

# Local Clustering of the Non-Zero Set of Functions in $W^{1,1}(E)$

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## A Measure Theoretical Lemma

For  $\rho > 0$ , denote by  $K_\rho(y) \subset \mathbb{R}^N$  a cube of edge  $\rho$  centered at  $y$  and with faces parallel to the coordinate planes. If  $y$  is the origin on  $\mathbb{R}^N$ , we write  $K_\rho(0) = K_\rho$ .

**Lemma** *Let  $u \in W^{1,1}(K_\rho)$  satisfy*

$$\|u\|_{W^{1,1}(K_\rho)} \leq \gamma \rho^{N-1} \quad \text{and} \quad |[u > 1]| \geq \alpha |K_\rho| \quad (1)$$

*for some  $\gamma > 0$  and  $\alpha \in (0, 1)$ . Then, for every  $\delta \in (0, 1)$  and  $0 < \lambda < 1$  there exist  $x_o \in K_\rho$  and  $\eta = \eta(\alpha, \delta, \gamma, \lambda, N) \in (0, 1)$ , such that*

$$|[f > \lambda] \cap K_{\eta\rho}(x_o)| > (1 - \delta) |K_{\eta\rho}(x_o)|. \quad (2)$$

Roughly speaking the Lemma asserts that if the set where  $u$  is bounded away from zero occupies a sizable portion of  $K_\rho$ , then there exists at least one point  $x_o$  and a neighborhood  $K_{\eta\rho}(x_o)$  where  $u$  remains large in a large portion of  $K_{\eta\rho}(x_o)$ . Thus the set where  $u$  is positive clusters about at least one point of  $K_\rho$ .

The Lemma was established in [1] for  $u \in W^{1,p}(K_\rho)$  and  $p > 1$ . Such a limitation on  $p$  was essential to the proof. We give a new proof which includes the case  $p = 1$  and is simpler.

**Proof:** It suffices to establish the Lemma for  $u$  continuous and  $\rho = 1$ . For  $n \in \mathbb{N}$  partition  $K_1$  into  $n^N$  cubes, with pairwise disjoint interior and each of edge  $1/n$ . Divide these cubes

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into two finite subcollections  $\mathbf{Q}^+$  and  $\mathbf{Q}^-$  by

$$\begin{aligned} Q_j \in \mathbf{Q}^+ &\iff |[u > 1] \cap Q_j| > \frac{\alpha}{2}|Q_j| \\ Q_i \in \mathbf{Q}^- &\iff |[u > 1] \cap Q_i| \leq \frac{\alpha}{2}|Q_i| \end{aligned}$$

and denote by  $\#(\mathbf{Q}^+)$  the number of cubes in  $\mathbf{Q}^+$ . By the assumption

$$\sum_{Q_j \in \mathbf{Q}^+} |[u > 1] \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |[u > 1] \cap Q_i| > \alpha|K_1| = \alpha n^N |Q|$$

where  $|Q|$  is the common measure of the  $Q_\ell$ . From the definitions of the classes  $\mathbf{Q}^\pm$ ,

$$\alpha n^N < \sum_{Q_j \in \mathbf{Q}^+} \frac{|[u > 1] \cap Q_j|}{|Q_j|} + \sum_{Q_i \in \mathbf{Q}^-} \frac{|[u > 1] \cap Q_i|}{|Q_i|} < \#(\mathbf{Q}^+) + \frac{\alpha}{2}(n^N - \#(\mathbf{Q}^+)).$$

Therefore

$$\#(\mathbf{Q}^+) > \frac{\alpha}{2 - \alpha} n^2. \quad (3)$$

Fix  $\delta, \lambda \in (0, 1)$ . The integer  $n$  can be chosen depending upon  $\alpha, \delta, \lambda, \gamma$  and  $N$ , such that

$$|[u > \lambda] \cap Q_j| \geq (1 - \delta)|Q_j| \quad \text{for some } Q_j \in \mathbf{Q}^+. \quad (4)$$

This would establish the Lemma for  $\eta = 1/n$ . Let  $Q \in \mathbf{Q}^+$  satisfy

$$|[u > \lambda] \cap Q| < (1 - \delta)|Q|. \quad (5)$$

Then, there exists a constant  $c = c(\alpha, \delta, \gamma, \eta, N)$  such that

$$\|u\|_{W^{1,1}(Q)} \geq c(\alpha, \delta, \gamma, \lambda, N) \frac{1}{n^{N-1}}. \quad (6)$$

From the assumptions

$$|[u \leq \lambda] \cap Q| \geq \delta|Q| \quad \text{and} \quad \left| \left[ u > \frac{1 + \lambda}{2} \right] \cap Q \right| > \frac{\alpha}{2}|Q|.$$

For fixed  $x \in [u \leq \lambda] \cap Q$  and  $y \in [u > (1 + \lambda)/2] \cap Q$ ,

$$\frac{1 - \lambda}{2} = u(y) - u(x) = \int_0^{|y-x|} Du(x + t\omega) \cdot \omega dt \quad \text{where} \quad \omega = \frac{y - x}{|x - y|}.$$

Let  $R(x, \omega)$  be the polar representation of  $\partial Q$  with pole at  $x$ , for the solid angle  $\omega$ . Integrate the previous relation in  $dy$  over  $[u > (1 + \lambda)/2] \cap Q$ . Minorize the resulting left hand side, by using the lower bound on the measure of such a set, and majorize the resulting integral

on the right hand side by extending the integration over  $Q$ . Expressing such integration in polar coordinates with pole at  $x \in [u \leq \lambda] \cap Q$  gives,

$$\begin{aligned} \frac{\alpha(1-\lambda)}{4}|Q| &\leq \int_{|\omega|=1} \int_0^{R(x,\omega)} r^{N-1} \int_0^{|y-x|} |Du(x+t\omega)| dt dr d\omega \\ &\leq N^{N/2}|Q| \int_{|\omega|=1} \int_0^{R(x,\omega)} |Du(x+t\omega)| dt d\omega \\ &= N^{N/2}|Q| \int_Q \frac{|Du(z)|}{|z-x|^{N-1}} dz. \end{aligned}$$

Integrate now in  $dx$  over  $[u \leq \lambda] \cap Q$ . Minorize the resulting left hand side by using the lower bound on the measure of such a set, and majorize the resulting right hand side, by extending the integration to  $Q$ . This gives

$$\frac{\alpha\delta(1-\lambda)}{4N^{N/2}}|Q| \leq \|u\|_{W^{1,1}(K_1)} \sup_{z \in Q} \int_Q \frac{1}{|z-x|^{N-1}} dx \leq C(N)|Q|^{1/N} \|u\|_{W^{1,1}(K_1)}$$

for a constant  $C(N)$  depending only upon  $N$ .

If (4) does not hold for any cube  $Q_j \in \mathbf{Q}^+$ , then (6) is verified for all such  $Q_j$ . Adding over such cubes and taking into account (3),

$$\frac{\alpha}{2-\alpha} c(\alpha, \delta, \gamma, N) n \leq \|u\|_{W^{1,1}(K_1)} \leq \gamma.$$

[1] E. DiBenedetto, V. Vespri, On the singular equation  $\beta(u)_t = \Delta u$ , *Arch. Rat. Mech. Anal.* **132**(3), 1995, 247–309.