Sums of reciprocals of the central binomial coefficients

SUMS OF RECIPROCALS OF THE CENTRAL BINOMIAL COEFFICIENTS

Renzo Sprugnoli
Dipartimento di Sistemi e Informatica, Viale Morgagni, 65 - Firenze, Italy
sprugnoli@dsi.unifi.it

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Abstract

We consider a set of combinatorial sums involving the reciprocals of the central binomial coefficients and try to solve (or close) them by means of generating functions. We obtain a number of results for infinite sums, in some of which the golden ratio ϕ appears. Besides, we close some finite sums by applying the method of coefficients to the generating functions previously obtained.

Key words: reciprocals of binomial coefficients; combinatorial sums; generating functions; method of coefficients; Riordan arrays.

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1. Introduction

Recently, several papers have appeared dealing with finite and infinite sums related to the reciprocals of binomial coefficients (see [1], [2], [8], [12], [15], [16], [17], [21]). Actually, these kinds of sums have been studied for a very long time (see, e.g., [7] and [20]). Table 2 in Gould's collection [6] (1972 edition) is dedicated to these sums, and the author reports twenty six identities. In particular, Trif [17] ascribes to Rockett [11] the identity:

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k},$$

but this is identity (2.25) in Gould's collection, where a paper of Staver [14], as old as 1947, is quoted.

The paper [21] concentrates on the reciprocals of the central binomial coefficients. Gould only gives two identities of this sort, namely (2.9) and (2.20), the latter having been proved by Euler. Actually, in his book [4, p. 89], as part of an exercise, Comtet proposes to prove the four identities:

$$\sum_{k=1}^{\infty} \binom{2k}{k}^{-1} = \frac{1}{3} + \frac{2\pi\sqrt{3}}{27} \qquad \sum_{k=1}^{\infty} \frac{1}{n} \binom{2k}{k}^{-1} = \frac{\pi\sqrt{3}}{9}$$

$$\sum_{k=1}^{\infty} \frac{1}{n^2} \binom{2k}{k}^{-1} = \frac{\pi^2}{18} \qquad \sum_{k=1}^{\infty} \frac{1}{n^4} \binom{2k}{k}^{-1} = \frac{17\pi^4}{3240}.$$

He does not give any hint or any reference, and when the present author encountered this exercise (many years ago), he was only able to prove the first three identities, but failed on the last (the interested reader can find a non-elementary proof in Borwain and Borwain [3], who quote the paper of De Doelder [5]. See also van der Poorten [18]).

However, I am still convinced that generating functions are a good approach for the other cases. Therefore I reconsidered my old results, which are comparable to (and perhaps are more flexible than) the method proposed in these recent papers, which are mostly based on the basic identity:

$$\binom{n}{k}^{-1} = (n+1) \int_0^1 y^k (1-y)^{n-k} \mathrm{d}y,$$

a particular case of the Beta function and a very useful formula. We will not adopt this formula here; instead, we freely use the theory of generating functions and Riordan arrays, as part of the general *method of coefficients*, as exposed in [9], and maintaining the exposition at an elementary level.

The paper is divided into three parts. In Section 2, we compute the generating functions related to some sequences involving the reciprocals of the central binomial coefficients. In Section 3, these generating functions are used to evaluate infinite sums, some of which are new. Finally, Section 4 contains the most original part of the paper: it is dedicated to showing some examples of finite sums, in which the method of coefficients (Riordan arrays) is applied to the generating functions previously found.

2. Generating Functions

In this section, we consider seven sequences related to the reciprocals of the central binomial coefficients; the sequences are defined by means of their general term:

$$Z_n = 4^n \binom{2n}{n}^{-1} \qquad W_n = \frac{4^n}{n} \binom{2n}{n}^{-1} \qquad T_n = \frac{4^n}{n^2} \binom{2n}{n}^{-1}$$

$$A_n = \frac{4^n}{2n+1} \binom{2n}{n}^{-1} \qquad P_n = \frac{4^n}{n+1} \binom{2n}{n}^{-1}$$

$$M_n = \frac{4^n}{n-1}\binom{2n}{n}^{-1} \qquad N_n = \frac{4^n}{(n-1)^2}\binom{2n}{n}^{-1}.$$

We do not consider sequences of the form $Y_n^{(m)} = 4^n n^m \binom{2n}{n}^{-1}$, which can be approached as in Lehmer [7], by the operator $\theta = t\mathrm{d}/\mathrm{d}t$, rule K3 in [9].

The first important step in our development is to find the corresponding generating functions. We present these functions in terms of the arctangent function; several authors have proposed related generating functions written in terms of arcsine, but the difference is negligible and in our opinion our approach is a bit more direct (see, e.g., Lehmer [7] and his proofs).

Theorem 2.1 *The ordinary generating function of the sequence $(Z_n)_{n\in N}$ is:*

$$Z(t) = \mathcal{G}\left(4^n\binom{2n}{n}^{-1}\right) = \frac{1}{1-t}\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}} + \frac{1}{1-t}.$$

(Note: $\mathcal{G}$ denotes the ordinary generating function of a sequence)

Proof. We can express Z_{n+1} in terms of Z_n:

$$4^{n+1}\binom{2n+2}{n+1}^{-1} = 4^{n+1}\frac{(n+1)^2}{2(n+1)(2n+1)}\binom{2n}{n}^{-1} = \frac{2(n+1)}{2n+1}4^n\binom{2n}{n}^{-1}.$$

This relation corresponds to the recurrence $(2n+1)Z_{n+1} = 2(n+1)Z_n$, which can be written:

$$2(n+1)Z_{n+1} - Z_{n+1} = 2nZ_n + 2Z_n.$$

By passing to the generating function $Z(t)$:

$$2Z'(t) - \frac{Z(t)-1}{t} = 2tZ'(t) + 2Z(t),$$

that is, the ordinary differential equation:

$$2t(1-t)Z'(t) - (1+2t)Z(t) + 1 = 0.$$

The solution is standard:

$$Z(t) = \sqrt{\frac{t}{(1-t)^3}}\left(-\int\sqrt{\frac{1-z}{z}}\cdot\frac{\mathrm{d}z}{2z}\right).$$

The integral can be solved by the change of variable $y = \sqrt{z/(1-z)}$, which gives $z = y^2/(1+y^2)$ and:

$$1 - z = \frac{1}{1+y^2}, \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{2y}{(1+y^2)^2}.$$

The integral now becomes:

$$\frac{1}{2}\int \frac{1+y^2}{y^2}(1+y^2)\frac{2y\mathrm{d}y}{y(1+y^2)^2} = \int \frac{\mathrm{d}y}{y^2(1+y^2)} =$$

$$= \int \frac{\mathrm{d}y}{y^2} - \int \frac{\mathrm{d}y}{1+y^2} = -\frac{1}{y} - \arctan y + C = -\arctan\sqrt{\frac{t}{1-t}} - \sqrt{\frac{1-t}{t}} + C.$$

This gives:

$$Z(t) = \frac{1}{1-t}\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}} + \frac{1}{1-t} + C;$$

since $Z(0) = Z_0 = 1$, we derive $C = 0$ and this concludes the proof. □

This result can be iterated:

Theorem 2.2 *The ordinary generating function of the sequence* $(W_n)_{n\in N}$ *is:*

$$W(t) = \mathcal{G}\left(\frac{4^n}{n}\binom{2n}{n}^{-1}\right) = 2\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}}.$$

Proof. In general, given a sequence $(f_n)_{n\in N}$ and its generating function $f(t) = \mathcal{G}(f_n)$, the generating function of the sequence $\mathcal{G}\left(\frac{1}{n}f_n\right)_{n\in N}$ (with $\frac{1}{0}f_0 = 0$) is:

$$\mathcal{G}\left(\frac{1}{n}f_n\right) = \int_0^t \frac{f(z)-f_0}{z}\mathrm{d}z,$$

as can be checked immediately. Therefore, we have:

$$W(t) = \int_0^t \sqrt{\frac{z}{(1-z)^3}}\arctan\sqrt{\frac{z}{1-z}}\frac{\mathrm{d}z}{z} + \int_0^t \frac{z}{1-z}\frac{\mathrm{d}z}{z},$$

where we have subtracted $Z_0 = 1$. We use again the substitution $y = \sqrt{z/(1-z)}$ and the first integral becomes:

$$\int_0^{\sqrt{t/(1-t)}} y\arctan(y)\frac{1+y^2}{y^2}(1+y^2)\frac{2y\mathrm{d}y}{(1+y^2)^2} = 2\int_0^{\sqrt{t/(1-t)}}\arctan(y)\mathrm{d}y =$$

$$= \Big[2y\arctan(y) - \ln(1+y^2)\Big]_0^{\sqrt{t/(1-t)}} = 2\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}} - \ln\frac{1}{1-t}.$$

Since the second integral is just $\ln(1/(1-t))$, we conclude with the asserted identity. □

We can now go on in the same manner:

Theorem 2.3 *The ordinary generating function of the sequence* $(T_n)_{n\in N}$ *is:*

$$T(t) = \mathcal{G}\left(\frac{4^n}{n^2}\binom{2n}{n}^{-1}\right) = 2\left(\arctan\sqrt{\frac{t}{1-t}}\right)^2 .$$

Proof. Proceeding as in the two previous theorems, we have:

$$T(t) = 2\int_0^t \sqrt{\frac{z}{1-z}}\arctan\sqrt{\frac{z}{1-z}}\frac{\mathrm{d}z}{z} = 2\int_0^{\sqrt{t/(1-t)}} y\arctan(y)\frac{1+y^2}{y^2}\frac{2y\mathrm{d}y}{(1+y^2)^2} =$$

$$= 4\int_0^{\sqrt{t/(1-t)}} \frac{\arctan y}{1+y^2}\mathrm{d}y = 2\Big[(\arctan y)^2\Big]_0^{\sqrt{t/(1-t)}} = 2\left(\arctan\sqrt{\frac{t}{1-t}}\right)^2$$

and this concludes the proof. □

Unfortunately, we have not been able to go further (see Lehmer [7, p. 453]), and we now consider another generating function:

Theorem 2.4 *The ordinary generating function of the sequence* $(A_n)_{n\in N}$ *is:*

$$A(t) = \mathcal{G}\left(\frac{4^n}{2n+1}\binom{2n}{n}^{-1}\right) = \frac{W(t)}{2t} = \frac{1}{t}\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}}.$$

Proof. Note that:

$$\frac{4^n}{2n+1}\binom{2n}{n}^{-1} = \frac{4^n}{2n+1}\frac{n!^2}{(2n)!}\frac{2(n+1)^2}{2(n+1)^2} = \frac{2\cdot 4^n}{n+1}\frac{(n+1)!^2}{(2n+2)!} = \frac{4^{n+1}}{2(n+1)}\binom{2n+2}{n+1}^{-1}.$$

This implies $A_n = W_{n+1}/2$ and the relation in the assert follows immediately. □

Another interesting generating function is the following.

Theorem 2.5 *The ordinary generating function of the sequence* $(P_n)_{n\in N}$ *is:*

$$P(t) = \mathcal{G}\left(\frac{4^n}{n+1}\binom{2n}{n}^{-1}\right) = \frac{2W(t)-T(t)}{2t} =$$

$$= \frac{2}{t}\sqrt{\frac{t}{1-t}}\arctan\left(\sqrt{\frac{t}{1-t}}\right) - \frac{1}{t}\left(\arctan\sqrt{\frac{t}{1-t}}\right)^2.$$

Proof. We proceed as follows:

$$\frac{4^n}{n+1}\binom{2n}{n}^{-1} = \frac{4^n}{n+1}\frac{n!^2}{(2n)!}\frac{(n+1)^2}{(n+1)^2}\frac{2(2n+1)}{2(2n+1)} = \frac{2\cdot 4^n(2n+1)}{(n+1)^2}\frac{(n+1)!^2}{(2n+2)!} =$$

$$= \frac{4^n(4(n+1)-2)}{(n+1)^2}\binom{2n+2}{n+1}^{-1} = \frac{4^{n+1}}{n+1}\binom{2n+2}{n+1}^{-1} - \frac{4^{n+1}}{2(n+1)^2}\binom{2n+2}{n+1}^{-1}.$$

Hence $P_n = W_{n+1} - T_{n+1}/2$, and passing to the generating functions $P(t)$, $W(t)$ and $T(t)$ we get the desired result. □

We now consider another situation:

Theorem 2.6 *The ordinary generating function of the sequence* $(M_n)_{n\in N}$ *is:*

$$M(t) = \mathcal{G}\left(\frac{4^n}{n-1}\binom{2n}{n}^{-1}\right) = 2tW(t) - 2tA(t) + 2t =$$

$$= 2\ (2\,t-1)\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}} + 2t.$$

(Note: this generating function gives $M_0 = M_1 = 0$*.)*

Proof. We have:

$$\frac{4^{n+1}}{n}\binom{2n+2}{n+1}^{-1} = \frac{4^{n+1}}{n}\frac{(n+1)!^2}{(2n+2)!} = \frac{(n+1)^2 4^{n+1}}{2n(n+1)(2n+1)}\binom{2n}{n}^{-1} =$$

$$= \frac{4^{n+1}(n+1)}{2n(2n+1)}\binom{2n}{n}^{-1} = \frac{4}{2}\frac{4^n}{n}\binom{2n}{n}^{-1} - \frac{4}{2}\frac{4^n}{2n+1}\binom{2n}{n}^{-1},$$

which implies $M_{n+1} = 2W_n - 2A_n$, except for $n = 0$ and $n = 1$. After adjusting these two values, we obtain the desired result. □

We conclude this section with the following result.

Theorem 2.7 *The ordinary generating function of the sequence* $(N_n)_{n\in N}$ *is:*

$$N(t) = \mathcal{G}\left(\frac{4^n}{(n-1)^2}\binom{2n}{n}^{-1}\right) = t(4A(t) - 2W(t) + 2T(t) - 4) =$$

$$= 4\ (1-t)\sqrt{\frac{t}{1-t}}\arctan\left(\sqrt{\frac{t}{1-t}}\right) + 4\,t\left(\arctan\sqrt{\frac{t}{1-t}}\right)^2 - 4t.$$

(Note: this generating function gives $N_0 = N_1 = 0$*.)*

Proof. We have:

$$\frac{4^{n+1}}{n^2}\binom{2n+2}{n+1}^{-1} = \frac{4^{n+1}}{n^2}\frac{(n+1)!^2}{(2n+2)!}\binom{2n}{n}^{-1} = \frac{(n+1)^2 4^{n+1}}{2n^2(n+1)(2n+1)}\binom{2n}{n}^{-1} =$$

$$= \frac{2\cdot 4^n(n+1)}{n^2(2n+1)}\binom{2n}{n}^{-1} = \frac{4}{2n+1}4^n\binom{2n}{n}^{-1} - \frac{2}{n}4^n\binom{2n}{n}^{-1} + \frac{2}{n^2}4^n\binom{2n}{n}^{-1},$$

after expansion into partial fractions. This implies $N_{n+1} = 4A_n - 2W_n + 2T_n$, except for $n = 0$ and $n = 1$. After adjusting these two values, the generating function follows immediately. □

3. Infinite Sums

The generating functions, obtained in the previous section, allow us to compute several infinite sums. The quoted papers contain evaluations of infinite sums of this sort; therefore, many of the listed results are already known. In our opinion, the important point is the unitary approach we give to the subject. First of all, we observe that:

$$\binom{2n}{n}^{-1} \sim \frac{\sqrt{\pi n}}{4^n},$$

and consequently only two of the quantities considered correspond to converging infinite sums. In fact we have:

Theorem 3.1 *The following identities hold:*

$$\sum_{n=1}^{\infty} \frac{4^n}{n^2}\binom{2n}{n}^{-1} = \frac{\pi^2}{2} \approx 4.93480220\ldots$$

$$\sum_{n=2}^{\infty} \frac{4^n}{(n-1)^2}\binom{2n}{n}^{-1} = \pi^2 - 4 \approx 5.86960440\ldots.$$

Proof. All of the generating functions of the previous section have a unique singularity at $t = 1$, so their radius of convergence is 1. Therefore, the first sum is obtained by setting $t = 1$ in $T(t)$. Clearly

$$\lim_{t\to 1} \arctan\sqrt{\frac{t}{1-t}} = \pi/2$$

and the result follows immediately. For the second sum, we observe that by Theorem 2.4, the generating function for $(N_n)_{n\in N}$ is:

$$N(t) = t(4A(t) - 2W(t) + 2T(t) - 4) = 2(1-t)W(t) + 2tT(t) - 4t.$$

The $(1-t)$ annihilates the $W(t)$ so that only the two last terms remain and the result follows, by the first sum. □

The analogous sums with alternating signs converge, except when we consider $(Z_n)_{n\in N}$, whose terms diverge in modulus. For example we have:

Theorem 3.2 *The following alternating, infinite sums converge:*

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{n}\binom{2n}{n}^{-1} = \sqrt{2}\ln(\sqrt{2}-1) \approx -1.24645048\ldots$$

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{n^2}\binom{2n}{n}^{-1} = -2(\ln(\sqrt{2}-1))^2 \approx -1.55363879\ldots$$

Proof. This time we have to compute $W(-1)$, which is:

$$\left[2\sqrt{\frac{t}{1-t}}\arctan\sqrt{\frac{t}{1-t}}\ \middle|\ t=-1\right] = 2\sqrt{-\frac{1}{2}}\arctan\sqrt{-\frac{1}{2}}.$$

The arctangent of a complex number can be computed by using the transformation into a natural logarithm:

$$\arctan z = -i\ln\frac{1+iz}{\sqrt{1+z^2}}; \tag{1}$$

for $z = i\sqrt{2}/2$ this formula gives $-i\ln(\sqrt{2}-1)$ and our result is quite immediate. The second sum is performed in the same way. □

For the sake of completeness, we also give the other values:

$$\sum_{n=0}^{\infty}\frac{(-4)^n}{2n+1}\binom{2n}{n}^{-1} = -\frac{\sqrt{2}}{2}\ln(\sqrt{2}-1) \approx 0.623225240\ldots$$

$$\sum_{n=0}^{\infty}\frac{(-4)^n}{n+1}\binom{2n}{n}^{-1} = -\sqrt{2}\ln(\sqrt{2}-1) - (\ln(\sqrt{2}-1))^2 \approx 0.469631080\ldots$$

$$\sum_{n=2}^{\infty}\frac{(-4)^n}{n-1}\binom{2n}{n}^{-1} = -3\sqrt{2}\ln(\sqrt{2}-1) - 2 \approx 1.739351440\cdots$$

$$\sum_{n=2}^{\infty}\frac{(-4)^n}{(n-1)^2}\binom{2n}{n}^{-1} = 4(\sqrt{2}\ln(\sqrt{2}-1) + (\ln(\sqrt{2}-1))^2 + 1) \approx 2.121475678\ldots$$

Actually, the generating functions can be used to find the infinite sums every time we substitute for 4 any number x such that $-4 \le x \le 4$. Many sums are reported by Lehmer [7] and can be easily checked by our generating functions. Here we present only the cases $x = 2$ and $x = -2$.

Theorem 3.3 *The following two identities hold:*

$$\sum_{n=0}^{\infty}2^n\binom{2n}{n}^{-1} = \frac{\pi}{2} + 2 \approx 3.57079632\ldots$$

$$\sum_{n=0}^{\infty}(-2)^n\binom{2n}{n}^{-1} = -\frac{2\sqrt{3}}{9}\ln\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) + \frac{2}{3} \approx 0.413218001\ldots.$$

Proof. We only have to compute $Z(1/2)$ and $Z(-1/2)$. In the former case we obtain:

$$\left[\arctan\sqrt{\frac{t}{1-t}}\ \middle|\ t=\frac{1}{2}\right] = \arctan 1 = \frac{\pi}{4}$$

and the result of the first sum follows directly. The second sum corresponds to $Z(-1/2)$, and here we have to compute $\arctan(\sqrt{-1/3})$; by using (1) we obtain $-i\ln((\sqrt{3}+1)/\sqrt{2})$. From this quantity the result follows after a few computations. □

Naturally, the most important situation is $x = 1$, corresponding to some classic sums. In that case the generating functions have to be evaluated for $t = 1/4$ and $t = -1/4$, and the only relevant point is the computation of the arctangent:

Theorem 3.4 *The following identities hold:*

$$
\begin{aligned}
\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} &= \frac{2\pi\sqrt{3}}{27} + \frac{4}{3} \approx 1.73639985\ldots \\
\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n}{n}^{-1} &= \frac{\sqrt{3}\pi}{9} \approx 0.604599788\ldots \\
\sum_{n=1}^{\infty} \frac{1}{n^2}\binom{2n}{n}^{-1} &= \frac{\pi^2}{18} \approx 0.548311355\ldots \\
\sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{2n}{n}^{-1} &= \frac{2\sqrt{3}\pi}{9} \approx 1.20919957\ldots \\
\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}^{-1} &= \frac{4\sqrt{3}}{9}\pi - \frac{\pi^2}{9} \approx 1.32177644\ldots \\
\sum_{n=2}^{\infty} \frac{1}{n-1}\binom{2n}{n}^{-1} &= \frac{1}{2} - \frac{\sqrt{3}}{18}\pi \approx 0.197700105\ldots \\
\sum_{n=2}^{\infty} \frac{1}{(n-1)^2}\binom{2n}{n}^{-1} &= \frac{\sqrt{3}}{6}\pi + \frac{1}{36}\pi^2 - 1 \approx 0.1810553599\ldots
\end{aligned}
$$

Proof. In this case we use the fact that:

$$\left[\arctan\sqrt{\frac{t}{1-t}} \;\Big|\; t = 1\right] = \arctan(\infty) = \frac{\pi}{2}$$

and perform the routine computations. □

We now consider the sequences with alternating signs. When we developed these sums for the first time, we were surprised by the appearance of the golden ratio $\phi = (\sqrt{5}+1)/2$. See [21] for the third sum.

Theorem 3.5 *The following identities hold true:*

$$\sum_{n=0}^{\infty} (-1)^n \binom{2n}{n}^{-1} = \frac{4}{5} - \frac{4\sqrt{5}}{25}\ln\phi \approx 0.627836423\ldots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \binom{2n}{n}^{-1} = -\frac{2\sqrt{5}}{5} \ln \phi \approx -0.430408940\ldots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \binom{2n}{n}^{-1} = -2(\ln \phi)^2 \approx -0.463129641\ldots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \binom{2n}{n}^{-1} = \frac{4\sqrt{5}}{5} \ln \phi \approx 0.860817881\ldots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \binom{2n}{n}^{-1} = \frac{8\sqrt{5}}{5} \ln \phi - 4(\ln \phi)^2 \approx 0.795376481\ldots$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \binom{2n}{n}^{-1} = \frac{3\sqrt{5}}{5} \ln \phi - \frac{1}{2} \approx 0.145613414\ldots$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)^2} \binom{2n}{n}^{-1} = 1 - \sqrt{5} \ln \phi + (\ln \phi)^2 \approx 0.155542468\ldots$$

Proof. In this case, according to (1), we have:

$$\left[\arctan \sqrt{\frac{t}{1-t}} \;\middle|\; t = -\frac{1}{4}\right] = \arctan\left(\sqrt{\frac{-1}{5}}\right) = i \ln(\phi),$$

and the identities then follow easily. □

At this point, we are in a position to prove a rather general theorem:

Theorem 3.6 *Let $(Y_n)_{n \in N}$ be any sequence of the type:*

$$Y_n = \frac{p(n)}{q(n)} \binom{2n}{n}^{-1}$$

where $p(n), q(n)$ are polynomials over $\mathbb{Z}$ with $\deg p(n) \leq \deg q(n)$, and where $q(n)$ contains only the factors $n, n^2, 2n+1, n+1, n-1, (n-1)^2$. Then it is possible to determine the value of the infinite sum (whenever it converges):

$$\sum_{n=k}^{\infty} x^n Y_n$$

($k = 0, 1, 2$ according to the case) where $-4 \leq x \leq 4$.

Proof. It is enough to convert $p(n)/q(n)$ by a partial fraction expansion and then apply the previous results. □

This result includes the last case in [21], where $q(n) = n^2(n+1)$. The method used in that paper seems rather specific and (it seems) cannot be extended in a simple way to other cases. We illustrate some special cases of Theorem 3.6 with the following theorem.

Theorem 3.7 *The following identities hold true:*

$$\sum_{n=2}^{\infty} \frac{4^n}{n^2(n^2-1)} \binom{2n}{n}^{-1} = 4 - \frac{3}{8}\pi^2 \approx 0.298898349\ldots$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2(n^2-1)} \binom{2n}{n}^{-1} = \frac{11}{8} - \frac{\sqrt{3}}{4}\pi \approx 0.01465047\ldots$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2(n^2-1)} \binom{2n}{n}^{-1} = 4(\ln\phi)^2 - \frac{\sqrt{5}}{2}\ln\phi - \frac{3}{8} \approx 0.01324810\ldots$$

$$\sum_{n=2}^{\infty} \frac{(-4)^n}{n^2(n^2-1)} \binom{2n}{n}^{-1} = \frac{5}{2}(\ln(\sqrt{2}-1))^2 - \sqrt{2}\ln(\sqrt{2}-1) - 3 \approx 0.188498980\ldots$$

Proof. By partial fraction expansion we find:

$$\frac{1}{n^2(n^2-1)} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)} - \frac{1}{n^2}.$$

Therefore, we have the generating function:

$$Q(t) = \mathcal{G}\left(\frac{4^n}{n^2(n^2-1)}\binom{2n}{n}^{-1}\right) = \frac{1}{2}M(t) - \frac{1}{2}P(t) - T(t) + \frac{1}{2} + \frac{5}{2}t,$$

where the last two terms are a simple correction to have $Q_0 = Q_1 = 0$. Proceeding as in the previous examples, we obtain the desired results. □

We conclude this section with two sums; the first one is well-known (see [17]), while the second appears to be new.

Theorem 3.8 *The following identities hold:*

$$\sum_{n=0}^{\infty} \binom{4n}{2n}^{-1} = \frac{16}{15} + \frac{\sqrt{3}}{27}\pi - \frac{2\sqrt{5}}{25}\ln\phi \approx 1.18211814\ldots, \quad \textit{and}$$

$$\sum_{n=0}^{\infty} (-1)^n \binom{4n}{2n}^{-1} \approx .846609430\ldots,$$

where this last constant is:

$$\frac{16}{17} + \frac{4\sqrt{34}(\sqrt{17}-2)}{289\sqrt{\sqrt{17}-1}} \arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right) + \frac{2\sqrt{34}(\sqrt{17}+2)}{289\sqrt{\sqrt{17}+1}} \ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right).$$

Proof. In order to find the first sum, we can observe that it is sufficient to compute $(Z(t)+Z(-t))/2$ at $t=1/4$. The computation is a bit complex, but by hand or using a Computer Algebra System we arrive at the desired result. For the second sum, a Computer Algebra System is necessary to be sure that the computation is performed correctly. First we use the bisection formula:

$$\mathcal{G}\left(Z_{2n}\right)=\frac{Z(\sqrt{t})+Z(-\sqrt{t})}{2},$$

and then we compute it at $t=-1/16$. The result is a rather complicated formula, containing complex numbers; however, it can be reduced to the expression given in the assertion, containing only real quantities. By again using the Computer Algebra System, we can check the result by computing partial sums of the series, which converges rapidly. □

4. Some Finite Sums

As we have shown, the generating function approach easily gives us a number of infinite sums. The method of coefficients can now be used to obtain many finite sums. In particular, the Riordan array theory suggests several interesting developments. Although the theory was devised to be used in finding (closed) solutions to combinatorial sums in a constructive (and human or computer-free) way, it can also be used to find new identities. This fact is shown, in an elementary fashion, by the following examples.

Let $(f_n)_{n\in N}$ be a sequence and $f(t)=\sum_{n=0}^{\infty} f_n t^n$ its ordinary generating function. If we define $g_n=\sum_{k=0}^{n} f_k$, the partial sum of the first n terms of the original sequence, it is easily proven that the generating function $\mathcal{G}\left(g_n\right)$ is:

$$\mathcal{G}\left(g_n\right)=\frac{f(t)}{1-t} \qquad \mathcal{G}\left((-1)^n g_n\right)=\frac{f(-t)}{1+t}.$$

This fact is known as Partial Sums Theorem, and by using it, we prove:

Theorem 4.1 *The following identity holds:*

$$\sum_{k=1}^{n}\frac{4^k}{k}\binom{2k}{k}^{-1}=2\cdot 4^n\binom{2n}{n}^{-1}-2.$$

Proof. The identity in the assertion of Theorem 2.1 can be written:

$$Z(t)=\frac{1}{2}\frac{1}{1-t}W(t)+\frac{1}{1-t}.$$

Therefore, we can extract the coefficient of t^n:

$$[t^n]Z(t)=\frac{1}{2}[t^n]\frac{1}{1-t}W(t)+[t^n]\frac{1}{1-t}.$$

Hence

$$Z_n = \frac{1}{2}\sum_{k=0}^{n} W_k + 1,$$

and after rearranging terms, this is equivalent to the identity in the assertion. □

In a similar way we prove the following result, which requires some care with indices:

Theorem 4.2 *The following identity holds:*

$$\sum_{k=2}^{n-1} \frac{4^k}{k(k-1)}\binom{2k}{k}^{-1} = 4 - \frac{(2n-1)4^n}{n(n-1)}\binom{2n}{n}^{-1}.$$

Proof. The formula in Theorem 2.6 can be written:

$$M(t) = -(1-t-t)W(t) + 2t.$$

Dividing by $1-t$, we have:

$$\frac{M(t)}{1-t} = -W(t) + \frac{t}{1-t}W(t) + 2\frac{t}{1-t}.$$

We can now extract the coefficient of t^n and obtain:

$$\sum_{k=2}^{n} \frac{4^k}{k-1}\binom{2k}{k}^{-1} - \sum_{k=1}^{n-1} \frac{4^k}{k}\binom{2k}{k}^{-1} = 2 - \frac{4^n}{n}\binom{2n}{n}^{-1}.$$

This is equivalent to:

$$\frac{4^n}{n-1}\binom{2n}{n}^{-1} + \sum_{k=2}^{n-1}\left(\frac{1}{k-1} - \frac{1}{k}\right)4^k\binom{2k}{k}^{-1} - \frac{4}{2} = 2 - \frac{4^n}{n}\binom{2n}{n}^{-1},$$

and, by rearranging terms, the desired result is obtained. □

We also prove:

Theorem 4.3 *The following identity holds:*

$$\sum_{k=1}^{n-1} \frac{4^k}{k(2k+1)}\binom{2k}{k}^{-1} = 2 - \frac{4^n}{n}\binom{2n}{n}^{-1}.$$

Proof. Let us consider the expression for $N(t)$ in Theorem 2.7. By using Theorem 2.4, it can be written:

$$N(t) = 2(1-t)W(t) + 2tT(t) - 4t,$$

or

$$\frac{N(t)}{1-t} = 2W(t) + \frac{2t}{1-t}T(t) - \frac{4t}{1-t}.$$

Extracting the coefficient of t^n, we have:

$$\sum_{k=2}^{n} N_k = 2W_n + 2\sum_{k=1}^{n-1} T_k - 4,$$

and hence:

$$\sum_{k=1}^{n-1}(N_{k+1} - 2T_k) = 2W_n - 4.$$

Note that:

$$N_{k+1} - 2T_k = \frac{4^{k+1}(k+1)^2}{k^2(2k+1)2(k+1)}\binom{2k}{k}^{-1} - \frac{1}{2}\frac{4^{k+1}}{k^2}\binom{2k}{k}^{-1} = -\frac{2\cdot 4^k}{k(2k+1)}\binom{2k}{k}^{-1},$$

and from this we obtain the result. □

We leave to the interested reader the proof of:

$$\sum_{k=0}^{n} 4^k \binom{2k}{k}^{-1} = \frac{2n+1}{3}4^{n+1}\binom{2n+2}{n+1}^{-1} + \frac{1}{3}.$$

Another general theorem in the method of coefficients is the so-called *Euler transformation*. If $f(t)$ is as above, we have:

$$\sum_{k=0}^{n}\binom{n}{k} f_k = [t^n]\frac{1}{1-t} f\left(\frac{t}{1-t}\right) \qquad \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} f_k = [t^n]\frac{1}{1+t} f\left(\frac{t}{1+t}\right).$$

With this result in mind, we prove the next two results:

Theorem 4.4 *The following identity holds:*

$$\sum_{k=1}^{n}\binom{n}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{k} = H_n - 2H_{2n}.$$

(note: H_n is the nth harmonic number, that is the sum of the reciprocals of the first n positive integers).

Proof. The proof consists in a string of equalities:

$$\sum_{k=1}^{n}\binom{n}{k}\binom{2k}{k}^{-1}\frac{(-4)^k}{k} = (-1)^n[t^n]\frac{1}{1+t}\left[2\sqrt{\frac{y}{1-y}}\arctan\sqrt{\frac{y}{1-y}}\ \Big|\ y = \frac{t}{1+t}\right] =$$

$$= (-1)^n 2[t^n]\frac{1}{1+t}\sqrt{t}\arctan\sqrt{t} = (-1)^n 2\sum_{k=1}^{n}(-1)^{n-k}[t^k]\sqrt{t}\arctan\sqrt{t} =$$

$$= 2\sum_{k=1}^{n}(-1)^k[t^{2k-1}]\arctan(t) = 2\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}[t^{2k-2}]\frac{1}{1+t^2} = 2\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}(-1)^{k-1} =$$

$$= -2\sum_{k=1}^{n}\frac{1}{2k-1} = -2\left(H_{2n} - \frac{1}{2}H_n\right) = H_n - 2H_{2n}.$$

□

Theorem 4.5 *The following identity holds for $n \geq 0$:*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1}(-4)^k = \frac{1}{1-2n}.$$

Proof. We proceed as follows:

$$(-1)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}4^k\binom{2k}{k}^{-1} =$$

$$= (-1)^n[t^n]\frac{1}{1+t}\left[\frac{1}{1-y}\sqrt{\frac{y}{1-y}}\arctan\sqrt{\frac{y}{1-y}} + \frac{1}{1-y} \;\middle|\; y = \frac{t}{1+t}\right] =$$

$$= (-1)^n[t^n]\left(\sqrt{t}\arctan\sqrt{t} + 1\right) = (-1)^n[t^{2n+1}]\arctan(t) = (-1)^n\frac{(-1)^{n-1}}{2n-1} = \frac{1}{1-2n}.$$

□

Now, we can invert the last two sums:

Theorem 4.6 *The following two identities hold:*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{1-2k} = 4^n\binom{2n}{n}^{-1}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(H_k - 2H_{2k}) = \frac{4^n}{n}\binom{2n}{n}^{-1}.$$

Proof. These are simple consequences of the general inversion theorem:

$$a_n = \sum_{k=0}^{n}\binom{n}{k}b_k \qquad \text{iff} \qquad b_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a_k.$$

where $(a_k)_{k\in N}$ and $(b_k)_{k\in N}$ are two sequences, which can be proved easily with the method of coefficients. If $a(t), b(t)$ are the generating functions of the two sequences, by the Euler transformation the identity on the left can be written as:

$$a(t) = \frac{1}{1-t} b\left(\frac{t}{1-t}\right).$$

By applying the change of variable: $y = t/(1-t)$ or $t = y/(1+y)$ we find:

$$a\left(\frac{y}{1+y}\right) = (1+y)b(y) \quad \text{or} \quad b(y) = \frac{1}{1+y} a\left(\frac{y}{1+y}\right)$$

which is exactly equivalent to the set of identities on the right of the *iff*. □ For a proof of the theorem mentioned in the previous proof, we direct the reader to [10].

Another application of the method of coefficients is connected with sums of the following kind:

$$\sum_{k=1}^{n} \binom{n-1}{k-1} f_k = [t^n] f\left(\frac{t}{1-t}\right) \quad \text{and} \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{n-1}{k-1} f_k = [t^n] f\left(\frac{t}{1+t}\right).$$

Let us consider the following result:

Theorem 4.7

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \binom{2k}{k}^{-1} \frac{(-4)^k}{k^2} = \frac{H_n - 2H_{2n}}{n}.$$

Proof. We have

$$\sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{n-k} \frac{4^k}{k^2} \binom{2k}{k}^{-1} = [t^n]\left[2\left(\arctan\sqrt{\frac{y}{1-y}}\right)^2 \;\middle|\; y = \frac{t}{1+t}\right] = 2[t^n]\left(\arctan\sqrt{t}\right)^2 =$$

$$= \frac{2}{n}[t^{n-1}] \frac{1}{1+t} \frac{\arctan\sqrt{t}}{\sqrt{t}} = \frac{2}{n} \sum_{k=0}^{n-1} (-1)^{n-k-1} [t^{2k+1}] \arctan(t) =$$

$$= \frac{2}{n} \sum_{k=0}^{n-1} \frac{(-1)^{n-1}}{2k+1} = \frac{2(-1)^{n-1}}{n} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) = \frac{2(-1)^{n-1}}{n} \left(H_{2n} - \frac{1}{2} H_n\right).$$

The result is now obtained by dividing by $(-1)^n$. □

Let us now extract a coefficient:

$$[t^n] \frac{1}{\sqrt{t}} \arctan\sqrt{\frac{t}{1-t}} = [t^{n+1/2}] \arctan\sqrt{\frac{t}{1-t}} = \frac{1}{n+1/2} [t^{n-1/2}] \frac{\mathrm{d}}{\mathrm{d}t} \arctan\sqrt{\frac{t}{1-t}} =$$

$$= \frac{1}{2n+1} [t^{n-1/2}] \frac{1}{\sqrt{t}} \frac{1}{\sqrt{1-t}} = \frac{1}{2n+1} \frac{1}{4^n} [t^n] \frac{1}{\sqrt{1-4t}} = \frac{1}{(2n+1)4^n} \binom{2n}{n}.$$

Here normal central binomial coefficients appear, and consequently we have a relation between central binomial coefficients and their reciprocals:

Theorem 4.8 *The following holds:*

$$\sum_{k=0}^{n} \frac{1}{2k+1}\binom{2k}{k}\frac{1}{2n-2k+1}\binom{2n-2k}{n-k} = \frac{16^n}{(n+1)(2n+1)}\binom{2n}{n}^{-1}.$$

Proof. The sum on the left is the convolution:

$$[t^n]\left(\frac{1}{\sqrt{t}}\arctan\sqrt{\frac{t}{1-t}}\right)^2 = \frac{1}{2}[t^{n+1}]2\left(\arctan\sqrt{\frac{t}{1-t}}\right)^2.$$

We now apply the generating function $T(t)$ and the result follows immediately. □

An analogous result is identity (3.93) in Gould's collection [6], a proof of which can be found in [13].

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