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On partial polynomial interpolation / C. Brambilla; G. Ottaviani. - In: LINEAR ALGEBRA AND ITS APPLICATIONS. - ISSN 0024-3795. - STAMPA. - 435:(2011), pp. 1415-1445. [10.1016/j.laa.2011.03.024]

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DOI: 10.1016/j.laa.2011.03.024

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## On partial polynomial interpolation

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## ARTICLE INFO

## Article history:

Received 11 June 2007

Accepted 15 March 2011

Available online xxxx

Submitted by V. Mehrmann

## AMS classification:

14C20

15A72

65D05

Q132E30

## ABSTRACT

The Alexander–Hirschowitz theorem says that a general collection of  $k$  double points in  $\mathbf{P}^n$  imposes independent conditions on homogeneous polynomials of degree  $d$  with a well known list of exceptions. We generalize this theorem to arbitrary zero-dimensional schemes contained in a general union of double points. We work in the polynomial interpolation setting. In this framework our main result says that the affine space of polynomials of degree  $\leq d$  in  $n$  variables, with assigned values of any number of general linear combinations of first partial derivatives, has the expected dimension if  $d \neq 2$  with only five exceptional cases. If  $d = 2$  the exceptional cases are fully described.

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## 24 1. Introduction

Let  $R_{d,n} = K[x_1, \dots, x_n]_d$  be the vector space of polynomials of degree  $\leq d$  in  $n$  variables over an infinite field  $K$ . Note that  $\dim R_{d,n} = \binom{n+d}{d}$ . Let  $p_1, \dots, p_k \in K^n$  be  $k$  general points and assume that over each of these points a general affine proper subspace  $A_i \subset K^n \times K$  of dimension  $a_i$  is given. Assume that  $a_1 \geq \dots \geq a_k$ . Let  $\Gamma_f \subseteq K^n \times K$  be the graph of  $f \in R_{d,n}$  and  $T_{p_i} \Gamma_f$  be its tangent space at the point  $(p_i, f(p_i))$ . Note that  $\dim T_{p_i} \Gamma_f = n$  for any  $i$ . Consider the conditions

$$A_i \subseteq T_{p_i} \Gamma_f, \quad \text{for } i = 1, \dots, k \quad (1)$$

25 When  $a_i = 0$ , the assumption (1) means that the value of  $f$  at  $p_i$  is assigned. When  $a_i = n$ , (1) means  
 26 that the value of  $f$  at  $p_i$  and the values of all first partial derivatives of  $f$  at  $p_i$  are assigned. In the  
 27 intermediate cases, (1) means that the value of  $f$  at  $p_i$  and the values of some linear combinations of  
 28 first partial derivatives of  $f$  at  $p_i$  are assigned.

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Consider now the affine space

$$V_{d,n}(p_1, \dots, p_k, A_1, \dots, A_k) = \{f \in R_{d,n} \mid A_i \subseteq T_{p_i} \Gamma_f, i = 1, \dots, k\} \quad (2)$$

29 The polynomials in this space solve a partial polynomial interpolation problem. The conditions in (1)  
30 correspond to  $(a_i + 1)$  affine linear conditions on  $R_{d,n}$ . Our main result describes the codimension of  
31 the above affine space. Since the description is different for  $d = 2$  and  $d \neq 2$ , we divide the result in  
32 two parts.

**Theorem 1.1.** *Let  $d \neq 2$  and  $\text{char}(K) = 0$ . For a general choice of points  $p_i$  and subspaces  $A_i$ , the affine space  $V_{d,n}(p_1, \dots, p_k, A_1, \dots, A_k)$  has codimension in  $R_{d,n}$  equal to*

$$\min \left\{ \sum_{i=1}^k (a_i + 1), \dim R_{d,n} \right\}$$

33 with the following list of exceptions

(a)  $n = 2, d = 4, k = 5, a_i = 2$  for  $i = 1, \dots, 5$

(b)  $n = 3, d = 4, k = 9, a_i = 3$  for  $i = 1, \dots, 9$

(b')  $n = 3, d = 4, k = 9, a_i = 3$  for  $i = 1, \dots, 8$  and  $a_9 = 2$

(c)  $n = 4, d = 3, k = 7, a_i = 4$  for  $i = 1, \dots, 7$

(d)  $n = 4, d = 4, k = 14, a_i = 4$  for  $i = 1, \dots, 14$

34 In particular when  $\sum_{i=1}^k (a_i + 1) = \binom{n+d}{d}$  there is a unique polynomial  $f$  in  $V_{d,n}(p_1, \dots, p_k, A_1, \dots,$   
35  $A_k)$ , with the above exceptions (a), (b'), (c), (d). In the exceptional cases the space  $V_{d,n}(p_1, \dots, p_k, A_1, \dots,$   
36  $A_k)$  is empty.

37 The “general choice” assumption means that the points can be taken in a Zariski open set (i.e.  
38 outside the zero locus of a polynomial) and for each of these points the space  $A_i$  can be taken again in  
39 a Zariski open set. On the real numbers this assumption means that the choices can be done outside a  
40 set of measure zero. Our result is not constructive but it ensures that in the case  $\sum_{i=1}^k (a_i + 1) = \binom{n+d}{d}$   
41 the linear system computing the interpolating polynomial with general data has a unique solution.  
42 Hence any algorithm solving linear systems can be successfully applied. Actually our proof shows that  
43 Theorem 1.1 holds on any infinite field, with the possible exception of finitely many values of  $\text{char} K$   
44 (see the appendix). For finite fields the genericity assumption is meaningless.

45 The case in which  $a_i = n$  for all  $i$  was proved by Alexander and Hirschowitz in [1,2], see [4] for  
46 a survey. The most notable exception is the case of seven points with seven tangent spaces for cubic  
47 polynomials in four variables, as in c). This example was known to classical algebraic geometers and  
48 it was rediscovered in the setting of numerical analysis in [11]. The case of curvilinear schemes was  
49 proved as a consequence of a more general result by [5] on  $\mathbf{P}^2$  and by [8] in general.

50 The case  $d = 1$  follows from elementary linear algebra. The case  $n = 1$  is easy and well known: in  
51 this case the statement of Theorem 1.1 is true with the only requirement that the points  $p_i$  are distinct  
52 and the spaces  $A_i$  are not vertical, that is their projections  $\pi(A_i)$  on  $K^n$  satisfy  $\dim A_i = \dim \pi(A_i)$ .

Assume now  $d = 2$ . We set  $a_i = -1$  for  $i > k$ . For any  $1 \leq i \leq n$  we denote

$$\delta_{a_1, \dots, a_k}(i) = \max \left\{ 0, \sum_{j=1}^i a_j - \sum_{j=1}^i (n + 1 - j) \right\}$$

53

**Theorem 1.2.** *Let  $K$  be an infinite field. For a general choice of points  $p_i$  and subspaces  $A_i$ , the affine space  $V_{2,n}(p_1, \dots, p_k, A_1, \dots, A_k)$  has codimension in  $R_{2,n}$  equal to*

$$\min \left\{ \sum_{i=1}^k (a_i + 1), \dim R_{2,n} \right\}$$

54 if and only if one of the following conditions takes place:

- 55 (1) either  $\delta_{a_1, \dots, a_k}(i) = 0$  for all  $1 \leq i \leq n$ ;  
 56 (2) or  $\sum_i (a_i + 1) \geq \binom{n+2}{2} + \max\{\delta_{a_1, \dots, a_k}(i) : 1 \leq i \leq n\}$ .

In particular when  $\sum_{i=1}^k (a_i + 1) = \binom{n+2}{2}$  there is a unique polynomial  $f$  in  $V_{2,n}(p_1, \dots, p_k, A_1, \dots, A_k)$  if and only if, for any  $1 \leq i \leq n$ , we have

$$\sum_{j=1}^i a_j \leq \sum_{j=1}^i (n+1-j).$$

57 The first nontrivial example which explains Theorem 1.2 is the following. Consider  $k = 2$  and  
 58  $(a_1, a_2) = (n, n)$ . Then the affine space  $V_{2,n}(p_1, p_2, A_1, A_2)$  is given by quadratic polynomials with  
 59 assigned tangent spaces  $A_1, A_2$  at two points  $p_1, p_2$ . This space is not empty if and only if the inter-  
 60 section space  $A_1 \cap A_2$  is not empty and its projection on  $K^n$  contains the midpoint of  $p_1 p_2$ , which is a  
 61 codimension one condition. In order to prove this fact restrict to the line through  $p_1$  and  $p_2$  and use

62 a well known property of the tangent lines to the parabola. In this case  $\delta_{n,n}(i) = \begin{cases} 0 & i \neq 1 \\ 1 & i = 1 \end{cases}$  and the

63 two conditions of Theorem 1.2 are not satisfied. In Section 3 we will explain these two conditions in  
 64 graphical terms, **with more details**

Let  $\pi(A_i)$  be the projection of  $A_i$  on  $K^n$ . For  $i = 1, \dots, k$  we consider the ideal

$$I_i = \left\{ f \in K[x_1, \dots, x_n] \mid f(p_i) + \sum_{j=1}^n (x_j - (p_i)_j) \frac{\partial f}{\partial x_j}(p_i) = 0 \text{ for any } x \in \pi(A_i) \right\}$$

65 Notice that we have  $m_{p_i}^2 \subseteq I_i \subseteq m_{p_i}$  and the ring  $K[x_1, \dots, x_n]/I_i$  corresponds to a zero-dimensional  
 66 scheme  $\xi_i$  of length  $a_i + 1$ , supported at  $p_i$  and contained in the double point  $p_i^2$ . When  $V_{d,n}(p_1, \dots, p_k,$   
 67  $A_1, \dots, A_k)$  is not empty, its associated vector space (that is its translate containing the origin) consists  
 68 of the hypersurfaces of degree  $d$  through  $\xi_1, \dots, \xi_k$ . Moreover, when this vector space has the expected  
 69 dimension, it follows that  $V_{d,n}(p_1, \dots, p_k, A_1, \dots, A_k)$  has the expected dimension too.

70 The space  $K^n$  can be embedded in the projective space  $\mathbf{P}^n$ . Since the choice of points is general,  
 71 we can always avoid the “hyperplane at infinity”. In order to prove the above two theorems, we will  
 72 reformulate them in the projective language of hypersurfaces of degree  $d$  through zero-dimensional  
 73 schemes. More precisely we refer to Theorem 3.2 for  $d = 2$ , Theorem 4.1 for  $d = 3$  and Theorem 5.6 for  
 74  $d \geq 4$ . This reformulation is convenient mostly to rely on the wide existing literature on the subject.  
 75 In this setting Alexander and Hirschowitz proved that a general collection of double points imposes  
 76 independent conditions on the hypersurfaces of degree  $d$  (with the known exceptions) and our result  
 77 generalizes to a general zero-dimensional scheme contained in a union of double points. It is possible  
 78 to degenerate such a scheme to a union of double points only in few cases, in such cases of course our  
 79 result is trivial from [1].

80 Our proof of Theorem 5.6, and hence of Theorem 1.1, is by induction on  $n$  and  $d$ . Since it is enough  
 81 to find a particular zero-dimensional scheme which imposes independent condition on hypersurfaces  
 82 of degree  $d$ , we specialize some of the points on a hyperplane, following a technique which goes back  
 83 to Terracini. We need a generalization of the Horace method, like in [1], that we develop in the proof of  
 84 Theorem 5.6. The case of cubics, which is the starting point of the induction, is proved by generalizing  
 85 the approach of [4], where we restricted to a codimension three linear subspace. This case is the crucial  
 86 step which allows to prove the Theorem 1.1. Section 4 is devoted to this case, which requires a lot of  
 87 effort and technical details, in the setting of discrete mathematics. Compared with the quick proof we  
 88 gave in [4], here we are forced to divide the proof in several cases and subcases. While the induction  
 89 argument works quite smoothly for  $n, d \gg 0$ , it is painful to cover many of the initial cases. In the case  
 $d = 3$  we need the help of a computer, by a Montecarlo technique explained in the appendix.

90 A further remark is necessary. In [1,4] the result about the independence of double points was  
 91 shown to be equivalent, through Terracini lemma, to a statement about the dimension of higher secant  
 92 varieties of the Veronese varieties, which in turn is related to the Waring problem for polynomials. Here  
 93 the assumption that  $K$  is algebraically closed of zero characteristic is necessary to translate safely the  
 94 results, see also Theorem 6.1 and Remark 6.3 in [10]. For example, on the real numbers, the closure in  
 95 the euclidean topology of the locus of secants to the twisted cubic is a semi-algebraic set, corresponding  
 96 to the cubic polynomials which have not three distinct real roots, and it is defined by the condition  
 97 that the discriminant is nonpositive. Indeed a real cubic polynomial can be expressed as the sum of  
 98 two cubes of linear polynomials (Waring problem) if and only if it has two distinct complex conjugate  
 99 roots or a root of multiplicity three.

## 100 2. Preliminaries

101 Let  $X$  be a scheme contained in a collection of double points of  $\mathbf{P}^n$ . We say that the type of  $X$  is  
 102  $(m_1, \dots, m_{n+1})$  if  $X$  contains exactly  $m_i$  subschemes of a double point of length  $i$ , for  $i = 1, \dots, n+1$ .  
 103 For example the type of  $k$  double points is  $(0, \dots, 0, k)$ . The degree of  $X$  is  $\deg X = \sum im_i$ . A scheme  
 104 of type  $(m_1, \dots, m_{n+1})$  corresponds to a collection of  $m_i$  linear subspaces  $L_i \subseteq \mathbf{P}^n$  with  $\dim L_i = i - 1$   
 105 and with a marked point on each  $L_i$ .

106 Algebraic families of such schemes can be defined over any field  $K$  with the Zariski topology.

107 Any irreducible component  $\zeta$  of length  $k$  contained in a double point supported at the point  $p$   
 108 corresponds to a linear space  $L$  of projective dimension  $k - 1$  passing through  $p$ . The hypersurfaces  
 109 containing  $\zeta$  are exactly the hypersurfaces  $F$  such that  $T_p F \supseteq L$ .

110 This description allows to consider a *degeneration* (or collision) of two components as the span of  
 111 the corresponding linear spaces. More precisely, consider two irreducible schemes  $\zeta_0, \zeta_1$ , supported  
 112 respectively at  $p_0, p_1$ , of length respectively  $k_0, k_1$  and consider the space  $V(\zeta_0, \zeta_1)$  of the hypersurfaces  
 113 containing  $\zeta_0$  and  $\zeta_1$ . Let  $L_i$  be the space corresponding to  $\zeta_i$ . By the above remark this space consists  
 114 of the hypersurfaces  $F$  such that  $T_{p_0} F \supseteq L_0$  and  $T_{p_1} F \supseteq L_1$ .

115 Let  $L = \langle L_0, L_1 \rangle$  be the projective span of  $L_0$  and  $L_1$ , that is the smallest projective space containing  
 116  $L_0$  and  $L_1$ . If  $L_0$  and  $L_1$  are general, and if moreover  $k_0 + k_1 - 1 \leq n$ , then  $\dim L = k_0 + k_1 - 1$  and  $L$   
 117 corresponds to an irreducible scheme  $\zeta$  of length  $k_0 + k_1$  supported at  $p_0$  (or at  $p_1$ ). It is not difficult  
 118 to construct a degeneration of  $\zeta_0 \cup \zeta_1$  which has  $\zeta$  as limit.

119 This implies, by semicontinuity, that  $\dim V(\zeta_0 \cup \zeta_1) \leq \dim V(\zeta)$ .

120 In particular if we prove that  $V(\zeta)$  has the expected dimension, the same is true for  $V(\zeta_0 \cup \zeta_1)$ . We  
 121 will use often this remark through the paper.

122 We recall now some notation and results from [4].

Given a zero-dimensional subscheme  $X \subseteq \mathbf{P}^n$ , the corresponding ideal sheaf  $\mathcal{I}_X$  and a linear system  
 $\mathcal{D}$  on  $\mathbf{P}^n$ , the Hilbert function is defined as follows:

$$h_{\mathbf{P}^n}(X, \mathcal{D}) := \dim H^0(\mathcal{D}) - \dim H^0(\mathcal{I}_X \otimes \mathcal{D}).$$

123 If  $h_{\mathbf{P}^n}(X, \mathcal{D}) = \deg X$ , we say that  $X$  is  $\mathcal{D}$ -independent, and in the case  $\mathcal{D} = \mathcal{O}_{\mathbf{P}^n}(d)$ , we say  $d$ -  
 124 independent.

125 A zero-dimensional scheme is called curvilinear if it is contained in the smooth part of a curve.  
 126 Notice that a curvilinear scheme contained in a double point has length 1 or 2.

127 **Lemma 2.1** (Curvilinear Lemma [6,4]). *Let  $X$  be a zero-dimensional scheme of finite length contained in  
 128 a union of double points of  $\mathbf{P}^n$  and  $\mathcal{D}$  a linear system on  $\mathbf{P}^n$ . Then  $X$  is  $\mathcal{D}$ -independent if and only if every  
 129 curvilinear subscheme of  $X$  is  $\mathcal{D}$ -independent.*

130 For any scheme  $X \subset L$  in a projective space  $L$ , we denote  $\mathcal{I}_X(d) = \mathcal{I}_X \otimes \mathcal{O}_L(d)$  and  
 131  $I_{X,L}(d) = H^0(\mathcal{I}_X(d))$ . The expected dimension of the vector space  $I_{X,\mathbf{P}^n}(d)$  is  $\text{exdim}(I_{X,\mathbf{P}^n}(d)) =$   
 132  $\max\left(\binom{n+d}{n} - \deg X, 0\right)$ .

For any scheme  $X \subseteq \mathbf{P}^n$  and any hyperplane  $H \subseteq \mathbf{P}^n$ , the residual of  $X$  with respect to  $H$  is denoted by  $X : H$  and it is defined by the ideal sheaf  $\mathcal{I}_{X:H} = \mathcal{I}_X : \mathcal{I}_H$ . We have, for any  $d$ , the well known *Castelnuovo sequence*

$$0 \rightarrow I_{X:H, \mathbf{P}^n}(d-1) \rightarrow I_{X, \mathbf{P}^n}(d) \rightarrow I_{X \cap H, H}(d).$$

133 **Remark 2.2.** If  $Y \subseteq X \subseteq \mathbf{P}^n$  are zero-dimensional schemes, then

- 134 • if  $X$  is  $d$ -independent, then so is  $Y$ ,  
135 • if  $h_{\mathbf{P}^n}(Y, d) = \binom{d+n}{n}$ , then  $h_{\mathbf{P}^n}(X, d) = \binom{d+n}{n}$ .

136 It follows that if any zero-dimensional scheme  $X \subseteq \mathbf{P}^n$  with  $\deg X = \binom{d+n}{n}$  is  $d$ -independent, then  
137 any scheme contained in  $X$  imposes independent conditions on hypersurfaces of degree  $d$  in  $\mathbf{P}^n$ .

138 **Remark 2.3.** Fix  $n \geq 2$  and  $d \geq 3$ . Assume that if a scheme  $X$  with degree  $\binom{d+n}{n}$  does not impose  
139 independent conditions on hypersurfaces of degree  $d$  in  $\mathbf{P}^n$ , then it is of type  $(m_1, \dots, m_{n+1})$  for some  
140 given  $m_i$ . It follows that any subscheme of  $X$  is  $d$ -independent. Indeed any proper subscheme  $Y$  of  $X$  is  
141 also a subscheme of a scheme  $X'$  with degree  $\binom{d+n}{n}$  and of type  $(m'_1, \dots, m'_{n+1}) \neq (m_1, \dots, m_{n+1})$ ,  
142 for some  $m'_i$  and since  $X'$  is  $d$ -independent, so is  $Y$ . Moreover any scheme  $Z$  containing  $X$  impose  
143 independent conditions on hypersurfaces of degree  $d$  if it contains a scheme  $X''$  with degree  $\binom{d+n}{n}$   
144 and of type  $(m''_1, \dots, m''_{n+1}) \neq (m_1, \dots, m_{n+1})$  for some  $m''_i$ . Indeed since  $X''$  imposes independent  
145 conditions on hypersurfaces of degree  $d$ , also  $Z$  does.

### 146 3. Quadratic polynomials

Assume that  $X$  is a general scheme of type  $(m_1, \dots, m_{n+1})$ . Let us fix an order on the irreducible components  $\xi_1, \dots, \xi_m$  of  $X$  (where  $m = \sum m_i$ ) such that

$$\text{length}(\xi_1) \geq \dots \geq \text{length}(\xi_m)$$

and for any  $1 \leq i \leq m$  let us denote by  $l_i$  the length of  $\xi_i$  and by  $p_i$  the point where  $\xi_i$  is supported. Set  $l_i = 0$  for  $i > m$ . For any  $1 \leq i \leq n$  let us denote

$$\delta_X(i) = \max \left\{ 0, \sum_{j=1}^i l_j - \sum_{j=1}^i (n+2-j) \right\}.$$

147 Note that  $\delta_X(1) = 0$  for any scheme  $X$ . Clearly  $\delta_X(2) = 0$  unless  $X$  is the union of two double points  
148 and in this case  $\delta_X(2) = 1$ . If  $\delta_X(2) = 0$ , then  $\delta_X(3) = 0$  unless  $l_1 = n+1, l_2 = l_3 = n$ , where  
149  $\delta_X(3) = 1$ . If  $\delta_X(2) = \delta_X(3) = 0$ , then  $\delta_X(4) = 0$  unless either  $l_1 = n+1, l_2 = n, l_3 = l_4 = n-1$ ,  
150 where  $\delta_X(4) = 1$ , or  $l_1 = l_2 = l_3 = n$  and  $l_4 \geq n-1$ , where  $1 \leq \delta_X(4) \leq 2$ .

151 **Lemma 3.1.** If  $\delta_X(i) > 0$  for some  $1 \leq i \leq n$ , then the quadrics containing  $\{\xi_1, \dots, \xi_i\}$  are exactly the  
152 quadrics singular along the linear space spanned by  $p_1, \dots, p_i$ .

153 **Proof.** Let us denote  $\mathbf{P}^n = \mathbf{P}(V)$ , fix a basis  $\{e_0, \dots, e_n\}$  of  $V$  and assume that  $p_j = [e_{n+2-j}]$  for all  
154  $j = 1, \dots, i$ . Let  $A$  be the symmetric matrix defining a quadric  $\mathcal{Q}$  in  $\mathbf{P}(V)$  passing through the scheme  
155  $\{\xi_1, \dots, \xi_i\}$ . Therefore  $\mathcal{Q}$  is defined in  $V$  by the equation  $\{v \in V : v^T A v = 0\}$  and the condition that  
156 the quadric contains  $\xi_j$  means that  $e_{n+2-j}^T A w = 0$  for any  $w \in W$ , where  $W$  is a general subspace of  $V$   
157 of dimension  $l_j$ . Then, it is easy to see that the condition  $\sum_{j=1}^i l_j \geq \sum_{j=1}^i (n+2-j)$  implies that the

158 elements of the last  $i$  columns and rows of the matrix  $A$  are all equal to 0. This implies that the quadric  
159  $\mathcal{Q}$  is singular along the linear space spanned by  $\{p_1, \dots, p_i\}$ .  $\square$

160 From the previous lemma it follows that if  $\delta_X(i)$  is positive for some  $1 \leq i \leq n$ , then the scheme  
161  $\{\xi_1, \dots, \xi_i\}$  does not impose independent conditions on quadrics. Indeed the scheme  $\{\xi_1, \dots, \xi_i\}$  has  
162 degree  $\sum_{j=1}^i l_j$ , but imposes only  $\sum_{j=1}^i (n+2-j) = \binom{n+2}{2} - \binom{n-i+2}{2}$  conditions on quadrics.

163 The following result describes all the schemes which impose independent conditions on quadrics,  
164 giving necessary and sufficient conditions.

165 **Theorem 3.2.** *A general zero-dimensional scheme  $X \subset \mathbf{P}^n$  contained in a union of double points of type*  
166  *$(m_1, \dots, m_{n+1})$  imposes independent conditions on quadrics if and only if one of the following conditions*  
167 *takes place:*

- 168 (1) either  $\delta_X(i) = 0$  for all  $1 \leq i \leq n$ ;  
169 (2) or  $\deg X \geq \binom{n+2}{2} + \max\{\delta_X(i) : 1 \leq i \leq n\}$ .

**Proof.** First we prove that if  $X$  does impose independent conditions on quadrics, then either condition 1 or 2 hold. Assume that both conditions are false and let us prove that  $I_X(2)$  has not the expected dimension  $\max\{0, \binom{n+2}{2} - \deg(X)\}$ . In particular assume that there is an index  $i \in \{1, \dots, n\}$  such that  $\delta_X(i) > 0$  and  $\deg(X) < \binom{n+2}{2} + \delta_X(i)$ . Consider the family  $\mathcal{C}$  of quadratic cones with vertex containing the linear space  $\mathbf{P}^{i-1}$  spanned by  $p_1, \dots, p_i$ . Of course we have

$$\dim I_X(2) \geq \dim(\mathcal{C}) - \left( \deg(X) - \sum_{j=1}^i l_j \right) = \binom{n-i+2}{2} - \deg(X) + \sum_{j=1}^i l_j =: c$$

Now, using  $\binom{n+2}{2} - \binom{n-i+2}{2} = \sum_{j=1}^i (n+2-j)$ , we compute

$$\dim I_X(2) - \text{expdim} I_X(2) \geq \min \left\{ c, \sum_{j=1}^i l_j - \binom{n+2}{2} + \binom{n-i+2}{2} \right\} = \min\{c, \delta_X(i)\}$$

By assumption  $\delta_X(i) > 0$  and

$$c > \binom{n-i+2}{2} - \binom{n+2}{2} - \delta_X(i) + \sum_{j=1}^i l_j = \sum_{j=1}^i l_j - \sum_{j=1}^i (n+2-j) - \delta_X(i) = 0$$

170 Hence the dimension of  $I_X(2)$  is higher than the expected dimension and we have proved that  $X$  does  
171 not impose independent conditions on quadrics.

172 Now we want to prove that if either condition 1 or condition 2 hold, then  $X$  imposes independent  
173 conditions on quadrics. We work by induction on  $n \geq 2$ . If  $n = 2$  it is easy to check directly our claim.

Consider a scheme  $X$  in  $\mathbf{P}^n$  which satisfies condition 1 and fix a hyperplane  $H \subset \mathbf{P}^n$ . We specialize all the components of  $X$  on  $H$  in such a way that the residual of each of the components  $\xi_1, \dots, \xi_n$  is 1 (if the component is not empty) and the residual of the remaining components is zero. Indeed the vanishing  $\delta_X(2) = 0$  implies that  $l_j \leq n$  for all  $j \geq 2$ , and so such a specialization is possible. Then we get the Castelnuovo sequence

$$0 \rightarrow I_{X:H, \mathbf{P}^n}(1) \rightarrow I_{X, \mathbf{P}^n}(2) \rightarrow I_{X \cap H, H}(2)$$

174 where  $X : H$  is the residual given by at most  $n$  simple points and  $X \cap H$  is the trace in  $H$ . Hence we  
175 conclude by induction once we have proved that the trace  $X \cap H$  satisfies condition 1 or 2.

Note that in order to compute  $\delta_{X \cap H}(i)$  we need to choose an order on the components  $\xi_i \cap H$  of  $X \cap H$  such that the sequence of their lengths is not increasing. If

$$\text{length}(\xi_n) - 1 = \text{length}(\xi_n \cap H) \geq \text{length}(\xi_{n+1} \cap H) = \text{length}(\xi_{n+1}) \quad (3)$$

then we can choose the same order on the components of  $X \cap H$  chosen for the components of  $X$ . In this case it is easy to prove that  $X \cap H$  satisfies condition 1. Indeed for any  $i \geq 1$ , let us denote by  $l'_i = \text{length}(\xi_i \cap H)$ . Recall that  $m$  is the number of components of  $X$ . By construction we have that  $l'_i = l_i - 1$  for any  $1 \leq i \leq \min\{n, m\}$ . Then for all  $1 \leq i \leq \min\{n - 1, m\}$  we have

$$\sum_{j=1}^i l'_j - \sum_{j=1}^i (n + 1 - j) = \sum_{j=1}^i l_j - i - \sum_{j=1}^i (n + 2 - j) + i = \sum_{j=1}^i l_j - \sum_{j=1}^i (n + 2 - j)$$

from which we have

$$\delta_{X \cap H}(i) = \max \left\{ 0, \sum_{j=1}^i l'_j - \sum_{j=1}^i (n + 1 - j) \right\} = \max \left\{ 0, \sum_{j=1}^i l_j - \sum_{j=1}^i (n + 2 - j) \right\} = \delta_X(i) = 0$$

176 Now assume that (3) does not hold. This implies in particular that  $l_n = l_{n+1}$ , and so when we  
 177 compute  $\delta_{X \cap H}(i)$  we have to change the order on the components. In order to better understand the  
 178 situation, let us consider the following example:  $X$  in  $\mathbf{P}^5$  given by 9 components of length 4. Note that  
 179  $\delta_X(i) = 0$  for any  $1 \leq i \leq 5$ . After the specialization described above we get a scheme  $X \cap H$  in  
 180  $H \cong \mathbf{P}^4$  given by 5 components of length 3 and 4 components of length 4. We easily compute that  
 181  $\delta_{X \cap H}(4) = 2 > 0$ .

182 Now we will prove that if  $X \cap H$  does not satisfy condition 1, then it satisfies 2. Assume that for  
 183  $X$  in  $\mathbf{P}^n$  we have  $\delta_X(i) = 0$  for all  $1 \leq i \leq n$ , while for  $X \cap H$  in  $H$  we have  $\delta_{X \cap H}(i) > 0$  for some  
 184  $1 \leq i \leq n - 1$ .

185 Let us denote  $l := l_n = l_{n+1}$  and let  $1 \leq k < n$  be the index such that  $l_k > l_{k+1} = \dots = l_n =$   
 186  $l_{n+1} = l$ . Let  $h$  be the index such that  $\delta_{X \cap H}(h) = \max\{\delta_{X \cap H}(i)\}$  and note that  $h > k$ .

As above we denote by  $l'_i$  the lengths of the components of  $X \cap H$  ordered in a not increasing way. Hence we have,

$$l'_1 = l_1 - 1, \dots, l'_k = l_k - 1, l'_{k+1} = l, \dots, l'_h = l, \dots$$

187 and this implies that  $l_k > l_{k+1} = \dots = l_n = \dots = l_{n+h-k} = l$ .

Now since  $\delta_{X \cap H}(h) > 0$  we obtain

$$\sum_{i=1}^h l'_i = \sum_{i=1}^k l_i - k + (h - k)l > \sum_{i=1}^h (n + 1 - i) = \sum_{i=1}^h (n + 2 - i) - h$$

188 and since  $\delta_X(k) = 0$  we have  $\sum_{i=1}^k l_i \leq \sum_{i=1}^k (n + 2 - i)$ , and combining these two inequalities we  
 189 have

$$(h - k)l > \sum_{i=k+1}^h (n + 2 - i) - (h - k) > (h - k)(n + 1 - h)$$

from which it follows:

$$l \geq (n + 2 - h). \quad (4)$$

Now in order to prove that  $X \cap H$  satisfies 2 we need to show that

$$\text{deg}(X \cap H) \geq \binom{n + 1}{2} + \delta_{X \cap H}(h).$$

Notice that

$$\deg(X \cap H) \geq \deg X - n \geq \sum_{i=1}^{n+h-k} l_i - n,$$

hence if we prove the following inequality we are done:

$$\sum_{i=1}^{n+h-k} l_i - n \geq \binom{n+1}{2} + \delta_{X \cap H}(h)$$

i.e.

$$\sum_{i=1}^k l_i + (n+h-2k)l - n \geq \binom{n+1}{2} + \sum_{i=1}^k l_i - k + (h-k)l - \sum_{i=1}^h (n+1-i)$$

which reduces to

$$(n-k)l \geq \binom{n+1}{2} + n - k - h(n+1) + \binom{h+1}{2}$$

By using inequality (4) it is enough to prove, for any  $n \geq 2$ , any  $1 \leq k < h \leq n-1$ , the inequality

$$(n-k)(n+2-h) \geq \binom{n+1}{2} + (n-k) - h(n+1) + \binom{h+1}{2} \tag{5}$$

and we prove this inequality by induction on  $h \leq n-1$ . First fix  $n, k$  and choose  $h = n-1$ . In this case (5) becomes

$$3(n-k) \geq \binom{n+1}{2} + (n-k) - (n^2-1) + \binom{n}{2} = n-k+1$$

190 which is true. Now if we assume that (5) is verified for  $h' \leq n-1$ , it is easy to check it for  $h = h' - 1$ ,  
191 thus completing the proof of (5).

192 It remains to prove that if  $X$  satisfies condition 2, then the system of quadrics  $|\mathcal{I}_X(2)|$  containing  $X$   
193 is empty. If  $\delta_X(i) = 0$  for all  $1 \leq i \leq n$  then we are in the previous case. We may assume that there  
194 exists  $i$  such that  $\delta_X(i) > 0$ .

195 Assume that the sequence  $\{\delta_X(i)\}$  is nondecreasing Then  $\delta_X(n) > 0$  and by Lemma 3.1 we know  
196 that the quadrics containing the first  $n$  components  $\{\xi_1, \dots, \xi_n\}$  are singular along the hyperplane  
197  $H = \langle p_1, \dots, p_n \rangle$ , so the only existing quadric is the double hyperplane  $H^2$ . By assumption  $\deg X >$   
198  $\left[\binom{n+2}{2} - 1\right] + \delta_X(n) = \sum_{j=1}^n l_j$ , hence there is at least an extra condition given by another component  
199  $\xi_{n+1}$  of  $X$  and so  $|\mathcal{I}_X(2)| = \emptyset$  as we wanted.

Therefore we may assume that there exists  $1 \leq i < n$  such that  $\delta_X(i+1) < \delta_X(i)$  and we pick the first such  $i$ . In particular it follows

$$l_{i+1} < n+1-i \tag{6}$$

200 As above, by Lemma 3.1 all the quadrics containing  $X_0 = \{\xi_1, \dots, \xi_i\}$  are singular along the linear  
201 space  $L_0 = \langle p_1, \dots, p_i \rangle$ . Let  $X_1 = X \setminus X_0$ . By definition  $\deg X_0 = \sum_{j=1}^i l_j = \binom{n+2}{2} - \binom{n+2-i}{2} + \delta_X(i)$ .

202 Let  $\pi$  be a general projection from  $L_0$  on a linear space  $L_1 \simeq \mathbf{P}^{n-i}$ . By (6) we have  $\deg X_1 =$   
203  $\deg \pi(X_1)$ . Hence there is a bijective correspondence between  $|\mathcal{I}_X(2)|$  and  $|\mathcal{I}_{\pi(X_1)}(2)| \subseteq |\mathcal{O}_{L_1}(2)|$ . By  
204 generality we may assume that  $X_1$  is supported outside  $L_0$ .

Note that

$$\deg \pi(X_1) - \binom{n-i+2}{2} = \deg X - \binom{n+2}{2} - \delta_X(i) \geq \max_h \{\delta_X(h)\} - \delta_X(i) \geq 0$$

hence if  $\delta_{\pi(X_1)}(h) = 0$  for  $h = 1, \dots, n - i$  we conclude again by the first case. If there exists  $1 \leq j \leq n - i$  such that  $\delta_{\pi(X_1)}(j) > 0$ , notice that in this case we have

$$\delta_{\pi(X_1)}(j) = \delta_X(j + i) - \delta_X(i),$$

hence

$$\max_p \{\delta_{\pi(X_1)}(p)\} = \max_h \{\delta_X(h)\} - \delta_X(i).$$

So we proved that

$$\deg \pi(X_1) - \binom{n+2-i}{2} \geq \max_p \{\delta_{\pi(X_1)}(p)\}$$

205 This means that  $\pi(X_1)$  satisfies the assumption 2 on  $L_1$  and then by (complete) induction on  $n$  we  
206 get that  $|\mathcal{I}_{\pi(X_1)}(2)| = \emptyset$  as we wanted.  $\square$

207 A straightforward consequence of the previous theorem is the following corollary.

208 **Corollary 3.3.** *A general zero-dimensional scheme  $X \subset \mathbf{P}^n$  contained in a union of double points with*  
209  *$\deg X = \binom{n+2}{2}$  imposes independent conditions on quadrics if and only if  $\delta_X(i) = 0$  for all  $1 \leq i \leq n$ .*

210 Theorem 3.2 provides a classification of all the types of general subschemes  $X$  of a collection of  
211 double points of  $\mathbf{P}^n$  which do not impose independent conditions on quadrics. For example in  $\mathbf{P}^2$ , the  
212 only case is  $X$  given by two double points. In  $\mathbf{P}^3$  and in  $\mathbf{P}^4$  we have the following lists of subschemes  
213 which do not impose independent conditions on quadrics.

#### 214 4. Cubic polynomials

215 In this section we generalize the approach of [4, Section 3] to the following  
216 result.

217 **Theorem 4.1.** *A general zero-dimensional scheme  $X \subset \mathbf{P}^n$  consisting of  $n$  double points imposes*  
218 *independent conditions on cubics with the only exception of  $n = 4$  and  $X$  given by 7 double points.*

219 First we give the proof of the previous theorem in cases  $n = 2, 3, 4$ .

220 **Lemma 4.2.** *Let be  $n = 2, 3$  or 4. Then a general zero-dimensional scheme  $X \subset \mathbf{P}^n$  contained in a union*  
221 *of double points imposes independent conditions on cubics with the only exception of  $n = 4$  and  $X$  given*  
222 *by 7 double points.*

223 **Proof.** By Remark 2.2 it is enough to prove the statement for  $X$  with degree  $\binom{n+3}{3}$ . Note that if  $X$  is a  
224 union of double points the statement is true by the Alexander–Hirschowitz theorem.

Let  $n = 2$  and  $X$  a subscheme of a collection of double points with  $\deg X = 10$ . Fix a line  $H$  in  $\mathbf{P}^2$  and consider the Castelnuovo exact sequence

$$0 \rightarrow I_{X:H, \mathbf{P}^2}(2) \rightarrow I_{X, \mathbf{P}^2}(3) \rightarrow I_{X \cap H}(3)$$

225 It is easy to prove that it is always possible to specialize some components of  $X$  on  $H$  so that  $\deg(X \cap$   
226  $H) = 4$  and that the residual  $X : H$  does not contain two double points. The last condition ensures  
227 that  $\delta_{X:H}(i) = 0$  for  $i = 1, 2$ . Hence we conclude by Corollary 3.3.

228 In the case  $n = 3$ , the scheme  $X$  has degree 20. Since there are no cubic surfaces with five singular  
229 points (in general position) we can assume that  $X$  contains at most three double points. Indeed if  $X$   
230 contains 4 double points we can degenerate it to a collection of 5 double points, in general position.

In Tables 1 and 2 below we list the subschemes which do not impose independent conditions on quadrics in  $\mathbf{P}^3$  and  $\mathbf{P}^4$ .

**Table 1**  
List of exceptions in  $\mathbf{P}^3$ .

$X$	$\deg X$	$\max\{\delta_X(i)\}$	$(m_1, \dots, m_4)$	$\dim I_X(2)$
4, 4, 4	12	3	(0, 0, 0, 3)	1
4, 4, 3	11	2	(0, 0, 1, 2)	1
4, 4, 2	10	1	(0, 1, 0, 2)	1
4, 4, 1, 1	10	1	(2, 0, 0, 2)	1
4, 4, 1	9	1	(1, 0, 0, 2)	2
4, 4	8	1	(0, 0, 0, 2)	3
4, 3, 3	10	1	(0, 0, 2, 1)	1

231 We fix a plane  $H$  in  $\mathbf{P}^3$  and we want to specialize some components of  $X$  on  $H$  so that  $\deg(X \cap H) = 10$   
 232 and that the residual  $X : H$  imposes independent conditions on quadrics. By looking at Table 1, since  
 233  $\deg(X : H) = 10$ , it is enough to require that  $X : H$  is not of the form  $(0, 1, 0, 2)$ ,  $(2, 0, 0, 2)$  or  
 234  $(0, 0, 2, 1)$ . It is easy to check that this is always possible: indeed specialize on  $H$  the components  
 235 of  $X$  starting from the ones with higher length and keeping the residual as minimal as possible until  
 236 the degree of the trace is 9 or 10. If the degree of the trace is 9 and there is in  $X$  a component with  
 237 length 1 or 2 we can obviously complete the specialization. The only special case is given by  $X$  of type  
 238  $(0, 0, 4, 2)$  and in this case we specialize on  $H$  the two double points and two components of length  
 239 3 so that each of them has residual 1.

240 If  $n = 4$  the case of 7 double points is exceptional. Assume that  $X$  has degree 35 and contains at  
 241 most 6 double points. We fix a hyperplane  $H$  of  $\mathbf{P}^4$  and we want to specialize some components of  $X$  on  
 242  $H$  so that  $\deg(X \cap H) = 20$  and that the residual  $X : H$  imposes independent conditions on quadrics.  
 243 By looking at Table 2, it is enough to require that  $X : H$  does not contain two double points, does  
 244 not contain one double point and two components of length 4 and it is not of the form  $(0, 0, 1, 3, 0)$ .  
 245 It is possible to satisfy this conditions by specializing the components of  $X$  in the following way: we  
 246 specialize the components of  $X$  on  $H$  starting from the ones with higher length and keeping the residual  
 247 as minimal as possible until the degree of the trace is maximal and does not exceed 20. Then we add  
 248 some components allowing them to have residual 1 in order to reach the degree 20. It is possible to  
 249 check that this construction works, except for the case  $(0, 0, 5, 0, 4)$  where we have to specialize on  $H$   
 250 all the double points and 2 of the components with length 3 so that both have residual 1. It is easy also  
 251 to check that following the construction above the residual has always the desired form, except for  $X$   
 252 of the form  $(0, 0, 1, 8, 0)$ , where the above rule gives a residual of type  $(0, 0, 1, 3, 0)$ . In this case we  
 253 make a specialization ad hoc: for example we can put on  $H$  six components of length 4 and the unique  
 254 component of length 3 in such a way that all them have residual 1 and we obtain a residual of type  
 255  $(7, 0, 0, 2, 0)$  which is admissible.

256 Now we have to check the schemes either contained in 7 double points or containing 7 double  
 257 points. But this follows immediately by Remark 2.3 as in [3]

We want to restrict a zero dimensional scheme  $X$  of  $\mathbf{P}^n$  to a given subvariety  $L$ . We could define the  
 residual  $X : L$  as a subscheme of the blow-up of  $\mathbf{P}^n$  along  $L$  [3], but we prefer to consider  $\deg(X : L)$   
 just as an integer associated to  $X$  and  $L$ . More precisely given a subvariety  $L \subset \mathbf{P}^n$ , we denote  $\deg(X : L) = \deg X - \deg(X \cap L)$ . In particular we will use this notion in the following cases:

$$\deg(X : L), \quad \deg(X : (L \cup M)), \quad \deg(X : (L \cup M \cup N))$$

where  $L, M, N \subset \mathbf{P}^n$  are three general subspaces of codimension three. We also recall that

$$\deg(X \cap (L \cup M)) = \deg(X \cap L) + \deg(X \cap M) - \deg(X \cap (L \cap M))$$

258 and

$$\begin{aligned} \deg(X \cap (L \cup M \cup N)) &= \deg(X \cap L) + \deg(X \cap M) + \deg(X \cap N) - \deg(X \cap L \cap M) \\ &\quad - \deg(X \cap L \cap N) - \deg(X \cap M \cap N) + \deg(X \cap L \cap M \cap N). \end{aligned}$$

259 The proof of Theorem 4.1 relies on a preliminary description, which is inspired to the approach of  
 260 [4]. More precisely the proof is structured as follows:

**Table 2**  
List of exceptions in  $\mathbf{P}^4$ .

$X$	$\deg X$	$\max\{\delta_X(i)\}$	$(m_1, \dots, m_5)$	$\dim I_X(2)$
5,5,5,5	20	6	(0, 0, 0, 0, 4)	1
5,5,5,4	19	5	(0, 0, 0, 1, 3)	1
5,5,5,3	18	4	(0, 0, 1, 0, 3)	1
5,5,5,2	17	3	(0, 1, 0, 0, 3)	1
5,5,5,1,1	17	3	(2, 0, 0, 0, 3)	1
5,5,5,1	16	3	(1, 0, 0, 0, 3)	2
5,5,5	15	3	(0, 0, 0, 0, 3)	3
5,5,4,4	18	4	(0, 0, 0, 2, 2)	1
5,5,4,3	17	3	(0, 0, 1, 1, 2)	1
5,5,4,2	16	2	(0, 1, 0, 1, 2)	1
5,5,4,1,1	16	2	(2, 0, 0, 1, 2)	1
5,5,4,1	15	2	(1, 0, 0, 1, 2)	2
5,5,4	14	2	(0, 0, 0, 1, 2)	3
5,5,3,3	16	2	(0, 0, 2, 0, 2)	1
5,5,3,2	15	1	(0, 1, 1, 0, 2)	1
5,5,3,1,1	15	1	(2, 0, 1, 0, 2)	1
5,5,3,1	14	1	(1, 0, 1, 0, 2)	2
5,5,3	13	1	(0, 0, 1, 0, 2)	3
5,5,2,2,1	15	1	(1, 2, 0, 0, 2)	1
5,5,2,2	14	1	(0, 2, 0, 0, 2)	2
5,5,2,1,1,1	15	1	(3, 1, 0, 0, 2)	1
5,5,2,1,1	14	1	(2, 1, 0, 0, 2)	2
5,5,2,1	13	1	(1, 1, 0, 0, 2)	3
5,5,2	12	1	(0, 1, 0, 0, 2)	4
5,5,1,1,1,1,1	15	1	(5, 0, 0, 0, 2)	1
5,5,1,1,1,1	14	1	(4, 0, 0, 0, 2)	2
5,5,1,1,1	13	1	(3, 0, 0, 0, 2)	3
5,5,1,1	12	1	(2, 0, 0, 0, 2)	4
5,5,1	11	1	(1, 0, 0, 0, 2)	5
5,5	10	1	(0, 0, 0, 0, 2)	6
5,4,4,2	15	1	(0, 1, 0, 2, 1)	1
5,4,4,1,1	15	1	(2, 0, 0, 2, 1)	1
5,4,4,1	14	1	(1, 0, 0, 2, 1)	2
5,4,4	13	1	(0, 0, 0, 2, 1)	3
4,4,4,4	16	2	(0, 0, 0, 4, 0)	1
4,4,4,3	15	1	(0, 0, 1, 3, 0)	1

- 261 – in Proposition 4.3 below we generalize [4, Proposition 5.2],  
 262 – in Proposition 4.7 and Proposition 4.8 we generalize [4, Proposition 5.3],  
 263 – the analogue of [4, Proposition 5.4] is contained in Lemma 4.9, Lemma 4.10, Lemma 4.11 and Propo-  
 264 sition 4.12.

265 **Proposition 4.3.** Let  $n \geq 8$  and let  $L, M, N \subset \mathbf{P}^n$  be general subspaces of codimension 3. Let  $X =$   
 266  $X_L \cup X_M \cup X_N$  be a general scheme contained in a union of double points, where  $X_L$  (resp.  $X_M, X_N$ ) is  
 267 supported on  $L$  (resp.  $M, N$ ), such that the triple  $(\deg(X_L : L), \deg(X_M : M), \deg(X_N : N))$  is one of the  
 268 following

- 269 (i) (6, 9, 12)  
 270 (ii) (3, 12, 12)  
 271 (iii) (0, 12, 15)  
 272 (iv) (6, 6, 15)  
 273 (v) (0, 9, 18)

274 then there are no cubic hypersurfaces in  $\mathbf{P}^n$  which contain  $L \cup M \cup N$  and which contain  $X$ .

275 **Proof.** For  $n = 8$  it is an explicit computation, which can be easily performed with the help of a  
 276 computer (see the appendix).

For  $n \geq 9$  the statement follows by induction on  $n$ . Indeed if  $n \geq 8$  it is easy to check that there are no quadrics containing  $L \cup M \cup N$ . Then given a general hyperplane  $H \subset \mathbf{P}^n$  the Castelnuovo sequence induces the isomorphism

$$0 \longrightarrow I_{L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{(L \cup M \cup N) \cap H, H}(3) \longrightarrow 0$$

hence specializing the support of  $X$  on the hyperplane  $H$ , since the space  $I_{L \cup M \cup N, \mathbf{P}^n}(2)$  is empty, we get

$$0 \longrightarrow I_{X \cup L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{(X \cup L \cup M \cup N) \cap H, H}(3)$$

277 then our statement immediately follows by induction.  $\square$

278 **Remark 4.4.** It seems likely that the previous proposition holds with much more general assumption.  
279 Anyway the general assumption  $\deg(X_L : L) + \deg(X_M : M) + \deg(X_N : N) = 27$  is too weak,  
280 indeed the triple  $(0, 6, 21)$  cannot be added to the list of the Proposition 4.3. Indeed there are two  
281 independent cubic hypersurfaces in  $\mathbf{P}^8$ , containing  $L, M, N$ , two general double points on  $M$  and seven  
282 general double points on  $N$ , as it can be easily checked with the help of a computer (see the appendix).  
283 Quite surprisingly, the triple  $(0, 0, 27)$  could be added to the list of the Proposition 4.3, and we think  
284 that this phenomenon has to be better understood. In Proposition 4.3 we have chosen exactly the  
285 assumptions that we will need in the following propositions, in order to minimize the number of the  
286 initial checks.

287 For the specialization technique we need the following two easy remarks.

288 **Remark 4.5.** Let  $L, N$  be two codimension three subspaces of  $\mathbf{P}^n$ , for  $n \geq 5$ . Let  $\xi$  be a general  
289 scheme contained in a double point  $p^2$  supported on  $L$  such that  $\deg(\xi : L) = a$ ,  $0 \leq a \leq 3$ .  
290 Then there is a specialization  $\eta$  of  $\xi$  such that the support of  $\eta$  is on  $L \cap N$ ,  $\deg(\eta : L) = a$  and  
291  $\deg((\eta \cap N) : (L \cap N)) = a$ .

292 **Remark 4.6.** Let  $L$  be a codimension three subspaces of  $\mathbf{P}^n$ . Let  $X$  be a scheme contained in a double  
293 point  $p^2$ .

- 294 (i) If  $\deg X = n + 1$  then there is a specialization  $Y$  of  $X$  which is supported at  $q \in L$  such that  
295  $\deg(Y : L) = 3$ .  
296 (ii) If  $\deg X = n$  then there are two possible specializations  $Y$  of  $X$  which are supported at  $q \in L$   
297 such that  $\deg(Y : L) = 3$  or  $2$ .  
298 (iii) If  $\deg X = n - 1$  then there are three possible specializations  $Y$  of  $X$  which are supported at  
299  $q \in L$  such that  $\deg(Y : L) = 3, 2$  or  $1$ .  
300 (v) If  $\deg X \leq n - 2$  then there are four possible specializations  $Y$  of  $X$  which are supported at  $q \in L$   
301 such that  $\deg(Y : L) = 3, 2, 1$  or  $0$ .

302 **Proposition 4.7.** Let  $n \geq 5$  and let  $L, M \subset \mathbf{P}^n$  be subspaces of codimension three. Let  $X = X_L \cup X_M \cup X_O$   
303 be a scheme contained in a union of double points such that  $X_L$  (resp.  $X_M$ ) is supported on  $L$  (resp.  $M$ ) and it  
304 is general among the schemes supported on  $L$  (resp.  $M$ ) and  $X_O$  is general. Assume that the following further  
305 conditions hold:

$$\deg(X_L : L) + \deg(X_M : M) + \deg X_O = 9(n - 1),$$

$$n - 2 \leq \deg(X_L : L) \leq \deg(X_M : M) \leq 4n - 6,$$

$$3n + 3 \leq \deg X_O \leq 3n + 6.$$

306 Then there are no cubic hypersurfaces in  $\mathbf{P}^n$  which contain  $L \cup M$  and which contain  $X$ .

307 **Proof.** For  $n = 5, 6, 7$  it is an explicit computation (see the appendix).

For  $n \geq 8$ , the statement follows by induction from  $n - 3$  to  $n$ . Indeed given a third general codimension three subspace  $N$ , we get the exact sequence

$$0 \longrightarrow I_{L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{L \cup M, \mathbf{P}^n}(3) \longrightarrow I_{(L \cup M) \cap N, N}(3) \longrightarrow 0$$

308 where the dimensions of the three spaces in the sequence are respectively 27,  $9(n - 1)$  and  $9(n - 4)$ .

309 We will specialize now some components of  $X_L$  on  $L \cap N$  and some components of  $X_M$  on  $M \cap N$ .  
310 We denote by  $X'_L$  the union of the components of  $X_L$  supported on  $L \setminus N$  and by  $X''_L$  the union of the  
311 components of  $X_L$  supported on  $L \cap N$ . Since  $n \geq 5$  we may assume also that  $\deg(X''_L : (L \cup N)) = 0$ .  
312 Analogously let  $X'_M$  and  $X''_M$  denote the corresponding subschemes of  $X_M$ . Now we describe more  
313 explicitly the specialization.

From the assumption

$$3n + 3 \leq \deg X_0 \leq 3n + 6$$

314 it follows that in particular  $X$  has at least three irreducible components and so we may specialize all  
315 the components of  $X_0$  on  $N$  in such a way that  $\deg(X_0 : N) = 9$ .

Notice that the degree of the trace  $X_0 \cap N = \deg X_0 - 9$  satisfies the same inductive hypothesis

$$3(n - 3) + 3 \leq \deg(X_0 \cap N) \leq 3(n - 3) + 6$$

and we have

$$6n - 15 \leq \deg(X_L : L) + \deg(X_M : M) \leq 6n - 12$$

If  $\deg(X_M : M) \leq 3n$ , by using that

$$\deg(X_L : L) \leq \frac{1}{2} (\deg(X_L : L) + \deg(X_M : M)) \leq \deg(X_M : M)$$

we get

$$3n - 7 \leq \deg(X_M : M) \leq 3n$$

$$3n - 15 \leq \deg(X_L : L) \leq 3n - 6$$

then we can specialize  $X_M$  and  $X_L$  in such a way that  $\deg(X'_M : M) = 12$  and  $\deg(X'_L : L) = 6$ , indeed the conditions

$$n - 5 \leq \deg(X_M : M) - 12 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) - 6 \leq 4n - 18$$

316 are true for  $n \geq 8$  and guarantee that the inductive assumptions are true on the trace.

Now if  $\deg(X_M : M) \geq 3n + 1$ , we have

$$3n + 1 \leq \deg(X_M : M) \leq 4n - 6$$

$$2n - 9 \leq \deg(X_L : L) \leq 3n - 13$$

and we can specialize in such a way that  $\deg(X'_L : L) = 0$  and  $\deg(X'_M : M) = 18$ . Indeed we have, for  $n \geq 6$

$$n - 5 \leq \deg(X_M : M) - 18 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) \leq 4n - 18$$

In any of the previous cases, the residual satisfies the assumptions of Proposition 4.3, while the trace  $(X \cup L \cup M) \cap N$  satisfies the inductive assumptions on  $N = \mathbf{P}^{n-3}$ . In conclusion by using the sequence

$$0 \longrightarrow I_{X \cup L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{X \cup L \cup M, \mathbf{P}^n}(3) \longrightarrow I_{(X \cup L \cup M) \cap N, N}(3)$$

317 we complete the proof.  $\square$

318 The following proposition is analogous to the previous one, with a different assumption on  $\deg X_0$ .  
 319 In this case we need an extra assumption on  $X_L$  and  $X_M$ , namely that in one of them there are enough  
 320 irreducible components with residual different from 2. The reason for this choice is that it makes  
 321 possible to find a suitable specialization with residual 3, 9 or 15, by the Remark 4.5 (if all the components  
 322 have residual 2, this should not be possible).

323 From now on we denote by  $X_L^i$  (resp.  $X_M^i$ ) for  $i = 1, 2, 3$  the union of the irreducible components  $\xi$   
 324 of  $X_L$  (resp.  $X_M$ ) such that  $\deg(\xi : L) = i$  (resp.  $\deg(\xi : M) = i$ ).

325 **Proposition 4.8.** *Let  $n \geq 5$  and let  $L, M \subset \mathbf{P}^n$  be subspaces of codimension three. Let  $X = X_L \cup X_M \cup X_0$   
 326 be a scheme contained in a union of double points such that  $X_L$  (resp.  $X_M$ ) is supported on  $L$  (resp.  $M$ ) and  
 327 it is general among the schemes supported on  $L$  (resp.  $M$ ) and  $X_0$  is general. Assume that either the number  
 328 of the irreducible components of  $X_L^1 \cup X_L^3$ , or that the number of the irreducible components of  $X_M^1 \cup X_M^3$  is  
 329 at least  $\frac{n-2}{3}$ . Assume that the following further conditions hold:*

$$\deg(X_L : L) + \deg(X_M : M) + \deg X_0 = 9(n - 1),$$

$$n - 2 \leq \deg(X_L : L) \leq \deg(X_M : M) \leq 4n - 6,$$

$$3n + 7 \leq \deg X_0 \leq 5n + 2.$$

330 Then there are no cubic hypersurfaces in  $\mathbf{P}^n$  which contain  $L \cup M$  and which contain  $X$ .

331 **Proof.** For  $n = 5, 6, 7$  it is an explicit computation (see the appendix), and the thesis is true even  
 332 without the assumption on  $X_L^1 \cup X_L^3$ .

For  $n \geq 8$  the statement follows by induction from  $n - 3$  to  $n$ , by using possibly also Proposition 4.7.  
 As in the previous proof, given a third general codimension three subspace  $N$ , we get the exact sequence

$$0 \longrightarrow I_{L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{L \cup M, \mathbf{P}^n}(3) \longrightarrow I_{(L \cup M) \cap N, N}(3) \longrightarrow 0$$

333 We will specialize now some components of  $X_L$  on  $L \cap N$  and some components of  $X_M$  on  $M \cap N$ .  
 334 We use the same notations as in the previous proof, and we describe more precisely the specialization  
 335 in the following two cases.

1. Assume first that

$$3n + 7 \leq \deg X_0 \leq 4n + 7$$

In particular  $X$  has at least four irreducible components and we may specialize all the compo-  
 nents of  $X_0$  on  $N$  in such a way that

$$\deg((X_0 \cap N) : N) = 12$$

and so we have

$$5n - 16 \leq \deg(X_L : L) + \deg(X_M : M) \leq 6n - 16$$

In particular it follows

$$\frac{5n}{2} - 8 \leq \deg(X_M : M) \leq 4n - 6$$

$$n - 2 \leq \deg(X_L : L) \leq 3n - 8$$

336 We divide into two subcases.

337 In the first one we assume that the number of the irreducible components of  $X_L^1 \cup X_L^3$  is at  
 338 least  $\frac{n-2}{3}$ . In this case we can specialize  $X_M$  and  $X_L$  in such a way that  $\deg(X_M' : M) = 12$  and  
 339  $\deg(X_L' : L) = 3$ . Moreover there exists a specialization such that  $X_L''$  has at least  $\frac{n-5}{3} = \frac{n-2}{3} - 1$   
 340 components with residual 1 or 3. Indeed in  $X_L'$  we keep at most one of these components, and if

341 we are forced to keep three components of length one, it means that there are no components  
342 of length 2 in  $X_L$ , which implies our claim.

Notice that the conditions

$$n - 5 \leq \deg(X_M : M) - 12 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) - 3 \leq 4n - 18$$

343 are true for  $n \geq 10$ . They are also true for  $n \geq 8$  as soon as  $\deg(X_M : M) \geq n + 7$ , so we need only  
344 to check the cases  $8 \leq n \leq 9$  and  $\deg(X_M : M) \leq n + 6$ , which implies  $\deg(X_L : L) \geq 4n - 22$ .

In this case we specialize  $X_M$  and  $X_L$  in such a way that  $\deg(X'_M : M) = 6$ ,  $\deg(X'_L : L) = 9$  and  $X''_L$  has at least  $\frac{n-5}{3} = \frac{n-2}{3} - 1$  components with residual 1 or 3. The conditions

$$n - 5 \leq \deg(X_M : M) - 6 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) - 9 \leq 4n - 18$$

345 are true if  $n = 9$  or if  $n = 8$  and  $\deg(X_L : L) \geq n + 4$ .

So the remaining cases to be considered are when  $n = 8$ ,  $\deg(X_M : M) \leq n + 6 = 14$ , and  $\deg(X_L : L) \leq n + 3 = 11$ , that is when the triple

$$(\deg(X_L : L), \deg(X_M : M), \deg X_O)$$

346 is one of the following: (10, 14, 39), (11, 13, 39), (11, 14, 38), which have been checked with  
347 random choices (see the appendix) with a computer.

348 In the second subcase, we know that the number of the irreducible components of  $X_M^1 \cup X_M^3$   
349 is at least  $\frac{n-2}{3}$ . Then we can specialize  $X_M$  and  $X_L$  in such a way that  $\deg(X'_M : M) = 9$  and  
350  $\deg(X'_L : L) = 6$ . As above it is easy to check that there exists a specialization such that  $X''_M$  has  
351 at least  $\frac{n-5}{3} = \frac{n-2}{3} - 1$  components with residual 1 or 3.

Notice that the conditions

$$n - 5 \leq \deg(X_M : M) - 9 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) - 6 \leq 4n - 18$$

352 are true for  $n \geq 8$  as soon as one of the following conditions is satisfied

353 (a)  $\deg(X_M : M) \leq 4n - 17$ , which implies  $\deg(X_L : L) \geq n + 1$ .

354 (b)  $n = 8$ ,  $\deg(X_L : L) \geq n + 1 = 9$ , which implies  $\deg(X_M : M) \leq 5n - 17 = 23$

355 Assume then that (a) and (b) are not satisfied.

We have  $4n - 16 \leq \deg(X_M : M) \leq 4n - 6$  and we specialize  $X_M$  and  $X_L$  in such a way that  $\deg(X'_M : M) = 15$  and  $\deg(X'_L : L) = 0$ . The conditions

$$n - 5 \leq \deg(X_M : M) - 15 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) \leq 4n - 18$$

356 are true for  $n \geq 9$  or if  $n = 8$  and  $\deg(X_M : M) \geq n + 10$ .

So the remaining cases to be considered are when  $n = 8$ ,  $4n - 16 = 16 \leq \deg(X_M : M) \leq$   
 $n + 9 = 17$  and (by case (b))  $\deg(X_L : L) \leq 8$ . The only remaining case are cases

$$(\deg(X_L : L), \deg(X_M : M), \deg X_O) = (7, 17, 39), (8, 16, 39), (8, 17, 38)$$

357 which we have checked with a computer.

2. Assume now that

$$4n + 8 \leq \deg X_O \leq 5n + 2$$

which implies

$$4n - 11 \leq \deg(X_L : L) + \deg(X_M : M) \leq 5n - 17$$

358 In particular  $X$  has at least five irreducible components and we may specialize all the components  
359 of  $X_0$  on  $N$  in such a way that  $\deg((X_0 \cap N) : N) = 15$ .

In this case we have

$$2n - 5 \leq \deg(X_M : M) \leq 4n - 6$$

$$n - 2 \leq \deg(X_L : L) \leq \frac{5n - 17}{2}$$

and we can specialize  $X_M$  and  $X_L$  in such a way that  $\deg(X'_M : M) = 12$  and  $\deg(X'_L : L) = 0$ .  
Notice that the conditions

$$n - 5 \leq \deg(X_M : M) - 12 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) \leq 4n - 18$$

360 are true for  $n \geq 12$  and also for  $n \geq 8$  as soon as  $\deg(X_M : M) \geq n + 7$ .

361 Assume now that  $8 \leq n \leq 11$  and  $\deg(X_M : M) \leq n + 6$ , which implies  $\deg(X_L : L) \geq 3n - 17$ .

In this case we specialize  $X_M$  and  $X_L$  in such a way that  $\deg(X'_M : M) = 6$  and  $\deg(X'_L : L) = 6$ .  
The conditions

$$n - 5 \leq \deg(X_M : M) - 6 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) - 6 \leq 4n - 18$$

362 are true for  $n \geq 9$  and also for  $n = 8$  if  $\deg(X_L : L) \geq n + 1$ .

363 The only remaining cases to be considered are then

$n = 8, 7 \leq \deg(X_L : L) \leq 8$ , and  $\deg(X_M : M) \leq n + 6 = 14$  that is when the triple

$$(\deg(X_L : L), \deg(X_M : M), \deg(X_0))$$

364 is one of the following:  $(7, 14, 42)$ ,  $(8, 13, 42)$ ,  $(8, 14, 41)$  which we have checked with a  
365 computer.

In conclusion in any previous case we conclude by using the sequence

$$0 \longrightarrow I_{X \cup L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{X \cup L \cup M, \mathbf{P}^n}(3) \longrightarrow I_{(X \cup L \cup M) \cap N, N}(3)$$

366 since the trace  $(X \cup L \cup M) \cap N$  satisfies the inductive assumptions on  $N = \mathbf{P}^{n-3}$  and the residual  
367 satisfies the hypotheses of Proposition 4.3.  $\square$

368 Let  $X_0 \subset \mathbf{P}^n$  be a scheme, contained in a union of double points, of degree  $(n + 1)^2 + \alpha$  with  
369  $0 \leq \alpha \leq n - 1$  and  $M$  be a subspace of codimension three. Assume that  $n \geq 8$  and that  $X_0$  contains  
370 at most one component of degree  $\leq 3$ . Let  $h_i$  be the number of components of  $X_0$  of degree  $i$  for  
371  $i = 4, \dots, n + 1$  and let  $h$  ( $0 \leq h \leq 3$ ) be the degree of the component of  $X_0$  of degree  $\leq 3$ . Note that  
372  $\sum_{i=4}^{n+1} ih_i + h = (n + 1)^2 + \alpha$ . Let us choose an order on the irreducible components of  $X_0$  in such a  
373 way the length of any component is non increasing.

374 We consider one of the following two specializations  $X_0 = X'_0 \cup X_M$  where  $X_M$  is supported on  $M$   
375 and it contains the possible component of degree  $\leq 3$ , and  $X'_0$  is supported outside  $M$ :

(a) we choose as  $X'_0$  the union of the irreducible components of  $X_0$ , starting from the ones with  
maximal length, in such a way that  $\deg X'_0 = 3(n + 1) + \beta \geq 3(n + 1) + \alpha$  and it is minimal. By  
construction  $0 \leq \beta - \alpha \leq n$ . Let  $a_i$  be the number of components of  $X_M = X_0 \setminus X'_0$  of degree  $i$  for  
 $i = 4, \dots, n + 1$ . Then

$$\sum_{i=4}^{n+1} ia_i + h = \deg(X_M) = (n + 1)(n - 2) + \alpha - \beta$$

(\hat{a}) we choose as  $X'_0$  the union of the irreducible components of  $X_0$ , starting from the ones with  
maximal length, in such a way that  $\deg X'_0 = 3(n + 1) + \hat{\beta} \geq 3(n + 1)$  and it is minimal. By construction

0 ≤ β̂ ≤ n − 1. Let â<sub>i</sub> be the number of components of X<sub>M</sub> = X<sub>O</sub> \ X'<sub>O</sub> of degree i for i = 4, . . . , n + 1. Then

$$\sum_{i=4}^{n+1} i\hat{a}_i + h = \deg(X_M) = (n + 1)(n - 2) + \alpha - \hat{\beta}$$

376 In both the specializations let us denote: γ = deg(X<sub>M</sub> ∩ M) − (n − 2)<sup>2</sup> and note that we have some  
 377 freedom to specialize X<sub>M</sub> on M, according to Remark 4.6. If we have a specialization with deg(X<sub>M</sub> ∩ M) =  
 378 p and another specialization with deg(X<sub>M</sub> ∩ M) = q then for any value between p and q there is a  
 379 suitable specialization such that deg(X<sub>M</sub> ∩ M) attains that value. We will use often this technique by  
 380 evaluating the maximum (resp. the minimum) possible value of deg(X<sub>M</sub> ∩ M) under a specialization.

**Lemma 4.9.** *If in the specialization (a) we have*

$$a_n + 2a_{n-1} + 3 \sum_{i=4}^{n-2} a_i \leq 1$$

381 then we have a<sub>n+1</sub> ≠ 0 and there exists a specialization of type (a) such that γ = α ≤ n − 4.

382 **Proof.** From the assumptions it follows that a<sub>j</sub> = 0 for any j = 4, . . . , n − 1 and a<sub>n</sub> = i with 0 ≤ i ≤ 1.  
 383 Then X<sub>M</sub> consists of points of maximal length n + 1 with at most one component of length h and  
 384 at most one component of length n. Hence X'<sub>O</sub> consists only of double points and this implies that  
 385 β is a multiple of n + 1. Hence we have a<sub>n+1</sub> =  $\frac{(n+1)(n-2)+\alpha-\beta-h-in}{n+1}$ , which is an integer, so that  
 386  $\frac{\alpha-h-i(n+1)+i}{n+1}$  is an integer, so that α = h − i ≤ n − 4.

387 It follows that a<sub>n+1</sub> = n − 2 − i, hence the maximum degree of X<sub>M</sub> ∩ M is (n − 2)<sup>2</sup> + h, the  
 388 minimum degree is (n − 2 − i)(n − 2) + i(n − 3) + (h − 1) = (n − 2)<sup>2</sup> + (h − i − 1), and we can  
 389 choose γ = h − i = α. □

**Lemma 4.10.** *If in the specialization (a) we have*

$$3a_{n+1} + 2a_n + a_{n-1} \geq 3n - 7 + \alpha - \beta$$

390 then there exists a specialization of type (â) such that either γ = α ≤ n − 4 or γ = α − 3 ≤ n − 4.

**Proof.** Assume first a<sub>n+1</sub> = 0. Since α − β ≥ −n, from the assumption it follows

$$2a_n + a_{n-1} \geq 2n - 7$$

Notice also that

$$a_n + a_{n-1} \leq \frac{(n + 1)(n - 2) + \alpha - \beta}{n} \leq n - 2 + \frac{n - 2}{n}$$

hence

$$a_n + a_{n-1} \leq n - 2.$$

These two conditions imply that we have only the following possibilities:

$$(a_n, a_{n-1}) \in \{(n - 2, 0)(n - 3, 0), (n - 4, 1), (n - 3, 1), (n - 4, 2), (n - 5, 3)\}$$

In all these cases, by performing the specialization of type (â), we have n − 3 ≤ β̂ ≤ n − 1 or β̂ = 0. Moreover it is easy to check that â<sub>n</sub> = a<sub>n</sub> if α ≤ β̂, â<sub>n</sub> = a<sub>n</sub> + 1 if α > β̂, and â<sub>j</sub> = a<sub>j</sub> for any j ≤ n − 1. In any case the difference δ between the maximum degree of the trace X<sub>M</sub> ∩ M and the minimum degree satisfies

$$\delta \geq \hat{a}_n + 2\hat{a}_{n-1} + 3 \sum_{i=4}^{n-2} \hat{a}_i + \max\{h - 1, 0\}.$$

We have  $\deg(X_M) = \sum_{i=4}^n \widehat{a}_i + h = (n+1)(n-2) + \alpha - \widehat{\beta}$  and so

$$\sum_{i=4}^{n-2} \widehat{a}_i + h \geq (n+1)(n-2) - \widehat{\beta} - n\widehat{a}_n - (n-1)\widehat{a}_{n-1}.$$

In the first two cases, where  $(a_n, a_{n-1}) = (a, 0)$  and  $n-3 \leq a \leq n-2$ , we assume first  $n-3 \leq \widehat{\beta}$ , then the maximal degree of the trace  $X_M \cap M$  is

$$(n-2)^2 + \alpha + 1 \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n \leq (n-2)^2 + \alpha + 3$$

since  $\widehat{a}_n \geq n-3$ , moreover  $\delta \geq n-2 \geq 6$  and so we have that either  $\gamma = \alpha$ , or  $\gamma = \alpha - 3$  work. It remains the case  $\widehat{\beta} = 0$  where we get that in  $X'_0$  we have three points of length  $n+1$ , then either  $\beta = 0$  and  $\alpha = 0$ , or  $\beta = n$  and  $\alpha > 0$ . By substituting in the hypothesis of our lemma the values  $(a_{n+1}, a_n, a_{n-1}) = (0, a, 0)$  we get  $\beta = n$  and  $0 < \alpha \leq 3$ . In this case the maximal degree  $\mathcal{M}$  of the trace  $X_M \cap M$  satisfies

$$(n+1)(n-2) + \alpha + (n-4) \leq \mathcal{M} \leq (n-2)^2 + \alpha + (n-2)$$

391 and, since  $\delta \geq n-2$ , the choice  $\gamma = \alpha$  works.

Now consider the case  $(a_n, a_{n-1}) = (a, 1)$ , where  $n-4 \leq a \leq n-3$ . Assume first  $n-3 \leq \widehat{\beta}$ , then the maximal degree of the trace  $X_M \cap M$  is

$$(n-2)^2 + \alpha \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n - 1 \leq (n-2)^2 + \alpha + 4$$

since  $n-4 \leq \widehat{a}_n \leq n-2$ , moreover  $\delta \geq n-1 \geq 7$  so that either  $\gamma = \alpha$ , or  $\gamma = \alpha - 3$  work. It remains the case  $\widehat{\beta} = 0$ , where we have either  $\beta = 0$  and  $\alpha = 0$ , or  $\beta = n$  and  $\alpha > 0$ . By substituting in the hypothesis of our lemma the values  $(a_{n+1}, a_n, a_{n-1}) = (0, a, 1)$ , for  $n-4 \leq a \leq n-3$ , we get  $\beta = n$  and  $0 < \alpha \leq 2$ . Then we have  $\widehat{a}_n = n-2$  and so the maximal degree of the trace  $X_M \cap M$  is

$$(n+1)(n-2) + \alpha - 2(n-2) - 1 = (n-2)^2 + \alpha + (n-3)$$

392 and since the difference  $\delta \geq n-1$ , the choice  $\gamma = \alpha$  works.

In the case  $(a_n, a_{n-1}) = (n-4, 2)$ , if  $n-3 \leq \widehat{\beta}$ , then the maximal degree of the trace  $X_M \cap M$  is

$$(n-2)^2 + \alpha + 1 \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n - 2 \leq (n-2)^2 + \alpha + 3$$

and since  $\delta \geq n \geq 6$  it follows that either  $\gamma = \alpha$ , or  $\gamma = \alpha - 3$  work. It remains the case  $\widehat{\beta} = 0$  where  $\beta = 0$  or  $\beta = n$ . By substituting in the hypothesis of our lemma the values  $(a_{n+1}, a_n, a_{n-1}) = (0, n-4, 2)$  we get  $\beta = n$  and  $\alpha = 1$ . In this case the maximal degree of the trace  $X_M \cap M$  is

$$(n+1)(n-2) + 1 - 2(n-3) - 2 = (n-2)^2 - 1 + n$$

393 and since  $\delta \geq n+1$  we can choose  $\gamma = \alpha = 1$ .

In the last case  $(a_n, a_{n-1}) = (n-5, 3)$ , if  $n-3 \leq \widehat{\beta}$ , then the maximal degree of the trace  $X_M \cap M$  is

$$(n-2)^2 + \alpha \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n - 3 \leq (n-2)^2 + \alpha + 4$$

394 and since  $\delta \geq n \geq 7$  it follows that either  $\gamma = \alpha$ , or  $\gamma = \alpha - 3$  work. It remains the case  $\widehat{\beta} = 0$   
395 where  $\beta = 0$  or  $\beta = n$ . By substituting in the hypothesis of our lemma the values  $(a_{n+1}, a_n, a_{n-1}) =$   
396  $(0, n-5, 3)$  we get  $\beta = n$  and  $\alpha = 0$ , which is a contradiction. Then this case is impossible.

Now assume that  $a_{n+1} \neq 0$ . In this case we have also  $\beta = 0$ , hence it follows  $\widehat{\beta} = 0$  and  $\widehat{a}_j = a_j$  for any  $4 \leq j \leq n+1$ . By assumption we have

$$3a_{n+1} + 2a_n + a_{n-1} \geq 3n - 7$$

and, as in the first case, we also have

$$a_{n+1} + a_n + a_{n-1} \leq n - 2$$

397 These two inequalities imply that  $(a_{n+1}, a_n, a_{n-1})$  lies in the tetrahedron with vertices  $(n-2, 0, 0)$ ,  
 398  $(n-3, 1, 0)$ ,  $(n-\frac{7}{3}, 0, 0)$ ,  $(n-\frac{5}{2}, 0, \frac{1}{2})$ . The only integer points in this tetrahedron are  $(n-2, 0, 0)$   
 399 and  $(n-3, 1, 0)$ .

In the case  $(n-2, 0, 0)$  the maximal degree of the trace  $X_M \cap M$  is

$$(n+1)(n-2) + \alpha - 3(n-2) = (n-2)^2 + \alpha$$

and clearly the minimal degree is  $(n-2)^2$ , thus one of the choices  $\gamma = \alpha$  or  $\gamma = \alpha - 3$  works. In the case  $(n-3, 1, 0)$  the maximal degree of the trace  $X_M \cap M$  is

$$(n+1)(n-2) + \alpha - 3(n-3) - 2 = (n-2)^2 + \alpha + 1$$

400 and the minimal degree is obviously  $(n-2)^2$ , so that one of the choices  $\gamma = \alpha$  or  $\gamma = \alpha - 3$  works.  $\square$

401 **Lemma 4.11.** *If all the assumptions of Lemma 4.9 and Lemma 4.10 are not satisfied, then there exists  $\gamma' \geq 0$*   
 402 *satisfying  $\gamma' + 2 \leq n - 4$ , and every  $\gamma \in [\gamma', \gamma' + 2]$  can be attained by a convenient specialization of*  
 403 *type (a).*

**Proof.** The maximal degree of the trace  $X_M \cap M$  is

$$\mathcal{M} := (n+1)(n-2) + \alpha - \beta - 3a_{n+1} - 2a_n - a_{n-1}$$

404 Since the assumption of Lemma 4.10 are not satisfied, we have  $\mathcal{M} \geq (n-2)^2 + 2$ .

405 The minimal possible degree of the trace  $X_M \cap M$  is

$$\begin{aligned} m &:= \sum_{i=4}^{n+1} (i-3)a_i + \min\{1, h\} = (n+1)(n-2) + \alpha - \beta - 3 \sum_{i=4}^{n+1} a_i + \min\{1-h, 0\} \\ &\leq (n+1)(n-2) - 3 \sum_{i=4}^{n+1} a_i \leq (n+1)(n-2) - 3(n-2) = (n-2)^2 \end{aligned}$$

where we use the fact that  $\sum_{i=4}^{n+1} a_i \geq n-2$ . This is true because either  $a_{n+1} = n-2$  or  $a_{n+1} \leq n-3$  and we have

$$\sum_{i=4}^n a_i \geq \frac{(n+1)(n-2 - a_{n+1}) + \alpha - \beta}{n} > n-2 - a_{n+1} - 1.$$

406 Hence if  $\mathcal{M} \leq n-4$  we choose  $\gamma' = \mathcal{M} - (n-2)^2 - 2$ . Otherwise if  $\mathcal{M} \geq n-3$  we choose  
 407  $\gamma' = n-6$ .

Both cases work because of the assumption

$$\mathcal{M} - m = a_n + 2a_{n-1} + 3 \sum_{i=4}^{n-2} a_i - \min\{1-h, 0\} \geq 2. \quad \square$$

408 We can now prove the last preliminary proposition. Recall that we denote by  $X_L^i$  for  $i = 1, 2, 3$  the  
 409 union of the irreducible components  $\xi$  of  $X_L$  such that  $\deg(\xi : L) = i$ .

410 **Proposition 4.12.** *Let  $n \geq 5$  and let  $L \subset \mathbf{P}^n$  be a subspace of codimension three. Let  $X = X_L \cup X_0$  be a*  
 411 *scheme contained in a union of double points such that  $X_L$  is supported on  $L$  and is general among the schemes*  
 412 *supported on  $L$  and  $X_0$  is general. Assume that  $\deg(X_L : L) + \deg X_0 = \binom{n+3}{3} - \binom{n}{3} = \frac{3}{2}n^2 + \frac{3}{2}n + 1$ ,*  
 413 *and that  $\deg X_0 = (n+1)^2 + \alpha$ , for  $0 \leq \alpha \leq n-1$ . We also assume that the number of the irreducible*  
 414 *components of  $X_L^1 \cup X_L^3$  is  $\geq \frac{n}{3}$ . Then there are no cubic hypersurfaces in  $\mathbf{P}^n$  which contain  $L$  and which*  
 415 *contain  $X$ .*

416 **Proof.** For  $n = 5, 6, 7$  it is a direct computation (see the appendix).

For  $n \geq 8$  the statement follows by induction, and by the sequence

$$0 \longrightarrow I_{L \cup M, \mathbf{P}^n}(3) \longrightarrow I_{L, \mathbf{P}^n}(3) \longrightarrow I_{L \cap M, M}(3) \longrightarrow 0$$

where  $M$  is a general codimension three subspace. We get

$$0 \longrightarrow I_{X \cup L \cup M, \mathbf{P}^n}(3) \longrightarrow I_{X \cup L, \mathbf{P}^n}(3) \longrightarrow I_{(X \cup L) \cap M, M}(3).$$

First by Lemmas 4.9, 4.10, 4.11 we can specialize  $X_0 = X'_0 \cup X_M$  in such a way that  $\deg X'_0 = 3(n+1) + \beta$  we will call in the following  $\widehat{\beta} = \beta$ ,  $X_M$  is supported on  $M$  and  $\deg(X_M \cap M) = (n-2)^2 + \gamma$ , where  $0 \leq \beta \leq 2n-1$ ,  $0 \leq \gamma \leq n-4$ ,  $\gamma = \alpha \pmod{3}$  and  $\alpha - \beta - n \leq \gamma \leq \alpha$ . Notice also that we have  $\alpha - \beta - \gamma \geq -2n + 4$ . It follows that

$$n-2 \leq \deg(X_M : M) = 3(n-2) + \alpha - \beta - \gamma \leq 4n-6$$

Moreover let us specialize  $X_L = X'_L \cup X''_L$  where  $X'_L$  is supported on  $L \setminus M$  and  $X''_L$  is supported on  $L \cap M$ . We may also assume that the number of irreducible components of  $(X''_L)^1 \cup (X''_L)^3$  is  $\geq \frac{n-3}{3}$ . We may assume that

$$2n-5 \leq \deg(X'_L : L) = 3(n-2) + \gamma - \alpha \leq 3(n-2)$$

indeed note that  $3(n-2) + \gamma - \alpha = 0 \pmod{3}$  and there exist at least  $\frac{n}{3}$  irreducible component in  $(X'_L)^1 \cup (X'_L)^3$ . Note that by using the minimal number of irreducible component in  $(X'_L)^1 \cup (X'_L)^3$ , at least  $\frac{n}{3} - 1$  components remain in  $X''_L$ , preserving our inductive assumption. It follows that

$$\deg(X'_L : L) + \deg(X_M : M) + \deg X'_0 = 9(n-1)$$

moreover we have clearly

$$4n-11 \leq \deg(X'_L : L) + \deg(X_M : M) \leq 6n-12$$

and we may apply Proposition 4.7 and Proposition 4.8, since the scheme  $X'_L \cup X_M \cup X'_0$  satisfies the corresponding assumptions. Then we conclude by induction, indeed the scheme  $(X_M \cup X'_L) \cap M$  satisfies our assumptions with respect to the spaces  $M$  and  $M \cap L \subset M$ . Precisely we have (by subtraction)

$$\deg((X''_L \cap M) : (L \cap M)) + \deg(X_M \cap M) = \frac{3}{2}(n-3)^2 + \frac{3}{2}(n-3) + 1,$$

417 and  $\deg(X_M \cap M) = (n-2)^2 + \gamma$ , where  $0 \leq \gamma \leq n-4 \quad \square$

418 We are finally in position to give the proof of the main theorem.

*Proof of Theorem 4.1.* We fix a codimension three linear subspace  $L \subset \mathbf{P}^n$  and we prove the statement by induction by using the exact sequence

$$0 \longrightarrow I_{L, \mathbf{P}^n}(3) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^n}(3)) \longrightarrow H^0(\mathcal{O}_L(3)).$$

419 We prove the claim by induction on  $n$  from  $n-3$  to  $n$ . By Lemma 4.2 we know that the theorem  
420 holds for  $n = 2, 3, 4$ . Let  $X$  be a general scheme contained in a collection of double points and with  
421  $\deg X = \binom{n+3}{3}$

Since  $n \geq 5$  we can assume that  $X$  contains at most one component of length  $\leq 3$ . Fix a codimension three linear subspace  $L \subset \mathbf{P}^n$  and consider the exact sequence

$$0 \longrightarrow I_{X \cup L, \mathbf{P}^n}(3) \longrightarrow I_{X, \mathbf{P}^n}(3) \longrightarrow I_{X \cap L, L}(3) \tag{7}$$

422 We want to specialize on  $L$  some components of  $X$  so that  $\deg(X \cap L) = \binom{n}{3}$  and apply Proposition 4.12.

423 We keep outside  $L$  the irreducible components of  $X$  starting from the ones with maximal length in  
424 such a way that  $\deg X_0 = (n+1)^2 + \alpha \geq (n+1)^2$  and it is minimal. We get by construction that

425  $\alpha \leq n - 1$ . Let  $a_i$  be the number of components of  $X_L = X \setminus X_0$  of degree  $i$  for  $i = 4, \dots, n + 1$  and  
 426 let  $h$  be the degree of the component of  $X$  of length  $\leq 3$ . Then  $\sum_{i=4}^{n+1} ia_i + h = \binom{n+3}{3} - (n+1)^2 - \alpha$ .

After the specialization, the minimum degree of the trace  $X_L \cap L$  is

$$\sum_{i=4}^{n+1} (i-3)a_i + 1 = \binom{n+3}{3} - (n+1)^2 - \alpha - h - 3 \sum_{i=4}^{n+1} a_i + 1$$

if  $h \geq 1$  or

$$\sum_{i=4}^{n+1} (i-3)a_i = \binom{n+3}{3} - (n+1)^2 - \alpha - 3 \sum_{i=4}^{n+1} a_i$$

if  $h = 0$ . On the other hand the maximum degree of the trace  $X_L \cap L$  is

$$\binom{n+3}{3} - (n+1)^2 - \alpha - 3a_{n+1} - 2a_n - a_{n-1}$$

We want to prove that  $\binom{n}{3}$  belongs to the range between the minimum and the maximum of  $\deg(X_L \cap L)$ . This is implied by the inequalities

$$\alpha + 3a_{n+1} + 2a_n + a_{n-1} \leq \frac{n(n-1)}{2} \tag{8}$$

and

$$\frac{n(n-1)}{2} \leq \alpha + h + 3 \sum_{i=4}^{n+1} a_i - 1, \quad \text{or} \quad \frac{n(n-1)}{2} \leq \alpha + 3 \sum_{i=4}^{n+1} a_i \tag{9}$$

427 In order to prove the inequality (8), consider first the case  $a_{n+1} \neq 0$ . Then  $\alpha = 0$  and we have

$$\begin{aligned} a_{n+1} + \frac{2}{3}a_n + \frac{1}{3}a_{n-1} &\leq \frac{1}{n+1} \sum_{i=4}^{n+1} ia_i = \frac{1}{n+1} \left[ \binom{n+3}{3} - (n+1)^2 - h \right] \\ &= \frac{n(n-1)}{6} - \frac{h}{n+1} \leq \frac{n(n-1)}{6} \end{aligned}$$

428 as we wanted. If  $a_{n+1} = 0$  we get

$$\begin{aligned} 2a_n + a_{n-1} + \alpha &\leq \frac{2}{n} \sum_{i=4}^{n+1} ia_i + \alpha = \frac{2}{n} \left[ \binom{n+3}{3} - (n+1)^2 - h - \alpha \right] \\ + \alpha &\leq \frac{2}{n} \left[ \binom{n+3}{3} - (n+1)^2 \right] + (n-1) \left( 1 - \frac{2}{n} \right) \end{aligned}$$

which is  $\leq \frac{n(n-1)}{2}$  if  $n \geq 6$ , as we wanted. In order to prove the inequality (9), notice that

$$\sum_{i=4}^{n+1} a_i \geq \frac{1}{n+1} \sum_{i=4}^{n+1} ia_i = \frac{n(n-1)}{6} - \frac{\alpha + h}{n+1}$$

429 then if  $h = 0$  we conclude since  $\alpha \left( 1 - \frac{3}{n+1} \right) \geq 0$ , while if  $h \geq 1$  we conclude by the inequality  
 430  $(\alpha + h) \left( 1 - \frac{3}{n+1} \right) \geq 1$ , which is true if  $\alpha + h \geq 2$ , in particular if  $\alpha \geq 1$ .

431 Consider the last case  $\alpha = 0$  and  $h \geq 1$ . If  $n \not\equiv 2 \pmod{3}$ , so that  $\frac{n(n-1)}{6}$  is an integer, then  $X \setminus X_0$   
 432 contains at least  $\frac{n(n-1)}{6} + 1$  irreducible components and this confirms the inequality. If  $n \equiv 2 \pmod{3}$ ,

433 even  $\lfloor \frac{n(n-1)}{6} \rfloor$  double points and one component of length 3 are not enough to cover all  $X \setminus X_0$ . Then  
 434  $X \setminus X_0$  contains at least  $\lfloor \frac{n(n-1)}{6} \rfloor + 2$  irreducible components and again the inequality is confirmed.

435 Then a suitable specialization of  $X_L$  exists such that  $\deg(X_L \cap L) = \binom{n}{3}$ . We denote again by  $X_L^i$  for  
 436  $i = 1, 2, 3$  the union of irreducible components  $\xi$  of  $X_L$  such that  $\deg(\xi : L) = i$ .

437 In order to apply Proposition 4.12 we need only to show that the irreducible components of  $X_L^1 \cup X_L^3$   
 438 are at least  $\frac{n}{3}$ . If this condition is not satisfied, we show now that it is possible to choose another  
 439 suitable specialization such that again  $\deg(X_L \cap L) = \binom{n}{3}$  but the number of irreducible components  
 440 of  $X_L^1 \cup X_L^3$  is  $\geq \frac{n}{3}$ . We assume that the number of irreducible components of  $X_L^1 \cup X_L^3$  is  $\leq \frac{n}{3}$ . Indeed  
 441 we may perform the following operations, that leave the degree of the trace and of the residual both  
 442 constant.

- 443 • Pull out a component from  $X_L^2$  to  $X_L^3$  and push down another component from  $X_L^2$  to  $X_L^1$ .
- 444 • Pull out a component from  $X_L^2$  to  $X_L^3$  and push down a component of  $X_L^1$ .
- 445 • Pull out two components from  $X_L^2$  to  $X_L^3$  and push down a component from  $X_L^3$  to  $X_L^1$ .

446 After such operations have been performed, we get that  $X_L$  is still a specialization of a subscheme  
 447 of  $X$ , allowing our semicontinuity argument.

448 If none of the above operations can be performed, then  $X_L^1$  contains only  $a_{n-1}$  components of length  
 449  $n - 1$ ,  $X_L^2$  contains only  $a'_n$  components of length  $n$ ,  $X_L^3$  contains only  $a''_n$  components of length  $n$  and  
 450  $a_{n+1}$  components of length  $n + 1$ .

Then we get

$$\deg(X_L : L) = a_{n-1} + 2a'_n + 3a''_n + 3a_{n+1} = \frac{n(n-1)}{2} - \alpha$$

hence

$$a'_n = \frac{n(n-1)}{4} - \frac{\alpha}{2} - \frac{a_{n-1}}{2} - \frac{3a''_n}{2} - \frac{3a_{n+1}}{2} \geq \frac{n(n-1)}{4} - \frac{\alpha}{2} - \frac{3}{2}(a_{n-1} + a''_n + a_{n+1})$$

451 On the other hand, we have also

$$\begin{aligned} \deg(X_L \cap L) &= \binom{n}{3} \geq (n-2)(a_{n-1} + a'_n + a''_n + a_{n+1}) \\ &\geq (n-2) \left[ \frac{n(n-1)}{4} - \frac{\alpha}{2} - \frac{1}{2}(a_{n-1} + a''_n + a_{n+1}) \right] \\ &> (n-2) \left[ \frac{n(n-1)}{4} - \frac{n-1}{2} - \frac{n-1}{6} \right] \geq \binom{n}{3} \end{aligned}$$

452 where the last inequality is true for  $n \geq 8$ . This contradiction concludes the proof.  $\square$

### 453 5. Induction

454 In order to prove Theorem 1.1 we will work by induction on the dimension and the degree. In the  
 455 following lemmas we describe case by case the initial and special instances, while in Theorem 5.6  
 456 below we present the general inductive procedure, which involves the differential Horace method.

457 **Lemma 5.1.** *A general zero-dimensional scheme  $X \subset \mathbf{P}^2$  contained in a union of double points imposes*  
 458 *independent conditions on  $\mathcal{O}_{\mathbf{P}^2}(d)$  for any  $d \geq 4$ , with the only exception of  $d = 4$  and  $X$  given by the*  
 459 *collection of 5 double points.*

460 **Proof.** Assume that  $X$  is a general subscheme of a union of double points with  $\deg(X) = \binom{d+2}{2}$ . If  $X$   
461 is a collection of double points the statement follows from the Alexander–Hirschowitz theorem on  $\mathbf{P}^2$   
462 (for an easy proof see for example [4, Theorem 2.4]).

If  $X$  is not a collection of double points, fix a hyperplane  $\mathbf{P}^1 \subset \mathbf{P}^2$ . Note that since  $\deg(X) = \binom{d+2}{2}$   
and  $d \geq 4$ , then  $X$  has at least  $d + 1$  components. Since  $X$  contains at least a component of length 1 or 2,  
it is clearly always possible to find a specialization of  $X$  such that the trace has degree exactly  $d + 1$ .  
Then we conclude by induction from the Castelnuovo sequence:

$$0 \rightarrow I_{X, \mathbf{P}^1, \mathbf{P}^2}(d-1) \rightarrow I_{X, \mathbf{P}^2}(d) \rightarrow I_{X \cap \mathbf{P}^1, \mathbf{P}^1}(d).$$

463 Notice that any subscheme of 5 double points and any scheme containing 5 double points impose  
464 independent conditions on quartics, by Remark 2.3.  $\square$

465 We give now an easy technical lemma that we need in the following.

466 **Lemma 5.2.** Assume that  $X$  is a general zero-dimensional scheme contained in a union of double points  
467 of  $\mathbf{P}^n$ , which contains at least  $n - 1$  components of length less than or equal to  $n$ . Then if  $\deg(X) =$   
468  $\binom{n+d}{n}$  it is possible to specialize some components of  $X$  on a fixed hyperplane  $\mathbf{P}^{n-1}$  in such a way that  
469  $\deg(X \cap \mathbf{P}^{n-1}) = \binom{n-1+d}{n-1}$ .

**Proof.** By assumption there exist at least  $n - 1$  components  $\{\eta_1, \dots, \eta_{n-1}\}$  with  $\text{length}(\eta_i) \leq n$ .  
Specialize  $\eta_1, \dots, \eta_{n-1}$  on the hyperplane  $\mathbf{P}^{n-1}$  in such a way that the residual of each component is  
zero. Then specialize other components so that

$$\delta = \binom{n-1+d}{n-1} - \deg(X \cap \mathbf{P}^{n-1}) \geq 0$$

470 is minimal. If  $\delta = 0$  the claim is proved, so assume  $\delta \geq 1$ . Obviously we have  $\delta < k - 1 \leq n$ , where  $k$  is  
471 the minimal length of the components of  $X$  which lie outside  $\mathbf{P}^{n-1}$ . Let  $\zeta$  be a component with length  
472  $k$ . Now we make the first components  $\eta_1, \dots, \eta_{k-1-\delta}$  having residual 1 with respect to  $\mathbf{P}^{n-1}$  and we  
473 specialize  $\zeta$  on  $\mathbf{P}^{n-1}$  with residual 1. Notice that this is possible since  $0 < k - 1 - \delta \leq n - 1$ .  $\square$

474 **Lemma 5.3.** Fix  $3 \leq n \leq 4$ . A general zero-dimensional scheme  $X \subset \mathbf{P}^n$  contained in a union of double  
475 points imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^n}(4)$ , with the following exceptions:

- 476 •  $n = 3$  and either  $X$  is the union of 9 double points, or  $X$  is the union of 8 double points and a component  
477 of length 3;
- 478 •  $n = 4$  and  $X$  is the union of 14 double points.

479 **Proof.** If  $X$  is a collection of double points, the statement holds by the Alexander–Hirschowitz theorem.  
480 We may assume that  $X$  is a scheme with degree  $\binom{n+4}{4}$  which is not a union of double points. Let us  
481 denote by  $D$  the number of double points in  $X$  and by  $C$  the number of the components with length  
482 less than or equal to  $n$ .

If  $n = 3$  and  $C = 1$ , then  $D = 8$  and  $X$  is one exceptional case of the statement. If  $n = 3$  and  $C = 2$ ,  
then  $D = 8$  and the two components  $\eta_1$  and  $\eta_2$  with length less than or equal to 3 have necessarily  
length 1 and 2. In this case we specialize  $X$  on  $\mathbf{P}^2$  in such a way that the trace is given exactly by the  
union of  $\eta_1, \eta_2$  and by the intersection of 4 of the 8 double points with  $\mathbf{P}^2$ . Hence we conclude by the  
Castelnuovo sequence

$$0 \rightarrow I_{X, \mathbf{P}^2, \mathbf{P}^3}(3) \rightarrow I_{X, \mathbf{P}^3}(4) \rightarrow I_{X \cap \mathbf{P}^2, \mathbf{P}^2}(4) \quad (10)$$

483 and by induction. If  $C \geq 3$ , then we denote by  $\eta$  the component of  $X$  with minimal length. We specialize  
484  $\eta$  on  $\mathbf{P}^2$  in such a way that its residual is 1 if  $\text{length}(\eta) \geq 2$ , and 0 if  $\eta$  is a simple point. Then we apply

485 the construction of Lemma 5.2 on  $X \setminus \eta$  (which has at least two components with length less than  
486 or equal to 3) and we obtain a trace different from 5 double points. Hence we conclude again by the  
487 Castelnuovo sequence (10) and by induction.

488 If  $n = 4$  and  $C = 2$ , then  $X$  is given either by the union of 13 double points, a component of length  
489 3 and one of length 2, or by the union of 13 double points, a component of length 4 and a simple  
490 point. In the first case we specialize  $X$  obtaining a trace given by 8 double points, a component of  
491 length 2 and a simple point. Then we conclude by induction as before. In the second case we cannot  
492 use the Castelnuovo sequence since we would obtain an exceptional case. In order to conclude we  
493 prove that a general union of 13 double points and a component of length 4 imposes independent  
494 conditions on quartics. Indeed we know by the Alexander–Hirschowitz theorem that there exists a  
495 unique quartic hypersurface through 14 double points supported at  $p_1, \dots, p_{14}$ . This implies that for  
496 any  $i = 1, \dots, 14$  there is a unique line  $r_i$  through  $p_i$  such that  $r_1, \dots, r_{14}$  are contained in a hyperplane.  
497 Then we consider the scheme  $Y$  given by the union of 13 double points supported at  $\{p_1, \dots, p_{13}\}$  and  
498 the component of length 4 corresponding to a linear space of dimension 3 which does not contain  $r_{14}$ .  
499 It is clear that the scheme  $Y$  imposes independent conditions on quartics, then also the scheme given  
500 by the union of  $Y$  and a general simple point does the same.

501 Assume now that  $n = 4$  and  $C = 3$ . If  $D = 13$ , then we can degenerate  $X$  to one of the previous  
502 cases where the components with length less than or equal to 4 are two. If  $D = 12$ , then the remaining  
503 three components have length either 3, 3, 4, or 2, 4, 4. In these cases we can obtain as a trace 7 double  
504 points and three components of length either 2, 2, 3, or 1, 3, 3, and we conclude by the Castelnuovo  
505 sequence.

506 If  $n = 4$  and  $C \geq 4$ , we denote by  $\eta$  the component of  $X$  with minimal length. If  $\text{length}(\eta) = 1$  we  
507 can degenerate  $X$  to a scheme  $X'$  where the components with length less than or equal to 4 are one  
508 less and we apply the argument to  $X'$ . If  $2 \leq \text{length}(\eta) \leq 3$ , then we specialize  $\eta$  on  $\mathbf{P}^3$  in such a way  
509 that the residual of  $\eta$  is 1. Then we apply the construction of Lemma 5.2 on  $X \setminus \eta$  (which has at least  
510 three components with length less than or equal to 3) and we obtain a trace different from 8 double  
511 points and a component of length 3. Moreover with this construction we always avoid a residual given  
512 by 7 double points. Hence we conclude by the Castelnuovo sequence. If  $\text{length}(\eta) = 4$ , we have only  
513 the following possibilities: 5 components of length 4 and 10 double points, 10 components of length  
514 4 and 6 double points, 15 components of length 4 and 2 double points. In the first two cases we can  
515 obtain trace on  $\mathbf{P}^3$  given by 5 components of length 3 and 5 double points, while in the third case we  
516 can obtain a trace equal to 9 components of length 3 and 2 double points. Then we conclude by the  
517 Castelnuovo sequence.  $\square$

518 **Lemma 5.4.** Fix  $5 \leq n \leq 9$ . A general zero-dimensional scheme  $X \subset \mathbf{P}^n$  contained in a union of double  
519 points imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^n}(4)$ .

520 **Proof.** If  $X$  is a collection of double points, the statement holds by the Alexander–Hirschowitz theorem.  
521 We may assume that  $X$  is a scheme with degree  $\binom{n+4}{4}$  which is not a union of double points. Let us  
522 denote by  $D$  the number of double points in  $X$  and by  $C$  the number of the components with length  
523 less than or equal to  $n$ .

524 If  $n \in \{5, 6, 8\}$  and  $C = 2$ , then we conclude by degenerating  $X$  to a union of double points, avoiding  
525 special cases.

526 If  $n = 5$  and  $C = 3$ , then we get either  $D = 20$ , or  $D = 19$ . In the first case we conclude degenerating  
527  $X$  to the union of 21 double points. In the second case the remaining three components have length  
528 2, 5, 5, or 3, 4, 5, or 4, 4, 4. Then we can obtain a trace equal to 12 double points and three components  
529 of length respectively 2, 4, 4 in the first case, or 3, 3, 4 in the second and third cases. Then we conclude  
530 by induction.

531 If  $n = 5$  and  $C = 4$ , then we have  $D \in \{20, 19, 18\}$ . In the first case we can degenerate  $X$  to a union  
532 of 21 double points. If  $X$  can be degenerate to a scheme which contains only three components with  
533 length less than or equal to 5, we conclude by using the previous results. Then we have to consider only  
534 the two cases where  $X$  is given by 18 double points and four components of length either 3, 5, 5, 5,  
535 or 4, 4, 5, 5. In these cases we can obtain a trace equal to 12 double points and three components

536 of length respectively 2, 4, 4 in the first case, and 3, 3, 4 in the second case. Hence we conclude by  
537 induction.

538 If  $n = 5$  and  $C \geq 5$ , we denote by  $\eta$  the component with minimal length. Then we specialize  $\eta$  on  
539  $\mathbf{P}^4$  in such a way that the residual of  $\eta$  is 1 if  $\text{length}(\eta) \geq 2$ , and 0 if  $\eta$  is a simple point. Then we  
540 apply the construction of Lemma 5.2 on  $X \setminus \eta$  (which has at least four components with length less  
541 than or equal to 5) and we obtain a trace different from 14 double points. Hence we conclude by the  
542 Castelnuovo sequence and by induction.

543 If  $n = 6$  and  $D \geq 21$ , we specialize 21 double points on  $\mathbf{P}^5$  and we conclude by the Castelnuovo  
544 sequence. If  $D < 21$ , then we have  $C \geq 5$  and we can apply Lemma 5.2, concluding by the Castelnuovo  
545 sequence.

546 If  $n = 7$  and  $D \geq 30$ , we specialize 30 double points on  $\mathbf{P}^6$  and we conclude by the Castelnuovo  
547 sequence. If  $D < 30$ , then we have  $C \geq 6$  and we can apply Lemma 5.2.

548 If  $n = 8$  and  $C = 3$ , then either  $D = 58$  and  $X$  can be degenerated to the union of 59 double points,  
549 or  $D = 57$ . In this case the remaining three components can have length 5, 5, 8, or 5, 6, 7, or 6, 6, 6.  
550 In all these case we can obtain a trace on  $\mathbf{P}^7$  given by 40 double points and two components of total  
551 degree 10.

552 If  $n = 8$  and  $C = 4$  and  $X$  can be degenerated to a scheme with less than 4 components with length  
553 less than or equal to 8, then we conclude. Then we have only to consider the case where  $D = 56$  and  
554 the remaining four components of  $X$  have length 3, 8, 8, 8, or 4, 7, 8, 8, or 5, 6, 8, 8, or 5, 7, 7, 8, or  
555 6, 6, 7, 8, or 6, 7, 7, 7. In all these cases we obtain a trace on  $\mathbf{P}^7$  given by 40 double points and two  
556 components of total degree 10, with the exception of the last case, where we can obtain a trace given  
557 by 39 double points and three components of total degree 18.

558 If  $n = 8$  and  $C = 5$  and  $X$  can be degenerated to a scheme with less than 5 components with length  
559 less than or equal to 8, then we conclude. Hence we have only to consider the cases  $D = 56$  or  $D = 55$ .  
560 Listing all the possible lengths of the remaining five components we easily notice that we can always  
561 obtain a trace on  $\mathbf{P}^7$  given either by 40 double points and two components of total degree 10, or by 39  
562 double points and three components of total degree 18.

563 If  $n = 8$  and  $C = 6$  and  $X$  can be degenerated to a scheme with less than 6 components with length  
564 less than or equal to 8, then we conclude. Hence we have only to consider the cases  $D = 55$  or  $D = 54$ .  
565 Listing all the possible lengths of the remaining six components, we easily notice, as before, that we  
566 can always obtain a trace on  $\mathbf{P}^7$  given either by 40 double points and two components of total degree  
567 10, or by 39 double points and three components of total degree 18.

568 If  $n = 8$  and  $C \geq 7$ , we apply Lemma 5.2 and we conclude by the Castelnuovo sequence.

569 If  $n = 9$  and  $D \geq 59$ , we specialize 59 double points on  $\mathbf{P}^8$  and we conclude by the Castelnuovo  
570 sequence. If  $D < 59$ , then we get  $C \geq 8$  and we conclude by applying Lemma 5.2 and by the Castelnuovo  
571 sequence.  $\square$

572 **Lemma 5.5.** Fix  $3 \leq n \leq 4$  and  $5 \leq d \leq 6$ . A general zero-dimensional scheme  $X \subset \mathbf{P}^n$  contained in a  
573 union of double points imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^n}(d)$ .

574 **Proof.** If  $X$  is a collection of double points, the statement holds by the Alexander–Hirschowitz theorem.  
575 Assume that  $X$  is a scheme with degree  $\binom{n+d}{n}$  which is not a union of double points.

576 If  $(n, d) \neq (4, 5)$  and  $X$  has only 2 components with length less than or equal to  $n$ , we conclude by  
577 degenerating  $X$  to a union of double points.

578 If  $(n, d) = (3, 5)$  and  $X$  contains at least 7 double points, we specialize them on the trace and we  
579 conclude by the Castelnuovo sequence, since the residual contains 7 simple points. If  $X$  has less than  
580 7 double points, then  $X$  has obviously at least 3 components with length less than or equal to 3. In  
581 this case we specialize a component with minimal length making it having residual 1, then we apply  
582 the construction of Lemma 5.2 on the remaining components and we conclude by the Castelnuovo  
583 sequence, since the residual contains at least a simple point.

584 If  $(n, d) = (4, 5)$  and  $X$  contains at least 14 double points, we specialize them on the trace and we  
585 conclude by the Castelnuovo sequence, since the residual contains 14 simple points. If  $X$  has less than  
586 14 double points, then  $X$  has obviously at least 4 components with length less than or equal to 4. In

587 this case we specialize a component with minimal length making it having residual 1, then we apply  
 588 the construction of Lemma 5.2 on the remaining components and we conclude by the Castelnuovo  
 589 sequence, since the residual contains at least a simple point.

590 If either  $(n, d) = (3, 6)$ , or  $(n, d) = (4, 6)$  and  $X$  has at least 3 components with length less than  
 591 or equal to 3, we conclude by Lemma 5.2 and by induction.  $\square$

592 We are now in position to give the general inductive argument which completes the proof of The-  
 593 orem 1.1.

594 Given a scheme  $X \subseteq \mathbf{P}^n$  of type  $(m_1, \dots, m_{n+1})$  and a fixed hyperplane  $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$ , we denote for  
 595 any  $1 \leq i \leq n + 1$ :

- 596 • by  $m_i^{(1)}$  the number of component of length  $i$  completely contained in  $\mathbf{P}^{n-1}$ ,
- 597 • by  $m_i^{(2)}$  the number of component of length  $i$  supported on  $\mathbf{P}^{n-1}$  and with residual 1 with respect  
 598 to  $\mathbf{P}^{n-1}$ , and
- 599 • by  $m_i^{(3)}$  the number of component of length  $i$  whose support does not lie in  $\mathbf{P}^{n-1}$ .

600 Obviously we have  $m_i^{(1)} + m_i^{(2)} + m_i^{(3)} = m_i$ , and  $m_{n+1}^{(1)} = 0, m_1^{(2)} = 0$ . We denote  $t_i = m_i^{(1)} + m_{i+1}^{(2)}$ ,  
 601 for  $i = 1, \dots, n + 1, r_1 = m_1^{(3)} + \sum m_i^{(2)}$ , and  $r_i = m_i^{(3)}$  for  $i = 2, \dots, n + 1$ . Note that, for any  $i, t_i$  is  
 602 the number of components of length  $i$  in the scheme  $X \cap \mathbf{P}^{n-1}$ , while  $r_i$  is the number of components  
 603 of length  $i$  in the scheme  $X : \mathbf{P}^{n-1}$ .

604 **Theorem 5.6.** Fix the integers  $n \geq 2$  and  $d \geq 4$ . A general zero-dimensional scheme  $X \subset \mathbf{P}^n$  contained  
 605 in a union of double points imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^n}(d)$  with the following exceptions

- 606 •  $n = 2, d = 4$  and  $X$  is the union of 5 double points;
- 607 •  $n = 3$  and either  $X$  is the union of 9 double points, or  $X$  is the union of 8 double points and a component  
 608 of length 3;
- 609 •  $n = 4$  and  $X$  is the union of 14 double points.

610 **Proof.** We prove the statement by induction on  $n$  and  $d$ . In Lemma 5.1 we have proved the statement  
 611 for  $n = 2, d \geq 4$ , in Lemma 5.3 and Lemma 5.4 for  $d = 4, 3 \leq n \leq 9$  and in Lemma 5.5 for  
 612  $d = 5, n = 3, 4$  and  $d = 6, n = 3, 4$ . Then we need to prove the remaining cases. Assume  $n \geq 3$  and  
 613 in particular when  $d = 4$  assume  $n \geq 10$ , and when  $5 \leq d \leq 6$  assume  $n \geq 5$ .

614 The proof by induction is structured as follows:

- 615 • for  $d = 4$  and  $n \geq 10$ , we assume that any scheme in  $\mathbf{P}^n$  imposes independent conditions on  
 616  $\mathcal{O}_{\mathbf{P}^{n-1}}(4)$ . Recall that any scheme in  $\mathbf{P}^n$  imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^n}(3)$  (by Theorem  
 617 4.1) and any scheme of degree greater than or equal to  $(n + 1)^2$  imposes independent conditions  
 618 on  $\mathcal{O}_{\mathbf{P}^n}(2)$  (by Theorem 3.2). Then we prove the statement for  $d = 4, n \geq 10$ ;
- 619 • for  $d \geq 5$  we assume that any scheme in  $\mathbf{P}^d$  imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^d}(b)$  for  $(a, b) \in$   
 620  $\{(n - 1, d), (n, d - 1), (n, d - 2)\}$  and we prove it for  $(a, b) = (n, d)$ .

621 It is enough to prove the statement for a scheme  $X$  with degree  $\deg X = \binom{d+n}{n}$ .

622 Let  $X \subseteq \mathbf{P}^n$  be a scheme of type  $(m_1, \dots, m_{n+1})$  contained in a union of double points and suppose  
 623  $\deg X = \sum im_i = \binom{d+n}{n}$ . Fix a hyperplane  $\mathbf{P}^{n-1}$  in  $\mathbf{P}^n$ . In order to apply induction, we want to  
 624 degenerate  $X$  so that some of the components fall in the hyperplane  $\mathbf{P}^{n-1}$ . By abuse of notation we call  
 625 again  $X$  the scheme after the degeneration.

Now if there exists a degeneration such that

$$\deg(X \cap \mathbf{P}^{n-1}) = \sum it_i = \binom{d+n-1}{n-1}$$

where  $m_i^{(1)}$ ,  $m_i^{(2)}$ ,  $m_i^{(3)}$  and  $t_i$ ,  $r_i$  are defined as above, then we can conclude by the Castelnuovo sequence

$$0 \rightarrow I_{X;\mathbf{P}^{n-1}}(d-1) \rightarrow I_X(d) \rightarrow I_{X \cap \mathbf{P}^{n-1}}(d)$$

and by induction. Then we may assume that such a degeneration does not exist. Let us choose a degeneration of  $X$  such that  $\binom{d+n-1}{n-1} - \sum it_i > 0$  is minimal and define

$$\varepsilon := \binom{d+n-1}{n-1} - \sum it_i. \tag{11}$$

626 Obviously  $0 < \varepsilon < n$  and  $\varepsilon < \min \{i : m_i^{(3)} \neq 0\} - 1$ . By the minimality assumption we have

627  $m_1^{(3)} = m_2^{(3)} = 0$  and we have also  $m_i^{(2)} = 0$  for all  $i \neq n+1$ .

Now let us define

$$\varepsilon_{n+1} = \min \{ \varepsilon, m_{n+1}^{(3)} \}, \quad \varepsilon_n = \min \{ \varepsilon - \varepsilon_{n+1}, m_n^{(3)} \}$$

and, for any  $i = n-1, \dots, 1$ ,

$$\varepsilon_i = \min \left\{ \varepsilon - \sum_{k=i+1}^{n+1} \varepsilon_k, m_i^{(3)} \right\}.$$

628 Obviously we have  $\varepsilon_1 = \varepsilon_2 = 0$  and  $\sum_{i=3}^{n+1} \varepsilon_i = \varepsilon$ .

629 *Step 1:* Let  $\Gamma \subseteq \mathbf{P}^{n-1}$  be a general scheme of type  $(0, \varepsilon_3, \dots, \varepsilon_{n+1}, 0)$  supported on a collection

630  $\{\gamma_1, \dots, \gamma_\varepsilon\} \subseteq \mathbf{P}^{n-1}$  of points and  $\Sigma \subseteq \mathbf{P}^n$  a general scheme of type  $(0, 0, m_3^{(3)} - \varepsilon_3, \dots, m_{n+1}^{(3)} -$

631  $\varepsilon_{n+1})$  supported at points which are not contained in  $\mathbf{P}^{n-1}$ .

By induction we know that

$$h_{\mathbf{P}^n}(\Gamma \cup \Sigma, d-1) = \min \left( \deg(\Gamma \cup \Sigma), \binom{n+d-1}{n} \right)$$

632 where  $\deg(\Gamma \cup \Sigma) = \sum (i-1)\varepsilon_i + \sum i(m_i^{(3)} - \varepsilon_i) = \sum im_i^{(3)} - \varepsilon$ .

633 Recall that  $\binom{n+d-1}{n} = \binom{n+d}{n} - \binom{n+d-1}{n-1}$ . From the definition of  $\varepsilon$  it follows that  $\binom{n+d-1}{n} =$

634  $\binom{n+d}{n} - \sum it_i - \varepsilon = m_{n+1}^{(2)} + \sum im_i^{(3)} - \varepsilon$  and since of course  $m_{n+1}^{(2)} \geq 0$ , we obtain  $h_{\mathbf{P}^n}(\Gamma \cup \Sigma, d-1) =$

635  $\sum im_i^{(3)} - \varepsilon$

*Step 2:* Now we want to add a collection  $\Phi$  of  $m_{n+1}^{(2)}$  simple points in  $\mathbf{P}^{n-1}$  to the scheme  $\Gamma \cup \Sigma$  and we want to obtain a  $(d-1)$ -independent scheme. From the previous step it is clear that  $\dim I_{\Gamma \cup \Sigma}(d-1) = m_{n+1}^{(2)}$ . Hence we have only to prove that there exist no hypersurfaces of degree  $d-2$  through  $\Sigma$ . Let us show that for  $d \geq 5$  we have

$$\deg(\Sigma) = \sum i(m_i^{(3)} - \varepsilon_i) \geq \binom{n+d-2}{n} \tag{12}$$

and for  $d = 4$  and  $n \geq 10$  we have

$$\deg(\Sigma) = \sum i(m_i^{(3)} - \varepsilon_i) \geq (n+1)^2 \geq \binom{n+2}{n} \tag{13}$$

Indeed by definition of  $\varepsilon$ , we have

$$\sum i(m_i^{(3)} - \varepsilon_i) = \binom{n+d-1}{n} + \varepsilon - \sum i\varepsilon_i - m_{n+1}^{(2)}$$

and since

$$\sum i\varepsilon_i - \varepsilon = \sum (i-1)\varepsilon_i \leq n\varepsilon \leq (n-1)n \quad \text{and} \quad m_{n+1}^{(2)} \leq \frac{1}{n} \binom{n+d-1}{n-1}$$

we obtain

$$\sum i(m_i^{(3)} - \varepsilon_i) \geq \binom{n+d-1}{n} - (n-1)n - \frac{1}{n} \binom{n+d-1}{n-1} =: S(n, d).$$

636 It is easy to check that for any  $d \geq 5$  and  $n \geq 3$  we have  $S(n, d) > \binom{n+d-2}{n}$ , which proves inequality  
 637 (12). On the other hand one can also check that  $S(n, 4) > (n+1)^2$  for any  $n \geq 10$ , proving thus  
 638 inequality (13).

Then by induction we know that  $\Sigma$  imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^n}(d-2)$ , and so we get  $\dim I_{\Sigma}(d-2) = 0$ . Thus we obtain

$$h_{\mathbf{P}^n}(\Gamma \cup \Sigma \cup \Phi, d-1) = \sum im_i^{(3)} - \varepsilon + m_{n+1}^{(2)} = \binom{n+d-1}{n}.$$

639 *Step 3:* Let us choose a family of general points  $\{\delta_{t_1}^1, \dots, \delta_{t_\varepsilon}^\varepsilon\} \subseteq \mathbf{P}^n$ , with parameters  $(t_1, \dots, t_\varepsilon) \in$   
 640  $K^\varepsilon$ , such that for any  $i = 1, \dots, \varepsilon$  we have  $\delta_0^i = \gamma_i \in \mathbf{P}^{n-1}$  and  $\delta_{t_i}^i \notin \mathbf{P}^{n-1}$  for any  $t_i \neq 0$ .

641 Now let us consider a family of schemes  $\Delta_{(t_1, \dots, t_\varepsilon)}$  of type  $(\varepsilon_2, \dots, \varepsilon_{n+1}, 0)$  supported at the points  
 642  $\{\delta_{t_1}^1, \dots, \delta_{t_\varepsilon}^\varepsilon\}$ . Note that  $\Delta_{(0, \dots, 0)}$  is the scheme  $\Gamma$  defined in Step 1. Moreover let  $\Psi \subseteq \mathbf{P}^{n-1}$  be a scheme  
 643 of type  $(m_1^{(1)}, \dots, m_n^{(1)}, 0)$  supported at general points of  $\mathbf{P}^{n-1}$ , and recall that in Step 2 we have in-  
 644 troduced the scheme  $\Phi \subset \mathbf{P}^{n-1}$ . Let  $\Phi^2$  be the union of double points, supported on the scheme  $\Phi$ .

By induction the scheme  $(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup \Gamma) \subseteq \mathbf{P}^{n-1}$  has Hilbert function

$$h_{\mathbf{P}^{n-1}}(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup \Gamma, d) = \sum im_i^{(1)} + nm_{n+1}^{(2)} + \varepsilon = \sum it_i + \varepsilon = \binom{d+n-1}{n-1}$$

645 i.e. it is  $d$ -independent.

646 We will work now with the following schemes:

- 647 •  $\Delta_{(t_1, \dots, t_\varepsilon)}$  the family of schemes introduced in Step 3, of type  $(\varepsilon_2, \dots, \varepsilon_{n+1}, 0)$  supported at  
 648 the points  $\{\delta_{t_1}^1, \dots, \delta_{t_\varepsilon}^\varepsilon\}$  and such that  $\Delta_{(0, \dots, 0)} = \Gamma$ ;
- 649 •  $\Psi \subseteq \mathbf{P}^{n-1}$  the scheme introduced in Step 3, of type  $(m_1^{(1)}, \dots, m_n^{(1)}, 0)$  supported at general  
 650 points of  $\mathbf{P}^{n-1}$ ;
- 651 •  $\Phi^2$  of type  $(0, \dots, 0, m_{n+1}^{(2)})$ , that is the union of double points supported on the scheme  
 652  $\Phi \subset \mathbf{P}^{n-1}$  introduced in Step 2;
- 653 •  $\Sigma \subseteq \mathbf{P}^n$ , the scheme defined in Step 1, of type  $(0, 0, m_3^{(3)} - \varepsilon_3, \dots, m_{n+1}^{(3)} - \varepsilon_{n+1})$ .

654 In order to prove that  $X$  imposes independent conditions on  $\mathcal{O}_{\mathbf{P}^n}(d)$ , it is enough to prove the fol-  
 655 lowing claim.

656 **Claim.** There exist  $(t_1, \dots, t_\varepsilon)$  such that the scheme  $\Delta_{(t_1, \dots, t_\varepsilon)}$  is  $\mathcal{D}$ -independent, where  $\mathcal{D}$  is the linear  
 657 system determined by the vector space  $I_{\Psi \cup \Phi^2 \cup \Sigma}(d)$ .

658 Assume by contradiction that the claim is false. Then by Lemma 2.1 for any  $(t_1, \dots, t_\varepsilon)$  there exist  
 pairs  $(\delta_{t_i}^i, \eta_{t_i}^i)$  for all  $i = 1, \dots, \varepsilon$ , with  $\eta_{t_i}^i$  a curvilinear scheme supported at  $\delta_{t_i}^i$  and contained in  
 $\Delta_{(t_1, \dots, t_\varepsilon)}$  such that

$$h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup \eta_{t_1}^1 \cup \dots, \eta_{t_\varepsilon}^\varepsilon, d) < \binom{d+n}{n} - \sum (i-2)\varepsilon_i. \tag{14}$$

659 Let  $\eta_0^i$  be the limit of  $\eta_{t_i}^i$ , for  $i = 1, \dots, \varepsilon$ .

660 Suppose that  $\eta_0^i \notin \mathbf{P}^{n-1}$  for  $i \in F \subseteq \{1, \dots, \varepsilon\}$  and  $\eta_0^i \in \mathbf{P}^{n-1}$  for  $i \in G = \{1, \dots, \varepsilon\} \setminus F$ .

661 Given  $t \in K$ , let us denote  $Z_t^F = \cup_{i \in F} (\eta_t^i)$  and  $Z_t^G = \cup_{i \in G} (\eta_t^i)$ . Denote by  $\widetilde{\eta}_0^i$  for  $i \in F$  the residual  
662 of  $\eta_0^i$  with respect to  $\mathbf{P}^{n-1}$  and by  $f$  and  $g$  the cardinalities respectively of  $F$  and  $G$ .

By the semicontinuity of the Hilbert function and by (14) we get

$$h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup Z_0^F \cup Z_t^G, d) \leq h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup Z_t^F \cup Z_t^G, d) < \binom{d+n}{n} - \sum (i-2)\varepsilon_i.$$

On the other hand, by the semicontinuity of the Hilbert function there exists an open neighborhood  $O$  of  $0$  such that for any  $t \in O$

$$h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F} \widetilde{\eta}_0^i) \cup Z_t^G, d-1) \geq h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F} \widetilde{\eta}_0^i) \cup Z_0^G, d-1)$$

Since the scheme  $\Phi \cup \Sigma \cup (\cup_{i \in F} \widetilde{\eta}_0^i) \cup Z_0^G$  is contained in  $\Phi \cup \Sigma \cup \Gamma$ , which is  $(d-1)$ -independent by Step 2, we have

$$h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F} \widetilde{\eta}_0^i) \cup Z_0^G, d-1) = m_{n+1}^{(2)} + \sum i(m_i^{(3)} - \varepsilon_i) + f + 2g.$$

Since  $\Psi \cup \Phi^2|_{\mathbf{P}^{n-1} \cup (\cup_{i \in F} \gamma_i)}$  is a subscheme of  $\Psi \cup \Phi^2|_{\mathbf{P}^{n-1} \cup \Gamma}$ , which is  $d$ -independent by Step 3, it follows that

$$h_{\mathbf{P}^{n-1}}(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1} \cup (\cup_{i \in F} \gamma_i)}, d) = \sum im_i^{(1)} + nm_{n+1}^{(2)} + f$$

663 Hence for any  $t \in O$ , by applying the Castelnuovo exact sequence to the scheme  $\Psi \cup \Phi^2 \cup \Sigma \cup Z_0^F \cup Z_t^G$ ,  
664 we get

$$\begin{aligned} & h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup Z_0^F \cup Z_t^G, d) \\ & \geq h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F} \widetilde{\eta}_0^i) \cup Z_t^G, d-1) + h_{\mathbf{P}^{n-1}}(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1} \cup (\cup_{i \in F} \gamma_i)}, d) \\ & \geq (m_{n+1}^{(2)} + \sum i(m_i^{(3)} - \varepsilon_i) + f + 2g) + (\sum im_i^{(1)} + nm_{n+1}^{(2)} + f) \\ & = \sum im_i - \sum i\varepsilon_i + 2\varepsilon = \binom{d+n}{n} - \sum (i-2)\varepsilon_i \end{aligned}$$

665 contradicting (14). This completes the proof of the claim.  $\square$

## 666 6. Appendix

Here we explain how to compute the dimension of the space

$$V_{d,n}(p_1, \dots, p_k, A_1, \dots, A_k)$$

667 defined in (2) in the introduction.

668 These computations are performed in characteristic 31991 using the program Macaulay2 [9], and  
669 consist essentially in checking that several square matrices, randomly chosen, have maximal rank.  
670 We underline that if an integer matrix has maximal rank in positive characteristic, then it has also  
671 maximal rank in characteristic zero. Very likely Theorem 1.1 should be true on any infinite field, but a  
672 finite number of values for the characteristic (not including 31991) require further and tedious checks,  
673 that we have not performed.

674 Assume that  $\dim A_i = a_i$  are given and that  $\sum_{i=1}^k (a_i + 1) = \binom{n+d}{n} = \dim R_{d,n}$ . Consider the  
675 monomial basis for  $R_{d,n}$  as a matrix  $T$  of size  $\binom{n+d}{n} \times 1$ . Consider the jacobian matrix  $J$  computed at  
676  $p_i$ , which has size  $\binom{n+d}{n} \times (n+1)$ . Choose a random  $(n+1) \times a_i$  integer matrix  $A$ . We concatenate  $T$

677 computed at  $p_i$  with  $J \cdot A$ . It results a matrix of size  $\binom{n+d}{n} \times (a_i + 1)$ . When  $a_i = n$  (this is the case of  
 678 Alexander–Hirschowitz theorem) there is no need to use a random matrix, and by Euler identity we  
 679 can simply take the jacobian matrix  $J$  computed at  $p_i$ . By repeating this construction for every point,  
 680 and placing side by side all these matrices, we get a square matrix of order  $\binom{n+d}{n}$ . This is the matrix of  
 681 coefficients of the system (1), which corresponds to our interpolation problem. Then there is a unique  
 682 polynomial  $f$  satisfying (1) if and only if the above matrix has maximal rank. We emphasize that this  
 683 Montecarlo technique provides a proof, and not only a probabilistic proof. Indeed consider the subset  
 684  $\mathcal{S}$  of points  $(p_1, \dots, p_k, A_1, \dots, A_k)$  (lying in a Grassmann bundle, which locally is isomorphic to the  
 685 product of affine spaces and Grassmannians, hence irreducible) such that the corresponding matrix  
 686 has maximal rank. The subset  $\mathcal{S}$  is open and if it is not empty, because it contains a random point, then  
 687 it is dense.

688 In Proposition 4.3, Proposition 4.7, Proposition 4.8, Proposition 4.12 we need a modification of the  
 689 above strategy, since the points are supported on some given codimension three subspaces.

690 As a sample we consider the case considered in Proposition 4.8 where  $n = 8, 1 = \deg(X_L : L) = 10,$   
 691  $m = \deg(X_M : M) = 14,$  and  $F = \deg(X_O) = 39$  and we list below the Macaulay2 script. Given  
 692 monomial subspaces  $L$  and  $M$ , we first compute the cubic polynomials containing  $L$  and  $M$ , finding a  
 693 basis of 63 monomials. Then we compute all the possible partitions of 10 and 14 in integers from 1 to  
 694 3 (which are the possible values of  $\deg(\xi : L)$ , resp.  $\deg(\xi : M)$ , where  $\xi$  is an irreducible component  
 695 of  $X_L$ , resp.  $X_M$ ), and of 39 in integers from 1 to 9 (which are the possible lengths of a subscheme of a  
 696 double point in  $\mathbf{P}^8$ ), by excluding the cases which can be easily obtained by degeneration. We collect  
 697 the results in the matrices `tripleL`, `tripleM` and `XO`, each row corresponds to a partition. Then for  
 698 any combination of rows of the three matrices the program computes a matrix `mat` of order 63 and its  
 699 rank. If the rank is different from 63 the program prints the case. Running the script we see that the  
 700 output is empty, as we want.

```

701 KK=ZZ/31991;
702 E=KK[e_0..e_8];
703 --coordinates in P8
704
705 f=ideal(e_0..e_8);
706 g=ideal(e_0..e_2);
707 h=ideal(e_3..e_5);
708 T1=f*g*h;
709 T=gens gb(T1)
710 --basis for the space of cubics containing
711 --L (e_0=e_1=e_2=0) and M (e_3=e_4=e_5=0)
712 --T is a (63x1) matrix
713
714 J=jacobian(T);
715 -- J is a (63x9) matrix
716
717 --first case: for the other cases of Proposition 4.8 it is enough
718 --to change to following line
719 l=10;m=14;F=39;
720
721 ---start program
722 tripleL=matrix{{0,0,0}};
723 for t from 0 to ceiling(1/3) do
724 for d from 0 to ceiling(1/2) do
725 for u from 0 to 1 do
726 (if (3*t+2*d+u==1) then tripleL=(tripleL|matrix>{{t,d,u}}));
727
728 tripleM=matrix{{0,0,0}},
729 for t from 0 to ceiling(m/3) do
730 for d from 0 to ceiling(m/2) do
731 for u from 0 to 1 do
732 (if (3*t+2*d+u==m) then tripleM=(tripleM|matrix>{{t,d,u}}));
733
734 XO=matrix{{0,0,0,0,0,0,0,0,0}};
735 for n from 0 to ceiling(F/9) do
736 (if (9*n+1==F) then XO=(XO|matrix>{{n,0,0,0,0,0,0,0,1}}));
737 (for n from 0 to ceiling(F/9) do
738 (for o from 0 to ceiling(F/8) do
739 (if (9*n+8*o+2==F) then XO=(XO|matrix>{{n,o,0,0,0,0,0,1,0}}));));
740 (for n from 0 to ceiling(F/9) do
741 (for o from 0 to ceiling(F/8) do
742 (for s from 0 to ceiling(F/7) do

```

```
743 (if (9*n+8*o+7*s+3==F) then XO=(XO|matrix({{n,o,s,0,0,0,1,0,0}})));
744 (for n from 0 to ceiling(F/9) do
745 (for o from 0 to ceiling(F/8) do
746 (for s from 0 to ceiling(F/7) do
747 (for e from 0 to ceiling(F/6) do
748 (for c from 0 to ceiling(F/5) do
749 (if (9*n+8*o+7*s+6*e+5*c==F)
750 then XO=(XO|matrix({{n,o,s,e,c,0,0,0}})));
751
752 k=1;
753 for a from 1 to (numgens(target(tripleL))-1) do
754 for b from 1 to (numgens(target(tripleM))-1) do
755 for c from 1 to (numgens(target(XO))-1) do
756 (k=k+1,
757 mat=random(E^1,E^63)*o,
758 for i from 1 to tripleL(a,0) do
759 (q1=(matrix(E,{{0,0,0}})|random(E^1,E^6)), mat=mat||random(E^3,E^9)*sub(J,q1),
760 for i from 1 to tripleL(a,1) do
761 (q1=(matrix(E,{{0,0,0}})|random(E^1,E^6)), mat=mat||random(E^2,E^9)*sub(J,q1),
762 for i from 1 to tripleL(a,2) do
763 (q1=(matrix(E,{{0,0,0}})|random(E^1,E^6)), mat=mat||random(E^1,E^9)*sub(J,q1),
764 for i from 1 to tripleM(b,0) do
765 (r1=(random(E^1,E^3)|matrix(E,{{0,0,0}})|random(E^1,E^3)),mat=mat||random(E^3,E^9)*sub(J,r1),
766 for i from 1 to tripleM(b,1) do
767 (r1=(random(E^1,E^3)|matrix(E,{{0,0,0}})|random(E^1,E^3)),mat=mat||random(E^2,E^9)*sub(J,r1),
768 for i from 1 to tripleM(b,2) do
769 (r1=(random(E^1,E^3)|matrix(E,{{0,0,0}})|random(E^1,E^3)),mat=mat||random(E^1,E^9)*sub(J,r1),
770 for i from 1 to XO(c,0) do
771 (p1=random(E^1,E^9), mat=mat||sub(J,p1),
772 for i from 1 to XO(c,1) do
773 (p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(8-1),E^9)*sub(J,p1),
774 for i from 1 to XO(c,2) do
775 (p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(7-1),E^9)*sub(J,p1),
776 for i from 1 to XO(c,3) do
777 (p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(6-1),E^9)*sub(J,p1),
778 for i from 1 to XO(c,4) do
779 (p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(5-1),E^9)*sub(J,p1),
780 for i from 1 to XO(c,5) do
781 (p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(4-1),E^9)*sub(J,p1),
782 for i from 1 to XO(c,6) do
783 (p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(3-1),E^9)*sub(J,p1),
784 for i from 1 to XO(c,7) do
785 (p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(2-1),E^9)*sub(J,p1),
786 for i from 1 to XO(c,8) do mat=mat||sub(T,random(E^1,E^9)),
787 if (rank(mat)!=63)
788 then (print(tripleL(a,0),tripleL(a,1),tripleL(a,2),tripleM(b,0),tripleM(b,1),tripleM(b,2),
789 XO(c,0),XO(c,1),XO(c,2),XO(c,3),XO(c,4),XO(c,5),XO(c,6),XO(c,7),XO(c,8)),
790 if (mod(k,29)=0) then print(k);
```

791 All the others scripts are available at the page <<http://web.math.unifi.it/users/brambill/homepage/>  
792 [macaulay.html](#)>.

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