysis, Springer, Berlin, 1985, pp. 65–73. MR 780036 (86g:20065)

[2] _____, *On the fixed points of Möbius transformations in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 15–27. MR 802464 (87c:20086)

[3] _____, *Cross-ratios and Schwarzian derivatives in* $\mathbb{R}^n$, Complex Analysis (J. . Hersch and A. . Huber, eds.), Birkhäuser, Basel, 1988, pp. 1–15. MR 981397 (90a:30055)

[4] H. Aslaksen, *Quaternionic determinants*, Math. Intelligencer **18** (1996), 57–65, http://dx.doi.org/10.1007/BF03024312. MR 1412993 (97j:16028)

[5] C. Cao and P.L. Waterman, *Conjugacy invariants of Möbius groups*, Quasiconformal Mappings and Analysis (Ann Arbor, MI), Springer, New York, 1998, pp. 109–139. MR 1488448 (98k:30058)

[6] W. Cao, J.R. Parker, and X. Wang, *On the classification of quaternionic Möbius transformations*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 349–361, http://dx.doi.org/10.1017/S0305004104007868. MR 2092064 (2005f:30082)

[7] H. Cartan, *Cours de Maitrise des Sciences Mathématiques*, 1969 (unpublished).

[8] N. Cohen and S. De Leo, *The quaternionic determinant*, Electron. J. Linear Algebra **7** (2000), 100–111 (electronic). MR 1779435 (2001g:15006)

[9] I. Gelfand, V. Retakh, and R.L. Wilson, *Quaternionic quasideterminants and determinants*, Lie Groups and Symmetric Spaces, Amer. Math. Soc. Transl. Ser. 2, vol. 210, Amer. Math. Soc., Providence, RI, 2003, pp. 111–123. MR 2018356 (2004j:15010)

[10] G. Gentili and D.C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math. **216** (2007), 279–301, http://dx.doi.org/10.1016/j.aim.2007.05.010. MR 2353257 (2008h:30052)

[11] V.V. Gorbatsevich, A.L. Onishchik, and É.B. Vinberg, *Lie Groups and Lie Algebras, III. Structure of Lie Groups and Lie Algebras* (A.L. Onishchik and É.B. Vinberg, editors), Encyclopaedia of Mathematical Sciences, vol. 41, Springer-Verlag, Berlin, 1994, ISBN 3-540-54683-9, A translation of *Current problems in mathematics. Fundamental directions.* Volume 41 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990 [MR1056485 (91b:22001)]. MR 1349140 (96d:22001)

[12] I.N. Herstein, *Topics in Algebra*, 2nd ed., Xerox College Publishing, Lexington, Mass., 1975. MR 0356988 (50 #9456)

[13] R. Heidrich and G. Jank, *On the iteration of quaternionic Moebius transformations*, Complex Variables Theory Appl. **29** (1996), 313–318. MR 1390616 (97e:30079)

[14] R. Kellerhals, *Collars in* PSL(2, $\mathbb{H}$), Ann. Acad. Sci. Fenn. Math. **26** (2001), 51–72. MR 1816562 (2002f:57036)

[15] R.M. Porter, *Quaternionic Möbius transformations and loxodromes*, Complex Variables Theory Appl. **36** (1998), 285–300. MR 1671478 (99m:20121)

[16] V. Retakh and R.L. Wilson, *Advanced Course on Quasideterminants and Universal Localization*, Centre de Recerca Matematica (CRM), Bellaterra (Barcelona), Spain, 2007, notes of the course.

[17] C.L. Siegel, *Topics in Complex Function Theory. Vol. II. Automorphic Functions and Abelian Integrals*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988, ISBN 0-471-60843-2, Translated from the German by A. Shenitzer and M. Tretkoff; With a preface by Wilhelm Magnus; Reprint of the 1971 edition; A Wiley-Interscience Publication. MR 1008931 (90h:30003)

[18] E. Vesentini, *Capitoli Scelti della Teoria delle Funzioni Olomorfe*, Unione Matematica Italiana, 1984.

[19] X. Wang, *Quaternionic Möbius transformations and subgroups*, Complex Var. Theory Appl. **48** (2003), 599–606. MR 1988686 (2004e:30072)

[20] J.B. Wilker, *The quaternion formalism for Möbius groups in four or fewer dimensions*, Linear Algebra Appl. **190** (1993), 99–136, http://dx.doi.org/10.1016/0024-3795(93)90222-A. MR 1230354 (94c:20086)

C. Bisi:
Dipartimento di Matematica
Universitá della Calabria
Ponte Bucci, Cubo 30b
Arcavacata di Rende (CS), 87036, Italy
E-mail: bisi@math.unifi.it; bisi@mat.unical.it

G. Gentili:
Dipartimento di Matematica
Universitá di Firenze
Viale Morgagni 67/A
50134 Firenze, Italy
E-mail: gentili@math.unifi.it