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CONTINUOUS DIFFERENTIAL INCLUSIONS**

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BAIRE CATEGORY AND THE WEAK BANG-BANG PROPERTY FOR CONTINUOUS DIFFERENTIAL INCLUSIONS

F. S. DE BLASI AND G. PIANIGIANI

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ABSTRACT. For continuous differential inclusions the classical bang-bang property is known to fail, yet a weak form of it is established here, in the case where the right hand side is a multifunction whose values are closed convex and bounded sets with nonempty interior contained in a reflexive and separable Banach space. Our approach is based on the Baire category method.

1. INTRODUCTION

The bang-bang property for differential inclusions has been studied by many authors from different points of view. For recent contributions see, among others, Papageorgiou [14], Tolstonogov [18], Donchev, Farkhi, Mordukhovich [10]. A comprehensive account on differential inclusions and additional references can be found in the monographs by Aubin and Cellina [1], Hu and Papageorgiou [12], Mordukhovich [13], and Tolstonogov [17]. Usually in the investigation of the bang-bang property a crucial role is played by the assumption that the differential inclusion satisfies a globally Lipschitz condition. Recently, it has been shown that the bang-bang property remains valid even under a locally Lipschitz condition [9], while it is known to be false under the mere assumption of continuity in view of an example of Plis [16]. The aim of the present paper is to show that, under appropriate assumptions, a somewhat weaker form of the bang-bang property is valid for continuous differential inclusions (Theorem 1). Our method of approach is based on the Baire category as developed in [6]-[9]. To apply it we need some technical results, among which is a suitable infinite dimensional version of the classical Carathéodory theorem concerning compact convex sets in \mathbb{R}^n (Proposition 4). For further details on the Baire method, see [4], [12], [15].

Let (M, ρ) be a metric space. The interior and the closure of a set $X \subset M$ are denoted by $\text{int } X$ and \overline{X} . For $a \in M$ and $X \subset M$, $X \neq \emptyset$, we set $d(a, X) = \inf_{x \in X} \rho(a, x)$.

Throughout the paper \mathbb{E} is a reflexive and separable real Banach space with norm $\|\cdot\|$ and $\mathcal{B}(\mathbb{E})$ (resp. $\mathcal{C}(\mathbb{E})$) is the space of all subsets of \mathbb{E} which are closed convex bounded with nonempty interior (resp. closed convex bounded nonempty). The spaces $\mathcal{B}(\mathbb{E})$, $\mathcal{C}(\mathbb{E})$ are equipped with the Pompeiu-Hausdorff metric

$$h(X, Y) = \max\left\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\right\}.$$

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If $X \subset \mathbb{E}$, then coX denotes the convex hull of X . If $X \subset \mathbb{E}$ is convex, then $ext X$ denotes the set of all extreme points of X . For $X \subset \mathbb{E}$, $X \neq \emptyset$, we put $\|X\| = \sup\{\|x\| \mid x \in X\}$. Moreover $B = \{x \in \mathbb{E} \mid \|x\| \leq 1\}$.

As usual $C(I, \mathbb{E})$, where $I \subset \mathbb{R}$ is a compact interval, denotes the space of all continuous functions $x : I \rightarrow \mathbb{E}$ with norm $\|x\|_I = \max\{t \in I \mid \|x(t)\|\}$.

Set $I = [t_0, t_1]$, $t_0 < t_1$. A multifunction $F : I \times \mathbb{E} \rightarrow \mathcal{C}(\mathbb{E})$ is said to satisfy assumption (H) if :

- (j) F is continuous on $I \times \mathbb{E}$,
- (jj) F is bounded, i.e. $\|F(t, x)\| < M$ for every $(t, x) \in I \times \mathbb{E}$, M a positive constant.

For F satisfying (H) and $a \in \mathbb{E}$, consider the following Cauchy problems:

$$\begin{aligned} (C_{F,a}) \quad & \dot{x}(t) \in F(t, x(t)), \quad x(t_0) = a, \\ (C_{\text{int } F,a}) \quad & \dot{x}(t) \in \text{int } F(t, x(t)), \quad x(t_0) = a, \\ (C_{\text{ext } F,a}) \quad & \dot{x}(t) \in \text{ext } F(t, x(t)), \quad x(t_0) = a. \end{aligned}$$

A function $x : I \rightarrow \mathbb{E}$ is said to be a *solution* of the Cauchy problem $(C_{F,a})$ (resp. $(C_{\text{int } F,a})$, $(C_{\text{ext } F,a})$) if x is Lipschitzian on I , with $x(t_0) = a$, and satisfies $(C_{F,a})$ (resp. $(C_{\text{int } F,a})$, $(C_{\text{ext } F,a})$), $t \in I$ a.e.

For F satisfying (H) and $a \in \mathbb{E}$ set:

$$\begin{aligned} \mathcal{M}_{F,a} &= \{x : I \rightarrow \mathbb{E} \mid x \text{ is a solution of } (C_{F,a})\}, \\ \mathcal{M}_{\text{int } F,a} &= \{x : I \rightarrow \mathbb{E} \mid x \text{ is a solution of } (C_{\text{int } F,a})\}, \\ \mathcal{M}_{\text{ext } F,a} &= \{x : I \rightarrow \mathbb{E} \mid x \text{ is a solution of } (C_{\text{ext } F,a})\}. \end{aligned}$$

The space $\mathcal{M}_{F,a}$ is equipped with the metric induced by the norm of $C(I, \mathbb{E})$, i.e. the metric of uniform convergence.

It is evident that $\mathcal{M}_{\text{ext } F,a}$ and $\mathcal{M}_{\text{int } F,a}$ are contained in $\mathcal{M}_{F,a}$. Furthermore $\mathcal{M}_{F,a}$ can be empty if \mathbb{E} is infinite dimensional and F is merely continuous.

Under the assumption that $F : I \times \mathbb{E} \rightarrow \mathcal{C}(\mathbb{E})$ is continuous, locally Lipschitzian in the x -variable and bounded, the set $\mathcal{M}_{F,a}$ is nonempty and moreover the following bang-bang property holds (see [9]):

$$\overline{\mathcal{M}}_{\text{ext } F,a} = \mathcal{M}_{F,a},$$

where the closure is in $C(I, \mathbb{E})$. Whenever F is only continuous, then the bang-bang property is no longer valid and one has

$$\overline{\mathcal{M}}_{\text{ext } F,a} \subset \mathcal{M}_{F,a},$$

where the inclusion can be strict, in view of the Plis example [16].

The aim of this paper is to show that if $F : I \times \mathbb{E} \rightarrow \mathcal{B}(\mathbb{E})$ is continuous and bounded, then $\mathcal{M}_{\text{int } F,a} \neq \emptyset$ and the following weak form of the bang-bang property holds:

$$(1.1) \quad \overline{\mathcal{M}}_{\text{ext } F,a} \supset \mathcal{M}_{\text{int } F,a}.$$

Remark 1. If F takes on values in $\mathcal{C}(\mathbb{E})$ and \mathbb{E} is infinite dimensional, then one can have $\mathcal{M}_{\text{ext } F,a} = \mathcal{M}_{F,a} = \emptyset$, by virtue of Godunov's theorem [11].

Remark 2. The inclusion (1.1) can be strict, as is shown by an example presented in Section 3.

For any set $J \subset I$ we denote by $|J|$ and χ_J , respectively, the Lebesgue measure and the characteristic function of J .

By a *regular partition* of the interval $I = [t_0, t_1]$ we mean a finite or denumerable infinite family $\mathcal{P} = \{I_i\}$ of pairwise disjoint nondegenerate intervals $I_i \subset I$ such that, setting $N_0 = I \setminus \bigcup_i I_i$, one has $|N_0| = 0$.

Definition 1. A map $u : I \rightarrow \mathbb{E}$ is said to be *piecewise constant* if u is given by

$$(1.2) \quad u(t) = \sum_i u_i \chi_{I_i}(t) + u_0(t) \chi_{N_0}(t), \quad t \in I,$$

where $\{I_i\}$ is a regular partition of I , $\{u_i\} \subset \mathbb{E}$ is a bounded sequence, and $u_0(t) \in \mathbb{E}$ for every $t \in N_0 = I \setminus \bigcup_i I_i$.

Definition 2. A solution $x \in \mathcal{M}_{F,a}$ is said to be *regular* if there exist a regular partition $\mathcal{P} = \{I_i\}$ of I and corresponding sequences $\{u_i\} \subset \mathbb{E}$ and $\{\sigma_i\} \subset (0, +\infty)$ such that, denoting by $u : I \rightarrow \mathbb{E}$ a piecewise constant map given by (1.2), one has:

$$(j) \quad x(t) = a + \int_{t_0}^t u(s) ds \text{ for each } t \in I,$$

$$(jj) \quad \dot{x}(t) + \sigma_i B = u_i + \sigma_i B \subset F(t, x(t)) \text{ for each } t \in \text{int } I_i \text{ and } I_i \in \mathcal{P}.$$

Set

$$\mathcal{M}_{F,a}^0 = \{x : I \rightarrow \mathbb{E} \mid x \text{ is a regular solution of } (C_{\text{int } F,a})\},$$

and define

$$\mathcal{M} = \overline{\mathcal{M}_{F,a}^0},$$

where the closure is in $C(I, \mathbb{E})$. The space \mathcal{M} is equipped with the metric induced by the norm of $C(I, \mathbb{E})$.

The Choquet function, which we now introduce, plays a crucial role in the proof of our main result.

Denote by \mathbb{E}^* the topological dual of \mathbb{E} . Let $\{l_n\}$, $\|l_n\| = 1$, be a sequence dense in the unit sphere of \mathbb{E}^* . Let F satisfy assumption (H). Following Choquet [5], Vol. II, Ch. 6, we define $\varphi_F : I \times \mathbb{E} \times \mathbb{E} \rightarrow [0, +\infty]$ by

$$\varphi(t, x, v) = \begin{cases} \sum_{n=1}^{\infty} \frac{(l_n(v))^2}{2^n}, & v \in F(t, x), \\ +\infty, & v \in \mathbb{E} \setminus F(t, x). \end{cases}$$

Let \mathcal{A} be the set of all continuous affine functions $a : \mathbb{E} \rightarrow \mathbb{R}$. Let $\overline{\varphi}_F : I \times \mathbb{E} \times \mathbb{E} \rightarrow [-\infty, +\infty)$ be given by

$$\overline{\varphi}_F(t, x, v) = \inf\{a(v) \mid a \in \mathcal{A} \text{ and } a(z) > \varphi_F(t, x, z) \text{ for every } z \in F(t, x)\}.$$

We define $d_F : I \times \mathbb{E} \times \mathbb{E} \rightarrow [-\infty, +\infty)$ by

$$d_F(t, x, v) = \overline{\varphi}_F(t, x, v) - \varphi_F(t, x, v).$$

In the next proposition we review some properties of d_F , the Choquet function associated to F (see Choquet [5], Castaing and Valadier [2]).

Proposition 1. *Let $F : I \times \mathbb{E} \rightarrow \mathcal{B}(\mathbb{E})$ satisfy (H). Then:*

- (i) *for each $(t, x) \in I \times \mathbb{E}$ and $v \in F(t, x)$ we have $0 \leq d_F(t, x, v) \leq M^2$. Moreover $d_F(t, x, v) = 0$ if and only if $v \in \text{ext } F(t, x)$;*
- (ii) *for each $(t, x) \in I \times \mathbb{E}$, the function $d_F(t, x, \cdot)$ is concave on \mathbb{E} and strictly concave on the set $F(t, x)$;*
- (iii) *d_F is upper semicontinuous on $I \times \mathbb{E} \times \mathbb{E}$;*

- (iv) for each $x \in \mathcal{M}_{F,a}$, the function $t \rightarrow d_F(t, x(t), \dot{x}(t))$ is nonnegative, bounded and integrable on I ;
- (v) if $\{x_n\} \subset \mathcal{M}_{F,a}$ converges uniformly to x , then

$$\limsup_{n \rightarrow \infty} \int_I d_F(t, x_n(t), \dot{x}_n(t)) dt \leq \int_I d_F(t, x(t), \dot{x}(t)) dt.$$

2. AUXILIARY RESULTS

In this section we prove some results which will be useful in what follows.

Proposition 2. *Let $F : I \times \mathbb{E} \rightarrow \mathcal{B}(\mathbb{E})$ satisfy (H). Let $x \in \mathcal{M}_{\text{int } F,a}$ and $\varepsilon > 0$. Then there exists a regular solution $y : I \rightarrow \mathbb{E}$ of the Cauchy problem $(C_{\text{int } F,a})$ such that $\|y - x\|_I < \varepsilon$.*

Proof. Let $x \in \mathcal{M}_{\text{int } F,a}$ and $\varepsilon > 0$ be given. Set

$$J = \{t \in (t_0, t_1) \mid \dot{x}(t) \text{ exists and } \dot{x}(t) \in \text{int } F(t, x(t))\}.$$

Let $\tau \in J$ be arbitrary. Hence for some $\sigma > 0$ we have

$$\dot{x}(\tau) + 3\sigma B \subset F(\tau, x(\tau)).$$

Since F is continuous, there exists $\delta_\tau > 0$, with $[\tau, \tau + \delta_\tau] \subset (t_0, t_1)$, such that
(2.1)

$$0 < h < \delta_\tau, |t - \tau| < \delta_\tau, t \in I, \|z - x(\tau)\| < \delta_\tau \Rightarrow \frac{x(\tau + h) - x(\tau)}{h} + \sigma B \subset F(t, z).$$

For any δ with

$$0 < \delta < \min\left\{\frac{\delta_\tau}{M + 1}, \frac{\varepsilon}{2M + 1}\right\},$$

set $I_{\tau,\delta} = [\tau, \tau + \delta]$ and define $y_{\tau,\delta} : I_{\tau,\delta} \rightarrow \mathbb{E}$ by

$$y_{\tau,\delta}(t) = x(\tau) + \frac{x(\tau + \delta) - x(\tau)}{\delta}(t - \tau), \quad t \in I_{\tau,\delta}.$$

Claim 1. $y_{\tau,\delta} : I_{\tau,\delta} \rightarrow \mathbb{E}$ is a regular solution of the following boundary value problem:

$$(B_{\tau,\delta}) \quad \dot{y}(t) = \text{int } F(t, y(t)), \quad y(\tau) = x(\tau), \quad y(\tau + \delta) = x(\tau + \delta).$$

Evidently $y(\tau) = x(\tau)$ and $y(\tau + \delta) = x(\tau + \delta)$. Moreover for $t \in I_{\tau,\delta}$ we have

$$(2.2) \quad \|y_{\tau,\delta}(t) - x(\tau)\| = \frac{\|x(\tau + \delta) - x(\tau)\|}{\delta}(t - \tau) < \delta_\tau,$$

because $\|x(\tau + \delta) - x(\tau)\| < M\delta$ and $t - \tau \leq \delta < \delta_\tau/(M + 1)$. From (2.1) (with $h = \delta$), in view of (2.2) it follows that

$$\dot{y}_{\tau,\delta}(t) + \sigma B = \frac{x(\tau + \delta) - x(\tau)}{\delta} + \sigma B \subset F(t, y_{\tau,\delta}(t)), \quad t \in (\tau, \tau + \delta),$$

and hence Claim 1 holds.

Now the family

$$\mathcal{F} = \{I_{\tau,\delta}, \tau \in J, 0 < \delta < \delta_\tau\}$$

of closed intervals $I_{\tau,\delta}$ covers J in the sense of Vitali. Hence there exists a finite or denumerable infinite family $\mathcal{F}_0 = \{I_{\tau_j,\delta_j}\} \subset \mathcal{F}$ of pairwise disjoint closed intervals

$I_{\tau_j, \delta_j} \in \mathcal{F}$ such that $J \setminus \bigcup_j I_{\tau_j, \delta_j}$ has measure zero. As $I \setminus J$ has measure zero it follows that $N_0 = I \setminus \bigcup_j I_{\tau_j, \delta_j}$ has measure zero. Evidently

$$(2.3) \quad I = \bigcup_j I_{\tau_j, \delta_j} \cup N_0,$$

and thus \mathcal{F}_0 is a regular partition of I . Now define $u : I \rightarrow \mathbb{E}$ and $y : I \rightarrow \mathbb{E}$ as follows:

$$u(t) = \sum_j \dot{y}_{\tau_j, \delta_j}(t) \chi_{I_{\tau_j, \delta_j}}(t), \quad t \in I,$$

$$y(t) = a + \int_{t_0}^t u(s) ds, \quad t \in I.$$

Claim 2. $y : I \rightarrow \mathbb{E}$ is a regular solution of the Cauchy problem $(C_{\text{int } F, a})$ satisfying $\|y - x\|_I < \varepsilon$.

Since the functions y_{τ_j, δ_j} and x agree at τ_j and $\tau_j + \delta_j$, the end points of I_{τ_j, δ_j} , then in view of the definition of y it is easy to show that $y(t) = y_{\tau_j, \delta_j}(t)$ for each $t \in I_{\tau_j, \delta_j}$ and $I_{\tau_j, \delta_j} \in \mathcal{F}_0$. By virtue of Claim 1 and (2.3) it follows that y is a regular solution of the Cauchy problem $(C_{\text{int } F, a})$. Furthermore for any $I_{\tau_j, \delta_j} \in \mathcal{F}_0$ and all $t \in I_{\tau_j, \delta_j}$ we have

$$\|y(t) - x(t)\| \leq \|y(t) - y(\tau_j)\| + \|x(\tau_j) - x(t)\| < 2M\delta_j.$$

As $\delta_j < \varepsilon/(2M+1)$, in view of (2.3) it follows that $\|y - x\|_I < \varepsilon$. Therefore Claim 2 holds. This completes the proof. \square

Proposition 3. *Let $F : I \times \mathbb{E} \rightarrow \mathcal{B}(\mathbb{E})$ satisfy (H). Then \mathcal{M} is a nonempty complete metric space (under the induced metric of $C(I, \mathbb{E})$) and $\mathcal{M} \subset \mathcal{M}_{F, a}$.*

Proof. Since the multifunction $\text{int } F$ admits locally Lipschitzian selections, we have $\mathcal{M}_{\text{int } F, a} \neq \emptyset$ and thus $\mathcal{M}_{F, a}^0 \neq \emptyset$, by virtue of Proposition 2. Hence $\mathcal{M} \neq \emptyset$. Evidently \mathcal{M} is complete for $C(I, \mathbb{E})$ is so. As F is a continuous and bounded multifunction with closed convex values contained in \mathbb{E} , a reflexive Banach space, the uniform limit of solutions is also a solution, and hence $\mathcal{M} \subset \mathcal{M}_{F, a}$. \square

Proposition 4. *Let $F : I \times \mathbb{E} \rightarrow \mathcal{B}(\mathbb{E})$ satisfy (H). Let $(t, x) \in I \times \mathbb{E}$ and let $u \in \text{int } F(t, x)$ and $\alpha > 0$ be given. Then for some $n \in \mathbb{N}$ there exist points $a_i \in \text{int } F(t, x)$, with $d_F(t, x, a_i) < \alpha$, $i = 1, \dots, n$, and numbers $\lambda_i > 0$, with $\lambda_1 + \dots + \lambda_n = 1$, such that*

$$\sum_{i=1}^n \lambda_i a_i = u.$$

Proof. Fix $\theta > 0$ so that $u + \theta B \subset F(t, x)$. For some $p \in \mathbb{N}$ there exist points $e_i \in \text{ext } F(t, x)$ and numbers $\mu_i > 0$ with $\mu_1 + \dots + \mu_p = 1$, such that, setting $c = \mu_1 e_1 + \dots + \mu_p e_p$, one has

$$0 < \|c - u\| < \frac{\theta}{2}.$$

By Proposition 1, $d_F(t, x, e_i) = 0$, $i = 1, \dots, p$, and thus sufficiently close to each e_i there exists a point $a_i \in \text{int } F(t, x)$ such that

$$(2.4) \quad d_F(t, x, a_i) < \frac{\alpha}{2}, \quad i = 1, \dots, p \quad \text{and} \quad 0 < r = \|a - u\| < \frac{\theta}{2},$$

where $a = \mu_1 a_1 + \dots + \mu_p a_p$. By virtue of Proposition 1 and (2.4) there exists an $\varepsilon > 0$ such that each ball $B_i = a_i + \varepsilon B$, $i = 1, \dots, p$, is contained in $F(t, x)$ and moreover,

$$(2.5) \quad d_F(t, x, v) < \alpha \quad \text{for every } v \in B_i, i = 1, \dots, p.$$

It is evident that

$$(2.6) \quad \sum_{i=1}^p \mu_i B_i = B_\varepsilon(a), \quad \text{where } B_\varepsilon(a) = a + \varepsilon B.$$

Let $b \in \text{int } F(t, x)$ be such that $u = \frac{a+b}{2}$. As above, for some $q \in \mathbb{N}$ there exist points $a'_i \in \text{int } F(t, x)$, with $d_F(t, x, a'_i) < \alpha$, $i = 1, \dots, q$, and numbers $\nu_i > 0$, with $\nu_1 + \dots + \nu_q = 1$, such that setting

$$(2.7) \quad b' = \sum_{i=1}^q \nu_i a'_i$$

one has $\|b' - b\| < \frac{\varepsilon}{4}$. As $u + \frac{\varepsilon}{2}B \subset \text{co}\{b, B_\varepsilon(a)\}$ and $h(\text{co}\{b, B_\varepsilon(a)\}, \text{co}\{b', B_\varepsilon(a)\}) \leq \|b' - b\| < \frac{\varepsilon}{4}$, it follows that $u + \frac{\varepsilon}{2}B \subset \text{co}\{b', B_\varepsilon(a)\} + \frac{\varepsilon}{4}B$, which implies $u + \frac{\varepsilon}{4}B \subset \text{co}\{b', B_\varepsilon(a)\}$. Hence there exist $d \in B_\varepsilon(a)$ and $t \in [0, 1]$ so that

$$(2.8) \quad u = tb' + (1 - t)d.$$

As $d \in B_\varepsilon(a)$, by virtue of (2.6) and (2.5) there exist points $d'_i \in B_i$, with $d_F(t, x, d'_i) < \alpha$, $i = 1, \dots, p$, such that

$$(2.9) \quad d = \sum_{i=1}^p \mu_i d'_i.$$

From (2.8), in view of (2.7) and (2.9), it follows that

$$u = t \sum_{i=1}^q \nu_i a'_i + (1 - t) \sum_{i=1}^p \mu_i d'_i.$$

This completes the proof. □

3. THE WEAK BANG-BANG PROPERTY

In this section we shall prove the weak bang-bang property. To this end, for $F : I \times \mathbb{E} \rightarrow \mathcal{B}(\mathbb{E})$ satisfying (H) and $\alpha > 0$, we set

$$\mathcal{N}_\alpha = \{x \in \mathcal{M} : \int_I d_F(t, x(t), \dot{x}(t)) dt < \alpha |I|\}.$$

Proposition 5. \mathcal{N}_α is dense in \mathcal{M} .

Proof. It suffices to show that given $x \in \mathcal{M}_{F,a}^0$ and $\varepsilon > 0$ there exists $y \in \mathcal{N}_\alpha$ such that $\|y - x\|_I \leq \varepsilon$.

By hypothesis x is a regular solution of $(C_{\text{int } F,a})$ and thus, with the notation of Definition 2, for some piecewise constant map $u : I \rightarrow \mathbb{E}$, given by (1.2), we have:

$$x(t) = a + \int_{t_0}^t u(s) ds, \quad t \in I,$$

$$\dot{x}(t) + \sigma_i B = u_i + \sigma_i B \subset F(t, x(t)), \quad t \in \text{int } I_i, I_i \in \mathcal{P}.$$

Consider an interval $I_i \in \mathcal{P}$ with end points $\alpha_i < \beta_i$ and let $\tau \in (\alpha_i, \beta_i)$ be arbitrary. Evidently

$$u_i + \sigma_i B \subset F(\tau, x(\tau)),$$

and thus by Proposition 4, for some $n \in \mathbb{N}$ there exist points $e_k \in \text{int } F(\tau, x(\tau))$, with $d_F(\tau, x(\tau), \dot{x}(\tau)) < \alpha$, $k = 1, \dots, n$, and corresponding numbers $\lambda_k > 0$, with $\lambda_1 + \dots + \lambda_n = 1$, such that

$$(3.1) \quad u_i = \sum_{k=1}^n \lambda_k e_k.$$

Clearly for some $\gamma_\tau > 0$,

$$(3.2) \quad \bigcup_{k=1}^n (e_k + \gamma_\tau B) \subset F(\tau, x(\tau)).$$

Since F is continuous and satisfies (3.2) there exists ρ_0 , with

$$(3.3) \quad 0 < \rho_0 < \min\{\varepsilon, \tau - \alpha_i, \beta_i - \tau\},$$

such that

$$(3.4) \quad t \in [\tau - \rho_0, \tau + \rho_0], \|y - x(\tau)\| < \rho_0 \quad \Rightarrow \quad \bigcup_{k=1}^n (e_k + \frac{\gamma_\tau}{2} B) \subset F(t, y).$$

Furthermore as d_F is upper semicontinuous at $(\tau, x(\tau), e_k)$ and $d_F(\tau, x(\tau), e_k) < \alpha$ there exists a ρ , with $0 < \rho < \min\{\rho_0, \gamma_\tau/2\}$, such that for $k = 1, \dots, n$ we have

$$(3.5) \quad t \in [\tau - \rho, \tau + \rho], \|y - x(\tau)\| < \rho, \|v - e_k\| < \rho \quad \Rightarrow \quad d_F(t, y, v) < \alpha.$$

Let δ_τ satisfy

$$(3.6) \quad 0 < \delta_\tau < \frac{\rho}{4M + 1}$$

and, for $0 < \delta < \delta_\tau$, set

$$I_{\tau, \delta} = [\tau - \delta, \tau + \delta].$$

Let $\{J_{\tau, \delta}^k\}_{k=1}^n$ be a partition of $I_{\tau, \delta}$ into n pairwise disjoint nondegenerate subintervals $J_{\tau, \delta}^k$ of length

$$(3.7) \quad |J_{\tau, \delta}^k| = \lambda_k |I_{\tau, \delta}|, \quad k = 1, \dots, n.$$

Now define $u_{\tau, \delta} : I \rightarrow \mathbb{E}$ and $y_{\tau, \delta} : I_{\tau, \delta} \rightarrow \mathbb{E}$ as follows:

$$(3.8) \quad u_{\tau, \delta}(t) = \sum_{k=1}^n e_k \chi_{J_{\tau, \delta}^k}(t), \quad t \in I_{\tau, \delta},$$

$$(3.9) \quad y_{\tau, \delta}(t) = x(\tau - \delta) + \int_{\tau - \delta}^t u_{\tau, \delta}(s) ds, \quad t \in I_{\tau, \delta}.$$

We have

- (j) $\dot{y}_{\tau,\delta}(t) + \frac{\gamma\tau}{2}B \subset F(t, y_{\tau,\delta}(t)), \quad t \in \bigcup_{k=1}^n \text{int } J_{\tau,\delta}^k,$
- (jj) $y_{\tau,\delta}(\tau \pm \delta) = x(\tau \pm \delta),$
- (jjj) $\|y_{\tau,\delta}(t) - x(t)\| < \varepsilon, \quad t \in I_{\tau,\delta},$
- (jv) $d_F(t, y_{\tau,\delta}(t), \dot{y}_{\tau,\delta}(t)) < \alpha, \quad t \in \bigcup_{k=1}^n \text{int } J_{\tau,\delta}^k.$

(j) Let $t \in \text{int } J_{\tau,\delta}^k$, where $1 \leq k \leq n$. Clearly

$$(3.10) \quad t \in [\tau - \rho, \tau + \rho]$$

since $|t - \tau| \leq \delta < \delta_\tau < \rho$ by (3.6). Moreover,

$$\begin{aligned} \|y_{\tau,\delta}(t) - x(\tau)\| &= \|x(\tau - \delta) + \int_{\tau-\delta}^t u_{\tau,\delta}(s)ds - a - \int_{t_0}^\tau u(s)ds\| \\ &= \left\| \int_{\tau-\delta}^t u_{\tau,\delta}(s)ds - \int_{\tau-\delta}^\tau u(s)ds \right\| \\ &\leq \int_{J_{\tau,\delta}^k} (\|u_{\tau,\delta}(s)\| + \|u(s)\|)ds \leq 4M\delta_\tau, \end{aligned}$$

for $|J_{\tau,\delta}^k| \leq |I_{\tau,\delta}| = 2\delta < 2\delta_\tau$ and $u_{\tau,\delta}$ and u are bounded by M . Since $\delta_\tau < \rho/(4M + 1)$ by (3.6), it follows that

$$(3.11) \quad \|y_{\tau,\delta}(t) - x(\tau)\| < \rho.$$

From (3.4), in view of (3.10) and (3.11), as $\rho < \rho_0$ and $\dot{y}_{\tau,\delta}(t) = e_k$, one has

$$\dot{y}_{\tau,\delta}(t) + \frac{\gamma\tau}{2}B \subset F(t, y_{\tau,\delta}(t))$$

and thus (j) holds.

(jj) From (3.9), in view of (3.8), (3.7) and (3.1) we have:

$$\begin{aligned} y_{\tau,\delta}(\tau + \delta) &= x(\tau - \delta) + \int_{\tau-\delta}^{\tau+\delta} \left(\sum_{k=1}^n e_k \chi_{J_{\tau,\delta}^k}(s) \right) ds \\ &= x(\tau - \delta) + \sum_{k=1}^n e_k |J_{\tau,\delta}^k| = x(\tau - \delta) + \sum_{k=1}^n \lambda_k e_k |I_{\tau,\delta}| \\ &= x(\tau - \delta) + u_i |I_{\tau,\delta}| = x(\tau - \delta) + \int_{\tau-\delta}^{\tau+\delta} u(s)ds = x(\tau + \delta). \end{aligned}$$

Clearly $y_{\tau,\delta}(\tau - \delta) = x(\tau - \delta)$, and thus (jj) holds.

(jjj) For any $t \in I_{\tau,\delta}$ we have:

$$\begin{aligned} \|y_{\tau,\delta}(t) - x(t)\| &= \left\| x(\tau - \delta) + \int_{\tau-\delta}^t u_{\tau,\delta}(s)ds - a - \int_{t_0}^t u(s)ds \right\| \\ &= \left\| \int_{\tau-\delta}^t u_{\tau,\delta}(s)ds - \int_{\tau-\delta}^t u(s)ds \right\| \leq \int_{I_{\tau,\delta}} (\|u_{\tau,\delta}(s)\| + \|u(s)\|)ds \leq 4M\delta. \end{aligned}$$

From the latter, (jjj) follows at once since $\delta < \delta_\tau < \rho/(4M + 1)$ by (3.6), and $\rho < \rho_0 < \varepsilon$ by (3.3).

(jv) Let $t \in \text{int } J_{\tau,\delta}^k$ be arbitrary, where $1 \leq k \leq n$. From (3.5), in view of (3.10) and (3.11), as $\dot{y}_{\tau,\delta}(t) = e_k \in F(t, y_{\tau,\delta}(t))$, one has $d_F(t, y_{\tau,\delta}(t), \dot{y}_{\tau,\delta}(t)) < \alpha$, and thus (jv) holds.

It is evident that the family

$$\mathcal{F} = \{I_{\tau,\delta} \mid \tau \in \bigcup_i \text{int } I_i, 0 < \delta < \delta_\tau\}$$

of closed intervals $I_{\tau,\delta} \subset I$ covers the set $\bigcup_i \text{int } I_i$ in the sense of Vitali. Hence there is a finite or denumerable infinite family $\mathcal{F}_0 = \{I_{\tau_j,\delta_j}\} \subset \mathcal{F}$ of pairwise disjoint closed intervals I_{τ_j,δ_j} such that

$$(3.12) \quad I = \left(\bigcup_j I_{\tau_j,\delta_j}\right) \cup N_0,$$

where $N_0 = I \setminus \bigcup_j I_{\tau_j,\delta_j}$ has measure zero. Now define $v : I \rightarrow \mathbb{E}$ and $y : I \rightarrow \mathbb{E}$ as follows:

$$v(t) = \sum_j u_{\tau_j,\delta_j}(t) \chi_{I_{\tau_j,\delta_j}}(t), \quad t \in I,$$

$$y(t) = a + \int_{t_0}^t v(s) ds, \quad t \in I.$$

Since by (jj) $y_{\tau_j,\delta_j}(\tau_j \pm \delta_j) = x(\tau_j \pm \delta_j)$, it is easy to see that for each $I_{\tau_j,\delta_j} \in \mathcal{F}_0$ we have

$$(3.13) \quad y(t) = y_{\tau_j,\delta_j}(t) \quad \text{for every } t \in I_{\tau_j,\delta_j}.$$

In view of (3.13), (3.12) and (j) it follows that $y : I \rightarrow \mathbb{E}$ is a regular solution of the Cauchy problem $(C_{\text{int } F,a})$; hence $y \in \mathcal{M}_{\text{int } F,a}^0$ and so a fortiori $y \in \mathcal{M}$. Furthermore, by virtue of (3.13), (3.12) and (jv),

$$\int_I d_F(t, y(t), \dot{y}(t)) dt < \alpha |I|$$

and hence $y \in \mathcal{N}_\alpha$. Finally from (3.13), (3.12) and (jjj) it follows that $\|y - x\|_I \leq \varepsilon$. This completes the proof. \square

Proposition 6. \mathcal{N}_α is open in \mathcal{M} .

Proof. Let $\{x_n\} \subset \mathcal{M} \setminus \mathcal{N}_\alpha$ be a sequence which converges uniformly to $x \in \mathcal{M}$. Then, by Proposition 1,

$$\int_I d_F(t, x(t), \dot{x}(t)) dt \geq \limsup_{n \rightarrow \infty} \int_I d_F(t, x_n(t), \dot{x}_n(t)) dt \geq \alpha |I|.$$

Thus $x \in \mathcal{M} \setminus \mathcal{N}_\alpha$, completing the proof. \square

We are now ready to prove the following weak form of the bang-bang property.

Theorem 1. Let $F : I \times \mathbb{E} \rightarrow \mathcal{B}(\mathbb{E})$ satisfy assumption (H). Then $\mathcal{M}_{\text{int } F,a} \neq \emptyset$ and

$$(3.14) \quad \overline{\mathcal{M}_{\text{ext } F,a}} \supset \mathcal{M}_{\text{int } F,a}.$$

Proof. Under our assumptions, $\mathcal{M}_{\text{int } F,a} \neq \emptyset$. To prove (3.14) set

$$\mathcal{M}^* = \bigcap_{n \in \mathbb{N}} \mathcal{N}_{1/n}.$$

By virtue of Propositions 5 and 6 each $\mathcal{N}_{1/n}$ is open and dense in \mathcal{M} , which is a complete metric space, by Proposition 3. Consequently \mathcal{M}^* is dense in \mathcal{M} .

Let $x \in \mathcal{M}^*$. Then for every $n \in \mathbb{N}$,

$$\int_I d_F(t, x(t), \dot{x}(t))dt < (1/n)|I|,$$

and thus, by Proposition 1, $d_F(t, y(t), \dot{y}(t)) = 0$, $t \in I$ a.e., which implies that $x \in \mathcal{M}_{\text{ext } F,a}$. Therefore $\mathcal{M}_{\text{ext } F,a} \supset \mathcal{M}^*$ and hence,

$$\overline{\mathcal{M}_{\text{ext } F,a}} \supset \mathcal{M}.$$

Since, by Proposition 2, $\mathcal{M} = \overline{\mathcal{M}_{F,a}^0} \supset \mathcal{M}_{\text{int } F,a}$, it follows that $\overline{\mathcal{M}_{\text{ext } F,a}} \supset \mathcal{M}_{\text{int } F,a}$. This completes the proof. \square

The following example shows that in (3.14) the inclusion can be strict.

Example 3.1. Set $f(y) = \sqrt{|y|}$ if $|y| \leq 1$ and $f(y) = 1$ if $|y| > 1$. For $F : \mathbb{R}^2 \rightarrow \mathcal{B}(\mathbb{R}^2)$ given by

$$F(x, y) = \{(u, v) \in \mathbb{R}^2 \mid u \in [0, 1], v \in [f(y), 2]\}, \quad (x, y) \in \mathbb{R}^2,$$

consider the following Cauchy problems:

$$\begin{aligned} (C_{\text{int } F,0}) \quad & (\dot{x}(t), \dot{y}(t)) \in \text{int } F(x(t), y(t)), \quad (x(0), y(0)) = (0, 0), \\ (C_{\text{ext } F,0}) \quad & (\dot{x}(t), \dot{y}(t)) \in \text{ext } F(x(t), y(t)), \quad (x(0), y(0)) = (0, 0). \end{aligned}$$

Put $I = [0, 1/2]$. Clearly

$$\text{ext } F(x, y) = \{(0, f(y)), (0, 2), (1, f(y)), (1, 2)\}$$

and thus $(0, 0) \in \text{ext } F(0, 0)$, which shows that $(x_0(t), y_0(t)) = (0, 0)$, $t \in I$, is a solution of $(C_{\text{ext } F,0})$. Let $(x(t), y(t))$, $t \in I$, be an arbitrary solution of $(C_{\text{int } F,0})$. As $x(0) = y(0) = 0$ and $x(t)$ and $y(t)$ satisfy

$$(3.15) \quad \dot{x}(t) \in (0, 1), \quad \dot{y}(t) \in (f(y(t)), 2),$$

it follows that $x(t)$ and $y(t)$ are strictly positive for every $t \in (0, 1/2]$. Clearly $\dot{y}(t) \in (\sqrt{y(t)}, 2)$ for $t \in I$ a.e. since, by (3.15), $y(t) \leq 1$ for every $t \in I$. Let $0 < \varepsilon < 1/4$. Then for $t \in [\varepsilon, 1/2]$ a.e. we have $\dot{y}(t) > \sqrt{y(t)}$, and hence $\sqrt{y(t)} > \sqrt{y(\varepsilon)} + \frac{t-\varepsilon}{2}$, $t \in [\varepsilon, 1/2]$, from which letting $\varepsilon \rightarrow 0$ one has

$$y(t) \geq \frac{t^2}{4}, \quad t \in I.$$

Consequently

$$d((x_0(\cdot), y_0(\cdot)), \mathcal{M}_{\text{int } F,0}) \geq \frac{1}{8},$$

as $(x(t), y(t))$, $t \in I$, is an arbitrary solution of $(C_{\text{int } F,0})$. This shows that the inclusion $\overline{\mathcal{M}_{\text{ext } F,0}} \supset \mathcal{M}_{\text{int } F,0}$ is strict.

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