On a variational problem modelling necking deformation

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This volume provides the texts of lectures given by L. Ambrosio, L. Caffarelli, M. Crandall, L.C. Evans, N. Fusco at the Summer course held in Cetraro (Italy) in 2005. These are introductory reports on current research by world-leaders in the fields of calculus of variations and partial differential equations. The topics discussed are transport equations for non smooth vector fields, homogenization, viscosity methods for the infinite Laplacian, weak KAM theory and geometrical aspects of symmetrization. Besides there is also an historical overview of all CIME courses on the calculus of variations and partial differential equations contributed by Elvira Mascolo.

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Calculus of Variations and Non-Linear Partial Differential Equations

CIME

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Calculus of Variations and Non-Linear Partial Differential Equations

Cetraro, Italy 2005

Editors: B. Dacorogna, P. Marcellini

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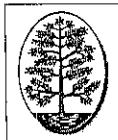

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INTERNATIONAL MATHEMATICAL SUMMER CENTER

C.I.M.E. means Centro Internazionale Matematico Estivo, that is, International Mathematical Summer Center. Conceived in the early fifties, it was born in 1954 and made welcome by the world mathematical community where it remains in good health and spirit. Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities during the past years.

So they already know what the C.I.M.E. is all about. For the benefit of future potential users and co-operators the main purposes and the functioning of the Centre may be summarized as follows: every year, during the summer, Sessions (three or four as a rule) on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. Each session is generally based on three or four main courses (24–30 hours over a period of 6-8 working days) held from specialists of international renown, plus a certain number of seminars.

A C.I.M.E. Session, therefore, is neither a Symposium, nor just a School, but maybe a blend of both. The aim is that of bringing to the attention of younger researchers the origins, later developments, and perspectives of some branch of live mathematics.

The topics of the courses are generally of international resonance and the participation of the courses cover the expertise of different countries and continents. Such combination, gave an excellent opportunity to young participants to be acquainted with the most advance research in the topics of the courses and the possibility of an interchange with the world famous specialists. The full immersion atmosphere of the courses and the daily exchange among participants are a first building brick in the edifice of international collaboration in mathematical research.

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Luigi Ambrosio · Luis Caffarelli
Michael G. Crandall · Lawrence C. Evans
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Calculus of Variations and Nonlinear Partial Differential Equations

Lectures given at the
C.I.M.E. Summer School
held in Cetraro, Italy
June 27–July 2, 2005

With a historical overview by Elvira Mascolo

Editors: Bernard Dacorogna, Paolo Marcellini

Springer

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CIME Courses on Partial Differential Equations and Calculus of Variations

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David Hilbert used to say

> every real progress walks **hand in hand** with the discovery of more and more rigorous tools and simpler methods which meanwhile make easier the understanding of previous theories.

Nevertheless Augustus De Morgan used to say:

> The mental attitude which stimulate the mathematical invention is not only a sharp reasoning but rather a deep imagination.

The "progress" Hilbert was talking about is based – in mathematics more than in other scientific fields – upon teaching and collaboration and the "imagination" De Morgan was referring to, must be stimulated through a progressive and gradual learning.

Both history and everybody personal experience show that mathematical learning and its improvement is not just a matter of studying books and original articles, but rather that of a continuous and effective relationships with our own teacher(s) and collegues, rising new questions and discussing together their possible answers.

In the Fifties a group of outstanding Italian mathematicians, all member of the Scientific Committee of UMI (the Union of Italian Mathematicians) under the presidency of Enrico Bompiani, decided that it was the moment to rise the mathematical research in Italy to the level it was before the Second Worldly War and that it should be done through the organisation of high level courses. They realized the importance of providing the young researchers with the possibility of learning the new theories, subjects and themes which were appearing in those years and of mastering the new techniques and tools.

It was right in those years that the CIME was founded and the first course was held in Varenna (a charming small city on the Como lake) in 1954. The subject was on Functional Analysis, which can be considered at that time a new subject. More precisely:

Funzionali Analitici ed Anelli Normati
Varenna (Como), June 9–18, 1954

Lectures: L. Amerio (Politecnico Milano), L. Fantapié (Univ. Roma), E.R. Lorch (Columbia Univ.)
Seminars: M. Cugiani (Univ. Milano), F. Pellegrino (INAM, Roma), G.B. Rizza (Univ. Genova).

The second was also held in Varenna in the month of August of the same year. That was on

Quadratura Delle Superficie e Questioni Connesse
Varenna (Como), August 16–25, 1954

Lectures: R. Caccioppoli (Univ. Napoli), L. Cesari, (Univ. Bologna, Purdue Univ.), Chr.Y. Pauc (Univ. Rennes).
Seminars: A. Finzi (Technion, Haifa), A. Zygmund (Univ. Chicago).

The exceptional personalities were the teachers chosen in that occasion: Renato Caccioppoli, Lamberto Cesari and Antoni Zygmund. How can we recall without the suspect of being limited the contributions given for example by Caccioppoli to the development of the modern Functional Analysis and the Geometric Theory of Measure.

In the subsequent fifty-one years the CIME organized 163 courses covering basically every aspect of mathematics, both pure and applied, thus playing a crucial role in promoting and developing the mathematical research, not only in Italy. In fact the CIME activities have favoured and promoted personal contacts among distinguished scientists and young researchers, coming from all over the world.

The mathematics in last years has known a nearly explosive development and the organization of courses is an exceptional instrument of formation for the young investigators and a real support for the most mature ones.

The *full immersion*, permitted by a common location, is the right preamble to develop new subjects, to suggest new methods, to learn how to apply old methods to new problems, to start joint papers.

One main reason for the success of CIME courses was in particular the fact that they have been all published and the C.I.M.E. Sessions are an essential mean of diffusion of the mathematical culture.

The texts of lectures and seminars of each Session were all published:

- The volumes of Sessions 1-39 are actually out of print,
- The volumes of Sessions 39-70 are on the Catalogue of Edizioni Cremonese, Firenze, Italy
- The volumes of Sessions 71-83 are on the Catalogue of Liguori Editore, Napoli, Italy
- Since 1981 all courses notes are being published by Springer Verlag in a Subseries (Fondazione C.I.M.E.) of the Lectures Notes in Mathematics

My aim today is to guide you in a ideal journey through the development of the Calculus of Variation ad Nonlinear Differential Equation via the CIME courses held on these topics in the past fifty years of his history.

For the majority of younger people these arguments and techniques are to be considered as standard; however in these fifty years the developments have been so fast than the so-called "variational questions" raised by Hilbert at the beginning of the past century, have blowed up during the entire century (particularly the second half), in so many and different directions that Hilbert himself could have never imagined.

Quoting James Serrin we could say that

> the relevant field of investigation is nowadays so spread and wide that only a few years ago would have thought of as unbelivable..... more new ideas and results appeared from the end of the Second Worldly War until the present day than from the time of Talete until the year 1945.

The CIME courses on these subjects are all worthy of remark for the high scientific level of the directors and lectures.

In the following I will give a short presentation of each of them which allows to appreciate the significant role that the CIME Foundation played in the last 51 years.

- In 1958 Alessandro Faedo was the Scientific Direction of a session, strictly related with the variational questions and in particular devoted to the minima principles and their applications. v

Principio di Minimo e le sue Applicazioni in Analisi Funzionale
Pisa, September 1–10, 1958

Director: S. Faedo (Univ. Pisa).
Lectures: L. Bers (Courant Institute), Ch. B. Morrey Jr. (Univ. Of California, Berkeley), L. Nirenberg (Courant Institute).
Seminars: S. Agmon (Hebrew Univ. Jerusalem), G. Fichera (Univ. Roma), G. Stampacchia (Univ. Genova).
Notice the presence as lectures of Charles Jr. Morrey. and Louis Niremberg, two of the most important among the personalities in the Calculus of Variations and Partial Differential Equations of past century. The notes are published on Annali Scuola Normale di Pisa.

- In 1961 Enrico Bompiani was the Scientific Director of a session which can be considered a first approach to the geometric methods in the Calculus of Variations.

Geometria del Calcolo delle Variazioni
Saltino (Firenze), August 21–30, 1961

Director: E. Bompiani (Univ. Roma).

Lectures: H. Busemann (Univ. of Southern Calif., Los Angeles), E.T. Davies (Univ. Southampton), D. Laugwitz (Technische Hochschule, Darmstadt).

- In 1964, Guido Stampacchia was the Scientific Director of a very interesting session, in which we note the relevance not only the lecturers but also of the young researchers which gave a seminar.

Equazioni Differenziali non Lineari
Varenna (Como), August 30 – September 8, 1964

Director: G. Stampacchia (Univ. Pisa).
Lectures: P. Lax (New York Univ.), J. Leray (Collège De France), J. Moser (New York Univ.).
Seminars: R. Courant (New York Univ.), E. Degiorgi (Scuola Normale Superiore, Pisa), J. Friberg (Univ. Lund), J. Necas (CSAV, Praha), I. Segal (M.I.T.), G. Stampacchia (Univ. Pisa) O. Vejvoda (CSAV, Praha). Unfortunately there is not the text of the seminar given by Ennio de Giorgi.

- In 1966 Roberto Conti, one of the most important Italian mathematician and in some sense the *father* of CIME (he was Scientific Secretary since 1954 to 1974 and the Director of CIME Foundation since 1975 to 1998) was the Scientific Director of a session dedicated to the applications of the methods of Calculus of Variation to Control Theory that at that times was called "modern" Calculus of Variations.

Calculus of Variations, Classical and Modern
Bressanone (Bolzano), June 10–18, 1966

Director: R. Conti (Univ. Firenze).
Lectures: A. Blaquiere (Univ. Paris-Orsay), L. Cesari (Univ. Michigan), E. Rothe (Univ. Michigan), E. O. Roxin (Univ. Buenos Aires).
Seminars: C. Castaing (Univ. Caen), H. Halkin (Univ. California, La Jolla), C. Olech (Univ. Krakow).

- In 1972 Enrico Bombieri was the Scientific Director of a session dedicated to the Geometric measures. The lecturers are one of the most important in the field: Enrico Giusti, Frederik Almgren and Mario Miranda.

Geometric Measure Theory and Minimal Surfaces Varenna (Como), August 25 – September 2, 1972

Director: E. Bombieri (Univ. Pisa).
Lectures: W.K. Allard (Princeton Univ.), F.J. Almgren (Princeton Univ.), E. Bombieri, E. Giusti (Univ. Pisa), M. Miranda (Univ. Ferrara).
Seminars: J. Guckenheimer (Princeton Univ.), D. Kinderlehrer (Univ. Minnesota), L. Piccinini (SNS, Pisa).

- In 1973 Guido Stampacchia and Gianfranco Capriz were the Scientific Directors of an important session on the Variational Methods in Mathematical Physics, which at that time were called "new".

New Variational Techniques in Mathematical Physics
Bressanone (Bolzano), June 17–26, 1973

Directors: G. Capriz (Univ. Pisa), G. Stampacchia (SNS, Pisa).
Lectures: G. Duvaut (Univ. Paris XIII), J.J. Moreau (Univ. Languedoc, Montpellier), B. Nayroles (Univ. Poitiers).
Seminars: C. Baiocchi (Univ. Pavia), Ch. Castaing (Univ. Languedoc, Montpellier), D. Kinderlehrer (Univ. Minnesota), H. Lanchon (Univ. Essex), J.M. Lasry (Univ. Paris-Dauphine), W. Noll (Carnegie Mellon Univ.), W. Velte (Univ. Wurzburg).

- In 1984 Enrico Giusti was the Scientific Director of a session devoted to the Harmonic Mapping.

Harmonic Mappings and Minimal Immersions
Montecatini Terme (Pistoia), June 24 – July 3, 1984

Director: E. Giusti (Univ. Firenze).
Lectures: S. Hildebrandt (Univ. Bonn), J. Jost (Univ. Bonn), L. Simon (Australian Nat. Univ., Canberra).
Seminars: J.H. Sampson (Johns Hopkins Univ.), M. Seppala (Univ. Helsinki).

- In 1987 Mariano Giaquinta was the Scientific Ddirector of a session in which different aspects of the Calculus of Variations were presented.

Topics in Calculus of Variations
Montecatini Terme (Pistoia), July 20–28, 1987

Director: M. Giaquinta (Univ. Firenze).
Lectures: L. Caffarelli (IAS, Princeton), A. J. Moser (ETH, Zurich), L. Nirenberg (Courant Inst.), R. Schoen (Univ. California, San Diego), A. Tromba (Max-Planck Inst., Bonn).
In this session for the first time Luis Caffarelli was a lecture. It is the begin of his valuable collaboration with C.I.M.E..

- In 1989 Arrigo Cellina was the Scientific Director of a session devoted to non convex problems and the methods for studying different subjects. In particular the non convex functionals of Calculus of Variations were considered.

Methods of Nonconvex Analysis
Varenna (Como), June 15–23, 1989

Director: A. Cellina (SISSA, Trieste).
Lectures: I. Ekeland (Univ. Paris, Dauphine), P. Marcellini (Univ. Firenze), A. Marino (Univ. Pisa), C. Olech (PAN, Warszawa), G. Pianigiani (Univ. Siena), T.R. Rockafellar (Univ. Washington, Seattle), M. Valadier (USTL, Montpellier).

- In 1995 Italo Capuzzo Dolcetta and Piere Louis Lions were the Scientific Directors of a session devoted to the viscosity solution and their applications in several fields of Partial Differential Equations.

Viscosity Solutions and Applications
Montecatini Terme (Pistoia), June 12–20, 1995

Directors: I. Capuzzo Dolcetta (Univ. Roma, La Sapienza), P.L. Lions (Univ. Paris Dauphine).
Lectures: M. Bardi (Univ. Padova), M.G. Crandall (Univ. California, Santa Barbara), L.C. Evans (Univ. California, Berkeley), M.H. Soner (Carnegie Mellon Univ.), P.E. Souganidis (Univ. Wisconsin).
One of the lecturers was L. C. Evans, which in the subsequent years has partecipated several times in the CIME activities.

- In 1996 Stefan Hildebrandt and Michael Struwe were Scientific Directors of a session, devoted to the contributions of the variational methods for the Ginzburg-Landau equations, the microstructure and phase transitions and the Plateau Problem.

Calculus of Variations and Geometric Evolution Problems
Cetraro (Cosenza), June 15–22, 1996

Directors: S. Hildebrandt (Univ. Bonn), M. Struwe (ETH, Zurich).
Lectures: F. Bethuel (ENS, Cachan), R. Hamilton (Univ. Califonia, San Diego), S. Muller (ETH, Zurich), K. Steffen (Univ. Dusseldorf).

- In 2001 Luis Caffarelli with Sandro Salsa were the Scientific Directors of a session which represents a basic guide on Optimal Transportation, by considering different point of views and perspectives,

Optimal Transportation and Applications
Martina Franca (Taranto), September 2–8, 2001

Directors: L. Caffarelli (Univ. Texas, Austin), S. Salsa (Politecnico Milano).
Lectures: L. Caffarelli (Univ. Texas, Austin), G. Buttazzo (Univ. Pisa), L.C. Evans (Univ. California, Berkeley), Y. Brenier (LAN-UPMC, Paris VI), C. Villani (Ecole Normale Sup., Lyon).

- Bernard Dacorogna and Paolo Marcellini are the Scientific Directors of this session, with almost 150 partecipants, the most attended course of the history of C.I.M.E.. Such wide partecipation is a clear indication that the Calculus of Variations is still an interesting and alive subject.

Calculus of Variations and NonLinear Partial Differential Equations
Cetraro (Cosenza), June 27 – July 2, 2005

Directors: Bernard Dacorogna (EPLF, Lausanne) Paolo Marcellini (Univ. Firenze).

Lectures L. Ambrosio (SNS Pisa), L.A. Caffarelli (Univ. Texas, Austin), M. Crandall (Univ. California, Santa Barbara), L.C. Evans (Univ. California, Berkeley, Usa), G. Dal Maso (SISSA, Trieste), N. Fusco (Univ. Napoli).

In these last years social and human sciences offered to the theory on Nonlinear PDE a new field to be added to the traditional ones coming from physics and natural sciences.

Even Hilbert in his famous conference in Paris in 1900, stressed the fruitfulness and opportunity of the interactions between *reason and experience* for the development of the mathematical theories.

However, nowadays it is a widespread opinion that pure mathematical research is important and absolutely necessary also without direct applications. Let me conclude by recalling the opinion of Cantor:

> The essence of mathematics is freedom and *independence*....
> freedom expressed as driving curiosity of a bright child....
> freedom to pursue innocent fascination until it finally touched the world we all live in.