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SOME PROPERTIES OF THE SOLUTION SET
FOR INTEGRAL DIFFERENTIAL EQUATIONS

1 Introduction and Notations

In this paper we are concerned with the solution sets for Volterra integral equation and integrodifferential equations like:

$$\begin{cases} x(t) &= h(t) + \int_0^t k(t, s)g(s, x(s))ds \\ x(0) &= x_0, \end{cases} \quad (1)$$

or

$$\begin{cases} x(t) &= f(t, x(t), \int_0^t k(t, s)g(s, x(s))ds) \\ x(0) &= x_0, \end{cases} \quad (2)$$

where $h : I = [0, T) \rightarrow \mathbb{R}^n$, $k : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous functions, x_0 is a given vector of \mathbb{R}^n , I a (possible unbounded) interval of \mathbb{R} .

In the following $B(x_0, r)$ will denote an r -ball (in the metric space (X, d)) i.e. the set $\{x \in X : d(x, x_0) < r\}$ where x_0 is any point in X ; $\bar{B}(0, r)$ will denote the closed ball centered in $x_0 = 0$.

Let now consider the (Hilbert) space $L^2(I, \mathbb{R}^n)$ normed, as usually, by $\|x\|_2 = (\int_I x^2(t)dt)^{\frac{1}{2}}$ and its (affine) subspace $E = \{x \in L^2(I, \mathbb{R}^n) : x(0) = x_0\}$. Let X be some Banach space; if $V \subset X$ is some subset then \bar{V} will denote its (topological) closure and V^c will denote the complement of V . Finally $\mathcal{B}(X)$ will denote the set of all nonempty and bounded subsets of X .

Definition 1 : Let X be a Banach space and $A \subset X$ a subset. A measure $\mu : \mathcal{B}(X) \rightarrow \mathbb{R}^+$ defined by $\mu(V) = \inf\{\epsilon > 0 : V \in \mathcal{B}(X) \text{ admits a finite cover by sets of diameter } \leq \epsilon\}$ where diameter of V is the $\sup\{\|x - y\| : x \in V, y \in V\}$, is called the (Kuratowski) measure of noncompactness.

A measure like μ has interesting properties, some of which are listed in the sequel:

a) $\mu(V) = 0$ if and only if \bar{V} is compact;

b) $\mu(V) = \mu(\bar{V})$; $\mu(\text{conv}(V)) = \mu(V)$; ($\text{conv}(V) = \text{convex hull of } V$);

c) $\mu(\alpha(V_1) + (1 - \alpha)V_2) \leq \alpha\mu(V_1) + (1 - \alpha)\mu(V_2)$, $\alpha \in [0, 1]$;

d) if $V_1 \subset V_2$ then $\mu(V_1) \leq \mu(V_2)$;

e) if $\{V_n\}$ is a nested sequence of closed sets of $B_d(X)$

and if $\lim_{n \rightarrow +\infty} \mu(V_n) = 0$ then $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$.

The analogous measure of noncompactness for an operator is defined by $\mu(F(V)) = \inf\{k > 0 : \mu(F(V)) \leq k\mu(V)\}$ for all bounded subsets $V \subset X$.

When X is a complete metric space and $f : X \rightarrow X$ is a continuous mapping f is called an μ -set contraction if there exists $k \in [0, 1)$ such that, for all bounded noncompact subsets V of X , the following relation holds: $\mu(f(V)) \leq k\mu(V)$ ([?], pag 160).

A continuous operator $F : X \rightarrow X$ such that $\mu(F(V)) < \mu(V)$, for any bounded $V \subset X$, is called *condensing* or *densifying*.

(The concept of measure of noncompactness is considerably dealt with in the references [?], [?] or [?].)

Let S and S_1 be topological spaces and let $f : S \rightarrow S_1$. Then f is said to be *proper* if, whenever K_1 is a compact subset of S_1 , $f^{-1}(K_1)$ is a compact set in S . It is also known ([?], pag 160) that if X is a Banach space and $f : X \rightarrow X$ is a continuous k -set contraction, then $I - f$ is a proper mapping.

The following result, due to R.K. Juberg ([?]), will be useful in the proof of our main result:

Proposition 1 : *Let (a, b) be any real (possibly unbounded) interval and let $L^p(a, c)$, $1 \leq p \leq +\infty$ be the Lebesgue's space of (the power p) summable*

functions over (a, c) for every $c \in (a, b)$. For $u \in L^p(c, b)$, $v \in L^q(a, c)$, where $\frac{1}{p} + \frac{1}{q} = 1$, we set

$$\rho = \limsup_{\epsilon \rightarrow 0} \left\{ \left[\int_x^{a+\epsilon} |u(y)|^p dy \right]^{\frac{1}{p}} \left[\int_a^x |v(y)|^q dy \right]^{\frac{1}{q}}, a < x \leq a + \epsilon \right\} + \\ + \limsup_{\delta \rightarrow 0} \left\{ \left[\int_x^b |u(y)|^p dy \right]^{\frac{1}{p}} \left[\int_{b-\delta}^x |v(y)|^q dy \right]^{\frac{1}{q}}, b - \delta \leq x < b \right\}.$$

Let D be the linear operator defined by: $D(f(y))(x) = \int_0^x u(x)v(y)f(y)dy$; in the sequel wh shall assume that D is a bounded operator in the space $L^p(0, T)$. We want to recall that the operator D is bounded (in the $L^p(a, b)$ space) if and only if the function

$\psi(x) = \left[\int_x^b |u(y)|^p dy \right]^{\frac{1}{p}} \left[\int_a^x |v(y)|^q dy \right]^{\frac{1}{q}}$ is bounded on (a, b) . This operator is not necessarily a compact operator; as matter of fact it is well known (see [?], for instance), that D is a compact operator if the functions $u(\cdot)$ e $v(\cdot)$ belongs to $L^2(a, b)$.

Furthermore the measure of noncompactness of D , i.e. $\mu(D)$ satisfies $(\frac{1}{2})^{1+\frac{1}{p}} \leq \mu(D) \leq p^{\frac{1}{q}} q^{\frac{1}{p}} \rho$; in the special case when $p = q = 2$, i.e. when the (Lebesgue) space L^p is a Hilbert space L^2 , we obtain $\rho \sqrt{\frac{1}{8}} \leq \mu(D) \leq 2\rho$.

Definition 2 : An R_δ -set is the intersection of a decreasing sequence $\{A_n\}$ of compact AR (metric absolute retracts; see [?] or [?], for a reference.) Moreover it is known (see [?] for instance) that an R_δ -set is an acyclic set in the Čech homology.

The following result also will be crucially used in teh sequel:

Proposition 2 : ([?], pag 159). Let X be a space and let $Y, \|\cdot\|$ be a Banach space and $f : X \rightarrow Y$ be a proper mapping. Assume further that for each $\epsilon_n > 0$, $n > 0 \in \mathbb{N}$ a proper mapping $f_n : X \rightarrow Y$ is given and the couple of conditions is satisfied:

- $\|f_n(x) - f(x)\| < \epsilon_n, \forall x \in X$;
- for any $\epsilon_n > 0$ and $y \in Y$ such that $\|y\| \leq \epsilon_n$, the equation $f_{\epsilon_n}(x) = y$ has exactly one solution.

Then the set $S = f^{-1}(0)$ is an R_δ -set.

Remark: a sequence f_{ϵ_n} is called an ϵ_n approximation (of the function f).

Proposition 3 : ([?], pag ???). Let $F, F_n : \overline{B}(0, r) \rightarrow Y$ be condensing operators such that

- $\delta_n = \sup\{ \|F_n(x) - F(x)\|, x \in \overline{B}(0, r)\} \rightarrow 0$, as $n \rightarrow +\infty$;
- the equation $x = F_n(x) + y$ has at most one solution if $\|y\| \leq \delta_n$.

Then the set of fixed points of F is an R_δ -set.

Main result

We are ready to establish out (main) existence result for the (initial value problems for) integral equations of the type here introduced.

First of all let $F : B(0, r) \rightarrow E$ be defined as follows:

$$F(y) = h(t) + \int_0^t k(t, s)g(s, y(s))ds$$

where r is a real number (suitably defined below) and put $m_0 = \|F(0)\|_2$.

Theorem 1 : Let ρ the number defined in Proposition 1; then we assume that:

1. *i)* there are functions $\alpha, \phi, : I \rightarrow \mathbb{R}^n$ belonging to $L^2(I)$ such that $k(t, s) = \alpha\phi(s)$ for every $(t, s) \in I \times I$; moreover we assume that $\|k\|_2 < 2\rho$;
2. *ii)* $\|g(t, x)\| \leq \frac{1}{2\rho}\|x\| + b(t)$, for $(t, x) \in I \times \mathbb{R}^n$, $b \in L^2(I)$, $b(t) \geq 0$;
3. *iii)* there is a ball $B(0, r)$ such that $r > \frac{2m_0\rho}{2\rho - \|k\|_2}$.

Then the set of solution of the integral problem (??) is an R_δ -set.

Remark: The first part of the assumption *i)* is satisfied in many cases: for instance when $k(t, s)$ is a Green function; see, for instance, [?] for similar cases.

Proof: Clearly the above operator F is a single value mapping and a possible fixed point of F is a solution of the integral problem (??).

In order to prove the theorem the following steps in the proof have to be established:

- a) F has a closed graph;
- b) F is a condensing mapping;
- c) The set of fixed point of F is R_δ -set.

Proof of Step a): in fact, let $y_n \rightarrow y_0$ and put $G(y)(t) = g(t, y(t))$. Now, from assumption *ii*), it follows that the superposition operator G mapping the space L^2 into L^2 is condensing (see [?]); thus we have $\lim_n \|G(y_n) - G(y_0)\|_2 = 0$. By using the Holder inequality, we get:

$$\begin{aligned} \|F(y_n) - F(y_0)\|_2 &= [\int_I |F(y_n)(s) - F(y_0)(s)|^2 ds]^{\frac{1}{2}} = \\ &= [\int_I [\int_0^t (k(t, s)g(s, y_n(s)) - k(t, s)g(s, y_0(s))) ds]^2 dt]^{\frac{1}{2}} \leq \|k\|_2 \| \|G(y_n) - G(y_0)\|_2 \end{aligned}$$

and this quantity is going to zero whenever $n \rightarrow +\infty$.

Proof of Step b): Always working from $B(0, r)$ into E , we have $F(y) = (H \circ G)(y)$, where

$$H(y)(t) = \int_0^t \phi(s)\alpha(t)y(s)ds + h(t).$$

Now, by assumptions *i*) and *ii*), we have (see [?]) $\mu(G(V)) \leq \frac{1}{2\rho}\mu(V)$, for any bounded set $V \subset L^2(I \times \mathbb{R}^n)$ and also $\mu(H) < 2\rho$; so (see [?]) $\mu(F) = \mu(H \circ G)(y) \leq \mu(H)\mu(G) < 1$.

Proof of Step c): Finally we have to prove that the set of fixed points of the operator F is an R_δ -set (in the sequel we assume that $(a, b) = (0, T)$.)

Let us consider the mappings $F_n : L^2(0, T) \rightarrow L^2(0, T)$ defined as:

$$F_n(x)(t) = \begin{cases} h(t) & = \text{if } 0 \leq t \leq \frac{T}{n}; \\ h(t) + \int_0^{t-\frac{T}{n}} \phi(s)\alpha(s)g(s, y(s))ds & = \text{if } \frac{T}{n} \leq t \leq T. \end{cases} \quad (3)$$

The mappings F_n are continuous mappings; by assumption *i*) and *ii*) we have that they are also condensing. The intervals $[0, \frac{T}{n}]$, $[\frac{T}{n}, \frac{2T}{n}]$, \dots , $[\frac{kT}{n}, \frac{(k+1)T}{n}]$, \dots , $[\frac{(n-1)T}{n}, T]$ are now coming in one after the other: each time the mappings F_n are bijective and their inverses F_n^{-1} are continuous. Moreover we have $\|F_n - F\|_2 \rightarrow 0$ as $n \rightarrow +\infty$. The latter fact allows us to say that the mappings $I - F_n$ and $I - F$ are proper maps. Finally we can conclude that the set of fixed points of F is an R_δ -set.

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