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A MATHEMATICAL MODEL OF A CRIMINAL-PRONE SOCIETY

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Abstract. Criminals are common to all societies. To fight against them the community takes different security measures as, for example, to bring about a police. Thus, crime causes a depletion of the common wealth not only by criminal acts but also because the cost of hiring a police force. In this paper, we present a mathematical model of a criminal-prone self-protected society that is divided into socio-economical classes. We study the effect of a non-null crime rate on a free-of-criminals society which is taken as a reference system. As a consequence, we define a criminal-prone society as one whose free-of-criminals steady state is unstable under small perturbations of a certain socio-economical context. Finally, we compare two alternative strategies to control crime: (i) enhancing police efficiency, either by enlarging its size or by updating its technology, against (ii) either reducing criminal appealing or promoting social classes at risk.

1. **Introduction.** The techniques of population dynamics and compartment models have been extensively used to study the influence of sociological and economical factors on the evolution of criminality [17], [3], [18], [1], [10]. Such studies have also provided a background to look for possible strategies aimed at reducing and controlling crime [6], [8]. Indeed, although it is clear that most of the parameters and constants appearing in the proposed equations have been hardly measured or estimated as yet, it is widely accepted [5] that simulations based on such models can yield useful hints to decision makers. A reason for that is the possibility of quickly exploring different social scenarios, corresponding to different parameter choices, by means of computer simulation.

Here we present a model of a criminal-prone self-protected society. We say that a society is criminal-prone if the free-of-criminals stationary state is unstable under small perturbations of the given socio-economic conditions. The term self-protected refers to the fact that the societies under consideration bring about a specialized set of individuals (the police) devoted to fight criminals. The model incorporates economical variables that influence criminality. For our current purposes, this last is assumed to consist only in crimes against property (e.g. burglary or car thefts). We consider a closed population that is homogeneous, so that space variations can be neglected and the only independent variable is time, t. Society is divided into subclasses according to their contribution to the total wealth of the population; we thus suppose this society to be formed by

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tax-payers. For the sake of simplicity, we assume further that individuals with criminal tendencies come mainly from the lowest social class. Therefore, fostering social promotion is a valid strategy to reduce crime. The model includes this policy together with the classical action of police against crime.

Concerning the structure of the society under consideration, the following subpopulations will be distinguished:

1): The *criminals*, whose number at time t is represented by $Y(t)$.
2): The *guards* that are hired to control criminality. The number of guards at time t is given by $G(t)$.
3): The *prisoners*, former criminals that are in jail. The number of individuals that are in prison at time t is $J(t)$.
4): The *non-criminal* population, that is divided in n subclasses, X_i, $i = 1, 2, \ldots, n$. Each class contributes in a different way (say, increasing with index i) to the total wealth of the whole population at time t, $W(t)$:

$$W(t) = \sum_{i}^{n} W_i(t) \tag{1}$$

In this approximation guards and criminals are not assumed to contribute to the legal economy.

The mathematical model derived below, and on which our forthcoming discussion is based, is written in terms of Ordinary Differential Equations (ODEs) and qualitatively studied in the following sections. In particular, criminal free societies are defined and analyzed in the next section. Here some simple but interesting situations are analytically solved. Section 3 introduces a general model that includes criminals and guards within the social framework previously defined. A 3-dimensional model is considered in Section 4 to compare different measures to check criminality, namely: reducing criminal appeal, fostering social promotion and rising the size of the police forces. Finally, a general discussion is presented in Section 5.

2. Free-of-criminals societies.

Let us consider first a simplified model consisting in a closed society without criminals nor guards. In this simple case, we model the evolution over time of each subpopulation according to the following additional assumptions:

(1): The flow of individuals through the social classes is given by:

$$\dot{X}_i(t) = \alpha_{i-1}\, X_{i-1}(t) - (\alpha_i + \beta_i)\, X_i(t) + \beta_{i+1}\, X_{i+1}(t) \tag{2}$$

where α_i, β_i are, respectively, the rate of social promotion and relegation for the ith class. By definition, $\alpha_0 = \alpha_n = \beta_1 = \beta_{n+1} = 0$. In particular, we can assume that α_i are non-negative increasing functions of the population wealth at each time, $W(t)$. Similarly, we suppose that β_i are non-negative decreasing functions with respect to $W(t)$. Nonetheless, they can also be taken to depend monotonically on X_i, which would correspond to a sort of intraspecific competition.

(2): The wealth of each subclass is proportional to its population, i.e $W_i = c_i\, X_i$. So, the total wealth of the population at time t, $W(t)$, is a linear combination of the subclasses population at this time:

$$W(t) = \sum_{i}^{n} c_i\, X_i(t) \tag{3}$$

Besides, it is assumed that $c_i \leq c_{i+1}$ for all $i = 1, 2, \ldots, n-1$ and $c_1 \geq 0$.

Under assumptions (1) and (2), the equilibrium distribution of individuals among the different classes corresponds to the solutions of:

$$\begin{array}{lcll} -\alpha_1\, X_1 + \beta_2\, X_2 & = & 0 & \\ \alpha_{i-1}\, X_{i-1} - (\alpha_i + \beta_i)\, X_i + \beta_{i+1}\, X_{i+1} & = & 0 & \quad \text{for} \quad i = 2, \ldots, n-1 \\ \alpha_{n-1}\, X_{n-1} - \beta_n\, X_n & = & 0 & \end{array}$$

Besides, for any t

$$X_1(t) + X_2(t) + \ldots + X_{n-1}(t) + X_n(t) = N \tag{4}$$

where N is the whole population. Notice that, in general, this is not a linear system since the kinetic parameters α_i and β_i may depend on $X_1, \ldots, X_n$ through W. It can be proved (see [15]) that, under general assumptions, at least one non-trivial equilibrium point exists.

We next consider a few simple but hopefully illustrative situations. To begin with, let us assume that the kinetic coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ are independent of W. In this case, the system 2 is linear with constant coefficients, and it can be straightforwardly solved. Besides the trivial solution ($X_i = 0$ for all $i = 1, 2, \ldots, n$), there is a unique stationary solution $\{\bar{X}_i\}$ such that:

$$\bar{X}_k = \frac{\alpha_1}{\beta_2}\frac{\alpha_2}{\beta_3}\ldots\frac{\alpha_{k-1}}{\beta_k}\bar{X}_1 \equiv \gamma_{k-1}\,\bar{X}_1 \tag{5}$$

where we set $\gamma_1 = 1$ and

$$\bar{X}_1 = \frac{\sum_{k=1}^{n}\bar{X}_k}{\sum_{k=1}^{n-1}\gamma_k} = \frac{N}{\gamma} \tag{6}$$

and N is as in 4 total population.

The situation becomes more complex when coefficients α_k and β_k depend on W. In this case, social mobility is a function of the total wealth of the society. Recalling equation (2.2) we can say that there are as many stationary solutions as fixed points of the transformation $\tilde{W} = \mathcal{T}(W)$ defined by:

$$\tilde{W} = N\frac{\sum_i c_i\,\gamma_{i-1}(W)}{\sum_i \gamma_{i-1}(W)} \tag{7}$$

In order to explore further this problem, let us consider the simplest situation when only two classes exist. Recalling that the total population is constant, let us write:

$$u = \frac{X_2}{N};\; v = \frac{X_1}{N} \tag{8}$$

so that $v + u = 1$. The wealth of the society can be written, upon replacing W by $\tilde{W} = \frac{W}{c_2\,N}$ as:

$$W = u + a\,v \tag{9}$$

where we are dropping the superscript in $\tilde{W}$ for convenience, and $a = \frac{c_1}{c_2}$ so that $0 \leq a < 1$ by the monotonicity assumption in 3. From 2 it follows that:

$$\dot{u} = \alpha - (\alpha + \beta)\,u \tag{10}$$

Assume that $\alpha \neq 0$ and $\beta \neq 0$. If α and β are constants, there is a unique equilibrium state:

$$\bar{u} = \frac{\alpha}{\alpha + \beta} \tag{11}$$

which is globally asymptotically stable because the associated eigenvalue is strictly negative since $\alpha + \beta > 0$.

The case when both parameters α and β are functions of the total wealth can be analyzed by setting:

$$\varphi(W) \equiv \frac{\alpha(W(u))}{\alpha(W(u)) + \beta(W(u))} \tag{12}$$

where $W(u) = a + (1-a)\,u$. Now, we have that the steady state solutions $\bar{u}$ are such that

$$\bar{u} = \varphi(\bar{u}) \tag{13}$$

It is clear from the continuity of φ that, since $0 \leq \varphi(u) \leq 1$, there is at least one solution of 13. If we further assume $\varphi'(u) < 1$, which implies:

$$\alpha'(u)\,\beta(u) - \alpha(u)\beta'(u) < (\alpha(u) + \beta(u))^2 \tag{14}$$

then the solution of 13 is unique and $\bar{u}$ is globally asymptotically stable since 14 guarantees that the corresponding eigenvalues have negative real parts. Sufficient conditions that assure 14 when α and β are monotonically dependent on u are the following

$$\alpha(u) > \alpha'(u); \qquad \beta(u) > -\beta'(u) \tag{15}$$

which roughly mean that the variation of these parameters with respect to W is small.

Let us next consider a three-classes system with constant total population, i.e. $\sum_{i=1}^{3} X_i = N$. Let us define:

$$u = \frac{X_3}{N}; v = \frac{X_2}{N}; w = \frac{X_1}{N} \tag{16}$$

and

$$W = u + a\,v + b\,(1 - u - v) \tag{17}$$

where $a = \frac{c_2}{c_3}$ and $b = \frac{c_1}{c_3}$ with $c_3 > 0$. Again, this W is actually $\tilde{W} = \frac{W}{c_3\,N}$.

Notice that

$$0 \leq b < a < 1 \tag{18}$$

and

$$b \leq W \leq 1 \tag{19}$$

Of course, the largest wealth, $W = 1$, is obtained when $u = 1$ (and, consequently, $v = w = 0$) and the lowest value $W = b$ corresponds to $u = v = 0$ and $w = 1$. Assume further that the resistance to upward mobility increases with higher positions in the social hierarchy, as the so-called Peter principle states [12]. A simple way of implementing this principle is to define:

$$\alpha_i = W^i; \qquad \beta_{i+1} = 1 - W^{3-i} \qquad \text{for } i = 1, 2 \tag{20}$$

Taking into account 17 these kinetic parameters are then given by:

$$\alpha_1 = (1-b)\,u + (a-b)\,v + b;\ \alpha_2 = ((1-b)\,u + (a-b)\,v + b)^2; \tag{21}$$

$$\beta_2 = 1 - ((1-b)\,u + (a-b)\,v + b)^2;\ \beta_3 = 1 - b - (1-b)\,u - (a-b)\,v \tag{22}$$

Under these assumptions the dynamical system (2.1) reduces to:

$$\begin{array}{rcl} \dot{u} &=& ((1-b)\,u + (a-b)\,v + b)^2\,v - (1 - b - (1-b)\,u - (a-b)\,v)\,u \\ \dot{v} &=& (1 - 2\,(u + av + b(1-u-v)))\,u - \\ &-& (1 + (u + a\,v + b\,(1-u-v)))\,v + u + a\,v + b\,(1-u-v) \end{array} \tag{23}$$

It is straightforward to show that the wealthiest state ($u = 1, v = 0$) always exists and is asymptotically stable if $a > 0$ (the corresponding eigenvalues are -1 and $-a$). Other steady states can appear depending on the values of the coefficients a and b (see figure 1).

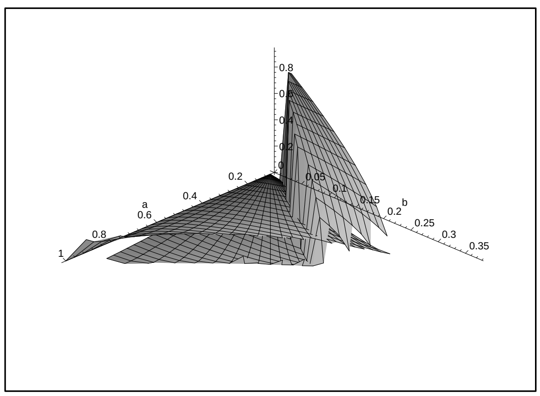

FIGURE 1. (u, v) components of the stationary state of coexistence for the system 23 as a function of a and b. While u behaves smoothly, notice the sharp change in behaviour for values of v large and small, which is related to different regimes

A particular, but interesting, situation occurs when the lowest class does not contributes to the total wealth, i.e. $b = 0$. In this case, three fixed points appear: the wealthiest state ($u = 1, v = 0$), the poorest state ($u = 0, v = 0$) and one that corresponds to coexistence of the three classes (see figure 2).

The first two are always asymptotically stable and the latter is a saddle-node. Therefore, a bistable transition occurs between the largest (that corresponds to the steady state ($u = 1, v = 0$)) and the lowest (attained in the state $u = 0$ and $v = 0$) value of the total wealth of the system (see figure 3). This bistable behaviour might corresponds to an economical crisis marking the transition from a rich-dominant to a poor-dominant society.

It could be of some interest to consider the case in which the value of a approaches 0, i.e. the limit case when only the highest class contributes to the total wealth. Then, it can be easily seen that the coexistence state coalesces with the unstable one ($u = 1, v = 0$), while the point ($u = 0, v = 0$) becomes asymptotically stable (see figure 4). This might be interpreted as an economical crisis by which society as a whole becomes poorer.

We conclude this section by observing that a mathematical analysis of the general case corresponding to n subpopulations in 2 is to be found in [15]. In particular, the existence of at least one

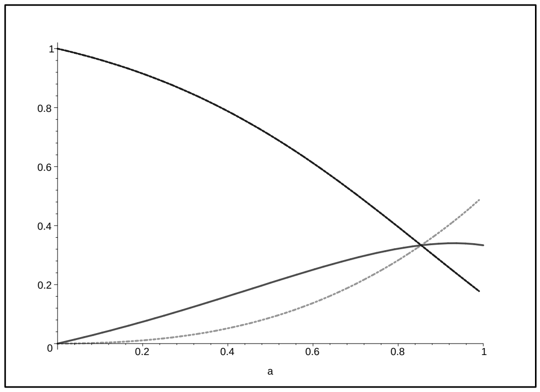

FIGURE 2. Concentrations of v (solid) and u (dash) and w (dash-dot) at the coexistence state when $b = 0$.

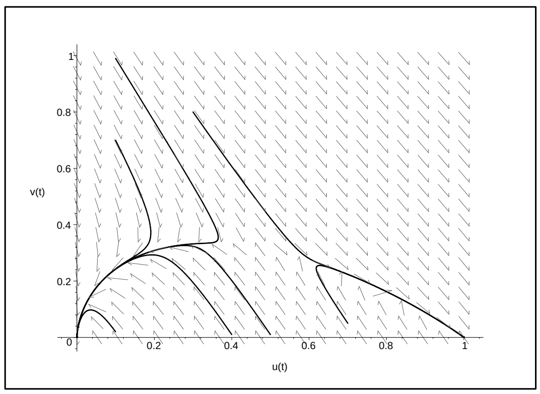

FIGURE 3. Phase portrait for the three-classes criminal free system. Parameter values are $a = 0.8$ and $b = 0$, which means that there is no contribution to the total wealth from the lowest social class.

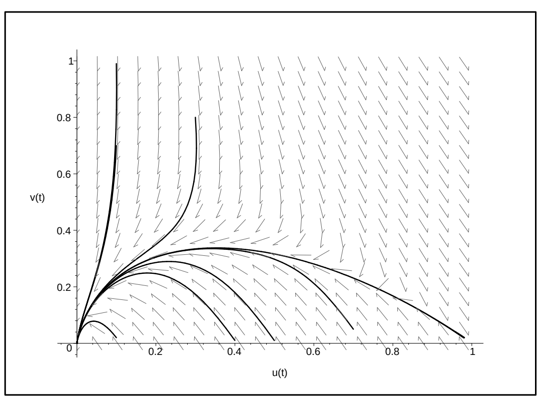

FIGURE 4. Phase portrait for the limit case $a = b = 0$, that is when only the wealthiest class contributes to the total wealth W.

non-trivial equilibrium point and the stability conditions is discussed therein under rather general assumptions on the system parameters.

3. **Criminal-prone self-protected societies.** In this Section we discuss the effect of the presence of criminals and guards in the social dynamics previously considered. As before, we assume that there is neither input nor output flux of individuals, so that the system is closed. Criminals thus come exclusively from any of the social classes that form the society. To proceed further, we assume the following additional hypotheses concerning crime and security (all the constants that will appear are meant to be non-negative):

(1): The crime rate $K(t)$ will be considered to be proportional to the number of "encounters" between criminals Y and targets, X_i. Thus,

$$K(t) = \sum_{i}^{n} \theta_i \, X_i(t) \, Y(t) \tag{24}$$

If one wants to take into account the effectiveness of the "collisions", the term θ_i can be assumed to be a decreasing function of X_i, e.g.

$$\theta_i(t) = \frac{a_i}{b_i + X_i(t)}, \tag{25}$$

so that the number of crimes committed per unit time against the $i-th$ population is simply proportional to the number of criminals when the number of targets is large enough. Such a choice of θ_i takes also into account the time spent by criminals in identifying the victim and preparing the attack (similar functional responses are used in Ecology and are said of Holling type [14], [2]).

(2): There will be a recruitment of criminals from the population of the classes X_i which we suppose to be relevant only for lower values of i. To be specific, we can gather in the class X_1 all individuals that are "at risk" of becoming criminals for any reason (e.g. social, familiar, housing or educational). Under such assumption, this is the only class that will be connected with Y and J (the prisoners). The criminal recruitment term $R(t)$ is assumed to be given by:

$$R(t) = k_1 \; X_1(t) \, Y(t) \tag{26}$$

It is reasonable to argue that the appeal of becoming a criminal depends on the wealth of the possible targets. To our acknowledge, there are no conclusive data about the impact of the total wealth of the society on criminal recruitment. Here we make the simplifying assumption that:

$$k_1 = k \, W(t) \tag{27}$$

essentially meaning that the total wealth of the society is proportional to the wealth of the targets. It must be said that the recruitment of criminals depends on other important factors, namely inequality among social classes, social promotion and age. Nevertheless, they will be taken into consideration in future investigations. Therefore, in the absence of guards, the equation for $Y(t)$ will be:

$$\dot{Y}(t) = k \, W(t) \, X_1(t) \, Y(t) - \rho Y(t) - \nu Y^2(t) \tag{28}$$

The last two negative terms incorporate a natural decay and the effect of an intraspecific competition. Notice that if initially there are no criminals, i.e. $Y(0) = 0$, the system is always free of them, i.e. $Y(t) = 0$ for all $t > 0$. However, we will see in the next Section that this situation is unstable.

(3): The number of arrests (i.e. removing criminals) per unit time $A(t)$ is proportional to the numbers of criminals and to the efficiency of guards. An argument similar to that used in point (1) above leads to the assumption:

$$A(t) = \frac{m \, Y(t) \, G(t)}{M + Y(t)} \tag{29}$$

(4): The dynamics of the population in jail is given by:

$$\dot{J}(t) = A(t) - \tau \, J(t) \tag{30}$$

We assume that all individuals leaving class J go to class X_1.

(5): Concerning the evolution of the number of guards we consider it to be governed by the following factors:
(i) a physiological term of retirement that can be modelled mathematically as:

$$P(t) = -q\, G(t) \tag{31}$$

(ii) The rate of hiring new guards is thought to depend on the fear perceived by the non-criminal population, and this last is assumed to be proportional to the number of crimes $K(t)$:

$$H(t) = h \sum_i^n \frac{a_i}{b_i + X_i(t)}\, X_i(t)\, Y(t) \tag{32}$$

(iii) The number of casualties suffered by guards is assumed to be proportional to the number of arrests, i.e.

$$L(t) = l \frac{Y(t)\, G(t)}{M + Y(t)} \tag{33}$$

Note that L may include the neutralization of guards by corruption.

(6): Finally, we have to model the effect on the wealth W of the system of crimes and that of the cost of maintaining the security service. To this end, we assume
(i) The negative effect of the crimes on W will be proportional to the sum of the number of crimes committed per unit time and to the total wealth of the system

$$C(t) = -\lambda\, W(t)\, K(t) = -\lambda\, W(t)\, Y(t) \sum_i^n \frac{a_i}{b_i + X_i(t)}\, X_i \tag{34}$$

(ii) The cost of maintaining the guards will be modelled as:

$$\gamma(t) = -g(W)\, G(t) \tag{35}$$

where g is the unit cost per unit time that the population is capable to bear and that, in general, depends on the total wealth W.

Putting together the assumptions made above and those in section 2, we are thus led to the following system of $(n+3)$-ODEs describing the population dynamics of our model:

$$\begin{array}{rcll} \dot{Y} & = & k\, W\, X_1\, Y - \rho\, Y - \nu\, Y^2 - m \frac{Y\,G}{M+Y} & \\ \dot{G} & = & -q\, G + h\, Y \sum_i^n \frac{a_i}{b_i + X_i(t)}\, X_i - l \frac{Y\,G}{M+Y} & \\ \dot{J} & = & m \frac{Y\,G}{M+Y} - \tau\, J & \\ \dot{X}_1 & = & -\alpha_1\, X_1 + \beta_2\, X_2 + \tau\, J - k\, W\, X_1\, Y & \\ \dot{X}_i & = & \alpha_{i-1}\, X_{i-1} - (\alpha_i + \beta_i)\, X_i + \beta_{i+1}\, X_{i+1} & i = 2, \ldots, n-1 \\ \dot{X}_n & = & \alpha_{n-1}\, X_{n-1} - \beta_n\, X_n & \end{array} \tag{36}$$

To this system we must add an equation describing the evolution of the total wealth:

$$\dot{W} = (\sum_i^n c_i X_i - W) - \lambda\, Y\, W \sum_i^n \frac{a_i}{b_i + X_i(t)}\, X_i - g(W)\, G \tag{37}$$

where c_i are nonnegative constants for all $i = 2, \ldots, n$ (see 3). Notice that the total reference wealth

$$\bar{W} = \sum_i^n c_i\, X_i \tag{38}$$

corresponds to the equilibrium state of 37 when neither criminals nor guards are present in the system, i.e. $Y = 0$ and $G = 0$. Equations 36-37 are to be completed with initial conditions:

$$Y(0) = Y_0 \geq 0;\ G(0) = G_0 \geq 0;\ J(0) = J_0;\ W(0) = W_0 \tag{39}$$

and

$$X_i(0) = X_{i0} \geq 0 \tag{40}$$

for all $i = 1, 2, \ldots, n$. It is easy to check that system 36 satisfies the requirement of preserving non-negativity of each component of the solution i.e. solutions do not leave the positive cone given by $X_i \geq 0$, $Y \geq 0$, $G \geq 0$ and $J \geq 0$, which is shown to be an invariant one. The total number of individuals is not conserved for the following reasons:

(a1) We have not specified where the guards leaving the subpopulation go, nor where the new guards are recruited from.

(a2) No assumption has been made concerning where criminals leaving the system go.

Point (a1) could be easily solved with formal complications that we decided to avoid for the sake of simplicity. In any case, if we add in the equation of $\dot{X}_i$ in system 36 the term:

$$- hK(t)\frac{X_i}{\sum_{j=2}^{n} X_j}, \tag{41}$$

that measures the recruitment of guards from the $i-th$ subpopulation, one obtains that solutions are bounded by

$$N_0 = Y_0 + G_0 + J_0 + \sum_{i=1}^{n} X_{io} \tag{42}$$

and thus, they exist globally (though some of the subpopulations may go extinct).

When the social structure is simplified in the sense that it can be assumed that the dimension of each social class is a given fraction of the total population N, the problem becomes similar to the so-called triangle model that we have considered previously [10]. This reduced model retains the main ingredients of the Routine Activity Theory introduced by Felson and Cohen [7], [6]: targets, criminals and guards. As stated before, guards act as predators of both criminals and owners, that have to bear the cost of their salaries. As shown in [10], the interaction of these three populations yields a complex dynamics as the efficiency of the police forces varies.

4. **Social measures versus police action.** Once a model for a criminal-prone self-protected society has been derived, a natural question that arises is that of comparing on it the two main strategies that are commonly considered to control crime, namely:

(i) improving the strength of police to catch criminals and remove them from the society and

(ii) hindering the recruitment of new criminals from the society, mainly from the lower classes at risk.

The first approach requires either an increase of the police size or a larger effectiveness or both. The second one can be achieved either by social promotion of the class at risk (i.e. increasing α_1 and decreasing β_2) or by reducing the appeal of criminal behavior (i.e. by decreasing the value of k, say by improving education and social consciousness, etc). To ascertain which strategy is better in a given socio-economical context is an open problem of great interest for policy makers.

To shed some light on the related effect of social promotion and police repression, we next discuss a particular situation in which only two classes exist and the police size is kept constant, G. Also, jailed population is not taken into account. Essentially, we assume that the time scale we consider is an intermediate one between the average time spend in jail by prisoners and the period over which the policy of guards recruitment changes. Instead of equations 36 we now have:

$$\begin{array}{rcl} \dot{X}_1 & = & -\alpha_1\, X_1 + \beta_2\, X_2 - k\, W\, X_1\, Y + \rho\, Y + \nu\, Y^2 + m\, \frac{Y\, G}{M+Y} \\ \dot{X}_2 & = & \alpha_1\; X_1 - \beta_2\, X_2 \\ \dot{Y} & = & k\, W\, X_1\, Y - \rho\, Y - \nu\, Y^2 - m\, \frac{Y\, G}{M+Y} \end{array} \tag{43}$$

The additional terms added to the equation of $\dot{X}_1$ in 43 with respect to the corresponding equation in 36 come from the criminal population (likely, after passing some time inactive). They have been introduced in order to close the dynamic system, so that $X_1 + X_2 + Y = N$. Besides, it is assumed that α_1 and β_2 are independent of W. The equation for the total wealth reads now:

$$\frac{d\, W}{d\, t} = c_1\, X_1 + c_2\, X_2 - W - \lambda\, Y\, W\, \frac{a_2}{b_2 + X_2}\, X_2 - g\, W\, G \tag{44}$$

where a linear relation for $g(W) = g\, W$, $g > 0$ has been assumed.

In order to compare the influence of different strategies to control criminality, we choose a reference parameter state and leave free the parameters k and G that measure the appearance of criminals and the size of the police forces. To discuss a particular example we set, upon normalization, $N = 1$ and $\lambda = 1$ and we choose the following set of reasonable values for the remaining parameters:

$$\begin{gathered} M = 1;\, \rho = 0.1;\, g = 0.1;\, a_2 = 1;\, b_2 = 1;\, m = 0.1 \\ c_1 = 0.1;\, c_2 = 1;\; g = 0.1;\, \alpha_1 = 0.1;\, \beta_2 = 0.01;\, \nu = 0. \end{gathered} \tag{45}$$

Under this choice, it can be shown that system 43-44 has at least the free-of-criminals stationary state:

$$Y = 0;\, X_1 \approx 0.091; \tag{46}$$

with a total wealth that depends on the value of G:

$$W \approx \frac{9.18}{10+G} \tag{47}$$

The stability of the free-of-criminals steady state can be also studied analytically by simple linearization. The Jacobian matrix evaluated in this point is given by:

$$J = \begin{pmatrix} -1-0.1\,G & -1-\frac{4.372}{10+G} & -0.9 \\ 0 & 0.835\frac{k}{10+G}-0.01-0.1\,G & 0 \\ 0 & -0.835\frac{k}{10+G}+0.1\,G & -0.11 \end{pmatrix} \tag{48}$$

whose eigenvalues are:

$$\lambda_1 = -0.11;\ \lambda_2 = -1-0.1\,G \tag{49}$$

and

$$\lambda_3 = -\frac{-0.835\,k+0.1+0.01\,G+G+0.1\,G^2}{10+G} \tag{50}$$

As it can be seen, for this third eigenvalue to be negative either k must be small or G must be large. However, notice that the effect of the size of the security forces in the sign of this eigenvalue is quadratic, whereas that of k is linear. If one assumes that the size of the security forces is fixed ($G = G_0$) then, in order to avoid a rise of the criminal population k must be smaller than

$$kc = 0.120 + 1.21\,G_0 + 0.120\,G_0^2 \tag{51}$$

Whenever k exceeds k_c an additional equilibrium with non-vanishing criminal population appears, although the explicit dependence of this state with respect to the control parameters is rather involved. Figure 5 depicts how the corresponding stationary criminal population depends on k and G.

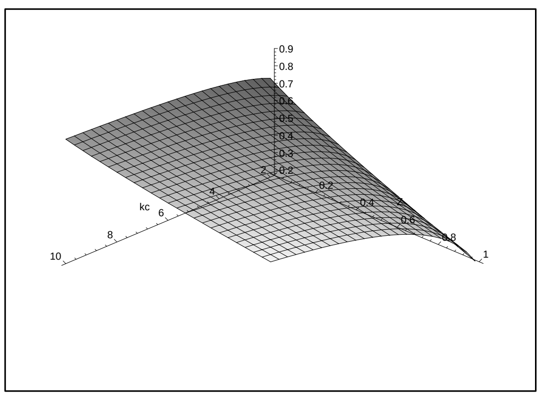

FIGURE 5. Stationary criminal population as a function of k and G. The police efficiency is fixed to $m = 0.1$ and the rest of the parameter values are: $N = 1.0$; $\nu = 0$; $M = 1$; $\rho = 0.1; g := 0.1$; $a_2 = 1$; $b_2 := 1; c_1 = 0.1$; $c_2 = 1$; $g = 0.1$; $\alpha_1 = 0.1$; $\beta_2 = 0.01$.

Social promotion is tuned by the parameters α_1 and β_2. In our previous analysis, these values have been fixed to $\alpha_1 = 0.1$ and $\beta_2 = 0.01$. Now, we want to study the effect of changing α_1 on the evolution of the criminal population and compare this effect with the variation of the police size, G. For this purpose, we now keep constant the value of k, say $k = 0.5$. The rest of the parameters is as before (see 45). In this situation, a free-of-criminals stationary state always exists:

$$Y = 0;\ x_1 = \frac{1}{100\,\alpha_1+1};\ W = \frac{1+1000\,\alpha_1}{1000\,\alpha_1+100\,\alpha_1\,G+10+G)} \tag{52}$$

The stability of this criminal free state depends on the sign of the following eigenvalue (the other two are negative for all values of α_1 and G (see figure 7 below).

$$\lambda = -\frac{0.50-3\,10^2\,\alpha_1+10^4\,\alpha_1^2+G(2\,10^3\,\alpha_1^2+40\alpha_1+0.20)+G^2(10^2\,\alpha_1^2+2\,\alpha_1+0.01)}{10^5\,\alpha_1^2+2\,10^3\,\alpha_1+10+G(10^4\,\alpha_1^2+2\,10^2\,\alpha_1+1)} \tag{53}$$

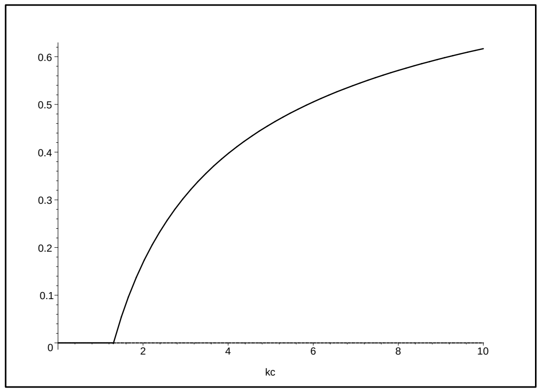

FIGURE 6. Bifurcation diagram of the criminal free steady state as a function of k. The size of the police is taken to be $G = 0.08$. The rest of the parameter values as in the previous figure. The bifurcation point is $k_c \approx 1.30$.

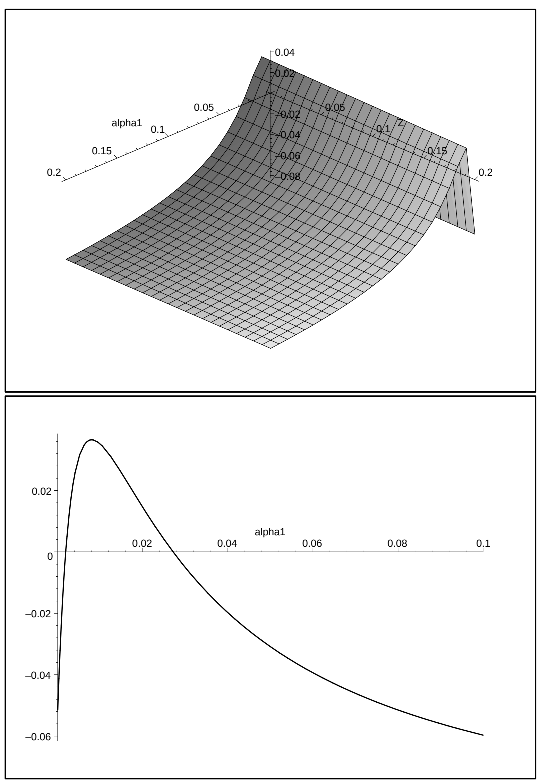

FIGURE 7. Eigenvalue λ corresponding to a criminal-free stationary state. (a) Graph of λ as a function of α_1 and G. (b) Curve obtained from the previous surface for $G = 0.1$. As before, parameter values are: $k = 0.5$; $m = 0.1$; $N = 1.0$; $\nu = 0$; $M = 1$; $\rho = 0.1; g := 0.1$; $a_2 = 1$; $b_2 := 1; c_1 = 0.1$; $c_2 = 1$; $g = 0.1$; $\alpha_1 = 0.1$; $\beta_2 = 0.01$.

It turns out that, for a given police size G this eigenvalue is positive if $A_1 < \alpha_1 < A_2$. This means that criminality is prevented when either social promotion is too low or too large. These critical values of α depend only slightly on the police size in the range $G \in [0, 0.2]$. For instance, for $G = 0.1$, it is straightforward to show that:

$$A_1 \approx 0.002;\ A_2 \approx 0.027 \tag{54}$$

Since our criminal-free society is unstable for $\alpha_1 \in (A_1, A_2)$, we seek for other stationary solutions with non-null criminal population. It can be shown that, for this range of variation of α_1, an additional steady state exist that assures the existence of a stable steady state in a society with criminality. In particular, for $G = 0.1$ the steady values are:

$$x_1 \approx 0.315;\ y \approx 0.055 \tag{55}$$

with a total wealth

$$W \approx 0.641 \tag{56}$$

5. **Concluding remarks.** Human societies are generally assumed to be criminal-prone [13], [4], [9]. To the best of our knowledge, there are no examples of human communities free of individuals intent to break the common law. The same can be said about other animal societies; even at the lowest levels of animal development, individuals that violate the rules are present (see, for instance, [16]). As a response to these offences, the group self-organizes to protect itself. The mechanisms of self-protection are manifold. A common one consists in the specialization of some individuals to defend law-abiding people against criminal attacks, which brings about a police.

The social model we have presented here takes into account a division of the tax-payer population according to their contribution to the total wealth of the system. Besides, a certain mobility is assumed to exist among the classes as measured by the coefficients α_k (that can be understood as a social promotion) and β_k (measuring the rate of relegation from highest social classes) that, in principle, are functions of the total wealth. This last changes over time due to the social mobility, as well as a consequence of the action of criminals and the cost of hiring a security service (see equation 37). It seems plausible that in the absence of criminals the total wealth of the system is a linear combination of the population of each of the subclasses. When criminals perturb the system this assumption is no longer valid.

Unfortunately, little is known about data on the the dependence of social promotion and social relegation on the total wealth. In general, a larger wealth not only fosters a larger flux to richer classes, but also a lower flux to poorer classes. Besides, the Peter principle states that social promotion is more difficult for richer classes. A similar principle for relegation has been considered in our model: the richer the class, the lower the probability of relegation for a given total wealth.

A relevant conclusion that can be derived from our study is that the kind of systems under consideration are criminal-prone, in the sense that criminal-free steady states are unstable under small perturbations in the socio-economical context. We have also seen that for the simplified system 17 a bistability phenomenon appears. This, in particular, means that an unstable threshold exists separating two different stable states, respectively corresponding to predominance of the higher or lower class. For the same simplified situation, in the limit case where only the upper class contributes to the global wealth, the whole society becomes impoverished.

In section 4 we have studied a simplified model where two different strategies (increasing police forces versus increasing social measures) are compared. In agreement with the common sense, we have shown that the free-of-criminals steady state becomes asymptotically stable when either the likelihood of being a criminal is decreased (i.e to decrease the k-value) or social promotion is increased (e.g. by rising α_1) or when the size of the police forces is also increased. However, a remarkable fact is that criminality may decrease even when low social promotion exist ($\alpha_1 < A_1$) . It is also seen that strategies enlarging police forces could be less effective that enhancing social policies due to quadratic dependence on G of the determinant eigenvalues (cf. 50).

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