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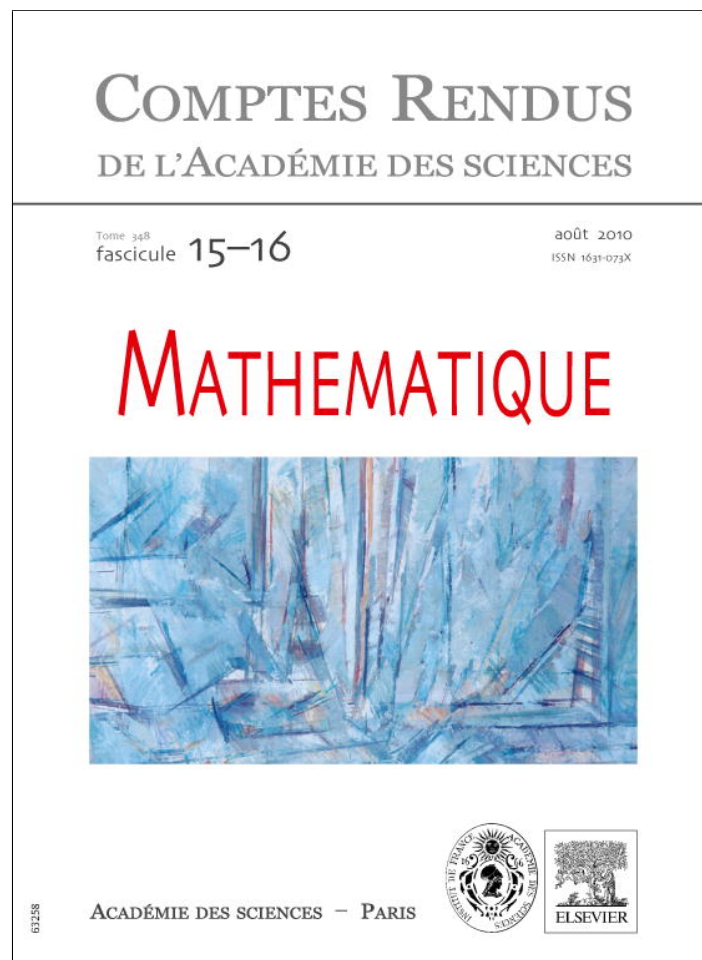
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Partial Differential Equations

# Liouville-type theorems for certain degenerate and singular parabolic equations

*Théorèmes de type Liouville pour quelques équations paraboliques singulières dégénérées*

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## ABSTRACT

Relying on recent results on Harnack inequalities for equations of  $p$ -Laplacian type, we prove Liouville-type estimates for solutions to these equations, both in the degenerate ( $p > 2$ ), and in the singular ( $1 < p < 2$ ) range.

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## RÉSUMÉ

En utilisant des résultats récents sur l'inégalité de Harnack pour les équations type  $p$ -laplacien, on établit des théorèmes de type Liouville pour les solutions de ces équations, dans le cas dégénéré  $p > 2$ , ainsi bien que dans le cas singulier  $1 < p < 2$ .

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## Version française abrégée

On sait que, pour les solutions de l'équation de la chaleur, des limitations unilatérales ne sont pas suffisantes pour garantir qu'elles sont constantes.

Des résultats analogues sont valables pour les solutions faibles des équations (1), (2) dans le cas  $p > 2$ .

Le résultat fondamental de cette Note montre que, dans l'intervalle singulier sur-critique (3), les solutions faibles de (1), (2), définies dans tout  $\mathbb{R}^{N+1}$  et bornées inférieurement (ou supérieurement) sont en fait constantes (Théorème 1.2).

Ce théorème n'est plus vrai quand  $p$  est dans l'intervalle singulier critique et sous-critique (4), comme on peut voir, grâce à certaines solutions explicites de (1)' dans cet intervalle.

Dans le cas dégénéré ( $p > 2$ ) il est nécessaire de supposer des limitations soit inférieures soit supérieures, pour pouvoir garantir que la solution est constante. Il est possible de formuler ces limitations bilatérales de différentes façons, comme on le montre dans le Théorème 1.1 et dans les Propositions 1.1, 1.2.

## 1. Liouville-type theorems

For  $T \in \mathbb{R}$  let  $S_T$  denote the semi-infinite strip

$$S_T = \mathbb{R}^N \times (-\infty, T).$$

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Let  $u$  be a non-negative, local, weak solution to the quasi-linear parabolic equation

$$\begin{aligned} u &\in C_{loc}(-\infty, T; L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(-\infty, T; W^{1,p}_{loc}(\mathbb{R}^N)), \\ u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) &= 0 \quad \text{weakly in } S_T, \end{aligned} \tag{1}$$

for  $p > 1$ , where  $\mathbf{A}: S_T \rightarrow \mathbb{R}^N$ , is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, Du) \cdot Du \geq C_0 |Du|^p \\ |\mathbf{A}(x, t, u, Du)| \leq C_1 |Du|^{p-1} \end{cases} \quad \text{a.e. in } S_T \tag{2}$$

where  $C_0$  and  $C_1$  are given positive constants. The prototype is

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0, \quad \text{in } S_T. \tag{1'}$$

The modulus of ellipticity of this class of equation is  $|Du|^{p-2}$  and accordingly they are degenerate for  $p > 2$  and singular for  $1 < p < 2$ .

Harmonic functions in  $\mathbb{R}^N$  with one-sided bound, are constant. This, known as the Liouville theorem, is solely a consequence of the Harnack inequality. As such it extends to solutions to homogeneous, quasi-linear, elliptic partial differential equations in  $\mathbb{R}^N$  with one-sided bound.

This property does not extend to caloric functions in  $\mathbb{R}^N \times \mathbb{R}$ , as a one-sided bound is not sufficient to imply that they are constant. The function

$$\mathbb{R} \times \mathbb{R} \ni (x, t) \rightarrow u(x, t) = e^{x+t}$$

is a non-negative, non-constant solution of the heat equation in  $\mathbb{R} \times \mathbb{R}$ . The Liouville theorem continues to be false for non-negative solutions to degenerate  $p$ -Laplacian type equations ( $p > 2$ ). The one-parameter family of non-negative functions defined in the whole  $\mathbb{R} \times \mathbb{R}$

$$u(x, t; c) = A(1 - x + ct)^{\frac{p-1}{p-2}}_+, \quad \text{where } A = c^{\frac{1}{p-2}} \left( \frac{p-2}{p-1} \right)^{\frac{p-1}{p-2}}$$

is a non-negative, non-constant, weak solution to (1)' in  $\mathbb{R}^2$ .

The main result of this note is that the Liouville property while false for  $p$  in the degenerate range  $p > 2$ , it does actually holds true for  $p$  in the singular, super-critical range

$$\frac{2N}{N+1} < p < 2 \tag{3}$$

and then it is false again for  $p$  in the singular, critical, and sub-critical range

$$1 < p \leq \frac{2N}{N+1}. \tag{4}$$

While some results appear in the literature for linear and coercive equations ( $p = 2$ ) (see for example [4–6]), to our knowledge, no results are known for degenerate ( $p > 2$ ) or singular ( $1 < p < 2$ ) quasi-linear equations of the type of (1)–(2).

### 1.1. Two-sided bounds and Liouville-type theorems in the degenerate range $p > 2$

Henceforth we let  $u$  be a continuous, local, weak solution to (1)–(2) in  $S_T$  for  $p > 2$ .

**Theorem 1.1.** *If  $u$  is bounded in  $S_T$ , then  $u$  is constant.*

The next proposition asserts that if a one-sided bound is available, then it suffices to verify the two-sided bound only at some time level.

**Proposition 1.1.** *Let  $u$  be bounded below in  $S_T$  and assume that*

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s < +\infty \quad \text{for some } s < T.$$

*Then  $u$  is constant in  $S_s$ .*

It has been observed that a one-sided bound on  $u$  is not sufficient to infer that  $u$  is constant in  $S_T$ . Such a conclusion however holds if  $u$  has a two-sided bound as indicated by Theorem 1.1. Consider the family of functions

$$u(x, t) = C(N, p) \left( \frac{|x|^p}{T-t} \right)^{\frac{1}{p-2}}$$

defined in  $S_T$ , where

$$C(N, p) = \left[ \frac{1}{\lambda} \left( \frac{p-2}{p} \right)^{p-1} \right]^{\frac{1}{p-2}} \quad \text{and} \quad \lambda = N(p-2) + p.$$

One verifies that this solves the prototype equation (1)' in  $S_T$ , for any  $p > 2$ , and it blows up as  $t \rightarrow T$ , for all  $x \in \mathbb{R}^N - \{0\}$ . This suggests that if  $u$  is defined in the whole  $\mathbb{R}^N \times \mathbb{R}$ , a condition weaker than a two-sided bound might imply that  $u$  is constant. The next proposition is in this direction; it asserts that it suffices to check the two-sided boundedness of  $u$  at a single point  $y \in \mathbb{R}^N$ , for large times, to conclude that  $u$  is constant.

**Proposition 1.2.** *Let  $u$  be defined and bounded below in  $\mathbb{R}^N \times \mathbb{R}$ . If*

$$\lim_{s \rightarrow +\infty} u(y, s) = \alpha \quad \text{for some } y \in \mathbb{R}^N \text{ and some } \alpha \in \mathbb{R},$$

then  $u$  is constant.

### 1.2. One-sided bounds and Liouville-type theorems in the singular super-critical range (3)

**Theorem 1.2.** *Let  $u$  be a continuous, local, weak solution to the singular, quasi-linear equation (1)–(2) in  $\mathbb{R}^N \times \mathbb{R}$ , for  $p$  in the singular, super-critical range (3). If  $u$  has a one-sided bound, then it is constant.*

The theorem is false for  $p$  in the singular, critical and sub-critical range (4). Consider the two-parameter family of functions

$$u(x, t) = (T - t)_+^{\frac{N+2}{4}} (a + b|x|^{\frac{2N}{N-2}})^{-\frac{N}{2}},$$

$$N > 2, \quad p = \frac{2N}{N+2} < \frac{2N}{N+1}$$

where  $a > 0$  and  $T$  are parameters, and

$$b = b(N, a) = \frac{N-2}{N^2} \left( \frac{N+2}{4Na} \right)^{\frac{N+2}{N-2}}.$$

They are non-negative, non-constant, locally bounded, weak solutions to the prototype  $p$ -Laplacian equation (1)' in  $\mathbb{R}^N \times \mathbb{R}$ . For the critical value  $\frac{2N}{N+1}$  the function

$$u(x, t) = (|x|^{\frac{2N}{N-1}} + e^{bt})^{-\frac{N-1}{2}},$$

$$b = \frac{2N^{\frac{2N}{N+1}}}{N-1}, \quad N \geq 2, \quad p = \frac{2N}{N+1}$$

is a non-negative, non-constant solution to (1)' in  $\mathbb{R}^N \times \mathbb{R}$ .

## 2. Intrinsic Harnack estimates [1,3]

Let  $u$  be a continuous, non-negative, local, weak solution to (1)–(2). Fix  $(x_0, t_0) \in S_T$  such that  $u(x_0, t_0) > 0$  and construct the cylinders

$$(x_0, t_0) + Q_\rho^\pm(\theta) = B_\rho(x_0) \times (t_0 \pm \theta \rho^p) \tag{5}$$

where  $B_\rho(x_0)$  is the ball in  $\mathbb{R}^N$  centered at  $x_0$  and of radius  $\rho$ , and

$$\theta = \delta u(x_0, t_0)^{2-p} \tag{6}$$

for a constant  $\delta > 0$ . These cylinders are “intrinsic” to the solution since their height is determined by the value of  $u$  at  $(x_0, t_0)$ . The point  $(x_0, t_0)$  and the constant  $\delta$  being determined, we let  $\rho > 0$  be so that

$$(x_0, t_0) + Q_{8\rho}^\pm(\theta) \subset S_T. \tag{7}$$

**Theorem 2.1.** (See [1,2].) *Let  $u$  be a continuous, non-negative, local, weak solution to the degenerate equations (1)–(2) in  $S_T$ , for  $p > 2$ . There exist constants  $\delta, \gamma > 1$  depending only upon the data  $\{p, N, C_0, C_1\}$ , such that for all intrinsic cylinders  $v$  as in (5)–(7), there*

holds

$$\gamma^{-1} \sup_{B_\rho(x_0)} u(\cdot, t_0 - \theta\rho^p) \leq u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(\cdot, t_0 + \theta\rho^p). \tag{8}$$

The constants  $\gamma$  and  $\delta$  deteriorate as  $p \rightarrow \infty$ , but they are “stable” as  $p \rightarrow 2$ . Thus by formally letting  $p \rightarrow 2$  in (8) one recovers the classical Moser’s Harnack inequality [7].

**Theorem 2.2.** (See [3,2].) *Let  $u$  be a continuous, non-negative, local, weak solution to the singular equations (1)–(2), in  $S_T$ , for  $p$  in the super-critical range (3). There exist constants  $\delta \in (0, 1)$  and  $\gamma > 1$ , depending only upon the data  $\{p, N, C_0, C_1\}$ , such that for all intrinsic cylinders as in (5)–(7), there holds*

$$\gamma^{-1} \sup_{B_\rho(x_0)} u(\cdot, \sigma) \leq u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(\cdot, \tau) \tag{9}$$

for any pair of time levels  $\sigma, \tau$  in the range

$$t_0 - \delta u(x_0, t_0)^{2-p} \rho^p \leq \sigma, \quad \tau \leq t_0 + \delta u(x_0, t_0)^{2-p} \rho^p. \tag{10}$$

The constants  $\delta$  and  $\gamma^{-1}$  tend to zero as either  $p \rightarrow 2$  or  $p \rightarrow \frac{2N}{N+1}$ .

Both, right and left inequalities in (9) are insensitive to the times  $\sigma, \tau$ , provided they range within the time-intrinsic geometry of (5)–(7). For  $\sigma = \tau = t_0$  the theorem yields

**Corollary 2.1** (The Elliptic Harnack Inequality [3]). *Let  $u$  be a continuous, non-negative, local, weak solution to the singular equations (1)–(2) for  $p$  in the super-critical range (3). Then for all intrinsic cylinders as in (5)–(7), there holds*

$$\gamma^{-1} \sup_{B_\rho(x_0)} u(\cdot, t_0) \leq u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(\cdot, t_0). \tag{11}$$

The right and left inequalities in (9) are simultaneously forward, backward and elliptic Harnack estimates. Inequalities of this type are false for non-negative solutions to the heat equation [7]. This is reflected in that the constants  $\delta$  and  $\gamma^{-1}$  tend to zero as  $p \rightarrow 2$ . These inequalities loose meaning also as  $p$  tends to the critical value  $\frac{2N}{N+1}$ . The range (3) of  $p$  is optimal for Theorem 2.2 and Corollary 2.1 to hold [3].

### 3. Proofs of the Liouville-type statements

Assume  $p > 2$ . If  $u$  is bounded above (below) in  $S_T$  set

$$M = \sup_{S_T} u \quad \left( m = \inf_{S_T} u \right)$$

and for points  $(y, s) \in S_T$  for which  $M > u(y, s)$ ,  $(u(y, s) > m$  respectively) construct the intrinsic, backward  $p$ -paraboloid(s)

$$P_M(y, s) = \{ (x, t) \in S_T \mid t - s \leq -\delta [M - u(y, s)]^{2-p} |x - y|^p \}$$

$$(P_m(y, s) = \{ (x, t) \in S_T \mid t - s \leq -\delta [u(y, s) - m]^{2-p} |x - y|^p \}),$$

where  $\delta$  is the constant in the intrinsic Harnack inequality of Theorem 2.1. The proof of Theorem 1.1 is an immediate consequence of the following:

**Lemma 3.1.** *Let  $u$  be bounded below (above) in  $S_T$ . Then for all  $x \in \mathbb{R}^N$*

$$\lim_{t \rightarrow -\infty} u(x, t) = \inf_{S_T} u \quad \left( \lim_{t \rightarrow -\infty} u(x, t) = \sup_{S_T} u \right)$$

and the limit is uniform in any  $p$ -paraboloid  $P_m(y, s)$  ( $P_M(y, s)$  respectively).

**Proof.** Having fixed  $\varepsilon > 0$ , there exists  $(y_\varepsilon, s_\varepsilon) \in S_T$ , such that

$$u(x_\varepsilon, t_\varepsilon) - m = \frac{\varepsilon}{\gamma}$$

where  $\gamma$  is the constant in the intrinsic, backward Harnack inequality in (8). Applying such inequality to  $(u - m)$ , gives

$$m \leq u(y, s) \leq m + \varepsilon, \quad \text{for all } (y, s) \in P_m(y_\varepsilon, s_\varepsilon).$$

Now, for all fixed  $x \in \mathbb{R}^N$ , the half-line  $[t < T] \times \{x\}$  enters the  $p$ -paraboloid  $P_m(y_\varepsilon, s_\varepsilon)$  for some  $t$ .  $\square$

**Proof of Proposition 1.1.** May assume  $m = 0$ . The assumption implies

$$0 \leq u(y, s) \leq M_s < \infty \quad \text{for all } y \in \mathbb{R}^N.$$

By the backward, intrinsic Harnack inequality (8)

$$0 \leq u \leq \gamma M_s \quad \text{in } P_m(y, s) \text{ for all } y \in \mathbb{R}^N.$$

Hence  $0 \leq u \leq \gamma M_s$  in  $S_s$ , and by Theorem 1.1  $u$  is constant in  $S_s$ .  $\square$

**Proof of Proposition 1.2.** Assume  $m = 0$ , and  $\alpha > 0$ . There exists a sequence  $\{s_n\} \rightarrow \infty$ , such that for all arbitrary but fixed  $\varepsilon > 0$ , there exists  $n_\varepsilon \in \mathbb{N}$  such that

$$\alpha - \varepsilon < u(y, s_n) < \alpha + \varepsilon, \quad \text{for all } n \geq n_\varepsilon.$$

Fix  $s > s_{n_\varepsilon}$ , and define a sequence of radii  $\{\rho_n\}$ , such that

$$s_n - \left(\frac{c}{\alpha + \varepsilon}\right)^{p-2} \rho_n^p = s \quad \Rightarrow \quad \rho_n = \left[ (s_n - s) \left(\frac{\alpha + \varepsilon}{c}\right)^{p-2} \right]^{\frac{1}{p}}.$$

By the intrinsic, backward Harnack inequality in (8)

$$\sup_{B_{\rho_n}} u \left( \cdot, s_n - \left(\frac{c}{u(y, s_n)}\right)^{p-2} \rho_n^p \right) \leq \gamma u(y, s_n) \leq \gamma(\alpha + \varepsilon)$$

which we rewrite as

$$\sup_{B_{\rho_n}} u(\cdot, s) \leq \gamma(\alpha + \varepsilon).$$

Now let  $n \rightarrow \infty$  by keeping  $s > s_{n_\varepsilon}$  fixed. Then  $\rho_n \rightarrow \infty$  and the previous inequality implies

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s \leq \gamma(\alpha + \varepsilon).$$

The conclusion follows from Proposition 1.1, since  $s > s_{n_\varepsilon}$  is arbitrary.  $\square$

**Remark 3.1.** Assuming  $\alpha > 0$  for simplicity, the same argument continues to hold, if there exists a sequence  $\{(y_n, s_n)\} \subset \mathbb{R}^N \times \mathbb{R}$  and  $s \in \mathbb{R}$ , such that  $s_n \rightarrow +\infty$ ,

$$s_n - s = \left(\frac{c}{\alpha}\right)^{p-2} |y_n|^p,$$

and  $\lim_{n \rightarrow +\infty} u(y_n, s_n) = \alpha$ .

**Proof of Theorem 1.2.** It suffices to assume that  $u$  is non-negative and non-constant. Fix  $(x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$  such that  $u(x_0, t_0) > 0$ . By the Harnack inequality (11), for any  $\rho > 0$ ,

$$u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(\cdot, t_0).$$

Now let  $\rho \rightarrow +\infty$  and deduce that  $u(x, t_0) = 0$  for all  $x \in \mathbb{R}^N$ . The left-hand side, intrinsic Harnack inequality (9)–(10) now implies that  $u \equiv 0$ .  $\square$

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