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DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **12**, Number **3**, October **2009**

doi:10.3934/dcdsb.2009.12.xx

pp. **1–XX**

RECENT PROGRESS ON PARTICLE TRAJECTORIES IN STEADY WATER WAVES

MATS EHRNSTRÖM AND GABRIELE VILLARI

Abstract. We survey recent results on particle trajectories within steady two-dimensional water waves. Particular emphasis is placed on the linear and exact mathematical theory of periodic and symmetric waves, and the effects of a (possibly rotational) background current. The different results vindicate and detail the classical Stokes drift, and also show the transition of orbits when waves propagate into running water. The classical approximation, depicting the trajectories as closed ellipses, is shown to be a mathematical rarity.

2000 *Mathematics Subject Classification.* 35Q35, 37N10, 76B15, 76F10.
Key words and phrases. Steady water waves; Stokes waves; Particle trajectories; Vorticity; Background current.

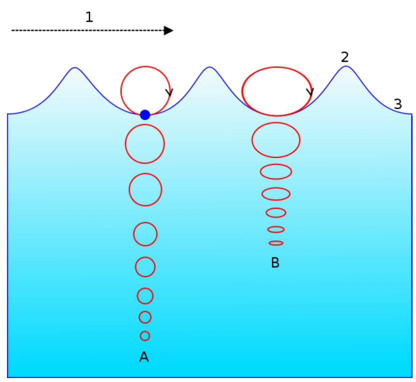

Figure 1. The first approximation of particle trajectories in linear water waves. A is in deep water, B in shallow; 1, 2 and 3 denote the direction of the wave, the crest, and the trough, respectively. The picture is taken from [45], and is licensed under the *GNU Free Documentation License*.

1. **Introduction.** Given that the study of wave motion is one of the oldest, largest and most active fields of science, it is remarkable that the most wide-spread image of the motion within water waves is erroneous. It holds that the particles within periodic wave-trains travelling across the sea move in closed orbits, elliptic or circular depending on the depth (see Figure 1). Since such waves move with more or less steady shape and speed, this may at first come as a surprise. A wave, however, is not the water through which it is travelling.

Indeed, in one of the few explicit solutions of the exact water wave problem, all particles move in closed circles [6, 25]. Constructed in 1809 by Gerstner [22], this example has provided good support for the notion of closed particle trajectories. It has also been extended to three-dimensional edge-waves with circular particle paths [46, 5]. Moreover, taking the first approximation of linear water waves, the same circular/elliptic pattern is obtained. That approach is a standard part of text-books on fluid dynamics (e.g. [31, 2, 33]), and the resulting picture is widespread [45].

The survey at hand has come about as a result of recent progress in the understanding of particle paths within progressive water waves. These results decisively point to the fact that the full picture is much more complicated than the circular approximation suggests, and that the case of closed orbits is a rarity rather than a general pattern. That the notion of closed orbits persists is surprising, especially when one considers that already in 1847 Stokes presented a geometrical argument for a net mass drift [39]. Notable work was later done by Ursell [40] and Longuet-Higgins [34, 35] in order to improve upon Stokes' argument. The work by Longuet-Higgins also encompasses many related papers. Nevertheless, judging from the subsequent mathematics literature much of this work unfortunately never reached a wider audience (or at least was not given proper attention).

In 2005 a reignition of this research area was sparkled by the then preprint of [17]. In that paper the authors took a new approach towards these matters. It was proved that even within the linear regime, the forward drift is obtained (the size of the drift is superlinear, but it arises already in the linear approximation). A lot of work followed, from which three main themes are discernible: the investigation of different types of waves and wave equations, the detailed study of orbits in exact water waves, and the incorporation of background currents and vorticity.

The aim of this survey is to describe the analytical progress that has been made in those recent years. In particular, we shall see that both the generality and the details of the theory has been advanced. That said, there is also other work on these matters, most of it in the physical and fluid-dynamics literature. The interested reader may for example consult [3, 4, 47]. While those investigations are in line with the analytical results, some of the recent mathematical findings cover cases which are new, or at least not well-investigated, from a more applied point of view. Similarly, many of the applied studies on mass-drift do not lend themselves to analytical studies.

The disposition is presented in the table of contents on page 1, wherefrom we hope that the reader shall be able to find the part that mostly interests him. As much as possible, the same notation and conventions have been used throughout the different sections. In this way one should be able to quickly jump from one setting to another. Since the notation may therefore differ from the original papers, we have tried to explain the differences in the cases when the they could possibly lead to misunderstandings. Last but not least: no survey can be complete. If—in spite of our efforts to cover this part of the field—we have mistakenly missed some recent contribution, we apologize. A more comprehensive survey, including also numerical and experimental work, would be most welcome.

2. **The governing equations.**

2.1. **Physical background.**

The setting is that of two-dimensional waves, i.e. we consider a cross-section of the flow perpendicular to the crest line. Choose Cartesian coordinates (x, y) with the y-axis pointing vertically upwards and the x-axis being the direction of wave propagation, with the origin at the average surface height. Let $(u(t, x, y),\ v(t, x, y))$ be the velocity field of the flow, and $y = \eta(t, x)$ the water's free surface. For waves propagating over a flat bed at $y = -d \in \mathbb{R}$, we let $h := d > 0$ denote the depth, while for for deep-water waves (for which $d = \infty$), we set $h := 1$. This is a notational convention which enables a single framework throughout the presentation. Homogeneity (constant density) is a physically reasonable assumption for water waves [33], and it implies the equation of mass conservation

$$u_x + v_y = 0 \tag{2.1a}$$

throughout the fluid. Appropriate for gravity waves is also the assumption of inviscid flow [33], so that the equation of motion is Euler's equation

$$\begin{aligned} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g, \end{aligned} \tag{2.1b}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration. In addition to gravity we include the effects of surface tension. The free surface decouples the motion of the water from that of the air so that the dynamic boundary condition

$$P = P_0 - \frac{\alpha \eta_{xx}}{(1 + \eta_x^2)^{3/2}} \qquad \text{on} \quad y = \eta(t, x), \tag{2.1c}$$

must hold. Here $\alpha > 0$ is the capillarity constant, and P_0 is the constant atmospheric pressure [31] . Moreover, since the same particles always form the free surface, we have the kinematic boundary condition

$$v = \eta_t + u\eta_x \qquad \text{on} \quad y = \eta(t,x). \tag{2.1d}$$

The fact that water cannot penetrate the rigid bed at $y = 0$ yields the other kinematic boundary condition,

$$v = 0 \qquad \text{on} \quad y = -d, \tag{2.1e}$$

which for deep-water waves (2.1e) is exchanged for

$$(u,v) \to (0,0) \quad \text{uniformly as} \quad y \to -\infty. \tag{2.1f}$$

The vorticity, ω, of the flow is captured by the curl,

$$v_x - u_y = \omega. \tag{2.1g}$$

The equations (2.1) govern the motion within two-dimensional inviscid water waves of constant density.

2.2. **Non-dimensionalization.** To non-dimensionalize we introduce scaling parameters. Let a denote the typical *amplitude*. For periodic waves we denote by λ the typical *wavelength*. The approximate speed of irrotational long waves [31], $\sqrt{gh}$, shall be the scale of the horizontal velocity. In addition, c denotes the wave speed, and we let

$$c \mapsto \frac{c}{\sqrt{gh}}$$

be the starting point of the non-dimensionalization:

$$x \mapsto \frac{x}{\lambda}, \quad y \mapsto \frac{y}{h}, \quad t \mapsto \frac{\sqrt{gh}\,t}{\lambda}, \quad u \mapsto \frac{u}{\sqrt{gh}}, \quad v \mapsto \frac{\lambda\, v}{h\sqrt{gh}}, \quad \eta \mapsto \frac{\eta}{a}.$$

Having made these transformations, define furthermore a new pressure function, $p(t,x,y)$, by the equality

$$P =: P_0 + gh(p-y).$$

Here $-ghy$ is the hydrostatic pressure distribution, describing the pressure change within a stationary fluid; the new variable p thus measures the pressure perturbation induced by a passing wave. Natural scalings for the vorticity and the capillarity are given by the mappings

$$\omega \mapsto \sqrt{\frac{h}{g}}\,\omega, \qquad \text{and} \qquad \alpha \mapsto \frac{\alpha}{gh\lambda^2}.$$

The water wave problem (2.1) then transforms into the equations

$$\begin{aligned} u_x + v_y &= 0,\\ u_t + uu_x + vu_y &= -p_x,\\ v_t + uv_x + vv_y &= -\frac{\lambda^2}{h^2}p_y,\\ \frac{h^2}{\lambda^2}v_x - u_y &= \omega, \end{aligned} \tag{2.2a}$$

valid in the fluid domain $-\frac{d}{h} < y < \frac{a}{h}\eta$, and

$$\begin{aligned} v &= \frac{a}{h}\,(\eta_t + u\eta_x),\\ p &= \frac{a}{h}\left(\eta - \alpha\eta_{xx}\left(1 + \frac{a^2}{h^2}\frac{\eta_x^2}{\lambda^2}\right)^{-3/2}\right), \end{aligned} \tag{2.2b}$$

valid at the surface $y = \frac{a}{h}\eta$, in conjunction with one of the boundary conditions (2.1e) and (2.1f), for finite ($y = -1$) and infinite ($y = -\infty$) depth , respectively. Here appear naturally the parameters

$$\varepsilon := \frac{a}{h}, \qquad \delta := \frac{h}{\lambda},$$

called the *amplitude parameter*, and the *shallowness parameter*, respectively. Small δ models long/shallow water waves, and small ε models a small disturbance of the underlying flow.

The next step is to restrict attention to steady (or travelling) waves. They are characterized by a space-time dependence of the form $x - ct$ in the original variables, corresponding to $\lambda(x-ct)$ in the equations (2.2). The change of variables $(x, y) \mapsto (x - ct, y)$ yields the problem

$$\begin{aligned} u_x + v_y &= 0, \\ (u-c)u_x + vu_y &= -p_x, \\ (u-c)v_x + vv_y &= -\frac{p_y}{\delta^2}, \\ \delta^2 v_x - u_y &= \omega, \end{aligned} \tag{2.3a}$$

valid in the fluid domain $-\frac{d}{h} < y < \varepsilon\eta$,

$$\begin{aligned} v &= \varepsilon(u-c)\eta_x, \\ p &= \varepsilon\left(\eta - \alpha\eta_{xx}\left(1 + \frac{\varepsilon^2}{\lambda^2}\eta_x^2\right)^{-3/2}\right) \end{aligned} \tag{2.3b}$$

valid at the surface $y = \varepsilon\eta$. In addition,

$$v = 0 \qquad \text{at} \qquad y = -1, \tag{2.3c}$$

in the case of finite depth, which for deep-water waves is replaced by (2.1f).

3. **Mathematical formulations.**

In this section we deduce from the non-dimensionalized steady water-wave problem (2.3) the different mathematical models that shall follow us through our analysis. In Subsection 3.1 we consider what happens when the small-amplitude parameter ε, and the shallowness parameter δ, respectively, are infinitesimally small. This yields us linear equations on finite and infinite depth, as well as linearized shallow water-wave equations, which can all be solved on the level of the velocity field. In Subsection 3.2 we introduce two alternative formulations to the governing equations (2.3). These are both based on the streamlines, being integral curves in the steady frame.

3.1. **Small-amplitude and shallow-water waves.**

To enable the study of explicit solutions, we shall study perturbations of laminar flows (characterized by a flat surface $\eta = 0$). Assuming that the vorticity is constant, (2.3) yields the one-parameter family of laminar flows,

$$U(y;\kappa) := \kappa - \omega y, \tag{3.1}$$

with $\eta = 0$, $p = 0$, $v = 0$. By writing a general solution as a perturbation of such a solution U, i.e.

$$u = U + \varepsilon\tilde{u}, \quad v = \varepsilon\tilde{v}, \quad p = \varepsilon\tilde{p},$$

dropping the tildes, and letting $\varepsilon \to 0$, we obtain the linearized problem

$$\begin{aligned}
u_x + v_y &= 0, && -d/h < y < 0,\\
(U-c)u_x + vU_y &= -p_x, && -d/h < y < 0,\\
\delta^2 (U-c)v_x &= -p_y, && -d/h < y < 0,\\
v &= (U-c)\eta_x, && y = 0,\\
p &= \eta - \alpha\eta_{xx}, && y = 0,
\end{aligned} \tag{3.2}$$

valid for $y = 0$. This is the linearization found in [17] for irrotational gravity waves, [8] for the corresponding deep-water waves, [24] when also capillarity is added, and [20] for gravity waves over finite depth with constant vorticity. Since the original wavelength λ and the original amplitude a have both been non-dimensionalized to unit length, a natural *Ansatz* is $\eta(x) = \cos(2\pi x)$. That allows for solutions (u, v, p, η) of (3.2) in closed form. In particular, for deep-water irrotational waves the velocity field is given by

$$\begin{aligned}
u(x,y) &= 2\pi\delta c \cos(2\pi x)\exp(2\pi\delta y),\\
v(x,y) &= 2\pi c \sin(2\pi x)\exp(2\pi\delta y),
\end{aligned} \tag{3.3a}$$

where the wave speed fulfills the dispersion relation

$$c = \frac{1}{\sqrt{2\pi\delta}} = \sqrt{\frac{\lambda}{2\pi}}. \tag{3.3b}$$

Notice that capillarity does not influence the form of the linearized velocity field. The same is true for a shear flow: the pressure is influenced but not the linearized steady velocity field *per se*. Thus, for waves over finite depth with constant vorticity, we have

$$\begin{aligned}
u(x,y) &= \frac{2\pi\delta c_{\kappa,\omega}}{\sinh(2\pi\delta)}\cos(2\pi x)\cosh\left(2\pi\delta(y+1)\right),\\
v(x,y) &= \frac{2\pi c_{\kappa,\omega}}{\sinh(2\pi\delta)}\sin(2\pi x)\sinh\left(2\pi\delta(y+1)\right),
\end{aligned} \tag{3.4a}$$

where $c_{\kappa,\omega} := (c - \kappa + \omega)$, and it is required that

$$c_{\kappa,\omega}\big(2\pi\delta c_{\kappa,\omega}\coth(2\pi\delta) - \omega\big) = 1. \tag{3.4b}$$

It can be seen that the limit of (3.4) as $\delta \to 0$ is well-defined, yielding shallow-water waves which are solutions of (3.2) for infinitesimally small δ:

$$\begin{aligned}
u(x,y) &= c_{\kappa,\omega}\cos(2\pi x),\\
v(x,y) &= 2\pi c_{\kappa,\omega}\sin(2\pi x)(y+1).
\end{aligned} \tag{3.5a}$$

with the dispersion relation

$$(c-\kappa+\omega)(c-\kappa) = 1. \tag{3.5b}$$

Those were studied in [28, 29].

Remark 3.1. The approach here adopted is slightly different than that of [28, 29]. For the sake of a uniform framework within this survey there is a running stream (3.1) to which a linear disturbance is added. That method is the same as in [20, 32], and in the limit $\varepsilon \to 0$ it yields a perfect match with the quantities κ and ω for exact water waves [14, 16, 44]. In the investigations [28, 29] the linearization is around still water, and the uniform current as well as the vorticity are both parts of the small disturbance. The dispersion relation $|c| = 1$ found in [28] is therefore completely in line with (3.5b), since it is the result of a linearization around still water for which $\kappa = \omega = 0$.

3.2. **The streamfunction and height formulations.** In this section we consider typical gravity water waves, whence we set $\varepsilon = 1$, $\delta = 1$, and $\alpha = 0$. We let Ω_η denote the *fluid domain* and define it as the interior of its boundary

$$\partial\Omega_\eta := \{y = -d/h\} \cup \{x, \eta(x))\}_{x\in\mathbb{R}}.$$

Since the velocity is divergence-free, we may then introduce a *stream function* ψ, defined (up to a constant) by

$$\psi_x := -v, \qquad \psi_y := u - c. \tag{3.6}$$

Provided that

$$u - c = \psi_y \le -\mu < 0, \tag{3.7}$$

the problem (2.3) can be transformed into a nonlinear elliptic problem for the stream function [13]. A solution of the water wave problem is then defined as a function $\psi \in C^2(\overline{\Omega}_\eta)$ such that

$$\begin{aligned} \Delta\psi &= -\omega(\psi), && (x,y) \in \Omega_\eta \\ |\nabla\psi|^2 + 2gy &= C, && y = \eta(x) \\ \psi &= 0, && y = \eta(x) \end{aligned} \tag{3.8a}$$

which in addition satisfies

$$\begin{aligned} &\psi = -p_0 && \text{at} \quad y = -1, && \text{(finite depth)} \\ &\nabla\psi \to (0,-c) && \text{uniformly as} \quad y \to -\infty && \text{(infinite depth)} \end{aligned} \tag{3.8b}$$

In (3.8) p_0 is called the *relative mass flux*, the *vorticity function* $\omega\colon [0,-p_0] \to \mathbb{R}$ is continuously differentiable, $g > 0$ is the gravitational constant, and C is a constant related to the energy.

The problem (3.8) includes periodic as well as solitary waves. When the period is finite it we shall consider the period $L := 2\pi$, while for solitary waves we define the period to be infinite, $L := \infty$. In both cases we shall require $\eta(0) = \max_{x\in\mathbb{R}}\{\eta(x)\}$ to be the vertical coordinate of the crest, unique within a period, and that

$$\eta(x) = \eta(-x), \quad \text{and} \quad \eta'(x) < 0 \text{ for } x \in (0, L/2). \tag{3.9}$$

For large classes of waves, this a priori guaranteed. For example, in the absence of stagnation points, the assumption (3.9) holds if the surface profile is monotone from trough to crest [36, 10, 11, 9, 18], or if every streamline attains its minimum below the trough [27]. Notice, however, that (3.9) might hold even when (3.7) does not (cf. [44]). The system (3.8) is the *streamfunction formulation* of the water wave problem, and its periodic solutions fulfilling (3.9) are called *Stokes waves*. This expression is usually reserved for the case whithout vorticity, but to emphasize this fact we shall write irrotational Stokes waves.

It is often convenient to work in a fixed domain. A partial hodograph transform converts the free boundary problem (3.8) into a problem with a fixed boundary. The new space variables are

$$q := x, \qquad p := -\psi, \tag{3.10}$$

with the corresponding fixed domain

$$\Omega := \left\{(q,p) \in \mathbb{R}^2 \colon p_0 < p < 0\right\},$$

where, consistent with (3.8), we have set $p_0 := -\lim_{y\to -d}\psi(x,y) < 0$. Let us then introduce a *height function*

$$h(q,p) := y(q,p) + const.$$

The inequality (3.7) guarantees that (3.10) is a local change of variables, and we have

$$h_q = -\frac{\psi_x}{\psi_y} = \frac{v}{u-c}, \qquad h_p = -\frac{1}{\psi_y} = \frac{1}{c-u}.$$

The local coordinate transform (3.10) is actually a global change of variables [13], and we may transform the problem (3.8) into these variables to obtain

$$\begin{aligned} (1+h_q^2)\,h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} + \omega(-p)\,h_p^3 &= 0 \quad \text{for} \quad p \in (p_0, 0),\\ 1 + h_q^2 + (2gh - Q)\,h_p^2 &= 0 \quad \text{on} \quad p = 0, \end{aligned} \tag{3.11a}$$

Here $Q > 0$ is determined by the constant C in (3.8), and the height function $h \in C^2(\overline{\Omega})$. Since for any physical solution the derivative h_p is strictly bounded away from zero, the first equation in (3.11) is uniformly elliptic. In the case of finite depth, if one lets $h := y + d$ be the height above the flat bottom, then $h(q,0) = \eta(q) + d$. For deep-water waves it is suitable to use $h(q,p) := y(q,p) - Q/(2g)$. We then get the additional boundary condition(s):

$$\begin{aligned} h &= 0 & &\text{at} \quad p = p_0, & &\text{(finite depth)}\\ \nabla h &\to (0, 1/c) & \text{uniformly as} \quad &p \to -\infty & &\text{(infinite depth)} \end{aligned} \tag{3.11b}$$

As assumed above, the height function h is of period L in the q-variable, with (cf. (3.9))

$$h(q,0) = h(-q,0), \quad \text{and} \quad h_q(q,0) < 0 \text{ for } q \in (0, L/2). \tag{3.12}$$

We investigate (3.11) in the fixed, possibly infinite, rectangle

$$R := (-L/2, L/2) \times (p_0, 0).$$

The equations (3.11) constitute the *height formulation* of the water-wave problem.

4. **Streamlines — particle trajectories in a steady frame.** This section is devoted to streamlines, and we consider separately the cases of linear, shallow linear, and exact waves. Since, as we shall see in Section 5, the streamlines virtually determines the particle paths, a thorough understanding of the former is often the key to understanding the latter.

4.1. **Linear theory.** Recall that in the original physical variables, we have $(\dot{x}, \dot{y}) = (u, v)$. Thus, if

$$(X(t), Y(t)) := (kx(t) - ft, ky(t)),$$

are the corresponding steady variables (here $f = kc$ is the frequency), it follows that

$$(\dot{X}, \dot{Y}) = (k\dot{x} - f, ky).$$

Using this transformation one obtains the following dynamical system for (3.3a):

$$\begin{aligned} \dot{X} &= M(\varepsilon)\cos(X)\exp(Y) - f,\\ \dot{Y} &= M(\varepsilon)\sin(X)\exp(Y), \end{aligned} \tag{4.1}$$

where M depends linearly on the small amplitude parameter ε. While the exact solution of this system remains unknown, it can be studied with methods from the theory of dynamical systems. In[8] the phase-portrait for (4.1) was deduced (see Figure 2). Similar methods had previously been used for the case of finite depth

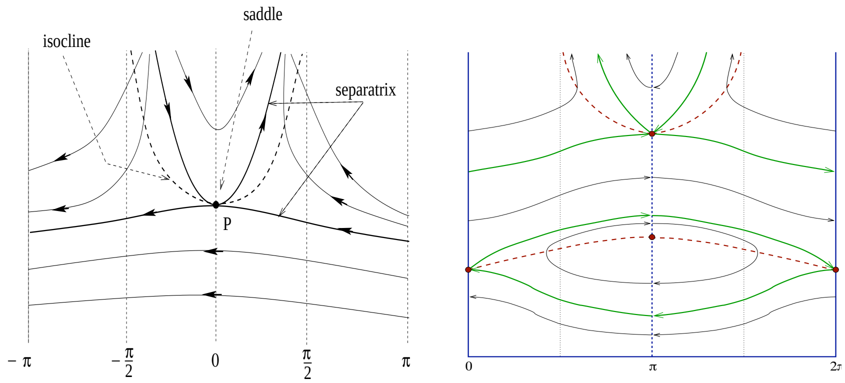

FIGURE 2. Qualitative phase portraits for linear steady water waves. Left: irrotational waves over infinite and finite depth. Right: waves with negative vorticity.

[17], and later also in the presence of capillarity [24] and a linear background current (constant vorticity) [20]. That is, when $\kappa = 0$ in (3.1), a transformation as above yields the dynamical system

$$\begin{cases} \dot{X}(t) & = M(\varepsilon)\cos(X)\cosh(Y) - \omega Y - f \\ \dot{Y}(t) & = M(\varepsilon)\sin(X)\sinh(Y). \end{cases} \tag{4.2}$$

Albeit different from the case when $\omega = 0$, a study of this system yields that when the vorticity is positive, the qualitative behaviour of the streamlines is the same as for the irrotational waves depicted to the left in Figure 2. If, however, the vorticity is negative and large enough, then a totally new situation may arise (cf. Figure 2, right). The reason for this is the dispersion relation (3.4b), which allows for two different signs of the modified speed $c_{\kappa,\omega}$. In the physical variables (3.4b) reads as

$$c - \kappa\sqrt{gh} + h\omega = \frac{1}{2k}\left(\omega\tanh(kh) \pm \sqrt{4gk\tanh(kh) + \omega^2\tanh^2(kh)}\right), \tag{4.3}$$

so that, depending on the background flow, there may be a minus sign in front of the square root. When $\kappa = 0$ we thus have the following result summarizing the results for linear waves (cf. [8, 20]):

Proposition 4.1. *For linear irrotational deep-water waves (4.1), and linear water waves over finite depth (4.2) with nonnegative vorticity, the qualitative phase portrait is depicted to the left in Figure 2, while for (4.2) with negative vorticity and small amplitude $a << 1$ it is given to the right in Figure 2. In the latter case the crest is at $X = 0$ for $h\omega > -c$, while for $h\omega < -c$ the crest is at $X = \pi$.*

4.2. **Small-amplitude shallow-water waves.** If one instead applies the change of variables

$$X(t) := 2\pi x\left(\frac{(c-\kappa)t}{2\pi}\right), \qquad Y(t) := y\left(\frac{(c-\kappa)t}{2\pi}\right) + 1,$$

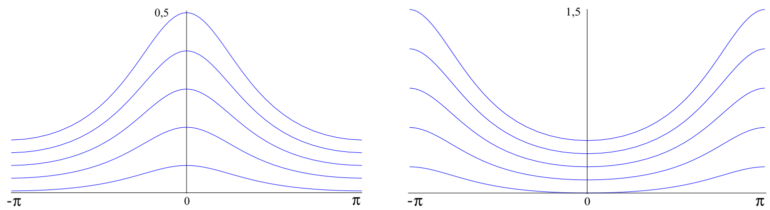

FIGURE 3. The explicit steady form of small-amplitude shallow-water waves with an affine background current. Left: $\Lambda = 2$. Right: $\Lambda = -2$.

to the linearized shallow water-wave solutions (3.5), they can be written in the steady form

$$\dot{X} = \cos(X) - c(c-\kappa),$$
$$\dot{Y} = Y\sin(X).$$

The term $c(c-\kappa)$ is due to the fact that (u, v) is the first time-derivative of the physical space variables. In this reference frame the streamlines $(X(t), Y(t))$ depends only on the parameter $\Lambda := c(c-\kappa)$, and for initial data $(X(0), Y(0)) = (0, Y_0)$ they can be explicitly calculated as

$$\begin{aligned} X(t) &= -2\arctan\left(\frac{\sqrt{\Lambda^2-1}}{1+\Lambda}\tan\left(\frac{\sqrt{\Lambda^2-1}}{2}t\right)\right), \\ Y(t) &= \frac{Y_0}{1+\Lambda}\left(\Lambda - 1 + 2\cos^2\left(\frac{\sqrt{\Lambda^2-1}}{2}t\right)\right). \end{aligned} \tag{4.4}$$

Although the formula for $X(t)$ is local in nature—it has jump-discontinuities of size 2π—it can easily be globally extended in time. The formula for $Y(t)$ is global in time, and it is immediate that unless $\Lambda^2 \geq 1$, the solutions become arbitrarily large. Hence, the physical intepretation of these solutions implies that

$$|c(c-\kappa)| \geq 1.$$

For such waves, one sees from (4.4) that $X(t)$ is strictly monotone, $Y(t)$ a perturbed cos-function, and the system describes a slightly perturbed graph $y = \cos^2 x$ for $\Lambda > 1$ ($y = \pi + \cos^2 x$ for $\Lambda < 1$). Two examples of such orbits (X, Y) are given in Figure 3.

4.3. **Exact theory.** Let $(x, \sigma(x))$ denote the parametrization of a (general) streamline

$$\{(x, y) \colon \psi(x, y) = -p\}.$$

Notice that since $\psi_y < 0$ the above definition is sensible, and we have

$$\sigma'(x) = -\frac{\psi_x(x, \sigma(x))}{\psi_y(x, \sigma(x))} = h_q(q, p).$$

In the exact theory, obtaining detailed information about the streamlines themselves is very difficult. For symmetric waves away from stagnation, the picture so far fits with the linear case:

Proposition 4.2. *Every streamline satisfies $\sigma' < 0$ for $x \in (0, \pi)$, and the maximal steepness of the streamlines is a strictly monotone function of depth. For negative* *vorticity the surface can be bounded below by a parabola, and for irrotational waves the surface is convex below a horizontal line.*

The first result comes from a maximal principle established in [13]; an observation from [19] then yields that the steepness is monotone with respect to depth. In [41] the convexity result for irrotational deep-water waves was proved. If, as in our case, the wave is symmetric with a monotone surface profile between crest and trough, then the surface is convex between the points on the surface where $|\nabla\psi| = 1$. In [42] the same author furthermore proved that for non-positive vorticity, the horizontal velocity decreases strictly from crest to trough along the surface:

$$\mathrm{D}_x u(x, \eta(x)) \leq 0 \quad \text{for} \quad x \in (0, \pi). \tag{4.5}$$

In that case one can bound the surface from below by a parabola [19]. Let $\nu := g/(C - 2g\eta(0)) = g/(c - u(0, \eta(0)))^2$. Then, for all $x \in (0, \pi)$, we have that

$$\eta''(x) \geq -\nu, \qquad \eta'(x) \geq -\nu x, \quad \text{and} \quad \eta(x) \geq \eta(0) - \tfrac{1}{2}\nu x^2.$$

Remark 4.3. Proposition 4.2 hold for a large class of waves. It should however be noted that there exist also other types of solution to the periodic water-wave problem (3.8). Those include waves with closed streamlines [44], waves with a peak of angle $2\pi/3$ [1, 43], and waves with that same kind of peak that additionally have a convex surface between the adjacent crests [37]. Those waves all share the property that $\nabla\psi$ vanishes at some point; the presence of such stagnation points has been excluded in our analysis.

5. **The drift of particles.** The main idea linking particle trajectories to the steady formulations of the problem was described in [17]. Since that approach is valid for both linear and exact waves over finite as well as infinite depth, and since the results surveyed in this section are the same for all those waves, we shall in this section make no difference between the different kinds of waves (except when explicitly stated).

Let $(x(t), y(t))$ be any physical trajectory, so that

$$(X(t), Y(t)) = (x(t) - ct, y(t))$$

is the corresponding path along a streamline in the steady variables. We have $\dot{X}(t) = u - c \leq -\delta < 0$. Thus $(X(t), Y(t))$ passes any $X \in \mathbb{R}$, and there is no loss of generality in choosing $X(0) = \pi$. We may also define τ through

$$X(\tau) := -\pi.$$

The following lemma (proved below, but see also [17]) then essentially reduces the investigation of particle paths to a certain time in the steady reference frame.

Lemma 5.1. *A physical particle trajectory is closed if and only if $\tau = 2\pi/c$. If $\tau > 2\pi/c$, then the particle displays a mean forward drift, and contrariwise.*

The main and consistent finding throughout the literature is the following:

Proposition 5.2. *Assume that the mean horizontal velocity over a period vanishes as $y \to -d$. For waves with nonpositive vorticity we have that $\tau > 2\pi/c$, and the particles thus display a forward drift.*

This proposition covers a lot of different settings, some of which will be dealt with in detail below. The fact that $\dot{X}$ is strictly negative implies a one-to-one correspondence between time and horizontal displacement, and we note that

$$\tau = -\int_{-\pi}^{\pi} \frac{dX}{\dot{X}} = \int_{-\pi}^{\pi} \frac{dX}{c-u}, \tag{5.1}$$

where the integration is along the streamline $(X, Y(X))$ corresponding to $(X(t), Y(t))$. Thus, to describe particle paths, one needs to understand what happens with the horizontal velocity along streamlines, and in particular to evaluate the integral in (5.1) in comparison to $2\pi/c$. After presenting the elementary, but important, proof of Lemma 5.1, we shall return to the evaluation of τ in different contexts.

Proof of Lemma 5.1. Since

$$\sigma' = \frac{v}{u-c} = \frac{\dot{Y}}{\dot{X}},$$

the streamlines describe the flow of the particles in the steady reference frame. By symmetry we thus have $Y(\tau) = Y(0)$. It moreover follows from Proposition 4.2 that $Y(0)$ is the lowest point of the trajectory, attained below the trough, and in view of symmetry $Y(\tau/2)$ is the highest, attained below the crest. In between $\dot{Y}(t) \neq 0$. Hence

$$Y(T) = Y(0) \quad \text{implies} \quad X(T) - X(0) = 2\pi n,$$

for some $n \in \mathbb{Z}$. In particular $n = -1$ for $T = \tau$.

Returning to the physical variables, this means that any new time a particle $(x(t), y(t))$ attains its lowest (or highest) position it has moved a distance of

$$x(\tau) - x(0) = c\tau - 2\pi$$

units in the horizontal direction. We infer that a physical particle trajectory is closed if and only if $\tau = 2\pi/c$. We also see from this reasoning that if $\tau > 2\pi/c$, then the particle displays a mean forward drift, and contrariwise. □

5.1. **The irrotational case with no background current.** The following result holds for irrotational waves adhering to Stokes' definition of wave speed (i.e. with a vanishing mean of the horizontal velocity along any horizontal line within the fluid, cf. (5.2) and (5.3)).

Proposition 5.3. *All fluid particles display a forward drift. The forward drift is strictly decreasing with depth, and for deep-water waves it vanishes as $y \to -\infty$.*

A proof of the forward drift for exact waves was given in [7]. A variant of that proof (but with vorticity and background current) will be presented in the next subsection. For linear waves, what complicates the calculation of τ through the formula (5.1) is that Y varies with X in a way not explicitly known (except from the case of shallow-water waves, cf. (3.5)). The trick in the linear case is to compare the streamline given by $Y(X)$ with the horizontal line passing through $Y(\pi/2)$. One then gets that

$$\begin{cases} Y(X) > Y(\pi/2), & X \in (0, \pi/2), \\ Y(X) < Y(\pi/2), & X \in (\pi/2, \pi). \end{cases}$$

This enables the elimination of $Y(X)$, and it is the main ingredient in the two different variants of proofs for the forward drift for linear waves [17, 8].

For linear waves the proof of the monotonicity comes from [8], whereas for exact waves it was later proved in [16] and [19] using different methods. There are proofs for this also in [40] and [34], the latter simplifying the proof from the first. There, however, the conclusion is drawn that the drift could be negative at the bottom, a possibility that is proved to be impossible by the modern analysis. In any case, in the investigations [16] and [19] the monotone increase of the forward drift is a consequence of a much stronger property (see Lemma 6.2), for which we know of no earlier evidence.

5.2. **Adding vorticity and a background current.** For very natural reasons, the results for rotational waves and waves with a uniform background current are less transparent. We do not aim at giving a complete description of the motion within such waves, but a general understanding of how the different quantities influence the drift and motion of the fluid particles. To this aim, we shall use the exact theory for Stokes waves, but in the presence of vorticity and a constant background flow. For a more complete description of small-amplitude waves with vorticity cf. [20, 44], and for small-amplitude shallow-water waves with vorticity and background flow [28, 29]. The analysis here presented is based on [7], with appropriate elements from [19] (vorticity) and [16] (background flow). The results are captured in Lemma 5.4 and Corollary 5.5, and essentially state that the forward drift is preserved for positive background flow and negative vorticity.

To denote an orbit in the steady variables we write

$$\gamma_\sigma(t) := (x(t), \sigma(x(t))),$$

corresponding to a physical trajectory

$$\Gamma_\sigma(t) := \gamma_\sigma(t) + (ct, 0).$$

Then let

$$u_\sigma(x) := u(x, \sigma(x)) \quad \text{and} \quad \kappa_\sigma := \frac{1}{2\pi}\int_{-\pi}^{\pi} u_\sigma(x)\, dx. \tag{5.2}$$

Traditionally, one has assumed that the mean horizontal velocity vanishes at the flat bed, and at great depths for deep-water waves. In other words, it has been required that

$$\kappa := \lim_{\sigma \to -d/h} \kappa_\sigma \tag{5.3}$$

vanishes. This is a consequence of Stokes' definition of wave speed, which, for irrotational waves, holds that the mean horizontal velocity vanishes along *any* horizontal line beneath the surface. In [16] the authors generalizes this notion, and let κ denote a uniform background current. Before adding vorticity, let us investigate how τ as above relates to κ.

Lemma 5.4. *Within the fluid domain of an irrotational Stokes wave holds*

$$\tau_\sigma \geq \frac{2\pi}{c - \kappa_\sigma} > \frac{2\pi}{c - \kappa}.$$

In particular, the particles display a forward drift whenever $\kappa \geq 0$.

Proof of Lemma 5.4. According to Young's inequality, we have

$$1 \leq \frac{1}{2}\left(\frac{c - u_\sigma}{c - \kappa} + \frac{c - \kappa}{c - u_\sigma}\right),$$

which after integration from $x = -\pi$ to $x = \pi$ reads as

$$2\pi \leq \pi + \frac{(c - \kappa_\sigma)\tau_\sigma}{2}.$$

Hence $\tau_\sigma \geq \frac{2\pi}{c-\kappa_\sigma}$. Too see that $\kappa_\sigma \geq \kappa$, let

$$F(\sigma) := \int_{-\pi}^{\pi} (c - u_\sigma)\left(1 + {\sigma'}^2\right) dx,$$

and, for two streamlines γ_{σ_1} and γ_{σ_2},

$$\Sigma := \{(x,y) \colon -\pi < x < \pi,\ \sigma_1(x) < y < \sigma_2(x)\}.$$

According to the Divergence theorem we have

$$\begin{aligned} 0 &= \int_\Sigma \nabla \cdot \nabla\psi \, dA = \int_{\gamma_{\sigma_1}} \nabla\psi \cdot \frac{\nabla\psi}{|\nabla\psi|}\, ds - \int_{\gamma_{\sigma_1}} \nabla\psi \cdot \frac{\nabla\psi}{|\nabla\psi|}\, ds \\ &= \int_{-\pi}^{\pi} \left(|\nabla\psi_{\sigma_1}| \sqrt{1+{\sigma_1'}^2} - |\nabla\psi_{\sigma_2}| \sqrt{1+{\sigma_2'}^2} \right) dx \qquad (5.4) \\ &= \int_{-\pi}^{\pi} \left((c-u_{\sigma_1})\left(1+{\sigma_1'}^2\right) - (c-u_{\sigma_2})\left(1+{\sigma_2'}^2\right) \right) dx. \end{aligned}$$

By letting $\sigma_1 \to -d/h$ we obtain that $F(\sigma) \equiv 2\pi(c-\kappa)$. That $\kappa_\sigma \geq \kappa$ is then a consequence of that $F(\sigma) \geq \int_{-\pi}^{\pi}(c-u_\sigma)\, dx = 2\pi(c-\kappa_\sigma)$. ☐

The effect of κ is very intuitive: since it describes a background current moving in the same direction as the wave, its positivity implies that the particles are moving faster forward than for a vanishing κ. It turns out that a similar relationship is valid for the vorticity:

Corollary 5.5. *If the vorticity is nonpositive, then the assertions of Lemma 5.4 still hold.*

Proof. Notice that the proof of Lemma 5.4 is actually based on the fact that $F(\sigma_2) \leq F(\sigma_1)$ in (5.4). When vortiticity is present, the calculation (5.4) yields that

$$-\int_\Sigma \omega(\psi)\, dA = F(\sigma_1) - F(\sigma_2),$$

so that $F(\sigma_2) \leq F(\sigma_1)$ whenever $\omega \leq 0$. ☐

When $\kappa < 0$ or when $\omega > 0$ things are more complicated. This is so since not only is the current working against the wave, but in a nonuniform way. Recall also that the undisturbed forward drift itself is not uniform. For large enough $\kappa < 0$ and/or $\omega > 0$, one obtains particles with a backward drift [16, 20, 29], but the complete picture—especially for a general vorticity distribution—remains an open problem.

6. **The particle paths.** The relation between streamlines and physical variables, $X = k(x-ct)$ and $Y = ky$, essentially enables us to pass from streamlines to particle paths: the particle orbits are just ct-disturbances of the streamlines. But what is c?

The issue is that c could be just about anything. This is most easily seen from the governing equations (2.3), since one could always add a constant to u and subtract it from c. It is also evident from the dispersion relation (4.3): even when all other physical variables are fixed, c depends on the background current κ in an arbitrarily way. One could deal with this in two ways: either normalize κ, or accept that it could be any real number. For irrotational waves, *Stokes' definition of the wave speed means prescribing* $\kappa = 0$ (as described in relation to (5.2)). This normalization is therefore an implicit part of the results [7, 23] on irrotational Stokes

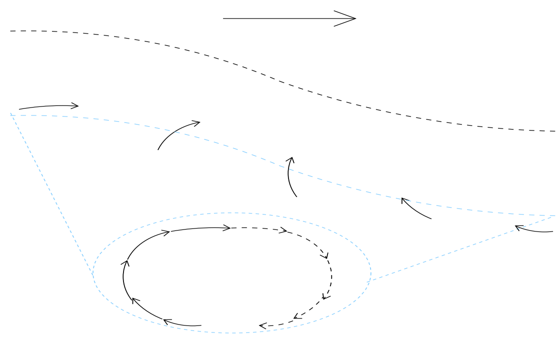

FIGURE 4. The horizontal velocity is everywhere decreasing along the streamlines, and it is increasing from bottom and up beneath the crest, whilst decreasing beneath the trough.

waves, and it is the explicit choice made in the papers [20, 19, 44] on waves with vorticity. For transparency we shall start with the particle paths within irrotational Stokes waves, whereafter we shall describe what differs when vorticity is added, or when one considers the nonvanishing background current κ introduced in [16].

6.1. **The irrotational case in still water.** Since for this case the exact results known today are that detailed, we shall confine ourselves to the exact equations. The main result, combined from [7, 16, 19, 23, 26], is the following:

Proposition 6.1. *The fluid particles within irrotational Stokes waves traverse non-closed oval orbits as in Figure 4. A particle moves upwards starting from the time a trough passes until the next crest passes. At the top of its orbit the particle attains its maximal horizontal velocity, after which the particle begins its descent with the horizontal speed strictly decreasing until it reaches its minimal value as the next trough passes. This motion is symmetric with respect to the highest point on the orbit, and the forward drift displayed by all particles is strictly increasing from bottom to surface.*

The proof of Proposition 6.1 is based on the forward drift proved in Section 5, the antisymmetry of the vertical velocity component with respect to the crest (cf. Proposition 4.2), and a detailed study of the horizontal velocity. As was discussed in relation to (5.1), it is this last component that in detail enables us to control the shape of the orbit. Using two different methods, the following result was proved independently in [16] and [19].

Lemma 6.2. $\mathrm{D}_x u(x, \sigma(x)) < 0$ *for* $x \in (0, \pi)$ *within irrotational Stokes waves.*

Both proofs are based on the fact that the flow is irrotational, and they both require a bound on the slope of the surface (45 and 30 degrees, respectively) that arises naturally in the proof. Notice that the interior angle of the highest irrotational Stokes wave is 120 degrees, whence 30 degrees is a matching upper bound. The proofs that the mean forward drift is strictly increasing from bed to surface are then based on Lemma 6.2. All taken together, a detailed analysis yields Proposition 6.1.

6.2. **Adding a uniform background current.** To see what happens when a current is added, imagine a circular orbit $(x(t), y(t)) = (\cos(t), \sin(t))$. According

FIGURE 5. A circular orbit under the influence of a uniform current κ. To the left and backmost: a favourable current ($\kappa > 0$). The the right and foremost: a current working against the wave ($\kappa < 0$). Notice that the case of irrotational Stokes waves corresponds to the loop just behind the closed circle, in which case the drift is rightward. As κ decreases the drift becomes negative and leftward on this side of the closed orbit.

.

to the arguments given above, a change of the uniform current κ corresponds to a change of c. Thus

$$(x(t), y(t)) \quad \mapsto \quad (x(t) + \kappa t, y(t))$$

maps the circular orbit to the corresponding orbit in the presence of a current of strength κ. The different topological variants of this transformation in the case of the circle are given in Figure 5.

While this picture does describe all possible scenarios also for irrotational gravity waves with a uniform background current, there are two difficulties that separates the circular case in Figure 5 from the general one:

i) The undisturbed orbit in an irrotational Stokes wave already displays a forward drift, the magnitude of which is in general unknown. Hence the influence of κ is not symmetric as in Figure 5, and we do not know what κ corresponds to a closed orbit.
ii) The undisturbed forward drift differs between different streamlines, and the orbits resulting from the current may therefore be qualitatively different at different heights within the wave.

As a consequence of those remarks, a complete analysis of the particle paths within waves with a current presupposes knowledge about the size of the horizontal velocity along the different streamlines. Even then it is not easy to distinguish and characterize the different cases. For a thorough analysis of these matters we refer to [16]. There, the following is proved.

Proposition 6.3. *Within exact gravitational irrotational Stokes waves in the presence of a uniform background current, all forms of the trajectories in Figure 5 are possible. The particular orbits depend on the strength κ of the underlying current, and more than one type of orbit may be present in the same wave.*

Due to the intractability of the exact equations, another way forward is to consider the linearized shallow-water equations (3.5). There are two main reasons for this, the main being that the limit $\delta \to 0$ nullifies the depth-dependence of the drift. The other reason is that via (4.4) one even has closed expressions for the solutions (orbits like those in Figure 5 may be plotted whenever $|c(c-\kappa)| > 1$).

In spite of this, the easiest way to find the critical values of κ is to consider the realization of (3.5) in the non-dimensionalized physical variables. There,

$$\dot{x} = \kappa + \varepsilon(c - \kappa + \omega)\cos(2\pi(x - ct)),$$

and the dispersion relation dictates that $c-\kappa+\omega = 1/(c-\kappa)$. We thus see that the transitions take place exactly when $\kappa(c-\kappa) = \pm\varepsilon$. These are the values for which the horizontal velocity $\dot{x}$ vanishes somewhere. When $|\kappa(c-\kappa)| > \varepsilon$ all particles are moving only in one direction constantly through time. Recall that the system (3.5) describes a physical wave only when $|c(c-\kappa)| > 1$, which gives us the other two critical values of κ. Together these values determine for which κ the transitions between the different type of waves as in Figure 5 take place.

Recall from Remark 3.1 that we have introduced κ slightly differently from the constant c_0 in [28, 29]. Returning to the non-dimensionalized physical variables in those investigations shows that $c_0 = \varepsilon\kappa$ is the normalized background current relative to the linear disturbance (a similiar scaling holds for the vorticity). Unless working entirely within the linearization, the amplitude parameter ε is an artefact of linear theory which cannot be bypassed. The study [28] however describes the transitions between the different shapes in Figure 5 in terms of the relative background current c_0.

6.3. **Adding vorticity.** When vorticity is added, things are even worse. We have already seen in Corollary 5.5 that for negative vorticity all particles drift along with the wave. In that case it can also be shown that, for waves bifurcating from the laminar rotational stream, the mean forward drift is strictly increasing from bed to surface [19], at least as long as the waves are small enough. But since the proof of Corollary 5.5 relies on an inequality it might be that the orbits are oval-shaped, undular, or both (at different levels in the water).

Except for the facts just mentioned there are no general results on the particle paths for exact rotational waves. The following proposition however indicates that the picture is not as uniform as for irrotational waves:

Proposition 6.4.
i) *In the case of linear shallow-water waves of constant vorticity with Stokes' definition of the wave speed there does not exist a single pattern for all particles* [29].
ii) *Dropping the shallowness assumption, one and the same linear wave may contain all types of orbits in Figure 5 at different depths* [20].
iii) *There exists an explicit solution of the full deep-water wave problem (Gerstner's wave), in which the particles are moving in closed circles at all depths* [6]*, but an extension of this family contains uni-directional trochoidal trajectories* [15].

The extension [15] of the famous Gerstner family of exact Lagrangian and (generally) rotational solutions of the gravitational deep-water wave problem is very recent. It is indicative that the trochoidal solutions reside upon flows of negative vorticity, a fact in line with the results of Section 5 and Corollary 5.5.

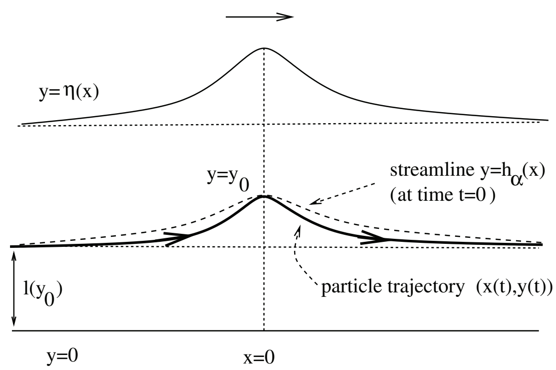

FIGURE 6. The particle trajectories within an irrotational solitary wave. The particles move along with the wave, and each orbit is contained within the height of the corresponding streamline. Reprinted from [12] with kind permission.

7. **Solitary and standing waves.** The core of this survey has been dedicated to periodic waves, and even in that case it is impossible to incorporate all recent results. In this section we very briefly describe some approaches differing from those previously accounted for. The cases of standing and solitary waves are of particular importance, due to their natural connections with periodic waves.

7.1. **Solitary waves.** Solitary waves can be considered periodic waves of infinite wavelength. The fluid motion within such waves was recently investigated in [12]. There, a pattern quite different from that of periodic waves was discovered (see also Figure 6).

Proposition 7.1. *The streamlines within gravitational finite-depth solitary waves are symmetric graphs with a single maximum. Any particle path share this property, and it is confined in the vertical gap between a streamline's highest point (beneath the crest) and its lowest (at infinity). All particles move in the same direction as the wave with a speed not exceeding the travelling-wave speed c.*

As far as we know, there are no known upper bounds on the drift for exact waves. It is possible that the drift increases with the wavelength L, and that the trajectories become uni-directional in the limit $L \to \infty$. If so, the process would resemble the foremost part of Figure 5, but with L substituted for κ as the parameter. It is in any case clear that the qualitative behaviour of the trajectories within solitary waves is captured by one period of the foremost orbit in that picture. An investigation of the transition of particle paths between periodic and solitary waves has not yet been carried out.

7.2. **Standing waves.** Another interesting transition takes place when two steady periodic wave-trains of the same speed counteract each other. This happens, for example, when a periodic wave is reflected by a horizontal wall. The resulting phenomenon is known as standing waves, and the existence of such waves has been

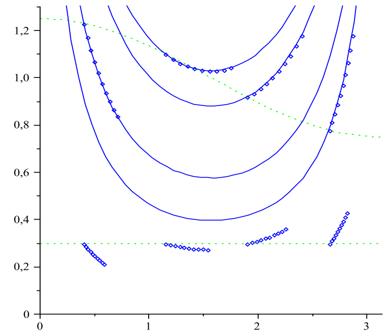

FIGURE 7. The particle trajectories within standing waves. The solid curves are streamlines, the hollow circles particle paths, and the dashed curves are the surface at time $t = 0$, and the line $y = 0.3$, respectively.

established in various forms by the author's of [30]. There is a beautiful picture series showing the experimentally established transition between periodic and standing waves [38]. The resulting orbits in the standing waves resemble the foremost paths in Figure 5, with the difference that the particles are moving back and forth in a periodic pattern as the waveform changes above. There is thus no mean drift of the particles in a period. In [21] it is shown that, even within the linear approximation of standing waves, this pattern can be mathematically established (see Figure 7). Indeed, though the water is not shallow, the particle trajectories are given by closed formulas in terms of elliptic functions.

Proposition 7.2. *The particle trajectories within linear standing waves over finite depth are periodic in time, while convex and symmetric in space. In particular, all particle paths are closed and there is no forward (or backward) drift.*

As in the case of solitary waves, we do not know of any mathematical study describing the passage from periodic to standing waves.

Acknowledgement. The authors would like to thank the referee for suggesting several improvements in the presentation.

REFERENCES

[1] C. J. Amick, L. E. Fraenkel, and J. F. Toland, *On the Stokes conjecture for the wave of extreme form*, Acta Math., **148** (1982), 193–214.
[2] R. Caflisch and H. Lamb, Hydrodynamics (Cambridge Mathematical Library), Cambridge University Press, 1993.
[3] H.-K. Chang, Y.-Y. Chen, and J.-C. Liou, *Particle trajectories of nonlinear gravity waves in deep water*, Ocean Engineering, **36** (2009), 324–329.
[4] H.-K. Chang, J.-C. Liou, and M.-Y. Su, *Particle trajectory and mass transport of finite-amplitude waves in water of uniform depth*, Eur. J. Mech. B Fluids, **26** (2007), 385–403.
[5] A. Constantin, *Edge waves along a sloping beach*, J. Phys. A, **34** (2001), 9723–9731.
[6] ———, *On the deep water wave motion*, J. Phys. A, **34** (2001), 1405–1417.
[7] ———, *The trajectories of particles in Stokes waves*, Invent. Math., **166** (2006), 523–535.
[8] A. Constantin, M. Ehrnström, and G. Villari, *Particle trajectories in linear deep-water waves*, Nonlinear Anal. Real World Appl., **9** (2008), 1336–1344.
[9] A. Constantin, M. Ehrnström, and E. Wahlén, *Symmetry for steady gravity water waves with vorticity*, Duke Math. J., **140** (2007), 591–603.
[10] A. Constantin and J. Escher, *Symmetry of steady deep-water waves with vorticity*, European J. Appl. Math., **15** (2004), 755–768.
[11] ———, *Symmetry of steady periodic surface water waves with vorticity*, J. Fluid Mech., **498** (2004), 171–181.
[12] ———, *Particle trajectories in solitary water waves*, Bull. Amer. Math. Soc. (N.S.), **44** (2007), 423–431.
[13] A. Constantin and W. A. Strauss, *Exact steady periodic water waves with vorticity*, Comm. Pure Appl. Math., **57** (2004), 481–527.
[14] ———, *Rotational steady water waves near stagnation*, Phil. Trans. R. Soc. Lond. A, **365** (2007), 2227–2239.
[15] ———, *Trochoidal solutions to the incompressible two-dimensional Euler equations*, J. Math. Fluid Mech., (2008), doi:10.1007/s00021-008-0281-2.
[16] ———, *Pressure beneath a Stokes wave*, preprint, (2009).
[17] A. Constantin and G. Villari, *Particle trajectories in linear water waves*, J. Math. Fluid Mech, **10** (2008), 1–18.
[18] M. Ehrnström, *Deep-water waves with vorticity: symmetry and rotational behaviour*, Discrete Contin. Dyn. Syst., **19** (2007), 483–491.
[19] ———, *On the streamlines and particle paths of gravitational water waves*, Nonlinearity, **21** (2008), 1141–1154.
[20] M. Ehrnström and G. Villari, *Linear water waves with vorticity: rotational features and particle paths*, J. Differential Equations, **244** (2008), 1888–1909.
[21] M. Ehrnström and E. Wahlén, *On the fluid motion in standing waves*, J. Nonlinear Math. Phys., **15** (2008), 74–86.
[22] F. Gerstner, *Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile*, Ann. Phys., **2** (1809), 412–445.
[23] D. Henry, *The trajectories of particles in deep-water Stokes waves*, International Mathematics Research Notices, **2006** (2006), 1–13.
[24] ———, *Particle trajectories in linear periodic capillary and capillary-gravity deep-water waves*, J. Nonlinear Math. Phys., **14** (2007), 1–7.
[25] ———, *On Gerstner's water wave*, J. Nonlinear Math. Phys., **15** (2008), 87–95.
[26] ———, *On the deep-water Stokes wave flow*, Int. Math. Res. Not., **2008** (2008), 1–7.
[27] V. M. Hur, *Symmetry of steady periodic water waves with vorticity*, Philos. Trans. R. Soc. Lond. Ser. A, **365** (2007), 2203–2214.
[28] D. Ionescu-Kruse, *Particle trajectories in linearized irrotational shallow water flows*, J. Nonlinear Math. Phys., **15** (2008), 13–27.
[29] ———, *Particle trajectories beneath small amplitude shallow water waves in constant vorticity flows*, Nonlinear Anal., (2009), In press.
[30] G. Iooss, P. I. Plotnikov, and J. F. Toland, *Standing waves on an infinitely deep perfect fluid under gravity*, Arch. Ration. Mech. Anal., **177** (2005), 367–478.
[31] R. S. Johnson, A modern introduction to the mathematical theory of water waves, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1997.

[32] ———, *The Camassa–Holm equation for water waves moving over a shear flow*, Fluid Dyn. Res., **33** (2003), 97–111.
[33] J. Lighthill, Waves in fluids, Cambridge University Press, Cambridge, 1978.
[34] M. S. Longuet-Higgins, *On the decrease of velocity with depth in an irrotational water wave*, Proc. Cambridge Philos. Soc., **49** (1953), 552–560.
[35] ———, *The trajectories of particles in steep, symmetric gravity waves*, J. Fluid Mech., **94** (1979), 497–517.
[36] H. Okamoto and M. Shōji, The mathematical theory of permanent progressive water-waves, vol. 20 of Advanced Series in Nonlinear Dynamics, World Scientific Publishing Co. Inc., River Edge, NJ, 2001.
[37] P. I. Plotnikov and J. F. Toland, *Convexity of Stokes waves of extreme form*, Arch. Ration. Mech. Anal., **171** (2004), 349–416.
[38] F. Ruellan and A. Wallet, *Trajectoires internes dans un clapotis partiel*, La Houille Blanche, **5** (1950), 483–489.
[39] G. G. Stokes, *On the theory of oscillatory waves*, Trans. Cambridge Phil. Soc., **8** (1847), 441–455.
[40] F. Ursell, *Mass transport in gravity waves*, Proc. Cambridge Philos. Soc., **49** (1953), 145–150.
[41] E. Varvaruca, *Some geometric and analytic properties of solutions of Bernoulli free-boundary problems*, Interfaces Free Bound., **9** (2007), 367–381.
[42] ———, *On some properties of traveling water waves with vorticity*, SIAM J. Math. Anal., **39** (2008), 1686–1692.
[43] ———, *On the existence of extreme waves and the Stokes conjecture with vorticity*, J. Differential Equations, **246** (2009), 4043–4076.
[44] E. Wahlén, *Steady water waves with a critical layer*, J. Differential Equations, **246** (2009), 2468–2483.
[45] Wikipedia, *Wave — Wikipedia, the free encyclopedia*, 2009. [Online; accessed 9-July-2009].
[46] C.-S. Yih, *Note on edge waves in a stratified fluid*, J. Fluid Mech., **24** (1966), 765–767.
[47] ———, *The role of drift mass in the kinetic energy and momentum of periodic water waves and sound waves*, J. Fluid Mech., **331** (1997), 429–438.

Institut für Angewandte Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany.
E-mail address: `ehrnstrom@ifam.uni-hannover.de`

Dipartimento di Matematica, Viale Morgagni 67/A, 50134 Firenze, Italy.
E-mail address: `villari@math.unifi.it`