# Some properties of the solution set for integral differential equations

SOME PROPERTIES OF THE SOLUTION SET
FOR INTEGRAL DIFFERENTIAL EQUATIONS

# 1 Introduction and Notations

In this paper we are concerned with the solution sets for Volterra integral equation and integrodifferential equations like:

$$\begin{cases} x(t) &= h(t) + \int_0^t k(t,s)g(s,x(s)ds \\ x(0) &= x_0, \end{cases} \tag{1}$$

or

$$\begin{cases} x(t) &= f(t, x(t), \int_0^t k(t,s)g(s,x(s))ds) \\ x(0) &= x_0, \end{cases} \tag{2}$$

where $h : I = [0,T) \longrightarrow \mathbb{R}^n$, $k : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous functions, $x_0$ is a given vector of $\mathbb{R}^n$, $I$ a (possible unbounded) interval of $\mathbb{R}$.

In the following $B(x_0, r)$ will denote an $r-$ ball (in the metric space $(X,d)$) i.e. the set $\{x \in X : d(x,x_0) < r\}$ where $x_0$ is any point in $X$; $B(0,r)$ wil denote the closed ball centered in $x_0 = 0$.

Let now consider the (Hilbert) space $L^2(I, \mathbb{R}^n)$ normed, as usually, by $||x||_2 = (\int_I x^2(t)dt)^{\frac{1}{2}}$ and its (affine) subspace $E = \{x \in L^2(I,\mathbb{R}^n) : x(0) = x_0\}$. Let $X$ be some some Banach space; f $V \subset X$ is some subset then $\bar{(}V)$ will denote its (topological) closure and $V^c$ will denote the complement of $V$. Finally $\mathcal{B(X)}$ will denote the set of all nonempty and bounded subsets of $X$.

**Definition 1** *:* Let $X$ be a Banach space and $A \subset$ a subset. A measure $\mu : B_d(X) \longrightarrow \mathbb{R}^+$ defined by $\mu(V) = \inf\{\epsilon > 0 : V \in \mathcal{B(X)}$ admits a finite cover by sets of diameter $\leq \epsilon\}$ where diameter of $V$ is the $\sup\{||x-y|| : x \in V, y \in V\}$, is called the (Kuratowski) measure of noncompactness.

A measure like $\mu$ has interesting properties, some of which are listed in the sequel:

$$a) \quad \mu(V) = 0 \text{if and only if } \overline{V} \text{ is compact;}$$

$$b) \quad \mu(V) = \mu(\overline{V}); \quad \mu(conv(V)) = \mu(V); (conv(V) = \text{ convex hull of} V);$$

$$c) \quad \mu(\alpha(V_1) + (1-\alpha)V_2) \leq \alpha\mu(V_1) + (1-\alpha)\mu(V_2), \quad \alpha \in [0,1];$$

$$d) \quad \text{if} V_1 \subset V_2 \quad \text{then} \quad \mu(V_1) \leq \mu(V_2);$$

$$e) \quad \text{if} \{V_n\} \quad \text{is a nested sequence of closed sets of } B_d(X)$$
$$\text{and if } \lim_{n\to+\infty} \mu(V_n) = 0 \quad \text{then} \quad \cap_{n=1}^{\infty} V_n \neq \emptyset.$$

The analogous measure of noncompactness for an operator is defined by $\mu(F(V)) = \inf\{k > 0 : \mu(F(V)) \leq k\mu(V)\}$ for all bounded subsets $V \subset X$.

When $X$ is a complete metric space and $f : X \longrightarrow X$ is a continuous mapping $f$ is called an
$mu-$set contraction if there exists $k \in [0,1)$ such that, for all bounded non-compact subsets $V$ of $X$, the following relation holds: $\mu(f(V)) \leq k\alpha(V)$ ( [?], pag 160).

A continuous operator $F : X \longrightarrow X$ such that $\mu(F(V)) < \mu(V)$, for any bounded $V \subset X$, is called *condensing* or *densifying*.

(The concept of measure of noncompactness is considerably dealed with in the references [?], [?] or [?].)

Let $S$ and $S_1$ be topological spaces and let $f : S \longrightarrow S_1$. Then $f$ is said to be *proper* if, whenever $K_1$ is a compact subset of $S_1$, $f^{-1}(K_1)$ is a compact set in $S$. It is also known ( [?], pag 160) that if $X$ is a Banach space and $f : X \longrightarrow X$ is a continuous $k-$set contraction, then $I - f$ is a proper mapping.

The following result, due to R.K. Juberg ([?]), will be useful in the proof of our main result:

**Proposition 1** *: Let $(a,b)$ be any real (possible unbounded) interval and let $L^p(a,c)$, $1 \leq p \leq +\infty$ be the Lebesgue's space of (the power $p$) summable* *functions over* $(a,c)$ *for every* $c \in (a,b)$. *For* $u \in L^p(c,b)$, $\quad v \in L^q(a,c)$, *where* $\frac{1}{p}+\frac{1}{q}=1$, *we set*

$$\rho = \lim_{\epsilon \to 0} \sup\{[\int_x^{a+\epsilon} |u(y)|^p dy]^{\frac{1}{p}} [\int_a^x |v(y)|^q dy]^{\frac{1}{q}}, a < x \leq a+\epsilon\} +$$

$$+ \lim_{\delta \to 0} \sup\{[\int_x^{b} |u(y)|^p dy]^{\frac{1}{p}} [\int_{b-\delta}^x |v(y)|^q dy]^{\frac{1}{q}}, b-\delta \leq x < b\}.$$

Let $D$ be the linear operator defined by: $D(f(y))(x) = \int_0^x u(x)v(y)f(y)dy$; in the sequel wh shall assume that $D$ is a bounded operator in the space $L^p(0,T)$. We want to recall that the operator $D$ is bounded (in the $L^p(a,b)$ space) if and only if the function

$\psi(x) = [\int_x^b |u(y)|^p dy]^{\frac{1}{p}} [\int_a^x |v(y)|^q dy]^{\frac{1}{q}}$ is bounded on $(a,b)$. This operator is not necessarily a compact operator; as matter of fact it is well known (see [?], for istance), that $D$ is a compact operator if the functions $u(\cdot)$ e $v(\cdot)$ belongs to $L^2(a,b)$.

Furthermore the measure of noncompactness of $D$, i.e. $\mu(D)$ satisfies $(\frac{1}{2})^{1+\frac{1}{p}} \leq \mu(D) \leq p^{\frac{1}{q}} q^{\frac{1}{p}} \rho$; in the special case when $p=q=2$, i.e. when the (Lebesgue) space $L^p$ is a Hilbert space $L^2$, we obtain $\rho\sqrt{\frac{1}{8}} \leq \mu(D) \leq 2\rho$.

**Definition 2** *: An* $R_\delta -$*set is the intersection of a decreasing sequnce* $\{A_n\}$ *of compact AR (metric absolute retracts; see [?] or [?], for a reference.) Moreover it is known (see [?] for istance) that an* $R_\delta -$*set is an acyclic set in the Cęch homology.*

The following result also will be crucially used in teh sequel:

**Proposition 2** *: ([?], pag 159). Let* $X$ *be a space and let* $Y, ||\cdot||$ *be a Banach space and* $f: X \longrightarrow X$ *be a proper mapping. Assume further that for each* $\epsilon_n > 0$, $n > 0 \in \mathbb{N}$ *a proper mapping* $f_n : X \longrightarrow X$ *is given and the couple of conditions is satisfied:*

- $||f_n(x) - f(x)|| < \epsilon_n, \ \forall\, x \in X$;
- *for any* $\epsilon_n > 0$ *and* $y \in E$ *such that* $||y|| \leq \epsilon_n$, *the equation* $f_{\epsilon_n}(x) = y$ *has exactly one solution.*

*Then the set $S = f^{-1}(0)$ is an $R_\delta-$set.*

*Remark*: a sequence $f_{\epsilon_n}$ is called an $\epsilon_n$ approximation (of the function $f$).

**Proposition 3** *: ([**?**], pag ???). Let $F, F_n : \overline{B}(0,r) \longrightarrow Y$ be condensing operators such that*

- $\delta_n = \sup\{$
  $F_n(x) - F(x)||,\ x \in \overline{B}(0,r)\} \to 0,$ *as* $n \to +\infty$;
- *the equation $x = F_n(x) + y$ has at most one solution if $||y|| \le \delta_n$.*

*Then the set of fixed points of $F$ is an $R_\delta-$set.*

**Main result**

We are ready to establish out (main) existence result for the (initial value problems for) integral equations of the type here introduced.

First of all let $F : B(0,r) \to E$ be defined as follows:
$F(y) = h(t) + \int_0^t k(t,s)g(s,y(s)ds$
where $r$ is a real number (suitably defined below) and put $m_0 = ||F(0)||_2$.

**Theorem 1** *: Let $\rho$ the number defined in Proposition 1; then we assume that:*

1. *i) there are functions $\alpha, \phi, : I \to \mathbb{R}^n$ belonging to $L^2(I)$ such that $k(t,s) = \alpha\phi(s)$ for every $(t,s) \in I \times I$; moreover we assume that $||k||_2 < 2\rho$;*
2. *ii) $||g(t,x)|| \le \frac{1}{2\rho}||x|| + b(t)$, for $(t,x) \in I \times \mathbb{R}^n$, $b \in L^2(I)$, $b(t) \ge 0$;*
3. *iii) there is a ball $B(0,r)$ such that $r > \frac{2m_0\rho}{2\rho - ||k||_2}$.*

*Then the set of solution of the integral problem (**??**) is an $R_\delta-$set.*

*Remark:* The first part of the assumption $i$) is satisfied in many cases: for istance when $k(t,s)$ is a Green function; see, for istance, [**?**] for similar cases.

*Proof*: Clearly the above operator $F$ is a single value mapping and a possible fixed point of $F$ is a solution of the integral problem (**??**).

In order to prove the theorem the following steps in the proof have to be established:

**a )** $F$ has a closed graph;

**b)** $F$ is a condensing mapping;

**c)** The set of fixed point of $F$ is $R_\delta-$set.

*Proof of Step a)*: in fact, let $y_n \to y_0$ and put $G(y)(t) = g(t, y(t))$. Now, from assumption *ii)*, it follows that the superposition operator $G$ mapping the space $L^2$ into $L^2$ is condensing (see [?]); thus we have $\lim_n ||G(y_n) - G(y_0)||_2 = 0$. By using the Holder inequality, we get:

$||F(y_n) - F(y_0)||_2 = [\int_I |F(y_n)(s) - F(y_0)(s)|^2 ds]^{\frac{1}{2}} =$

$= [\int_I [\int_)^t (k(t,s)g(s, y_n(s)) - k(t,s)g(s, y_0(s))ds]^2 dt]^{\frac{1}{2}} \leq ||k||_2 || ||G(y_n) - G(y_0)||_2$

and this quantity is gioing to zero whenever $n \to +\infty$.

*Proof of Step b)*: Always working from $B(0,r)$ into $E$, we have $F(y) = (H \circ G)(y))$, where

$H(y)(t) = \int_0^t \phi(s)\alpha(t)y(s)ds + h(t)$.

Now, by assumptions $i)$ and $ii)$, we have (see [?]) $\mu(G(V)) \leq \frac{1}{2\rho}\mu(V)$, for any bounded set $V \subset L^2(I \times \mathbb{R}^n)$ and also $\mu(H) < 2\rho$; so (see [?]) $\mu(F) = \mu(H \circ G)(y)) \leq \mu(H)\mu(G) < 1$.

*Proof of Step c)*: Finally we have to prove that the set of fixed points of the operator $F$ is an $R_\delta-$set (in the sequel we assume that $(a,b) = (0,T)$.)

Let us consider the mappings $F_n : L^2(0,T) \to L^2(0,T)$ defined as:

$$F_n(x)(t) = \begin{cases} h(t) & = \quad \text{if } 0 \leq t \leq \frac{T}{n}; \\ h(t) + \int_0^{t-\frac{T}{n}} \phi(s)\alpha(s)g(s,y(s))ds & = \quad \text{if } \frac{T}{n} \leq t \leq T. \end{cases} \tag{3}$$

The mappings $F_n$ are continuous mappings; by assumption $i)$ and $ii)$ we have that they are also condensing. The intervals $[0, \frac{T}{n}], [\frac{T}{n}, \frac{2T}{n}], \cdots [\frac{kT}{n}, \frac{(k+1)T}{n}], \cdots [\frac{(n-1)T}{n}, T]$ are now coming in one after the other: each time the mappings $F_n$ are bijective and their inverses $F_n^{-1}$ are continuous. Moreover we have $||F_n - F||_2 \to 0$ as $n \to +\infty$. The latter fact allows us to say that the mappings $I - F_n$ and $I - F$ are proper maps. Finally we can conclude that the set of fixed points of $F$ is an $R_\delta-$set.

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