

Optimal dynamics for quantum-state and entanglement transfer through homogeneous quantum systems

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The capability of faithfully transmit quantum states and entanglement through quantum channels is one of the key requirements for the development of quantum devices. Different solutions have been proposed to accomplish such a challenging task, which, however, require either an *ad hoc* engineering of the internal interactions of the physical system acting as the channel or specific initialization procedures. Here we show that optimal dynamics for efficient quantum-state and entanglement transfer can be attained in generic quantum systems with homogeneous interactions by tuning the coupling between the system and the two attached qubits. We devise a general procedure to determine the optimal coupling, and we explicitly implement it in the case of a channel consisting of a spin- $\frac{1}{2}$ XY chain. The quality of quantum-state and entanglement transfer is found to be very good and, remarkably, almost independent of the channel length.

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I. INTRODUCTION

One of the most commonly requested conditions in quantum communication and computation protocols is that two distant parties (Alice and Bob) share a couple of entangled qubits. When the physical objects encoding the qubits can travel, as in the case of photons, the goal can be accomplished by creating the entangled couple in a limited region of space and then letting the qubits fly where necessary. On the other hand, when qubits are realized via intrinsically localized physical objects, as in the case of $S = \frac{1}{2}$ spins or atomic systems, different strategies must be adopted (see, e.g., Refs. [1,2], and references therein). One such strategy is the following: First, two neighboring qubits (A and A') are prepared in an entangled state by means of a short-range interaction; then, the mixed state of one of the two qubits (say A) is transferred to a third distant qubit via a quantum channel. If the state transfer is perfect, the procedure results in a pair of distant entangled qubits A' and B , as requested.

For this strategy to make sense, one has to equip oneself with a quantum channel capable of transferring mixed states. Different proposals have been put forward [2–9] in order to obtain such a channel, some based on the idea of specifically engineering the internal interactions of the channel, and others on that of intervening in the initialization process by preparing the channel in a suitable configuration. In both cases, a severe external action on the physical system constituting the quantum channel, hereafter referred to as the *wire*, is required.

Here a different point of view is adopted: The purpose is that of devising conditions for “optimal dynamics,” i.e., a time evolution of the state of the wire such that the mixed state of A is transferred with high fidelity to a distant party B at some given later time. In particular, we define a procedure for maximizing the quality of the transmission process by acting only on the coupling between the qubits and the wire, which is supposed to be homogeneous.

In what follows, the main guideline is the fact that excitations characterized by a linear dispersion play a crucial role in determining effective transfer of quantum states. Since

this idea has been put forward [4,10], it had to face the unfortunate evidence that a quantum channel with a linear dispersion relation over the whole Brillouin zone is not only difficult to design [5] but perhaps just a chimera to realize, so far. On the other hand, most physical systems are characterized by excitations whose dispersion relation has zones of linearity, typically near inflection points: If one were able to induce a dynamical evolution of the wire essentially ruled by excitations belonging to such a zone, a coherent propagation should result, leading to an effective transfer of quantum states.

A schema for quantum information transfer based on the use of a uniform chain with different end-point interactions was adopted in earlier papers (see, e.g., Refs. [7,11]), where it was shown that when the end-bond strength is considerably reduced, a high quality quantum-state transfer is possible, but it requires very long transmission times. On the contrary, resorting to the quite different mechanism of coherent wave-packet propagation, we find the optimal end-bond strength to be comparable with the intrawire interaction, so that a quasiperfect quantum-state transfer is obtained in a much shorter time, set by the group velocity of the relevant excitations; moreover, the quality of the quantum-state and entanglement transfer is not substantially affected by the wire length.

In Sec. II the overall transmission scheme is depicted and a general procedure is introduced for determining the end-point interaction parameters in such a way as to induce the optimal dynamical evolution of the channel. The procedure is implemented for a possible physical realization of the wire in Sec. III, where several results for the quantities characterizing quantum-state and entanglement transmission are reported for a channel modeled by a spin- $\frac{1}{2}$ XY chain. Finally, some conclusions and perspectives are outlined in Sec. IV.

II. OPTIMIZED DYNAMICAL SETTINGS

The general setup we consider for quantum-state transfer is illustrated in Fig. 1. Each one of two separated parties, Alice

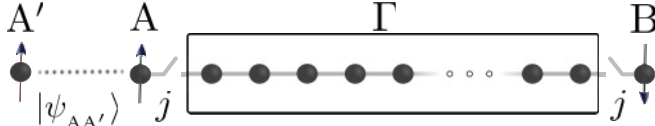


FIG. 1. The end points of a quantum channel Γ are coupled to the qubits A and B , via an interaction j ; A can be entangled with an external qubit A' .

and Bob, owns a qubit, A and B , respectively. Each qubit may be subjected to a local interaction, $h_Q \mathcal{H}_A$ and $h_Q \mathcal{H}_B$, respectively, with h_Q possibly tunable. The wire constituting the quantum channel is realized by an extended physical system Γ , which is assumed to be made of N interacting particles on a discrete lattice. The internal dynamics of the wire is ruled by the Hamiltonian \mathcal{H}_Γ , while the end points of the wire can be put in contact with the qubit A (B) via an interaction $j\mathcal{H}_{A\Gamma}$ ($j\mathcal{H}_{B\Gamma}$) that can be fixed to a proper value. The overall system has mirror symmetry and the Hamiltonian which rules its dynamics is $\mathcal{H} = \mathcal{H}_\Gamma + j(\mathcal{H}_{A\Gamma} + \mathcal{H}_{B\Gamma}) + h_Q(\mathcal{H}_A + \mathcal{H}_B)$. Before the process starts, A , B , and Γ do not interact with each other; the wire is in its ground state $|\Omega_\Gamma\rangle$, while Alice and Bob prepare their qubits in the initial states $\rho_A(0)$ and $\rho_B(0)$, respectively. At time $t = 0$ the interaction between each qubit and the wire is switched on.

We assume that (i) \mathcal{H} can be written (either exactly or via a reasonable approximation) as a quadratic form of $N + 2$ local, either fermionic or bosonic, operators $\{\eta_i, \eta_i^\dagger\}$, where the index $i = 0, \dots, N + 1$ labels the sites of the lattice, and (ii) the bulk of the wire is translation symmetric. Condition (i) guarantees the existence of a unitary transformation $\{\eta_i, \eta_i^\dagger\} \rightarrow \{\eta_k, \eta_k^\dagger\}$, which diagonalizes the total Hamiltonian, $\mathcal{H} = \sum_k \omega_k \eta_k^\dagger \eta_k + E_0$, E_0 being the ground-state energy. Condition (ii) entails that this transformation is close to a Fourier transform when i corresponds to sites in the bulk of the wire, so that k can be approximately considered as a quasimomentum. For the sake of clarity, let us consider the qubits A and B as initially prepared in the pure states $|\alpha\rangle$ and $|\beta\rangle$, respectively; the overall system at time $t = 0$ is then described by $|\Psi_0\rangle = |\alpha\rangle|\Omega_\Gamma\rangle|\beta\rangle$, which generally is a nonequilibrium state. Its dynamics is ruled by quasiparticle excitations with density in k space $n(k) = \langle \Psi_0 | \eta_k^\dagger \eta_k | \Psi_0 \rangle$, which evolve according to \mathcal{H} . If the dispersion relation were linear, $\omega_k \simeq \omega_0 + vk$, the bulk dynamics would consist in a coherent wave packet traveling from A to B (and vice versa) at velocity v along the wire. Under the further assumption of mirror symmetry, such a wave packet would perfectly rebuild the initial state of A on the qubit B , after a time $t \simeq N/v$, and again after regular time intervals $\simeq 2N/v$.

Despite dispersion relations of real interacting systems being typically nonlinear (see, e.g., Fig. 2), they can exhibit an approximately linear region around an inflection point k_0 . By varying the local parameters h_Q and j one can change the position of the peak of $n(k)$ and its variance Δ^2 . By setting the peak position at k_0 , the dynamically relevant excitations are those whose frequency can be expanded as

$$\omega_k \simeq \omega_{k_0} + v(k - k_0) + \frac{2}{3}a(k - k_0)^3, \quad (1)$$

where the main dispersive term is retained. Under these conditions, as shown explicitly for Gaussian packets in

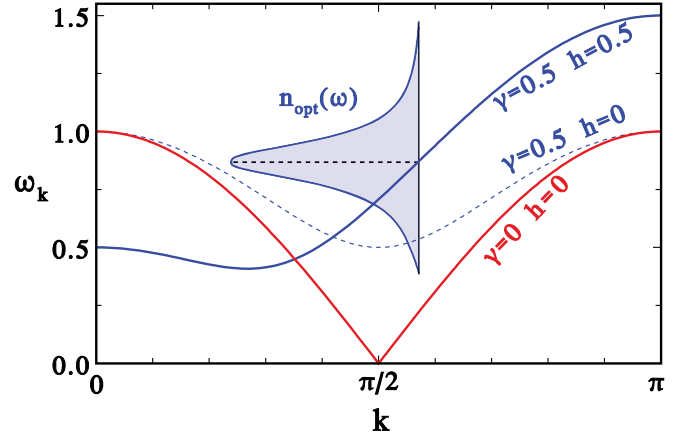


FIG. 2. (Color online) Dispersion relation of the spin- $\frac{1}{2}$ XY chain for different interaction parameters. The bell-shaped curve edging the shaded area, $n_{\text{opt}}(\omega)$, is the optimal density of excitations vs ω , as obtained for $\gamma = \gamma_{0,N} = 0.5$ and $h = 0.5$.

Refs. [4,12], the dynamics of the wire can still be described by a traveling spatial wave packet centered at the position $\simeq (v + 2a\Delta^2)t$, with a variance that increases according to $\sigma^2(t) \simeq \sigma^2 + 8a^2\Delta^4 t^2$, where σ^2 is the initial variance.

Since $n(k)$ and the spatial wave packet are related by a Fourier transformation, we cannot choose σ^2 and Δ^2 independently, as they must fulfill the condition $4\sigma^2\Delta^2 \gtrsim 1$. Therefore, taking into account such a constraint (the equality holding for Gaussian wave packets), we can state that the optimal dynamics is obtained by further choosing σ^2 so as to minimize $\sigma(t)$ at the arrival time $t_N \simeq N/v$, leading to

$$\sigma_{\text{opt}} = \sqrt[3]{|a|t} \simeq (|a|N/v)^{1/3}. \quad (2)$$

Based on this reasoning, we define the following procedure for inducing optimal dynamics: Choose a wire whose Hamiltonian parameters grant the existence of a wide enough interval in k with almost linear dispersion, as described by Eq. (1) (should this be unfeasible, the wire is unlikely to behave as a good quantum channel); tune h_Q so as to peak $n(k)$ at an inflection point of ω_k ; then set σ to its optimal value, Eq. (2), by varying j . The value of the coupling corresponding to σ_{opt} will hereafter be referred to as the *optimal coupling*, j_{opt} . It is of absolute relevance that $\sigma(t_N) \simeq \sqrt{3/2}\sigma_{\text{opt}}$; that is, the final optimal packet is just about 22 % wider than at the start, irrespective of the wire length, which suggests that the quality of the transfer can also be independent of N . It is worth mentioning that the optimization of σ can also be obtained by creating a finite-size excitation in space as the starting state [4,6,13,14]; however, this strategy results in a much less versatile scheme.

III. OPTIMAL TRANSFER IN THE XY CHAIN

Let us now describe a specific implementation of the procedure just described. We consider a system defined on a one-dimensional discrete lattice of $N + 2$ sites, labeled by the index $i = 0, 1, \dots, N + 1$. The qubits A and B sit at sites 0 and $N + 1$, respectively. The wire is physically realized by a chain of interacting $S = \frac{1}{2}$ spins, taking up sites from 1 to N : Neighboring spins interact via a homogeneous XY exchange coupling (which sets the energy scale) and are

possibly subjected to a uniform external magnetic field. The total Hamiltonian is

$$\begin{aligned} \mathcal{H} = & - \sum_{i=1}^{N-1} [(1+\gamma)S_i^x S_{i+1}^x + (1-\gamma)S_i^y S_{i+1}^y] - h \sum_{i=1}^N S_i^z \\ & - j \sum_{i=0,N} [(1+\gamma_0)S_i^x S_{i+1}^x + (1-\gamma_0)S_i^y S_{i+1}^y] \\ & - h_Q(S_0^z + S_{N+1}^z). \end{aligned} \quad (3)$$

For vanishing anisotropies, $\gamma = \gamma_0 = 0$, one has the so-called XX model. A Jordan-Wigner transformation casts \mathcal{H} into a quadratic form of $N + 2$ fermionic operators $\{\eta_i, \eta_i^\dagger\}$ defined on the lattice sites, and a Bogoliubov transformation diagonalizes it: $\mathcal{H} = \sum_k \omega_k \eta_k^\dagger \eta_k + E_0$, where E_0 is the ground-state energy [15].

The optimization procedure begins with the analysis of the dispersion relation in the infinite chain limit, $\omega_k = [(h - \cos k)^2 + \gamma^2 \sin^2 k]^{1/2}$, displayed in Fig. 2. One can easily spot the existence of regions of linearity in the neighborhood of the inflection point(s) k_0 , where Eq. (1) holds. By tuning $h_Q \simeq \omega_{k_0}$, the peak of $n(k)$ is made to sit at k_0 . The coupling j is then tuned to the optimal value j_{opt} by looking at the numerically determined excitation density $n(k)$, in such a way that its variance $1/(4\sigma^2)$ fulfills Eq. (2). The region of linear dispersion shrinks by switching on the anisotropy γ , but it can be extended again by increasing h . In addition, in the Ising limit ($\gamma = 1$), for $h = 0$ one has $\omega_k = 1$, which does not allow for propagation; this explains the result in Ref. [16], where it is observed that in such a limit no entanglement propagation takes place over the chain. However, applying a finite h on Γ fixes the problem by inducing a finite group velocity. Therefore, one can act on the field so as to fulfill the conditions for optimal dynamics.

As a first example of the optimization procedure, we consider the XX model with $h = 0$, as this choice makes more analytical expressions available, which in turn allows for a more detailed analysis. The inflection point $k_0 = \pi/2$ corresponds to $\omega_{k_0} = 0$ and we hence set $h_Q = 0$. Figure 3 shows the time evolution of the magnetization parallel to the quantization axis all along the wire, $\langle S_i^z(t) \rangle, i = 0, 1, \dots, N, N + 1$, for $N = 50$ and initial states of the qubits $|\alpha\rangle = |\beta\rangle = |\uparrow\rangle$. The upper panel corresponds to a generic value of the coupling, $j = 1$, while in the lower one, $j = j_{\text{opt}} = 0.58$, a value determined from analytical expressions holding for the XX model [11], which also show $j_{\text{opt}} \propto N^{-1/6}$ at leading order. The difference is striking: Indeed, the induced nondispersive wave-packet propagation makes $\langle S_B^z(t) \rangle$, at the arrival time $t \simeq N$, an almost perfect reproduction of the initial magnetization $\langle S_A^z(0) \rangle$ of the qubit A . These results also confirm our recipe for determining j_{opt} .

But does this peculiar dynamical evolution also affect the quality of quantum-state transfer? To answer this question, we now deal with the time evolution of quantities which are used to monitor such quality. In particular, we analyze the quantum-state transfer process in terms of the fidelity between an initial state $|\alpha\rangle$ of A and the evolved state $\rho_B(t)$ of B , $\mathcal{F}_{AB}(t) = \langle \alpha | \rho_B(t) | \alpha \rangle$, where initially $\rho_B(0) = |\uparrow\rangle\langle\uparrow|$. As for the entanglement, we refer to the time evolution of the concu-

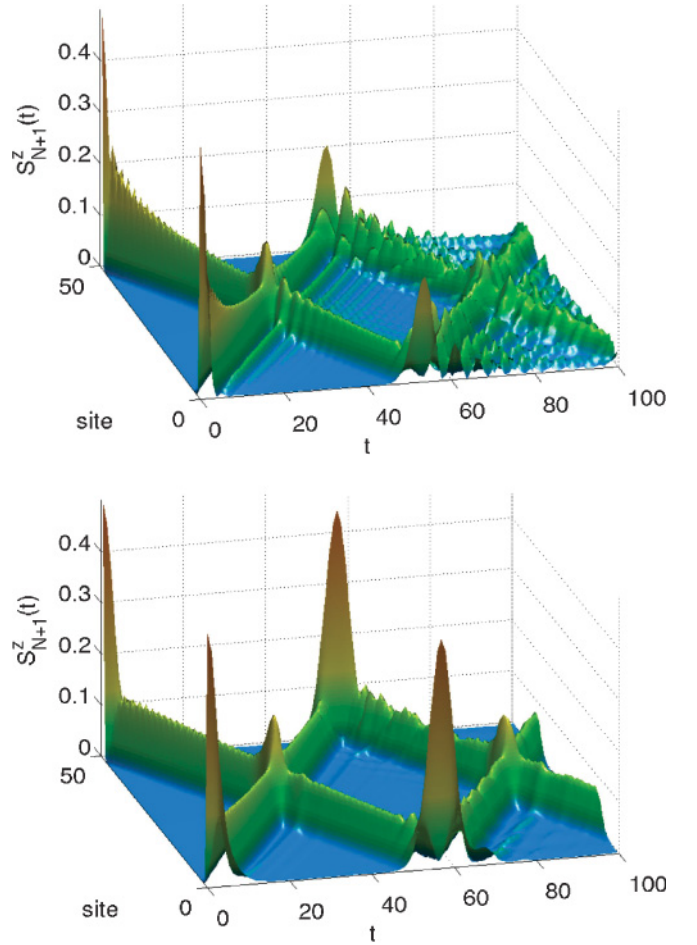


FIG. 3. (Color online) Time evolution of the on-site z magnetization for $N = 50$ and an initial state $|\uparrow\rangle|\Omega_F\rangle|\uparrow\rangle$; the qubit-wire coupling is $j = 1$ (top) and $j = j_{\text{opt}} = 0.58$ (bottom).

rence between A' and B , $\mathcal{C}_{BA'}(t) \equiv \mathcal{C}(\rho_{BA'}(t))$ [17]. In the case of the XX model we can also evaluate the *minimum* fidelity over any possible $|\alpha\rangle$, hereafter indicated by $\mathcal{F}_{AB}^{\min}(t)$: The quality of the entanglement transmission from AA' to BA' can then be checked by the known lower bound [18,19] for the fidelity of entanglement, $\mathcal{F}(|\psi_{AA'}\rangle\langle\psi_{AA'}|, \rho_{BA'}(t)) \geq \frac{3}{2}\mathcal{F}_{AB}^{\min}(t) - \frac{1}{2}$. The entangled state $|\psi_{AA'}\rangle$ can be any Bell state, since a local operation on A' does not change the concurrence dynamics.

All these quantities essentially depend on the components of the magnetization of B , $\langle \Psi_0 | S_{N+1}^\alpha(t) | \Psi_0 \rangle$. These can be obtained by writing S_{N+1}^α in terms of the operators $\{\eta_k, \eta_k^\dagger\}$, whose dynamics simply follows from $\mathcal{H} = \sum_k \omega_k \eta_k^\dagger \eta_k$. Inverse Bogoliubov and Jordan-Wigner transformations, and a further expansion of $S_{N+1}^\alpha(t)$ in terms of the spin operators of the extremal qubits and of the fermionic operators defined by the diagonalization of the sole \mathcal{H}_Γ , eventually yields the required expectation values. In this way we devise a numerical procedure for determining, for any model described by the Hamiltonian (3), the Kraus operators [18] $M_\mu(t)$, in terms of which we get $\rho_B(t) = \sum_\mu M_\mu(t) \rho_A(0) M_\mu^\dagger(t)$ and $\rho_{BA'}(t) = \sum_\mu [M_\mu(t) \otimes 1] \rho_{AA'}(0) [M_\mu^\dagger(t) \otimes 1]$, i.e., the necessary tools for our analysis.

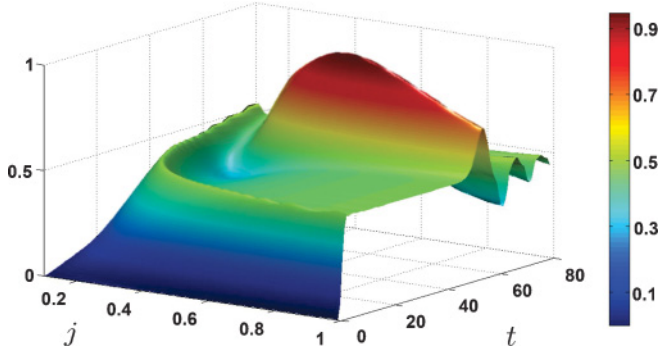


FIG. 4. (Color online) Time evolution of the minimum fidelity $\mathcal{F}_{AB}^{\min}(t)$ vs j , for $N=50$, $\gamma=\gamma_0=\gamma_N=0$ (XX chain), and $h=h_Q=0$.

We first consider the fidelity $\mathcal{F}_{AB}(t)$ as just defined. In the case of the XX chain, we put ourselves in the worst possible case and evaluate $\mathcal{F}_{AB}^{\min}(t)$: A high value of such a quantity ensures a very good transfer of *any* initial state, modulo a local operation. In Fig. 4 we show $\mathcal{F}_{AB}^{\min}(t)$ for the same model of Fig. 3 as a function of the coupling j . For the same value $j_{\text{opt}} \simeq 0.58$ of the lower panel of Fig. 3, the minimum fidelity reaches its maximum value simultaneously with that of $\langle S_B^z(t) \rangle$, i.e., at the expected arrival time $t \simeq N$, as can be analytically proven in the XX case; the peak value of the minimum fidelity is just slightly below unity, an outcome that is also confirmed for N as large as 500. We underline that the aforementioned lower bound also implies an optimal transmission of entanglement.

Let us now consider the entanglement transfer in the more general XY model. By setting $\gamma=\gamma_0=0.5$ and $h=0.5$ the inflection point of ω_k is found at $k_0 \simeq 1.795$. Finite-size and boundary effects require a fine tuning of $h_Q \simeq 0.85$, slightly

different from ω_{k_0} . Finally, from Eq. (2) we numerically determine $j_{\text{opt}} = 0.49, 0.39$, and 0.34 , for $N = 50, 250$, and 500 , respectively. Despite the XY case being complicated by the presence of several energy scales, a very good state transfer is also obtained for large N , as testified by the fidelity, averaged over all possible initial states of A , $\bar{\mathcal{F}}_{AB}$, shown in Fig. 5(a). The precession induced by the magnetic field h_Q on the qubit B yields oscillations of $\bar{\mathcal{F}}_{AB}$, but it does not affect the entanglement transfer. The entanglement between B and A' is monitored by the concurrence $\mathcal{C}_{BA'}(t)$, shown in Fig. 5(b) for different values of N : For each N , this is characterized by a well-defined peak of height > 0.8 , even for chains as long as 500 sites. This peak occurs simultaneously with the maximum of $\bar{\mathcal{F}}_{AB}$ and has a finite but small width. The first echo (shown in Fig. 5 only for $N=50$) is seen to be considerably intense and localized, attesting to a long-lasting quasi-nondispersive dynamics of the wire.

IV. CONCLUSIONS

Based on a general picture of the dynamical evolution of quantum wave packets, we have devised a procedure for inducing optimal dynamics through a homogeneous quantum wire by tuning its coupling with the two end-point qubits. Indications about the best setting of other, possibly tunable, parameters of the system have been given. For the procedure to apply, a few very simple conditions must be fulfilled, and there is no need for a specific design of the wire, or of its initial state. By implementing the approach with the spin- $\frac{1}{2}$ XY chain, extremely good quantum-state and entanglement transfer is obtained, which is close to perfect, and, in particular, quite a bit higher than the “classical” ones, as required by quantum communication protocols. The transfer time scale is considerably shorter than in previous works concerning quantum-state transmission over Heisenberg models [2,11,20–22]. Moreover, the quality of the state and entanglement transfer that we obtain only weakly deteriorates as the length of the wire increases.

In the case of the XY model the excitations are noninteracting, which guarantees their independent propagation along the chain as well as their infinite lifetime. In the case of a nonlinear wire the quasiparticle interactions can possibly affect the quantum-state transfer over long distances. Indeed, recent results for the XXZ model [16] show that the quality of quantum-state transfer decreases with the chain length: This can be due both to the lack of optimization of the end-point interaction and to the nonlinearity of the wire. This issue deserves further investigation and is currently under study.

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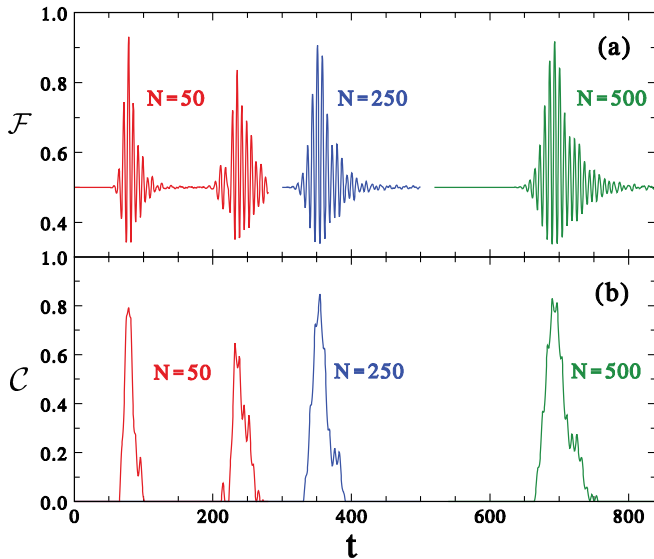


FIG. 5. (Color online) Time evolution of $\bar{\mathcal{F}}_{AB}$ (upper panel) and $\mathcal{C}_{BA'}(t)$ (lower panel) for different channel lengths; $\gamma=\gamma_0=0.5$, $h=0.5$, $h_Q=0.85$, and $j_{\text{opt}}=0.49, 0.39$, and 0.34 for $N=50, 250$, and 500 , respectively. The second signal observed in both panels is the first echo for $N=50$.

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