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Proof Theory of Epistemic Logics

Tesi di Dottorato – ciclo XIV

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Preface

The present dissertation is the result of the research done at the universities of Firenze and Helsinki, and it has been jointly supervised by Prof. Pierluigi Minari (Firenze) and Ph.D. Sara Negri (Helsinki), according to a co-supervision agreement established between the departments of Philosophy of the two institutions. At the end of my first year of doctoral studies in Firenze, I had the opportunity to visit the university of Helsinki from January 2010 to June 2010 and from September 2010 to January 2011, where most of the material of Ch. 4 and Ch. 5 of this work has been prepared. In Helsinki, I was involved in the research project led by Sara Negri on the proof theory of formal epistemology, and under her patient supervision, I developed a labelled sequent calculus for the logic of public announcements. The results achieved appeared in two international conference series (Maffezioli and Negri 2010, 2011) and are discussed in Ch. 4. Afterwards, in collaboration with Sara Negri and Alberto Naibo, I contributed to the proof-theoretic analysis of the Church-Fitch paradox of knowability, resulted in journal publication (Maffezioli *et al.* 2011), and reproduced with minor modifications in the last chapter of this work. At the university of Firenze, Pierluigi Minari introduced me to the complexity of cut elimination and supervised the analysis of the Ch. 2 of this work. Pierluigi Minari also carefully reviewed and provided valuable comments on the other chapters.

Acknowledgments. It is difficult to say how much this work benefited from the co-supervision. I wish to thank both my supervisors for all the efforts to read the manuscript. Sara Negri constantly supported my work and, besides her scientific contribution to this dissertation, introduced me to the concrete practice of doing research: working in collaboration, learning by trial and error and the international approach to the scientific research are the most important lessons I learned from her teaching. Pierluigi Minari oriented me with his vast knowledge and with a tireless dedication to making this dissertation more complete and accurate. I thank them also for running the logic seminar in their respective institutions, and Jan von Plato and Gabriel Sandu in Helsinki and Andrea Cantini in Firenze who contributed to the organization of these seminars. Among colleagues and friends, I mention Alberto Naibo who is at the same time a valuable colleague and a good friend: around him I always felt philosophically stimulated and highly-esteemed as a person. I am deeply grateful to Michael and Julia von Boguslawski and Viktor Granö, they made me feel at home in Helsinki. I also thank my family who has every confidence in me, and for several other reasons – which I am not able to list here – I wish to dedicate this work to Virginia.

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Introduction

Modal logic has been mostly developed from a semantic perspective. Although the adjective “semantic” can be interpreted in various ways, in the standard literature on modal logic it is usually a synonymous of “model-theoretic” (see Blackburn *et al.* 2001). The model theory of modal logic studies the interplay between the modal language and the models for that language. Thus, modal logic is considered an useful tool designed for talking about a certain kind of mathematical structures by means of which many concrete situations can be formally described. Flows of time, states of knowledge, transitions between computational states can be all represented as **relational structures**, that is, non-empty sets together with an accessibility relation on their members. From this perspective, modal logic is a language equipped with a suitable relational semantics, rather than a set of axioms and inference rules. Even when modal logics are presented in an axiomatic style, formal derivations have a little role to play. On the other hand, when the purpose is to find derivations or the analysis of their structural properties, **sequent calculi** have been preferred to the axiomatic Hilbert-style approach. However, the traditional sequent systems for modal logic fail to satisfy most of the properties usually required on sequent calculi and the difficulties of finding cut-free sequent systems are encountered already for quite simple modal systems such as S5. The problem of a satisfactory proof-theoretic account to modal logic can be partially solved by generalizing the notion of sequent in a more rich and complex syntax, and various attempts will be reviewed in the later chapters. Nevertheless, the conceptual unification made possible by the relational semantics for modal logics has been not achieved yet at the corresponding syntactic level. In the present work that unification is obtained through the direct internalization of the relational semantic into the syntax of sequent rules, following the labelled approach to modal logic of Negri (2005). The internalization of the semantics makes it possible to talk about relational structures in proof-theoretic terms. Our attention will focus mainly on **epistemic modal logic** (see Fagin *et. al.* 1995) and, more generally, on the large variety of problems concerning the logical notions of knowledge and belief. Sequent systems for epistemic logic are obtained as modular extensions of a basic modal calculus, through the addition of appropriate mathematical rules that correspond to the properties of epistemic frames. All the calculi enjoy remarkable structural properties, in particular they are contraction and cut free.

The first chapter provides a general background on labelled sequent systems and offers an inferentialist justification of the logical rules through a system of natural deduction with general elimination rules. The chapter can also be read as a general introduction to the problems that **structural proof theory** generally deals with, in particular the admissibility of the structural rules and cut elimination.

The second chapter is entirely devoted to the **cut-elimination theorem**, and the complexity of cut-elimination methods are discussed: a numerical bound on cut-free derivations is calculated following the pattern of the proof of the same theorem for first-order logic. The aim is to provide a labelled sequent system in which all the structural rules are admissible, the logical rules are invertible, and the cut rule can be dispensed with. The system so obtained allows a systematic proof-search procedure and can be effectively used for finding derivation in basic modal logic.

In the third chapter is shown how to extend the techniques and results of the previous parts in order to get sequent systems for logics that extend basic modal logic. The problem of how to treat axioms in sequent calculus is introduced and the solution of Negri and von Plato (2001, 2011) is applied to the multi-modal logic of knowledge and belief. Sequent systems for epistemic and doxastic logic are obtained by adding suitable inference rules for the accessibility relation that express the properties of the corresponding relations in Kripke frames.

The last two chapters constitute the core of the research project on the proof theory of modal epistemic logic. The fourth chapter presents a labelled sequent system for the **logic of public announcements** (see Plaza 1989 and van Ditmarsch *et. al.* 2007). The formal study of the dynamics of knowledge and of process of information are nowadays among the most prominent developments of epistemic modal logic. Nevertheless, model-theoretic aspects have been dominant and an adequate proof-theoretic treatment is still an open question. Most of the material of this chapter has been presented in Maffezioli and Negri (2010, 2011).

The last chapter consists in a proof-theoretic analysis of the **Church-Fitch paradox of knowability** (see Fitch 1963). By exploiting the semantic features of a labelled sequent system it is shown how to give a cut-free reconstruction of the Fitch derivation and to isolate the semantic frame condition that correspond to the principle at the base of the paradox. The aim of this analysis is to provide an adequate proof theory governing the interaction among the modalities involved in Fitch's proof and to give a logical framework for dealing with the Fitch paradox (knowability logic). Moreover, it is argued in favor of the use of intuitionsitic logic as a solution of the paradox and it is shown that the paradoxical conclusion is only classically derivable, but neither intuitionistically derivable nor intuitionistically admissible. The material presented in this chapter can be found, with minor modifications, in Maffezioli *et al.* (2011).

We conclude this introduction by recalling the fundamentals of the language and of the semantics of epistemic logic, and we briefly discuss the basic notions we will deal with in what follows.

The language

The **language** of epistemic logic consists of a countable set of atomic formulas $\mathcal{P}$ and a finite set of (names for) agents $\mathcal{A}$. From $\mathcal{P}$ it is possible to form compound formulas by the usual propositional connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), so that if A and B are formulas, so are $A \wedge B$, $A \vee B$, $A \supset B$. The language contains also the symbol $\perp$ which stands for any contradiction. In addition, there is a **knowledge operator** K_a for each agent a in $\mathcal{A}$. Intuitively, $\mathsf{K}_a A$ means: "the agent a knows that A". In the following we shall use some notational conventions: $\neg A$ will be an abbreviation for $A \supset \perp$ and $A \supset\subset B$ a shorthand for implication in both directions, that is, $(A \supset B) \wedge (B \supset A)$. We omit the outermost parentheses when this does not lead to confusion. Despite its simplicity, this language permits to express rather complex information about what agents know about other agents' knowledge. For instance, the formula $P \wedge \neg \mathsf{K}_a P$ says that P is true but a doesn't know it, whereas $\neg \mathsf{K}_b K_c P \wedge \neg \mathsf{K}_b \neg K_c P$ says that b does not know whether c knows P.

The formal semantics

The most influential model of knowledge is the well-known **possible-world semantics** or, **relational semantics**. The idea is that besides the actual state of affairs there are a number of other possible states which describe the world as it could be, if the things were different. An agent may have access to some possible state, whereas some others are inaccessible. Among the accessible states, an agent a may not be able to tell the difference with respect to the actual one, since in both the same proposition P holds. When P holds in every state that a considers possible it is said that a knows that P. Thus, the basic components of the formal semantics for knowledge are a set of possible states and a collection of arrows between states.

Definition (Epistemic Frame)**.** *An* epistemic frame *is a structure* $\mathfrak{F} = \langle \mathsf{X}, \mathsf{R}_a \rangle$ *where*

X *is a non-empty set and* R_a *is a collection of binary relations on* X*, one for each* $a \in \mathcal{A}$*. Furthermore, each* R_a *is an equivalence relation. The standard notation for* $\langle x, y \rangle \in \mathsf{R}_a$ *is* $x\mathsf{R}_a y$*.*

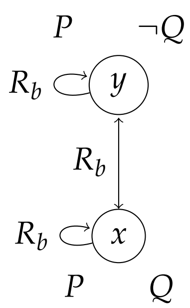
In an epistemic frame, the elements in X may be intuitively interpreted as possible worlds. However, there is no reason for limiting oneself to a specific interpretation and in the following we shall generally speak of **possible states**. In the standard mono-modal logic the binary relation gets interpreted as an accessibility relation between states, but, in the field of multi-modal epistemic logic, relations are better understood when they are interpreted as **indistinguishability relations**: an agent a cannot distinguish between two states x and y when they are related by R_a. Thus, $x\mathsf{R}_a y$ can be read as: "as far as a concerns, the state x might be y as well". The idea is that the fewer states an agent considers possible, the less is his uncertainty, and more he knows. Finally, in order to describe formally the properties of K_a it is assumed that each relation is an equivalence relation, that is, it is reflexive, symmetric and transitive. In fact, no agent is supposed to distinguish a state x from itself, and if x is indistinguishable from y so is y from x; finally, if x and y cannot be distinguished and also y cannot be distinguished from z then x cannot be distinguished from z. However, weaker notions of knowledge are possible and in the last chapter we shall assume only the reflexivity of the accessibilities relations. An epistemic frame becomes an epistemic model when atomic formulas receive an evaluation. This can be obtained by adding a new relation, indicated by $\Vdash$ and called **forcing**, between possible states and atoms. Intuitively, $x \Vdash P$ says that the formula P is true at the state x.

Definition (Epistemic Model)**.** *An* epistemic model *is a structure* $\mathfrak{M} = \langle \mathfrak{F}, \Vdash \rangle$ *where* $\mathfrak{F}$ *is an epistemic frame and* $\Vdash$ *is a binary relation between elements in* X *and atomic formulas* P*. The standard notation for* $\langle x, P \rangle \in \Vdash$ *is* $x \Vdash P$*.*

The relation $\Vdash$ is extended in a unique way to arbitrary formulas by means of the following inductive clauses.

$$\begin{array}{lll}
x \Vdash \bot & \text{for no } x & \\
x \Vdash A \wedge B & \text{if and only if} & x \Vdash A \text{ and } x \Vdash B \\
x \Vdash A \vee B & \text{if and only if} & x \Vdash A \text{ or } x \Vdash B \\
x \Vdash A \supset B & \text{if and only if} & x \Vdash A \text{ implies } x \Vdash B \\
x \Vdash \mathsf{K}_a A & \text{if and only if} & \text{for all } y,\ x\mathsf{R}_\mathsf{a} y \text{ implies } y \Vdash A
\end{array}$$

Epistemic models get easily represented with diagrams. In the picture below, agent b considers possible at x the state x itself and the state y. However, while he knows that P, since P is true at all states R_b-accessible, he does not not know Q, since y is R_b-accessible but the atom Q is not forced at y.



A similar semantics can be given for the **belief operator** $\mathsf{B}_a A$: “the agent a believes that A”. In contrast with knowledge, a belief is not necessarily true and we must modify the semantics accordingly. In fact, knowledge is supposed to imply truth, whereas it is natural to think that agents may believe something even if it is false. In other words, although we may believe something false, if we know something then it must be true. Given that what is known is true semantically corresponds to the reflexivity of the accessibility relations, in a frame for belief the reflexivity of the accessibility relations will not be assumed. Despite believing something is weaker than knowing we still assume that our beliefs are consistent, so that contradictions are not believed. Semantically, this is equivalent to imposing seriality of the accessibility relation R_a, for each agent a: for each state x there is a state y such that xR_ay.

Properties of knowledge

The properties of knowledge are described in terms of validity in an epistemic frame. A formula A is valid in an epistemic frame $\mathfrak{F}$ when it is forced at every state x in every model $\mathfrak{M}$ based on $\mathfrak{F}$. An important property of knowledge is the **distributivity** of the K_a operator over implicative formulas, that is, the validity of the formula $\mathsf{K}_a(A \supset B) \supset (\mathsf{K}_aA \supset \mathsf{K}_aB)$. This suggests that agents are very powerful reasoners since distributivity implies that agents know all the logical consequences of their knowledge: if an agent a knows both $A \supset B$ and A, then $A \supset B$ and A hold in all possible states he considers possible, so also B must holds in all these states. But if B holds in every possible state then B is known by the agent. Another important property is expressed by the knowledge generalization rule (or, **necessitation**): if A is valid then K_aA is valid too. This is not to say that the formula $A \supset \mathsf{K}_aA$ is valid. Agents do not necessarily know all the true facts, and in the last chapter we will consider this formula as a form of paradox. Conversely, it is instead the case that if an agent knows something that this fact must be true, a principle expressed by the formula $\mathsf{K}_aA \supset A$. As we have already said, this formula has been taken to be characteristic of knowledge as distinguished from belief. The property of **factivity** follows from the reflexivity of each accessibility relation in an epistemic frame: since the actual state is always accessible from itself, if A holds at every state accessible then, in particular, it holds at the actual one. Finally, the agents have complete introspection concerning their knowledge, that is, they know what they know and what they do not know. In terms of validity, **positive and negative introspection** correspond to the formulas $\mathsf{K}_aA \supset \mathsf{K}_a\mathsf{K}_aA$

and $\neg \mathsf{K}_a A \supset \mathsf{K}_a \neg \mathsf{K}_a A$. Validity of positive introspection follows from transitivity of R_a, whereas the negative introspection follows from transitivity and symmetry.

Historically, the modern epistemic modal logic originated with the work of the Finnish logicians H.G. von Wright and J. Hintikka, especially with the pioneering contributions von Wright (1951) and Hintikka (1962). The former is one of the earliest attempt to formalize the properties of knowledge and belief in terms of axiomatic systems. The latter is the most influential treatment of the modal logic of knowledge and helped to carry the subject of epistemic logic into mainstream epistemology, game theory, economics, and computer science. Since Hintikka's book epistemic notions have been strictly connected with the familiar possible-world semantics. The applications of epistemic logic to computer science (see Meyer and van der Hoek 2004) and the modern extensions of it with dynamic modal operators (see van Ditmarsch *et. al.* 2007) are still in the tradition of that early studies.

Chapter 1

Labelled Sequent Systems

In this chapter, basic modal logic is formulated as a **labelled sequent system** through an internalization of Kripke semantics within the syntax of the rules. In a labelled system, each formula A receives a label x, and this is indicated by $x : A$. The labels are interpreted as possible states and the labeling specifies the state at which a formula is true. Moreover, labels may occur also in expressions for accessibility relation as xRy. The rules of a labelled system operate on the labels and on the relations between them. More specifically, the language of sequents in enriched in such a way that in a sequent $\Gamma \to \Delta$ two kinds of expression may occur: labelled formulas $x : A$ and relational atom xRy. A labelled formula corresponds to forcing relation $x \Vdash A$, whereas xRy is the accessibility relation between worlds. As usual, Γ and Δ are multisets (lists without order) of labelled formulas or relational atoms. The sequent rules are found from (and justified by) the corresponding rules of natural deduction: the introduction rules get translated directly into right sequent rules, whereas the elimination rules, written in the manner of disjunction elimination, are converted into a left sequent rules by cut.

In this chapter, and also in the next one, we take into account the case of basic mono-modal logic: the language contain only two modal operators $\Box$ and $\Diamond$, interpreted as necessity and possibility operators, respectively. Intuitively, a formula as

$\Box A$ means that A is necessarily true, whereas $\Diamond A$ means that A is possibly true. The semantics is modified accordingly, and we assume that there just a single accessibility relation R.

1.1 Labelled natural deduction

The sequent rules for each connective and for the modalities $\Box$ and $\Diamond$ are presented as a formalization of the derivability relation of their corresponding natural deduction rules, following the pattern of Negri and von Plato (2001). In turn, the rules of natural deduction are found from the meaning explanation of connectives and modalities in terms of Kripke semantics and an inversion principle. First, we consider the inductive definition of forcing for a modal formula $\Box A$

$x \Vdash \Box A$ *if and only if* *for all* y, xRy *implies* $y \Vdash A$

This equivalence gives at once the sufficient and necessary conditions for $\Box A$ to be forced at an arbitrary state x. By considering only the if-direction of this definition, the sufficient condition is found:

If for all y, xRy *implies* $y \Vdash A$ *then* $x \Vdash \Box A$

In terms of proof system, this part corresponds to a derivability condition of the form

If for all y, xRy *derives* $y : A$ *then* $x : \Box A$ *can be derived*

The latter gives an introduction rule of natural deduction for the $\Box$ operator (see also Simpson 1994, p.66 and Viganò 2000, p. 20). If on the assumption that y is an arbitrary world accessible from x, we can show that A holds at y then we can conclude that A holds at x. Formally,

$$\dfrac{\begin{array}{c}\overset{1}{[xRy]}\\ \vdots \\ y : A\end{array}}{x : \Box A}\ \Box I_1$$

Note that in this formulation of rule $\Box I$ only the active assumption xRy is displayed. $I\Box$ combines the features of introduction rules for $\forall$ and $\supset$. As the introduction rule for universal quantifier of first order logic, $\Box I$ must meet the condition that the label y is different from x. Moreover, for implication introduction, the assumption xRy is discharged. The introduction rules for propositional connectives are found similarly. As a conjunction $A \wedge B$ is forced at x when both A and B are forced at x, we find

$$\dfrac{x : A \quad x : B}{x : A \wedge B}\ \wedge I$$

A disjunction $A \vee B$ is forced at x if either A or B is forced at x, so we have two introduction rules for $x : A \vee B$

$$\dfrac{x : A}{x : A \vee B}\ \vee I1 \qquad\qquad \dfrac{x : B}{x : A \vee B}\ \vee I2$$

Finally, the rule for introducing an implication is

$$\dfrac{\begin{array}{c}\overset{1}{[x : A]}\\ \vdots \\ x : B\end{array}}{x : A \supset B}\ \supset I_1$$

Elimination rules are found from introduction rules. The idea of a justification of elimination rules in terms of the introduction rules was already present in the work of Gentzen (see Gentzen 1969, p. 80), when he noted that

> It should be possible to display the E-inferences [elimination rules] as unique functions of their corresponding I-inferences [introduction rules], on the basis of certain requirements.

The requirement was made explicit by D. Prawitz in his monograph on natural deduction (see Prawitz 1965), and it is nowadays known as **inversion principle**. This idea is that nothing is gained if an introduction rule is followed by an elimination rule, or more precisely

Inversion Principle (Prawitz). The conclusion of an elimination rule R with major premise $A \star B$ is already contained in the assumptions used to derive $A \star B$ from $\star$-introduction rules, together with the minor premises of the rule.

However, Gentzen's and Prawitz's principle justifies but does not uniquely determine the elimination rules. We consider here a generalization of the inversion principle, one that leads to elimination rules that are more general than the usual ones. The general inversion principle and general elimination rules were introduced to obtain a simpler proof of normalization theorem and to achieve a full correspondence between natural deduction and sequent calculus. However, we use them here only to justify the sequent calculus rules. In an elimination rule, the formula $x : \Box A$ occurs as major premise and we ask what conditions are needed to satisfy the following (see Negri and von Plato 2001, p. 6).

Inversion Principle (General). Whatever follows from the direct grounds for deriving a proposition must follows from that proposition.

The elimination rules for propositional connectives are the general elimination rules of von Plato (2001), with the exception that here formulas are labelled. However, in systems for classical logic, propositional rules do not change the labels, when applied on formulas with a propositional connective as principal connective. Thus, the general elimination rule for conjunction is

$$\dfrac{x : A \wedge B \qquad \begin{matrix} \overset{1}{[x : A, x : B]} \\ \vdots \\ u : C \end{matrix}}{u : C} \wedge E_1$$

The standard rules of conjunction elimination are special cases of the general one, when u is x and C is either A or B.

$$\dfrac{x : A \wedge B}{x : A} \wedge E1 \qquad \dfrac{x : A \wedge B}{x : B} \wedge E2$$

The general elimination of disjunction was found already by Gentzen. The labelled version of the rule is as follows

$$\dfrac{x : A \vee B \qquad \begin{matrix} \overset{1}{[x : A]} \\ \vdots \\ u : C \end{matrix} \qquad \begin{matrix} \overset{1}{[x : B]} \\ \vdots \\ u : C \end{matrix}}{u : C} \vee E_1$$

The elimination of an implication is more complicated because the direct ground for deriving $x : A \supset B$ is not a formula, but, in turn, a derivation of $x : B$ from the assumption $x : A$. In fact, in Schroeder-Heister (1984, 2010) the rule gets formulated as an higher-level unlabelled rule

$$\dfrac{A \supset B \qquad \begin{matrix} \overset{1}{[A \vdash B]} \\ \vdots \\ C \end{matrix}}{C} \supset E_1'$$

where the symbol $\vdash$ denotes the derivability relation and expresses the fact that B is derivable from A. However, the existence of this derivation can be expressed by saying that if C follows from B then it already follows from A. In the labelled formalism, this gives the following elimination rule for $x : A \supset B$

$$\dfrac{x : A \supset B \quad x : A \quad \begin{matrix} [x : B]^1 \\ \vdots \\ u : C \end{matrix}}{u : C} \supset E_1$$

When u is x and B is C, the special elimination rule obtained is the labelled version rule of *modus ponens*

$$\dfrac{x : A \supset B \quad x : A}{x : B} \supset E$$

Finally, the zero-ary connective $\bot$ has only an elimination rule. Given that there is no ground for deriving $\bot$, from the inversion principle we obtain an elimination rule that has only the major premise $x : \bot$. The rule is also known as rule of *ex falso quodlibet*

$$\dfrac{x : \bot}{u : C} \text{ EFQ}$$

The same reasoning of general elimination of $\supset$ holds for the general elimination rule of $\Box$. In Read (2008) is proposed the heigher-level general elimination rule for $\Box$

$$\dfrac{x : \Box A \qquad \begin{matrix} [xRy \vdash y : A]^1 \\ \vdots \\ u : C \end{matrix}}{u : C} \Box E'_1$$

Analogously to the case of implication elimination, the direct ground for deriving $x : \Box A$ is the existence of a hypothetical derivation of $y : A$ from xRy. However, our rule $\Box E$ is justified, as above, by the fact that the existence of such derivation can be expressed by saying that if $u : C$ follows from $y : A$, then it already follows from xRy. Thus,

$$\dfrac{x : \Box A \quad xRy \quad \begin{array}{c}[y \overset{1}{:} A] \\ \vdots \\ u : C\end{array}}{u : C}\ \Box E_1$$

The rule specializes in the standard elimination rule of Simpson (1994) when u is y and C is A

$$\dfrac{x : \Box A \quad xRy}{y : A}\ \Box E$$

The same considerations apply, *mutatis mutandis*, to the possibility operator $\Diamond$. From the semantic clause we get the sufficient condition for a formula as $\Diamond A$ to be forced, that is,

If for some y, xRy *and* $y \Vdash A$ *then* $x \Vdash \Diamond A$

In the labelled system, this condition is expressed in terms of derivability and we have that

If for some y, xRy *and* $y : A$ *can be derived then* $x : \Diamond A$ *can be derived*

Thus, the same introduction rule for $\Diamond$ of Simpson (1994) is found

$$\dfrac{xRy \quad y : A}{x : \Diamond A}\ \Diamond I$$

The corresponding elimination rule is obtained through the inversion principle

$$\dfrac{x : \Diamond A \qquad \begin{array}{c}[xRy, \overset{1}{y} : A] \\ \vdots \\ u : C\end{array}}{u : C}\ \Diamond E_1$$

where again y must be different form x and u and must not appear in any assumption other than xRy and $y : A$.

1.2 From natural deduction to sequent calculus

Sequent calculus is designed for keeping track locally of open assumptions, a feature that natural deduction lacks: in natural deduction only active formulas are shown, leaving implicit the other assumptions. Sequent calculus can be understood as a formal theory of derivability relation $\vdash$ in the corresponding system of natural deduction. As usual, we use two different symbols $\vdash$ and $\to$ in order to keep separated the metalevel expressions as $\Gamma \vdash u : C$ in natural deduction from the sequent $\Gamma \to u : C$ which is part of object language of sequent calculus. In this way, a sequent

$$\Gamma \to u : C$$

is interpreted as the assertion in natural deduction

$$\Gamma \vdash u : C$$

where formulas in Γ are the assumptions $u : C$ on which depends. In a sequent $\Gamma \to u : C$ the multiset Γ is called the antecedent and the formula $u : C$ the succedent. A translation from natural deduction to sequent calculus was already present in Gentzen's original work and it is discussed deeply in the context of general elimination rules in von Plato (2001, 2003). Each introduction rule is translated into a sequent rule that introduces the principal formula in the antecedent. The elimination rules are translated in two phases and a left sequent rule for each elimination rule in natural deduction is found by cut.

Translation of propositional rules

In natural deduction, a derivation can start with any formula A as assumption and assumptions, in general, can be discharged. However, it may happen that the formula assumed is the same formula that is discharged. In other words, the same formula can be an assumption and conclusion in a derivation. For instance, the law of identity needs that $x : A$ can act as both assumption and conclusion of an application of $L\supset$. The same behavior is encountered in a labelled natural deduction system, so we have

$$\frac{\overset{1}{[x : A]}}{x : A \supset A} \supset I_1$$

In sequent calculus, the fact that the same formula can be assumed and derived at the same time gives **initial sequents** $A \to A$. In labelled systems, initial sequents are of the form

$$x : A \to x : A \qquad xRy \to xRy$$

Often, initial sequents are called logical axioms, and the derivation of the law of identity is immediate

$$\frac{x : A \to x : A}{\to x : A \supset A} R\supset$$

Thus, discharge in natural deduction corresponds to the application of a sequent calculus rule that has an active formula in the antecedent of a premise.

The introduction rules of natural deduction get translated into **right rules** in sequent calculus, where the comma replaces the set-theoretic union. Conjunction introduction can be written with the assumptions made explicit as

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ x : A\end{matrix} \quad \begin{matrix}\Delta\\ \vdots\\ x : B\end{matrix}}{x : A \wedge B}\ {\scriptstyle \wedge I}$$

and it is converted to the following sequent calculus rule

$$\dfrac{\Gamma \rightarrow x : A \quad \Delta \rightarrow x : B}{\Gamma, \Delta \rightarrow x : A \wedge B}\ {\scriptstyle R\wedge}$$

Note that the rule has independent contexts, that is, Γ and Δ need not be the same multiset. On the other hand, from general elimination rules of natural deduction, **left rules** of sequent calculus are found. With explicit assumptions, the general elimination rule for $\wedge$ becomes

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ x : A \wedge B\end{matrix} \qquad \begin{matrix}\overset{1}{[x : A, x : B]}, \Delta\\ \vdots\\ u : C\end{matrix}}{u : C}\ {\scriptstyle \wedge E_1}$$

We want to translate it into a left sequent rule of the form

$$\dfrac{x : A, x : B, \Gamma, \Delta \rightarrow u : C}{x : A \wedge B, \Gamma, \Delta \rightarrow u : C}\ {\scriptstyle L\wedge}$$

The translation is in two steps. The general elimination rule is immediately rewritten into

$$\dfrac{\Gamma \rightarrow x : A \wedge B \quad x : A, x : B, \Delta \rightarrow u : C}{\Gamma, \Delta \rightarrow u : C}\ {\scriptstyle L\wedge'}$$

Then, it is shown that $L\wedge'$ is derivable from $L\wedge$ by cut.

$$\dfrac{\Gamma \rightarrow x : A \wedge B \quad \dfrac{x : A, x : B, \Delta \rightarrow u : C}{x : A \wedge B, \Delta \rightarrow u : C}\ {\scriptstyle L\wedge}}{\Gamma, \Delta \rightarrow u : C}\ {\scriptstyle \text{CUT}}$$

As to disjunction introduction we consider only one of the two cases. The rule

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ x : A\end{array}}{x : A \vee B}\ \vee I1$$

is immediately converted into a sequent calculus right rule

$$\dfrac{\Gamma \to x : A}{\Gamma \to x : A \vee B}\ R\vee$$

The corresponding elimination rule, that is,

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ x : A \vee B\end{array} \quad \begin{array}{c}[x \overset{1}{:} A], \Delta\\ \vdots\\ u : C\end{array} \quad \begin{array}{c}[x \overset{1}{:} B], \Theta\\ \vdots\\ u : C\end{array}}{u : C}\ \vee E_1$$

is first translated in sequent calculus

$$\dfrac{\Gamma \to x : A \vee B \quad x : A, \Delta \to u : C \quad x : B, \Theta \to u : C}{\Gamma, \Delta, \Theta \to u : C}\ L\vee'$$

Then, the labelled version of the standard context-independent left rule for disjunction is considered:

$$\dfrac{x : A, \Gamma \to u : C \quad x : B, \Delta \to u : C}{x : A \vee B, \Gamma, \Delta \to u : C}\ L\vee$$

Finally, $L\vee'$ is proved to be derivable in presence of the latter as follows

$$\dfrac{\Gamma \to x : A \vee B \quad \dfrac{x : A, \Delta \to u : C \quad x : B, \Theta \to u : C}{x : A \vee B, \Delta, \Theta \to u : C}\ L\vee}{\Gamma, \Delta, \Theta \to u : C}\ \text{CUT}$$

Now, the case of implication. Its introduction rule with explicit assumptions is

$$\dfrac{\dfrac{\overset{1}{[x : A]}, \Gamma}{\vdots}}{\dfrac{x : B}{x : A \supset B}} \supset I$$

and gets translated immediately into a right sequent calculus rule

$$\frac{\Gamma \rightarrow x : A \supset B}{x : A, \Gamma \rightarrow x : B} R\supset$$

Consider now the corresponding elimination, that is,

$$\frac{\begin{array}{c}\Gamma\\ \vdots\\ x : A \supset B\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ x : A\end{array} \quad \begin{array}{c}\overset{1}{[x : B]}, \Theta\\ \vdots\\ u : C\end{array}}{u : C} \supset E_1$$

The immediate translation gives

$$\frac{\Gamma \rightarrow x : A \supset B \quad \Delta \rightarrow x : A \quad x : B, \Theta \rightarrow u : C}{\Gamma, \Delta, \Theta \rightarrow u : C} L\supset'$$

Once again, the latter is shown to be derivable from the following $L \supset$, by cut

$$\frac{\Gamma \rightarrow x : A \quad x : B, \Delta \rightarrow u : C}{x : A \supset B, \Gamma, \Delta \rightarrow u : C} L\supset$$

In fact,

$$\frac{\Gamma \rightarrow x : A \supset B \quad \dfrac{\Delta \rightarrow x : A \quad x : B, \Theta \rightarrow u : C}{x : A \supset B, \Delta, \Theta \rightarrow u : C} L\supset}{\Gamma, \Delta, \Theta \rightarrow u : C} \text{CUT}$$

As we already said, the logical symbol $\perp$ has no introduction rule because there are no grounds for asserting $\perp$. Thus, $\perp$ has only an elimination rule, known as rule of *ex falso quodlibet*, and written with explicit assumptions

$$\cfrac{\begin{array}{c}\Gamma\\ \vdots\\ x:\bot\end{array}}{u:C}\ \text{EFQ}$$

First, it is translated into

$$\frac{\Gamma \to x:\bot}{\Gamma \to u:C}\ L\bot'$$

Then, it is shown that this rule in derivable in presence of the following zero-ary rule $L\bot$

$$\overline{x:\bot \to u:C}\ L\bot$$

Again, using cut,

$$\frac{\Gamma \to x:\bot \quad \overline{x:\bot \to u:C}\ L\bot}{\Gamma \to u:C}\ \text{CUT}$$

Observation. Along with initial sequents, also sequents $x : \bot \to u : C$ are considered as initial rather than a zero-ary inference rule. However, such sequents cannot be properly taken as initial from the perspective of the translation from natural deduction because $L\bot$, being a translation of an inference rule, must be, in turn, an inference rule. Thus, they are considered here as a zero-ary inference rules as in Negri and von Plato (2001) and in Negri (2005).

Negation is not primitive, but it is defined in terms of $\supset$ and $\bot$ so that $x : \neg A$ stands for $x : A \supset \bot$. In this way, the rules for negation are derived from those for $\supset$, accordingly. For negation introduction we have

$$\dfrac{\begin{array}{c}[x \overset{1}{:} A], \Gamma \\ \vdots \\ x : \bot\end{array}}{x : \neg A}\ \neg I_1$$

An immediate translation into a sequent rule gives

$$\dfrac{x : A, \Gamma \to x : \bot}{\Gamma \to x : \neg A}\ R\neg'$$

The general elimination rule for negation takes the form

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ x : \neg A\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ x : A\end{array} \quad \begin{array}{c}[x \overset{1}{:} \bot], \Theta \\ \vdots \\ u : C\end{array}}{u : C}\ \neg E_1$$

The direct translation into sequent calculus gives

$$\dfrac{\Gamma \to x : \neg A \quad \Delta \to x : A \quad x : \bot, \Theta \to u : C}{\Gamma, \Delta, \Theta \to u : C}\ L\neg'$$

Now, consider the left rule for negation

$$\dfrac{\Gamma \to x : A}{x : \neg A, \Gamma \to}\ L\neg$$

Thus, $L\neg'$ is derivable in presence of $L\neg$ as follows

$$\dfrac{\dfrac{\Gamma \to x : \neg A \quad \dfrac{\dfrac{\Delta \to x : A}{\Delta \to x : A, x : \bot}\ \text{R-W}}{x : \neg A, \Delta \to x : \bot}\ L\neg}{\Gamma, \Delta \to x : \bot}\ \text{CUT} \qquad x : \bot, \Theta \to u : C}{\Gamma, \Delta, \Theta \to u : C}\ \text{CUT}$$

Translation of modal rules

We turn now to the translation from natural deduction to sequent calculus for modal rules. The rule $I\Box$ with explicit assumptions

$$\dfrac{\begin{matrix}\overset{1}{[xRy]}, \Gamma \\ \vdots \\ y : A\end{matrix}}{x : \Box A}\ \Box I$$

corresponds to the right sequent rule

$$\dfrac{xRy, \Gamma \to y : A}{\Gamma \to x : \Box A}\ R\Box 0$$

$R\Box$ must meet the usual condition that the label y is different from x and must not occur in Γ, that is, y must not be in the conclusion of the rule. It is said that y is the *eigenvariable* of the rule. The general elimination rule is

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ x : \Box A\end{matrix} \quad \begin{matrix}\Delta \\ \vdots \\ xRy\end{matrix} \quad \begin{matrix}\overset{1}{[y : A]}, \Theta \\ \vdots \\ u : C\end{matrix}}{u : C}\ \Box E$$

It is translated into a left sequent rule of the form

$$\dfrac{\Gamma \to x : \Box A \quad \Delta \to xRy \quad y : A, \Theta \to u : C}{\Gamma, \Delta, \Theta \to u : C}\ L\Box'$$

Consider the left sequent rule with independent contexts

$$\dfrac{\Gamma \to xRy \quad y : A, \Delta \to u : C}{x : \Box A, \Gamma, \Delta \to u : C}\ L\Box 0$$

As above, $L\Box'$ can be derived from $L\Box 0$ by cut

$$\dfrac{\Gamma \to x : \Box A \qquad \dfrac{\Delta \to xRy \quad y : A, \Theta \to u : C}{x : \Box A, \Delta, \Theta \to u : C}\, L\Box 0}{\Gamma, \Delta, \Theta \to u : C}\,\text{CUT}$$

Finally, we deal with the $\Diamond$ operator. The rule of $\Diamond$ introduction is

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ xRy\end{matrix} \quad \begin{matrix}\Delta \\ \vdots \\ y : A\end{matrix}}{x : \Diamond A}\,\Diamond I$$

It is translated into a right sequent rule straightforwardly

$$\dfrac{\Gamma \to xRy \quad \Delta \to y : A}{\Gamma, \Delta \to x : \Diamond A}\, R\Diamond 0$$

The $\Diamond$-elimination with explicit contexts gets formulated as

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ x : \Diamond A\end{matrix} \qquad \begin{matrix}\overset{1}{[xRy, y : A]}, \Delta \\ \vdots \\ u : C\end{matrix}}{u : C}\,\Diamond E_1$$

where y is different from x and u and it does not appear in Γ, Δ. Its immediate translation is

$$\dfrac{\Gamma \to x : \Diamond A \quad xRy, y : A, \Delta \to u : C}{\Gamma, \Delta \to u : C}\, L\Diamond'$$

$L\Diamond'$ is derivable in presence of the standard left rule

$$\dfrac{xRy, y : A, \Gamma, \Delta \to u : C}{x : \Diamond A, \Gamma, \Delta \to u : C}\, L\Diamond$$

by cut as follows

$$\dfrac{\Gamma \rightarrow x : \Diamond A \qquad \dfrac{xRy, y : A, \Delta \rightarrow u : C}{x : \Diamond A, \Delta \rightarrow u : C}\, L\Diamond}{\Gamma, \Delta \rightarrow u : C}\, \text{CUT}$$

Structural rules

Strictly speaking, natural deduction has no **structural rules**. This means that in natural deduction no structural rule is explicitly assumed, but it does not mean that it is not possible to manage assumptions. Assumptions can be discharged and the discharge is optional: it is possible to leave an assumption open, even if it could be discharged. The way in which assumptions are managed in natural deduction has a correspondence in the usual structural rules of sequent calculus. In particular, it is possible to discharge assumptions which have been not made as in the derivation of the *a fortiori* law,

$$\dfrac{\dfrac{\overset{1}{[x : A]}}{x : B \supset A}\, {\supset}I}{x : A \supset (B \supset A)}\, {\supset}I_1$$

In sequent calculus, the vacuous discharge corresponds to the structural rule of **weakening**, that is,

$$\dfrac{\Gamma \rightarrow u : C}{x : A, \Gamma \rightarrow u : C}\, \text{L-W}$$

In fact, by weakening, we obtain a sequent calculus derivation of the *a fortiori* law as follows

$$\dfrac{\dfrac{\dfrac{x : A \rightarrow x : A}{x : A, x : B \rightarrow x : A}\, \text{L-W}}{x : A \rightarrow x : B \supset A}\, R{\supset}}{\rightarrow x : A \supset (B \supset A)}\, R{\supset}$$

However, in some cases we may also need to discharge more than one occurrence of the same assumption, as in the following derivation

$$
\dfrac{\dfrac{\dfrac{[x : A \supset \overset{2}{(}A \supset B)] \quad [x \overset{1}{:} A]}{x : A \supset B}\supset E \qquad [x \overset{1}{:} A]}{\dfrac{x : B}{x : A \supset B}\supset I_1}\supset E}{x : (A \supset (A \supset B)) \supset (A \supset B)}\supset I_2
$$

The multiple discharge of the assumption $x : A$ corresponds, in sequent calculus, to the rule of **contraction**

$$
\dfrac{x : A, x : A, \Gamma \to u : C}{x : A, \Gamma \to u : C}\text{ L-C}
$$

In fact, the above formula is derivable in the presence of contraction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{x : A \to x : A \qquad \dfrac{x : A \to x : A \quad x : B \to x : B}{x : A \supset B, x : A \to x : B} L\supset}{x : A \supset (A \supset B), x : A, x : A \to x : B} L\supset}{x : A \supset (A \supset B), x : A \to x : B}\text{ L-C}}{x : A \supset (A \supset B) \to x : A \supset B} R\supset}{\to x : (A \supset (A \supset B)) \supset (A \supset B)} R\supset
$$

The rule of **exchange** was primitive in the original Gentzen's systems **LK** and **LJ**, but the use of multisets of formulas instead of lists makes it superfluous. Finally, in natural deduction, derivations can be composed. If $x : A$ has been derived from the open assumptions Γ and $u : C$ has been derived from $x : A$ along with the open assumptions Δ then $u : C$ can be derived from the open assumptions Γ, Δ. This corresponds, in sequent calculus, to the rule of **cut**

$$
\dfrac{\Gamma \to x : A \quad x : A, \Delta \to u : C}{\Gamma, \Delta \to u : C}\text{ CUT}
$$

Cut is the only rule that makes a formula disappear in a derivation. This feature has the consequence that when we want to determine whether a sequent $\Gamma \to u : C$ is derivable we could always try to reduce the task into $\Gamma \to v : A$ and $v : A, \Delta \to u : C$, where $v : A$ is an arbitrary new formula, with no end. Because of this lack of determinism introduced by cut, the main task of structural proof theory is to prove that the rule of cut is redundant in a given system of rules. The redundancy of the cut rule is expressed formally in terms of rule admissibility: for every cut-free derivation of the premises of cut there exists a derivation of its conclusion that uses only primitive rules or rules already proved to be admissible. Moreover, the proof of cut admissibility we shall give in the following is constructive: we effectively show how to find a derivation of the conclusion of cut from all derivations of its premises. In this sense, we can also say that the redundancy of cut for a system of rules means that the system is closed under cut in a strong sense. Else, it can be also said that cut is eliminable: if cut is considered as a primitive inference rule, cut admissibility reduces to the proof of cut elimination. The latter version corresponds to the celebrated main theorem, or *Hauptsatz*, of Gerhard Gentzen who gave its first proof for systems **LJ** and **LK** of intuitionistic and classical logic.

Subformula property

Among the consequences of cut elimination is the **subformula property**: every formula in a derivation is subformula of the formulas in the endsequent. As already noted by Gentzen (see Gentzen 1934, p. 88)

> Intuitively speaking, these properties of derivations without cuts may be expressed as follows: the S–formulas [formulas in sequents] become longer as we descend lower down in derivation, never shorter. The final result is, as it were, gradually built up from its constituent elements. The proof represented by the derivation is not round-about in that it contains only concepts which recur in the final result.

Here, the notion of subformula is generalized in order to match the setting of labelled formulas. In particular, we consider for an arbitrary label y, a formula such a $y : A$ as a (proper) subformula of $x : \Box A$.

Definition (Subformula Set)**.** *Let* L *be the set of labels. The* subformula set SF *of a formula* $x : B$ *is defined inductively.*

$SF(x : P) = \{x : P\}$*;*

$SF(xRy) = \{xRy\}$*;*

$SF(x : \neg B) = SF(x : B) \cup \{x : \neg B\}$*;*

$SF(x : B \circ C) = SF(x : B) \cup SF(x : C) \cup \{x : B \circ C\}$*, if* $\circ$ *is* $\wedge$*,* $\vee$*,* $\supset$*;*

$SF(x : \Box B) = \bigcup\limits_{y \in L} SF(y : A) \cup \{x : \Box B\}$.

Consequently, $x : A$ *is a* subformula *of* $x : B$ *when* $x : A \in SF(x : B)$*. Finally,* proper subformulas *of a formula* $x : B$ *are all the subformulas of* $x : B$*, except* $x : B$ *itself.*

The subformula property is usually the main consequence of cut elimination. However, in labelled systems is not any longer so. In fact, the modal rules do not satisfy subformula property, since the relational atom xRy occurring in the premise of $R\Box$ and $L\Diamond$ disappears in the conclusion. The lack of the subformula property is surely an unpleasant feature because it constitutes a serious obstacle to the possibility of ensuring decidability. Nevertheless, a closer inspection of the modal rules reveals that when a relational atom as xRy disappears by an application of $R\Box$ or $L\Diamond$, what is irremediably lost is the *eigenvariable* y, whereas x still occurs in the conclusion as the label of principal formula $x : \Box A$ or $x : \Diamond A$. The same happens with the rules $R\forall$ and $L\exists$ of **LK**. Therefore, no variable, except for *eigenvariables*, disappears and a more refined result can be given in the form of the **subterm property**: all labels in a derivation are either *eigenvariables* or labels in the endsequent.

1.3 A system for basic modal logic

Modal logic is mostly presented as based on classical logic. More precisely, axiomatic systems of modal logic consist of the axioms of the classical propositional calculus together with specific axioms concerning the modal operators. However, all sequent rules we have considered so far have at most one formula in the succedent of the sequent, so they give a system which is weaker than classical logic. In natural deduction, classical logic can be obtained by adding the rule of excluded middle, or rule of *tertium non datur*.

$$\frac{\begin{array}{c}[x \overset{1}{:} A], \Gamma \\ \vdots \\ u : C\end{array} \quad \begin{array}{c}[x \overset{1}{:} \neg A], \Delta \\ \vdots \\ u : C\end{array}}{u : C}\ \mathrm{EM}_1$$

In the presence of EM, the law of excluded middle is derivable as follows

$$\frac{\dfrac{[x \overset{1}{:} A]}{x : A \vee \neg A}\ \vee I_1 \quad \dfrac{[x \overset{1}{:} \neg A]}{x : A \vee \neg A}\ \vee I_1}{x : A \vee \neg A}\ \mathrm{EM}_1$$

EM generalizes the rule of indirect proof, or rule of *reductio ad absurdum*, considered in Prawitz (1965)

$$\frac{\begin{array}{c}[x \overset{1}{:} \neg A] \\ \vdots \\ u : \bot\end{array}}{x : A}\ \mathrm{RAA}_1$$

The translation of EM in sequent calculus gives the rule

$$\frac{x : A, \Gamma \to u : C \quad x : \neg A, \Delta \to u : C}{\Gamma, \Delta \to u : C}\ LR\neg$$

The rules for negation we considered so far do not give a derivation of the law of excluded middle, whereas $LR\neg$ does.

$$\dfrac{\dfrac{x : A \to x : A}{x : A \to x : A \lor \neg A}\,R\lor \qquad \dfrac{x : A \to x : A}{x : \neg A \to x : A \lor \neg A}\,R\lor}{\to x : A \lor \neg A}\,LR\neg$$

However, rule $LR\neg$ is not the only way to get classical logic in sequent calculus. Alternatively, we may extend the notion of sequent so that a sequent can have an arbitrary multiset Δ as succedent, instead of a single formula $u : C$. By allowing multi-succedent sequents, the rules for negation become the labelled version of the rules already considered in Gentzen's original work, that is,

$$\dfrac{\Gamma \to \Delta, x : A}{x : \neg A, \Gamma \to \Delta}\,L\neg \qquad\qquad \dfrac{x : A, \Gamma \to \Delta}{\Gamma \to \Delta, x : \neg A}\,R\neg$$

Using contraction, the law of excluded middle can now be derived

$$\dfrac{\dfrac{\dfrac{\dfrac{x : A \to x : A}{\to x : A, x : \neg A}\,R\neg}{\to x : A \lor \neg A, x : \neg A}\,R\lor}{\to x : A \lor \neg A, x : A \lor \neg A}\,R\lor}{\to x : A \lor \neg A}\,\text{R-C}$$

With multi-succedent sequents, we obtain a labelled sequent calculus with the same propositional rules of **G0c** (without quantifiers) of Negri and von Plato (2001), p. 95. In addition, here we have rules for $\Box$ and $\Diamond$. We shall refer to this system as **G0K**, where **K** stands for Kripke. **G0K** is a labelled sequent calculus for the basic modal logic **K**. All the two-premise rules of **G0K** have independent contexts because they are translated from the rules of natural deduction. This feature makes **G0K** not suitable for the systematic search of derivations, since two-premise rules are not invertible and they cannot be applied backwards. Later, only context sharing two-premise rules will be used, in order to have contraction admissible.

Logical rules of **G0K**

$$x : A \to x : A \qquad xRy \to xRy$$

$$\frac{x : A, x : B, \Gamma \to \Delta}{x : A \wedge B, \Gamma \to \Delta}\, L\wedge \qquad \frac{\Gamma \to \Delta, x : A \quad \Gamma' \to \Delta', x : B}{\Gamma, \Gamma' \to \Delta', \Delta, x : A \wedge B}\, R\wedge$$

$$\frac{x : A, \Gamma \to \Delta \quad x : B, \Gamma' \to \Delta'}{x : A \vee B, \Gamma, \Gamma' \to \Delta', \Delta}\, L\vee \qquad \frac{\Gamma \to \Delta, x : A}{\Gamma \to \Delta, x : A \vee B}\, R\vee 1 \qquad \frac{\Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \vee B}\, R\vee 2$$

$$\frac{\Gamma \to \Delta, x : A \quad x : B, \Gamma' \to \Delta'}{x : A \supset B, \Gamma, \Gamma' \to \Delta', \Delta}\, L\supset \qquad \frac{x : A, \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \supset B}\, R\supset$$

$$\frac{}{x : \bot \to \Delta}\, L\bot$$

$$\frac{xRy, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\, R\Box 0 \qquad \frac{\Gamma \to \Delta, xRy \quad y : A, \Gamma' \to \Delta'}{x : \Box A, \Gamma, \Gamma' \to \Delta', \Delta}\, L\Box 0$$

$$\frac{\Gamma \to \Delta, xRy \quad \Gamma' \to \Delta', y : A}{\Gamma, \Gamma' \to \Delta', \Delta, x : \Diamond A}\, R\Diamond 0 \qquad \frac{xRy, y : A, \Gamma \to \Delta}{x : \Diamond A, \Gamma \to \Delta}\, L\Diamond 0$$

With negation defined in terms of $\supset$ and $\bot$, the corresponding rules are derived from those for $\supset$, and will be used only to shorten derivations. Note that initial sequent have an arbitrary labelled formula $x : A$ as principal. As in **G0c** of Negri and von Plato (2001), weakening and contraction are primitive, and not admissible, inference rules of **G0K**. They can have as active formulas either labelled formulas or relational atoms.

Structural rules of **G0K**

$$\frac{\Gamma \rightarrow \Delta}{x : A, \Gamma \rightarrow \Delta}\text{ L-W} \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, x : A}\text{ R-W} \qquad \frac{\Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}\text{ L-W}$$

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, xRy}\text{ R-W}$$

$$\frac{x : A, x : A, \Gamma \rightarrow \Delta}{x : A, \Gamma \rightarrow \Delta}\text{ L-C} \qquad \frac{\Gamma \rightarrow \Delta, x : A, x : A}{\Gamma \rightarrow \Delta, x : A}\text{ R-C} \qquad \frac{xRy, xRy, \Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}\text{ L-C}$$

$$\frac{\Gamma \rightarrow \Delta, xRy, xRy,}{\Gamma \rightarrow \Delta, xRy}\text{ R-C}$$

As we already said, the rule of cut

$$\frac{\Gamma \rightarrow \Delta, x : C \quad x : C, \Gamma' \rightarrow \Delta'}{\Gamma, \Gamma' \rightarrow \Delta', \Delta}\text{ CUT}$$

is not assumed as a primitive rule, but it can be proved to be **admissible** in **G0K**. In general, a rule R with premises $S_1, \ldots, S_n$ and conclusion S is admissible in a system **G** if, whenever an istance of $S_1, \ldots, S_n$ is derivable in **G**, the corresponding istance of S is derivable in **G**. The presence of contraction complicates the proof of cut admissibility. Already Gentzen met the problem of finding a suitable permutation of cut and contraction in the proof of the *Hauptsatz* for **LK**. If the right premise of cut is derived by contraction, the permutation of cut and contraction does not guarantees that the istance of cut is admissible by the inductive hypothesis. The solution proposed by Gentzen is to consider a version of cut that permits to eliminate $m \geqslant 1$ occurrences of the cut formula. Then it is proved that the calculus with multicut is equivalent to the calculus with cut, that is, they derive exactly the same sequents. For details of the *Hauptsatz* with multicut see Takeuti (1987) and for a proof without multicut see von Plato (2001a). The system **G0K** is strictly similar to the unlabelled system **G0c** of Negri and von Plato 2001. The translation of natural deduction rules into sequent calculus rules are discussed at length in the introductory chapter of Negri and von Plato (2011).

Chapter 2

Cut Elimination

The aim of this chapter is to find a labelled sequent calculus in which all the structural rules (weakening and contraction) are admissible and cut is eliminable. Sequent systems in which cut is eliminable permit to find derivations in a systematic way and to check whether a sequent $\Gamma \to \Delta$ is derivable by a root-first proof search procedure: given $\Gamma \to \Delta$, we can decompose its formulas and get simpler sequents until we arrive at sequents in which there is nothing to decompose left. However, the possibility of building a derivation starting from the sequent to be derived rests not only on cut elimination, but also on the possibility to apply logical rules backwards. This is to say that the logical rules must be **invertible**: from the derivability of the conclusion of an inference rule, the derivability of its premises follows. The property of inversion has been first isolated by Ketonen (see Ketonen 1944 and von Plato 2009 for historical backgrounds) and can be achieved for classical propositional logic by considering all the two-premise rule in their context-sharing formulation. The rules with independent context are similar to (and derived directly from) those of natural deduction but they do not support proof search. The context-independent rules impose that we know how the contexts in the conclusion should be divided in the premises. However, when we search for a derivation we do not divide the context at all but repeat it fully in both premises. Therefore rules

$R\wedge$, $L\vee$ and $L\supset$ become:

$$\frac{\Gamma \to \Delta, x : B \quad \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \wedge B} R\wedge \qquad \frac{x : A, \Gamma \to \Delta \quad x : B, \Gamma \to \Delta}{x : A \vee B, \Gamma \to \Delta} L\vee$$

$$\frac{\Gamma \to \Delta, x : A \quad x : B, \Gamma \to \Delta}{x : A \supset B, \Gamma \to \Delta} L\supset$$

Also for the two-premise modal rules $L\Box 0$ and $R\Diamond 0$ a context-sharing formulation is possible.

$$\frac{\Gamma \to \Delta, xRy \quad y : A, \Gamma \to \Delta}{x : \Box A, \Gamma \to \Delta} L\Box 0 \qquad \frac{\Gamma \to \Delta, xRy \quad \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Diamond A} R\Diamond 0$$

Moreover, instead of two rules for right disjunction we can consider a single rule, which restores the duality between $\wedge$ and $\vee$

$$\frac{\Gamma \to \Delta, x : A, x : B}{\Gamma \to \Delta, x : A \vee B} R\vee$$

In the presence of weakening, contraction, and the rules of **G0K**, these new rules are derivable. For instance, the new $R\vee$ is derivable by contraction and the previous $R\vee$, indicated as $R\vee''$.

$$\cfrac{\cfrac{\cfrac{\to x : A, x : B}{\to x : A \vee B, x : B} R\vee''}{\to x : A \vee B, x : A \vee B} R\vee''}{\to x : A \vee B} \text{R-C}$$

The latter derivation is similar to that of the law of excluded middle given in the previous chapter, but using $R\vee$, the law of excluded middle can be derived without any application of contraction and, more importantly, a systematic proof-search procedure from the conclusion becomes possible

$$\cfrac{\cfrac{x : A \to x : A}{\to x : A, x : \neg A} R\neg}{\to x : A \vee \neg A} R\vee$$

In fact, contraction can be as "bad" as cut as for as the problem of finding derivations as concerned: reading contraction bottom-up (from the conclusion to the premise), formulas in the antecedent are multiplied with no end. Proof search is irremediably lost as long as contraction is primitive and not admissible in our system. The rules of weakening become admissible when initial sequents and $L\bot$ are formulated in a form that allows both left and right contexts.

$$x : A, \Gamma \to \Delta, x : A \qquad\qquad xRy, \Gamma \to \Delta, xRy$$

$$\frac{}{x : \bot, \Gamma \to \Delta}\ L\bot$$

In this way, weakening is built into initial sequents: consider the derivation of the *a fortiori law* of the previous chapter and note that the application of weakening can be dispensed with because $x : A, x : B \to x : A$ is an initial sequent and should not be derived from $x : A \to x : A$. However, initial sequents should be modified further. Note that in the formulation above initial sequents can have $x : A$ and xRy as principal formulas, where $x : A$ is an arbitrary labelled formula and xRy a relational atom. Actually, the latter can be left out, provided that the modal rules with xRy in the succedent are replaced by rules in which xRy appears only in the antecedent. This can be achieved by considering the following rule $L\Box 1$ (resp. $R\Diamond 1$) instead of $L\Box 0$ (resp. $R\Diamond 0$)

$$\frac{y : A, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\ L\Box 1 \qquad\qquad \frac{\Gamma \to \Delta, y : A}{xRy, \Gamma \to \Delta, x : \Diamond A}\ R\Diamond 1$$

It easy to show that $L\Box 1$ is derivable in the presence of $L\Box 0$ and initial sequents with xRy as principal formulas, as follows

$$\frac{xRy, \Gamma \to \Delta, xRy \quad y : A, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\ L\Box 0$$

Analogously, $R\Diamond 1$ is derivable in the presence of $R\Diamond 0$ and relational initial sequents

$$\frac{xRy, \Gamma \to \Delta, xRy \quad \Gamma \to \Delta, y : A}{xRy, \Gamma \to \Delta, x : \Diamond A} \; R\Diamond 0$$

Viceversa, $L\Box 0$ is derivable in the presence of $L\Box 1$ and cut

$$\frac{\Gamma \to \Delta, xRy \quad \dfrac{y : A, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta} \; L\Box 1}{x : \Box A, \Gamma \to \Delta} \; \text{CUT}$$

Analogously, $R\Diamond 0$ is derivable in presence of $R\Diamond 1$ and cut

$$\frac{\Gamma \to \Delta, xRy \quad \dfrac{\Gamma \to \Delta, y : A}{xRy, \Gamma \to \Delta, x : \Diamond A} \; R\Diamond 1}{\Gamma \to \Delta, x : \Diamond A} \; \text{CUT}$$

When $L\Box 0$ is replaced with $L\Box 1$ and $R\Diamond 0$ with $R\Diamond 1$, no rule of **G0K** removes a relational atom xRy from the succedent and initial sequents with xRy as principal formulas can be left out. Moreover, there is no need to impose that initial sequents $x : A, \Gamma \to \Delta, x : A$ have an arbitrary labelled formula as principal: we can limit ourselves to atomic initial sequents and prove that arbitrary ones are derivable (Lemma 2.1.1). Thus, from now on the only initial sequents we will consider are those of the form $x : P, \Gamma \to \Delta, x : P$, where $x : P$ is a labelled atom. With such sequents as initial we will able to prove a stronger result concerning invertibility: not only the inverse rules are admissible but also their application does not increase the height of the derivation.

Contraction is more complicated to build in. Its admissibility requires the invertibility of the logical rules. Although explained in greater details in the following Lemma 2.1.5, the proof of contraction admissibility consists in showing that every application of contraction can be reduced to an application on smaller formulas, until it acts only on atoms. In order to see how invertibility permits the admissibility of contraction, consider a derivation in which the last step is by an application of contraction and one of the occurrences of the contracted formula is concluded by an invertible logical rule

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ x : A \supset B, \Gamma \to \Delta, x : A\end{array} \quad \begin{array}{c}\vdots\\ x : A \supset B, x : B, \Gamma \to \Delta\end{array}}{x : A \supset B, x : A \supset B, \Gamma \to \Delta}\, L\supset}{x : A \supset B, \Gamma \to \Delta}\, \text{L-C}
$$

Suppose that $L \supset$ is invertible. Therefore, from the derivability of its premises it follows that the sequents $\Gamma \to \Delta, x : A, x : A$ and $x : B, x : B, \Gamma \to \Delta$ are derivable, and contraction can be applied on the smaller formulas $x : A$ and $x : B$. Then, an application of $L \supset$ gives $x : A \supset B, \Gamma \to \Delta$. However, not every rule considered so far is invertible. In particular, $L\Box 1$ and $R\Diamond 1$ are not invertible. Thus, we follow the method adopted in Kleene (1952) for intuitionistic logic where $L \supset$ is not invertible with respect to its left premise, and we repeat the principal formulas in the premise. In this way, from $L\Box 1$ and $R\Diamond 1$ we obtain the rules $L\Box$ and $R\Diamond$ of Negri (2005)

$$
\frac{y : A, x : \Box A, xRy, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\, L\Box \qquad \frac{xRy, \Gamma \to \Delta, x : \Diamond A, y : A}{xRy, \Gamma \to \Delta, x : \Diamond A}\, R\Diamond
$$

Thus, $L\Box$ are and $R\Diamond$ are strictly cumulative and their invertibility follows trivially from admissibility of weakening. Following the terminology of Troelstra and Schwichtenberg (2000), we shall call the system just outlined **G3K**. Although both **G3K** and the system of Troelstra and Schwichtenberg (2000, pp. 284–8) are **G3**-systems for modal logic, **G3K** is labelled.

Logical rules of **G3K**

$$x : P, \Gamma \rightarrow \Delta, x : P \qquad \frac{}{x : \bot, \Gamma \rightarrow \Delta}\ L\bot$$

$$\frac{x : A, x : B, \Gamma \rightarrow \Delta}{x : A \wedge B, \Gamma \rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \rightarrow \Delta, x : B \quad \Gamma \rightarrow \Delta, x : B}{\Gamma \rightarrow \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, \Gamma \rightarrow \Delta \quad x : B, \Gamma \rightarrow \Delta}{x : A \vee B, \Gamma \rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \rightarrow \Delta, x : A, x : B}{\Gamma \rightarrow \Delta, x : A \vee B}\ R\vee$$

$$\frac{\Gamma \rightarrow \Delta, x : A \quad x : B, \Gamma \rightarrow \Delta}{x : A \supset B, \Gamma \rightarrow \Delta}\ L\supset \qquad \frac{x : A, \Gamma \rightarrow \Delta, x : B}{\Gamma \rightarrow \Delta, x : A \supset B}\ R\supset$$

$$\frac{y : A, x : \Box A, xRy, \Gamma \rightarrow \Delta}{x : \Box A, xRy, \Gamma \rightarrow \Delta}\ L\Box \qquad \frac{xRy, \Gamma \rightarrow \Delta, y : A}{\Gamma \rightarrow \Delta, x : \Box A}\ R\Box$$

$$\frac{xRy, y : A, \Gamma \rightarrow \Delta}{x : \Diamond A, \Gamma \rightarrow \Delta}\ L\Diamond \qquad \frac{xRy, \Gamma \rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \rightarrow \Delta, x : \Diamond A}\ R\Diamond$$

G3K has no structural rule as primitive because they are built in the logical rules. Therefore we do not assume any structural rule but we prove their admissibility in **G3K**. Given that relational atoms xRy can occur only in the antecedent, we need not take into account the structural rules with such atom as principal in the succedent, but only the following

$$\frac{\Gamma \rightarrow \Delta}{x : A, \Gamma \rightarrow \Delta}\ \text{L-W} \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, x : A}\ \text{R-W} \qquad \frac{\Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}\ \text{L-W}$$

$$\frac{x : A, x : A, \Gamma \rightarrow \Delta}{x : A, \Gamma \rightarrow \Delta}\ \text{L-C} \qquad \frac{\Gamma \rightarrow \Delta, x : A, x : A}{\Gamma \rightarrow \Delta, x : A}\ \text{R-C} \qquad \frac{xRy, xRy, \Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}\ \text{L-C}$$

In contrast with **G0K**, we assume cut as a primitive rule in **G3K** and we shall give a proof of **cut elimination**, rather than cut admissibility. Consequently, in the proof of the cut-elimination theorem we do not assume that in a given derivation there is at most one istance of cut, but we permit that the derivations of the premises of cut can contain, in turn, other applications of cut. When cut is explicitly present in the system, it can be formulated as the other two-premise rule, that is, with shared contexts.

Cut rule of **G0K**

$$\frac{\Gamma \to \Delta, x : C \quad x : C, \Gamma \to \Delta}{\Gamma \to \Delta}\ \text{CUT}$$

2.1 Admissibility of the structural rules

We said that the most important property of **G3K** is that all the structural rules are admissible in it. Recall that admissibility in a system **G** corresponds to the possibility of finding, for every derivation in **G** with some application of an inference rule R, a new derivation of the same conclusion in which all the applications of R can be dispensed with. In addition, weakening and contraction are height-preserving admissible, that is, whenever their premises are derivable, the conclusion is also derivable with derivation height bounded by the derivation height of the premise.

Measure of derivations

Before going into the details of the structural properties of **G3K**, we need to provide a precise definition of formal derivation in **G3K** and introduce the two main parameters, the **height** and the **rank**, by means of which they are measured.

Definition (**G3K**-derivation)**.** *A* derivation *in* **G3K** *is either an initial sequent, or an instance of* $L\bot$*, or an application of a logical rule to the derivation(s) concluding* *its premise(s). A sequent* $\Gamma \to \Delta$ *is* derivable *in* **G3K** *if there exists a derivation* d *for it. This is indicated by*

$$d \vdash \Gamma \to \Delta$$

In some cases it is useful to reason inductively on the the height of a formula which is the length of the longest branch of its construction tree, minus 1.

Definition (Formula-height)**.** *The* height h *of* A *is defined inductively.*

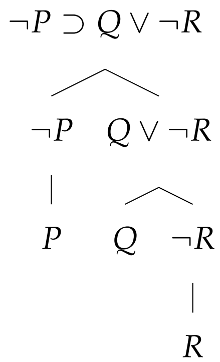
$h(P) = h(\perp) = 0$*;*

$h(\circ A) = h(A) + 1$*, when* $\circ$ *is* $\Box, \Diamond$*;*

$h(A \circ B) = max(h(A), h(B)) + 1$*, when* $\circ$ *is* $\wedge, \vee, \supset$.

The height of a labelled formula $x : A$ *is defined as the height of* A *and relational atoms* xRy *have height 0.*

Example. The height of $\neg P \supset (Q \vee \neg R)$ is 3 and its construction tree is



There is another parameter that measures derivations in **G3K** and we call it the rank of a derivation. Derivation rank measures the height of cut formulas and it is defined as the smallest $n \in \mathbb{N}$ such that every cut formula $x : C$ in d has height $< n$. It follows that a derivation with rank 0 is a derivation without cuts, and conversely. The inductive definitions of derivation height and rank are as follows:

Definition (Derivation-height)**.** *The* height h *of* d *is defined inductively.*

If d *is an initial sequent or a conclusion of* $L\perp$ *then* $h(d) = 0$*;*

If the last rule of d is a one-premise rule R then $h(d) = h(d') + 1$, where d' is the derivation of the premise of R;

If the last rule of d is a two-premise rule R then $h(d) = max(h(d'), h(d'')) + 1$, where d' and d'' are the derivations of the premises of R.

Definition (Derivation-rank)**.** *The* rank *r of d is defined by induction.*

If d is an initial sequent or a conclusion of $L\bot$ then $r(d) = 0$;

If the last rule of d is a one-premise rule R then $r(d) = r(d')$, where d' is the derivation of the premise of R;

If the last rule of d is a two-premise rule R other than cut then $r(d) = max(r(d'), r(d''))$, where d' and d'' are the derivations of the permises of R;

If the last rule of d is cut then $h(d) = max(r(d'), r(d''), (h(x : C) + 1))$, where d' and d'' are the derivations of the premises of cut and $x : C$ is the cut formula.

Notational convention. By writing

$$d \vdash \Gamma \xrightarrow[p]{n} \Delta$$

we shall indicate that d is a derivation of $\Gamma \to \Delta$ and that $h(d) \leqslant n$, $r(d) \leqslant p$. Moreover, $\vdash \Gamma \xrightarrow[p]{n} \Delta$ (or even $\Gamma \xrightarrow[p]{n} \Delta$) means that there is a derivation d such that $d \vdash \Gamma \xrightarrow[p]{n} \Delta$. Thus, the parameter n (resp. p) is considered as an upper bound of the height (resp. of the rank) of d. Note that according to this notational convention we have that for every $n \leqslant n'$ and $p \leqslant p'$, if $d \vdash \Gamma \xrightarrow[p]{n} \Delta$ then $d \vdash \Gamma \xrightarrow[p']{n'} \Delta$. In what follows we frequently make (tacit) use of this fact.

Example. Suppose $\vdash \xrightarrow[1]{2} x : \Box(P \wedge Q)$ and recall that $h(x : \Box(P \wedge Q)) = 2$. Then the following derivation has height 4 and rank 3.

$$
\dfrac{\dfrac{\dfrac{\dfrac{y : P, y : Q, xRy, x : \Box(P \wedge Q) \xrightarrow[0]{0} y : P}{y : P \wedge Q, xRy, x : \Box(P \wedge Q) \xrightarrow[0]{1} y : P} L\wedge}{xRy, x : \Box(P \wedge Q) \xrightarrow[0]{2} y : P} L\Box}{\xrightarrow[1]{2} x : \Box(P \wedge Q) \qquad x : \Box(P \wedge Q) \xrightarrow[0]{3} x : \Box P} R\Box}{\xrightarrow[3]{4} x : \Box P} \text{CUT}
$$

Arbitrary initial sequents

Derivations in **G3K** start with initial sequents with atoms as principal formulas. The reason why it is preferable to have atomic initial sequents is that it guarantees height-preserving invertibility of all the logical rules (cf. Lemma 2.1.5) and this is needed in order to prove that contraction is an admissible rule (cf. Theorem 2.1.6). For instance, by allowing general initial sequents as primitive, height-preserving invertibility of $R\Box$ would fail. However, even if we take as primitive initial sequents with atomic formulas as principal, it is possible to prove that initial sequents with arbitrary formulas are derivable in **G3K**.

Lemma 2.1.1. *In* **G3K** *it holds that*

$$\vdash x : A, \Gamma \xrightarrow[0]{2 \cdot h(A)} \Delta, x : A$$

for every labelled formula $x : A$.

Proof. By induction on h.

If $h = 0$ then A is P and the claim holds, since $x : P, \Gamma \to \Delta, x : P$ is initial. Else, A is $\perp$ and again $x : \perp, \Gamma \to \Delta, x : \perp$ is derivable because it is a conclusion of $L\perp$.

If $h = k + 1$ assume by inductive hypothesis (IH) that the claim holds for $h = k$ and prove that it holds also for $h = k + 1$. We argue by distinction of cases according to $x : A$.

If $x : A$ is $x : B \wedge C$ we find a derivation of $x : B \wedge C, \Gamma \rightarrow \Delta, x : B \wedge C$ as follows.

$$\dfrac{\dfrac{x : B, x : C, \Gamma \xrightarrow[0]{2 \cdot h(B)} \Delta, x : B \quad x : B, x : C, \Gamma \xrightarrow[0]{2 \cdot h(C)} \Delta, x : C}{x : B, x : C, \Gamma \xrightarrow[0]{max(2 \cdot h(B), 2 \cdot h(C))+1} \Delta, x : B \wedge C}\ R\wedge}{x : B \wedge C, \Gamma \xrightarrow[0]{max(2 \cdot h(B), 2 \cdot h(C))+2} \Delta, x : B \wedge C}\ L\wedge$$

where the topmost sequents are derivable by IH, since $h(B), h(C) < h(B \wedge C)$. Moreover, $max(2 \cdot h(B), 2 \cdot h(C)) + 2 = 2 \cdot (max(h(B), h(C)) + 1) = 2 \cdot h(B \wedge C)$.

If $x : A$ is $x : B \vee C$ then sequents as $x : B \vee C, \Gamma \rightarrow \Delta, x : B \vee C$ are derivable by

$$\dfrac{\dfrac{x : B, \Gamma \xrightarrow[0]{2 \cdot h(B)} \Delta, x : B, x : C \quad x : C, \Gamma \xrightarrow[0]{2 \cdot h(C)} \Delta, x : B, x : C}{x : x : B \vee C, \Gamma \xrightarrow[0]{max(2 \cdot h(B), 2 \cdot h(C))+1} \Delta, x : B, x : C}\ L\vee}{x : B \vee C, \Gamma \xrightarrow[0]{max(2 \cdot h(B), 2 \cdot h(C))+2} \Delta, x : B \vee C}\ R\vee$$

where the topmost sequents are derivable by IH, since $h(B), h(C) < h(B \vee C)$. As above, $max(2 \cdot h(B), 2 \cdot h(C)) + 2 = 2 \cdot h(B \vee C)$.

If $x : A$ is $x : B \supset C$ the sequent $x : B \supset C, \Gamma \rightarrow \Delta, x : B \supset C$ has the following derivation.

$$\dfrac{\dfrac{x : B, \Gamma \xrightarrow[0]{2 \cdot h(B)} \Delta, x : B, x : C \quad x : B, x : C, \Gamma \xrightarrow[0]{2 \cdot h(C)} \Delta, x : C}{x : B \supset C, x : B, \Gamma \xrightarrow[0]{max(2 \cdot h(B), 2 \cdot h(C))+1} \Delta, x : C}\ L\supset}{x : B \supset C, \Gamma \xrightarrow[0]{max(2 \cdot h(B), 2 \cdot h(C))+2} \Delta, x : B \supset C}\ R\supset$$

where the topmost sequents are derivable by IH, since $h(B), h(C) < h(B \supset C)$ and $max(2 \cdot h(B), 2 \cdot h(C)) + 2 = 2 \cdot h(B \supset C)$.

If A is $\Box B$ we have a derivation of $x : \Box B, \Gamma \rightarrow \Delta, x : \Box B$ as follows

$$\cfrac{\cfrac{y : B, xRy, x : \Box B, \Gamma \xrightarrow[0]{2\cdot h(B)} \Delta, y : B}{xRy, x : \Box B, \Gamma \xrightarrow[0]{2\cdot h(B)+1} \Delta, y : B}\ L\Box}{x : \Box B, \Gamma \xrightarrow[0]{2\cdot h(B)+2} \Delta, x : \Box B}\ R\Box$$

where the topmost sequent is derivable by IH, since $h(B) < h(\Box B)$. Moreover, $2 \cdot h(B) + 2 = 2 \cdot (h(B) + 1) = 2 \cdot h(\Box B)$.

If A is $\Diamond B$ we have a derivation of $x : \Diamond B, \Gamma \to \Delta, x : \Diamond B$ as follows

$$\cfrac{\cfrac{xRy, y : B, \Gamma \xrightarrow[0]{2\cdot h(B)} \Delta, x : \Diamond B, y : B}{xRy, y : B, \Gamma \xrightarrow[0]{2\cdot h(B)+1} \Delta, x : \Diamond B}\ R\Diamond}{x : \Diamond B, \Gamma \xrightarrow[0]{2\cdot h(B)+2} \Delta, x : \Diamond B}\ L\Diamond$$

where the topmost sequent is derivable by IH, since $h(B) < h(\Diamond B)$ and, as above, $h(B) + 2 = 2 \cdot h(\Diamond B)$.

$\boxtimes$

In the proof of the cut-elimination theorem we also need the following result which states that formulas such as $x : \bot$ can be freely removed when occurring in the succedent.

Lemma 2.1.2. *In* **G3K** *it holds that*

$$\textit{If} \quad \vdash \Gamma \xrightarrow[p]{n} \Delta, x : \bot \quad \textit{then} \quad \vdash \Gamma \xrightarrow[p]{n} \Delta$$

Proof. By induction on n.

If $n = 0$ then $\Gamma \to \Delta, x : \bot$ is initial or conclusion of $L\bot$, then either Γ and Δ have an atom in common, or $u : \bot$ is in Γ. In either case, $\Gamma \to \Delta$ is initial or conclusion of $L\bot$.

If $n = k + 1$ assume by inductive hypothesis (IH) that the claim holds for $n = k$ and prove that it holds also for $k + 1$. Consider the rule R that concluded $\Gamma \to \Delta, x : \bot$. Apply IH on its premise(s) and the conclusion follows by an application R. Note that $x : \bot$ is never principal because no rule introduces $x : \bot$ in the succedent. For instance, when R is $R\Box$,

$$\dfrac{\begin{array}{c}\vdots\\ uRv, \Gamma \xrightarrow{k} \Delta', x : \bot, v : A\end{array}}{\Gamma \xrightarrow{k+1} \Delta', x : \bot, u : \Box A}\ R\Box$$

where v does not appear in the conclusion. By IH on the premise $uRv, \Gamma \xrightarrow{k} \Delta', v : A$ and by $R\Box$ again $\Gamma \xrightarrow{k+1} \Delta', u : \Box A$. The other cases are analogous.

$\boxtimes$

Substitution of labels

Owing to the presence of labels in the language, there is a strong analogy between **G3K** and systems for predicate logic. Labels in **G3K**, as well as free variables in predicate logic, can be replaced and the replacement does not increase the height and the rank of the derivation (cf. the analogous result Lemma 4.1.2 in Negri and von Plato 2001). Moreover, substitution of labels is essential in the proof of admissibility of the necessitation rule of the basic modal logic.

Definition (Substitution)**.** *The* substitution of labels *in relational atoms and labelled formulas is defined by cases:*

$$
\begin{array}{rcll}
(xRy)[z/w] & \equiv & xRy & \textit{if } w \neq x \textit{ and } w \neq y \\
(xRy)[z/x] & \equiv & zRy & \textit{if } x \neq y \\
(xRy)[z/y] & \equiv & xRz & \textit{if } x \neq y \\
(xRx)[z/x] & \equiv & zRz & \\
(x : A)[z/y] & \equiv & x : A & \textit{if } y \neq x \\
(x : A)[z/x] & \equiv & z : A &
\end{array}
$$

and it is extended to multisets thereof componentwise.

Lemma 2.1.3. *The* substitution of labels *is height- and rank-preserving admissible in* **G3K**, i.e.

$$
\textit{If} \quad \vdash \Gamma \xrightarrow[p]{n} \Delta \quad \textit{then} \quad \vdash \Gamma[y/x] \xrightarrow[p]{n} \Delta[y/x]
$$

for every label x and y.

Proof. By induction on n.

If $n = 0$ then $\Gamma \to \Delta$ is initial or conclusion of $L\bot$ and so is $\Gamma[y/x] \to \Delta[y/x]$.

If $n = k + 1$ assume by inductive hypothesis (IH) that the claim holds for $n = k$ and prove that it holds also for $n = k + 1$. We distinguish the following cases, according to the last rule R of d. First, we deal with propositional rules and modal rules without variable condition, that is $L\Box$ and $R\Diamond$, and then with modal rules with *eigenvariable*. When R is not cut the parameter p is omitted in order to simplify the notation. As for the propositional rules we deal only with the $\wedge$-rules, the other cases being analogous, and we go into all the details when R is a modal rule.

If R is a propositional rule then the lemma is proved by applying IH on the premise of R and then R again. Suppose R is $L\wedge$ and its principal formula is labelled by u. Then the derivation ends with

$$\dfrac{\begin{array}{c}\vdots\\ u : B, u : C, \Gamma \xrightarrow{k} \Delta\end{array}}{u : B \wedge C, \Gamma \xrightarrow{k+1} \Delta}\ L\wedge$$

By IH on the premise of $L\wedge$ we obtain

$$(u : B)[y/x], (u : C)[y/x], \Gamma[y/x] \xrightarrow{k} \Delta[y/x]$$

An application of $L\wedge$ gives the desired conclusion. When R is $R\wedge$ and its principal formula is labelled by u, the last step of the derivation is

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \Gamma \xrightarrow{k} \Delta, u : B & \Gamma \xrightarrow{k} \Delta, u : C\end{array}}{\Gamma \xrightarrow{k+1} \Delta, u : B \wedge C}\ R\wedge$$

A new derivation is found by applying IH on both the premises of $R\wedge$

$$\Gamma[y/x] \xrightarrow{k} \Delta[y/x], (u : B)[y/x] \quad \text{and} \quad \Gamma[y/x] \xrightarrow{k} \Delta[y/x], (u : C)[y/x]$$

and by another application of $R\wedge$ we obtain the conclusion.

If R is a modal rule without variable condition then the case is similar to that of propositional rules. Suppose the last step of the derivation is by $L\Box$:

$$\dfrac{\begin{array}{c}\vdots\\ v : B, u : \Box B, uRv, \Gamma' \xrightarrow{k} \Delta\end{array}}{u : \Box B, uRv, \Gamma' \xrightarrow{k+1} \Delta}\ L\Box$$

An application of IH on the premise gives

$$(v : B)[y/x], (u : \Box B)[y/x], (uRv)[y/x], \Gamma'[y/x] \xrightarrow{k} \Delta[y/x]$$

and the claim holds by another application of $L\Box$.

When R is $R\Box$ or $L\Diamond$, that is, a modal rule with variable condition some care is needed in order to avoid clash of labels. If R is $R\Box$ we have several subcases, according to the *eigenvariable* of $R\Box$: it could be either x, or y, or else some z distinct from x and y. In the first case, the principal formula cannot be labelled by x and x does not appear in Γ and Δ because of the variable condition: therefore the substitution is vacuous because there is no occurrence of x. In the second case, if y is the *eigenvariable* and the principal formula is labelled by x we have a derivation the last step of which is

$$\dfrac{\begin{array}{c}\vdots\\ xRy, \Gamma \xrightarrow{k} \Delta, y : B\end{array}}{\Gamma \xrightarrow{k+1} \Delta, x : \Box B}\ {\scriptstyle R\Box}$$

where y does not appear in the conclusion. Replacing directly x with y would make $R\Box$ inapplicable, therefore we need to replace by IH the *eigenvariable* y with a new label z.

$$xRz, \Gamma \xrightarrow{k} \Delta, z : B$$

Note that by the variable condition this substitution does not involve formulas in Γ, Δ. Now by applying IH once again:

$$yRz, \Gamma[y/x] \xrightarrow{k} \Delta[y/x], z : B$$

and finally the rule $R\Box$ in order to conclude

$$\Gamma[y/x] \xrightarrow{k+1} \Delta[y/x], y : \Box B$$

The case in which the principal formula is labelled by a variable other that x is analogous. In the third case neither x nor y is the *eigenvariable*. Suppose the principal formula is labelled by u, so the derivation is

$$\dfrac{\begin{array}{c}\vdots\\ uRv, \Gamma \xrightarrow{k} \Delta, v : B\end{array}}{\Gamma \xrightarrow{k+1} \Delta, u : \Box B}\ R\Box$$

By IH we have

$$(uRv)[y/x], \Gamma[y/x] \xrightarrow{k} \Delta[y/x], (v : B)[y/x]$$

and, consequently, by $R\Box$, we obtain the desired conclusion. The case of $L\Diamond$ is similar to $R\Box$.

If R is cut with $u : C$ as principal formula and $h(u : C) < p$ then

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \xrightarrow[p]{k} \Delta, u : C\end{array} \quad \begin{array}{c}\vdots\\ u : C, \Gamma \xrightarrow[p]{k} \Delta\end{array}}{\Gamma \xrightarrow[p]{k+1} \Delta}\ \text{CUT}$$

By applying IH on its premises of cut

$$\Gamma[y/x] \xrightarrow[p]{k} \Delta[y/x], (u : C)[y/x] \quad \text{and} \quad \Gamma[y/x] \xrightarrow[p]{k} \Delta[y/x], (u : C)[y/x]$$

and the claim holds by another application of cut.

$$\Gamma[y/x] \xrightarrow[p]{k+1} \Delta[y/x]$$

$\boxtimes$

Admissibility of weakening

The calculus **G3K** is closed under weakening, that is if a sequent $\Gamma \to \Delta$ is derivable then $x : A, \Gamma \to \Delta$, $\Gamma \to \Delta, x : A$, and $xRy, \Gamma \to \Delta$ are derivable. In addition, the height and the rank of the derivation are preserved.

Theorem 2.1.4. Weakening *is height- and rank-preserving admissible in* **G3K**, i.e.

$$\begin{array}{lllll} i) & \textit{If} & \vdash \Gamma \overset{n}{\underset{p}{\to}} \Delta & \textit{then} & \vdash x : A, \Gamma \overset{n}{\underset{p}{\to}} \Delta \\ ii) & \textit{If} & \vdash \Gamma \overset{n}{\underset{p}{\to}} \Delta & \textit{then} & \vdash \Gamma \overset{n}{\underset{p}{\to}} \Delta, x : A \\ iii) & \textit{If} & \vdash \Gamma \overset{n}{\underset{p}{\to}} \Delta & \textit{then} & \vdash xRy, \Gamma \overset{n}{\underset{p}{\to}} \Delta \end{array}$$

Proof. By induction on n.

If $n = 0$ then $\Gamma \to \Delta$ is initial or conclusion of $L\bot$ and so are $x : A, \Gamma \to \Delta$ and $\Gamma \to \Delta, x : A$ and $xRy, \Gamma \to \Delta$.

If $n = k + 1$ assume by inductive hypothesis (IH) that the claim holds for $n = k$ and prove that it holds also for $k + 1$. We distinguish the following cases, according to the last rule R applied.

If R is propositional rule or a modal rule without variable condition, apply IH on the premise(s) of R and then R again. For instance, if R is $L\wedge$ then Γ is $u : B \wedge C, \Gamma'$ and the last step of the derivation is

$$\dfrac{\begin{array}{c}\vdots \\ u : B, u : C, \Gamma' \xrightarrow{k} \Delta\end{array}}{u : B \wedge C, \Gamma' \xrightarrow{k+1} \Delta}\ L\wedge$$

By the IH we have

$$\begin{array}{ll} \text{i)} & x : A, u : B, u : C, \Gamma' \xrightarrow{k} \Delta \\ \text{ii)} & u : B, u : C, \Gamma' \xrightarrow{k} \Delta, x : A \\ \text{iii)} & xRy, u : B, u : C, \Gamma' \xrightarrow{k} \Delta \end{array}$$

from which by $L\wedge$ conclude

$$\begin{array}{ll} \text{i)} & x : A, u : B \wedge C, \Gamma' \xrightarrow{k+1} \Delta \\ \text{ii)} & u : B \wedge C, \Gamma' \xrightarrow{k+1} \Delta, x : A \\ \text{iii)} & xRy, u : B \wedge C, \Gamma' \xrightarrow{k+1} \Delta \end{array}$$

The proof is analogous when R is one of the other one-premise propositional rule, *i.e.* when it is $R\vee$ and $R\supset$.

If R is $R\wedge$ then Δ is $\Delta', u : B \wedge C$ and the derivation ends with

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \Gamma \xrightarrow{k} \Delta', u : B & \Gamma \xrightarrow{k} \Delta', u : C \end{array}}{\Gamma \xrightarrow{k+1} \Delta', u : B \wedge C} \; R\wedge$$

By IH we have

$$\begin{array}{llll} \text{i)} & x : A, \Gamma \xrightarrow{k} \Delta', u : B & \text{and} & x : A, \Gamma \xrightarrow{k} \Delta', u : C \\ \text{ii)} & \Gamma \xrightarrow{k} \Delta', u : B, x : A & \text{and} & \Gamma \xrightarrow{k} \Delta', u : C, x : A \\ \text{iii)} & xRy, \Gamma \xrightarrow{k} \Delta', u : B & \text{and} & xRy, \Gamma \xrightarrow{k} \Delta', u : C \end{array}$$

and by $R\wedge$ we conclude

$$\begin{array}{ll} \text{i)} & x : A, \Gamma \xrightarrow{k+1} \Delta', u : B \wedge C \\ \text{ii)} & \Gamma \xrightarrow{k+1} \Delta', u : B \wedge C, x : A \\ \text{iii)} & xRy, \Gamma \xrightarrow{k+1} \Delta', u : B \wedge C \end{array}$$

Analogously, when R is a two-premise rule, *i.e.* $L\vee$, $L\supset$ and cut.

If R is a modal rule without the variable condition, say $L\Box$, then Γ is $u : \Box B, uRv, \Gamma'$ and last step of the derivation is

$$\dfrac{\begin{array}{c} \vdots \\ v : B, u : \Box B, uRv, \Gamma' \xrightarrow{k} \Delta \end{array}}{u : \Box B, uRv, \Gamma' \xrightarrow{k+1} \Delta} \; L\Box$$

By IH we have

$$
\begin{array}{ll}
\text{i)} & x : A, v : B, u : \Box B, uRv, \Gamma' \xrightarrow{k} \Delta \\
\text{ii)} & v : B, u : \Box B, uRv, \Gamma' \xrightarrow{k} \Delta, x : A \\
\text{iii)} & xRy, v : B, u : \Box B, uRv, \Gamma' \xrightarrow{k} \Delta
\end{array}
$$

and another application of $L\Box$ yields

$$
\begin{array}{ll}
\text{i)} & x : A, u : \Box B, uRv, \Gamma' \xrightarrow{k+1} \Delta \\
\text{ii)} & u : \Box B, uRv, \Gamma' \xrightarrow{k+1} \Delta, x : A \\
\text{iii)} & xRy, u : \Box B, uRv, \Gamma' \xrightarrow{k+1} \Delta
\end{array}
$$

When R is a modal rule with variable condition then we need to replace the *eigenvariable* with a label z not occurring in Γ, Δ' and distinct from x and u. Suppose that the last step of the derivation is by $R\Box$ and its eigenvariable is x. Then Δ is $\Delta', u : \Box B$, and the derivation ends with

$$
\dfrac{\begin{array}{c}\vdots \\ uRx, \Gamma \xrightarrow{k} \Delta', x : B\end{array}}{\Gamma \xrightarrow{k+1} \Delta', u : \Box B}\, R\Box
$$

We apply Lemma 2.1.3 on the premise in order to replace x with a new variable z.

$$
uRz, \Gamma \xrightarrow{k} \Delta', z : B
$$

and then IH

$$
\begin{array}{ll}
\text{i)} & x : A, uRz, \Gamma \xrightarrow{k} \Delta', z : B \\
\text{ii)} & uRz, \Gamma \xrightarrow{k} \Delta', z : B, x : A \\
\text{iii)} & xRy, uRz, \Gamma \xrightarrow{k} \Delta', z : B
\end{array}
$$

Given that z does not appears anywhere else but in the principal formulas of $R\Box$ we conclude by $R\Box$

i) $x : A, \Gamma \xrightarrow{k+1} \Delta', u : \Box B$

ii) $\Gamma \xrightarrow{k+1} \Delta', u : \Box B, x : A$

iii) $xRy, \Gamma \xrightarrow{k+1} \Delta', u : \Box B$

$\boxtimes$

Inversion Lemma

All the rules of **G3K** are invertible with the preservation of height and rank. Invertibility is needed in order to prove admissibility of contraction and it is fundamental for a systematic proof-search procedure.

Lemma 2.1.5. *All the rules of* **G3K** *are height- and rank-preserving invertible,* i.e.

i)	*If*	$\vdash x : A \wedge B, \Gamma \underset{p}{\overset{n}{\to}} \Delta$	*then*	$\vdash x : A, x : B, \Gamma \underset{p}{\overset{n}{\to}} \Delta$
ii)	*If*	$\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : A \wedge B$	*then*	$\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : A$ *and d* $\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : B$
iii)	*If*	$\vdash x : A \vee B, \Gamma \underset{p}{\overset{n}{\to}} \Delta$	*then*	$\vdash x : A, \Gamma \underset{p}{\overset{n}{\to}} \Delta$ *and d* $\vdash x : B, \Gamma \underset{p}{\overset{n}{\to}} \Delta$
iv)	*If*	$\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : A \vee B$	*then*	$\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : A, x : B$
v)	*If*	$\vdash x : B \supset C, \Gamma \underset{p}{\overset{n}{\to}} \Delta$	*then*	$\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : B$ *and* $\vdash x : C, \Gamma \underset{p}{\overset{n}{\to}} \Delta$
vi)	*If*	$\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : B \supset C$	*then*	$\vdash x : B, \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : C$
vii)	*If*	$\vdash x : \Box A, xRy, \Gamma \underset{p}{\overset{n}{\to}} \Delta$	*then*	$\vdash y : A, x : \Box A, xRy, \Gamma \underset{p}{\overset{n}{\to}} \Delta$
viii)	*If*	$\vdash \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : \Box A$	*then*	$\vdash xRy, \Gamma \underset{p}{\overset{n}{\to}} \Delta, y : A$, *for every y*
ix)	*If*	$\vdash xRy, \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : \Diamond A$	*then*	$\vdash xRy, \Gamma \underset{p}{\overset{n}{\to}} \Delta, x : \Diamond A, y : A$
x)	*If*	$\vdash x : \Diamond A, \Gamma \underset{p}{\overset{n}{\to}} \Delta$	*then*	$\vdash xRy, y : A, \Gamma \underset{p}{\overset{n}{\to}} \Delta$, *for every y*

Proof. For the propositional rules we consider in detail only case v, all the other being analogous. The proof is by induction on n. As usual, we leave out the parameter p when the cut rule is not explicitly applied.

If $n = 0$ then $x : A \supset B, \Gamma \to \Delta$ is initial or conclusion of $L\bot$; then both $\Gamma \to \Delta, x : A$ and $x : B, \Gamma \to \Delta$ are initial or conclusion of $L\bot$. Note that the claim holds for the base case only if initial sequents are assumed to have atoms as principal formulas. If $n = k + 1$ assume by inductive hypothesis (IH) that the claim holds for $n = k$ and prove that it holds also for $k + 1$. If $x : A \supset B, \Gamma \xrightarrow{k+1} \Delta$ has been derived by $L \supset$ with $x : A \supset B$ as principal formula then we have a derivation of $\Gamma \xrightarrow{k} \Delta, x : A$ and $x : B, \Gamma \xrightarrow{k} \Delta$, so the claim holds also for $n = k + 1$. If $x : A \supset B, \Gamma \xrightarrow{k+1} \Delta$ is conclusion of a rule different from $L \supset$ or has been concluded by $L \supset$ with principal formula distinct from the displayed occurrences of $x : A \supset B$, then we apply IH to the premise(s) $x : A \supset B, \Gamma' \xrightarrow{k} \Delta'$ (and $x : A \supset B, \Gamma'' \xrightarrow{k} \Delta''$) in order to obtain $\Gamma' \xrightarrow{k} \Delta', x : A$ and $x : B, \Gamma' \xrightarrow{k} \Delta'$ (and $\Gamma'' \xrightarrow{k} \Delta'', x : A$ and $x : B, \Gamma'' \xrightarrow{k} \Delta''$); then by an application of R we can conclude $\Gamma \xrightarrow{k+1} \Delta, x : A$ (and $x : B, \Gamma \xrightarrow{k+1} \Delta$).

For the modal cases ($vii - x$), we distinguish rules in which the principal formulas are repeated in the premise from those without repetition. $L\Box$ (resp.$R\Diamond$) of case vii (resp. ix) is invertible because the conclusion can be derived from its premise by an application of weakening which is admissible by Lemma 2.1.4. On the contrary, modal rules with variable condition as $R\Box$ (resp. $L\Diamond$) corresponding to $viii$ (resp. x) is proved to be height- and rank-preserving invertible by induction on n. Consider case $viii$. If $n = 0$ then $\Gamma \to \Delta, x : \Box A$ is initial or conclusion of $L\bot$; then so is $xRy, \Gamma \to \Delta, y : A$. If $n = k + 1$, assume by IH that the claim holds for $n = k$ and prove that it holds also for $k + 1$. If $\Gamma \xrightarrow{k+1} \Delta, x : \Box A$ is concluded by $R\Box$ with $x : \Box A$ as principal formula then there is a label z not occurring in Γ, Δ and different from x such that $xRz, \Gamma \xrightarrow{k} \Delta, z : A$. By an application of the Lemma 2.1.3 we obtain that for every y, it holds that $xRy, \Gamma \xrightarrow{k} \Delta, y : A$ and the claim holds by IH. On the other hand, if it has been derived by a rule R or by $R\Box$ with principal formula different from $x : \Box A$, then we can apply IH, but some care is needed when R is in turn a modal rule with variable condition. For instance, suppose that $\Gamma \xrightarrow{k+1} \Delta, x : \Box A$ is the conclusion of $L\Diamond$ with principal formula $u : \Diamond B$ then we have the following derivation

$$\dfrac{\vdots \atop uRv, v : B, \Gamma \xrightarrow{k} \Delta, x : \Box A}{u : \Diamond B, \Gamma' \xrightarrow{k+1} \Delta, x : \Box A} L\Diamond$$

We want to prove $xRy, x : \Diamond B, \Gamma' \xrightarrow{k+1} \Delta, y : A$ for every y. If v, the *eigenvariable* of $L\Diamond$, is different from y then we apply IH on the premise and obtain the sequent $xRy, uRv, v : B, \Gamma \xrightarrow{k} \Delta, y : A$ and the conclusion is obtained by $L\Diamond$ again. Otherwise, if v is y we need Lemma 2.1.3 on the premise of $L\Diamond$ in order to replace the *eigenvariable* y with a new z

$$uRz, z : B, \Gamma' \xrightarrow{k} \Delta, x : \Box A$$

Now, by IH we obtain

$$xRy, uRz, z : B, \Gamma' \xrightarrow{k} \Delta, y : A$$

and then by $L\Diamond$ again

$$xRy, u : \Diamond B, \Gamma' \xrightarrow{k+1} \Delta, y : A$$

$\boxtimes$

Admissibility of contraction

In this section we shall prove that contraction is admissible with the preservation of the height and the rank of derivations. This result, as we have already said, is fundamental for the proof-search in **G3K**.

Theorem 2.1.6. Contraction *is height- and rank-preserving admissible in* **G3K**, i.e.

i) *If* $\vdash x : A, x : A, \Gamma \xrightarrow[p]{n} \Delta$ *then* $\vdash x : A, \Gamma \xrightarrow[p]{n} \Delta$

ii) *If* $\vdash \Gamma \xrightarrow[p]{n} \Delta, x : A, x : A$ *then* $\vdash \Gamma \xrightarrow[p]{n} \Delta, x : A$

iii) *If* $\vdash xRy, xRy, \Gamma \xrightarrow[p]{n} \Delta$ *then* $\vdash xRy, \Gamma \xrightarrow[p]{n} \Delta$

Proof. By simultaneous induction on n.

If $n = 0$ then $x : A, x : A, \Gamma \to \Delta$ (resp. $\Gamma \to \Delta, x : A, x : A$) is initial or conclusion of $L\bot$. In both cases also $x : A, \Gamma \to \Delta$ (resp. $\Gamma \to \Delta, x : A$) is initial or conclusion of $L\bot$.

If $n = k + 1$ assume by inductive hypothesis (IH) that the claim holds for $n = k$ and prove that it holds also for $k + 1$. We distinguish two cases: if none of the contraction formulas is principal in the last rule, then both occurrences are in the premise(s) and we apply IH to the premise(s) and then the rule. If one of the contraction formulas is principal, we first apply Lemma 2.1.5 to the premise(s), IH and then the rule. The latter case has three subcases: if R is a propositional rule, say $L \supset$, then $x : A$ is $x : B \supset C$ and the derivation ends with

$$\dfrac{\begin{array}{c}\vdots\\ x : B \supset C, \Gamma \xrightarrow{k} \Delta, x : B\end{array} \quad \begin{array}{c}\vdots\\ x : B \supset C, x : C, \Gamma \xrightarrow{k} \Delta\end{array}}{x : B \supset C, x : B \supset C, \Gamma \xrightarrow{k+1} \Delta}$$

By applying Lemma 2.1.5, item v, we obtain

$$\Gamma \xrightarrow{k} \Delta, x : B, x : B \quad \text{and} \quad x : C, x : C, \Gamma \xrightarrow{k} \Delta$$

By IH for left and right contraction simultaneously we conclude

$$\Gamma \xrightarrow{k} \Delta, x : B \quad \text{and} \quad x : C, \Gamma \xrightarrow{k} \Delta$$

and by $L\supset$ we have a derivation of the desired conclusion

$$x : B \supset C, \Gamma \xrightarrow{k+1} \Delta$$

In the case of right contraction we start from

$$\dfrac{\vdots \atop x : B, \Gamma \xrightarrow{k} \Delta, x : C, x : B \supset C}{\Gamma \xrightarrow{k+1} \Delta, x : B \supset C, x : B \supset C}\ R\supset$$

By Lemma 2.1.5, item *vi*, we have

$$x : B, x : B, \Gamma \xrightarrow{k} \Delta, x : C, x : C$$

from which by IH simultaneously for left and right contraction and $L\supset$ we conclude

$$\Gamma \xrightarrow{k+1} \Delta, x : B \supset C$$

If R is $L\Box$ or $R\Diamond$ the proof is straightforward because the principal formula is repeated in the premises and IH can be applied directly without any appeal to invertibility. For instance, suppose we have a derivation of a sequent with two occurrences of $x : \Box A$ and one of them is the principal formula of an application of $L\Box$, *i.e.*

$$\dfrac{\vdots \atop y : B, x : \Box B, x : \Box B, xRy, \Gamma' \xrightarrow{k} \Delta}{x : \Box B, x : \Box B, xRy, \Gamma' \xrightarrow{k+1} \Delta}\ L\Box$$

Given that the principal formula $x : \Box B$ appears also in the premise we can apply directly IH

$$y : B, x : \Box B, xRy, \Gamma' \xrightarrow{k} \Delta$$

from which by $L\Box$ again

$$x : \Box B, xRy, \Gamma' \xrightarrow{k+1} \Delta$$

If R is a modal rule with variable condition, *i.e.* $R\Box$ or $L\Diamond$, the last step of the derivation is

$$\dfrac{\vdots \atop xRy, y : B, x : \Diamond B, \Gamma \xrightarrow{k} \Delta}{x : \Diamond B, x : \Diamond B, \Gamma \xrightarrow{k+1} \Delta}\ L\Diamond$$

By Lemma 2.1.5, item x, on the premise of $L\Diamond$ we obtain

$$xRy, y : B, xRy, y : B, \Gamma \xrightarrow{k} \Delta$$

in order to make IH applicable

$$xRy, y : B, \Gamma \xrightarrow{k} \Delta$$

Then, by $L\Diamond$ again we get

$$x : \Diamond B, \Gamma \xrightarrow{k+1} \Delta$$

$\boxtimes$

2.2 Cut elimination

In this section we shall indicate the system **G3K** with cut as $\mathbf{G3K}^{\mathrm{C}}$ and we shall prove that in $\mathbf{G3K}^{\mathrm{C}}$ the cut-elimination theorem holds. Furthermore, our aim is to take into account the problem of the rate of growth of derivations during cut elimination. In particular, this proof shows that there is a **hyperexponential upper bound** on growth of derivations under the procedure of cut elimination, that is, when a derivation is converted into a cut-free one the latter is at most hyperexponentially heigher than the former. This bound is calculated following the proof of cut elimination for first order logic in Schwichtenberg (1977). However, we argue that this bound is not sharp and that a better result might be achieved by a modification of the rule $L\Box$.

Definition (Hyp2). *Let* $2_k : \mathbb{N}^2 \longrightarrow \mathbb{N}$ *be a function defined recursively as*

$$2_0(m) = m \qquad \textit{and} \qquad 2_{k+1}(m) = 2^{2_k(m)}$$

2_k *is called* hyperexponential function (with base 2).

Observation. The argument k of 2_k refers to the height of the "exponentiation tower". In fact, $2_k(m)$ is $\underbrace{2^{2^{\cdot^{\cdot^{2^m}}}}}_{k \text{ times}}$. Moreover, it is easy to see that 2_k increases fast:

$2_0(0) = 0; \quad 2_1(0) = 1; \quad 2_2(0) = 2; \quad 2_3(0) = 4; \quad 2_4(0) = 16; \quad 2_5(0) = 65.536; \quad \ldots$

A property of the hyperexponential functions we shall use in the following is:

Proposition 2.2.1. *For any* $k, m \in \mathbb{N}$ *it holds that*

$$2_{k+1}(m) = 2_k(2^m)$$

Main Lemma

The following lemma shows that cuts can be permuted upward in a derivation until they reach an initial sequent or a conclusion of $L\bot$. The proof presents the derivation transformations that are at the core of the original Gentzen's *Hauptsatz* and in the proof of cut admissibility of Negri (2005), with two basic differences. First, the cut here considered is context sharing, that is, the context in premises of cut is the same. Secondly, in the proof of cut admissibility of Negri (2005) and in Negri and von Plato (2001, 2011) the topmost cut of a given derivation is considered, and it is shown that this choice is not restrictive. In the following proof instead we deal with the case in which derivations of the premises of cut can in turn contain other applications of cut.

Lemma 2.2.2 (Main Lemma). *Let d_1 and d_2 be two derivations in* $\mathbf{G3K}^{\mathrm{C}}$ *such that*

$$d_1 \vdash \Gamma \xrightarrow[p]{n} \Delta, x : C \quad \textit{and} \quad d_2 \vdash x : C, \Gamma \xrightarrow[p]{m} \Delta$$

and let $h(x : C) = p$. Then there is a derivation d $\mathbf{G3K}^{\mathrm{C}}$ *such that*

$$d \vdash \Gamma \xrightarrow[p]{n+m} \Delta$$

Observation. Obviously, the conclusion $\Gamma \rightarrow \Delta$ could be easily derived by cut. However, the derivation d we obtain in this way would be of height $max(n, m) + 1$ and rank $p + 1$, because $h(x : C) = p$ by hypothesis. The Lemma says that the rank can be reduced from $p + 1$ to p, provided that the height increases sufficiently. In fact, $max(n, m) + 1 \leqslant n + m$ for $n, m \neq 0$.

Proof. By induction on $n + m$. The proof follows the pattern:

1. Either d_1 or d_2 is initial or conclusion of $L\bot$:

(a) d_1 is initial or conclusion of $L\perp$;

(b) d_2 is initial or conclusion of $L\perp$.

2. Neither d_1 nor d_2 is initial or conclusion of $L\perp$ and:

   (a) $x : C$ is not principal in d_1;

   (b) $x : C$ is not principal in d_2;

   (c) $x : C$ is principal both in d_1 and d_2.

Case 1a

Suppose that d_1 is an initial sequent or a conclusion of $L\perp$. There are three sub-cases:

If $x : C$ is principal then C is an atom P and Γ is $x : P, \Gamma'$. In this case take d_2

$$x : P, x : P, \Gamma' \xrightarrow[p]{m} \Delta$$

The two occurrences of $x : P$ can be contracted by Theorem 2.1.6, thus

$$x : P, \Gamma' \xrightarrow[p]{m} \Delta$$

and so, also $x : P, \Gamma' \xrightarrow[p]{n+m} \Delta$.

If $x : C$ is not principal then Γ and Δ have an atom in common, say $x : P$. Therefore, the conclusion $x : P, \Gamma' \to \Delta', x : P$ has height 0 and so $x : P, \Gamma' \xrightarrow{n+m} \Delta', x : P$.

If d_1 is conclusion of $L\perp$ then $x : \perp$ is in Γ and also the conclusion $x : \perp, \Gamma' \to \Delta$ is derivable.

Case 1b

Suppose that d_2 is an initial sequent or a conclusion of $L\perp$ and consider the following subcases.

If d_2 is an initial sequent and $x : C$ is principal then C is atomic and the atom $x : P$ is in Δ. In this case, from d_1

$$\Gamma \underset{p}{\overset{n}{\to}} \Delta' x : P, x : P$$

we obtain

$$\Gamma \underset{p}{\overset{n}{\to}} \Delta', x : P$$

by admissibility of contraction (Theorem 2.1.6).

If $x : C$ is not principal then Γ and Δ have an atom in common and also the conclusion $x : P, \Gamma' \overset{0}{\to} \Delta' x : P$ is derivable because initial.

If d_2 is conclusion of $L\perp$ then either $x : \perp$ is in Γ or it is $x : C$. In the first case, the conclusion $x : \perp, \Gamma' \overset{0}{\to} \Delta$ is derivable because it is concluded by $L\perp$. In the second case take d_1 which is

$$\Gamma \underset{p}{\overset{n}{\to}} \Delta, x : \perp$$

and apply Lemma 2.1.2 in order to conclude $\Gamma \underset{p}{\overset{n}{\to}} \Delta$, and so also $\Gamma \underset{p}{\xrightarrow{n+m}} \Delta$.

Case 2a

If d_1 is neither an initial sequent nor a conclusion of $L\perp$, consider first of all the case in which d_1 is concluded by a rule R_1 with $x : C$ not principal. There are as many cases as istances of R_1.

If R_1 is a one-premise propositional rule, say $L\wedge$, then Γ is $u : A \wedge B, \Gamma'$ and d_1 is

$$\dfrac{\begin{array}{c}\vdots\\ u : A, u : B, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C\end{array}}{u : A \wedge B, \Gamma' \xrightarrow[p]{n} \Delta, x : C}\ L\wedge$$

First, take d_2

$$x : C, u : A \wedge B, \Gamma' \xrightarrow[p]{m} \Delta$$

and apply inversion of $L\wedge$ (Lemma 2.1.5, item i) in order to obtain

$$x : C, u : A, u : B, \Gamma' \xrightarrow[p]{m} \Delta$$

and then find a derivation d of $u : A \wedge B, \Gamma' \xrightarrow[p]{n+m} \Delta$ as follows

$$\dfrac{\dfrac{u : A, u : B, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C \quad x : C, u : A, u : B, \Gamma' \xrightarrow[p]{m} \Delta}{u : A, u : B, \Gamma' \xrightarrow[p]{(n-1)+m} \Delta}\ \text{IH}}{u : A \wedge B, \Gamma' \xrightarrow[p]{n+m} \Delta}\ L\wedge$$

The cases of the other propositional one-premise rules, *i.e.* $R\vee$ and $R\supset$, are analogous.

Let R_1 be a two-premise rule as $L\supset$. Then Γ is $u : A \supset B, \Gamma'$ and d_1 is

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma' \xrightarrow[p]{n-1} \Delta, u : A, x : C\end{array} \quad \begin{array}{c}\vdots\\ u : B, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C\end{array}}{u : A \supset B, \Gamma' \xrightarrow[p]{n} \Delta, x : C}\ L\supset$$

Also in this case, take d_2

$$x : C, u : A \supset B, \Gamma' \xrightarrow[p]{m} \Delta$$

and apply Lemma 2.1.5, item v, giving

$$x : C, \Gamma' \xrightarrow[p]{m} \Delta, u : A \quad \text{and} \quad x : C, u : B, \Gamma' \xrightarrow[p]{m} \Delta$$

Then find a derivation d of $u : A \supset B, \Gamma' \xrightarrow[p]{n+m} \Delta$ as follows

$$\dfrac{\dfrac{\Gamma' \xrightarrow[p]{n-1} \Delta, u : A, x : C \quad x : C, \Gamma' \xrightarrow[p]{m} \Delta, u : A}{\Gamma' \xrightarrow[p]{(n-1)+m} \Delta, u : A}\,\text{IH} \qquad \dfrac{u : B, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C \quad x : C, u : B, \Gamma' \xrightarrow[p]{m} \Delta}{u : B, \Gamma' \xrightarrow[p]{(n-1)+m} \Delta}\,\text{IH}}{u : A \supset B, \Gamma' \xrightarrow[p]{n+m} \Delta}\,L\supset$$

Analogously for other two-premise rules as $R\wedge$ and $L\vee$.

When R_1 is a modal rule the proof follows the pattern of other one-premise rules. For instance, suppose R_1 is $L\Box$ then Γ is $uRv, u : \Box B, \Gamma'$ and d_1 is

$$\dfrac{\begin{matrix}\vdots \\ v : B, uRv, u : \Box B, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C\end{matrix}}{uRv, u : \Box B, \Gamma' \xrightarrow[p]{n} \Delta, x : C}\,L\Box$$

Find a derivation d of $uRv, u : \Box B, \Gamma' \xrightarrow[p]{n+m}$ as follows

$$\dfrac{\dfrac{v : B, uRv, u : \Box B, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C \quad \dfrac{x : C, uRv, u : \Box B, \Gamma' \xrightarrow[p]{m} \Delta}{x : C, v : B, uRv, u : \Box B, \Gamma' \xrightarrow[p]{m} \Delta}\,\text{L-W}}{v : B, uRv, u : \Box B, \Gamma' \xrightarrow[p]{(n-1)+m} \Delta}\,\text{IH}}{uRv, u : \Box B, \Gamma' \xrightarrow[p]{n+m} \Delta}\,L\Box$$

If R_1 is $R\Box$ then Δ is $\Delta', u : \Box B$ and d_1 is

$$\dfrac{uRv, \Gamma \xrightarrow[p]{n-1} \Delta', v : B, x : C}{\Gamma \xrightarrow[p]{n} \Delta', u : \Box B, x : C}\; R\Box$$

with the condition that v is not in the conclusion of $R\Box$. Then we find a derivation d of $\Gamma \xrightarrow[p]{n+m} \Delta', u : \Box B$ as follows. By INV we refer to an application of Lemma 2.1.5.

$$\dfrac{\dfrac{uRv, \Gamma \xrightarrow[p]{n-1} \Delta', v : B, x : C \quad \dfrac{x : C, \Gamma \xrightarrow[p]{m} \Delta', u : \Box B}{x : C, uRv, \Gamma \xrightarrow[p]{m} \Delta', v : B}\;\text{INV}}{uRv, \Gamma \xrightarrow[p]{(n-1)+m} \Delta', v : B}\;\text{IH}}{\Gamma \xrightarrow[p]{n+m} \Delta', u : \Box B}\; R\Box$$

The last case is when R_1 is cut on a formula $u : B$ different from the displayed $x : C$ and $h(x : B) < p$. Then d_1 is

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \xrightarrow[p]{n-1} \Delta, x : C, u : B\end{array} \quad \begin{array}{c}\vdots \\ u : B, \Gamma \xrightarrow[p]{n-1} \Delta, x : C\end{array}}{\Gamma \xrightarrow[p]{n} \Delta, x : C}\;\text{CUT}$$

We find a derivation d of $\Gamma \xrightarrow[p]{n+m}$ as follows. Consider the following two derivations. The first one takes d_2 and applies admissibility of weakening (Lemma 2.1.4) in order to make IH applicable on the left premise of cut.

$$\dfrac{\Gamma \xrightarrow[p]{n-1} \Delta, x : C, u : B \quad \dfrac{x : C, \Gamma \xrightarrow[p]{m} \Delta}{x : C, \Gamma \xrightarrow[p]{m} \Delta, u : B}\;\text{R-W}}{\Gamma \xrightarrow[p]{(n-1)+m} \Delta, u : B}\;\text{IH}$$

The second derivation is similar. It takes d_2 and applies admissibility of weakening so that IH can be applied on the right premise of cut.

$$\dfrac{u : B, \Gamma \xrightarrow[p]{n-1} \Delta, x : C \quad \dfrac{x : C, \Gamma \xrightarrow[p]{m} \Delta}{u : B, x : C, \Gamma \xrightarrow[p]{m} \Delta}\text{ L-W}}{u : B, \Gamma \xrightarrow[p]{(n-1)+m} \Delta}\text{ IH}$$

Finally, take the conclusion of the two derivations and apply cut in order to get the conclusion $\Gamma \xrightarrow[p]{n+m} \Delta$.

$$\dfrac{\Gamma \xrightarrow[p]{(n-1)+m} \Delta, u : B \quad u : B, \Gamma \xrightarrow[p]{(n-1)+m} \Delta}{\Gamma \xrightarrow[p]{n+m} \Delta}\text{ CUT}$$

Case 2b

Similar to 2a.

Case 2c

When the cut formula $x : C$ is principal in both d_1 and d_2 we consider what is $x : C$.

If $x : C$ is $x : A \wedge B$ then d_1 and d_2 are

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \xrightarrow[p]{n-1} \Delta, x : A\end{array} \quad \begin{array}{c}\vdots \\ \Gamma \xrightarrow[p]{n-1} \Delta, x : B\end{array}}{\Gamma \xrightarrow[p]{n} \Delta, x : A \wedge B}\, R\wedge \qquad \dfrac{\begin{array}{c}\vdots \\ x : A, x : B, \Gamma \xrightarrow[p]{m-1} \Delta\end{array}}{x : A \wedge B, \Gamma \xrightarrow[p]{m} \Delta}\, L\wedge$$

Find d as follows

$$\dfrac{\Gamma \xrightarrow[p]{m-1} \Delta, x:B \qquad \dfrac{\dfrac{\Gamma \xrightarrow[p]{n-1} \Delta, x:A}{x:B,\Gamma \xrightarrow[p]{n-1} \Delta, x:A}\text{L-W} \qquad x:A, x:B, \Gamma \xrightarrow[p]{m-1} \Delta}{x:B,\Gamma \xrightarrow[p]{max(n,m)} \Delta}\text{CUT}}{\Gamma \xrightarrow[p]{max(m,max(n,m)+1)} \Delta}\text{CUT}$$

Note that $max(m, max(n,m)+1) = max(n,m)+1$ and then the height of the conclusion is $max(n,m)+1$. As noted above, when m and n are are greater than 0 then $max(n,m)+1 \leqslant n+m$. Therefore, we have also $\Gamma \xrightarrow{n+m} \Delta$. Moreover, the rank is p because $h(x:A), h(x:B) < p = h(x:A \wedge B)$.

If $x:C$ is $x:A \vee B$ then d_1 and d_2 are

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \xrightarrow[p]{n-1} \Delta, x:A, x:B\end{matrix}}{\Gamma \xrightarrow[p]{n} \Delta, x:A \vee B}R\vee \qquad \dfrac{\begin{matrix}\vdots\\ x:A,\Gamma \xrightarrow[p]{m-1} \Delta\end{matrix} \quad \begin{matrix}\vdots\\ x:B,\Gamma \xrightarrow[p]{m-1} \Delta\end{matrix}}{x:A \vee B, \Gamma \xrightarrow[p]{m} \Delta}L\vee$$

Find d as follows

$$\dfrac{\dfrac{\Gamma \xrightarrow[p]{n-1} \Delta, x:A, x:B \qquad \dfrac{x:A,\Gamma \xrightarrow[p]{m-1} \Delta}{x:A,\Gamma \xrightarrow[p]{m-1} \Delta, x:B}\text{R-W}}{\Gamma \xrightarrow[p]{max(n,m)} \Delta, x:B}\text{CUT} \qquad x:B,\Gamma \xrightarrow[p]{m-1} \Delta}{\Gamma \xrightarrow[p]{max(max(n,m)+1,m)} \Delta}\text{CUT}$$

As above, the height of derivation of the conclusion is $max(n,m)+1$ and therefore $\Gamma \xrightarrow[p]{n+m} \Delta$. Moreover, $h(x:A), h(x:B) < p = h(x:A \vee B)$ and so the rank is p.

If $x:C$ is $x:A \supset B$ then d_1 and d_2 are

$$\dfrac{\begin{matrix}\vdots\\ x:A,\Gamma \xrightarrow[p]{n-1} \Delta, x:B\end{matrix}}{\Gamma \xrightarrow[p]{n} \Delta, x:A\supset B}\,R\supset \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma \xrightarrow[p]{m-1} \Delta, x:A\end{matrix} \quad \begin{matrix}\vdots\\ x:B,\Gamma \xrightarrow[p]{m-1} \Delta\end{matrix}}{x:A\supset B,\Gamma \xrightarrow[p]{m} \Delta}\,L\supset$$

Find d as follows

$$\dfrac{\Gamma \xrightarrow[p]{m-1} \Delta, x:A \qquad \dfrac{x:A,\Gamma \xrightarrow[p]{n-1} \Delta, x:B \qquad \dfrac{x:B,\Gamma \xrightarrow[p]{m-1} \Delta}{x:B,x:A,\Gamma \xrightarrow[p]{m-1} \Delta}\,\text{L-W}}{x:A,\Gamma \xrightarrow[p]{max(n,m)} \Delta}\,\text{CUT}}{\Gamma \xrightarrow[p]{max(m,max(n,m)+1)} \Delta}\,\text{CUT}$$

Again, the height of derivation of the conclusion is $max(n,m)+1$, so $\Gamma \xrightarrow[p]{n+m} \Delta$. Furthermore, the rank of d is p because $h(x:A), h(x:B) < p = h(x:A\supset B)$.

If $x:C$ is $x:\Box A$ then Γ is xRy,Γ' and d_1 and d_2 are

$$\dfrac{\begin{matrix}\vdots\\ xRz, xRy,\Gamma' \xrightarrow[p]{n-1} \Delta, z:A\end{matrix}}{xRy,\Gamma' \xrightarrow[p]{n} \Delta, x:\Box A}\,R\Box \qquad \dfrac{\begin{matrix}\vdots\\ y:A, x:\Box A, xRy,\Gamma' \xrightarrow[p]{m-1} \Delta\end{matrix}}{x:\Box A, xRy,\Gamma' \xrightarrow[p]{m} \Delta}\,L\Box$$

where z is not in the conclusion of $R\Box$. First, consider the two following derivations. The first, has as premises the conclusion of $R\Box$ (with a weakening for matching the contexts) and the premise of $L\Box$. The derivation uses IH in order to keep p as rank. Note that by applying cut instead IH we would have rank $p+1$ because of the cut on $h(x:\Box A) = p$.

$$
\dfrac{\dfrac{xRy, \Gamma' \xrightarrow[p]{n} \Delta, x : \Box A}{y : A, xRy, \Gamma' \xrightarrow[p]{n} \Delta, x : \Box A}\ \text{L-W} \qquad y : A, x : \Box A, xRy, \Gamma' \xrightarrow[p]{m-1} \Delta}{y : A, xRy, \Gamma' \xrightarrow[p]{n+(m-1)} \Delta}\ \text{IH}
$$

In the second derivation, Lemma 2.1.3 is applied in order to replace z with y in the premise of $R\Box$. Note that z is *eigenvariable* and so the substitution does not affect the context.

$$
\dfrac{\dfrac{xRz, xRy, \Gamma' \xrightarrow[p]{n-1} \Delta, z : A}{xRy, xRy, \Gamma' \xrightarrow[p]{n-1} \Delta, y : A}\ y/z}{xRy, \Gamma' \xrightarrow[p]{n-1} \Delta, y : A}\ \text{L-C}
$$

Then, by applying cut on their conclusions we obtain

$$
\dfrac{xRy, \Gamma' \xrightarrow[p]{n-1}, \Delta, y : A \qquad y : A, xRy, \Gamma' \xrightarrow[p]{n+(m-1)} \Delta}{xRy, \Gamma' \xrightarrow[p]{max(n-1, n+(m-1))+1} \Delta}\ \text{CUT}
$$

Now, $max(n-1, n+(m-1)) + 1 = max(n, n+m) = n+m$ because $n, m \neq 0$. Moreover, the rank is p because cut applies on a formula smaller than $x : \Box A$. Therefore, we conclude $xRy, \Gamma' \xrightarrow[p]{n+m} \Delta$.

If $x : C$ is $x : \Diamond A$ then Γ is xRy, Γ' and d_1 and d_2 are

$$
\dfrac{\begin{array}{c}\vdots \\ xRy, \Gamma' \xrightarrow[p]{n-1} \Delta, x : \Diamond A, y : A\end{array}}{xRy, \Gamma' \xrightarrow[p]{n} \Delta, x : \Diamond A}\ R\Diamond \qquad \dfrac{\begin{array}{c}\vdots \\ z : A, xRz, xRy, \Gamma' \xrightarrow[p]{m-1} \Delta\end{array}}{x : \Diamond A, xRy, \Gamma' \xrightarrow[p]{m} \Delta}\ L\Diamond
$$

where z is not in the conclusion of $L\Diamond$. As above, consider the following partial derivations.

$$\dfrac{xRy, \Gamma' \xrightarrow[p]{n-1} \Delta, x : \Diamond A, y : A \qquad \dfrac{x : \Diamond A, xRy, \Gamma' \xrightarrow[p]{m} \Delta}{x : \Diamond A, xRy, \Gamma' \xrightarrow[p]{m} \Delta, y : A}\text{R-W}}{xRy, \Gamma' \xrightarrow[p]{(n-1)+m} \Delta, y : A}\text{IH}$$

and

$$\dfrac{\dfrac{z : A, xRz, xRy, \Gamma' \xrightarrow[p]{m-1} \Delta}{y : A, xRy, xRy, \Gamma' \xrightarrow[p]{m-1} \Delta}\, y/z}{y : A, xRy, \Gamma' \xrightarrow[p]{m-1} \Delta}\text{L-C}$$

By applying cut on the conclusions we obtain

$$\dfrac{xRy, \Gamma' \xrightarrow[p]{(n-1)+m} \Delta, y : A \qquad y : A, xRy, \Gamma' \xrightarrow[p]{m-1} \Delta}{xRy, \Gamma' \xrightarrow[p]{max((n-1)+m, m-1)+1} \Delta}\text{CUT}$$

Once again, height of the derivation is $max((n-1)+m, m-1)+1 = n+m$ and rank p because $h(y : A) < h(x : \Diamond A)$.

$\boxtimes$

Observation. In the proof we often use height and rank preserving invertibility of the logical rules. However, the invertibility can be avoided and the derivation conversions can be obtained by an application of weakening and contraction which are height and rank preserving admissible by Theorems 2.1.4 and 2.1.6. The choice of applying the Inversion Lemma 2.1.5 is due to the fact that in this way all the applications of contraction have atomic formulas $x : P$ and xRy as principal formulas. In fact, the only applications of contraction admissibility required are that of the case 1a when one of the premise of cut is an initial sequent and the cut formula $x : P$ is principal in it, and that of the case 2b in which the cut formula is a modal formula $x : \Box A$ and it is principal in both premises of cut. In the latter, contraction applies on two occurrences of xRy. Thus, there is no need to prove admissibility of contraction for arbitrary formulas and the proof of Theorem 2.1.6 can be restricted to the case in which contracted formulas are either propositional $x : P$ or relational atoms xRy.

Rank reduction Lemma

In the previous lemma, it is shown that if the premises of cut are derivable then the conclusion of cut is also derivable, and the height and the rank can be kept constant. Now, we prove that every derivation of a sequent $\Gamma \to \Delta$ can be transformed into a derivation of the same sequent in which the rank of derivations can be reduced. However, the height increases from m to 2^m.

Lemma 2.2.3 (Rank Reduction)**.** *Every derivation d in* $\mathbf{G3K}^{\mathrm{C}}$ *such that*

$$d \vdash \Gamma \xrightarrow[p+1]{m} \Delta$$

can be converted into a derivation d^* such that

$$d^* \vdash \Gamma \xrightarrow[p]{2^m} \Delta$$

Proof. By induction on m.

If $m = 0$ then d is an initial sequent or conclusion of $L\bot$. In both cases, we take $d^* := d$ and we have $d^* \vdash \Gamma \xrightarrow[p]{2^0=1} \Delta$.

If $m = k + 1$ assume by inductive hypothesis (IH) that the claim holds for $m = k$ and prove that it holds also for $k + 1$. We argue by distinction of cases according to the last rule R of d. In all cases d^* is found by applying IH on the premise(s) of

R and then R again, the only exception being that of cut rule with cut formula of rank p when we need Lemma 2.2.2.

If R is a one-premise rule, say $L\Box$, then Γ is $x : \Box A, xRy, \Gamma'$ and d ends with

$$\dfrac{\begin{array}{c}\vdots \\ y : A, x : \Box A, xRy, \Gamma' \xrightarrow[p+1]{k} \Delta\end{array}}{x : \Box A, xRy, \Gamma' \xrightarrow[p+1]{k+1} \Delta} L\Box$$

Find d^* as follows

$$\dfrac{\dfrac{y : A, x : \Box A, xRy, \Gamma' \xrightarrow[p+1]{k} \Delta}{y : A, x : \Box A, xRy, \Gamma' \xrightarrow[p]{2^k} \Delta} \text{IH}}{x : \Box A, xRy, \Gamma' \xrightarrow[p]{2^k+1} \Delta} L\Box$$

Given that $2^k + 1 \leqslant 2^{k+1}$, we can conclude also $x : \Box A, xRy, \Gamma' \xrightarrow[p]{2^{k+1}} \Delta$.

If R is $R\Box$ then Δ is $\Delta', x : \Box A$ and d is

$$\dfrac{\begin{array}{c}\vdots \\ xRy, \Gamma \xrightarrow[p+1]{k} \Delta, y : A\end{array}}{\Gamma \xrightarrow[p+1]{k+1} \Delta', x : \Box A} R\Box$$

where y is not in the conclusion of $R\Box$. Find d^* as follows

$$\dfrac{\dfrac{xRy, \Gamma \xrightarrow[p+1]{k} \Delta, y : A}{xRy, \Gamma \xrightarrow[p]{2^k} \Delta', y : A} \text{IH}}{\Gamma \xrightarrow[p]{2^k+1} \Delta', x : \Box A} R\Box$$

Therefore $d^* \vdash \Gamma' \xrightarrow[p]{2^{k+1}} \Delta, x : \Box A$.

If R is a two-premise rule as $L \supset$ we have that $x : A \supset B, \Gamma'$ and d is

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma' \xrightarrow[p+1]{k} \Delta, x : A\end{array} \quad \begin{array}{c}\vdots \\ x : B, \Gamma' \xrightarrow[p+1]{k} \Delta\end{array}}{x : A \supset B, \Gamma' \xrightarrow[p+1]{k+1} \Delta} \, L\supset$$

and it can be converted into d^* as follows

$$\dfrac{\dfrac{\Gamma' \xrightarrow[p+1]{k} \Delta, x : A}{\Gamma' \xrightarrow[p]{2^k} \Delta, x : A} \, \text{IH} \qquad \dfrac{x : B, \Gamma' \xrightarrow[p+1]{k} \Delta}{x : B, \Gamma' \xrightarrow[p]{2^k} \Delta} \, \text{IH}}{x : A \supset B, \Gamma' \xrightarrow[p]{2^k+1} \Delta} \, L\supset$$

As above, $d^* \vdash x : A \supset B, \Gamma' \xrightarrow[p]{2^{k+1}} \Delta$.

The case in which R is cut is straightforward when cut formula has height $< p$ and follows the same pattern of two-premise rules. The most important case is when R is cut and height of cut formula is p, *i.e.* $h(x : C) = p$

$$\dfrac{\Gamma \xrightarrow[p+1]{k} \Delta, x : C \quad x : C, \Gamma \xrightarrow[p+1]{k} \Delta}{\Gamma \xrightarrow[p+1]{k+1} \Delta} \, \text{CUT}$$

By applying IH on the premises of cut we obtain

$$\Gamma \xrightarrow[p]{2^k} \Delta, x : C \qquad \text{and} \qquad x : C, \Gamma \xrightarrow[p]{2^k} \Delta$$

At this point we use the Main Lemma 2.2.2 in order to get

$$\Gamma \xrightarrow[p]{2^{k+1}} \Delta$$

since $2^k + 2^k = 2(2^k) = 2^{k+1}$.

$\boxtimes$

Cut-free derivations

The final step of cut elimination consists in showing that any istance of cut can be dispensed with, that is, every derivation with rank p can be converted into a derivation with rank 0. We will show that the procedure of cut elimination has as consequence an hyperexponential growth of the derivation.

Theorem 2.2.4 (Cut elimination)**.** *In* **G3K**$^{\mathrm{C}}$ *cut is eliminable,* i.e. *every derivation d such that*

$$d \vdash \Gamma \xrightarrow[p]{n} \Delta$$

can be converted into a derivation d^- such that

$$d^- \vdash \Gamma \xrightarrow[0]{2_p(n)} \Delta$$

Proof. By induction on p.

If $p = 0$ then d has no cuts and we can take d as d^-.

If $p = k+1$ assume by inductive hypothesis (IH) that the claim holds for $p = k$ and prove that it holds also for $p = k+1$. Thus from

$$d \vdash \Gamma \xrightarrow[k+1]{n} \Delta$$

we find d^- by applying the rank reduction Lemma 2.2.3:

$$d^- \vdash \Gamma \xrightarrow[k]{2^n} \Delta$$

Now, by IH

$$d^{-} \vdash \Gamma \xrightarrow[0]{2_k(2^n)} \Delta$$

By Proposition 2.2.1 we have $2_k(2^n) = 2_{k+1}(n)$ and therefore

$$d^{-} \vdash \Gamma \xrightarrow[0]{2_{k+1}(n)} \Delta$$

$\boxtimes$

Observation. We return to the role of contraction in the proof of cut elimination. We have seen that rule $L\Box$ (resp. $R\Diamond$) is trivially invertible, once one has shown that weakening is admissible: since principal formulas $x : \Box A, xRy$ (resp. $xRy, x : \Diamond A$) are repeated, the inversion of $L\Box$ (resp. $R\Diamond$) holds by Theorem 2.1.4. Therefore, contraction is admissible (Theorem 2.1.6) without any use of inversion (Lemma 2.1.5) of $L\Box$ (resp. $R\Diamond$). This is to say, the repetition of principal formulas builds contraction into the logical rules. In fact, when the rule without repetition is considered, *i.e.* $L\Box 1$

$$\frac{y : A, \Gamma \rightarrow \Delta}{x : \Box A, xRy, \Gamma \rightarrow \Delta}\ L\Box 1$$

the standard $L\Box$ becomes derivable by a step of contraction. The double inference line indicates repeated applications of the structural rules of weakening or contraction.

$$\dfrac{\dfrac{y : A, x : \Box A, xRy, \Gamma \rightarrow \Delta}{x : \Box A, xRy, x : \Box A, xRy, \Gamma \rightarrow \Delta}\ L\Box 1}{\overline{\overline{x : \Box A, xRy, \Gamma \rightarrow \Delta}}}\ \text{L-C}$$

The difference between $L\Box$ and $L\Box 1$ is that $L\Box$ derives basic theorems of modal logic without any application of contraction, whereas with $L\Box 1$ these applications are unavoidable. For example, to derive $x : \Box(P \supset Q), x : \Box P \to x : \Box Q$ with $L\Box 1$ contraction is essential.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y : P \to y : Q, y : P \quad y : Q, y : P \to y : Q}{y : P \supset Q, y : P \to y : Q}\,L\supset}{xRy, y : P \supset Q, x : \Box P \to y : Q}\,L\Box 1}{xRy, xRy, x : \Box(P \supset Q), x : \Box P \to y : Q}\,L\Box 1}{xRy, x : \Box(P \supset Q), x : \Box P \to y : Q}\,\text{C}}{x : \Box(P \supset Q), x : \Box P \to x : \Box Q}\,R\Box
$$

The situation is analogous in first-order logic to the derivation of the sequent $\to \exists x(Px \supset \forall y Py)$, where a contraction on $\exists x(P(x) \supset \forall y P(y))$ is required if $R\exists$ is without repetition of the principal formula. On the contrary, using $L\Box$ any application of contraction can be dispensed with and the above derivation can be found by applying a systematic proof-search from the sequent to be derived. However, on a closer inspection the application of contraction in the above derivation has xRy as principal formula, whereas in $L\Box$ both $x : \Box A$ and xRy are repeated in the premise. Therefore, we consider a new left rule for $\Box$ where the relational atom xRy, but not $x : \Box A$, is repeated in the premise.

$$
\frac{xRy, y : A, \Gamma \to \Delta}{xRy, x : \Box A, \Gamma \to \Delta}\,L\Box 2
$$

Still, the standard $L\Box$ is derivable from $L\Box 2$ by contraction

$$
\cfrac{\cfrac{y : A, xRy, x : \Box A, \Gamma \to \Delta}{x : \Box A, xRy, x : \Box A, \Gamma \to \Delta}\,L\Box 2}{xRy, x : \Box A, \Gamma \to \Delta}\,\text{L-C}
$$

In contrast to $L\Box 1$, $L\Box 2$ proves $x : \Box(P \supset Q), x : \Box P \to x : \Box Q$ without any contraction. In fact,

$$\dfrac{\dfrac{\dfrac{\dfrac{y : P, xRy \to y : Q, y : P \quad y : Q, xRy, y : P \to y : Q}{xRy, y : P \supset Q, y : P \to y : Q}\, L\supset}{xRy, y : P \supset Q, x : \Box P \to y : Q}\, L\Box 2}{xRy, x : \Box(P \supset Q), x : \Box P \to y : Q}\, L\Box 2}{x : \Box(P \supset Q), x : \Box P \to x : \Box Q}\, L\supset$$

Therefore, if we take $L\Box 1$ as primitive, we need primitive contraction as well otherwise the calculus is not complete: as we have shown, there is a valid sequent which is not derivable without contraction, *i.e.* $x : \Box(P \supset Q), x : \Box P \to x : \Box Q$. However, we conjecture that the repetition of $x : \Box A$ in the the premise of $L\Box$ is not needed in **G3K**. In other words, we could take $L\Box 2$ as primitive instead of $L\Box$, and still have a cut-free and complete system. However, like $L\Box 1$, rule $L\Box 2$ is not invertible and this constitutes a serious obstacle to the proof of contraction admissibility: if the principal formula $x : \Box A$ is not available in the premise, there is no immediate method of converting a derivation of

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ xRy, x : \Box A, y : A, \Gamma \to \Delta\end{array}}{xRy, x : \Box A, x : \Box A, \Gamma \to \Delta}\, L\Box 2}{xRy, x : \Box A, \Gamma \to \Delta}\, \text{L-C}$$

into a derivation in which contraction is applied to smaller formulas.

2.3 Correspondence with an axiomatic system

The system **G3K** corresponds to the Hilbert system **K** of the basic modal logic. All the axioms of **K** are derivable and its rules are admissible in **G3K**, so **K** $\subseteq$ **G3K**. Along with completeness of **K**, admissibility of rules of **K** and derivability of **K** axioms give an indirect completeness proof for **G3K**. The full correspondence between **G3K** and **K** with the soundness of **G3K** rules (see Lemma 2.4.1). We recall that the standard presentation of an axiomatic system consists of all the axioms of the classical propositional logic (A1) together with the distributivity axiom (A2) and the axiom of duality between modal operators (A3). The rules are the *modus ponens* and the generalization of $\Box$. For a detailed exposition see from Hughes and Cresswell (1996).

A1	All the axioms of propositional logic **PC**	
A2	$\Box(A \supset B) \supset \Box A \supset \Box B$	Distributivity
A3	$\Diamond A \supset\subset \neg\Box\neg A$	Duality
R1	From $\Gamma \vdash A \supset B$ and $\Delta \vdash A$ infer $\Gamma, \Delta \vdash B$	Modus Ponens
R2	From $\vdash A$ infer $\vdash \Box A$	Necessitation

Since, the system **G3K** allows a systematic proof-search procedure, a derivation for each axiom of the Hilbert-style system can be systematically found.

Lemma 2.3.1. *All the axioms (rules) of* **K** *are derivable (resp. admissible) in* **G3K***.*

Proof. By a systematic proof-search procedure from the sequent to be derived. The axioms of **PC** are derivable straightforwardly. The distributivity axiom A2 has the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A, xRy, x : \Box(A \supset B), x : \Box A \to y : B, y : A \quad y : A, y : B, xRy, x : \Box(A \supset B), x : \Box A \to y : B}{y : A, y : A \supset B, xRy, x : \Box(A \supset B), x : \Box A \to y : B}\, L\supset}{xRy, x : \Box(A \supset B), x : \Box A \to y : B}\, L\Box}{x : \Box(A \supset B), x : \Box A \to x : \Box B}\, R\Box}{\to x : \Box(A \supset B) \supset \Box A \supset \Box B}\, R\supset$$

Note that topmost sequents are derivable by Lemma 2.1.1. The duality axiom A3 is derivable by

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{xRy, y : A, x : \Box\neg A \to y : A}{y : \neg A, xRy, y : A, x : \Box\neg A \to}\ L\neg}{xRy, y : A, x : \Box\neg A \to}\ L\Box}{x : \Diamond A, x : \Box\neg A \to}\ L\Diamond}{x : \Diamond A \to x : \neg\Box\neg A}\ R\neg}{\to x : \Diamond A \supset \neg\Box\neg A}\ R\supset
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y : A, xRy \to x : \Diamond A, y : A}{xRy \to x : \Diamond A, y : \neg A, y : A}\ R\neg}{xRy \to x : \Diamond A, y : \neg A}\ R\Diamond}{\to x : \Diamond A, x : \Box\neg A}\ R\Box}{x : \neg\Box\neg A \to x : \Diamond A}\ L\neg}{\to x : \neg\Box\neg A \supset \Diamond A}\ R\supset
$$

Once again, topmost sequents are derivable by Lemma 2.1.1. *Modus ponens* (R1) is proved to be admissible by cut as follows

$$
\cfrac{\cfrac{\Delta \to x : A}{\Gamma, \Delta \to x : B, x : A}\ \text{R-W} \qquad \cfrac{\cfrac{\Gamma \to x : A \supset B}{\Gamma, \Delta \to x : A \supset B}\ \text{L-W}}{x : A, \Gamma, \Delta \to x : B}\ \text{INV}}{\Gamma, \Delta \to x : B}\ \text{CUT}
$$

The admissibility of necessitation (R2) requires admissibility of substitution (Lemma 2.1.3).

$$
\cfrac{\cfrac{\cfrac{\to x : A}{\to y : A}\ y/x}{xRy \to y : A}\ \text{L-W}}{\to x : \Box A}\ R\Box
$$

Note that admissibility of necessitation requires essentially the use of substitution of labels which is admissible by Lemma 2.1.3. ☒

2.4 Completeness

There are three main methods for proving the completeness theorem of a sequent system: One is the indirect method that establishes an equivalence with an axiomatic system known to be complete with respect to a certain class of frames. The second is through Henkin sets with the canonical frame construction, and the third by a direct method that shows how root-first proof search in the sequent system either gives a proof or leads to a countermodel. The results of the previous section correspond to the first method for proving the completeness theorem: the sequent system we have presented for modal logic is closed under the rules of modus ponens and necessitation and permits to derive the axioms of a standard axiomatic presentation. In this section, we follow the proof of Negri (2009) and we prove that **G3K** is complete by the method that will permit proofs of underivability and constructions of countermodels. First, we recall the definitions of frame and model from previous sections, suitably adapted for the mono-modal logic.

Definition (Frame). *A* frame *is a structure* $\mathfrak{F} = \langle \mathsf{X}, \mathsf{R} \rangle$ *where* X *is a non-empty set and* R *is a binary relation on* X.

Definition (Model). *A* model *is a structure* $\mathfrak{M} = \langle \mathfrak{F}, \Vdash \rangle$ *where* $\mathfrak{F}$ *is a frame and* $\Vdash$ *is a binary relation between elements of* X *and atomic formulas* P.

The relation $\Vdash$ is extended in a unique way to arbitrary formulas by means of the following clauses

$x \Vdash \bot$	for no x	
$x \Vdash A \wedge B$	if and only if	$x \Vdash A$ and $x \Vdash B$
$x \Vdash A \vee B$	if and only if	$x \Vdash A$ or $x \Vdash B$
$x \Vdash A \supset B$	if and only if	$x \Vdash A$ implies $x \Vdash B$
$x \Vdash \Box A$	if and only if	for all x, $x\mathsf{R}y$ implies $y \Vdash A$
$x \Vdash \Diamond A$	if and only if	for some y, $x\mathsf{R}y$ and $y \Vdash A$

Definition (Interpretation). *Let* L *be the set of labels. An* interpretation *of labels in a frame* $\mathfrak{F}$ *is a function* $[\![\cdot]\!] : L \longrightarrow \mathsf{X}$ *that assigns a possible state* $[\![x]\!]$ *of* $\mathfrak{F}$ *to each label* x *in* L*, and the accessibility relation* R *of* $\mathfrak{F}$ *to the relational symbol* R.

Definition (Validity in a model). *A sequent* $\Gamma \to \Delta$ *is* valid in a model $\mathfrak{M}$ *if for every interpretation it holds that whenever for all labelled formulas* $x : A$ *and relational atoms* yRz *in* Γ, $[\![x]\!] \Vdash A$ *and* $[\![y]\!]\mathsf{R}[\![z]\!]$ *hold, then for some* $w : B$ *in* Δ, $[\![w]\!] \Vdash B$.

Definition (Validity). *A sequent* $\Gamma \to \Delta$ *is* valid *when it is valid in every model.*

Before proving the completeness of **G3K**, we show that it is sound, that is in **G3K** are derivable only valid sequents.

Theorem 2.4.1 (Soundness of **G3K**). *If* $\Gamma \to \Delta$ *is derivable* **G3K** *then it is valid.*

Proof. By induction on the derivation of $\Gamma \to \Delta$.

If $\Gamma \to \Delta$ is initial then Γ and Δ have an atom in common $x : P$ and the claim is obvious; similarly if $\Gamma \to \Delta$ is a conclusion of $L\bot$ since no x can force $\bot$.

If $\Gamma \to \Delta$ is a conclusion of a propositional or modal rule assume by inductive hypothesis (IH) that its premise(s) is (are) valid and prove that also the conclusion is. We distinguish the following cases according to the last rule applied. If it is $L\wedge$ then the derivation ends with

$$\dfrac{\begin{matrix}\vdots\\ x : A, x : B, \Gamma' \to \Delta\end{matrix}}{x : A \wedge B, \Gamma' \to \Delta}\, L\wedge$$

Assume by IH the validity of the premise, the validity of the conclusion follows since $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash B$ is equivalent to $[\![x]\!] \Vdash A \wedge B$.

If $\Gamma \to \Delta$ is a conclusion of $L\supset$ then the derivation ends with

$$\dfrac{\begin{matrix}\vdots\\ \Gamma' \to \Delta, x : A\end{matrix} \quad \begin{matrix}\vdots\\ x : B, \Gamma' \to \Delta\end{matrix}}{x : A \supset B, \Gamma' \to \Delta}\, L\supset$$

By IH the premises are valid. If the left premise is valid then either $[\![x]\!] \Vdash A$ or $[\![w]\!] \Vdash C$, for some $w : C$ in Δ. In the latter case, the conclusion is valid. If the right premise is valid then for all $v : D$ in Γ' $[\![v]\!] \Vdash D$ and $[\![x]\!] \Vdash B$. Then $[\![x]\!] \Vdash A \supset B$ and the conclusion is valid too.

The situation is analogous for the other propositional rules. If $\Gamma \to \Delta$ is a conclusion of a modal rule, say $L\Diamond$, then the last step of the derivation is

$$\dfrac{\begin{array}{c}\vdots\\ xRy, y : A, \Gamma' \to \Delta\end{array}}{x : \Diamond A, \Gamma' \to \Delta}\ L\Diamond$$

Assume by IH that the premise is valid. Let $[\![\cdot]\!]$ be an arbitrary interpretation that validates all the formulas in $\Gamma', x : \Diamond A$. We claim that one of the formulas in Δ is valid under this interpretation. Since $[\![x]\!] \Vdash \Diamond A$, we can choose an element k of X such that $[\![x]\!]\mathsf{R}k$ and $k \Vdash A$. Let $[\![\cdot]\!]'$ be the interpretation identical to $[\![\cdot]\!]$ except possibly on y, for which we set $[\![y]\!]' = k$. Clearly $[\![\cdot]\!]'$ validates all the formulas in the antecedent of the premise, so it validates one formula in Δ. Since y does not occur in Δ, also $[\![\cdot]\!]$ validates one formula in Δ.

$\boxtimes$

The completeness theorem is proved following the pattern of Negri (2009), in analogy with Kripke's original proof (see Kripke 1963). Instead of Kripke's proof, we do not look for a failed search of a countermodel, but directly for a proof: To see whether a formula is derivable, we check if it is universally valid, that is, if $x \Vdash A$ for an arbitrary state x. This is translated to a sequent $\to x : A$. The rules of **G3K** applied backwards give equivalent conditions until the atomic components of A are reached. It can happen that we find a proof, or that we find that a proof does not exist either because we reach a stage where no rule is applicable, or because we go on with the search forever. In the two latter cases the attempt proof itself gives a countermodel.

Theorem 2.4.2. *For all $\Gamma \to \Delta$ in* **G3K** *either $\Gamma \to \Delta$ is derivable or it has a countermodel.*

Proof. We define for an arbitrary $\Gamma \to \Delta$ of **G3K** a reduction tree by applying the rules of **G3K** root first in all possible ways. If the construction terminates we obtain a proof, else the tree becomes infinite. By König's lemma an infinite tree has an infinite branch that is used to define a countermodel to the end-sequent.

Construction of the reduction tree

The reduction tree is defined inductively in stages as follows: Stage 0 has $\Gamma \to \Delta$ at the root of the tree. Stage $n > 0$ has two cases:

CASE I: If every topmost sequent is initial or a conclusion of $L\bot$ the construction of the tree ends.

CASE II: If not every topmost sequent is initial or a conclusion of $L\bot$, we continue the construction of the tree by writing above those sequents that are not initial nor a conclusion of $L\bot$, other sequents that are obtained by applying root first the rules of **G3K** whenever possible, in a given order.

There are 10 different stages, 6 for propositional rules, 4 for modal rules. At stage $n = 11$ we repeat stage 1, at stage $n = 12$ we repeat stage 2, and so on for every n.

Case of $L\wedge$. For each topmost sequent of the form

$$x_1 : B_1 \wedge C_1, \ldots, x_m : B_m \wedge C_m, \Gamma' \to \Delta$$

where $B_1 \wedge C_1, \ldots, B_m \wedge C_m$ are all the formulas in Γ with a conjunction as the outermost logical connective, we write

$$x_1 : B_1, x_1 : C_1, \ldots, x_m : B_m, x_m : C_m, \Gamma' \to \Delta$$

on top of it. This step corresponds to applying root first m times rule $L\wedge$.

Case of $R\wedge$. For each topmost sequent of the form

$$\Gamma \to \Delta', x_1 : B_1 \wedge C_1, \dots, x_m : B_m \wedge C_m$$

where $B_1 \wedge C_1, \dots, B_m \wedge C_m$ are all the formulas in Γ with a conjunction as the outermost logical connective, we write on top of it the 2^m sequents

$$\Gamma \to \Delta', x_1 : D_1, \dots, x_m : D_m$$

where D_i is either B_i or C_i and all possible choices are taken.

Case of $L\vee$. Analogous to $R\wedge$.

Case of $R\vee$. Analogous to $L\wedge$.

Case of $L \supset$. For each topmost sequent of the form

$$x_1 : B_1 \supset C_1, \dots, x_m : B_m \supset C_m, \Gamma' \to \Delta$$

where $B_1 \supset C_1, \dots, B_m \supset C_m$ are all the formulas in Γ with a conjunction as the outermost logical connective, we write on top of it the 2^m sequents

$$x_{i_1} : C_{i_1}, \dots, x_{i_k} : C_{i_k}, \Gamma' \to \Delta, x_{j_{k+1}} : B_{j_{k+1}}, \dots, x_{j_m} : B_{j_m}$$

where $i_1, \dots i_k \in \{1, \dots, m\}$ and $k_{k+1}, \dots i_m \in \{1, \dots, m\} \smallsetminus \{i_1, \dots i_k\}$. Although less transparent, this step corresponds to the root-first application of rule $L \supset$ with principal formulas $B_1 \supset C_1, \dots, B_m \supset C_m$.

Case of $R \supset$. For each topmost sequent of the form

$$\Gamma \to \Delta', x_1 : B_1 \supset C_1, \dots, x_m : B_m \supset C_m$$

where $B_1 \supset C_1, \dots, B_m \supset C_m$ are all the formulas in Γ with a implication as the outermost logical connective, we write on top of it

$$x_1 : B_1, \ldots, x_m : B_m, \Gamma \rightarrow \Delta, x_1 : C_1, \ldots, x_m : C_m \supset C_m$$

that is, we apply m time the rule $R \supset$.

Case of $L\Box$. For each topmost sequent of the form

$$x_1 : \Box B_1, \ldots, x_m : \Box B_m, x_1 R y_1, \ldots, x_m R y_m, \Gamma' \rightarrow \Delta$$

where $\Box B_1, \ldots, \Box B_m$ are all the formulas with $\Box$ as the outermost logical connective, we write on top of it

$$y_1 : B_1, \ldots, y_m : B_m, x_1 : \Box B_1, \ldots, x_m : \Box B_m, x_1 R y_1, \ldots, x_m R y_m, \Gamma' \rightarrow \Delta$$

Case of $R\Box$. For each topmost sequent of the form

$$\Gamma \rightarrow \Delta', x_1 : \Box B_1, \ldots, x_m : \Box B_m$$

we write on top of it

$$x_1 R y_1, \ldots, x_m R y_m, \Gamma \rightarrow \Delta', y_1 : B_1, \ldots, y_m : B_m$$

where $y_1, \ldots, y_m$ are fresh variables, not yet used in the reduction tree.

Case of $L\Diamond$. Analogous to $R\Box$.

Case of $R\Diamond$. Analogous to $L\Box$.

For any n, for each sequent that is neither initial, nor conclusion of $L\bot$, nor treatable by any one of the above reductions, we write the sequent itself above it. If the reduction tree is finite, all its leaves are initial or conclusions of $L\perp$, and the tree, read from the leaves to the root, yields a derivation.

Construction of the countermodel

By König's lemma, if the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \rightarrow \Delta_0 \equiv \Gamma \rightarrow \Delta, \Gamma_1 \rightarrow \Delta_1, \ldots, \Gamma_i \rightarrow \Delta_i, \ldots$ be one such branch. Consider the set of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i \geq 0} \Gamma_i \quad \text{and} \quad \mathbf{\Delta} \equiv \bigcup_{i \geq 0} \Delta_i$$

We define a model that forces all formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \rightarrow \Delta$.

Consider the frame $\mathfrak{F}$ the elements of which are all the labels that appear in the relational atoms in $\mathbf{\Gamma}$, with their mutual relationship expressed by the xRy's in $\mathbf{\Gamma}$. The model is defined as follows: For all atomic formulas $x : P$ in $\mathbf{\Gamma}$, we stipulate that $x \Vdash P$ in the frame $\mathfrak{F}$, and for all atomic formulas $y : Q$ in $\mathbf{\Delta}$, we stipulate that $y \nVdash Q$ in $\mathfrak{F}$. Since no sequent in the infinite branch is initial, this choice can be coherently made, for if there were the same labelled atom in $\mathbf{\Gamma}$ and in $\mathbf{\Delta}$, then, since the sequents in the reduction tree are defined in a cumulative way, for some i there would be a labelled atom $x : P$ both in the antecedent and in the succedent of $\Gamma_i \rightarrow \Delta_i$.

We then show inductively on the structure of formulas that B is forced at x if $x : B$ is in $\mathbf{\Gamma}$ and B is not forced at x if $x : B$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the end-sequent $\Gamma \rightarrow \Delta$.

If B is $\perp$, it cannot be in $\mathbf{\Gamma}$ because no sequent in the branch contains $x : \perp$ in the antecedent, so it is not forced at any node of the model.

If $x : B \wedge C$ is in $\mathbf{\Gamma}$, there exists i such that $x : B \wedge C$ appears first in Γ_i, and therefore, for some $j \geqslant 0$, $x : B$ and $x : C$ are in Γ_{i+j}. By IH, $x \Vdash B$ and $x \Vdash C$ and therefore

$x \Vdash B \wedge C$.

If $x : B \wedge C$ is in $\mathbf{\Delta}$ consider the step i in of the reduction tree. This gives a branching, and one of the two branches belongs to the infinite branch, so either $x : B$ or $x : C$ is in $\mathbf{\Delta}$. By IH, $x \nVdash B$ or $x \nVdash C$ and therefore $x \nVdash B \wedge C$.

The case of $x : B \vee C$ is in $\mathbf{\Gamma}$ is analogous to the case of $x : B \wedge C$ is in $\mathbf{\Delta}$.

The case of $x : B \vee C$ is in $\mathbf{\Delta}$ is analogous to the case of $x : B \wedge C$ is in $\mathbf{\Gamma}$.

If $x : B \supset C$ is in $\mathbf{\Gamma}$, either $x : B$ is in $\mathbf{\Delta}$ or $x : C$ is in $\mathbf{\Gamma}$. By IH, in the former case $x \nVdash B$, and in the latter $x \Vdash C$, so in both cases $x \Vdash B \supset C$.

If $x : B \supset C$ is in $\mathbf{\Delta}$, for some i, $x : B$ is in Γ_i and $x : C$ is in Δ_i, so by IH $x \Vdash B$ and $x \nVdash C$, so $x \nVdash B \supset C$.

If $x : \Box B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms xRy that occur in $\mathbf{\Gamma}$. If there is no such atom, then the condition that for all y accessible from x in the frame, $y \Vdash B$ is vacuously satisfied, and therefore $x \Vdash \Box B$ in the model. Else, for any occurrence of xRy in $\mathbf{\Gamma}$ we find, by the construction tree, an occurrence of $y : B$ in $\mathbf{\Gamma}$. By IH $y \Vdash B$, and therefore $x \Vdash \Box B$ in the model.

If $x : \Box B$ is in $\mathbf{\Delta}$, consider the step at which the reduction for $x : \Box B$ applies. We find $y : B$ in $\mathbf{\Delta}$, for some y with xRy in $\mathbf{\Gamma}$. By IH $x \nVdash B$, and therefore $x \nVdash \Box B$.

⊠

Corollary 2.4.3 (Completeness of **G3K**). *If a sequent* $\Gamma \rightarrow \Delta$ *is valid then it is derivable in* **G3K**.

The proof of the completeness theorem given in this section is close to Kripke's original argument but without any appeal to a geometric intuition. In fact, Kripke's proof was criticized since it makes appeal to intuitive arguments on the geometry of tableau proofs and lacks the rigor of the alternative set-theoretic approach due to Henkin. The proof can be extended to systems with mathematical for the accessibility relation R as the logic of knowledge and belief, and also to systems with new modal operators as in the dynamic epistemic logic.

Chapter 3

Extensions of Labelled Sequent Systems

This chapter is devoted to the labelled sequent systems for logics that extend basic modal logic. In particular, our aim is to augment the set of the rules of **G3K** and find cut-free systems equivalent to well-known systems for modal logics such as **T**, **S4**, **B**, **S5**, **D**, etc. In the Hilbert-style approach various extensions of **K** are obtained by simply adding new axioms. Thus, the system **T** is **K** together with $\Box A \supset A$ (axiom T), system **S4** is **T** plus $\Box A \supset \Box\Box A$ (axiom 4), and, **B** is **T** with $A \supset \Box\Diamond A$ (axiom B). Finally, system **S5** is **T** with $\Diamond A \supset \Box\Diamond A$ (axiom 5). Many other systems and axioms are known and we shall deal with them in the following. For the time being, our aim is to follow the Gentzen-style tradition and present extensions of **G3K** by new inference rules, rather than new axioms. Sequent calculi equivalent to **K**, **T**, **S4** and **S5** are presented in Ono (1998) as extensions of the original Gentzen's system **LK** (without quantifiers rules). The system **GK** is **LK** with the new rule $\Box$

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\ \Box$$

$\Box\Gamma$ denotes the list of all the $\Box A$ for A in Γ. Observe that when Γ is empty, rule $\Box$ is the rule of necessitation. When Γ is not empty, with the application of $\Box$ all formulas in Γ get prefixed by $\Box$. This prevents the derivation of the invalid formula $A \supset \Box A$. A system for the modal logic **T** (**G3T**) is then obtained from **GK** by adding the rule $\Box \rightarrow$

$$\frac{A, \Gamma \rightarrow \Delta}{\Box A, \Gamma \rightarrow \Delta}\ \Box\!\rightarrow$$

The rule permits to derive the axiom $\Box A \supset A$. The system **G3S4** is **G3T** plus the following rule $\Box \rightarrow_1$ (or, equivalently, **LK** with $\Box \rightarrow$ and $\Box \rightarrow_1$)

$$\frac{\Box\Gamma \rightarrow A}{\Box\Gamma \rightarrow \Box A}\ \Box\!\rightarrow_1$$

With this addition the corresponding axioms $\Box A \supset \Box\Box A$ is derivable. Finally, **GS5** is obtained from **G3T** and the following rule $\Box \rightarrow_2$ (or, equivalently, **LK** with $\Box \rightarrow$ and $\Box \rightarrow_2$)

$$\frac{\Box\Gamma \rightarrow \Box\Delta, A}{\Box\Gamma \rightarrow \Box\Delta, \Box A}\ \Box\!\rightarrow_2$$

Note that $\Box \rightarrow_1$ is a special case of $\rightarrow_2$ with Δ empty. In fact, axiom 5, formulated as $\neg\Box\neg A \supset \Box\neg\Box\neg A$, is derivable as follows

$$\cfrac{\cfrac{\cfrac{\cfrac{\Box\neg A \rightarrow \Box\neg A}{\rightarrow \Box\neg A, \neg\Box\neg A}\ R\neg}{\rightarrow \Box\neg A, \Box\neg\Box\neg A}\ \Box\!\rightarrow_2}{\neg\Box\neg A \rightarrow \Box\neg\Box\neg A}\ L\neg}{\rightarrow \neg\Box\neg A \supset \Box\neg\Box\neg A}\ R\supset$$

Cut elimination holds for all the systems thus obtained (see also Ohnishi and Matsumoto 1957), with the exception of **G3S5**. A simple counterexample is given by the derivation of $A \supset \Box\Diamond A$ (axiom B) which is theorem of **S5** but not derivable without cut in **G3S5**,

$$\dfrac{\dfrac{\dfrac{\Box\neg A \rightarrow \Box\neg A}{\rightarrow \neg\Box\neg A, \Box\neg A}\, R\neg}{\rightarrow \Box\neg\Box\neg A, \Box\neg A}\, \Box{\rightarrow_2} \qquad \dfrac{\dfrac{A \rightarrow A}{\neg A, A \rightarrow}\, L\neg}{\Box\neg A, A \rightarrow}\, \Box{\rightarrow_1}}{\dfrac{A \rightarrow \Box\neg\Box\neg A}{\rightarrow A \supset \Box\neg\Box\neg A}\, R{\supset}}\, \text{CUT}$$

More recently and also in view of the applications to automated deduction, **G3**-systems have been preferred to the original Gentzen's system **LK** which has all the structural rules primitive. Thus, it is reasonable to start from the classical multi-succedent sequent calculus **G3c** of Troelstra and Schwichtenberg (2000) (see also Negri and von Plato 2001) in which all the structural rules are admissible and all the logical rules are invertible. A cut-free system for the basic modal logic is considered in Hakli and Negri (2011) and is obtained from **G3c** by the rule $LR\Box$.

$$\frac{\Gamma \rightarrow A}{\Phi, \Box\Gamma \rightarrow \Box A, \Psi}\, LR\Box$$

The calculus with $LR\Box$ is proved to be equivalent to the axiomatic system **K** and it is used to show that the standard argument in favor of the failure of deduction theorem in modal logic is untenable. A sequent system for the modal logic S4 is presented in Troelstra and Schwichtenberg (2000, ch. 9) by adding to **G3c** the following rules for $\Box$ and $\Diamond$, both taken as primitive:

$$\frac{\Gamma, A, \Box A \rightarrow \Delta}{\Gamma, \Box A \rightarrow \Delta}\, L\Box \qquad \frac{\Box\Gamma \rightarrow A, \Diamond\Delta}{\Gamma', \Box\Gamma \rightarrow \Box A, \Diamond\Delta, \Delta'}\, R\Box$$

$$\frac{\Box\Gamma, A \rightarrow \Diamond A}{\Gamma', \Box\Gamma, \Diamond A \rightarrow \Diamond\Delta, \Delta'}\, L\Diamond \qquad \frac{\Gamma \rightarrow A, \Diamond\Delta, \Delta}{\Gamma \rightarrow \Diamond A, \Delta}\, R\Diamond$$

The calculus is used to prove that a variant of the Gödel embedding of intuitionistic logic into modal logic S4 is faithful. However, having a weakening- and contraction-free calculus does not solve the long-standing problem of cut elimination for S5. Nowadays, it is a common opinion that a satisfactory account of the modal logic S5 cannot be given within the traditional Gentzen systems. As noted at the very beginning of Avron (1996, p. 3)

> The framework of ordinary sequents is not capable of handling all interesting logics. There are logics with nice, simple semantics and obvious interest for which no decent, cut-free formulation seems to exist.

Therefore, alternative proof systems have been recently proposed in which the syntax of the rules is enriched. The modifications come in two flavors. On the one hand, it is possible to generalize sequent calculus so that the semantics is made implicit part of a more structured syntax. Among the various proposal, there is the hypersequent approach (see Avron 1996 for an overview). Roughly speaking, hypersequents are multisets of sequents interpreted disjunctively. If $\Gamma_1 \to \Delta_1 \dots \Gamma_n \to \Delta_n$ are sequents, an hypersequent is a syntactic object of the form $\Gamma_1 \to \Delta_1 \mid \dots \mid \Gamma_n \to \Delta_n$, where the standard interpretation of the $\mid$ is disjunctive. In Poggiolesi (2008) the following rules for the $\Box$ operator, where G stands for an arbitrary hypersequent, are introduced

$$\frac{G \mid A, \Box A, \Gamma \to \Delta}{G \mid \Box A, \Gamma \to \Delta}\ \Box A_1 \qquad\qquad \frac{G \mid \Box A, \Gamma \to \Delta \mid A, \Gamma' \to \Delta'}{G \mid \Box A, \Gamma \to \Delta \mid \Gamma' \to \Delta'}\ \Box A_2$$

$$\frac{G \mid \Gamma \to \Delta \mid \ \to A}{G \mid \Gamma \to \Delta, \Box A}\ \Box K$$

Informally, the rules can be read in terms of relational semantics: the rule $\Box K$ says that if $\Box A$ is false at the actual state then there is some possible state at which A is false. The existence of such a state is achieved syntactically by inserting one disjunct on the top of the hypersequent of the conclusion. Conversely, if $\Box A$ is true at the actual state then A is true at every possible state (rule $\Box A_2$), including the actual one (rule $\Box A_1$). The rules for other propositional connectives are obvious and they do not change the hypersequents but only the formulas within the sequents. Moreover, the axiom $A \supset \Box\neg\Box\neg A$ which was problematic in the traditional approach, has a simple cut-free derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A \mid \Box\neg A \to}{A, \neg A \to \ \mid \Box\neg A \to}\ \neg A}{A \to \ \mid \Box\neg A \to}\ \Box A_1}{A \to \ \mid \ \to \neg\Box\neg A}\ \neg K}{A \to \Box\neg\Box\neg A}\ \Box K}{\to A \supset \Box\neg\Box\neg A}\ \supset K$$

Systems alternative to hypersequents are carefully surveyed in Wansing (2002) and they include Došen's heigher-level sequent systems (see Došen 1985), or 2-sequents systems (see Martini and Masini 1996), multiple-sequent systems of Indrzejczak (1998), and others. More recently, systems of tree-sequents (see Cerrato 1996), tree-hypersequents of Poggiolesi (2009, 2010) and systems of deep inference (see Stewart and Stouppa 2006, Brünnler 2009) have been proposed.

On the other hand, in the labelled approach we employed so far, the notion of sequent is left untouched and the modal content is achieved by an explicit internalization of the relational semantics into the syntax of the rules. We already discussed how the internalization works in the case of basic modal logic, and in the rest of the chapter we show how to get labelled systems for various modal logics. The underlying idea it that they are obtained by adding rules that correspond to the properties of the accessibility relation. Therefore, the new rules do not act on labelled formulas, but only on relational atoms and they do not directly correspond to modal axioms, but to first order conditions on the accessibility relation. For instance, instead of adding a sequent rule that corresponds directly to the axiom $\Box A \supset A$, we add a rule for reflexivity of R, $\forall x(xRx)$. However, the correspondence between modal axioms and frame properties does not solve at all the problem of finding cut-free extensions of **G3K**. In fact, it does not matter whether axioms are formulated in the language of modal logic or in that of first order logic. In either case, an immediate extension would cause the failure of cut elimination (see Girard 1987, p. 125). Therefore, before going into the details of modal systems, we shall shortly outline the general problem concerning axioms and cut elimination.

3.1 Axioms in sequent calculus

In general, an axiom A is added in sequent calculus by permitting derivations to start with the sequent $\rightarrow A$. For instance, a sequent system for first order logic with equality can be obtained by adding to **G3c** the reflexivity of equality and the replacement schema in the form of sequents $\rightarrow x = x$ and $t = s, P(t) \rightarrow P(s)$, where P is atomic. However, as a simple counterexample to cut elimination, consider that the cut applied in the derivation of the symmetry of $=$, that is, $\rightarrow t = s \supset s = t$ is not eliminable. Suppose that $P(x)$ is the atom $x = t$ (so, the instance of the replacement schema is $t = s, t = t \rightarrow s = t$), then there is no cut-free derivation of symmetry,

$$\dfrac{\dfrac{\rightarrow t = t \quad t = s, t = t \rightarrow s = t}{t = s \rightarrow s = t}\text{CUT}}{\rightarrow t = s \supset s = t}R\supset$$

Instead of axiomatic sequents, one may consider rules of inference. In particular, one can replace $\rightarrow x = x$ with the following rule $Ref_=$, and the rule schema $Repl$, instead of the axiom schema $t = s, P(t) \rightarrow P(s)$.

$$\dfrac{x = x, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}Ref_= \qquad\qquad \dfrac{P(s), \Gamma \rightarrow \Delta}{t = s, P(t), \Gamma \rightarrow \Delta}Repl'$$

Since the P can be any atomic formula, the latter rule schema specializes into the following rule when the atom $P(x)$ is $x = t$.

$$\dfrac{s = t, \Gamma \rightarrow \Delta}{t = s, t = t, \Gamma \rightarrow \Delta}Repl*$$

Note that the axioms are derivable from the rules, and *viceversa*. Thus, assume $\rightarrow x = x$ and the premise of $Ref_=$. Then the conclusion of $Ref_=$ is derivable by cut as follows

$$\dfrac{\rightarrow x = x \quad x = x, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\,\text{CUT}$$

In the other direction, find a derivation of $\rightarrow x = x$ by applying $Ref_=$ on the initial sequent $x = x \rightarrow x = x$:

$$\dfrac{x = x \rightarrow x = x}{\rightarrow x = x}\,Ref_=$$

Also the replacement schema and the rules of replacement are derivable from each other, as the following derivation show:

$$\dfrac{t = s, P(t) \rightarrow P(s) \quad P(s), \Gamma \rightarrow \Delta}{t = s, P(t), \Gamma \rightarrow \Delta}\,\text{CUT} \qquad \dfrac{P(s) \rightarrow P(s)}{t = s, P(t) \rightarrow P(s)}\,Repl'$$

However, axioms and rules are different with respect to the possibility of finding a cut-free derivations. This is clear in our example of symmetry of equality, since the sequent $\rightarrow t = s \supset s = t$ is derivable without any application of cut:

$$\dfrac{\dfrac{\dfrac{s = t \rightarrow s = t}{t = t, t = s \rightarrow s = t}\,Repl*}{t = s \rightarrow s = t}\,Ref_=}{\rightarrow t = s \supset s = t}\,R\supset$$

Examples of failure of cut elimination arise also in modal logic. Observe that it is possible to deal with the accessibility relation R of modal logic in the same way as equality in first-order logic. Similarly to equality, the relation R can be assumed to satisfy certain properties. Suppose that R is an equivalence relation, that is, it is reflexive and euclidean. When these properties are considered as axioms of the form $\rightarrow xRx$ and $xRy, xRz \rightarrow yRz$ the following derivation of the symmetry of R

$$\frac{\to xRx \quad xRy, xRx \to yRx}{xRy \to yRx}\ \text{CUT}$$

has a cut that cannot be eliminated. Therefore, we can start from **G3K** and find the sequent rule corresponding to the reflexivity and euclideaness of R

$$\frac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}\ \mathit{Ref} \qquad\qquad \frac{yRz, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}\ \mathit{Eucl'}$$

Once again, with Ref and $Eucl'$ the symmetry of R is derivable without cut:

$$\cfrac{\cfrac{yRx \to yRx}{xRy, xRx \to yRx}\ \mathit{Eucl'}}{xRy \to yRx}\ \mathit{Ref}$$

In contrast to Ref, rule $Eucl'$ reveals an important feature of the new rules, they are "logic free". The role of logical connectives in the axiom of euclideaness, that is, $\wedge$ and $\supset$, is absorbed into the combinatorics of the rule: the role of conjunction is played by the comma in the antecedent, that of implication by the inference step. When rules such as $Eucl'$ are deprived of logical content, only relational atoms appear as principal in them. This is the reason why these rules are called mathematical or, more generally, non-logical inference rules. Another important feature of mathematical rules is that they act only on one side of sequents. In this formulation principal formulas of relational rules appear only in the antecedent, but an equivalent system can be obtained with rules in which relational atoms occur only in the succedent.

3.2 The method of axioms-as-rules

The above examples show only the idea of how to get sequent rules from axioms, but this idea can be made precise and generalized so to get a systematic procedure.

In Negri and von Plato (1998) a general method of adding axioms to sequent calculus in the form of extra-logical inference rule while preserving cut elimination is introduced (see also Negri and von Plato 2001, Ch. 6, for a detailed survey and Negri and von Plato 2011 for further developments). This method covers specific mathematical theories (apartness, order and lattice theories) and geometric theories (affine and projective geometry) and, besides, it successfully applies to modal and non-classical logics in Negri (2005).

We start from the classical multi-succedent sequent calculus **G3c** and we use the existence of **conjunctive normal form** in classical logic: every quantifier-free formula is equivalent to some formula in conjunctive normal form, that is, to a conjunction of disjunctions of atomic formulas or negation of atomic formulas. Within each conjunct the positive atomic formulas can be separated from the negation of atomic formulas

$$\neg P_1 \vee \cdots \vee \neg P_m \vee Q_1 \vee \cdots \vee Q_n$$

and this can be converted into the classically equivalent implication

$$P_1 \wedge \cdots \wedge P_m \supset Q_1 \vee \cdots \vee Q_n \qquad \text{REG}$$

Special cases are with $m = 0$, where REG reduces to $Q_1 \vee \cdots \vee Q_n$, and with $n = 0$ where REG is $\neg(P_1 \wedge \cdots \wedge P_m)$. The universal closure of this implication is called a **regular formula**. Regular formulas can be converted into a sequent rule in two ways. One is based on the idea that if each Q_j together with other assumptions Γ is sufficient to derive Δ, then the P_i's together with Γ are sufficient to derive Δ. Formally, it corresponds to the following rule schema, where the multiset $P_1, \ldots, P_m$ (resp. $Q_1, \ldots, Q_n$) is abbreviated in $\overline{P}$ (resp. $\overline{Q}$).

$$\frac{Q_1, \Gamma \to \Delta \quad \dots \quad Q_n, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta} \text{ L-Reg}'$$

A dual schema is found if we start from the idea that if each P_i (together with Δ) can be derived from Γ then also each $\overline{Q}$ (together with Δ) can be derived from Γ.

$$\frac{\Gamma \to \Delta, P_1 \quad \dots \quad \Gamma \to \Delta, P_n}{\Gamma \to \Delta, \overline{Q}} \text{ R-Reg}'$$

In practice, regular rule schemata specialize in many rules, depending on the context of application. We shall give some example of theories that extend system **G3c** with the rules following the schema L-Reg$'$. We already considered the case of first order logic with equality, where the relation $=$ is reflexive. This corresponds to the axiom of reflexivity $\forall x(x = x)$ which is a special istance of REG, with $m = 0$ and $n = 1$. In the theory of strict linear order atoms are of the form $x < y$ and, unlike $=$, the relation $<$ is irreflexive. However, irreflexivity $\forall x \neg(x < x)$ is still a special case of a regular formula, with $m = 0$ and $n = 1$. Thus, the rule of irreflexivity of strict order is

$$\frac{}{x < x, \Gamma \to \Delta} \, Irref_{<}$$

The relation $\leqslant$ in the theory of linear order satisfies the property expressed by the axiom of linearity (or totality), $\forall x \forall y(x \leqslant y \vee y \leqslant x)$. Once again, linearity is a regular formula with $m = 0$ and $n = 2$. The corresponding rule is

$$\frac{x \leqslant y, \Gamma \to \Delta \quad y \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta} \, Lin_{\leqslant}$$

However, the schemata L-Reg$'$ and R-Reg$'$ do not satisfy the structural properties usually required to **G3**-systems and must be augmented in order to have contraction admissible. Once again, the method of Kleene (1952) is followed and the principal formulas in the the conclusion of L-Reg′ are repeated in the premises. The same holds for R-Reg′, *mutatis mutandis*, and the rule schemata take the form

$$\frac{Q_1, \overline{P}, \Gamma \to \Delta \quad \dots \quad Q_n, \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta} \text{ L-Reg}$$

$$\frac{\Gamma \to \Delta, \overline{Q}, P_1 \quad \dots \quad \Gamma \to \Delta, \overline{Q}, P_m}{\Gamma \to \Delta, \overline{Q}} \text{ R-Reg}$$

For the left schema, repetitions in the premises make left contraction commute with rules following the schema, whereas admissibility of right contraction is not problematic. This is reversed for the right rule schema. Moreover, it can happen that instantiation of labels in atoms produces a duplication, so two identical atoms are in the conclusion of a rule schema: $P_1, \dots, P_{m-2}, P, P, \Gamma \to \Delta$. As every formula in the conclusion is repeated in the premises, each premise has the duplication of P to ensure the admissibility of contraction. We must require that the rule with duplication P, P contracted into a single P is added to the system, that is, we impose that the system satisfies the following **closure condition**.

Proposition 3.2.1 (Closure Condition)**.** *Given a system with rule following the regular schema, if it has a rule where a substitution in the atoms produces a rule-instance of the form*

$$\frac{Q_1, P_1, \dots, P_{m-2}, P, P, \Gamma \to \Delta \quad \dots \quad Q_n, P_1, \dots, P_{m-2}, P, P, \Gamma \to \Delta}{P_1, \dots, P_{m-2}, P, P, \Gamma \to \Delta}$$

then it also contains the rule

$$\frac{Q_1, P_1, \dots, P_{m-2}, P, \Gamma \to \Delta \quad \dots \quad Q_n, P_1, \dots, P_{m-2}, P, \Gamma \to \Delta}{P_1, \dots, P_{m-2}, P, \Gamma \to \Delta}$$

Symmetrically, for the right rule schema, we have that a system containing a rule-

instance of the form

$$\frac{\Gamma \to \Delta, Q_1, \ldots, Q_{n-2}, Q, Q, P_1 \quad \Gamma \to \Delta, Q_1, \ldots, Q_{n-2}, Q, Q, P_m}{\Gamma \to \Delta, Q_1, \ldots, Q_{n-2}, Q, Q}$$

it also contains the rule

$$\frac{\Gamma \to \Delta, Q_1, \ldots, Q_{n-2}, Q, P_1 \quad \Gamma \to \Delta, Q_1, \ldots, Q_{n-2}, Q, P_m}{\Gamma \to \Delta, Q_1, \ldots, Q_{n-2}, Q}$$

The condition is not problematic, since the number of rules to be added to a given system is finite and often the closure condition is even superfluous, because the contracted rule is already a rule of the system or admissible in it. It is clear that universal axioms are derivable from the schema L-Reg (R-Reg), and that the schema L-Reg (resp. R-Reg) is derivable from the corresponding axiom, using cut. A detailed proof of admissibility of the structural rules and cut in the presence of rules following the schema R-Reg or L-Reg can be found in Negri and von Plato 2001, pp. 131–34.

By the same method, it is possible to convert into rules also existential axioms, or, more generally, axioms of the form of **geometric implications**. These are universal closures of implications $A \supset B$ in which A and B do not contain implications or universal quantifiers. Geometric implications can be turned in a useful normal form that consists of conjunctions of formulas of the form

$$\forall \overline{x}(P_1 \wedge \cdots \wedge P_m \supset \exists \overline{y}_1 M_1 \vee \cdots \vee \exists \overline{y}_n M_n) \qquad \text{GEOM}$$

In GEOM, each P_i is an atomic formula, each M_j a conjunction of a list of atomic formulas $\overline{Q}_j$, and none of the variables in the vectors $\overline{y}_j$ are free in P_i. In turn, each of these formulas can be turned into an inference rule of the following form:

$$\frac{\overline{Q}_1(\overline{z}_1/\overline{y}_1), \overline{P}, \Gamma \to \Delta \quad \dots \quad \overline{Q}_n(\overline{z}_n/\overline{y}_n), \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta} \text{ L-Gen}$$

The variables $\overline{y}_i$ are called the replaced variables of the schema, and the variables $\overline{z}_i$ the proper variables, or *eigenvariables*. In what follows, we shall consider for ease of notation the case in which the vectors of variables $\overline{y}_i$ consist of a single variable. The geometric rule schema is subject to the condition that the eigenvariables must not be free in the conclusion of the rule, $\overline{P}, \Gamma, \Delta$. In this way the rule expresses in a logic-free way the role of the existential quantifier in a geometric axiom. Cut elimination still holds in presence of rules following the general rule schema and a detailed proof can be found in Negri (2003). An example of geometric theory is Robinson arithmetic in which the induction schema of PA in replaced by a weaker axiom: every number is either zero or it is a successor of some number, that is, $\forall x(x = 0 \vee \exists y(x = s(y)))$. Robison's axiom is a special case of GEOM, with $m = 0$ and gets converted into

$$\frac{x = 0, \Gamma \to \Delta \quad x = s(y), \Gamma \to \Delta}{\Gamma \to \Delta} \text{ RA}$$

The rule must meet the variable condition that y must not appear in the conclusion. Geometric formulas arise naturally also in modal logic. For instance, the property of directedness of R, that is, $\forall x \forall y \forall z(xRy \wedge xRz \supset \exists w(yRw \wedge zRw))$, is a special case of GEOM with $m = 2$ and $n = 1$. The corresponding rule *Dir* must meet the condition that w is the *eigenvariable* and it cannot appear in the conclusion

$$\frac{yRw, zRw, xRy, xRz, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta} \textit{ Dir}$$

3.3 From frame properties to sequent rules

In the light of the method of axioms-as-rules, modal logic is viewed as a mathematical theory of the accessibility relation R and its axioms as the standard frame conditions. In this perspective, reflexivity of R can be converted to a left (right) rule following the schema L-Reg (resp. R-Reg).

$$\frac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}\ \textit{L-Ref} \qquad\qquad \frac{}{\Gamma \to \Delta, xRx}\ \textit{R-Ref}$$

Next, the modal axioms of the various axiomatic systems become theorems of the corresponding sequent calculus. Thus, in the presence of $L\text{-}Ref$ ($R\text{-}Ref$) the axiom `T` is derivable. Observe that when the mathematical rules follow R-Reg, the modal rules with relation atoms xRy in the succedent are needed, thus we shall use also $L\Box 0$ and $R\Diamond 0$ of **G0K**, accordingly:

$$\cfrac{\cfrac{\cfrac{x : A, xRx, x : \Box A \to x : A}{xRx, x : \Box A \to x : A}\ L\Box}{x : \Box A \to x : A}\ \textit{L-Ref}}{\to x : \Box A \supset A}\ R{\supset} \qquad\qquad \cfrac{\cfrac{\cfrac{}{\to xRx}\ \textit{R-Ref} \quad x : A \to x : A}{x : \Box A \to x : A}\ L\Box 0}{\to x : \Box A \supset A}\ R{\supset}$$

In the same way, transitivity R corresponds to the following left (right) rule,

$$\frac{xRz, xRy, yRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}\ \textit{L-Trans} \qquad\qquad \frac{\Gamma \to \Delta, xRz, xRy \quad \Gamma \to \Delta, xRz, yRz}{\Gamma \to \Delta, xRz}\ \textit{R-Trans}$$

And the axiom `4` has the following derivation by using $L\text{-}Trans$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{z : A, xRz, xRy, yRz, x : \Box A \to z : A}{xRz, xRy, yRz, x : \Box A \to z : A}\ L\Box}{xRy, yRz, x : \Box A \to z : A}\ \textit{L-Trans}}{xRy, x : \Box A \to y : \Box A}\ R\Box}{x : \Box A \to x : \Box\Box A}\ R\Box}{\to x : \Box A \supset \Box\Box A}\ R{\supset}$$

Otherwise, by $R\text{-}Trans$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{xRy, yRz \to xRz, xRy \quad xRy, yRz \to xRz, yRz}{xRy, yRz \to xRz}\,R\text{-}Trans \qquad z : A \to z : A}{xRy, yRz, x : \Box A \to z : A}\,L\Box 0}{xRy, x : \Box A \to y : \Box A}\,R\Box}{x : \Box A \to x : \Box\Box A}\,R\Box}{\to x : \Box A \supset \Box\Box A}\,R\supset
$$

Finally, symmetry of R gets converted into

$$
\dfrac{yRx, xRy, \Gamma \to \Delta}{xRy, \Gamma \to \Delta}\,L\text{-}Sym \qquad \dfrac{\Gamma \to \Delta, xRy, yRx}{\Gamma \to \Delta, xRy}\,R\text{-}Sym
$$

and axiom B has the following derivations

$$
\dfrac{\dfrac{\dfrac{\dfrac{yRx, xRy, x : A \to y : \Diamond A, x : A}{yRx, xRy, x : A \to y : \Diamond A}\,R\Diamond}{xRy, x : A \to y : \Diamond A}\,L\text{-}Sym}{x : A \to x : \Box\Diamond A}\,R\Box}{\to x : A \supset \Box\Diamond A}\,R\supset \qquad \dfrac{\dfrac{\dfrac{\dfrac{xRy \to yRx, xRy}{xRy \to yRx}\,R\text{-}Sym \qquad x : A \to x : A}{xRy, x : A \to y : \Diamond A}\,R\Diamond 0}{x : A \to x : \Box\Diamond A}\,R\Box}{\to x : A \supset \Box\Diamond A}\,R\supset
$$

Note that *L-Ref*, *L-Trans* and *L-Sym* (*R-Ref*, *R-Tran* and *R-Sym*), when added to **G3K**, give a system equivalent to **S5**. In fact, **S5** is sound and complete with respect to the class of reflexive, transitive, and symmetric frames. Equivalently, **S5** is characterized by the class of reflexive and euclidean frames. Therefore, we can drop *L-Trans* and *L-Sym* (resp. *R-Trans* and *R-Sym*), and add the rules corresponding to the property of euclideaness, $\forall x \forall y \forall z (xRy \wedge xRz \supset yRz)$

$$
\dfrac{yRz, xRy, xRz, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}\,L\text{-}Eucl \qquad \dfrac{\Gamma \to \Delta, yRz, xRy \quad \Gamma \to \Delta, yRz, xRz}{\Gamma \to \Delta, yRz}\,R\text{-}Eucl
$$

Axiom 5, can be derived by *L-Eucl*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{yRz, xRy, xRz, z : A \to y : \Diamond A, z : A}{yRz, xRy, xRz, z : A \to y : \Diamond A}\,R\Diamond}{xRy, xRz, z : A \to y : \Diamond A}\,L\text{-}Eucl}{xRy, x : \Diamond A \to y : \Diamond A}\,L\Diamond}{x : \Diamond A \to x : \Box\Diamond A}\,R\Box}{\to x : \Diamond A \supset \Box\Diamond A}\,R\supset
$$

or, equivalently, by R-*Eucl*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{xRy, xRz, \to yRz, xRy \quad xRy, xRz, \to yRz, xRz}{xRy, xRz, \to yRz}\,{\scriptstyle R\text{-}Eucl} \qquad z : A \to z : A}{xRy, xRz, z : A \to y : \Diamond A}\,{\scriptstyle R\Diamond 0}}{xRy, x : \Diamond A \to y : \Diamond A}\,{\scriptstyle L\Diamond}}{x : \Diamond A \to x : \Box\Diamond A}\,{\scriptstyle R\Box}}{\to x : \Diamond A \supset \Box\Diamond A}\,{\scriptstyle R\supset}$$

Besides the property of directedness taken into account in the previous section, another important geometric formula in modal logic is expressed by the seriality of R, that is, $\forall x \exists y (xRy)$, a special case of GEOM with $m = 0$ and $n = 1$. The role of existential quantifier is reflected by the variable condition that the label y is not in Γ, Δ.

$$\dfrac{xRy, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle L\text{-}Ser} \qquad\qquad \dfrac{}{\Gamma \to \Delta, xRy}\,{\scriptstyle R\text{-}Ser}$$

The axiom $\Box A \supset \Diamond A$ corresponding to seriality is known in the literature as D, from deontic logic. With the new rules it has the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A, xRy, x : \Box A \to x : \Diamond A, y : A}{xRy, x : \Box A \to x : \Diamond A, y : A}\,{\scriptstyle L\Box}}{xRy, x : \Box A \to x : \Diamond A}\,{\scriptstyle R\Diamond}}{x : \Box A \to x : \Diamond A}\,{\scriptstyle L\text{-}Ser}}{\to x : \Box A \supset \Diamond A}\,{\scriptstyle R\supset} \qquad \dfrac{\dfrac{\dfrac{}{\to xRy}\,{\scriptstyle R\text{-}Ser} \quad \dfrac{\dfrac{}{\to xRy}\,{\scriptstyle R\text{-}Ser} \quad y : A \to y : A}{x : \Box A \to y : A}\,{\scriptstyle L\Box 0}}{x : \Box A \to x : \Diamond A}\,{\scriptstyle R\Diamond 0}}{\to x : \Box A \supset \Diamond A}\,{\scriptstyle R\supset}$$

In the table below some well-known modal logic with its characteristic axioms and frame properties is presented

Name	Axiom	Frame property
T	$\Box A \supset A$	$\forall x(xRx)$
4	$\Box A \supset \Box\Box A$	$\forall x \forall y \forall z(xRy \wedge yRz \supset xRz)$
5	$\Diamond A \supset \Box\Diamond A$	$\forall x \forall y \forall z(xRy \wedge xRz \supset yRz)$
B	$A \supset \Box\Diamond A$	$\forall x \forall y(xRy \supset yRx)$
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall x \forall y \forall z(xRy \wedge xRz \supset zRy)$
D	$\Box A \supset \Diamond A$	$\forall x \exists y(xRy)$
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall x \forall y \forall z(xRy \wedge xRz \supset \exists w(yRw \wedge zRw))$

The corresponding Gentzen systems are obtained by adding combinations of the above rules to **G3K**.

G3T $=$ **G3K** $+$ *Ref*

G3K4 $=$ **G3K** $+$ *Trans*

G3B $=$ **G3T** $+$ *Sym*

G3S4 $=$ **G3T** $+$ *Trans*

G3S5 $=$ **G3K** $+$ *Ref* $+$ *Trans* $+$ *Sym* $=$ **G3K** $+$ *Ref* $+$ *Eucl*

G3D $=$ **G3K** $+$ *Ser*

3.4 Multi-modal epistemic logic

Starting from the cut-free calculus **G3K** we find a system for epistemic logic **G3Kn**. As we already said in the introduction, the language of alethic modal logic has a single modal operator $\Box$, whereas epistemic logic has one knowledge K operator for each agent a in a given set of agents $\mathcal{A}$. Formulas such as $\mathsf{K}_a A$ have to be read: "agent a knows that A". Consequently, in the corresponding epistemic frame we have as many accessibility relations R as agents in $\mathcal{A}$. Therefore, $x\mathsf{R}_a y$ says that the agent a can reach the state y from x, or y is R_a-accessible from x. In order to formally describe the properties of knowledge, each R_a is also assumed to be an equivalence relation, that is, it is reflexive, transitive and symmetrical. The knowledge formulas are evaluated as follows

$$x \Vdash \mathsf{K}_a B \quad \textit{if and only if} \quad \textit{for all } y,\ x\mathsf{R}_a y \textit{ implies } y \Vdash B$$

Consequently, **G3Kn** has the following rules for the K_a operator

Logical rules of **G3Kn**

$$\frac{y : A, x : \mathsf{K}_a A, xR_a y, \Gamma \to \Delta}{x : \mathsf{K}_a A, xR_a y, \Gamma \to \Delta}\, L\mathsf{K} \qquad \frac{xR_a y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathsf{K}_a A}\, R\mathsf{K}$$

with the usual restriction on $R\mathsf{K}$ that y must not appear in the conclusion. The first order conditions on R_a, that is, reflexivity, transitivity and symmetry of R_a, are regular formula and so can be converted into mathematical rules following the regular rule schema. The difference with respect to mono-modal system **G3K** is that here we have as many accessibility relations as agents in $\mathcal{A}$, with the following rules:

Mathematical rules of **G3Kn**

$$\frac{xR_a x, \Gamma \to \Delta}{\Gamma \to \Delta}\, Ref_a \qquad \frac{yR_a x, xR_a y, \Gamma \to \Delta}{xR_a y, \Gamma \to \Delta}\, Sym_a$$

$$\frac{xR_a z, xR_a y, yR_a z, \Gamma \to \Delta}{xR_a y, yR_a z, \Gamma \to \Delta}\, Trans_a$$

Starting from **G3Kn** we can also obtain a sequent system for belief. Belief is weaker than knowledge as it does not satisfy axiom `T`. Intuitively, it is natural to think that something is believed but not true, and thus, in contrast with knowledge, believing that something is true does not imply that it is indeed true. However, our beliefs are at least coherent and the only requirement that belief is suppose to satisfy is that contradictions are not believed. Thus, to obtain a system for belief we first replace the knowledge operators K_a with belief operators B_a, one for each agent a in $\mathcal{A}$, and we read a formula $\mathsf{B}_a A$ as "the agent a believes that A". Then, the rules for B_a are obvious:

Logical rules of **G3Be**

$$\frac{y : A, x : \mathsf{B}_a A, xR_a y, \Gamma \to \Delta}{x : \mathsf{B}_a A, xR_a y, \Gamma \to \Delta}\; L\mathsf{B} \qquad \frac{xR_a y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathsf{B}_a A}\; R\mathsf{B}$$

The axiomatic system for belief **G3D** is obtained from the multi-modal basic epistemic logic by adding the axiom $\neg\mathsf{B}_a(A \wedge \neg A)$. The system **D** is sound and complete with respect to the class of frames in which each accessibility relation R_a is serial. Therefore, a labelled sequent system **G3Be** is found by adding to **G3K** the geometric rule of seriality. Rule Ser_a must meet the condition that the label y does not appear in the conclusion.

Logical rule of **G3Be**

$$\frac{xR_a y, \Gamma \to \Delta}{\Gamma \to \Delta}\; Ser_a$$

In the next section we show the admissibility of all the structural rules and cut elimination for the systems **G3Kn** and **G3Be**.

Admissibility of the structural rules

In this section we show that in the systems **G3Kn** and **G3Be** all the structural rules are admissible and cut is eliminable. Often, the proofs are straightforward extensions of that of **G3K** and we shall refer to previous results.

Lemma 3.4.1. *In* **G3Kn** *and* **G3Be** *it holds that*

$$\vdash x : A, \Gamma \xrightarrow[0]{2\cdot h(A)} \Delta, x : A$$

for every labelled formula $x : A$.

Proof. See Lemma 2.1.1. ⊠

Lemma 3.4.2. *The* substitution of labels *is height- and rank-preserving admissible in* **G3Kn** *and* **G3Be**, i.e.

$$\textit{If} \quad \vdash \Gamma \xrightarrow[p]{n} \Delta \quad \textit{then} \quad \vdash \Gamma[y/x] \xrightarrow[p]{n} \Delta[y/x]$$

for every label x and y.

Proof. By induction on n. The proof is the same as Lemma 2.1.3 up to the case in which $\Gamma \rightarrow \Delta$ is concluded by a mathematical rule. All the mathematical rules of **G3Kn** follow the regular rule schema and the proof proceeds as for the propositional case: apply IH on the premise of the mathematical rule R that concluded $\Gamma \rightarrow \Delta$ and then another application of R gives the conclusion. In **G3Be** we need to be careful with the variable condition because Ser_a follows the geometric rule schema. The problematic case arises when the y is the *eigenvariable* of Ser_a.

$$\cfrac{\begin{array}{c}\vdots\\ xR_a y, \Gamma \xrightarrow{k} \Delta\end{array}}{\Gamma \xrightarrow{k+1} \Delta}\ Ser_a$$

We first replace y with a new z in order to have $\vdash xR_az, \Gamma \xrightarrow{k} \Delta$ and next apply IH in order to conclude $\vdash xR_az[y/x], \Gamma[y/x] \xrightarrow{k} \Delta[y/x]$. Finally, given that z appears only in yR_az, Ser_a can be correctly applied. ☒

Theorem 3.4.3. Weakening *is height- and rank-preserving admissible in* **G3Kn** *and* **G3Be***,* i.e.

i) *If* $\vdash \Gamma \xrightarrow[p]{n} \Delta$ *then* $\vdash x : A, \Gamma \xrightarrow[p]{n} \Delta$

ii) *If* $\vdash \Gamma \xrightarrow[p]{n} \Delta$ *then* $\vdash \Gamma \xrightarrow[p]{n} \Delta, x : A$

iii) *If* $\vdash \Gamma \xrightarrow[p]{n} \Delta$ *then* $\vdash xR_ay, \Gamma \xrightarrow[p]{n} \Delta$

Proof. By induction on n. Also in this case, the proof follows that of Theorem 2.1.4. The cases in which $\Gamma \to \Delta$ is concluded by mathematical rules of **G3Kn** are dealt with analogously to the propositional ones. When $\Gamma \to \Delta$ is concluded by Ser_a in **G3Be**, we apply Lemma 3.4.2 in order to avoid label clash, IH, and then Ser_a again. ☒

Lemma 3.4.4. *All the rules of* **G3Kn** *and* **G3Be** *are height- and rank-preserving invertible.*

Proof. The proof is straightforward because the mathematical rules have the repetition of the principal formulas in the premise, so the premise can be obtained from the conclusion by Theorem 3.4.3. ☒

Theorem 3.4.5. Contraction *is height- and rank-preserving admissible in* **G3Kn** *and* **G3Be***,* i.e.

i) *If* $\vdash x : A, x : A, \Gamma \xrightarrow[p]{n} \Delta$ *then* $\vdash x : A, \Gamma \xrightarrow[p]{n} \Delta$

ii) *If* $\vdash \Gamma \xrightarrow[p]{n} \Delta, x : A, x : A$ *then* $\vdash \Gamma \xrightarrow[p]{n} \Delta, x : A$

iii) *If* $\vdash xR_ay, xR_ay, \Gamma \xrightarrow[p]{n} \Delta$ *then* $\vdash xR_ay, \Gamma \xrightarrow[p]{n} \Delta$

Proof. By simultaneous induction on n. For the cases i and ii the proof is an immediate extension of Theorem 2.1.6. Consider now the case iii. If $xR_ay, xR_ay, \Gamma \rightarrow \Delta$ is initial or conclusion of $L\bot$ then none of the occurrences of xR_ay is principal and therefore also $xR_ay, \Gamma \rightarrow \Delta$ is initial or conclusion of $L\bot$. Else, if $xR_ay, xR_ay, \Gamma \rightarrow \Delta$ is concluded by a rule R of **G3Kn** and none of the occurrences of xRy is principal then IH is applied to the premise(s) and a derivation of the conclusion is found by R again. The same holds if R is a rule of **G3Be**. If one of the occurrences of xR_ay is principal then the derivation is concluded by a mathematical rule. This is the case that requires the repetition of the principal formulas in the premise. In fact, suppose that $xR_ay, xR_ay, \Gamma \rightarrow \Delta$ is concluded by Sym_a and one of the occurrences of xR_ay is principal

$$\dfrac{\begin{array}{c}\vdots\\ yR_ax, xR_ay, xR_ay, \Gamma \xrightarrow{k} \Delta\end{array}}{xR_ay, xR_ay, \Gamma \xrightarrow{k+1} \Delta}\ {\scriptstyle Sym_a}$$

Given that both the occurrences of xR_ay are in the premise, IH can be applied and the conclusion is obtained by Sym_a. However, there is another case to deal with: it may happen that both the occurrences of xR_ay are principal formulas of $Trans_a$. This is possible when both y and z are are one and the same variable.

$$\dfrac{xR_ax, xR_ax, xR_ax, \Gamma \rightarrow \Delta}{xR_ax, xR_ax, \Gamma \rightarrow \Delta}\ {\scriptstyle Trans_a}$$

In this case, IH can be applied twice and the conclusion follows. Else, $xR_ax, \Gamma \rightarrow \Delta$ can be derived by IH and Ref_a. Else, note that $Trans_a$ is subject to the closure condition and therefore if **G3Kn** contains $Trans_a$ it must contain also its contracted istance

$$\dfrac{xR_ax, xR_ax, \Gamma \rightarrow \Delta}{xR_ax, \Gamma \rightarrow \Delta}\ {\scriptstyle Trans_a}$$

$\boxtimes$

The latter case is an example of a general result concerning the role of the closure condition in labelled systems: instances of the closure condition that are just like contractions on relational atoms need not be added because they are admissible. More precisely, when R is a frame rule and $R^\star$ is the corresponding contracted instance that arises from the closure condition, it is possible to prove that if $R^\star$ is an instance of contraction, it is hp-admissible in the system extended with those rules arising from the closure condition that are not instances of contraction. The result has been proved for a labelled system for the logic of group acceptance in Hakli and Negri (2011a).

Cut elimination

The proof of the cut-elimination theorem for **G3K** can be extended to systems with mathematical rules following the regular and geometric rule schema. The proof with all the details can be found in Negri (2005). Here as a case study, we prove that the systems **G3Kn** and **G3Be** satisfy cut elimination. In addition we show that the upper bound on the growth of cut free derivations is maintained in these systems.

Lemma 3.4.6 (Main Lemma)**.** *Let d_1 and d_2 be two derivations in* $\mathbf{G3Kn}^{\mathrm{C}}$ *and* $\mathbf{G3Be}^{\mathrm{C}}$ *such that*

$$d_1 \vdash \Gamma \xrightarrow[p]{n} \Delta, x : C \quad \textit{and} \quad d_2 \vdash x : C, \Gamma \xrightarrow[p]{m} \Delta$$

and let $h(x : C) = p$. Then there is a derivation d in $\mathbf{G3Kn}^{\mathrm{C}}$ *and* $\mathbf{G3Be}^{\mathrm{C}}$ *such that*

$$d \vdash \Gamma \xrightarrow[p]{n+m} \Delta$$

Proof. By induction on $n+m$. The proof is to a large extent similar to the proof of Main Lemma 2.2.2 for **G3K**. In addition, we have to consider the cases in which either d_1 or d_2 is the conclusion of a mathematical rule following the regular (**G3Kn**$^{\mathrm{C}}$) or the geometric rule schema (**G3Be**$^{\mathrm{C}}$). Observe that when at least one of d_1, d_2 is initial or conclusion of $L\bot$, the proof is the same as in Main Lemma 2.2.2 because initial sequents and conclusion of $L\bot$ do not have relational atoms xR_ay as principal. If d_1 is neither an initial sequent nor a conclusion of $L\bot$, we first give the proof for **G3Kn**$^{\mathrm{C}}$. Suppose that d_1 is concluded by either Ref_a, or $Trans_a$, or else Sym_a. Therefore $x : C$ is not principal and d_1 is

$$\dfrac{\begin{array}{c}\vdots\\ uRu, \Gamma \overset{n-1}{\underset{p}{\longrightarrow}} \Delta, x : C\end{array}}{\Gamma \overset{n}{\underset{p}{\rightarrow}} \Delta, x : C}\,Ref_a \qquad \dfrac{\begin{array}{c}\vdots\\ uRw, uRv, vRw, \Gamma' \overset{n-1}{\underset{p}{\longrightarrow}} \Delta, x : C\end{array}}{uRv, vRw, \Gamma' \overset{n}{\underset{p}{\rightarrow}} \Delta, x : C}\,Trans_a$$

$$\dfrac{\begin{array}{c}\vdots\\ vRu, uRv, \Gamma' \overset{n-1}{\underset{p}{\longrightarrow}} \Delta, x : C\end{array}}{uRv, \Gamma' \overset{n}{\underset{p}{\rightarrow}} \Delta, x : C}\,Sym_a$$

The conclusion d is found from d_2 by admissibility of weakening (Theorem 3.4.3) and IH as follows. For Ref_a,

$$\dfrac{\dfrac{uRu, \Gamma \overset{n-1}{\underset{p}{\longrightarrow}} \Delta, x : C \qquad \dfrac{x : C, \Gamma \overset{m}{\underset{p}{\rightarrow}} \Delta}{x : C, uRu, \Gamma \overset{m}{\underset{p}{\rightarrow}} \Delta}\,\text{L-W}}{uRu, \Gamma \overset{(n-1)+m}{\underset{p}{\longrightarrow}} \Delta}\,\text{IH}}{\Gamma \overset{n+m}{\underset{p}{\longrightarrow}} \Delta}\,Ref_a$$

Similarly, for $Trans_a$ and Sym_a,

$$\dfrac{uRw, uRv, vRw, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C \qquad \dfrac{x : C, uRv, vRw, \Gamma' \xrightarrow[p]{m} \Delta}{x : C, uRw, uRv, vRw, \Gamma' \xrightarrow[p]{m} \Delta}\,\text{L-W}}{\dfrac{uRw, uRv, vRw, \Gamma'\Gamma \xrightarrow[p]{(n-1)+m} \Delta}{uRv, vRw, \Gamma' \xrightarrow[p]{n+m} \Delta}\,\mathit{Trans}_a}\,\text{IH}$$

$$\dfrac{vRu, uRv, \Gamma' \xrightarrow[p]{n-1} \Delta, x : C \qquad \dfrac{x : C, uRv, \Gamma' \xrightarrow[p]{m} \Delta}{x : C, vRu, uRv, \Gamma' \xrightarrow[p]{m} \Delta}\,\text{L-W}}{\dfrac{vRu, uRv, \Gamma'\Gamma \xrightarrow[p]{(n-1)+m} \Delta}{uRv, \Gamma' \xrightarrow[p]{n+m} \Delta}\,\mathit{Sym}_a}\,\text{IH}$$

For **G3Be**$^{\mathrm{C}}$ the proof is analogous. If d_1 is concluded by Ser_a,

$$\dfrac{\begin{matrix}\vdots \\ uRv, \Gamma \xrightarrow[p]{n-1} \Delta, x : C\end{matrix}}{\Gamma \xrightarrow[p]{n} \Delta, x : C}\,\mathit{Ser}_a$$

where v does not appear in the conclusion of Ser_a, a derivation d of the conclusion is obtained from d_2, Theorem 2.1.4 and IH as follows. Note that in this case there is no need to change the *eigenvariable* v of Ser_a because v does not appear in d_2 too:

$$\dfrac{uRv, \Gamma \xrightarrow[p]{n-1} \Delta, x : C \qquad \dfrac{x : C, \Gamma \xrightarrow[p]{m} \Delta}{x : C, uRv, \Gamma \xrightarrow[p]{m} \Delta}\,\text{L-W}}{\dfrac{uRv, \Gamma \xrightarrow[p]{(n-1)+m} \Delta}{\Gamma \xrightarrow[p]{n+m} \Delta}\,\mathit{Ser}_a}\,\text{IH}$$

The case in which d_2 is concluded by a mathematical rule of **G3Kn** or **G3Be** and the cut formula is not principal is analogous. The case in which both occurrences of the cut formula are principal in a mathematical rule of $\mathbf{G3Kn}^{\mathrm{C}}$ or $\mathbf{G3Be}^{\mathrm{C}}$ simply does not arise because no relational atom can occur in the succedent.

$\boxtimes$

Lemma 3.4.7 (Rank Reduction). *Every derivation d in $\mathbf{G3Kn}^{\mathrm{C}}$ and $\mathbf{G3Be}^{\mathrm{C}}$ such that*

$$d \vdash \Gamma \xrightarrow[p+1]{m} \Delta$$

can be converted into a derivation d^* such that

$$d^* \vdash \Gamma \xrightarrow[p]{2^m} \Delta$$

Proof. By induction on m. The proof extends the proof of Reduction Lemma 2.2.3 with new cases corresponding to mathematical rules. If $m = 0$ then d is an initial sequent or conclusion of $L\bot$ and the claim holds as above. If $m = k+1$ we argue by distinction of cases according to the last rule R of d. If R is a mathematical rule of $\mathbf{G3Kn}^{\mathrm{C}}$ or $\mathbf{G3Be}^{\mathrm{C}}$ then IH is applied to its premise and then another application of R gives the conclusion. For instance, if

$$\dfrac{\begin{array}{c}\vdots\\ xR_ax, \Gamma \xrightarrow[p+1]{k} \Delta\end{array}}{\Gamma \xrightarrow[p+1]{k+1} \Delta}\ \mathit{Ref}_a$$

by IH on the premise of Ref_a, we find that $xR_ax, \Gamma \xrightarrow[p]{2^k} \Delta$ and, by Ref_a, $\Gamma \xrightarrow[p]{2^{k+1}} \Delta$ since $2^k + 1 \leqslant 2^{k+1}$. Other mathematical rules of $\mathbf{G3Kn}^{\mathrm{C}}$ or $\mathbf{G3Be}^{\mathrm{C}}$ are analogously dealt with.

$\boxtimes$

Theorem 3.4.8 (Cut elimination)**.** *In* **G3Kn**$^{\mathrm{C}}$ *and in* **G3Be**$^{\mathrm{C}}$ *cut is eliminable,* i.e. *every derivation d such that*

$$d \vdash \Gamma \xrightarrow[p]{n} \Delta$$

can be converted into a derivation d^- such that

$$d^- \vdash \Gamma \xrightarrow[0]{2_p(n)} \Delta$$

Proof. By induction on p. The proof is the same of Theorem 2.2.4 of **G3K**$^{\mathrm{C}}$. $\boxtimes$

Completeness

In this section we shall prove the completeness of the systems **G3Kn** and **G3Be** through the equivalence with their corresponding Hilbert-style systems. In particular, we prove that all the axioms are derivable and all the rules are admissible. As a sound and complete axiomatization consider the system **Kn** of Fagin *et al.* (1995).

A1	All the axioms of modal logic **K**	
A2	$\mathsf{K}_a A \supset A$	Factual Knowledge
A3	$\mathsf{K}_a A \supset \mathsf{K}_a \mathsf{K}_a A$	Positive Introspection
A4	$\neg \mathsf{K}_a A \supset \mathsf{K}_a \neg \mathsf{K}_a A$	Negative Introspection
R1	From $\Gamma \vdash A \supset B$ and $\Delta \vdash A$ infer $\Gamma, \Delta \vdash B$	Modus Ponens
R2	From $\vdash A$ infer $\vdash \mathsf{K}_a A$	Necessitation

An adequate system for belief **Be** can be easily obtained from the basic modal system **K** together with the axiom of consistency for the belief operator B_a.

A1 All the axioms of modal logic **K**

A2 $\neg \mathsf{B}_a(A \wedge \neg A)$ Consistency

R1 From $\Gamma \vdash A \supset B$ and $\Delta \vdash A$ infer $\Gamma, \Delta \vdash B$ Modus Ponens

R2 From $\vdash A$ infer $\vdash \mathsf{B}_a A$ Necessitation

Lemma 3.4.9. *All the axioms (rules) of* **Kn** *are derivable (resp. admissible) in* **G3Kn**.

Proof. By root-first proof search from the sequent to be derived. For the derivations of **K** axioms see Lemma 2.3.1. For factivity of knowledge (A2), the derivation is

$$\dfrac{\dfrac{\dfrac{x : A, xR_a x, x : \mathsf{K}_a A \to x : A}{xR_a x, x : \mathsf{K}_a A \to x : A}\,{\scriptstyle L\mathsf{K}}}{x : \mathsf{K}_a A \to x : A}\,{\scriptstyle Ref_a}}{\to x : \mathsf{K}_a A \supset A}\,{\scriptstyle R\supset}$$

where the topmost sequents are derivable by Lemma 3.4.1. Positive introspection (A3) has the following derivations.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : A, xR_a z, xR_a y, yR_a z, x : \mathsf{K}_a A \to z : A}{xR_a z, xR_a y, yR_a z, x : \mathsf{K}_a A \to z : A}\,{\scriptstyle L\mathsf{K}}}{xR_a y, yR_a z, x : \mathsf{K}_a A \to z : A}\,{\scriptstyle Trans_a}}{xRy, x : \mathsf{K}_a A \to y : \mathsf{K}_a A}\,{\scriptstyle R\mathsf{K}}}{x : \mathsf{K}_a A \to x : \mathsf{K}_a \mathsf{K}_a A}\,{\scriptstyle R\mathsf{K}}}{\to x : \mathsf{K}_a A \supset \mathsf{K}_a \mathsf{K}_a A}\,{\scriptstyle R\supset}$$

Finally, for negative introspection (A4) we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : A, z : \mathsf{K}_a A, zR_a y, zR_a x, xR_a y, xR_a z \to y : A}{z : \mathsf{K}_a A, zR_a y, zR_a x, xR_a y, xR_a z \to y : A}\,{\scriptstyle L\mathsf{K}}}{z : \mathsf{K}_a A, zR_a x, xR_a y, xR_a z \to y : A}\,{\scriptstyle Trans_a}}{z : \mathsf{K}_a A, xR_a y, xR_a z \to y : A}\,{\scriptstyle Sym_a}}{xR_a y, xR_a z \to y : A, z : \neg \mathsf{K}_a A}\,{\scriptstyle R\neg}}{xR_a y \to y : A, x : \mathsf{K}_a \neg \mathsf{K}_a A}\,{\scriptstyle R\mathsf{K}}}{\to x : \mathsf{K}_a A, x : \mathsf{K}_a \neg \mathsf{K}_a A}\,{\scriptstyle R\mathsf{K}}}{x : \neg \mathsf{K}_a A \to x : \mathsf{K}_a \neg \mathsf{K}_a A}\,{\scriptstyle L\neg}}{\to x : \neg \mathsf{K}_a A \supset \mathsf{K}_a \neg \mathsf{K}_a A}\,{\scriptstyle R\supset}$$

In either case the topmost sequents are derivable by Lemma 3.4.1. The admissibility of the **Kn** rules in **G3Kn** is proved as in Lemma 2.3.1. ☒

Lemma 3.4.10. *All the axioms (rules) of* **Be** *are derivable (resp. admissible) in* **G3Be**.

Proof. The proof reduces to find a derivation of axiom A2 as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : A, xR_a y, x : \mathsf{B}_a(A \wedge \neg A) \rightarrow y : A}{y : A, y : \neg A, xR_a y, x : \mathsf{B}_a(A \wedge \neg A) \rightarrow}\,L\neg}{y : A \wedge \neg A, xR_a y, x : \mathsf{B}_a(A \wedge \neg A) \rightarrow}\,L\wedge}{xR_a y, x : \mathsf{B}_a(A \wedge \neg A) \rightarrow}\,L\mathsf{B}}{x : \mathsf{B}_a(A \wedge \neg A) \rightarrow}\,Ser_a}{\rightarrow x : \neg \mathsf{B}_a(A \wedge \neg A)}\,R\neg$$

where the topmost sequents are derivable by Lemma 3.4.1. Again, for admissibility of **Be** rules, see Lemma 2.3.1.

☒

3.5 Intuitionistic Logic

It is well known that the semantics of **S4** can be used to provide a direct interpretation of the intuitionistic connectives, the intuitionistic implication being a $\square$-type modality (see Kripke 1965). The intuitionistic accessibility relation is denoted by $\leqslant$ and satisfies the properties of reflexivity and transitivity. Therefore, it is possible to internalize the semantics of intuitionistic implication into the the syntax of sequent calculus analogously to the internalization of the modal operator $\square$ in **G3S4**. In fact, the inductive definition of validity of implicative formulas is:

$$x \Vdash A \supset B \quad \textit{if and only if} \quad \textit{for all } y,\ x \leqslant y \textit{ and } y \Vdash A \textit{ implies } y \Vdash B$$

Along with the clauses for the other connectives, the definition can be converted into a pair of sequent rules with the condition that the label y must not appear in the conclusion of the right rule for implication. In addition, the forcing relation has to be proved monotone with respect to the relation $\leqslant$. That is, for any arbitrary formula A the following has to hold:

$$x \leqslant y \text{ and } x \Vdash A \quad \textit{implies} \quad y \Vdash A$$

It is enough to impose monotonicity of forcing, in the form of an initial sequent, with respect to only atomic formulas. This is not a restriction because full monotonicity is then shown derivable. In this way, one of the design principles of **G3**-style calculi, namely the restriction of initial sequents to atomic formulas needed to guarantee the full range of structural properties, is respected. The following labelled sequent calculus **G3I** for intuitionistic logic is thus obtained. As usual, negation is defined in terms of $\perp$ and $\supset$, the formulas P are atomic, and $y \notin \Gamma, \Delta$ in rule $R\supset$.

Rule of **G3I**

$$x \leqslant y, x : P, \Gamma \rightarrow \Delta, y : P$$

$$\frac{x : A, x : B, \Gamma \rightarrow \Delta}{x : A \wedge B, \Gamma \rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \rightarrow \Delta, x : A \quad \Gamma \rightarrow \Delta, x : B}{\Gamma \rightarrow \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, \Gamma \rightarrow \Delta \quad x : B, \Gamma \rightarrow \Delta}{x : A \vee B, \Gamma \rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \rightarrow \Delta, x : A, x : B}{\Gamma \rightarrow \Delta, x : A \vee B}\ R\vee$$

$$\frac{x \leqslant y, x : A \supset B, \Gamma \rightarrow \Delta, y : A \quad x \leqslant y, x : A \supset B, y : B, \Gamma \rightarrow \Delta}{x \leqslant y, x : A \supset B, \Gamma \rightarrow \Delta}\ L\supset$$

$$\frac{}{x : \perp, \Gamma \rightarrow \Delta}\ L\perp \qquad \frac{x \leqslant y, y : A, \Gamma \rightarrow \Delta, y : B}{\Gamma \rightarrow \Delta, x : A \supset B}\ R\supset$$

If $\leqslant$ is assumed to be reflexive and transitive, system **G3I** must contain also the following rules for $\leqslant$.

Mathematical rules of **G3I**

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta}\ Ref_{\leqslant} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta}\ Trans_{\leqslant}$$

Full monotonicity of forcing is obtained by the following:

Lemma 3.5.1. *In* **G3I** *it holds that*

i) $\vdash x \leqslant y, x : A, \Gamma \to \Delta, y : A$

ii) $\vdash x : A, \Gamma \to \Delta, x : A$

for every labelled formula $x : A$.

Proof. By simultaneous induction on the height h of A. The proof of ii is done at each step of the induction by $Ref_{\leqslant}$ and the inductive hypothesis of i. The proof of (i) is trivial for A atomic and for $\perp$, whereas it uses the inductive hypothesis of (ii) and $Trans_{\leqslant}$ if A is $B \supset C$:

$$\cfrac{\cfrac{\cfrac{\ldots, z : B, \Gamma \to \Delta, z : C, z : B \qquad \ldots, z : C, \Gamma \to \Delta, z : C}{x \leqslant z, x \leqslant y, y \leqslant z, x : B \supset C, z : B, \Gamma \to \Delta, z : C}\ L\supset}{x \leqslant y, y \leqslant z, x : B \supset C, z : B, \Gamma \to \Delta, z : C}\ Trans_{\leqslant}}{x \leqslant y, x : B \supset C, \Gamma \to \Delta, y : B \supset C}\ R\supset$$

The cases in which A is a conjunction or a disjunction are handled by the inductive hypothesis of i.

$\boxtimes$

System **G3I** enjoys all the structural properties usually required of sequent systems and the same holds for each extension **G3I*** with rules that follow the regular or the geometric rule schema.

Theorem 3.5.2. *In* **G3I*** *it holds that:*

i) All the logical rules are hp-invertible;

ii) The rules of weakening and contraction are hp-admissible;

iii) Cut is admissible.

Proof. See the proofs of Theorems 12.27–12.29 of Negri and von Plato (2011). ☒

For our purposes, the most remarkable extension of **G3I** is obtained by imposing symmetry of the accessibility relation

$$\frac{y \leqslant x, x \leqslant y, \Gamma \rightarrow \Delta}{x \leqslant y, \Gamma \rightarrow \Delta} \; Sym_{\leqslant}$$

This extension gives a system equivalent to classical logic and we shall refer to it as **G3C**. Given that **G3C** is an extension of **G3I** with a rule that follows the regular rule schema, it admits cut elimination by Theorem 3.5.2. Systems **G3I** and **G3C** will play an important role in the last chapter.

Chapter 4

Dynamic Epistemic Logics

This chapter takes into account one of the most prominent development of epistemic modal logic, the **dynamics of knowledge**. So far, we have been considering epistemic logic as a formal investigation concerning what agents statically know or believe, in the sense that our analysis did not take into account the possibility that knowledge might change. In fact, epistemic logic traditionally builds on a model of knowledge which is not formally able to cope with the fact that agents that are ignorant about a certain P can eventually learn it. However, it is natural to think that they indeed may come to know that P is the case, once for instance they are informed to this end by other agents. From this perspective, knowledge is strictly connected with the practice of communication, and agents' comprehension of the world depends not only on what they know, but also on what they eventually may come to know in the process of information flow.

4.1 Public Announcement Logic

Dynamic epistemic logic (DEL) provides a general account of the problem of knowledge change. However, DEL is a large family and not a single logic, and in this chapter we shall focus on the simplest type of DEL, the **logic of public an-**

nouncements (PAL). Along with the standard epistemic modal operators K_a for each agent a, and propositional connectives, the language of PAL has formulas for announcements $[A]B$, intuitively read as: "after every announcement of A, B". It is clear that there is a strong analogy between announcements in PAL and programs in dynamic logic of programs (PDL, see Harel 2000). In the latter, the modality is a $\Box$-like operator such that $[\alpha]B$ stands for: after every terminating execution of the program α, B. Public announcements are specific form of programs in which α is in turn a formula. The dual operation is $\langle\ \rangle$ and $\langle A\rangle B$ is defined as $\neg[A]\neg B$ and read as: after some announcements of A, B. Although this perspective on the PAL language is nowadays dominant, it is not the only one. PAL originated with the seminal work of Plaza (1989) in which announcements are formalized by the binary non truth-functional connective $+$ in such a way that a formula as $A + B$ means: "A is true and after announcing that A, also B is true". Despite $A + B$ and $\langle A\rangle B$ can be considered equivalent, they reveal two different perspectives: the latter notation takes the operator [ ] as a unary operator applying to formulas that are postconditions of the program execution, where [ ] is relative to some program α. The former is simply a binary connective as conjunction or disjunction, with the basic difference that it is truth-functional.

Formal semantics

The basic idea behind the formal semantics of PAL is that agents can gain new information by the public announcement of some (true) fact. The consequence of an announcement is the update of the agents' knowledge: agents rule out some situations that are not any longer considered as possible because incompatible with the announcement. We have said that the standard presentation of PAL in van Ditmarsch *et al.* (2007) arises from the seminal work of Plaza (1989): despite the binary notation $A + B$ employed there, Plaza's announcements are formulas such as $\langle A\rangle B$ which is true whenever A is true and after A is announced B is true. Therefore, the dual formula $[A]B$ is satisfied whenever if A is true then after A is announced B is true. Thus, in Plaza's interpretation (P-interpretation) of announcements a formula can be announced only if it is true and hence announcements are considered as a completely truthful resource of information. The truthfulness of announcements is formally expressed by the formula $(A \supset [A]B) \supset [A]B$ which is is a theorem of PAL when Plaza's interpretation of announcements is considered. Thus, "announcement" means "truthful and public announcement". However, this is not the whole picture and alternative interpretations are possible if we drop the requirement that what is announced must be true and allow that every formula can be announced, no matter what its truth-value is. In contrast with P-announcements, it may happen that the agents do not assume the truth of what is announced and could correctly exclude as impossible also the situation in which the announcement is made. This approach, proposed by Gerbrandy and Groenveled (1997), modifies the original perspective on truthful announcements due to Plaza (1989). For a clear and compact presentation of the Gerbrandy and Groenveled announcements (GG-announcements) see Bucheli *et al.* (2010). In what follows we present the formal semantics for both P- and GG-interpretations of PAL, even though we shall go into the details only of the GG-interpretation. In either case, the semantics of PAL is based on the semantics of epistemic logic and consists into a modification of the standard epistemic frames and models by means of an operation of state and arrow restriction.

Definition (P-Restricted Model). *Let A be a formula and $\mathfrak{M} = \{\mathsf{X}, \mathsf{R}_a, \Vdash\}$ an epistemic model. The* P-restriction *of $\mathfrak{M}$ to A is the model $\mathfrak{M}^A = \langle \mathsf{X}^A, \mathsf{R}_a^A, \Vdash^A\rangle$ where*

$\mathsf{X}^A = \{x \in \mathsf{X} \mid x \Vdash A\}$;

$\mathsf{R}_a^A = \{\langle x, y\rangle \in \mathsf{X}^A \times \mathsf{X}^A \mid x\mathsf{R}_a y\}$ *for every agent $a \in \mathcal{A}$;*

$\Vdash^A = \{\langle x, P\rangle \mid x \in \mathsf{X}^A \text{ and } x \Vdash P\}$ *for every atom P.*

Definition (GG-Restricted Model). *Let A be a formula and $\mathfrak{M} = \{\mathsf{X}, \mathsf{R}_a, \Vdash\}$ an epistemic model. The* GG-restriction *of $\mathfrak{M}$ to A is the model $\mathfrak{M}^A = \langle \mathsf{X}^A, \mathsf{R}^A_a, \Vdash^A \rangle$ where*

$\mathsf{X}^A \equiv \mathsf{X}$;

$\mathsf{R}^A_a = \{\langle x, y\rangle \in \mathsf{X} \times \mathsf{X} \mid x\mathsf{R}_a y \text{ and } y \Vdash A\}$ *for every agent $a \in \mathcal{A}$;*

$\Vdash^A = \Vdash$.

Observation. From the above definitions we have that in the P-restricted model $\mathfrak{M}^A$, for every $x, y \in \mathsf{X}$,

$x\mathsf{R}^A_a y$ *if and only if* $x\mathsf{R}_a y$ and $\mathfrak{M}, x \Vdash A$ *and* $\mathfrak{M}, y \Vdash A$;

$\mathfrak{M}^A, x \Vdash P$ *if and only if* $\mathfrak{M}, x \Vdash A$ *and* $\mathfrak{M}, x \Vdash P$.

On the other hand, when $\mathfrak{M}^A$ is a GG-restricted model, we have that for every $x, y \in \mathsf{X}$,

$x\mathsf{R}^A_a y$ *if and only if* $x\mathsf{R}_a y$ *and* $\mathfrak{M}, y \Vdash A$;

$\mathfrak{M}^A, x \Vdash P$ *if and only if* $\mathfrak{M}, x \Vdash P$.

Notational convention. In order to simplify the notation we shall write $x \Vdash^A P$ instead of $\mathfrak{M}^A, x \Vdash P$.

In both P- and GG-announcements the forcing relation $\Vdash$ is given on atomic formulas and it is extended in a unique way to arbitrary formulas. The clauses for the propositional connectives and knowledge are the standard ones, whereas there are two different ways to evaluate announcements. In the P-interpretation the clause assumes explicitly that what is announced is true.

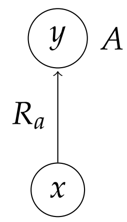
$x \Vdash [A]B$ *if and only if* $x \Vdash A$ *implies* $x \Vdash^A B$

Conversely, in GG-interpretation the assumption that A is true is left out.

$x \Vdash [A]B \quad$ *if and only if* $\quad x \Vdash^A B$

In order to understand the difference between P- and GG- announcements we use an example of a formula which is valid in the P-interpretation, but falsified in the GG-interpretation.

Example. Consider the formula $(A \supset [A]\mathsf{K}_a P) \supset [A]\mathsf{K}_a P$ and the following model: $A \supset [A]\mathsf{K}_a P$ holds but not $[A]\mathsf{K}_a P$, so the model is a countermodel to $(A \supset [A]\mathsf{K}_a P) \supset [A]\mathsf{K}_a P$.



In fact, $x \nVdash A$ and then $A \supset [A]\mathsf{K}_a P$ is trivially true at x. On the other hand, we have $x \nVdash [A]\mathsf{K}_a P$ if and only if $x \nVdash^A \mathsf{K}_a P$. This, in turn, holds if and only if there is some state s which is R_a^A-accessible from x and $s \nVdash^A P$. The latter conjunction is equivalent to $xR_a s$ and $s \Vdash A$ but $s \nVdash P$. It is clear that if y is such s and $y \nVdash P$ then we have that $x \nVdash^A \mathsf{K}_a P$. On the other hand, with P-announcements $(A \supset [A]\mathsf{K}_a P) \supset [A]\mathsf{K}_a P$ is true in the model because it is valid and this can be verified by applying the above semantic definitions for P-announcements without any appeal to the diagram. In fact, suppose that for an arbitrary x it holds that $x \Vdash A \supset [A]\mathsf{K}_a P$. This is equivalent to $x \Vdash A$ implies that if $x \Vdash A$ then $x \Vdash^A \mathsf{K}_a P$. This holds if and only if $x \Vdash A$ and $x \Vdash A$ implies $x \Vdash^A \mathsf{K}_a P$ by propositional reasoning. Therefore we have $x \Vdash A$ implies $x \Vdash^A \mathsf{K}_a P$ which is $x \Vdash [A]\mathsf{K}_a P$.

In the P- and GG-interpretation announcements can be composed: any effect of two consecutive assertions could also have been produced by making only one assertion. Two consecutive announcements that A and B are equivalent to the single announcements that A and after the announcement that A, B. In other words, two announcements $[A][B]$ can be reduced to the single announcement $[A \wedge [A]B]$. Another property is the associativity of public announcements, that is, the equivalence between $[A][B \wedge [B]C]D$ and $[A \wedge [A]B][C]D$.

Proposition 4.1.1 (Associativity). *The following are equivalent:*

i) $x \Vdash [A \wedge [A]B][C]D$

ii) $x \Vdash [A][B \wedge [B]C]C$

Proposition 4.1.2 (Compositionality). *The following are equivalent:*

i) $x \Vdash [A \wedge [A]B]C$

ii) $x \Vdash [A][B]C$

Both the properties are provable by induction on the formula announced and they are similar, so we prove only compositionality of GG-announcements.

Proof. By induction on C. If C is an atom P then by definition $x \Vdash [A \wedge [A]B]P$ if and only $x \Vdash^{A \wedge [A]B]} P$ if and only if $x \Vdash P$. In GG-restricted model, the latter is equivalent to $x \Vdash^{A} P$ and this, in turn, holds if and only if $x \Vdash^{A,B} P$, and, by definition, $x \Vdash [A][B]P$. If C is $D \wedge E$ we have that $x \Vdash [A \wedge [A]B](D \wedge E)$ if and only if $x \Vdash^{A \wedge [A]B} (D \wedge E)$. By definition of conjunction this is equivalent to $x \Vdash^{A \wedge [A]B} D$ and $x \Vdash^{A \wedge [A]B} E$. By inductive hypothesis (IH) we have that $x \Vdash^{A,B} D$ and $x \Vdash^{A,B} E$ which is equivalent to $x \Vdash^{A,B} (D \wedge E)$. Now, by definition, we conclude $x \Vdash [A][B](D \wedge E)$. The proof is analogous when C is another propositional formula. When C is of the form $\mathsf{K}_a D$, we have: $x \Vdash [A \wedge [A]B](\mathsf{K}_a D)$ if and only if $x \Vdash^{A \wedge [A]B}$ $\mathsf{K}_a D$. The latter holds if and only if for an arbitrary y, $xR_a y$ and $y \Vdash A \wedge [A]B$ implies $y \Vdash^{A \wedge [A]B} D$. By IH, we obtain: for all y, $xR_a y$ and $y \Vdash A \wedge [A]B$ implies $y \Vdash^{A,B} D$. Now, the antecedent of the conditional holds if and only if $xR_a y$ and $y \Vdash A$ and $y \Vdash^{A} B$. By definition of R_a^A in GG-models this is equivalent to $xR_a^A y$ and $y \Vdash^{A} B$. Now, we have that for all y, $xR_a^A y$ and $y \Vdash^{A} B$ implies $y \Vdash^{A,B} D$

and this is, by definition of K_a, $x \Vdash^{A,B} (\mathsf{K}_a D)$. Finally, when C is of the form $[C]D$ then we have $x \Vdash [A \wedge [A]B]([D]E)$ if and only if $x \Vdash^{A \wedge [A]B,D} E$. By associativity of public announcements, $x \Vdash^{A \wedge [A]B,D} E$ and $x \Vdash^{A,B \wedge [B]D} E$ are equivalent. By IH on the latter we obtain $x \Vdash^{A,B,D} E$ and from this by definition we conclude $x \Vdash [A][B]([D]E)$.

⊠

To prove the compositionality of public announcement an inductive argument is required. This is to say that the validity of the formula $[A \wedge [A]B]C \supset\subset [A][B]C$ cannot be proved schematically as other valid formulas like, for instance, the axioms of knowledge operator. However, in the labelled system we shall introduce in the next section compositionality and associativity of public announcements can be proved schematically and without any induction on formulas.

4.2 A labelled sequent calculus for PAL

In this section we present a labelled sequent system for PAL in which announcements are GG-interpreted and we shall refer to this system as **G3PAL**. When the semantics of announcements is internalized into sequent calculus, we have to consider the general case of models restricted to a (possibly empty) list of formulas, instead of a single formula. Given a list φ of formulas, we indicate by $\mathfrak{M}^\varphi$ the model restricted to φ

$$\mathfrak{M}^\varphi = \begin{cases} \mathfrak{M} & \text{if } \varphi = \epsilon \\ ((\mathfrak{M}^{A_1})^{A_2} \dots)^{A_n}) & \text{if } \varphi = A_1, \dots, A_n \end{cases}$$

Note that $\mathfrak{M}^{\varphi,A}$ should be written $\mathfrak{M}^{\varphi^A}$, but we prefer a linear notation where the comma has the same role of the concatenation operator • of Balbiani *et al.* (2010).

As usual, the logical rules of **G3PAL** are obtained by exploiting the definition of the forcing relation in the restricted model. A labelled formula of **G3PAL** is an expression of the form $x :^{\varphi} A$, where φ is an arbitrary list of formulas and indicates that A is true at the state x in the model restricted to (formulas in) φ. Propositional rules are immediate because their application leaves untouched the formulas in φ. In the case of atomic and knowledge formulas we must distinguish the case in which the list is empty from that in which it is of the form φ, A. In particular, if φ is empty, atoms $x : P$ can appear as principal only in initial sequents, whereas if the list is of the form φ, A we add two rules, $L0$ and $R0$, that reflect the definition of the forcing relation in GG-models

$$x \Vdash^{\varphi,A} P \quad \textit{if and only if} \quad x \Vdash^{\varphi} P$$

The rules corresponding to this definition are

$$\frac{x :^{\varphi} P, \Gamma \to \Delta}{x :^{\varphi,A} P, \Gamma \to \Delta} \; L0 \qquad\qquad \frac{\Gamma \to \Delta, x :^{\varphi} P}{\Gamma \to \Delta, x :^{\varphi,A} P} \; R0$$

Likewise, the rules for the knowledge operator are the standard ones when φ is empty. Instead, when φ is a non empty list the last element of which is the formula A, then the clause becomes

$$x \Vdash^{\varphi,A} \mathsf{K}_a B \quad \textit{if and only if} \quad \textit{for all } y,\ xR_a^{\varphi,A}y \textit{ implies } y \Vdash^{\varphi,A} B$$

This can be immediately converted in two logical rules following the usual method

$$\frac{y :^{\varphi,A} B, x :^{\varphi,A} y, \mathsf{K}_a B, xR_a^{\varphi,A}y, \Gamma \to \Delta}{x :^{\varphi,A} \mathsf{K}_a B, xR_a^{\varphi,A}y, \Gamma \to \Delta} \qquad\qquad \frac{xR_a^{\varphi,A}y, \Gamma \to \Delta, y :^{\varphi,A} B}{\Gamma \to \Delta, x :^{\varphi,A} \mathsf{K}_a B}$$

In the presence of such rules we need as primitive also two non-logical rules for formulas such as $xR_a^{\varphi,A}y$, in analogy to the rules for restricted propositional atoms $L0$ and $R0$. These are found from the definition of GG-models:

$$xR_a^{\varphi,A}y \textit{ if and only if } xR_a^{\varphi}y \textit{ and } y \Vdash^{\varphi} A$$

Therefore, we find the rules

$$\frac{xR^{\varphi}y, y : A, \Gamma \to \Delta}{xR^{\varphi,A}y, \Gamma \to \Delta} \qquad \frac{\Gamma \to \Delta, xR^{\varphi}y \quad \Gamma \to \Delta, y :^{\varphi} A}{\Gamma \to \Delta, xR^{\varphi,A}y}$$

However, these rules make problematic the proof of the admissibility of the structural rules, in particular the admissibility of cut because relational atoms appears active both in the antecedent and succedent of a mathematical rule. A possible way out is to rephrase the semantics of the knowledge operator so that the definition of restricted relational atoms is embedded in it.

$$x \Vdash^{\varphi,A} \mathsf{K}_a B \quad \textit{if and only if} \quad \textit{for all } y,\ xR_a^{\varphi}y \textit{ and } y \Vdash^{\varphi} A \textit{ implies } y \Vdash^{\varphi,A} B$$

The move is analogous to that for the treatment of Gödel-Löb provability logic in Negri (2005), where a modification in the definition of the $\Box$-operator allows to internalize a condition of the accessibility relation, the property of being Noetherian, that is not even first order expressible. Thus, from the the sufficient condition for $\mathsf{K}_a B$ to be forced at x in a model restricted to a list φ, A we have

$$\frac{xR_a^{\varphi}y, y :^{\varphi} A, \Gamma \to \Delta, y :^{\varphi,A} B}{\Gamma \to \Delta, x :^{\varphi,A} \mathsf{K}_a B} \, R\mathsf{K}'$$

As usual, $R\mathsf{K}'$ has the variable condition that y must not appear in the conclusion. Conversely, from the opposite direction of the definition the following rule is found

$$\frac{y :^{\varphi,A} B, x :^{\varphi,A} \mathsf{K}_a B, xR_a^{\varphi}y, y :^{\varphi} A, \Gamma \to \Delta}{x :^{\varphi,A} \mathsf{K}_a B, xR_a^{\varphi}y, y :^{\varphi} A, \Gamma \to \Delta} \, L\mathsf{K}'$$

Finally, we have also two rules corresponding to the property of compositionality of public announcements

$$\frac{x :^{\varphi,A,B} C, \Gamma \to \Delta}{x :^{\varphi,A \wedge [A]B} C, \Gamma \to \Delta}\ L_{cmp} \qquad \frac{\Gamma \to \Delta, x :^{\varphi,A,B} C}{\Gamma \to \Delta, x :^{\varphi,A \wedge [A]B} C}\ R_{cmp}$$

The system **G3PAL** can be easily modified in order to deal with the P-interpretation of announcements. The rules that must be changed are $L0$, $R0$, $L\mathsf{K}'$, $R\mathsf{K}'$, $L[\ ]$ and $R[\ ]$. In the P-interpretation, what is announced must be true. So, when atomic formulas are evaluated in the P-restricted model to a non empty list of formulas, the inductive clause in the semantic definition is

$x \Vdash^{\varphi,A} P$ *if and only if* $x \Vdash^{\varphi} A$ *and* $x \Vdash^{\varphi} P$

and the corresponding rules are

$$\frac{x :^{\varphi} A, x :^{\varphi} P, \Gamma \to \Delta}{x :^{\varphi,A} P, \Gamma \to \Delta}\ L0' \qquad \frac{\Gamma \to \Delta, x :^{\varphi} A \quad \Gamma \to \Delta, x :^{\varphi} P}{\Gamma \to \Delta, x :^{\varphi,A} P}\ R0'$$

Analogously, the semantics of announcements brings the two following rules

$$\frac{x :^{\varphi,A} B, x :^{\varphi} [A]B, x :^{\varphi} A, \Gamma \to \Delta}{x :^{\varphi} [A]B, x :^{\varphi} A, \Gamma \to \Delta}\ L[\,]' \qquad \frac{x :^{\varphi} A, \Gamma \to \Delta, x :^{\varphi,A} B}{\Gamma \to \Delta, x :^{\varphi} [A]B}\ R[\,]'$$

Finally, in the definition of $x \Vdash^{\varphi,A} \mathsf{K}_a B$ we need to take into account also the definition of restricted relational atom $xR_a^{\varphi,A}y$ in a P-restricted model: $xR_a^{\varphi}y$ and $x \Vdash^{\varphi} A$ and $y \Vdash^{\varphi} A$. The rules for the K_a operator become

$$\frac{y :^{\varphi,A} B, x :^{\varphi,A} \mathsf{K}_a B, xR_a^{\varphi}y, x :^{\varphi} A, y :^{\varphi} A, \Gamma \to \Delta}{x :^{\varphi,A} \mathsf{K}_a B, xR_a^{\varphi}y, x :^{\varphi} A, y :^{\varphi} A, \Gamma \to \Delta}\ L\mathsf{K}''$$

$$\frac{xR_a^{\varphi}y, x :^{\varphi} A, y :^{\varphi} A, \Gamma \to \Delta, y :^{\varphi,A} B}{\Gamma \to \Delta, x :^{\varphi,A} \mathsf{K}_a B}\ R\mathsf{K}''$$

Thus, the logical rules of **G3PAL** with GG-announcements are the following:

Logical rules of **G3PAL**

$$x : P, \Gamma \to \Delta, x : P \qquad\qquad \cfrac{}{x :^{\varphi} \bot, \Gamma \to \Delta}\ L\bot$$

$$\cfrac{x :^{\varphi} P, \Gamma \to \Delta}{x :^{\varphi, A} P, \Gamma \to \Delta}\ L0 \qquad\qquad \cfrac{\Gamma \to \Delta, x :^{\varphi} P}{\Gamma \to \Delta, x :^{\varphi, A} P}\ R0$$

$$\cfrac{x :^{\varphi} A, x :^{\varphi} B, \Gamma \to \Delta}{x :^{\varphi} A \wedge B, \Gamma \to \Delta}\ L\wedge \qquad\qquad \cfrac{\Gamma \to \Delta, x :^{\varphi} B \quad \Gamma \to \Delta, x :^{\varphi} B}{\Gamma \to \Delta, x :^{\varphi} A \wedge B}\ R\wedge$$

$$\cfrac{x :^{\varphi} A, \Gamma \to \Delta \quad x :^{\varphi} B, \Gamma \to \Delta}{x :^{\varphi} A \vee B, \Gamma \to \Delta}\ L\vee \qquad\qquad \cfrac{\Gamma \to \Delta, x :^{\varphi} A, x :^{\varphi} B}{\Gamma \to \Delta, x :^{\varphi} A \vee B}\ R\vee$$

$$\cfrac{\Gamma \to \Delta, x :^{\varphi} A \quad x :^{\varphi} B, \Gamma \to \Delta}{x :^{\varphi} A \supset B, \Gamma \to \Delta}\ L\supset \qquad\qquad \cfrac{x :^{\varphi} A, \Gamma \to \Delta, x :^{\varphi} B}{\Gamma \to \Delta, x :^{\varphi} A \supset B}\ R\supset$$

$$\cfrac{y : A, x : \mathsf{K}_a A, xR_a y, \Gamma \to \Delta}{x : \mathsf{K}_a A, xR_a y, \Gamma \to \Delta}\ L\mathsf{K} \qquad\qquad \cfrac{xR_a y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathsf{K}_a A}\ R\mathsf{K}$$

$$\cfrac{y :^{\varphi, A} B, x :^{\varphi, A} \mathsf{K}_a B, xR_a^{\varphi} y, y :^{\varphi} A, \Gamma \to \Delta}{x :^{\varphi, A} \mathsf{K}_a B, xR_a^{\varphi} y, y :^{\varphi} A, \Gamma \to \Delta}\ L\mathsf{K}' \qquad \cfrac{xR_a^{\varphi} y, y :^{\varphi} A, \Gamma \to \Delta, y :^{\varphi, A} B}{\Gamma \to \Delta, x :^{\varphi, A} \mathsf{K}_a B}\ R\mathsf{K}'$$

$$\cfrac{x :^{\varphi, A} B, \Gamma \to \Delta}{x :^{\varphi} [A]B, \Gamma \to \Delta}\ L[\,] \qquad\qquad \cfrac{\Gamma \to \Delta, x :^{\varphi, A} B}{\Gamma \to \Delta, x :^{\varphi} [A]B}\ R[\,]$$

$$\cfrac{x :^{\varphi, A, B} C, \Gamma \to \Delta}{x :^{\varphi, A \wedge [A]B} C, \Gamma \to \Delta}\ L_{cmp} \qquad\qquad \cfrac{\Gamma \to \Delta, x :^{\varphi, A, B} C}{\Gamma \to \Delta, x :^{\varphi, A \wedge [A]B} C}\ R_{cmp}$$

P- and GG- restricted models were introduced as based on standard epistemic models. Therefore, we still assume that each accessibility relation R_a is an equivalence relation and that the rules for R_a are those of **G3S5**, and we have

Mathematical rules of **G3PAL**

$$\frac{xR_a^{\varphi}x, \Gamma \to \Delta}{\Gamma \to \Delta}\ \mathit{Ref}$$

$$\frac{xR_a^{\varphi}z, xR_a^{\varphi}y, yR_a^{\varphi}z, \Gamma \to \Delta}{xR_a^{\varphi}y, yR_a^{\varphi}z, \Gamma \to \Delta}\ \mathit{Trans} \qquad \frac{xR_a^{\varphi}y, yR_a^{\varphi}x, \Gamma \to \Delta}{xR_a^{\varphi}y, \Gamma \to \Delta}\ \mathit{Sym}$$

4.3 Admissibility of the structural rules and cut

In this section we prove that the structural rules of weakening and contraction are hp-admissible and all the logical rules of **G3PAL** hp-invertible. Furthermore, the rule of cut is proved to be admissible. All this results hold also in the system for the P-interpretation of announcements and the proofs can be easily adapted. The structural rules of PAL are as follows:

$$\frac{\Gamma \to \Delta}{x :^{\varphi} A, \Gamma \to \Delta}\ \text{L-W} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, x :^{\varphi} A}\ \text{R-W} \qquad \frac{\Gamma \to \Delta}{xR_a^{\varphi}y, \Gamma \to \Delta}\ \text{L-W}$$

$$\frac{x :^{\varphi} A, x :^{\varphi} A, \Gamma \to \Delta}{x :^{\varphi} A, \Gamma \to \Delta}\ \text{L-C} \quad \frac{\Gamma \to \Delta, x :^{\varphi} A, x :^{\varphi} A}{\Gamma \to \Delta, x :^{\varphi} A}\ \text{R-C} \quad \frac{xR_a^{\varphi}y, xR_a^{\varphi}y, \Gamma \to \Delta}{xR_a^{\varphi}y, \Gamma \to \Delta}\ \text{L-C}$$

$$\frac{\Gamma \to \Delta, x :^{\varphi} A \quad x :^{\varphi} A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta', \Delta}\ \text{CUT}$$

Note that the cut rule is formulated as a context-independent rule, unlike the context-sharing formulation employed in **G3K**. In order to prove these results we need to extend the straightforward definition of **length** of a formula as follows:

Definition. *The* length ℓ *of a formula A is defined by induction.*

$\ell(\bot) = 0$*;*

$\ell(P) = 1$;

$\ell(A \circ B) = \ell(A) + \ell(B) + 1$, *when* $\circ$ *is* $\wedge, \vee, \supset$;

$\ell(\mathsf{K}_a A) = \ell(A) + 1$;

$\ell([A]B) = \ell(A) + \ell(B) + 1$.

For labelled formulas, $\ell(x : A) = \ell(A)$ *and* $\ell(xR_a y) = 1$. *Furthermore, if* φ *is the list* $A_1, \ldots, A_n$, *we define* $\ell(x :^{\varphi, A} B) = \ell(A_1) + \cdots + \ell(A_n) + \ell(A) + \ell(B)$.

Lemma 4.3.1. *In* **G3PAL** *it holds that:*

i) The substitution of labels is hp-admissible;

ii) Arbitrary initial sequents are derivable;

iii) All the rules are hp-invertible.

Proof. (i) Substitution of labels is proved to be hp-admissible by induction on the height h of the derivation of a sequent $\Gamma \to \Delta$. If $h = 0$ then the premise is initial or an istance of $L\bot$ and also the conclusion is initial or conclusion of $L\bot$. If $h = n + 1$, suppose that the claim holds for derivations of height n and consider the last rule applied in the derivation. If the last rule is a propositional rule or a modal rule without variable conditions, apply the inductive hypothesis (IH) to the premises and then the rule. If the last rule is a rule with a variable condition such as $R\mathsf{K}$ or $R\mathsf{K}'$, we must be careful with the cases in which either x or y is the *eigenvariable* of the rule, because a straightforward substitution may result in a violation of the restriction. In those cases we must apply IH to the premise and replace the eigenvariable with a fresh variable that does not appear in the derivation. Consider the case in which $\Gamma \to \Delta$ has been concluded by $R\mathsf{K}'$ with $x :^{\varphi, A} \mathsf{K}_a B$ as principal formula and y as *eigenvariable*. Thus the derivation is

$$\dfrac{\vdots \atop xR_a^{\varphi}y, y :^{\varphi} A, \Gamma \to \Delta', y :^{\varphi,A} B}{\Gamma \to \Delta', x :^{\varphi,A} \mathsf{K}_a B} \; R\mathsf{K}'$$

Given that y is the *eigenvariable* of $R\mathsf{K}'$ we must replace it with a fresh z. Note that if z is fresh the substitution does not affect the context Γ, Δ'. Then by IH and $R\mathsf{K}'$ we obtain the desired conclusion.

$$\dfrac{\dfrac{\dfrac{xR_a^{\varphi}y, y :^{\varphi} A, \Gamma \to \Delta', y :^{\varphi,A} B}{xR_a^{\varphi}z, z :^{\varphi} A, \Gamma \to \Delta', z :^{\varphi,A} B} \; \text{IH}}{xR_a^{\varphi}z, z :^{\varphi} A, \Gamma[y/x] \to \Delta'[y/x], z :^{\varphi,A} B} \; \text{IH}}{\Gamma[y/x] \to \Delta'[y/x], x :^{\varphi,A} \mathsf{K}_a B} \; R\mathsf{K}'$$

When $\Gamma \to \Delta$ is concluded by an announcement rule such as

$$\dfrac{\vdots \atop u :^{\varphi,A} B, \Gamma \to \Delta}{u :^{\varphi} [A]B, \Gamma \to \Delta} \; L[\,]$$

we apply IH to the premise of $L[\,]$ and then $L[\,]$ again

$$\dfrac{\vdots \atop u :^{\varphi,A} B, \Gamma[y/x] \to \Delta[y/x]}{u :^{\varphi} [A]B, \Gamma[y/x] \to \Delta[y/x]} \; L[\,]$$

(ii) Derivability of arbitrary initial sequents, that is, $\vdash x :^{\varphi} B, \Gamma \to \Delta, x :^{\varphi} B$ for an arbitrary list φ and an arbitrary formula B, is proved by induction on $\ell(x :^{\varphi} B)$. If B is atomic and φ has length zero, we have an initial sequent. If B is $\bot$ then we have an istance of $L\bot$. If B is an atomic formula P and φ is of the form φ, A, the rules $L0$ and $R0$ are used to reduce the length of the list of announcements and IH is applied:

$$\dfrac{\dfrac{x :^{\varphi} P, \Gamma \to \Delta, x :^{\varphi} P}{x :^{\varphi} P, \Gamma \to \Delta, x :^{\varphi, A} P} R0}{x :^{\varphi, A} P, \Gamma \to \Delta, x :^{\varphi, A} P} L0$$

The topmost sequent is derivable by IH because $\ell(x :^{\varphi} P) < \ell(x :^{\varphi, A} P)$. If B is a compound formula, apply root-first the appropriate rules and observe that similar sequents, of reduced length, appear in the premises. Then the claim holds by IH.

(iii) Invertibility with the preservation of the height is proved for all the logical rules by induction on the height h of the derivation. The proof of the cases corresponding to the rule for $\wedge$, $\vee$ and $\supset$ is similar to the proof of Lemma 2.1.5. Rules $L\mathsf{K}$ and $L\mathsf{K}'$ are trivially hp-invertible, since their premises are obtained by weakening from the conclusion (see Theorem 4.3.3 in the next). The proof for $R\mathsf{K}$ and $R\mathsf{K}'$ needs some care for the variable condition. Consider the case of $R\mathsf{K}'$: we need to prove that if $\Gamma \to \Delta, x :^{\varphi, A} \mathsf{K}_a B$ is derivable, also $xR^{\varphi}_{a} y, y :^{\varphi} A, \Gamma \to \Delta, y :^{\varphi, A} B$ is derivable. If $\Gamma \to \Delta, x :^{\varphi, A} \mathsf{K}_a B$ is an initial sequent or an istance of $L\bot$ then $x :^{\varphi} \mathsf{K}_a B$ is not principal and also $xR^{\varphi} y, y :^{\varphi} A, \Gamma \to \Delta, y :^{\varphi, A} B$ is initial or an istance of $L\bot$. If $\Gamma \to \Delta, x :^{\varphi, A} \mathsf{K}_a B$ is concluded by a derivation of height $h > 0$, we have to consider the rule that introduced it. If $x :^{\varphi, A} \mathsf{K}_a B$ is principal formula of $R\mathsf{K}'$ and y is its *eigenvariable* then the premise of $R\mathsf{K}'$ has a derivation of a lower height and the claim holds by IH. If instead, the *eigenvariable* of $R\mathsf{K}'$ is a z different from y then apply hp-substitution of z with y and, again, the claim holds by IH. If $x :^{\varphi, A} \mathsf{K}_a B$ is not principal and it has been introduced by a rule without variable condition, apply IH and then the rule. If $\Gamma \to \Delta, x :^{\varphi, A} \mathsf{K}_a B$ is a conclusion of $R\mathsf{K}'$ we apply first hp-substitution admissibility in order to avoid clash of variables and then IH and $R\mathsf{K}'$ again. The last step is

$$\dfrac{\begin{array}{c}\vdots\\ uR^{\psi} y, y :^{\psi} C, \Gamma \to \Delta, y :^{\psi, C} D, x :^{\varphi, A} \mathsf{K}_a B\end{array}}{\Gamma \to \Delta, x :^{\varphi, A} \mathsf{K}_a B, u :^{\psi, C} \mathsf{K}_a D} R\mathsf{K}'$$

By replacing y with a new z have $uR^{\psi}z, z :^{\psi} C, \Gamma \to \Delta, z :^{\psi,C} D, x :^{\varphi,A} \mathsf{K}_a B$. By IH we get $xR_a^{\varphi}y, y :^{\varphi} A, uR^{\psi}z, z :^{\psi} C, \Gamma \to \Delta, z :^{\psi,C} D, y :^{\varphi,A} B$ and an application of $R\mathsf{K}'$ gives $xR_a^{\varphi}, y :^{\varphi} A, \Gamma \to \Delta, y :^{\varphi,A} B$. The hp-inversion of $L[\,]$, $R[\,]$, $L0$, $R0$, L_{cmp} and R_{cmp} is proved exactly as for the propositional cases: apply IH on the premise of the last rule applied and then the rule. ☒

In **G3PAL** an analogous result of Lemma 2.1.2 of **G3K** holds. The result will be useful in the proof the admissibility of cut rule.

Lemma 4.3.2. *In* **G3PAL** *it holds that*

$$\textit{If} \quad \vdash \Gamma \xrightarrow{n} \Delta, x :^{\varphi} \bot \quad \textit{then} \quad \vdash \Gamma \xrightarrow{n} \Delta$$

Proof. Similar to the proof of the Lemma 2.1.2. ☒

Now we can turn to the the admissibility of the structural rules of weakening and contraction. Also in this case, we shall refer to previous results we the proofs can be easily adapted.

Theorem 4.3.3. Weakening *is height-preserving admissible in* **G3PAL***,* i.e.

$$\begin{array}{llllll} i) & \textit{If} & d \vdash \Gamma \xrightarrow{n} \Delta & \textit{then} & d \vdash x :^{\varphi} A, \Gamma \xrightarrow{n} \Delta \\ ii) & \textit{If} & d \vdash \Gamma \xrightarrow{n} \Delta & \textit{then} & d \vdash \Gamma \xrightarrow{n} \Delta, x :^{\varphi} A \\ iii) & \textit{If} & d \vdash \Gamma \xrightarrow{n} \Delta & \textit{then} & d \vdash xR_a^{\varphi}y, \Gamma \xrightarrow{n} \Delta \end{array}$$

Proof. By induction on n. The proof follows the pattern of Theorem 2.1.4 to which we add the following cases. Suppose the premise of d is the conclusion of $R0$, so Δ is $\Delta', u :^{\psi,B} P$ and

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \xrightarrow{n-1} \Delta', u :^{\psi} P\end{array}}{\Gamma \xrightarrow{n} \Delta', u :^{\psi,B} P}\, R0$$

By inductive hypothesis (IH) on a lower derivation we obtain

$$\begin{array}{ll} \text{i)} & x :^{\varphi} A, \Gamma \xrightarrow{n-1} \Delta', u :^{\psi} P \\ \text{ii)} & \Gamma \xrightarrow{n-1} \Delta', u :^{\psi} P, x :^{\varphi} A \\ \text{iii)} & xR^{\varphi}_{a}y, \Gamma \xrightarrow{n-1} \Delta', u :^{\psi} P \end{array}$$

and by $R0$ once again we get the conclusion. If d has been concluded by $R[\,]$ then it ends with

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma \xrightarrow{n-1} \Delta', u :^{\psi,B} C\end{array}}{\Gamma \xrightarrow{n} \Delta', u :^{\psi} [B]C}\ R[\,]$$

By IH we have

$$\begin{array}{ll} \text{i)} & x :^{\varphi} A, \Gamma \xrightarrow{n-1} \Delta', u :^{\psi,B} C \\ \text{ii)} & \Gamma \xrightarrow{n-1} \Delta', u :^{\psi,B} C, x :^{\varphi} A \\ \text{iii)} & xR^{\varphi}_{a}y, \Gamma \xrightarrow{n-1} \Delta', u :^{\psi,B} C \end{array}$$

from which by $R[\,]$ we conclude

$$\begin{array}{ll} \text{i)} & x :^{\varphi} A, \Gamma \xrightarrow{n} \Delta', u :^{\psi} [B]C \\ \text{ii)} & \Gamma \xrightarrow{n} \Delta', u :^{\psi} [B]C, x :^{\varphi} A \\ \text{iii)} & xR^{\varphi}_{a}y, \Gamma \xrightarrow{n} \Delta', u :^{\psi} [B]C \end{array}$$

The case in which d is concluded by $L[\,]$ is dealt with in analogous way.

$\boxtimes$

Theorem 4.3.4. Contraction *is height-preserving admissible in* **G3PAL**, i.e.

$$\begin{array}{llll} \textit{i)} & \textit{If} \quad d \vdash x :^{\varphi} A, x :^{\varphi} A, \Gamma \xrightarrow{n} \Delta & \textit{then} & d \vdash x :^{\varphi} A, \Gamma \xrightarrow{n} \Delta \\ \textit{ii)} & \textit{If} \quad d \vdash \Gamma \xrightarrow{n} \Delta, x :^{\varphi} A, x :^{\varphi} A & \textit{then} & d \vdash \Gamma \xrightarrow{n} \Delta, x :^{\varphi} A \\ \textit{iii)} & \textit{If} \quad d \vdash xR^{\varphi}_{a}y, xR^{\varphi}_{a}y, \Gamma \xrightarrow{n} \Delta & \textit{then} & d \vdash xR^{\varphi}_{a}y, \Gamma \xrightarrow{n} \Delta \end{array}$$

Proof. By simultaneous induction on n. The proof follows the pattern of Theorem 2.1.6. A new case arises when one of the contracted formulas $x :^{\varphi} A$ is principal in an announcement rule. We have

$$\dfrac{\begin{array}{c}\vdots\\ x :^{\varphi,B} C, x :^{\varphi} [B]C, \Gamma \xrightarrow{n-1} \Delta\end{array}}{x :^{\varphi} [B]C, x :^{\varphi} [B]C, \Gamma \xrightarrow{n} \Delta}\ L[\,]$$

From the hp-invertibility of the logical rules of **G3PAL** (Lemma 4.3.1) the premise becomes

$$x :^{\varphi,B} C, x :^{\varphi,B} C, \Gamma \xrightarrow{n-1} \Delta$$

and IH is applicable

$$x :^{\varphi,B} C, \Gamma \xrightarrow{n-1} \Delta$$

and we get the conclusion by $L[\,]$

$$x :^{\varphi} [B]C, \Gamma \xrightarrow{n} \Delta$$

⊠

Theorem 4.3.5 (Cut Admissibility)**.** *Cut in admissible in* **G3PAL**, i.e. *for every* d_1 *and* d_2 *derivations in* **G3PAL** *such that*

$$d_1 \vdash \Gamma \xrightarrow{n} \Delta, x :^{\varphi} C \quad \textit{and} \quad d_2 \vdash x :^{\varphi} C, \Gamma' \xrightarrow{m} \Delta'$$

there is a derivation d *in* **G3PAL** *such that*

$$d \vdash \Gamma, \Gamma' \to \Delta', \Delta$$

Proof. The proof has the same structure as the proof of admissibility of cut for the modal systems **G3K** of Negri (2005). The proof is by induction on the length of the cut formula with sub-induction on the sum of the heights of the derivations of the premises of cut. In the inductive step, we shall refer to the inductive hypothesis on the length of the cut cut formula as the main inductive hypothesis (MIH), and to the inductive hypothesis on the sum of the heights as the secondary inductive hypothesis (SIH). The proof follows the pattern:

1. Either d_1 or d_2 is initial or conclusion of $L\bot$:
   (a) d_1 is initial or conclusion of $L\bot$;
   (b) d_2 is initial or conclusion of $L\bot$.
2. Neither d_1 nor d_2 is initial or conclusion of $L\bot$ and:
   (a) $x : C$ is not principal in d_1;
   (b) $x : C$ is not principal in d_2;
   (c) $x : C$ is principal both in d_1 and d_2.

Case 1a

If d_1 is initial then Γ and Δ have an atom $u : P$ in common and so also $\Gamma, \Gamma' \to \Delta', \Delta$ is initial. Else, $x : C$ is atomic and Γ is $x : P, \Gamma''$. In this case take d_2 and apply hp-admissibility of weakening (Theorem 4.3.3)

$$\cfrac{x : P, \Gamma' \to \Delta'}{x : P, \Gamma'', \Gamma' \to \Delta', \Delta}\ \text{W}$$

If d_1 is conclusion of $L\bot$ then $u :^{\varphi} \bot$ is in Γ and also $\Gamma, \Gamma' \to \Delta', \Delta$ is conclusion of $L\bot$.

Case 1b

If d_2 is initial then either Γ' and Δ' have an atom $u : P$ in common and so also $\Gamma, \Gamma' \to \Delta', \Delta$ is initial, or else, $x : C$ is an atom $x : P$ and it is in Δ'. As above, take d_1 and apply hp-admissibility of weakening (Theorem 4.3.3).

$$\frac{\Gamma \to \Delta, x : P}{\Gamma', \Gamma \to \Delta, \Delta'', x : P}\ \mathrm{W}$$

where Δ' is $\Delta'', x : P$. If d_2 is conclusion of $L\bot$ then either $u :^{\varphi} \bot$ is in Γ' or $x :^{\varphi} C$ is $x :^{\varphi} \bot$. In the first case also the conclusion of cut is derived by $L\bot$. In the second case apply Lemma 4.3.2 and weakening on d_1.

$$\cfrac{\cfrac{\Gamma \to \Delta, x :^{\varphi} \bot}{\Gamma \to \Delta}}{\Gamma, \Gamma' \to \Delta', \Delta}\ \mathrm{W}$$

Case 2a

If none of the premises of cut is initial or conclusion of $L\bot$, consider the case in which d_1 is conclusion of a rule R_1 in which the cut formula $x :^{\varphi} C$ is not principal.

If R_1 is a propositional rule, say $L\wedge$, then Γ is $u :^{\psi} A \wedge B, \Gamma''$ and d_1 is

$$\frac{\begin{array}{c}\vdots\\ u :^{\psi} A, u :^{\psi} B, \Gamma'' \xrightarrow{n-1} \Delta, x :^{\varphi} C\end{array}}{u :^{\psi} A \wedge B, \Gamma'' \xrightarrow{n} \Delta, x :^{\varphi} C}\ L\wedge$$

From d_2 and the premise of $L\wedge$ we find d by applying IH and $L\wedge$

$$\dfrac{\dfrac{u :^{\psi} A, u :^{\psi} B, \Gamma'' \xrightarrow{n-1} \Delta, x :^{\varphi} C \quad x :^{\varphi} C, \Gamma' \xrightarrow{m} \Delta'}{u :^{\psi} A, u :^{\psi} B, \Gamma'', \Gamma' \to \Delta', \Delta} \text{IH}}{u :^{\psi} A \wedge B, \Gamma'', \Gamma' \to \Delta', \Delta} L\wedge$$

The proof is similar when R_1 is one of the other propositional rules or a modal rule without variable condition, that is $L\mathsf{K}$ and $L\mathsf{K}'$. Suppose that d_1 is the conclusion of an announcement rule like $L[\,]$ or $R[\,]$ with principal formula $u :^{\varphi} [A]B$

$$\dfrac{\begin{array}{c}\vdots \\ u :^{\psi,A} B, \Gamma'' \xrightarrow{n-1} \Delta, x :^{\varphi} C\end{array}}{u :^{\psi} [A]B, \Gamma'' \xrightarrow{n} \Delta, x :^{\varphi} C} L[\,]$$

A derivation d of the conclusion of cut can be found in this way

$$\dfrac{\dfrac{u :^{\psi,A} B, \Gamma'' \xrightarrow{n-1} \Delta, x :^{\varphi} C \quad x :^{\varphi} C, \Gamma' \xrightarrow{m} \Delta'}{u :^{\psi,A} B, \Gamma'', \Gamma' \to \Delta', \Delta} \text{SIH}}{u :^{\psi} [A]B, \Gamma'', \Gamma' \to \Delta', \Delta} L[\,]$$

The case in which d_1 is by $L0$ or $R0$ are similar. When the last step of d_1 is by a modal rule with variable condition, say $R\mathsf{K}'$, we have

$$\dfrac{\begin{array}{c}\vdots \\ uR_a^{\psi}v, v :^{\psi} A, \Gamma \xrightarrow{n} \Delta'', v^{\psi,A}B, x :^{\varphi} C\end{array}}{\Gamma \xrightarrow{n} \Delta'', u^{\psi,A}\mathsf{K}_a B, x :^{\varphi} C} R\mathsf{K}'$$

where v is the *eigenvariable* of the rule and does not appear in the conclusion. We find d by applying IH on the premise of $R\mathsf{K}'$ and d_2. We need hp-admissibility of substitution in order to avoid variable clash because v may occur in Γ', Δ'. So, let z be a fresh label

$$\dfrac{\dfrac{\dfrac{uR_a^{\psi}v, v:^{\psi} A, \Gamma \xrightarrow{(n-1)} \Delta'', v^{\psi,A}B, x:^{\varphi} C}{uR_a^{\psi}z, z:^{\psi} A, \Gamma \xrightarrow{(n-1)} \Delta'', z^{\psi,A}B, x:^{\varphi} C}\, z/v \qquad x:^{\varphi} C, \Gamma' \xrightarrow{m} \Delta'}{uR_a^{\psi}z, z:^{\psi} A, \Gamma, \Gamma' \to \Delta', \Delta'', z^{\psi,A}B}\,\text{SIH}}{\Gamma, \Gamma' \to \Delta', \Delta'', u^{\psi,A}\mathsf{K}_a B}\,RK'$$

Finally, we have the cases in which the left premise of cut is concluded by a mathematical rule, say Ref_a. In this case, d_1 is of the form

$$\dfrac{uR_au, \Gamma \xrightarrow{n-1} \Delta, x:^{\varphi} C}{\Gamma \xrightarrow{n} \Delta, x:^{\varphi} C}\,Ref_a$$

The conclusion of cut is found from d_2 and the premise of Ref_a

$$\dfrac{\dfrac{uR_au, \Gamma \xrightarrow{n-1} \Delta, x:^{\varphi} C \qquad x:^{\varphi} C, \Gamma' \xrightarrow{m} \Delta'}{uR_au, \Gamma, \Gamma' \to, \Delta', \Delta}\,\text{SIH}}{\Gamma, \Gamma' \to \Delta', \Delta}\,Ref_a$$

Analogously, when d_1 is concluded by Sym_a or $Trans_a$

$$\dfrac{\begin{matrix}\vdots\\ vR_au, uR_av, \Gamma'' \xrightarrow{n-1} \Delta, x:^{\varphi} C\end{matrix}}{uR_av, \Gamma'' \xrightarrow{n} \Delta, x:^{\varphi} C}\,Sym_a \qquad \dfrac{\begin{matrix}\vdots\\ uR_aw, uR_av, vR_aw, \Gamma'' \xrightarrow{n-1} \Delta, x:^{\varphi} C\end{matrix}}{uR_av, vR_aw, \Gamma'' \xrightarrow{n} \Delta, x:^{\varphi} C}\,Trans_a$$

In the first case, d is found as follows

$$\dfrac{\dfrac{vR_au, uR_av, \Gamma'' \xrightarrow{n-1} \Delta, x:^{\varphi} C \qquad x:^{\varphi} C, \Gamma' \xrightarrow{m} \Delta'}{vR_au, uR_av, \Gamma'', \Gamma' \to, \Delta', \Delta}\,\text{SIH}}{uR_av, \Gamma'', \Gamma' \to \Delta', \Delta}\,Sym_a$$

Else, if d_1 is concluded by $Trans_a$, d is

$$\dfrac{\dfrac{wR_au, uR_av, vR_aw, \Gamma'' \xrightarrow{n-1} \Delta, x:^{\varphi} C \qquad x:^{\varphi} C, \Gamma' \xrightarrow{m} \Delta'}{uR_aw, uR_av, vR_aw, \Gamma'', \Gamma' \to, \Delta', \Delta}\,\text{SIH}}{vR_au, uR_av, \Gamma'', \Gamma' \to \Delta', \Delta}\,Trans_a$$

Case 2b

Similar to case 2a.

Case 2c

Suppose now that the cut formula $x :^{\varphi} C$ is principal in both d_1 and d_2. We take into account only the cases arising from announcement and atomic rules, the other being analogous. We argue by distinction of cases according to the structure of $x :^{\varphi} C$. If C is atomic and φ non empty, that is, if $x : C$ is of the form $x :^{\varphi, A} P$, then d_1 and d_2 are

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \xrightarrow{n-1} \Delta, x :^{\varphi} P\end{array}}{\Gamma \xrightarrow{n} \Delta, x :^{\varphi, A} P}\ R0 \qquad \dfrac{\begin{array}{c}\vdots\\ x :^{\varphi} P, \Gamma' \xrightarrow{m-1} \Delta'\end{array}}{x :^{\varphi, A} P, \Gamma' \xrightarrow{m} \Delta'}\ L0$$

At this stage IH is applicable because $\ell(x :^{\varphi} P) < \ell(x :^{\varphi, A} P)$ and d is found as follows.

$$\dfrac{\Gamma \to \Delta, x :^{\varphi} P \quad x :^{\varphi} P, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta', \Delta}\ \mathrm{IH}$$

If the cut formula is of the form $x :^{\varphi} [B]C$ then from d_1 and d_2 of the form

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \xrightarrow{n-1} \Delta, x :^{\varphi, B} C\end{array}}{\Gamma \xrightarrow{n} \Delta, x :^{\varphi} [B]C}\ R[\,] \qquad \dfrac{\begin{array}{c}\vdots\\ x :^{\varphi, B} C, \Gamma' \xrightarrow{m-1} \Delta'\end{array}}{x :^{\varphi} [B]C, \Gamma' \xrightarrow{m} \Delta'}\ L[\,]$$

a derivation d is found as follows

$$\dfrac{\Gamma \to \Delta, x :^{\varphi, B} C \quad x :^{\varphi, B} C, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta', \Delta}\ \mathrm{IH}$$

$\boxtimes$

The admissibility of cut and other structural rules has as main consequence the possibility to use the system **G3PAL** for searching systematically derivations in PAL. This possibility is precluded with the Hilbert-style system of the next section.

4.4 Completeness

In the table below, we recall from Bucheli *et al*. (2010) the standard Hilbert-style system for PAL (**PAL**). We shall prove that all **PAL** axioms are derivable and all **PAL** rules are admissible in **G3PAL**. More specifically, **PAL** axioms can be proved to be derivable in **G3PAL** by applying a systematic proof-search procedure. **PAL** rules, *modus ponens* and the necessitation, are admissible in **G3PAL** by using the admissible rules of **G3PAL**. Derivability and admissibility in **G3PAL** of **PAL** axioms and rules give that every theorem of **PAL** is derivable, that is, **PAL** $\subseteq$ **G3PAL**. The completeness theorem for **PAL** proved in Gerbrandy and Groenveled (1997) and in Bucheli *et al.* (2010) permits to conclude that every valid sequent of PAL is derivable in **G3PAL**. The proof is indirect proof since it is based on the completeness of the axiomatic system **PAL**. The other direction of the inclusion **G3PAL** $\subseteq$ **PAL** can be proved following the pattern of Theorem 2.4.1 and gives the equivalence between **PAL** and **G3PAL**. However, completeness can be established also directly by extending the proof of the Theorem 2.4.3. The results of this section also exemplify how **G3PAL** is used for making proofs in PAL. In fact, **PAL** axioms are difficult to use in practice because they reduction axioms: Every formula that contains announcements can be rewritten into a formula without announcements. On the contrary, the admissibility of the structural rules in **G3PAL** allow a proof-search procedure for **G3PAL** derivations, that is, permit to construct a derivation starting from the conclusion: the end-sequent is analyzed in order to determine a last possible rule of inference and thus its premise(s). The procedure is iterated until a node at which no rule can be applied is reached: If every leaf is an initial sequent or a conclusion of $L\bot$, we obtain a derivation. Otherwise, the procedure fails if at least one of the leaves is not an initial sequent or a conclusion $L\bot$, or if the proof search does not terminate. The table below shows the axioms and rules of the axiomatic system given in Bucheli *et al.* (2010).

A1	All the axioms of modal logic **S5**	
A2	$[A]P \supset\subset P$	Atomic Independence
A3	$[A](B \supset C) \supset\subset ([A]B \supset [A]C)$	Normality
A4	$[A]\neg B \supset\subset \neg[A]B$	Functionality
A5	$[A]\mathsf{K}_a B \supset\subset \mathsf{K}_a(A \supset [A]B)$	Update
A6	$[A][B]C \supset\subset [A \wedge [A]B]C$	Announcements Composition
R1	From $\Gamma \vdash A \supset B$ and $\Delta \vdash A$ infer $\Gamma, \Delta \vdash B$	Modus Ponens
R2	From $\vdash A$ infer $\vdash \mathsf{K}_a A$	Necessitation

Through the admissibility of the structural rules and cut it is possible in **G3PAL** to find systematically a derivation for each axiom of the list above. The admissibility of the necessitation and *modus ponens* are proved as in Lemma 3.4.9.

Lemma 4.4.1. *All the axioms (rules) of* **PAL** *are derivable (resp. admissible) in* **G3PAL**.

Proof. By applying a systematic proof-search procedure from the sequent to be derived. First, axiom A2 is derivable as follows:

$$\dfrac{\dfrac{\dfrac{x : P \to x : P}{x :^A P \to x : P}\,L0}{x : [A]P \to x : P}\,L[\,]}{\to x : [A]P \supset P}\,R\supset \qquad \dfrac{\dfrac{\dfrac{x : P \to x : P}{x : P \to x :^A P}\,R0}{x : P \to x : [A]P}\,R[\,]}{\to x : P \supset [A]P}\,R\supset$$

The derivation of axiom A3 (left-to-right direction) is

$$\dfrac{\dfrac{\dfrac{\dfrac{x :^A B \to x :^A C, x :^A B \qquad x :^A C, x :^A B \to x :^A C}{x :^A B \supset C, x :^A B \to x :^A C}\,L\supset}{x :^A (B \supset C), x :^A B \to x : [A]C}\,R[\,]}{x : [A](B \supset C), x : [A]B \to x : [A]C}\,L[\,]}{\to x : [A](B \supset C) \supset ([A]B \supset [A]C)}\,R\supset$$

where the top sequents are derivable by Lemma 4.3.1. Axiom A3 (right-to-left direction) is derivable as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^A B \to x :^A C, x :^A B}{x :^A B \to x :^A C, x : [A]B}\,R[\,] \qquad \dfrac{x :^A C, x :^A B \to x :^A C}{x : [A]C, x :^A B \to x :^A C}\,L[\,]}{x : [A]B \supset [A]C, x :^A B \to x :^A C}\,L\supset}{x : [A]B \supset [A]C \to x :^A B \supset C}\,R\supset}{x : [A]B \supset [A]C \to x : [A](B \supset C)}\,R[\,]}{\to x : ([A]B \supset [A]C) \supset [A](B \supset C)}\,R\supset$$

where the top sequents are derivable by Lemma 4.3.1. Axiom A4 can be derived by the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{x :^A B \to x :^A B}{x :^A \neg B, x :^A B \to}\,L\neg}{x : [A]\neg B, x : [A]B \to}\,L[\,]}{x : [A]\neg B \to x : \neg[A]B}\,R\neg}{\to x : [A]\neg B \supset \neg[A]B}\,R\supset \qquad \dfrac{\dfrac{\dfrac{\dfrac{x :^A B \to x :^A B}{\to x :^A \neg B, x :^A B}\,R\neg}{\to x : [A]\neg B, x : \neg[A]B}\,R[\,]}{x : \neg[A]B \to x : [A]\neg B}\,L\neg}{\to x : \neg[A]B \supset [A]\neg B}\,R\supset$$

where the top sequents are derivable by Lemma 4.3.1. Axiom A5 (left-to-right direction) has the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y :^A B, xR_ay, x :^A \mathsf{K}_aB, y : A \to y :^A B}{y :^A B, xR_ay, x :^A \mathsf{K}_aB, y : A \to y : [A]B}\,R[\,]}{xR_ay, x :^A \mathsf{K}_aB, y : A \to y : [A]B}\,L\mathsf{K}'}{xR_ay, x :^A \mathsf{K}_aB \to y : A \supset [A]B}\,R\supset}{x :^A \mathsf{K}_aB \to x : \mathsf{K}_a(A \supset [A]B)}\,R\mathsf{K}}{x : [A]\mathsf{K}_aB \to x : \mathsf{K}_a(A \supset [A]B)}\,L[\,]}{\to x : [A]\mathsf{K}_aB \supset \mathsf{K}_a(A \supset [A]B)}\,R\supset$$

where the top sequent is derivable by Lemma 4.3.1. Axiom A5 (right-to-left direction) is derivable with the following derivation

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x : \mathsf{K}_a(A \supset [A]B), xR_a y, y : A \to y :^A B, y : A \qquad \cfrac{y :^A B, x : \mathsf{K}_a(A \supset [A]B), xR_a y, y : A \to y :^A B}{y : [A]B, x : \mathsf{K}_a(A \supset [A]B), xR_a y, y : A \to y :^A B}\, L[\,]}{y : A \supset [A]B, x : \mathsf{K}_a(A \supset [A]B), xR_a y, y : A \to y :^A B}\, L{\supset}}{x : \mathsf{K}_a(A \supset [A]B), xR_a y, y : A \to y :^A B}\, L\mathsf{K}}{x : \mathsf{K}_a(A \supset [A]B) \to x :^A \mathsf{K}_a B}\, R\mathsf{K}'}{x : \mathsf{K}_a(A \supset [A]B) \to x : [A]\mathsf{K}_a B}\, R[\,]}{\to x : \mathsf{K}_a(A \supset [A]B) \supset [A]\mathsf{K}_a B}\, R{\supset}
$$

where the top sequents are derivable by Lemma 4.3.1. The derivation of axiom A6 is found as follows

$$
\cfrac{\cfrac{\cfrac{\cfrac{x :^{A,B} C \to x :^{A,B} C}{x :^{A,B} C \to x :^{A \wedge [A]B} C}\, R_{cmp}}{x :^{A,B} C \to x : [A \wedge [A]B]C}\, R[\,]}{x : [A][B]C \to x : [A \wedge [A]B]C}\, L[\,]}{\to x : [A][B]C \supset [A \wedge [A]B]C}\, R{\supset}
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{x :^{A,B} C \to x :^{A,B} C}{x :^{A \wedge [A]B} C \to x :^{A,B} C}\, L_{cmp}}{x :^{A \wedge [A]B} C \to x : [A][B]C}\, R[\,]}{x : [A \wedge [A]B]C \to x : [A][B]C}\, L[\,]}{\to x : [A \wedge [A]B]C \supset [A][B]C}\, R{\supset}
$$

where the top sequents are derivable by Lemma 4.3.1. The derivation of compositionality axiom A6 requires the rules L_{cmp} and R_{cmp}. These rules make derivable also the other property of PAL, that is, the associativity of public announcements (see 4.1.1) by the following

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x :^{A,B,C} D \to x :^{A,B,C} D}{x :^{A,B \wedge [B]C} D \to x :^{A,B,C} D}\, L_{cmp}}{x : [A][B \wedge [B]C]D \to x :^{A,B,C} D}\, L[\,]}{x : [A][B \wedge [B]C]D \to x :^{A,B} [C]D}\, R[\,]}{x : [A][B \wedge [B]C]D \to x :^{A \wedge [A]B} [C]D}\, R_{cmp}}{x : [A][B \wedge [B]C]D \to x : [A \wedge [A]B][C]D}\, R[\,]}{\to x : [A][B \wedge [B]C]D \supset [A \wedge [A]B][C]D}\, R{\supset}
$$

and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x :^{A,B,C} D \to x :^{A,B,C} D}{x :^{A,B,C} D \to x :^{A,B\wedge[B]C} D}\, R_{cmp}}{x :^{A,B,C} D \to x : [A][B \wedge [B]C]D}\, R[\,]}{x :^{A,B} [C]D \to x : [A][B \wedge [B]C]D}\, L[\,]}{x :^{A\wedge[A]B} [C]D \to x : [A][B \wedge [B]C]D}\, L_{cmp}}{x : [A \wedge [A]B][C]D \to x : [A][B \wedge [B]C]D}\, L[\,]}{\to x : [A \wedge [A]B][C]D \supset [A][B \wedge [B]C]D}\, R\supset
$$

The proofs of admissibility of *modus ponens* and necessitation is the same as in **G3K**.

$\boxtimes$

It is possible to give also a direct completeness proof for **G3PAL** following the pattern of Theorem 2.4.3 of the previous section. The theorem has been proved for the P-interpretation of announcements in Maffezioli and Negri (2010) and we shall give here the corresponding proof for the GG-interpretation.

Theorem 4.4.2. *For all* $\Gamma \to \Delta$ *in* **G3PAL** *either* $\Gamma \to \Delta$ *is derivable or it has a countermodel.*

Proof. We define for an arbitrary $\Gamma \to \Delta$ of **G3PAL** a reduction tree by applying the rules of **G3PAL** root first in all possible ways. If the construction terminates we obtain a proof, else the tree becomes infinite. By König's lemma an infinite tree has an infinite branch that is used to define a countermodel to the end-sequent.

Construction of the reduction tree

The reduction tree is defined inductively in stages as follows: Stage 0 has $\Gamma \to \Delta$ at the root of the tree. Stage $n > 0$ has two cases:

CASE I: If every topmost sequent is initial or a conclusion of $L\bot$ the construction of the tree ends.

CASE II: If not every topmost sequent is initial or a conclusion of $L\bot$, we continue the construction of the tree by writing above those sequents that are not initial nor a conclusion of $L\bot$, other sequents that are obtained by applying root first the rules of **G3PAL** whenever possible, in a give order. There are 14 different stages, 8 for the propositional and atomic rules, 2 for the epistemic rules for each K, 2 for the announcement rules, and 2 for the composition of announcements. At stage $n = 15$ we repeat stage 1, at stage $n = 16$ we repeat stage 2, and so on for every n. We will not take into account the details of the proof when the topmost sequents have either a conjunction, or a disjunction, or an implication, or else an epistemic formula as principal formula, the proof being similar to the proof given in Negri (2009). The essentially new cases are as follows.

We start, for $n = 1$, with $L0$. For each topmost sequent of the form

$$x_1 :^{\varphi,A} P_1, \ldots, x_m :^{\varphi,A} P_m \Gamma' \rightarrow \Delta$$

where $P_1, \ldots P_m$ are all the formulas in Γ with an atom as the principal formula, we write

$$x_1 :^{\varphi} P_1, \ldots, x_m :^{\varphi} P_m, \Gamma' \rightarrow \Delta$$

on top of it. This corresponds to applying m times rule $L0$.

For $n = 2$, with $R0$. For each topmost sequent of the form

$$\Gamma \rightarrow \Delta' x_1 :^{\varphi,A} P_1, \ldots, x_m :^{\varphi,A} P_m$$

where $P_1, \ldots P_m$ are all the formulas in Δ with an atom as the principal formula, we write

$$\Gamma' \rightarrow \Delta' x_1 :^{\varphi} P_1, \ldots, x_m :^{\varphi} P_m$$

on top of it. This corresponds to applying m times rule $R0$. For the stages from $n = 3$ to $n = 10$, corresponding to propositional and epistemic cases, the proof is analogous to Negri (2009).

For $n = 11$, take all the topmost sequents with $x_1 :^{\varphi} [B_1]C_1, \ldots, x_m :^{\varphi} [B_m]C_m$ in the antecedent, and write on top of these sequents

$$x_1 :^{\varphi, B_1} C_1, \ldots, x_m :^{\varphi, B_m} C_m, \Gamma' \to \Delta$$

For $n = 12$, take all the topmost sequents with $x_1 :^{\varphi} [B_1]C_1, \ldots, x_m :^{\varphi} [B_m]C_m$ in the succedent, and write on top of these sequents

$$\Gamma \to \Delta', x_1 :^{\varphi, B_1} C_1, \ldots, x_m :^{\varphi, B_m} C_m$$

For $n = 13$, we consider all the topmost sequents with the multiset of formulas $x_1 :^{\varphi, A \wedge [A]B} C_1, \ldots, x_m :^{\varphi, A \wedge [A]B} C_m$ in the antecedent, and write on top of these sequents

$$x_1 :^{\varphi, A, B} C_1, \ldots, x_m :^{\varphi, A, B} C_m, \Gamma' \to \Delta$$

that is, apply m times L_{cmp}.

Likewise, for $n = 14$, take all the topmost sequents with the multiset of formulas $x_1 :^{\varphi, A \wedge [A]B} C_1, \ldots, x_m :^{\varphi, A \wedge [A]B} C_m$ in the succedent, and write on top of these sequents

$$\Gamma \to \Delta', x_1 :^{\varphi, A, B} C_1, \ldots, x_m :^{\varphi, A, B} C_m$$

that is, apply m times R_{cmp}.

For any n, for each sequent that is neither initial, nor conclusion of $L\bot$, nor treatable by any one of the above reductions, we write the sequent itself above it. If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot$, and the tree, read from the leaves to the root, yields a derivation.

Construction of the countermodel

By König's lemma, if the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \to \Delta_0 \equiv \Gamma \to \Delta, \Gamma_1 \to \Delta_1, \ldots, \Gamma_i \to \Delta_i, \ldots$ be one such branch. Consider the set of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i \geq 0} \Gamma_i \quad \text{and} \quad \mathbf{\Delta} \equiv \bigcup_{i \geq 0} \Delta_i$$

We define a restricted Kripke model that forces all formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \to \Delta$. The construction of the countermodel is similar to that given in Theorem 2.4.3 and in Negri (2009). The new cases are:

If $x :^{\varphi} [B]C$ is in $\mathbf{\Gamma}$, we find $x :^{\varphi,B} C$ in $\mathbf{\Gamma}$. By IH $x \Vdash^{\varphi,B} C$, and therefore $x \Vdash^{\varphi} [B]C$ in the model.

If $x :^{\varphi} [B]C$ is in $\mathbf{\Delta}$, consider the step at which the reduction for $x :^{\varphi} [B]C$ applies. We find $w :^{\varphi,B} C$ in $\mathbf{\Delta}$. By IH $x \nVdash^{\varphi,B} C$, and by definition of the semantics $x \nVdash^{\varphi} [B]C$.

If $x :^{\varphi,A\wedge[A]B} C$ is in $\mathbf{\Gamma}$, for some i, $x :^{\varphi,A,B} C$ is in Γ_i. By IH $x \Vdash^{\varphi,A,B} C$ and by Lemma 4.1.2 we conclude $x \Vdash^{\varphi,A\wedge[A]B} C$.

If $x :^{\varphi,A\wedge[A]B} C$ is in $\mathbf{\Delta}$, for some i, $x :^{\varphi,A,B} C$ is in Δ_i. By IH $x \nVdash^{\varphi,A,B} C$ and by Lemma 4.1.2 we conclude $x \nVdash^{\varphi,A\wedge[A]B} C$. ⊠

Corollary 4.4.3. *If a sequent* $\Gamma \to \Delta$ *is valid in every restricted Kripke model then it is derivable in* **G3PAL**.

4.5 Conclusions

Although we focused mostly on GG-announcements, in the literature on DEL the P-interpretation of announcement is dominant. In van Ditmarsch *et al.* (2007, p.

89) is proposed the following axiomatization for Plaza's announcements, where it is clear that it is assumed the truth of what is announced.

A1	All the axioms of modal logic **S5**	
A2	$[A]P \supset\subset (A \supset P)$	Atomic Permanence
A3	$[A](B \wedge C) \supset\subset ([A]B \wedge [A]C)$	Announcement and Conjunction
A4	$[A]\neg B \supset\subset (A \supset \neg[A]B)$	Announcement and Negation
A5	$[A]\mathsf{K}_a B \supset\subset (A \supset \mathsf{K}_a[A]B)$	Announcement and Knowledge
A6	$[A][B]C \supset\subset [A \wedge [A]B]C$	Announcements Composition
R1	From $\Gamma \vdash A \supset B$ and $\Delta \vdash A$ infer $\Gamma, \Delta \vdash B$	Modus Ponens
R2	From $\vdash A$ infer $\vdash \mathsf{K}_a A$	Necessitation

Obviously, axioms A1, A3 and A5 cannot be derived using the rules applied so far. However, if we consider the rules for P-announcements it is possible to find a derivation of all the axioms listed above. For instance, consider axiom A1 using and derive it by using the rules $L0'$, $R0'$, $L[\,]'$ and $R[\,]'$ of the previous section. We have:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : A, x : P, x : [A]P, x : A \to x : P}{x :^{A} P, x : [A]P, x : A \to x : P}\,{\scriptstyle L0'}}{x : [A]P, x : A \to x : P}\,{\scriptstyle L[\,]'}}{x : [A]P \to x : A \supset P}\,{\scriptstyle R\supset}}{\to x : [A]P \supset (A \supset P)}\,{\scriptstyle R\supset}$$

And, in the opposite direction

$$\dfrac{\dfrac{\dfrac{x : A \supset P, x : A \to x : A \qquad \dfrac{x : A \to x : P, x : A \quad x : P, x : A \to x : P}{x : A \supset P, x : A \to x : P}\,{\scriptstyle L\supset}}{x : A \supset P, x : A \to x :^{A} P}\,{\scriptstyle R0'}}{x : A \supset P \to x : [A]P}\,{\scriptstyle R[\,]'}}{\to x : (A \supset P) \supset [A]P}\,{\scriptstyle R\supset}$$

In this chapter, we introduced a sequent system for the logic of public announcements and proved that all the structural properties are satisfied. Moreover, we proved both indirectly, through equivalence with the axiomatic system, and directly, through the method of reduction trees, its completeness with respect to the semantics of restricted Kripke models. As we pointed out, **G3PAL** is not only a different formalism, alternative to the standard axiom systems: It is designed for making explicit the structure of proofs in PAL. The novelty of **G3PAL** is that the rules incorporate the notion of model change and the dynamics of information update through the internalization of semantics of restricted forcing into the syntax of the calculus. The next step should be that of adding rules to deal with the common knowledge operator (cf. van Ditmarsch *et al.* 2007) in order to formalize sentences such as: "After it is announced that A, it is a common knowledge among the agents that A". However, the proof theory of the logic of common knowledge (with or without public announcements) is problematic and requires a rule with an infinite number of premise. Thus, the possibility of mechanizing proofs is definitely lost. A closely related approach is presented in Balbiani *et al.* (2010) in which a tableau system for PAL is given. From the point of view of sequent systems, a tableau proof can be regarded as a single-sided sequent calculus proof, with formulas only in the antecedent, that aims at a check for satisfiability, whereas a sequent proof in a labelled system is a check for validity. By the duality in a classical framework between the unsatisfiability of a formula and the validity of its negation, the two approaches are duals to each other. The tableau system of Balbiani *et al.* (2010) operates on labelled formulas and accessibility relations: It has labels that range over natural numbers, which would seem to impose a restriction to linear orders, whereas our system does not assume any underlying implicit structure on the set of labels, but imposes it with suitable properties of the explicit accessibility relation. A closed tableau corresponds to a proof in our system, whereas an open tableau gives a countermodel.

Chapter 5

The Church-Fitch Paradox

In this chapter, the attention focuses on the **Church-Fitch paradox of knowability** (see Fitch 1963 and Salerno 2009 for an historical introduction). The Church-Fitch's paradox is a well-known semantic paradox that claims to threaten the anti-realist position about truth by deriving that every truth is actually known (omniscience thesis) from the assumption that every truth is possibly known (knowability principle). The principle gets formalized as $A \supset \Diamond \mathsf{K}A$ ("if A is true then it possible to know that A"), and along with minimal assumptions on K and the use of classical logic, makes it possible to derive the counter-intuitive conclusion $A \supset \mathsf{K}A$ ("if A is true that it is known"). We propose a Gentzen-style reconstruction of the Church-Fitch paradox following a labelled approach to sequent calculi. First, it is shown how to identify the semantic condition for $A \supset \Diamond \mathsf{K}A$ to be valid by exploiting cut elimination in labelled systems. This condition is then converted into non-logical inference rule by applying the method of-axioms-as-rules. Finally, when the rule is made part of logical system, it is possible to provide an adequate proof-theory governing the interaction among the modalities involved in Fitch's proof and to give a logical framework for dealing with Fitch's paradox (knowability logic). Moreover, it is argued in favor of the use of intuitionsitic logic as a solution of the paradox and it is shown that $A \supset \mathsf{K}A$ is only classically derivable, but neither intuitionistically derivable nor intuitionistically admissible.

5.1 Overview

According to the Dummettian tradition in the philosophy of language, realism / anti-realism debate can be characterized in terms of the notion of truth involved. Realism takes the notion of truth either as primitive or as defined over the notion of "fact", whereas anti-realism embraces an epistemic conception of truth. One possible version of this epistemic conception is the following:

(1E) A is true if and only if it is possible to exhibit a direct justification for A.

A justification is something connected to linguistic practice, therefore it is supposed not to transcend our epistemic capacities. This leads to:

(2E) If it is possible to exhibit a direct justification for A, then it is possible to know that A.

Putting (1E) and (2E) together we get what is called the **knowability principle**:

(3E) If A is true, then it is possible to know that A.

What is known as the Fitch or Church-Fitch paradox is an argument that threatens the anti-realist position: In the argument, it is concluded from the knowability principle that all truths are actually known, a paradoxical consequence, known as the principle of **omniscience**, that undermines the epistemic conception of truth. The paradox was presented in Fitch (1963) but, as recently discovered by Joe Salerno and Julien Murzi, it was actually suggested by Church in a series of referee's reports dating back to 1945 and now reproduced in Salerno (2009). The force of the argument lies in the fact that it is a formal argument, completely developed in a plainly faultless logical setting. More precisely, the knowability principle is formalized with a schema that uses two modal operators, K and $\Diamond$. The first is a zero-agent epistemic operator K to be read as "it is known that ...". The second is the possibility operator $\Diamond$ to be read as "it is possible that ...". In this formal language, the knowability principle takes the form of the schema

$$A \supset \Diamond \mathsf{K} A \qquad \textbf{KP}$$

In the same manner, omniscience is formalized by the schema

$$A \supset \mathsf{K} A \qquad \textbf{OP}$$

The Church-Fitch paradox consists in a formal derivation that starts from **KP**, passes through its instance with the Moore sentence[1] $A \wedge \neg \mathsf{K} A$, and then leads to **OP** by using only logical steps. We shall consider here only the definition of K as a primitive modal operator, and not the one, alternatively proposed by Fitch at the end of his paper (1963, p. 141) in which K is defined on the basis of a causation relation that allows to define knowledge in terms of justified true belief.

Many different ways to block the paradox have been proposed. They can be grouped into three categories of intervention:

1. Restriction on the possible instances of **KP** (Dummett 2001, Tennant 1997, 2009, Restall 2009);

2. Reformulation of the formalization of the knowability principle (Edington 1985, Rabinowicz and Segerberg 1994, Martin-Löf 1998, van Benthem 2009, Burgess 2009, Proietti and Sandu 2010, Artemov and Protopopescu 2011, Proietti 2011);

3. Revision of the logical framework in which the derivation is made (Williamson 1982, Beall 2000, 2009, Wansing 2002a, Dummett 2009, Giaretta 2009, Priest 2009).

[1] We extend here to knowledge the usual notion of Moore sentence, originally conceived for belief.

Even if some of the proposed solutions focus on the type of derivability relation that connects **OP** to **KP**, none of them has taken derivations themselves as objects of study or analyzes the structure of the derivation of **OP** from **KP**. Our precise aim, instead, is to focus on this analysis. Before proceeding, it is worth noting that the standard derivation of the Church-Fitch paradox is given in an axiomatic calculus (Beall 2000, Brogaard and Salerno 2009, Wansing 2002a). This calculus hides structural operations such as cut, weakening and contraction. For the purposes of an analysis that leaves no inferential passage implicit, it is therefore preferable to move to systems of sequent calculus that make these operations explicit, and, by a suitable design as achieved in the **G3**-systems, completely eliminable. We begin with a sequent calculus derivation of the Church-Fitch paradox, built by translating a natural deduction derivation. The calculus that it is used is contraction free and cut free, thus a good basis for the structural analysis of the paradox. However, the presence of an axiomatic assumption in the derivation results in a non-eliminable cut. As we shall see, the method of axioms-as-rules is not applicable here because the knowability principle cannot be reduced to its atomic instances. This fact is established syntactically by means of a failed proof search in the given sequent system. We turn therefore to the method of labelled calculi and present a bimodal extension of the system **G3I** of Ch. 3. We show that the system has all the structural rules admissible. The system is equivalent to a standard axiomatic system used in the analysis of the paradox, but the labelled approach allows a stronger completeness result: We prove completeness in a direct way by showing that for every sequent in the language of the logic in question, either there is a proof in the calculus, or a countermodel in a precisely defined frame class is found. The completeness result is used for showing that the classical standard form of the Church-Fitch paradox is not derivable intuitionistically: We consider the classically derivable sequent with **KP** instantiated with the Moore sentence $A \wedge \neg \mathsf{K} A$ as an antecedent and **OP** instantiated with A as succedent. Then, by the failed proof search, we extract a countermodel for **OP**. The argument suffices for blocking, within an intuitionistic bimodal system, the specific proof of the paradox, but it is not yet conclusive. To conclude that an intuitionistic system that incorporates **KP** as a derivation principle does not derive **OP**, it is not sufficient to shown that **OP** does not follow from a particular instance of **KP**. Therefore, we make clear, through an example from classical logic, that the notion to be considered when comparing principles of proof should be admissibility, rather than derivability. To clarify the relation between the two principles, it is necessary to make explicit the conditions that characterize their validity. The semantical assumption behind the axiom schema **KP** is determined, and the frame condition **KP-Fr** is then made part of the logical system in the form of a block of additional rules of inference, linked by a variable condition. By this addition, a complete contraction- and cut-free proof system for intuitionistic bimodal logic extended with the knowability principle is obtained. We show, using proof search and construction of countermodels, that **OP** is not derivable in the system, therefore not valid. We also discuss how an oversight on the variable condition could lead to an opposite conclusion. We then show how, by just adding symmetry of the preorder, **OP** becomes derivable. The latter is a cut-free derivation of the Church-Fitch paradox that uses **KP** as a derivation principle and that guarantees that the source of the paradox is to be found only in the assumption on which it depends. It is also shown that the same result can be obtained for belief-like notions of knowledge that do not assume factivity among their defining principles.

5.2 Towards a structural analysis of the Church-Fitch argument

The Church-Fitch paradox was originally presented in Fitch (1963) without using an explicit logical system, and it was later formalized using semantic arguments and various deductive systems for modal logic: linear derivations, natural deduction, sequent calculus. All these formalizations have contributed to single out a minimal logical ground that gives rise to the paradox. It consists in a basic bimodal logic that extends classical propositional logic with an alethic modality $\Diamond$ and an epistemic modality K. No requirement is made on the alethic modality, whereas the epistemic modality is supposed to satisfy **distributivity over conjunction**, $\mathsf{K}(A \wedge B) \supset \mathsf{K}A \wedge \mathsf{K}B$, and **factivity**, $\mathsf{K}A \supset A$. The former property is indeed derivable for any necessity-like modality in normal modal logic, so the only requirement added to a normal bimodal logic is factivity of K. A formalization of the Church-Fitch argument is the first step towards its analysis. We start with a derivation in natural deduction:

$$
\dfrac{\dfrac{\dfrac{[A \wedge \neg \mathsf{K}A]^{2} \qquad \overset{\mathbf{KP}}{A \wedge \neg\mathsf{K}A \supset \Diamond\mathsf{K}(A\wedge\neg\mathsf{K}A)}}{\Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)}\supset E \qquad \dfrac{\dfrac{[\mathsf{K}(A\wedge\neg\mathsf{K}A)]^{1}}{\mathsf{K}A}\,\mathsf{K}\wedge 1 \qquad \dfrac{\dfrac{[\mathsf{K}(A\wedge\neg\mathsf{K}A)]^{1}}{\mathsf{K}\neg\mathsf{K}A}\,\mathsf{K}\wedge 2}{\neg\mathsf{K}A}\,\mathsf{K}E}{\bot}\supset E}{\bot}\,\Diamond E_1}{\neg(A\wedge\neg\mathsf{K}A)}\supset I_2
$$

The conclusion is the weaker intuitionistic version of **OP**, and intuitionistically equivalent to $A \supset \neg\neg\mathsf{K}A$. We will call both of them **WOP** (for **weak omniscience principle**). The conclusion $A \supset \mathsf{K}A$ is obtained by classical propositional steps and leads, in conjunction with factivity, to the identification of truth and knowledge, $A \supset\subset \mathsf{K}A$. A closer inspection of the derivation above shows that we used the following rules

$$\frac{\mathsf{K}(A \wedge B)}{\mathsf{K}A}\ \mathsf{K}\wedge 1 \qquad \frac{\mathsf{K}(A \wedge B)}{\mathsf{K}B}\ \mathsf{K}\wedge 2$$

These are derivable in any system of normal epistemic modal logic. Rule $\mathsf{K}E$ corresponds to factivity of knowledge and rule $\Diamond E$ is the dual of the familiar necessitation rule: observe that the latter can be formulated in natural deduction as

$$\dfrac{\Diamond A \qquad \begin{matrix} \overset{1}{[A]} \\ \vdots \\ \bot \end{matrix}}{\bot}\ \Diamond E_1$$

The minor premise of the rule is $\bot$ and may depend on A, discharged by the rule, similarly to the rule of existence elimination, with falsity, rather than any formula not containing the *eigenvariable*, as the minor premise. With a sequent notation and an empty succedent in place of $\bot$, the rule becomes

$$\frac{A \rightarrow}{\Diamond A \rightarrow}$$

This rule is the dual of the rule of of necessitation:

$$\frac{\rightarrow A}{\rightarrow \Box A}$$

A further step in the analysis of derivations comes from sequent calculus that has several advantages over natural deduction. First, structural steps are explicit and not hidden in vacuous and multiple discharge and in non-normal instances of rules (see Negri and von Plato 2001, Ch. 1). Secondly, sequent calculus, contrary to natural deduction, is well suited for classical logic and its modal extensions. The sequent calculus that we shall use is obtained as an extension of the classical propositional contraction-free sequent calculus **G3c** with the following rules for the alethic and epistemic modalities, where $\mathsf{K}\Gamma$ denotes the multiset of all the $\mathsf{K}A$ for A in Γ:

Modal rules of **G3◊K**

$$\frac{\Gamma \to A}{\mathsf{K}\Gamma, \Theta \to \Delta, \mathsf{K}A}\ LR\text{-}\mathsf{K} \qquad \frac{A, \mathsf{K}A, \Gamma \to \Delta}{\mathsf{K}A, \Gamma \to \Delta}\ L\mathsf{K}$$

$$\frac{A \to \Delta}{\Diamond A, \Gamma \to \Theta, \Diamond\Delta}\ LR\text{-}\Diamond$$

The resulting system, called **G3◊K**, is an extension of the calculus **G3K** presented in section 4 of Hakli and Negri (2011), and the proof of its structural properties follows the lines of the proof for **G3K**:

Theorem 5.2.1. *In* **G3◊K** *it holds that:*

i) All sequents of the form $A, \Gamma \to \Delta, A$ *are derivable;*

ii) All the propositional rules are height-preserving invertible;

iii) Weakening and contraction are hp-admissible.

iv) Cut is admissible.

Proof. We show here only one extra case that arises in the proof of cut elimination because of the addition of rule $L\mathsf{K}$, with the cut formula principal in both premises of cut, the right one being $L\mathsf{K}$:

$$\dfrac{\dfrac{\Gamma \to A}{\Theta, \mathsf{K}\Gamma \to \Delta, \mathsf{K}A}\ LR\text{-}\mathsf{K} \qquad \dfrac{\mathsf{K}A, A, \Gamma' \to \Delta'}{\mathsf{K}A, \Gamma' \to \Delta'}\ L\mathsf{K}}{\Theta, \mathsf{K}\Gamma, \Gamma' \to \Delta, \Delta'}\ \text{CUT}$$

The cut is transformed as follows in two consecutive cuts, the upper of decreased derivation height, the lower of decreased cut formula height. Repeated applications of LK are denoted by a double inference line.

$$
\cfrac{\Gamma \to A \qquad \cfrac{\Theta, \mathsf{K}\Gamma \to \Delta, \mathsf{K}A \quad \mathsf{K}A, A, \Gamma' \to \Delta'}{A, \Theta, \mathsf{K}\Gamma, \Gamma' \to \Delta, \Delta'}\ \text{CUT}}{\cfrac{\Theta, \Gamma, \mathsf{K}\Gamma, \Gamma' \to \Delta, \Delta'}{\Theta, \mathsf{K}\Gamma, \Gamma' \to \Delta, \Delta'}\ L\mathsf{K}}\ \text{CUT}
$$

The conversions for a cut formula of the form $\Diamond A$ principal in both premises of cut in LR-$\Diamond$ is symmetric to the conversion of a cut formula of the form $\mathsf{K}A$ principal in both premises of cut in LR-K treated in the above mentioned article. ⊠

The sequent-style reconstruction of the Church-Fitch paradox calls for the following

Lemma 5.2.2. *The following rules*

$$
\frac{\to A \wedge \neg \mathsf{K}A \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)}{A, \neg \mathsf{K}A \to \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)}\ \mathit{Inv} \qquad \frac{\mathsf{K}A, \mathsf{K}\neg \mathsf{K}A \to}{\mathsf{K}(A \wedge \neg \mathsf{K}A) \to}\ \mathit{Distr}
$$

are derivable in **G3◊K** *with cut.*

Proof. By the two derivations

$$
\cfrac{\to (A \wedge \neg \mathsf{K}A) \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A) \qquad \cfrac{\cfrac{A \to A \quad \neg \mathsf{K}A \to \neg \mathsf{K}A}{A, \neg \mathsf{K}A \to A \wedge \neg \mathsf{K}A}\ R\wedge \qquad \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A) \to \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)}{A, \neg \mathsf{K}A, (A \wedge \neg \mathsf{K}A) \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A) \to \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)}\ L\supset}{A, \neg \mathsf{K}A \to \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)}\ \text{CUT}
$$

$$
\begin{prooftree}
\AxiomC{$A, \neg \mathsf{K}A \to \neg \mathsf{K}A$}
\RightLabel{$L\wedge$}
\UnaryInfC{$A \wedge \neg \mathsf{K}A \to \neg \mathsf{K}A$}
\RightLabel{LR-K}
\UnaryInfC{$\mathsf{K}(A \wedge \neg \mathsf{K}A) \to \mathsf{K}\neg \mathsf{K}A$}
\AxiomC{$A, \neg \mathsf{K}A \to A$}
\RightLabel{$L\wedge$}
\UnaryInfC{$A \wedge \neg \mathsf{K}A \to A$}
\RightLabel{LR-K}
\UnaryInfC{$\mathsf{K}(A \wedge \neg \mathsf{K}A) \to \mathsf{K}A$}
\AxiomC{$\mathsf{K}A, \mathsf{K}\neg \mathsf{K}A \to$}
\RightLabel{CUT}
\BinaryInfC{$\mathsf{K}(A \wedge \neg \mathsf{K}A), \mathsf{K}\neg \mathsf{K}A \to$}
\RightLabel{CUT}
\BinaryInfC{$\mathsf{K}(A \wedge \neg \mathsf{K}A) \to$}
\end{prooftree}
$$

The topmost sequents, except of the premises of the rules in question, are derivable by Theorem 5.2.1.

$\boxtimes$

A proof of the Church-Fitch paradox can now be obtained as a derivation in system **G3◊K** of the sequent $\to A \supset \mathsf{K}A$ from a special instance of the knowability principle **KP**, the sequent $\to (A \wedge \neg \mathsf{K}A) \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)$, as follows

$$
\begin{prooftree}
\AxiomC{$\mathbf{KP}$}
\noLine
\UnaryInfC{$\to A \wedge \neg \mathsf{K}A \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)$}
\RightLabel{Inv}
\UnaryInfC{$A, \neg \mathsf{K}A \to \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)$}
\AxiomC{$\mathsf{K}A \to \mathsf{K}A$}
\RightLabel{$L\neg$}
\UnaryInfC{$\mathsf{K}A, \neg \mathsf{K}A \to$}
\RightLabel{L-W}
\UnaryInfC{$\mathsf{K}A, \mathsf{K}\neg \mathsf{K}A, \neg \mathsf{K}A \to$}
\RightLabel{$L\mathsf{K}$}
\UnaryInfC{$\mathsf{K}A, \mathsf{K}\neg \mathsf{K}A \to$}
\RightLabel{$Distr$}
\UnaryInfC{$\mathsf{K}(A \wedge \neg \mathsf{K}A) \to$}
\RightLabel{LR-$\Diamond$}
\UnaryInfC{$\Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A) \to$}
\RightLabel{CUT}
\BinaryInfC{$A, \neg \mathsf{K}A \to$}
\RightLabel{$R\neg$}
\UnaryInfC{$A \to \neg\neg \mathsf{K}A$}
\AxiomC{$\mathsf{K}A \to \mathsf{K}A$}
\RightLabel{$R\neg$}
\UnaryInfC{$\to \mathsf{K}A, \neg \mathsf{K}A$}
\RightLabel{$L\neg$}
\UnaryInfC{$\neg\neg \mathsf{K}A \to \mathsf{K}A$}
\RightLabel{CUT}
\BinaryInfC{$A \to \mathsf{K}A$}
\RightLabel{$R\supset$}
\UnaryInfC{$\to A \supset \mathsf{K}A$}
\end{prooftree}
$$

Observe that the presence of the the sequent $\to A \wedge \neg \mathsf{K}A \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)$ from which the derivation starts makes the application of cut non-eliminable because, in general, cut elimination fails when cuts depend on proper axioms. By applying the method of axioms-as-rules introduced in the previous section, **KP** should be converted into a rule of the form

$$
\frac{\Diamond \mathsf{K}A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\ Kn
$$

This rule can be easily proved to be equivalent to the sequent $\to A \supset \Diamond\mathsf{K}A$. Then, it should be reduced to a rule that has only formulas devoid of logical structure as principal, *i.e.*, a reduction of the general knowability principle to the knowability principle for only atomic formulas. If such were the case, the rule in the above derivation could be turned into a left rule of sequent calculus with atomic principal formulas, of the form

$$\frac{\Diamond\mathsf{K}P, \Gamma \to \Delta}{P, \Gamma \to \Delta}\; {\scriptstyle Kn\text{-}At}$$

However, it can be proved that the knowability principle cannot be reduced to its atomic instances. By the following result, the knowability principle on a conjunction does not follow from the knowability on its conjuncts.

Lemma 5.2.3. *The sequent* $P \supset \Diamond\mathsf{K}P, Q \supset \Diamond\mathsf{K}Q \to P \wedge Q \supset \Diamond\mathsf{K}(P \wedge Q)$ *is not derivable in* **G3◊K**.

Proof. The result is obtained through a failed proof-search procedure: Start a derivation tree with the sequent $P \supset \Diamond\mathsf{K}P, Q \supset \Diamond\mathsf{K}Q \to P \wedge Q \supset \Diamond\mathsf{K}(P \wedge Q)$ as a root and apply backwards all the propositional rules:

$$\dfrac{\dfrac{Q \supset \Diamond\mathsf{K}Q, P, Q \to \Diamond\mathsf{K}(P \wedge Q), P \qquad \dfrac{\Diamond\mathsf{K}P, P, Q \to \Diamond\mathsf{K}(P \wedge Q), Q \qquad \begin{matrix}\vdots \\ \Diamond\mathsf{K}P, \Diamond\mathsf{K}Q, P, Q \to \Diamond\mathsf{K}(P \wedge Q)\end{matrix}}{\Diamond\mathsf{K}P, Q \supset \Diamond\mathsf{K}Q, P, Q \to \Diamond\mathsf{K}(P \wedge Q)}\, {\scriptstyle L\supset}}{\dfrac{P \supset \Diamond\mathsf{K}P, Q \supset \Diamond\mathsf{K}Q, P, Q \to \Diamond\mathsf{K}(P \wedge Q)}{P \supset \Diamond\mathsf{K}P, Q \supset \Diamond\mathsf{K}Q, P \wedge Q \to \Diamond\mathsf{K}(P \wedge Q)}\, {\scriptstyle L\wedge}}\, {\scriptstyle L\supset}}{P \supset \Diamond\mathsf{K}P, Q \supset \Diamond\mathsf{K}Q \to P \wedge Q \supset \Diamond\mathsf{K}(P \wedge Q)}\, {\scriptstyle R\supset}$$

Since the rules used are invertible, there is no need of backtracking. The left premises of the two steps of $L \supset$ are initial sequents, and therefore derivability of the sequent is equivalent to derivability of the rightmost sequent, $\Diamond\mathsf{K}P, \Diamond\mathsf{K}Q, P, Q \to$

$\Diamond\mathsf{K}(P \wedge Q)$. Proof search for the latter can be effected in two ways, depending on the choice of principal formula in LR-$\Diamond$, each followed by an application of LR-K. In one case it leads to the sequent $Q \to P$, in the other to $P \to Q$. Since both are underivable, the proof search fails.

$\boxtimes$

By Lemma 5.2.3 we conclude that the rule of knowability on arbitrary formulas does not follow from its restriction to atomic formulas. The method of conversion of axiom into rules, successfully employed elsewhere for extending structural proof analysis from standard sequent calculi to systems with added axioms (see Negri and von Plato 2001, 2011) thus cannot be applied in this case. We shall therefore use the more refined labelled deductive machinery.

5.3 Intuitionistic bimodal logic

We start from the cut-free labelled calculi **G3I** for intuitionistic logic of Ch. 3 and we consider its language augmented with two modalities K and $\Diamond$. The corresponding accessibility relations in Kripke semantics are R_{K} and $R_{\Diamond}$, and the behavior of these two modal operators is captured by the following valuation clauses:

$x \Vdash \mathsf{K}A$ *if and only if* *for all* y, $xR_{\mathsf{K}}y$ *implies* $y \Vdash A$

$x \Vdash \Diamond A$ *if and only if* *for some* y, $xR_{\Diamond}y$ *and* $y \Vdash A$

Each definition can be unfolded in the necessary and sufficient conditions and converted into the following sequent rules, with the condition $y \neq x$, $y \notin \Gamma, \Delta$ for $R\mathsf{K}$ and $L\Diamond$.

Logical rules of $\mathbf{G3I}_{\mathsf{K}\Diamond}$

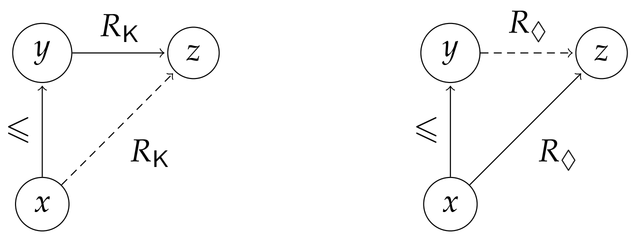
$$\frac{y : A, xR_{\mathsf{K}}y, x : \mathsf{K}A, \Gamma \to \Delta}{xR_{\mathsf{K}}y, x : \mathsf{K}A, \Gamma \to \Delta}\ L\mathsf{K} \qquad \frac{xR_{\mathsf{K}}y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \mathsf{K}A}\ R\mathsf{K}$$

$$\frac{xR_{\Diamond}y, y : A, \Gamma \to \Delta}{x : \Diamond A, \Gamma \to \Delta}\ L\Diamond \qquad \frac{xR_{\Diamond}y, \Gamma \to \Delta, x : \Diamond A, y : A}{xR_{\Diamond}y, \Gamma \to \Delta, x : \Diamond A}\ R\Diamond$$

Unlike for the extension with $Sym_{\leqslant}$, in the presence of the new rules it is not guaranteed that Theorem 3.5.2 is still valid. Moreover, we need to prove that the full monotonicity property (Lemma 3.5.1) extends also to modal formulas. Indeed, it is easy to see that if the standard rules for K and $\Diamond$ are used, Lemma 3.5.1 does not hold. A possible way out has been found in Božić and Došen (1984) by requiring that models satisfy the extra conditions

$$\forall x \forall y \forall z (x \leqslant y \wedge yR_{\mathsf{K}}z \supset xR_{\mathsf{K}}z) \quad \mathbf{Mon}_{\mathsf{K}}$$
$$\forall x \forall y \forall z (x \leqslant y \wedge xR_{\Diamond}z \supset yR_{\Diamond}z) \quad \mathbf{Mon}_{\Diamond}$$

Observe that these conditions state that the following diagrams can be completed (the completing arrows are the dotted ones):



Conditions $\mathbf{Mon}_{\mathsf{K}}$ and $\mathbf{Mon}_{\Diamond}$ are universal axioms and by applying the method of conversion of axioms into sequent rules they become:

Mathematical rules of $\mathbf{G3I}_{\mathsf{K}\Diamond}$

$$\frac{xR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, \Gamma \to \Delta}{x \leqslant y, yR_{\mathsf{K}}z, \Gamma \to \Delta}\ Mon_{\mathsf{K}} \qquad \frac{yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, \Gamma \to \Delta}{x \leqslant y, xR_{\Diamond}z, \Gamma \to \Delta}\ Mon_{\Diamond}$$

We shall call $\mathbf{G3I}_{\mathsf{K}\Diamond}$ the extension of **G3I** with rules $L\mathsf{K}$, $R\mathsf{K}$, $L\Diamond$, $R\Diamond$, Mon_{K} and $Mon_{\Diamond}$. With the new mathematical rules monotonicity of forcing can now be extended to cover arbitrary formula, including also modal formulas.

Lemma 5.3.1. *In* $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *it holds that*

i) $\vdash x \leqslant y, x : A, \Gamma \to \Delta, y : A$

ii) $\vdash x : A, \Gamma \to \Delta, x : A$

Proof. By simultaneous induction on the height h of A, as in the proof of Lemma 3.5.1. The proof of *ii* is done at each step of the induction by $Ref_{\leqslant}$ and the inductive hypothesis of *i*. The most relevant cases of the proof of *i* are the following, where the new mathematical rules are applied.

$$\cfrac{\cfrac{\cfrac{\cfrac{z \leqslant z, z : B, xR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, x : \mathsf{K}B, \Gamma \to \Delta, z : B}{z : B, xR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, x : \mathsf{K}B, \Gamma \to \Delta, z : B}\ Ref_{\leqslant}}{xR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, x : \mathsf{K}B, \Gamma \to \Delta, z : B}\ L\mathsf{K}}{x \leqslant y, yR_{\mathsf{K}}z, x : \mathsf{K}B, \Gamma \to \Delta, z : B}\ Mon_{\mathsf{K}}}{x \leqslant y, x : \mathsf{K}B, \Gamma \to \Delta, y : \mathsf{K}B}\ R\mathsf{K}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{z \leqslant z, yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B, z : B}{yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B, z : B}\ Ref_{\leqslant}}{yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B}\ R\Diamond}{x \leqslant y, xR_{\Diamond}z, z : B, \Gamma \to \Delta, y : \Diamond B}\ Mon_{\Diamond}}{x \leqslant y, x : \Diamond B, \Gamma \to \Delta, y : \Diamond B}\ L\Diamond$$

where the topmost sequents are derivable by the inductive hypothesis. ⊠

Finally, the system $\mathbf{G3I}_{\mathsf{K}\Diamond}$ satisfies the structural properties of **G3**-systems, that is, the admissibility of the structural rules and cut.

Theorem 5.3.2. *In* $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *it holds that*

i) The substitution of labels is hp-admissible;

ii) All the logical rules are hp-invertible;

iii) The rules of weakening and contraction are hp-admissible;

iv) Cut is admissible.

Proof. (i) By induction on the height h of the derivation of the premise. If $h = 0$ and the substitution is not vacuous then $\Gamma \to \Delta$ is $x \leqslant y, x : P, \Gamma' \to \Delta', y : P$ or $x : \bot, \Gamma' \to \Delta$. In each case, we obtain an initial sequent or a conclusion of $L\bot$. If $h = n + 1$, suppose by induction hypothesis (IH) that we have the conclusion for derivations of height n and consider the last rule applied. If it is a rule without a variable condition, apply IH to the premise(s) and then the rule. If the last rule applied is either $R\supset$, or $R\mathsf{K}$, or $L\Diamond$ we have to consider whether y is the *eigenvariable* or not. Consider the case of $L\Diamond$, the others being analogous. If y is the *eigenvariable* then the premise is $xR_{\Diamond}y, y : A, \Gamma' \to \Delta$ and we have to refresh by IH y with a new z in order to avoid a variable clash, and we obtain a derivation of $xR_{\Diamond}z, z : A, \Gamma' \to \Delta$. Again by IH, we replace x with y and thus obtain the sequent $yR_{\Diamond}z, z : A, \Gamma' \to \Delta$. Next, we are allowed to apply $L\Diamond$ to conclude $y : \Diamond A, \Gamma' \to \Delta$. Note that if the eigenvariable is x, the substitution is vacuous.

(ii) We prove the result for those rules that are not in common with **G3I**, the others have been already proved admissible by Theorem 3.5.2. $L\mathsf{K}$ and $R\Diamond$ are clearly invertible by hp-admissibility of weakening. We consider only the case of $L\Diamond$ because $R\mathsf{K}$ is analogous, and we proceed by induction on the height h of premise $x : \Diamond A, \Gamma \to \Delta$. If $h = 0$ and the premise is initial or a conclusion of $L\bot$, then so is $xR_{\Diamond}y, y : A, \Gamma \to \Delta$. If $h = n + 1$ then $x : \Diamond A, \Gamma \to \Delta$ has been concluded by a certain rule R. If $x : \Diamond A$ is principal, then R is $L\Diamond$ and its premise, that is $xR_{\Diamond}y, y : A, \Gamma \to \Delta$, has a derivation with height n. If on the contrary $x : \Diamond A$ is not principal, consider what rule R is. If it is a rule without variable condition, apply the IH to its premise(s) and then R again. If R is, for instance, $R\mathsf{K}$ with $x : \mathsf{K}B$ as principal formula, its premise is $xR_{\mathsf{K}}y, x : \Diamond A, \Gamma \to \Delta', y : B$. Apply first hp-admissibility of substitution and replace y with a new z, so to obtain $xR_{\mathsf{K}}z, x : \Diamond A, \Gamma \to \Delta', z : B$. Then by IH conclude $xR_{\mathsf{K}}z, xR_{\Diamond}y, y : A, \Gamma \to \Delta', z : B$ and by one application of $R\mathsf{K}$ obtain $xR_{\Diamond}z, y : A, \Gamma \to \Delta', x : \mathsf{K}B$.

(iii) Consider the case of weakening with a relational atom xRy. The proof is by induction on the height h of the derivation of the premise. The inductive step is straightforward if the premise(s) is concluded by a rule without a variable condition. If the last rule is a rule with a variable condition, say R_{K} with $x : \mathsf{K}B$ as principal formula, hp-admissibility of substitution is applied to its premise $xR_{\mathsf{K}}y, \Gamma \to \Delta', y : B$ in order to replace the eigenvariable y with a new z. Then by the IH and $R\mathsf{K}$, we obtain the conclusion $xRy, \Gamma \to \Delta', x : \mathsf{K}B$.

(iii) By simultaneous induction on the height h of the derivation. If $h = 0$, the premise is an initial sequent or has been concluded by $L\bot$. In each case the conclusion is initial or $L\bot$. If $h = n + 1$, suppose the claim holds for derivations of height n and distinguish what rule R is used to derive the premise. If the contraction formula is not principal in R, both occurrences are in the premise(s) of R and by IH we can contract the two occurrences and obtain a smaller derivation height of the conclusion. If contraction formula is principal of R, we distinguish two cases. The premise is concluded by a rule with the repetition of the principal formula, as $L\supset$, $L\mathsf{K}$, $R\Diamond$, and the mathematical rules. In this case the IH is applicable directly on the premise of R. For instance, if R is $R\Diamond$, the the last step of the derivation is, where Γ is $xR_{\Diamond}y, \Gamma'$):

$$\dfrac{\begin{array}{c}\vdots\\ xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A, x : \Diamond A, y : A\end{array}}{xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A, x : \Diamond A}\ R\Diamond$$

By IH on the premise we obtain $xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A, y : A$ and next by $R\Diamond$ again $xR_{\Diamond}y, \Gamma' \to \Delta, x : \Diamond A$. If R is without repetition of principal formulas we need hp-inversion on the premise(s), as in the standard proof for **G3c**. The crucial steps here are the cases in which R is either $R\supset$, or $R\mathsf{K}$, or $L\Diamond$, that is, rules with variable condition. Take for instance the case in which R is $L\Diamond$, the others being analogous. The premise has the following derivation:

$$\dfrac{\begin{array}{c}\vdots\\ xR_{\Diamond}y, y : A, x : \Diamond A, \Gamma \to \Delta\end{array}}{x : \Diamond A, x : \Diamond A, \Gamma \to \Delta}\ L\Diamond$$

By the invertibility of $L\Diamond$, we obtain $xR_{\Diamond}y, y : A, xR_{\Diamond}y, y : A, \Gamma \to \Delta$. Then by IH, $y : A, xR_{\Diamond}y, \Gamma \to \Delta$ and, by $L\Diamond$ again, we conclude $x : \Diamond A, \Gamma \to \Delta$.

(iv) By induction on the height of the cut formula with subinduction on the sum of the heights of the derivations of the premises of cut. We consider in detail only the case of cut formula principal in modal rules in both premises of cut and in mathematical rules. As for the latter, consider the case of left premise concluded by Mon_{K}.

$$\dfrac{\dfrac{xR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, \Gamma'' \to \Delta, x : A}{x \leqslant y, yR_{\mathsf{K}}z, \Gamma'' \to \Delta, x : A}\ Mon_{\mathsf{K}} \qquad x : A, \Gamma' \to \Delta'}{x \leqslant y, yR_{\mathsf{K}}z, \Gamma'', \Gamma' \to \Delta', \Delta}\ \text{CUT}$$

It converts to

$$\dfrac{\dfrac{xR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, \Gamma'' \to \Delta, x : A \qquad x : A, \Gamma' \to \Delta'}{xR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, \Gamma'', \Gamma' \to \Delta', \Delta}\ \text{CUT}}{x \leqslant y, yR_{\mathsf{K}}z, \Gamma'', \Gamma' \to \Delta', \Delta}\ Mon_{\mathsf{K}}$$

Likewise for other mathematical rules. If the cut formula is principal in a K-rule, it is of the form $x : \mathsf{K}B$ and the cut derivation is

$$\dfrac{\dfrac{xR_{\mathsf{K}}z, \Gamma \to \Delta, z : B}{\Gamma \to \Delta, x : \mathsf{K}B}\,{\scriptstyle R\mathsf{K}} \quad \dfrac{y : B, x : \mathsf{K}B, xR_{\mathsf{K}}y, \Gamma'' \to \Delta'}{x : \mathsf{K}B, xR_{\mathsf{K}}y, \Gamma'' \to \Delta'}\,{\scriptstyle L\mathsf{K}}}{xR_{\mathsf{K}}y, \Gamma'', \Gamma \to \Delta, \Delta'}\,\text{CUT}$$

It can be converted into

$$\dfrac{\dfrac{\dfrac{xR_{\mathsf{K}}z, \Gamma \to \Delta, z : B}{xR_{\mathsf{K}}y, \Gamma \to \Delta, y : B}\,{\scriptstyle z/y} \quad \dfrac{\dfrac{xR_{\mathsf{K}}z, \Gamma \to \Delta, z : B}{\Gamma \to \Delta, x : \mathsf{K}B}\,{\scriptstyle R\mathsf{K}} \quad y : B, x : \mathsf{K}B, xR_{\mathsf{K}}y, \Gamma'' \to \Delta'}{y : B, xR_{\mathsf{K}}y, \Gamma'', \Gamma \to \Delta, \Delta'}\,\text{CUT}}{xR_{\mathsf{K}}y, xR_{\mathsf{K}}y, \Gamma'', \Gamma, \Gamma \to \Delta, \Delta, \Delta'}\,\text{CUT}}{xR_{\mathsf{K}}y, \Gamma'', \Gamma \to \Delta, \Delta'}\,\text{C}$$

Note that the first cut reduced cut-height and the second is on a smaller formula. If the cut formula is principal in a $\Diamond$-rule, it is of the form $x : \Diamond B$ and the cut derivation is

$$\dfrac{\dfrac{xR_{\Diamond}y, \Gamma'' \to \Delta, x : \Diamond B, y : B}{xR_{\Diamond}y, \Gamma'' \to \Delta, x : \Diamond B}\,{\scriptstyle R\Diamond} \quad \dfrac{xR_{\Diamond}z, z : B, \Gamma' \to \Delta'}{x : \Diamond B, \Gamma' \to \Delta'}}{xR_{\Diamond}y, \Gamma'', \Gamma' \to \Delta', \Delta}\,\text{CUT}$$

and it can be converted into

$$\dfrac{\dfrac{\dfrac{xR_{\Diamond}y, \Gamma'' \to \Delta, x : \Diamond B, y : B \quad x : \Diamond B, \Gamma' \to \Delta'}{xR_{\Diamond}y, \Gamma'', \Gamma' \to \Delta', \Delta, y : B}\,\text{CUT} \quad \dfrac{xR_{\Diamond}z, z : B, \Gamma' \to \Delta'}{xR_{\Diamond}y, y : B, \Gamma' \to \Delta'}\,{\scriptstyle y/z}}{xR_{\Diamond}y, xR_{\Diamond}y, \Gamma'', \Gamma', \Gamma' \to \Delta', \Delta', \Delta}\,\text{CUT}}{xR_{\Diamond}y, \Gamma'', \Gamma' \to \Delta', \Delta}\,\text{C}$$

$\boxtimes$

Observe that all the above structural results that have been established for $\mathbf{G3I}_{\mathsf{K}\Diamond}$ hold also for any of its extensions with frame rules that follow the regular or geometric rule schema. The details can be easily spelled out following the general pattern of the parallel results of basic modal logic in Negri (2005).

We shall sketch the direct completeness proof, along the lines of Negri (2009), because this is the method that will permit proofs of underivability and constructions of countermodels in what follows. The proof is similar to the proof of the completeness theorem for **G3K** and we shall refer to it for the details. First, we adapt the semantic definition of validity for a sequent $\Gamma \to \Delta$ to the setting of the bimodal intuitionistic logic.

Definition. *Let* $\mathfrak{F} = \langle \mathsf{X}, \leqslant, R_{\mathsf{K}}, R_{\Diamond} \rangle$ *be a* frame *that satisfies the properties* $Ref_{\leqslant}$, $Trans_{\leqslant}$, Mon_{K}, $Mon_{\Diamond}$. *A* model $\mathfrak{M}$ *is a frame together with a binary relation* $\Vdash$ *between possible states and atomic formulas,* $x \Vdash P$*. The forcing is also monotone, that is, if* $x \Vdash P$ *and* $x \leqslant y$ *then* $y \Vdash P$*. Let* L *be the set of labels, an* interpretation *of the labels in a frame* $\mathfrak{F}$ *is a function* $[\![\cdot]\!] : L \longrightarrow \mathsf{X}$ *that assigns a possible state* $[\![x]\!]$ *of* $\mathfrak{F}$ *to each label* x *in* L*, and an accessibility relation of* $\mathfrak{F}$ *to the relational symbol* $\leqslant, R_{\mathsf{K}}, R_{\Diamond}$.

Forcing is extended in a unique way to arbitrary formulas by means of inductive clauses.

$x \Vdash \bot$	for no x	
$x \Vdash A \wedge B$	if and only if	$x \Vdash A$ and $x \Vdash B$
$x \Vdash A \vee B$	if and only if	$x \Vdash A$ or $x \Vdash B$
$x \Vdash A \supset B$	if and only if	for all y, $x \leqslant y$ and $y \Vdash A$ implies $y \Vdash B$
$x \Vdash \mathsf{K}A$	if and only if	for all y, $xR_{\mathsf{K}}y$ implies $y \Vdash A$
$x \Vdash \Diamond A$	if and only if	for some y, $xR_{\Diamond}y$ and $y \Vdash A$

Definition. *A sequent* $\Gamma \to \Delta$ *is* valid in a model $\mathfrak{M}$ *if for all labelled formulas* $x : A$ *and relational atoms* $y \leqslant z$, $y' R_{\mathsf{K}} z'$, $y'' R_{\Diamond} z''$ *in* Γ, *whenever* $[\![x]\!] \Vdash A$ *and* $[\![y]\!] \leqslant [\![z]\!]$, $[\![y']\!] R_{\mathsf{K}} [\![z']\!]$, $[\![y'']\!] R_{\Diamond} [\![z'']\!]$ *in* X, *then for some* $w : B$ *in* Δ, $[\![w]\!] \Vdash B$. *A sequent is* valid *if it is valid for every model.*

The rules of $\mathbf{G3I}_{\mathsf{K}\Diamond}$ are sound, that is, the conclusion is valid whenever their premise(s) are valid.

Theorem 5.3.3 (Soundness). *If the sequent* $\Gamma \to \Delta$ *is derivable in* $\mathbf{G3I}_{\mathsf{K}\Diamond}$, *it is valid in every frame with the properties* $Ref_{\leqslant}$, $Trans_{\leqslant}$, Mon_{K}, $Mon_{\Diamond}$.

Proof. By induction on the derivation of $\Gamma \to \Delta$ in $\mathbf{G3I}_{\mathsf{K}\Diamond}$. If it is an initial sequent, there is a labelled atom $x : P$ both in Γ and in Δ so the claim is obvious, and similarly if the sequent is a conclusion of $L\bot$, since for no valuation can $\bot$ be forced at any node. Moreover, if $\Gamma \to \Delta$ is of the form $x : P, x \leqslant y, \Gamma' \to \Delta', y : P$ then the claim holds by the monotonicity of forcing relation. If $\Gamma \to \Delta$ is the conclusion of either a propositional or modal rule the proof is similar to the proof of Theorem 2.4.1. If the sequent is a conclusion of a rule for the accessibility relations, let the rule be for instance $Mon_{\Diamond}$:

$$\frac{yR_{\Diamond}z, x \leqslant y, xR_{\Diamond}z, \Gamma \to \Delta}{x \leqslant y, xR_{\Diamond}z, \Gamma \to \Delta}\; Mon_{\Diamond}$$

Let $[\![x]\!] \leqslant [\![y]\!]$ and $[\![x]\!] R_{\Diamond} [\![z]\!]$. Since $\leqslant$ and $R_{\Diamond}$ satisfy $Mon_{\Diamond}$ by assumption, we have $[\![y]\!] R_{\Diamond} [\![z]\!]$, so validity of the premise gives validity of the conclusion.

$\boxtimes$

Next, we show that derivability of a formula in the calculus is equivalent to validity, that is, validity at an arbitrary world for an arbitrary valuation. The latter is expressed by $x \Vdash A$ where x is arbitrary, and it is translated into a sequent $\to x : A$

in our calculus. The rules of the calculus applied backwards give equivalent conditions until the atomic components of A are reached. It can happen that we find a proof, or that we find that a proof does not exist either because we reach a stage where no rule is applicable, or because we go on with the search forever. In the two latter cases the attempted proof itself gives directly a countermodel. The two following results establish the completeness of $\mathbf{G3I}_{\mathsf{K}\Diamond}$

Theorem 5.3.4. *Let* $\Gamma \rightarrow \Delta$ *be a sequent in the language of* $\mathbf{G3I}_{\mathsf{K}\Diamond}$*. Then either the sequent is derivable in* $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *or it has a countermodel with properties* $Ref_{\leqslant}$*,* $Trans_{\leqslant}$*,* Mon_{K}*,* $Mon_{\Diamond}$*.*

Proof. We define for an arbitrary sequent $\Gamma \rightarrow \Delta$ in the language of $\mathbf{G3I}_{\mathsf{K}\Diamond}$ a reduction tree by applying the rules of $\mathbf{G3I}_{\mathsf{K}\Diamond}$ root first in all possible ways. If the construction terminates we obtain a proof, else the tree becomes infinite. By König's lemma an infinite tree has an infinite branch that is used to define a countermodel to the endsequent. The reduction tree is constructed in the same way of Theorem 2.4.3. If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot$ and the tree, read from the leaves to the root, yields a derivation. Else, if the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \rightarrow \Delta_0$ $\Gamma \rightarrow \Delta \equiv \Gamma_1 \rightarrow \Delta_1 \ldots, \Gamma_i \rightarrow \Delta_i, \ldots$ be one such branch. Consider the sets of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i>0} \Gamma_i \qquad \mathbf{\Delta} \equiv \bigcup_{i>0} \Delta_i$$

We define a model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \rightarrow \Delta$. Consider the frame $\mathfrak{F}$ the nodes of which are all the labels that appear in the relational atoms in $\mathbf{\Gamma}$, with their mutual relationships expressed by the relational atoms in $\mathbf{\Gamma}$. In general, the construction of the reduction tree imposes the frame properties of the countermodel, which in this case are $Ref_{\leqslant}$, $Trans_{\leqslant}$, Mon_{K}, $Mon_{\Diamond}$. The model is defined as follows: For all atomic formulas $x : P$ in $\mathbf{\Gamma}$, we stipulate that $x \Vdash P$, and for all atomic formulas $y : Q$ in $\mathbf{\Delta}$ we stipulate that $y \nVdash Q$. Since no sequent in the infinite branch is initial, this choice can be coherently made. It can then be shown inductively on the height of formulas that A is forced in the model at node x if $x : A$ is in $\mathbf{\Gamma}$ and A is not forced at node x if $x : A$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the endsequent $\Gamma \to \Delta$. The details are similar to those in Negri (2009) and of Theorem 2.4.3 of the previous chapters.

⊠

Corollary 5.3.5 (Completeness)**.** *If a sequent $\Gamma \to \Delta$ is valid in every model with the frame properties $Ref_{\leqslant}$, $Trans_{\leqslant}$, Mon_{K}, $Mon_{\Diamond}$, it is derivable in* $\mathbf{G3I}_{\mathsf{K}\Diamond}$.

Digression: A conceptual analysis of accessibility relations

Before proceeding to the structural analysis of the Church-Fitch paradox by our labelled calculus, we shall outline a conceptual analysis of the accessibility relations introduced in the previous section. This will serve both as an explanation of the notions used, as well as a justification of the formal choices made in defining system $\mathbf{G3I}_{\mathsf{K}\Diamond}$. First, the relation $\leqslant$ is the standard accessibility relation for the semantics of intuitionistic logic. Its intuitive meaning is clarified in Kripke (1965, pp. 98–99). Because worlds in a model can be identified with the propositions true in them, the relation gets the following intuitive meaning: A world y is $\leqslant$-accessible from a world x if y is a possible development of the information contained in x. Under this interpretation, worlds are recognized as temporal states in a process of acquisition of information. The properties of reflexivity and transitivity of the preorder thus appear obvious, whereas monotonicity of forcing reflects the requirement that the acquisition of information is a cumulative process. When agents who can gain knowledge are added to the scenario, epistemic operators together with their accessibility relations are needed. Here we have considered just one (impersonal and generic) epistemic attitude, K, with the accessibility relation R_{K}. The question naturally arises of what the relation should be between R_{K} and $\leqslant$. A minimal requirement is that in the language extended with formulas as $\mathsf{K}A$ monotonicity of forcing is preserved: The perfect recall should apply to all formulas, not just to the purely propositional ones, and this is achieved by imposing the property $\mathbf{Mon}_{\mathsf{K}}$. On the other hand, factivity of knowledge, *i.e.*, axiom $\mathsf{K}A \supset A$, which is explicitly assumed in Fitch's derivation, corresponds to reflexivity of R_{K}. This axiom states that only true formulas can be known and separates knowledge from what is the mere belief. Monotonicity and reflexivity of R_{K} imply that what is temporally accessible is also epistemically accessible, *i.e.*, the condition $\forall x \forall y(x \leqslant y \supset x R_{\mathsf{K}} y)$ holds. Notice that this implication does not exclude the possibility of the existence of epistemically accessible states that are not future states. Our analysis will show that if this existence is explicitly imposed, *i.e.*, $\exists x \exists y(x R_{\mathsf{K}} y \wedge x \nleqslant y)$ holds, then the identification of truth and knowledge is avoided (cf. Proietti 2011). Moreover, **KP** fails if $Sym_{\leqslant}$ and $Ref_{\Diamond}$ are added. Similar formal requirements apply to the accessibility relation $R_{\Diamond}$, a relation that expresses logical possibility. A state y is $R_{\Diamond}$-accessible from x when y is logically compatible with x, in the sense that y is a state that can in principle be reachable from x, even if we cannot specify the nature of this access (temporal, causal, epistemic, etc.). Note that this relation is temporally upward closed: If a state z is possibly reachable from x, then z is possibly reachable from all the future states of x. We do not want to commit ourselves in any way to assuming more than the necessary properties of $R_{\Diamond}$, in particular we do not identify it with any other of the accessibility relations considered. A different choice is pursued in Proietti (2011) and Artemov and Protopopescu (2011), where the intuitionistic double negation gets interpreted as a possibility operator, leading to a reformulation of the knowability principle that employs only the epistemic modality. The above interpretations also allow to capture the temporal flavor ascribed to the knowledge operator in Fitch's original article. Its core result, Theorem 5, is based on the existence of "some true proposition which nobody knows (or has known or will know) to be true" (Fitch 1963, p. 139). The temporal interpretation of $\leqslant$ suggests that the statement $\mathsf{K}A$ has to be evaluated in all situations temporally accessible from x, where x can be considered as the actual world, but also as a past world, or better, x can be considered as any world in which A is true. More generally, the structural reconstruction of Fitch's derivation will reveal that every occurrence of $\mathsf{K}A$ is always in the scope of a negation or of an implication. Therefore, reasoning root first, the application of a K-rule is always preceded by an application of a rule that imposes a temporal-dependent evaluation of $\mathsf{K}A$.

5.4 Proof-theoretical analysis of the Church-Fitch paradox

We have now all the logical instruments needed for a structural proof analysis of the paradox. We start with the reconstruction of the standard derivation that uses the labelled sequent calculus introduced in the previous section. Our analysis made clear what the ingredients of the Church-Fitch paradox are: distributivity of K over conjunction, $\mathsf{K}(A \wedge B) \supset \mathsf{K}A \wedge \mathsf{K}B$, and the factivity of knowledge, $\mathsf{K}A \supset A$. Distributivity holds for operators that satisfy necessitation and the normality axiom in any system for normal modal logic. Factivity of knowledge corresponds to reflexivity of the accessibility relation, *i.e.*, $xR_{\mathsf{K}}x$, for all possible worlds x. Through the method of conversion of axioms into sequent rules we obtain the following:

$$\frac{xR_{\mathsf{K}}x, \Gamma \to \Delta}{\Gamma \to \Delta}\ Ref_{\mathsf{K}}$$

Axiom $\mathsf{K}A \supset A$ is shown derivable by this rule. Both properties are provable in $\mathbf{G3I}_{\mathsf{K}\Diamond}$ and in $\mathbf{G3I}_{\mathsf{K}\Diamond}$ with Ref_{K}, respectively. The following two lemmas single out the special instances that are needed in the proof of the Church-Fitch paradox:

Lemma 5.4.1. *In* $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *it holds that:*

i) $\vdash x : \mathsf{K}(A \wedge \neg\mathsf{K}A) \to x : \mathsf{K}A$

ii) $\vdash x : \mathsf{K}(A \wedge \neg\mathsf{K}A) \to x : \mathsf{K}\neg\mathsf{K}A$

Proof. By a systematic proof-search procedure from the sequent to be derived. ⊠

Lemma 5.4.2. *In* $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *with* Ref_{K} *it holds that:*

$\vdash x : \mathsf{K}\neg\mathsf{K}A \to x : \neg\mathsf{K}A$

Proof. Consider the derivation

$$\dfrac{\dfrac{x : \neg\mathsf{K}A, xR_{\mathsf{K}}x, x : \mathsf{K}\neg\mathsf{K}A \to x : \neg\mathsf{K}A}{xR_{\mathsf{K}}x, x : \mathsf{K}\neg\mathsf{K}A \to x : \neg\mathsf{K}A}\, L\mathsf{K}}{x : \mathsf{K}\neg\mathsf{K}A \to x : \neg\mathsf{K}A}\, Ref_{\mathsf{K}}$$

where the topmost sequent is derivable by Lemma 5.3.1. ⊠

No further explicit conditions beyond $R_{\Diamond}$, $L_{\Diamond}$, and $Mon_{\Diamond}$ need to be imposed on $\Diamond$ to reconstruct the standard proof of the Church-Fitch paradox. By a proof analogous to that of Lemma 5.2.2, we have:

Lemma 5.4.3. *The following rule*

$$\dfrac{\to x : A \wedge \neg\mathsf{K}A \supset \Diamond\mathsf{K}(A \wedge \neg\mathsf{K}A)}{x \leqslant y, y : A \wedge \neg\mathsf{K}A \to y : \Diamond\mathsf{K}(A \wedge \neg\mathsf{K}A)}\, Inv$$

is derivable in $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *with Cut.*

Proof. Analogous to the proof of Lemma 5.2.2. ⊠

The use of classical logic is a further requirement for obtaining the standard proof of **OP**. In particular the following lemma has to be proved.

Lemma 5.4.4. *In* $\mathbf{G3C}_{\mathsf{K}\Diamond}$ *it holds that:*

$\vdash x : \neg(A \wedge \neg\mathsf{K}A) \rightarrow x : A \supset \mathsf{K}A$

Proof. By root-first proof search from the sequent to be derived. Note that the proof is classical because it makes appeal to a non-eliminable application of rule $Sym_{\leqslant}$. ⊠

We have now all the information that is needed in order to reconstruct the standard derivation of the Church-Fitch paradox.

Theorem 5.4.5 (Fitch's Paradox)**.** *In* $\mathbf{G3C}_{\mathsf{K}\Diamond}$ *with* Ref_{K}*, Cut and* **KP** *it holds that:* $\vdash \rightarrow x : A \supset \mathsf{K}A$.

Proof. Consider the following derivation

$$
\begin{prooftree}
\AxiomC{\textbf{KP}}
\noLine
\UnaryInfC{$\rightarrow x : A \wedge \neg\mathsf{K}A \supset \Diamond\mathsf{K}(A \wedge \neg\mathsf{K}A)$}
\RightLabel{\scriptsize Inv}
\UnaryInfC{$x \leqslant y, y : A \wedge \neg\mathsf{K}A \rightarrow y : \Diamond\mathsf{K}(A \wedge \neg\mathsf{K}A)$}
\AxiomC{$z : \mathsf{K}(A \wedge \neg\mathsf{K}A) \rightarrow z : \mathsf{K}A$}
\RightLabel{\scriptsize R-W}
\UnaryInfC{$yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg\mathsf{K}A) \rightarrow z : \mathsf{K}A, y : \bot$}
\AxiomC{$z : \mathsf{K}(A \wedge \neg\mathsf{K}A) \rightarrow z : \mathsf{K}\neg\mathsf{K}A$}
\AxiomC{$z : \mathsf{K}\neg\mathsf{K}A \rightarrow z : \neg\mathsf{K}A$}
\RightLabel{\scriptsize CUT}
\BinaryInfC{$z : \mathsf{K}(A \wedge \neg\mathsf{K}A) \rightarrow z : \neg\mathsf{K}A$}
\RightLabel{\scriptsize INV}
\UnaryInfC{$z \leqslant y, z : \mathsf{K}(A \wedge \neg\mathsf{K}A), y : \mathsf{K}A \rightarrow y : \bot$}
\AxiomC{}
\RightLabel{\scriptsize $L\bot$}
\UnaryInfC{$y : \bot \rightarrow$}
\RightLabel{\scriptsize CUT}
\BinaryInfC{$z \leqslant y, z : \mathsf{K}(A \wedge \neg\mathsf{K}A), y : \mathsf{K}A \rightarrow$}
\RightLabel{\scriptsize z/y}
\UnaryInfC{$z \leqslant z, z : \mathsf{K}(A \wedge \neg\mathsf{K}A), z : \mathsf{K}A \rightarrow$}
\RightLabel{\scriptsize CUT}
\BinaryInfC{$z \leqslant z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg\mathsf{K}A) \rightarrow y : \bot$}
\RightLabel{\scriptsize $Ref_{\leqslant}$}
\UnaryInfC{$yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg\mathsf{K}A) \rightarrow y : \bot$}
\RightLabel{\scriptsize $L\Diamond$}
\UnaryInfC{$y : \Diamond\mathsf{K}(A \wedge \neg\mathsf{K}A) \rightarrow y : \bot$}
\RightLabel{\scriptsize CUT}
\BinaryInfC{$x \leqslant y, y : A \wedge \neg\mathsf{K}A \rightarrow y : \bot$}
\RightLabel{\scriptsize $R\supset$}
\UnaryInfC{$\rightarrow x : \neg(A \wedge \neg\mathsf{K}A)$}
\AxiomC{$x : \neg(A \wedge \neg\mathsf{K}A) \rightarrow x : A \supset \mathsf{K}A$}
\RightLabel{\scriptsize CUT}
\BinaryInfC{$\rightarrow x : A \supset \mathsf{K}A$}
\end{prooftree}
$$

Notice that the topmost sequents, except for **KP**, are derivable by Lemmas 5.4.1, 5.4.2 and 5.4.4. Furthermore, the applications of the rule of weakening are eliminable by pushing them up to the initial sequents of the derivations used for the proof of Lemma 5.4.1.

⊠

Observe that Theorem 5.4.5 states a derivability result: There is a derivation of **OP** from **KP** by means of the rules of $\mathbf{G3C}_{\mathsf{K}\Diamond}$, Ref_{K} and cut. In this result, **KP** plays the role of a derivation principle, similar to a zero-premise inference rule. Nonetheless, a crucial difference remains. On the one hand, the inference rules are valid in the sense that they respect the deductive harmony imposed by the inversion principle, as it stated in Negri and von Plato (2001, p. 6). On the other hand, the validity of **KP** is fixed by stipulation, because, at the syntactic level, there is nothing that differentiates **KP** from another sentence of the bimodal language under the analysis. A crucial step of our work will be to understand which class of relational structures **KP** singles out, so to determine as well in which class of models **KP** can be considered as formally true.

5.5 Structural analysis of the Church-Fitch paradox

There are two special aspects of the proof of Theorem 5.4.5. The instance of **KP** appears in the derivation as a sequent with an empty antecedent of the form $\rightarrow A$.[2] Moreover, the proof uses cuts. The presence of cuts makes it difficult to point out where the paradox arises from, in the first place because the structure of such derivations is not transparent. Secondly, by a thesis of Tennant, a paradox is a non-normal derivation the normalization of which enters into a loop (Tennant 1982). In sequent calculus, the notion of normalization is replaced by cut elimination that becomes the essential means for analyzing the precise nature of the paradox, and for distinguishing the case of a derivation without eliminability of cut from that of a fallacy, in which the latter assumption and the paradoxical conclusion are equivalent principles. Applying the cut elimination procedure for $\mathbf{G3C}_{\mathsf{K}\Diamond}$ with

[2] Cf. Definition 6.3.1 (a) in Negri and von Plato (2001), p. 134.

Ref_{K} to our derivation of **OP**, we obtain the following derivation in which, to save space, we have abbreviated with $KP(A)$ the formula $(A \wedge \neg \mathsf{K}A) \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)$:

$$
\dfrac{\overset{\textbf{KP}}{\to x : KP(A)} \qquad \dfrac{\dfrac{\dfrac{\overset{\vdots}{S_1} \qquad \overset{\vdots}{S_2}}{x \leqslant y, yR_{\mathsf{K}}z, x : KP(A), y : A \to z : A}\,L\supset}{x \leqslant y, x : KP(A), y : A \to y : \mathsf{K}A}\,R\mathsf{K}}{x : KP(A) \to x : A \supset \mathsf{K}A}\,R\supset}{\to x : A \supset \mathsf{K}A}\,\text{CUT}
$$

The right premise S_2 of $L \supset$ is derivable as follows

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, wR_{\mathsf{K}}w, wR_{\mathsf{K}}t, w : A, t : A, t : \neg\mathsf{K}A, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A, t : A}{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, wR_{\mathsf{K}}w, wR_{\mathsf{K}}t, w : A, t : A \wedge \neg\mathsf{K}A, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A, t : A}\,L\wedge}{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, wR_{\mathsf{K}}w, wR_{\mathsf{K}}t, w : A, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A, t : A}\,L\mathsf{K}}{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, wR_{\mathsf{K}}w, w : A, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A, w : \mathsf{K}A}\,R\mathsf{K}}{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, wR_{\mathsf{K}}w, w : A, w : \neg\mathsf{K}A, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A}\,L\supset}{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, wR_{\mathsf{K}}w, w : A \wedge \neg\mathsf{K}A, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A}\,L\wedge}{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, wR_{\mathsf{K}}w, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A}\,L\mathsf{K}}{x \leqslant y, yR_{\mathsf{K}}z, yR_{\Diamond}w, w : \mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A}\,Ref_{\mathsf{K}}}{x \leqslant y, yR_{\mathsf{K}}z, y : \Diamond\mathsf{K}(A \wedge \neg\mathsf{K}A), x : KP(A), y : A \to z : A}\,L\Diamond
$$

where the topmost sequent is derivable by Lemma 5.3.1. The right premise of $L \supset$ (in S_2) is derivable because it is an instance of $L\bot$, left unwritten here. The left premise S_1 is derivable:

$$
\dfrac{x \leqslant y, yR_{\mathsf{K}}z, y : A \to z : A, y : A \qquad \dfrac{\dfrac{\dfrac{\dfrac{rR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, y \leqslant r, r \leqslant y, y : A, r : \mathsf{K}A, z : A \to z : A, r : \bot}{rR_{\mathsf{K}}z, x \leqslant y, yR_{\mathsf{K}}z, y \leqslant r, r \leqslant y, y : A, r : \mathsf{K}A \to z : A, r : \bot}\,L\mathsf{K}}{x \leqslant y, yR_{\mathsf{K}}z, y \leqslant r, r \leqslant y, y : A, r : \mathsf{K}A \to z : A, r : \bot}\,Mon_{\mathsf{K}}}{x \leqslant y, yR_{\mathsf{K}}z, y \leqslant r, y : A, r : \mathsf{K}A \to z : A, r : \bot}\,Sym_{\leqslant}}{x \leqslant y, yR_{\mathsf{K}}z, y : A \to z : A, y : \neg\mathsf{K}A}\,R\supset}{x \leqslant y, yR_{\mathsf{K}}z, y : A \to z : A, y : A \wedge \neg\mathsf{K}A}\,R\wedge
$$

There remains one application of cut in the derivation. Unlike the other instances of cut, the last one is not eliminable because it depends on an instance of **KP** that behaves like a proper axiom. We shall discuss this aspect later. From the previous proof, just by ignoring the last step, we obtain the following result:

Proposition 5.5.1. *In* $\mathbf{G3C}_{\mathsf{K}\Diamond}$ *with* Ref_{K} *it holds that:*

$\vdash x : A \wedge \neg \mathsf{K}A \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A) \rightarrow x : A \supset \mathsf{K}A$

without any application of cut.

The result can be stated briefly as follows:

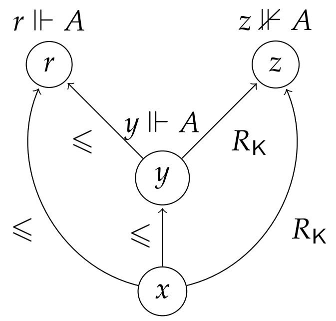
> **OP** *is derivable from the special instance* $KP(A)$ *of* **KP**.

Moreover, we notice that classical logic is used only in the step of symmetry in the right branch of the derivation S_1. Therefore that branch, pruned just before the application of $Sym_{\leqslant}$, suggests a countermodel to the sequent of Proposition 5.5.1 in the intuitionistic system $\mathbf{G3I}_{\mathsf{K}\Diamond}$ with Ref_{K}:

Theorem 5.5.2. *In* $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *with* Ref_{K} *it holds that:*

$\nvdash x : A \wedge \neg \mathsf{K}A \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A) \rightarrow x : A \supset \mathsf{K}A$

Proof. Consider the model $\mathfrak{G} = \langle W, \leqslant, R_{\mathsf{K}}, R_{\Diamond}, \Vdash \rangle$ where $W = \{x, y, z, r\}$, $x \leqslant y$, $y \leqslant r$, $x \leqslant r$, $yR_{\mathsf{K}}z$, $xR_{\mathsf{K}}z$, all the reflexivities for $\leqslant$ and R_{K} hold, and A is forced in y and in r but not in z. A diagrammatic representation, with the omission of the reflexive arrows, takes the form



In this model, we have that $x \Vdash A \wedge \neg \mathsf{K}A \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A)$ because $x, r \nVdash A \wedge \neg \mathsf{K}A$. To see why, just observe that $r \Vdash \mathsf{K}A$ and use the definitions to conclude that r, and therefore also y, do not force $\neg \mathsf{K}A$. On the other hand, $x \nVdash A \supset \mathsf{K}A$, because $y \Vdash A$, but $y \nVdash \mathsf{K}A$. $\boxtimes$

It is well known that one can obtain a derivation of the weak **OP** in the intuitionistic system. More precisely, a cut-free derivation of **WOP** from the assumption $KP(A)$ is obtained in our system as follows:

Theorem 5.5.3. *In* $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *with* Ref_{K} *it holds that:*

$\vdash x : A \wedge \neg \mathsf{K}A \supset \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A) \rightarrow x : \neg(A \wedge \neg \mathsf{K}A)$

Proof. The sequent is derived as follows:

$$
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{R_1}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{R_2}
\RightLabel{$L\supset$}
\BinaryInfC{$x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot$}
\RightLabel{$R\supset$}
\UnaryInfC{$x : KP(A) \rightarrow x : \neg(A \wedge \neg \mathsf{K}A)$}
\DisplayProof
$$

In the derivation R_1 is

$$
\AxiomC{$y \leqslant y, x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot, y : A \wedge \neg \mathsf{K}A$}
\RightLabel{$Ref_{\leqslant}$}
\UnaryInfC{$x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot, y : A \wedge \neg \mathsf{K}A$}
\DisplayProof
$$

whereas R_2 is

$$
\AxiomC{$zR_{\mathsf{K}}w, z \leqslant z, z : A, z : \neg \mathsf{K}A, zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), w : A, w : \neg \mathsf{K}A, x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot, w : A$}
\RightLabel{$L\wedge$}
\UnaryInfC{$zR_{\mathsf{K}}w, z \leqslant z, z : A, z : \neg \mathsf{K}A, zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), w : A \wedge \neg \mathsf{K}A, x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot, w : A$}
\RightLabel{$L\mathsf{K}$}
\UnaryInfC{$zR_{\mathsf{K}}w, z \leqslant z, z : A, z : \neg \mathsf{K}A, zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot, w : A$}
\RightLabel{$R\mathsf{K}$}
\UnaryInfC{$z \leqslant z, z : A, z : \neg \mathsf{K}A, zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot, z : \mathsf{K}A$}
\RightLabel{$L\supset$}
\UnaryInfC{$z \leqslant z, z : A, z : \neg \mathsf{K}A, zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot$}
\RightLabel{$Ref_{\leqslant}$}
\UnaryInfC{$z : A, z : \neg \mathsf{K}A, zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot$}
\RightLabel{$L\wedge$}
\UnaryInfC{$z : (A \wedge \neg \mathsf{K}A), zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot$}
\RightLabel{$L\mathsf{K}$}
\UnaryInfC{$zR_{\mathsf{K}}z, yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot$}
\RightLabel{Ref_{K}}
\UnaryInfC{$yR_{\Diamond}z, z : \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot$}
\RightLabel{$L\Diamond$}
\UnaryInfC{$y : \Diamond \mathsf{K}(A \wedge \neg \mathsf{K}A), x \leqslant y, x : KP(A), y : A \wedge \neg \mathsf{K}A \rightarrow y : \bot$}
\DisplayProof
$$

and the topmost sequent is derivable by Lemma 5.3.1. ☒

The countermodel of Theorem 5.5.2 shows, together with the completeness theorem, that the classical version of the Church-Fitch paradox is not derivable in an intuitionistic setting, thus seemingly confirming the thesis that intuitionistic logic saves anti-realism from the threat of the paradox (Williamson 1982). To say that **KP** implies **OP** does not require that there is a derivation from a special instance of **KP** to the conclusion **OP**, as in Proposition 5.5.1. In fact, the admission of the knowability principle corresponds to the assumption that **KP** is generally valid, instead of the assumption of just a particular instance. Therefore, the following admissibility statement should be put under analysis:

If **KP** *is valid, then also* **OP** *is valid.* (5.1)

Merely to show that **OP** does not follow intuitionistically from a particular instance of **KP** is not sufficient for establishing that **OP** is not derivable in an intuitionistic system that incorporates **KP** as a derivation principle. In other words, the countermodel given in the proof of Theorem 5.5.2 is not sufficient for showing that (5.1) does not hold in an intuitionistic setting. An analogy from propositional logic may clarify this point: The law of double negation $\neg\neg A \supset A$ follows from the principle of excluded middle, $A \vee \neg A$, in the sense that there is an intuitionistic derivation of $(A \vee \neg A) \supset (\neg\neg A \supset A)$. The converse $(\neg\neg A \supset A) \supset (A \vee \neg A)$ instead is not intuitionistically derivable even if the two principles give equivalent extensions of intuitionistic logic. However, $A \vee \neg A$ follows from a particular instance of the law of double negation, namely $\neg\neg(A \vee \neg A) \supset (A \vee \neg A)$. In conclusion, the cut-free analysis we have made suffices to establish intuitionistic underivability of **OP** from a particular instance of **KP**. The latter does not exclude, however, the derivability of **OP** from other instance of **KP**, a question to which a definite answer is give in the next section.

5.6 Proof analysis of KP

We proceed to find the necessary and sufficient frame property for the validity of **KP**. First, we use our calculus to single out frame rules that suffice for a derivation of **KP**. Then we extract from these rules a frame property and show that it is sufficient and necessary to validate **KP**. We start root first from the sequent to be derived. Observe that the only applicable rule as a first step is $R\supset$. Next, for the proof search to continue, to be able to apply $R\Diamond$ it is necessary to have an $R_\Diamond$-accessibility. The only rules that make available such relational atom in the absence of other $R_\Diamond$-atom are $Ref_\Diamond$ or $Ser_\Diamond$. Rule $Ser_\Diamond$ is derivable from $Ref_\Diamond$, and to make the set of assumptions on the accessibility relations minimal, we choose the latter. Notice that $Ser_\Diamond$ has the variable restriction that y must not occur in the conclusion. After that, the only applicable rule is $R\mathsf{K}$. An initial sequent is then obtained if a rule is used that adds the atom $y \leqslant w$, indicated by $\Diamond\mathsf{K}\text{-}Tr$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \leqslant y, y \leqslant w, yR_\Diamond z, zR_\mathsf{K} w, y : A \to y : \Diamond\mathsf{K}A, w : A}{x \leqslant y, yR_\Diamond z, zR_\mathsf{K} w, y : A \to y : \Diamond\mathsf{K}A, w : A}\ \Diamond\mathsf{K}\text{-}Tr}{x \leqslant y, yR_\Diamond z, y : A \to y : \Diamond\mathsf{K}A, z : \mathsf{K}A}\ R\mathsf{K}}{x \leqslant y, yR_\Diamond z, y : A \to y : \Diamond\mathsf{K}A}\ R\Diamond}{x \leqslant y, y : A \to y : \Diamond\mathsf{K}A}\ Ser_\Diamond}{\to x : A \supset \Diamond\mathsf{K}A}\ R\supset$$

This derivation would seem to suggest that the frame properties needed are those that correspond to the two extra-logical rules used, namely,

$$\dfrac{xR_\Diamond y, \Gamma \to \Delta}{\Gamma \to \Delta}\ Ser_\Diamond \qquad \dfrac{x \leqslant z, xR_\Diamond y, yR_\mathsf{K} z, \Gamma \to \Delta}{xR_\Diamond y, yR_\mathsf{K} z, \Gamma \to \Delta}\ \Diamond\mathsf{K}\text{-}Tr$$

Rule $Ser_\Diamond$ has the variable condition that $y \notin \Gamma, \Delta$, which corresponds to an existential condition, whereas rule $\Diamond\mathsf{K}$-Tr corresponds to a universal one:

$$\begin{array}{ll} \forall x \exists y (x R_\Diamond y) & \textbf{Ser}_\Diamond \\ \forall y \forall z \forall w (y R_\Diamond z \wedge z R_\mathsf{K} w \supset y \leqslant w) & \Diamond\mathsf{K}\textbf{-Tr} \end{array}$$

The universal frame property $\Diamond\mathsf{K}$-**Tr** is, however, too strong. The instance of rule $\Diamond\mathsf{K}$-Tr used in the derivation of **KP** is not applied, root first, to an arbitrary sequent, but to one in which the middle term is the *eigenvariable* introduced by $Ser_\Diamond$. The requirement that $\Diamond\mathsf{K}$-Tr has to be applied above $Ser_\Diamond$, and that the middle term of $\Diamond\mathsf{K}$-Tr is the eigenvariable of $Ser_\Diamond$, is the side condition of the rule. Thus the following frame property can be read off from the derivation of **KP**:

$$\forall x \exists y (x R_\Diamond y \wedge \forall z (y R_\mathsf{K} z \supset x \leqslant z)) \qquad \textbf{KP-Fr}$$

It is easy to show that **KP-Fr** is derivable in a **G3**-sequent system for intuitionistic first-order logic extended by the two rules $Ser_\Diamond$ and $\Diamond\mathsf{K}$-Tr:

$$\dfrac{\dfrac{\dfrac{\dfrac{x R_\Diamond y \to x R_\Diamond y \quad \dfrac{\dfrac{\dfrac{x \leqslant z, x R_\Diamond y, y R_\mathsf{K} z \to x \leqslant z}{x R_\Diamond y, y R_\mathsf{K} z \to x \leqslant z}\,\Diamond\mathsf{K}\text{-}Tr}{x R_\Diamond y \to y R_\mathsf{K} z \supset x \leqslant z}\,R\supset}{x R_\Diamond y \to \forall z (y R_\mathsf{K} z \supset x \leqslant z)}\,R\forall}{x R_\Diamond y \to x R_\Diamond y \wedge \forall z (y R_\mathsf{K} z \supset x \leqslant z)}\,R\wedge}{x R_\Diamond y \to \exists y (x R_\Diamond y \wedge \forall z (y R_\mathsf{K} z \supset x \leqslant z))}\,R\exists}{\to \exists y (x R_\Diamond y \wedge \forall z (y R_\mathsf{K} z \supset x \leqslant z))}\,Ser_\Diamond}{\to \forall x \exists y (x R_\Diamond y \wedge \forall z (y R_\mathsf{K} z \supset x \leqslant z))}\,R\forall$$

Observe that the side condition on the application of $\Diamond\mathsf{K}$-Tr is respected. Conversely, any derivation that uses the rules $Ser_\Diamond$ and $\Diamond\mathsf{K}$-Tr in compliance with the side condition, can be transformed into a derivation that uses cuts with **KP-Fr**. If rule $\Diamond\mathsf{K}$-Tr is used, it is followed by $Ser_\Diamond$ because of the side condition, and the derivation contains a subderivation of the form

$$\cfrac{\cfrac{x \leqslant z, xR_{\Diamond}y, yR_{\mathsf{K}}z\Gamma' \to \Delta'}{xR_{\Diamond}y, yR_{\mathsf{K}}z\Gamma' \to \Delta'}\ \Diamond\mathsf{K}\text{-}Tr}{\cfrac{\vdots\ \mathcal{D}}{\cfrac{xR_{\Diamond}y, \Gamma \to \Delta}{\Gamma \to \Delta}\ Ser_{\Diamond}}}$$

We transform it as follows:

$$\cfrac{\to \forall x \exists y(xR_{\Diamond}y \wedge \forall z(yR_{\mathsf{K}}z \supset x \leqslant z)) \qquad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{yR_{\mathsf{K}}z, yR_{\mathsf{K}}z \supset x \leqslant z \to x \leqslant z} \qquad x \leqslant z, xR_{\Diamond}y, yR_{\mathsf{K}}z, \Gamma' \to \Delta'}{xR_{\Diamond}y, yR_{\mathsf{K}}z, yR_{\mathsf{K}}z, yR_{\mathsf{K}}z \supset x \leqslant z, \Gamma', \to \Delta'}\ \text{CUT}}{xR_{\Diamond}y, yR_{\mathsf{K}}z, yR_{\mathsf{K}}z \supset x \leqslant z, \Gamma', \to \Delta'}\ \text{L-C}}{\cfrac{xR_{\Diamond}y, yR_{\mathsf{K}}z, \forall z(yR_{\mathsf{K}}z \supset x \leqslant z), \Gamma', \to \Delta'}{\vdots\ \mathcal{D}'}\ L\forall}}{xR_{\Diamond}y, \forall z(yR_{\mathsf{K}}z \supset x \leqslant z), \Gamma \to \Delta}}{xR_{\Diamond}y \wedge \forall z(yR_{\mathsf{K}}z \supset x \leqslant z), \Gamma \to \Delta}\ L\wedge}{\cfrac{\exists y(xR_{\Diamond}y \wedge \forall z(yR_{\mathsf{K}}z \supset x \leqslant z), \Gamma \to \Delta}{\forall x \exists y(xR_{\Diamond}y \wedge \forall z(yR_{\mathsf{K}}z \supset x \leqslant z), \Gamma \to \Delta}\ L\forall}\ L\exists}}{\Gamma \to \Delta}\ \text{CUT}$$

Here $\mathcal{D}'$ is obtained by adding $\forall z(yR_{\mathsf{K}}z \supset x \leqslant z)$ to all the antecedents of the sequents in $\mathcal{D}$. If rule $Ser_{\Diamond}$ is used alone, namely without occurrences of $\Diamond\mathsf{K}\text{-}Tr$ above it, the conversion is obtained through $L\exists$ applied on the premise of $Ser_{\Diamond}$ and a cut with $\to \forall x \exists y xR_{\Diamond}y$; the latter follows from $\to \forall x \exists y(xR_{\Diamond}y \wedge \forall z(yR_{\mathsf{K}}z \supset x \leqslant z))$. We can conclude:

Proposition 5.6.1. *The system with rules $\Diamond\mathsf{K}\text{-}Tr$ and $Ser_{\Diamond}$ that respect the side condition is a cut-free equivalent of the system that employs* **KP-Fr** *as an axiomatic sequent in addition to the structural rules.*

The rules corresponding to **KP-Fr** do not follow the geometric rule schema. However, all the structural rules are still admissible in presence of such rules. In particular, cut elimination holds and the proof follows the pattern of 5.3.2.

Theorem 5.6.2. *The rule of cut*

$$\frac{\Gamma \to \Delta, x : A \quad x : A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'} \text{CUT}$$

is admissible in $\mathbf{G3I}_{\mathsf{K}\Diamond}$ *with* Ref_{K} *and extended with* $Ser_{\Diamond}$ *and* $\Diamond\mathsf{K}\text{-}Tr$.

Proof. Suppose that one of the premises of cut has been derived by $\Diamond\mathsf{K}\text{-}Tr$ followed by $Ser_{\Diamond}$ and that the middle term of the former disappeared by an application of the latter. We have

$$\cfrac{\Gamma \to \Delta, x : A \qquad \cfrac{\begin{array}{c}\cfrac{y \leqslant z, xR_{\Diamond}y, yR_{\mathsf{K}}z, \Gamma'' \to \Delta''}{xR_{\Diamond}y, yR_{\mathsf{K}}z, \Gamma'' \to \Delta''} \Diamond\mathsf{K}\text{-}Tr \\ \vdots \\ x : A, xR_{\Diamond}y, \Gamma' \to \Delta' \end{array}}{x : A, \Gamma' \to \Delta'} Ser_{\Diamond}}{\Gamma, \Gamma' \to \Delta, \Delta'} \text{CUT}$$

Observe that by hp-admissibility of substitution (Lemma 5.3.2) we can assume without loss of generality that the variable y does not occur in the left premise of cut. The derivation is transformed into the following in which the application of cut is of a lower height and therefore eliminable by the inductive hypothesis

$$\cfrac{\cfrac{\Gamma \to \Delta, x : A \qquad \begin{array}{c}\cfrac{x \leqslant z, xR_{\Diamond}y, yR_{\mathsf{K}}z, \Gamma'' \to \Delta''}{xR_{\Diamond}y, yR_{\mathsf{K}}z, \Gamma'' \to \Delta''} \Diamond\mathsf{K}\text{-}Tr \\ \vdots \\ x : A, xR_{\Diamond}y, \Gamma' \to \Delta' \end{array}}{xR_{\Diamond}y, \Gamma, \Gamma' \to \Delta, \Delta'} \text{CUT}}{\Gamma, \Gamma' \to \Delta, \Delta'} Ser_{\Diamond}$$

$\boxtimes$

It is worth noting that the acceptance of **KP** as valid implicitly forces us to accept some properties of the operator $\Diamond$, in particular, the derivability of $A \supset \Diamond A$.

Proposition 5.6.3. *In* $\boldsymbol{G3I}_{\mathsf{K}\Diamond}$ *with* Ref_{K}, $Ser_{\Diamond}$ *and* $\Diamond\mathsf{K}\text{-}Tr$ *it holds that:*

$\vdash \rightarrow x : A \supset \Diamond A$.

Proof. We have the following derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \leqslant z, zR_{\mathsf{K}}z, yR_{\Diamond}z, x \leqslant y, y : A \rightarrow y : \Diamond A, z : A}{zR_{\mathsf{K}}z, yR_{\Diamond}z, x \leqslant y, y : A \rightarrow y : \Diamond A, z : A}\,\Diamond\mathsf{K}\text{-}Tr}{yR_{\Diamond}z, x \leqslant y, y : A \rightarrow y : \Diamond A, z : A}\,Ref_{\mathsf{K}}}{yR_{\Diamond}z, x \leqslant y, y : A \rightarrow y : \Diamond A}\,R\Diamond}{x \leqslant y, y : A \rightarrow y : \Diamond A}\,Ser_{\Diamond}}{\rightarrow x : A \supset \Diamond A}\,R\supset
$$

Observe that the side condition on $\Diamond\mathsf{K}\text{-}Tr$ is respected.

⊠

In monomodal systems, the axiom schema $A \supset \Diamond A$ is characterized by reflexive frames, *i.e.*, frames in which $\forall x(xR_{\Diamond}x)$ holds. This is not any longer the case in multimodal systems. The above proposition shows, in fact, that the reflexivity of $R_{\Diamond}$ is a sufficient, but not a necessary, condition for the validity of $A \supset \Diamond A$. We have a derivation of a purely alethic property that uses properties of the global system, in particular, of the epistemic accessibility relation. This is a non-conservativity of the whole system with respect to the system without K. In order to restore conservativity, we add to our set of rules the rule of reflexivity of the alethic accessibility relation:

$$
\dfrac{xR_{\Diamond}x, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\,Ref_{\Diamond}
$$

With $Ref_{\Diamond}$ at our disposal, it becomes clear why the unrestricted $\Diamond\mathsf{K}\text{-}Tr$ is too strong. In fact, together with reflexivity of $R_{\Diamond}$ it would collapse our intuitionistic system into a classical one because it would permit to derive symmetry of $\leqslant$, as in

$$\dfrac{\dfrac{\dfrac{\dfrac{y \leqslant x, yR_{\Diamond}x, xR_{\mathsf{K}}x, x \leqslant y, xR_{\Diamond}x \rightarrow y \leqslant x}{yR_{\Diamond}x, xR_{\mathsf{K}}x, x \leqslant y, xR_{\Diamond}x \rightarrow y \leqslant x}\ \Diamond\mathsf{K}\text{-}Tr}{yR_{\Diamond}x, x \leqslant y, xR_{\Diamond}x \rightarrow y \leqslant x}\ Ref_{\mathsf{K}}}{x \leqslant y, xR_{\Diamond}x \rightarrow y \leqslant x}\ Mon_{\Diamond}}{x \leqslant y \rightarrow y \leqslant x}\ Ref_{\Diamond}$$

The derivation of the knowability principle in $\mathrm{G3I}_{\mathsf{K}\Diamond}$ using Ref_{K}, $Ser_{\Diamond}$ and $\Diamond\mathsf{K}$-Tr guarantees that the two rules are strong enough to capture the force of **KP** but does not yet permit to conclude that **KP-Fr** is the characterizing frame property of **KP**. This latter is achieved by the following proposition:

Proposition 5.6.4. *The property* **KP-Fr** *is necessary and sufficient to validate* **KP** *in intuitionistic bimodal frames.*

Proof. For sufficiency, it is enough to use the standard definition of forcing in Kripke models. Let x be a world in a frame. To show $x \Vdash A \supset \Diamond\mathsf{K}A$, let y be such that $x \leqslant y$, and suppose $y \Vdash A$. By **KP-Fr** and monotonicity of forcing, $y \Vdash \Diamond\mathsf{K}A$.

For necessity, we reason by contraposition. Consider an arbitrary frame and suppose that **KP-Fr** does not hold in it. Thus,

$$\exists x \forall y (xR_{\Diamond}y \supset \exists z(yR_{\mathsf{K}}z \wedge x \nleqslant z))$$

Let P an atom and u an arbitrary state. We can define $\Vdash$ so that $u \Vdash P$ if and only if $x \leqslant u$. Therefore, in the resulting model, **KP** instantiated with P is not forced at x.

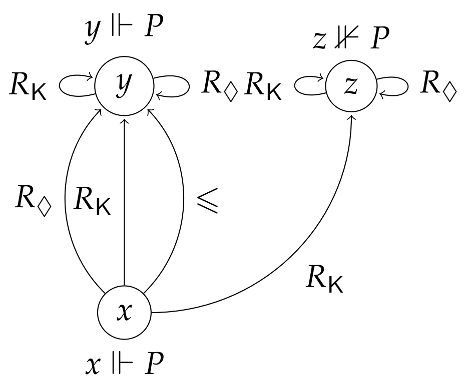
$\boxtimes$

What we have achieved by our analysis is a correspondence between the knowability principle in the form of the bimodal axiom **KP** and the frame property **KP-Fr**.

We have also shown that **KP-Fr** is equivalent to the two rules $\Diamond\mathsf{K}$-Tr and $Ser_{\Diamond}$ used in compliance with a side condition. By this equivalence, the system obtained by the addition of suitable combinations of these two rules provides a complete contraction- and cut-free system for intuitionistic bimodal logic extended with **KP**. We shall call it **knowability logic**, and indicate it with **G3KP**. Thus, **G3KP** is defined as $\mathbf{G3I}_{\mathsf{K}\Diamond} + Ref_{\mathsf{K}} + Ser_{\Diamond} + \Diamond\mathsf{K}$-$Tr$ with the proper side condition. By **KP-Fr** we can establish the following result:

Proposition 5.6.5. *There exists an intuitionistic frame that validates* **KP**, *but not* **OP**.

Proof. Take the frame with three worlds, x, y, and z such that $xR_{\Diamond}y$, $x \leqslant y$, and $xR_{\mathsf{K}}z$. The only R_{K} and $R_{\Diamond}$-accessibilities from y and z are the reflexive ones:



The frame respects condition **KP-Fr** and therefore validates **KP** (this can be checked also directly). On the other hand, the valuation defined by $x \Vdash P$, $x \Vdash y$ and $z \nVdash P$ shows that $x \nVdash P \supset \mathsf{K}P$.

$\boxtimes$

This result shows that the admissibility statement 5.1 does not hold for intuitionistic logic. Our proof system gives a confirmation for this semantic argument through a syntactic criterion, a failed exhaustive proof search.

To show that **OP** is not derivable in **G3KP**, care is needed with the use of labels. Consider the following attempt:

$$
\dfrac{\dfrac{\dfrac{\dfrac{y \leqslant z, x \leqslant y, yR_{\Diamond}y, yR_{\mathsf{K}}z, y : A \rightarrow z : A}{x \leqslant y, yR_{\Diamond}y, yR_{\mathsf{K}}z, y : A \rightarrow z : A}\,\Diamond\mathsf{K}\text{-}Tr}{x \leqslant y, yR_{\Diamond}y, y : A \rightarrow y : \mathsf{K}A}\,R\mathsf{K}}{x \leqslant y, y : A \rightarrow y : \mathsf{K}A}\,Ref_{\Diamond}}{\rightarrow x : A \supset \mathsf{K}A}\,R\!\supset
$$

This would seem to be derivation of **OP**, in contrast to what we would expect from the semantic argument above. Here, similarly to what happened in the derivation of symmetry, the application of $\Diamond\mathsf{K}$-*Tr* is not correct because the *eigenvariable* (here y) appears also where it should not, namely as a first argument of $R_{\Diamond}$, in the preorder atom, and in two labelled formulas. The variable condition expresses formally that rule $\Diamond\mathsf{K}$-*Tr* should consider the most general $R_{\Diamond}$ accessibility. By admitting only the reflexivity one, actuality and possibility are conflated with "the mystery of the disappearing diamond" (Jenkins 2009). Replacing reflexivity with seriality, the search turns into

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ x \leqslant y, yR_{\Diamond}z, yR_{\mathsf{K}}w, y : A \rightarrow w : A\end{array}}{x \leqslant y, yR_{\Diamond}z, y : A \rightarrow y : \mathsf{K}A}\,R\mathsf{K}}{x \leqslant y, y : A \rightarrow y : \mathsf{K}A}\,Ser_{\Diamond}}{\rightarrow x : A \supset \mathsf{K}A}\,R\!\supset
$$

Rule $\Diamond\mathsf{K}$-*Tr* is no longer applicable because the upper sequent in the attempted proof does not match its conclusion. The only applicable rule is Mon_{K} that adds $xR_{\mathsf{K}}w$. The search is exhaustive and we do not get what we would need to close it, namely the relational atom $y \leqslant w$. The failed search can be used instead to extract a countermodel to **OP**. The accessibilities are $xR_{\mathsf{K}}w$ in addition to those in the antecedent of the upper sequent; A, is forced at x and at y but not at w. Clearly $x \nVdash A \supset \mathsf{K}A$. By our analysis, the use of intuitionistic logic blocks the paradox in general, not only the specific derivation that uses a specific instance of the knowability principle (see Theorem 5.5.2).

It is only in classical logic that the paradox may arise. The question remains as to whether the Moore sentence $A \wedge \neg \mathsf{K}A$ is an essential ingredient of the paradox in its classical derivation. It is a natural question, because of the attempts at circumventing the paradox through a limitation of **KP** to certain classes of formulas that exclude seemingly pathological ones such as $A \wedge \neg \mathsf{K}A$ (as in Dummett 2001). Whether Moore sentences are indispensable in the derivation of **OP** can be determined by a root-first proof search. The search in our calculus leads to a sufficient condition for the derivation of **OP**, starting with the "compulsory" steps

$$\cfrac{\cfrac{\vdots \atop yR_{\mathsf{K}}z, x \leqslant y, y : A \to z : A}{x \leqslant y, y : A \to y : \mathsf{K}A}\, R\mathsf{K}}{\to x : A \supset \mathsf{K}A}\, R{\supset}$$

A correct derivation is obtained if the atom $y \leqslant z$ can be added, that is, if the K-accessibility implies the $\leqslant$-accessibility, or, in other words, if we are allowed to use the following rule:

$$\frac{x \leqslant y, xR_{\mathsf{K}}y, \Gamma \to \Delta}{xR_{\mathsf{K}}y, \Gamma \to \Delta}\, \mathit{Know}$$

The rule is the translation of the frame property

$$\forall x \forall y (xR_{\mathsf{K}}y \supset x \leqslant y) \qquad \textbf{Know}$$

As a diagram, it takes the form

$$\begin{tikzcd} y \\ x \arrow[u, "R_{\mathsf{K}}"] \arrow[u, dashed, bend right, "\leqslant"'] \end{tikzcd}$$

We then have

Proposition 5.6.6. *Rule* Know *is admissible in* $\mathbf{G3C}_{\mathsf{K}\Diamond} + Ref_{\mathsf{K}} + Ser_{\Diamond} + \Diamond\mathsf{K}\text{-}Tr$.

Proof. Using admissibility of weakening we have the following

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x \leqslant y, xR_{\mathsf{K}}y, \Gamma \to \Delta}{x \leqslant y, z \leqslant x, x \leqslant z, xR_{\Diamond}z, zR_{\mathsf{K}}y, zR_{\mathsf{K}}z, xR_{\mathsf{K}}y, \Gamma \to \Delta}\ \text{L-W}}{z \leqslant x, x \leqslant z, xR_{\Diamond}z, zR_{\mathsf{K}}y, zR_{\mathsf{K}}z, xR_{\mathsf{K}}y, \Gamma \to \Delta}\ \Diamond\mathsf{K}\text{-}Tr}{z \leqslant x, x \leqslant z, xR_{\Diamond}z, zR_{\mathsf{K}}z, xR_{\mathsf{K}}y, \Gamma \to \Delta}\ Mon_{\mathsf{K}}}{x \leqslant z, xR_{\Diamond}z, zR_{\mathsf{K}}z, xR_{\mathsf{K}}y, \Gamma \to \Delta}\ Sym_{\leqslant}}{xR_{\Diamond}z, zR_{\mathsf{K}}z, xR_{\mathsf{K}}y, \Gamma \to \Delta}\ \Diamond\mathsf{K}\text{-}Tr}{xR_{\Diamond}z, xR_{\mathsf{K}}y, \Gamma \to \Delta}\ Ref_{\mathsf{K}}}{xR_{\mathsf{K}}y, \Gamma \to \Delta}\ Ser_{\Diamond}}{}
$$

Observe that two applications of $\Diamond\mathsf{K}\text{-}Tr$, with the same *eigenvariable* y, are used. This is a licit use of the block of rules since a double use of $\Diamond\mathsf{K}\text{-}Tr$, followed by a step of seriality that removes the *eigenvariable*, corresponds to a multiple discharge of the minor assumption in the rule of elimination of the existential quantifier in natural deduction. ⊠

The derivation of **OP** can be given also directly in the system $\mathbf{G3C}_{\mathsf{K}\Diamond} + Ref_{\mathsf{K}} + Ser_{\Diamond} + \Diamond\mathsf{K}\text{-}Tr$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{K}}z, y \leqslant z, z \leqslant y, yR_{\mathsf{K}}w, zR_{\mathsf{K}}w, y \leqslant w, y : A \to w : A}{x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{K}}z, y \leqslant z, z \leqslant y, yR_{\mathsf{K}}w, zR_{\mathsf{K}}w, y : A \to w : A}\ \Diamond\mathsf{K}\text{-}Tr}{x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{K}}z, y \leqslant z, z \leqslant y, yR_{\mathsf{K}}w, y : A \to w : A}\ Mon_{\mathsf{K}}}{x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{K}}z, y \leqslant z, z \leqslant y, y : A \to y : \mathsf{K}A}\ R\mathsf{K}}{x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{K}}z, y \leqslant z, y : A \to y : \mathsf{K}A}\ Sym_{\leqslant}}{x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{K}}z, y : A \to y : \mathsf{K}A}\ \Diamond\mathsf{K}\text{-}Tr}{x \leqslant y, yR_{\Diamond}z, y : A \to y : \mathsf{K}A}\ Ref_{\mathsf{K}}}{x \leqslant y, y : A \to y : \mathsf{K}A}\ Ser_{\Diamond}}{\to x : A \supset \mathsf{K}A}\ R{\supset}
$$

There are no occurrences of Moore sentences in this derivation. Could we then conclude that it is not necessary the appeal to them for the derivation of **OP**? Actually, the absence of Moore sentences in this derivation is only fictitious, because **KP-Fr** has been identified by considering all possible instances of **KP** and so, *a fortiori*, also the instances with Moore sentences. On the contrary, allowing only a limited type of instances of **KP**, it could be the case that we are restricting also the class of frames validating **KP** and that these particular frames would not validate **OP**.

The cut-free derivation indicates that the source of the paradox has to be found in the joint use of **KP-Fr** and classical logic. This means that **KP** is not *per se* paradoxical, but it becomes so when used in a classical frame. Moreover, it is possible to show in classical logic, that if $Ref_{\Diamond}$ is included from the beginning in the derivation system, then **KP** corresponds to the frame condition **Know**.

Proposition 5.6.7. *The frame property* **Know** *is necessary and sufficient to validate* **KP** *in classical bimodal frames satisfying* $Ref_{\Diamond}$.

Proof. Necessity can be proved by the following chain of implications: Validity of **KP** implies the validity of **KP-Fr** (Proposition 5.6.4); validity of **KP-Fr** implies the admissibility of $Ser_{\Diamond}$ and $\Diamond\mathsf{K}$-Tr respecting the side condition (Proposition 5.6.1); $Ser_{\Diamond}$ and $\Diamond\mathsf{K}$-Tr respecting the side condition implies the admissibility of *Know* (Proposition 5.6.6).

For sufficiency, consider the derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y \leqslant z, yR_{\mathsf{K}}z, yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathsf{K}A, z : A}{yR_{\mathsf{K}}z, yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathsf{K}A, z : A}\ Know}{yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathsf{K}A, y : \mathsf{K}A}\ R\mathsf{K}}{yR_{\Diamond}y, x \leqslant y, y : A \to y : \Diamond\mathsf{K}A}\ R\Diamond}{x \leqslant y, y : A \to y : \Diamond\mathsf{K}A}\ Ref_{\Diamond}}{\to x : A \supset \Diamond\mathsf{K}A}\ R\supset$$

⊠

As we have seen, in classical logic **Know** is sufficient for deriving **OP** and even has the collapse of truth and knowledge as a consequence:

Proposition 5.6.8. *In* $\mathbf{G3C}_{\mathsf{K}\Diamond} + Ref_{\mathsf{K}} + Ref_{\Diamond} + Know$ *the relations* $\leqslant$ *and* R_{K} *collapse to the same relation.*

Proof. To preserve the monotonicity of $\leqslant$ in the presence of R_{K} we have assumed the validity of $\mathbf{Mon}_{\mathsf{K}}$. By reflexivity of R_{K}, $\mathbf{Mon}_{\mathsf{K}}$ implies that $\forall x \forall y(x \leqslant y \supset x R_{\mathsf{K}} y)$, *i.e.*, $\leqslant \subseteq R_{\mathsf{K}}$. The other direction of the inclusion, *i.e.*, $R_{\mathsf{K}} \subseteq \leqslant$, holds by **Know**. ⊠

We have thus shown that if $R_{\Diamond}$ is reflexive, truth and knowledge coincide in classical logic. Therefore, in the standard classical presentation of the Church-Fitch paradox, the assumption **KP** is semantically equivalent to **OP**.

Finally, let us consider the indispensability of the principle of factivity of knowledge in the derivation of the Church-Fitch paradox. Mackie (1980) and Tennant (1997) have maintained that the principle is not necessary, and that the paradox arises equally for belief-like notions. That such is the case is confirmed by our analysis as follows. First it is seen that a knowability principle for belief imposes the same frame condition as it did for knowledge: The characterization result employs never the rule of reflexivity for epistemic accessibility. Then it can be shown that a "belief omniscience" is derivable when reflexivity for knowledge accessibility replaced by seriality and transitivity for belief accessibility, as the following proposition shows (the names of the rules that have to respect the proper variable condition are obtained from those for K):

Proposition 5.6.9. *In* $\mathbf{G3C}_{\mathsf{B}\Diamond} + Ser_{\mathsf{B}} + Trans_{\mathsf{B}} + Ser_{\Diamond} + \Diamond\mathsf{B}\text{-}Tr$ *it holds that:*

$\vdash \to x : A \supset \mathsf{B}A.$

Proof. By the derivation

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y \leqslant t, w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{B}}w, wR_{\mathsf{B}}t, zR_{\mathsf{B}}t, yR_{\mathsf{B}}t, y : A \to t : A}{w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{B}}w, wR_{\mathsf{B}}t, zR_{\mathsf{B}}t, yR_{\mathsf{B}}t, y : A \to t : A}\ \Diamond\mathsf{K}\text{-}\mathit{Tr}}{w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{B}}w, wR_{\mathsf{B}}t, yR_{\mathsf{B}}t, y : A \to t : A}\ \mathit{Trans}_{\mathsf{B}}}{w \leqslant y, y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{B}}w, yR_{\mathsf{B}}t, y : A \to t : A}\ \mathit{Mon}_{\mathsf{B}}}{y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{B}}w, yR_{\mathsf{B}}t, y : A \to t : A}\ \mathit{Sym}_{\leqslant}}{y \leqslant w, x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{B}}w, y : A \to y : \mathsf{B}A}\ R\mathsf{B}}{x \leqslant y, yR_{\Diamond}z, zR_{\mathsf{B}}w, y : A \to y : \mathsf{B}A}\ \Diamond\mathsf{K}\text{-}\mathit{Tr}}{x \leqslant y, yR_{\Diamond}z, y : A \to y : \mathsf{B}A}\ \mathit{Ser}_{\mathsf{B}}}{x \leqslant y, y : A \to y : \mathsf{B}A}\ \mathit{Ser}_{\Diamond}}{\to x : A \supset \mathsf{B}A}\ R{\supset}
$$

$\boxtimes$

5.7 Conclusions

In Tennant (2009), the following is written about the prospect of a proof theory that covers the Church-Fitch paradox: “we are still a long way, of course, from having a fully adequate proof-theory governing the interaction among [the modalities involved] (let alone a formal semantics, with respect to which one might be able to establish the soundness and completeness of whatever proof system is devised)” (*ibid.*, p. 237). The proof systems $\mathbf{G3I}_{\mathsf{K}\Diamond}$ and $\mathbf{G3C}_{\mathsf{K}\Diamond}$ developed in this chapter, with the analysis of the accessibility relations $\leqslant$, R_{K} and $R_{\Diamond}$ and the way they interact in formal proofs, offer an answer to the first Tennant's issue. The completeness theorem with respect to Kripke semantics for these calculi answers Tennant's second issue. The results are here formulated for labelled sequent calculi but can be adapted also to proof systems based on natural deduction. Our work offers a new methodology for a general theory of knowability and, more broadly, of logical epistemology. We have determined the first-order correspondents of modal axioms on the basis of a root-first proof search in labelled sequent calculi for bimodal logic. The correspondence results have a standing independent of the use of labelled calculi. Extending a general Kripke completeness result, we have shown that the modal logic obtained by the addition of the knowability principle is complete with respect to the class of frames that satisfy the first-order frame condition which was determined by the procedure. The resulting calculi are complete proof systems for knowability logic, both in a classical and in an intuitionistic setting. The strong structural properties of these calculi make it possible to draw conclusions not only about questions of derivability, but also about underivability of the paradox in precisely defined formal systems of intuitionistic and classical bimodal logic. The crucial step here is the conversion of a non-geometric axiom, the frame condition corresponding to **KP**, into a system of rules so as to achieve full control over derivations in intuitionistic bimodal logic extended by the knowability principle. exploiting the frame property corresponding to **KP** our work goes a step further, namely it shows that the use of intuitionistic logic for blocking the paradox succeeds: Not only **OP** is intuitionistically underivable from **KP** instantiated with the Moore sentence, but **OP** is not even intuitionistically admissible from **KP**. On the other hand, the paradox is indeed derivable in classical logic: the standard proof is reconstructed in our analysis and converted into a cut-free form. Nonetheless, we claim that this derivation is nothing else than a fallacious argument in disguise: The reason is that **KP** and **OP** are semantically equivalent in a classical frame. We thus have an argument in favor of the anti-realist position, provided that the formalization of the knowability principle corresponds to **KP**. If anti-realism is conceived in a strict Dummettian sense, then intuitionistic logic is already sufficient for blocking Fitch's argument. Otherwise, if a weaker anti-realism is embraced and accordingly classical logic is allowed, the paradox gets reduced to a *petitio principii*. The conversion of the frame property **KP-Fr** into a combination of rules governed by a side condition follows the methodology of proof analysis in which universal and geometrical axioms have been treated so far. It is a first successful attempt to extract a general method for transforming a much wider type of mathematical axioms into a set of inference rules. From this perspective, the proof-theoretic analysis **KP** opens up promising possibilities also for a more traditional type of foundational study.

Observation. It has been observed[3] that in intuitionistic frames the weaker property $W\text{-}Ref_{\mathsf{K}}$ $\forall x \exists y(xR_{\mathsf{K}}y \wedge y \leqslant x)$ suffices to characterize factivity of K. In fact, the sequent $\rightarrow x : \mathsf{K}A \rightarrow A$ is derivable in the presence of the rule corresponding to the frame condition $W\text{-}Ref_{\mathsf{K}}$

$$\frac{xR_{\mathsf{K}}y, y \leqslant x, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ W\text{-}Ref_{\mathsf{K}}$$

Rule $W\text{-}Ref_{\mathsf{K}}$ has the variable condition that y does not appear in Γ, Δ. Thus, factivity of knowledge follows from the weaker frame condition as follows

$$\cfrac{\cfrac{\cfrac{z : A, yR_{\mathsf{K}}z, z \leqslant y, x \leqslant y, y : \mathsf{K}A \rightarrow y : A}{yR_{\mathsf{K}}z, z \leqslant y, x \leqslant y, y : \mathsf{K}A \rightarrow y : A}\ L\mathsf{K}}{x \leqslant y, y : \mathsf{K}A \rightarrow y : A}\ W\text{-}Ref_{\mathsf{K}}}{\rightarrow x : \mathsf{K}A \rightarrow A}\ R\supset$$

A similar weaker property $W\text{-}Ref_{\Diamond}$ $\forall x \exists y(xR_{\Diamond}y \wedge x \leqslant y)$ characterizes $A \supset \Diamond A$. We consider the following rule with y as *eigenvariable*

$$\frac{xR_{\Diamond}y, x \leqslant y, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}\ W\text{-}Ref_{\Diamond}$$

Thus, the sequent $\rightarrow x : A \supset \Diamond A$ can be derived without any application of $Ref_{\Diamond}$

$$\cfrac{\cfrac{\cfrac{yR_{\Diamond}z, y \leqslant z, x \leqslant y, y : A \rightarrow y : \Diamond A, z : A}{yR_{\Diamond}z, y \leqslant z, x \leqslant y, y : A \rightarrow y : \Diamond A}\ R\Diamond}{x \leqslant y, y : A \rightarrow y : \Diamond A}\ W\text{-}Ref_{\mathsf{K}}}{\rightarrow x : A \rightarrow \Diamond A}\ R\supset$$

The derivations above show that the conditions are sufficient for $\mathsf{K}A \supset A$ and

[3]Pierluigi Minari, personal communication.

$A \supset \Diamond A$ to be valid. They are also necessary. Suppose that $W\text{-}Ref_{\mathsf{K}}$ is not satisfied, that is, let x be such that for all y, $xR_{\mathsf{K}}y$ implies $y \nleqslant x$. It is possible to give a model in which $x \Vdash \mathsf{K}A$ but $x \nVdash A$, so a countermodel for $x \Vdash \mathsf{K}A \supset A$. The model is defined by imposing that for every atom P, $y \Vdash P$ if and only if there is an u such that $xR_{\mathsf{K}}u$ and $u \leqslant y$. Observe that in this model the monotonicity of $\Vdash$ with respect to $\leqslant$ is satisfied, that is, it holds that $y \Vdash P$ and $y \leqslant y'$ implies $y' \Vdash P$. In this model, for an arbitrary z, if $xR_{\mathsf{K}}z$ then $xR_{\mathsf{K}}z$ and $z \leqslant z$, since $\leqslant$ is reflexive. Therefore, there is an u such that $xR_{\mathsf{K}}u$ and $u \leqslant z$, and we conclude that $z \Vdash A$ by the definition of $\Vdash$. Thus, $x \Vdash \mathsf{K}A$. On the other hand, suppose by contradiction that $x \Vdash A$. Then, by the definition of $\Vdash$ we have that there is an u such that $xR_{\mathsf{K}}u$ and $u \leqslant x$, which is impossible since $xR_{\mathsf{K}}u$ implies $u \nleqslant x$ by hypothesis. By a similar reasoning we can prove that if $W\text{-}Ref_{\Diamond}$ is not satisfied then it is possible to find a countermodel for $A \supset \Diamond A$.

Chapter 6

Conclusions and related work

The idea underlying the labelled approach to the proof theory of modal logics is that the rules encode the explanation of modalities in terms of relational semantics. This allows to exploit the modularity of the relational semantics so that systems for various modal logics result by extending the basic system with rules corresponding to the properties of the accessibility relations. This idea has been largely developed: possible worlds semantics has been internalized in the form of tableaux in Fitting (1983), Catach (1991), Nerode (1991), Goré (1998), Masacci (2000), and in the form of natural deduction in Fitch (1966), Simpson (1994). Finally, Mints (1997), Kushida and Okada (2003), Castellini and Smaill (2002), Castellini (2005) provide an labelled approach based on sequent systems. The survey by Negri (2011) gives an overview of the method and references to its applications. In this final part, we compare our systems with the labelled approach of Viganò (2000) which is one of the most extensive and comprehensive contribution to the topic. In the first part (Ch. 2, Part I) a labelled natural deduction system **N(K)** for basic modal logic is introduced. System **N(K)** is basically the same system we introduced in the first chapter, with the exception of the falsity elimination rule which gets formulated as

$$
\cfrac{\begin{array}{c}[x : \overset{1}{A} \supset \bot] \\ \vdots \\ y : A\end{array}}{x : A}\ \bot E_1
$$

Another difference is the formulation of the elimination rules: only special elimination rules are considered in Viganò's book, whereas we adapted to labelled systems the general formulation of von Plato (2001). However, our main goal was not the analysis of the structural properties of the natural deduction system we have introduced, but the justification of the corresponding sequent calculus rules. Although the labelled systems are based on the semantics, it is possible to explain the meaning of a formula $x : A$ in terms of its use, that is, in terms of the rules that manipulate it. In fact, in Read (2008) the labelled natural deduction system of Simpson (1994) is indicated as a possible solution to the problem of finding an harmonic pair of rules for modal operators. Instead, the structural properties of derivations **N(K)** are deeply analyzed in Viganò (2000): the system is proved to be sound and complete with respect to its semantics, and a detailed proof of the normalization theorem is given. **N(K)** can be also extended with rules for accessibility relation: the rules correspond to **Horn relational formulas**, that is, formulas of the form $\forall x_1 \ldots \forall x_n((s_1 R t_1 \wedge, \ldots, \wedge s_m R t_m) \supset s_0 R t_0)$, where $m \geqslant 0$ and the s_i and t_j are terms built from labels $x_1 \ldots x_n$ and constant function symbols. Example of such rules are:

$$
\cfrac{}{xRf(x)}\ Ser \qquad \cfrac{}{xRx}\ Ref \qquad \cfrac{xRy}{yRx}\ Sym \qquad \cfrac{xRy \quad yRz}{xRz}\ Trans
$$

Observe that $f(x)$ in rule *Ser* is a Skolem function. Correspondence results between modal axioms and properties of R can be easily obtained by the relational rules. The proof of the completeness theorem can be extended so that any system $\mathbf{N}(\mathcal{L})$ obtained by extending **N(K)** with a Horn relational theory $\mathbf{N}(\mathcal{T})$ is sound and complete with respect to its semantics. Normalization is proved by considering several rules for $\perp$ (global, universal, local). Therefore, a possible direction for a future research could be to see how the techniques and results of Viganò's book can be adapted to setting of our system of natural deduction, and what kind of relationship can be stated between them.

In Viganò's work, labelled sequent systems are introduced because they allow a finer grained control on the structure of formal derivations, and complexity results and decidability are more easily established when logics are presented as sequent systems of rules. The system **S(K)** introduced in Ch. 6 is strictly related to our system **G0K** of the first chapter. Although the rules for the modal operator $\Diamond$ and those for conjunction, disjunction and negation are derived, all the two-premise rules are formulated as context-independent rules, as in **G0K**. Initial sequents have a relational atomic formula xRy as principal, so derivations may start with sequents $xRy \rightarrow xRy$. Like **G0K**, also **S(K)** has all the structural rules primitive, and a partial elimination of contraction is the major issue of the entire second part of the work, as noted in Negri (2005). An important difference with respect to our approach is that labelled formulas and the relational atoms occur in a sequent only separated. A sequent is either an expression of the form $\Delta \rightarrow xRy$ or $\Gamma, \Delta \rightarrow \Gamma'$, where Γ and Γ' are multisets of labelled formulas, and Δ is a multisets of relational atoms[1]. The two possible forms of sequents correspond to the **separation** of the the basic system from the relational theory: $\Delta \rightarrow xRy$ expresses that a relational atom follows only from other relational atoms, and $\Gamma, \Delta \rightarrow \Gamma'$ expresses that labelled formulas may follow from other labelled formulas or relational atoms (see *ibid.*, p. 139). The rules for the modal operator $\Box$ are then formulated as follows:

$$\frac{\Delta \rightarrow xRy \quad y : A, \Gamma, \Delta \rightarrow \Gamma'}{x : \Box A, \Gamma, \Delta \rightarrow \Gamma'} L\Box \qquad \frac{xRy, \Gamma, \Delta \rightarrow \Gamma', y : A}{\Gamma, \Delta \rightarrow \Gamma', x : \Box A} R\Box$$

[1]We maintain the use of the sequent symbol $\rightarrow$ instead of that used by the author $\vdash$ in order to avoid confusions.

Since in sequent as $\Delta \to xRy$ there is at most one relational atom in the succedent, the system **S(K)** does not contain rules for weakening and contraction for such formulas in the succedent. Thus, the structural rules of **S(K)** are:

$$\frac{\Gamma, \Delta \to \Gamma'}{xRy, \Gamma, \Delta \to \Gamma'} \text{ L-W} \qquad \frac{xRy, xRy, \Gamma, \Delta \to uRv}{xRy, \Gamma, \Delta \to uRv} \text{ L-C}$$

$$\frac{\Gamma, \Delta \to \Gamma'}{x : A, \Gamma, \Delta \to \Gamma'} \text{ L-W} \qquad \frac{\Gamma, \Delta \to \Gamma'}{\Gamma, \Delta \to \Gamma', x : A} \text{ R-W}$$

$$\frac{x : A, x : A, \Gamma, \Delta \to \Gamma'}{x : A, \Gamma, \Delta \to \Gamma'} \text{ L-C} \qquad \frac{x : A, x : A, \Gamma, \Delta \to \Gamma'}{\Gamma, \Delta \to \Gamma', x : A} \text{ R-C}$$

The separation between the basic system and the relational theory (labeling algebra, in the terminology of the author) is maintained in the derivations: in the relational theory only relational atoms are inferred, whereas in the basic systems both relational atoms and labelled formulas are used to derive other labelled formulas, "so that a derivation in the base system may depend on a derivation in the relational theory, but not *viceversa*" (see *ibid.*, p. 9). Several extensions of **S(K)** are then considered. The new rules for the accessibility relation introduce a relational formula in the succedent, so in our terminology, they follow the schema R-Reg. Examples of rules in the relational theory are:

$$\frac{}{\to xRf(x)} \textit{ Ser} \qquad \frac{}{\to xRx} \textit{ Ref}$$

$$\frac{\Delta \to xRy}{\Delta \to yRx} \textit{ Sym} \qquad \frac{\Delta \to xRy \quad \Delta \to yRz}{\Delta \to xRz} \textit{ Trans}$$

The admissibility of the cut rule is not given directly by a derivation conversion, but rather indirectly. It is shown (Theorem 6.3.1., p.149) that the labelled sequent and the corresponding natural deduction systems are equivalent, that is, **S**($\mathcal{L}$) $=$ **S(K)** $+$ **S**($\mathcal{T}$) and **N**($\mathcal{L}$) $=$ **N(K)** $+$ **N**($\mathcal{T}$). Moreover, the subformula property is satisfied: in any derivation of a sequent $\Gamma, \Delta \to \Gamma'$, only labelled subformulas of Γ and Γ' may occur. Observe that here the definition of subformula is restricted to the labelled formulas and does not include the relational atoms (see Definition 2.3.10, p. 46). When cut can be dispensed with and the subformula property holds, derivations can be built from the sequent to be derived, working towards the initial sequents. However, contraction duplicates formulas and it is always applicable, with the consequence that the proof-search procedure may not terminate. The entire second part (Ch. 8–11) is devoted to the problem of bounding the application of contraction in order to ensure the decidability of various modal logics. In fact, a large number of modal logics are known to be decidable and their decidability has been established by using model-theoretic techniques as the finite model property. The elimination of contraction, or, when it is not possible, a bound on its application, is then required when the question of decidability and complexity for modal logics is addressed in proof-theoretic terms. Contraction is indispensable for modal logics stronger than the basic modal logics: the sequent $\rightarrow \neg\Box\neg(A \supset \Box A)$ is not derivable in **S(T)** without contraction, and so is for $\rightarrow \Box\neg\Box A \supset \Box\neg\Box\Box A$ in **S(K4)**. The author also gives a list of sequents derivable in **S(T)** (**S(K4)**) with the specification of the the number of contraction required for its derivability in **S(T)** (resp. in **S(K4)**). Although application of left contraction with a modal formula $x : \Box A$ as principal cannot be eliminated while retaining the completeness, every derivable sequent in **S(T)** has a derivation in which there are no occurrences of contraction, except for application of left contraction with a modal formula $x : \Box A$ as principal; however, these applications are not needed more than $pbs(\Gamma, \Delta \rightarrow \Gamma')$ in each branch, where $pbs(\Gamma, \Delta \rightarrow \Gamma')$ is the number of positive boxed subformulas of $\Gamma, \Delta \rightarrow \Gamma'$ (Theorem 10.1.4, p. 210). A similar result (Theorem 11.2.5, p. 235) holds for the systems **S(K4)** and **S(S4)**. On the other hand, contraction is eliminable once and for all from the system **S(K)** for basic modal logic (Theorem 9.1.1, p. 187). The problem is strictly connected to that of the admissibility of contraction for our labelled system with rule $L\Box 2$. As we already said, our conjecture is that the system **G3K** where the standard rule $L\Box$ is replaced by the following

$$\frac{xRy, y : A, \Gamma \to \Delta}{xRy, x : \Box A, \Gamma \to \Delta}\ L\Box 2$$

is cut and contraction free, so it is possible to get rid of the repetition of $x : \Box A$ without losing the completeness of the system. However, an obstacle to the proof of contraction admissibility is given by the non-invertibility of $L\Box_2$. The eliminability result Viganò could cast light on the our problem.

Finally, we summarize our main contributions and discuss the directions for a future research. The methodology of the labelled systems has been motivated by the attempt to give a modular and uniform presentation for dealing with a large class of modal epistemic logics. Although the presence of the semantics in the syntax has been considered disputable from a conceptual point of view, it must be noted that it does not preclude the possibility of reasoning about modal logic in proof-theoretic terms: none of the proofs of the main results of this work (cut elimination and admissibility of the structural rules) makes appeal to the infinitary model-theoretic techniques typically used in modal logics, and they have all been established through the derivation conversion strategy which is essential in the proof theory studies. Through the use of labels, the relational semantics can be successfully employed to make formal derivations. Possible states and accessibilities between them play an important inferential role when we reason about modal logic: When we prove that a modal axiom characterizes a certain class of frames, we constantly deduce a property of R from another, or conclude that some formula A is forced at a state x because x is accessible from another state y. In the labelled systems, the inferential role of the relational semantics is precisely formalized and made explicit part of the syntax of sequent rules.

Through the chapters we have pointed out the possible developments of our work, we briefly summarize the most interesting ones. The material covered by the first chapter can be used to give a normalizing labelled system of natural deduction with general elimination rules for a large class of modal logics. The treatment of non-logical axioms in natural deduction has been recently investigated in Negri and von Plato (2011) but its application to modal logic is still material for a future research.

The second chapter should be further developed in order to show that a better result on the upper bound on cut-free derivations can be achieved: admissibility of contraction in the presence of non-invertible logical rules remains an open problem, and the possibility to find a semantic solution to it should be seriously taken into consideration. It has been also noted that the proof of the cut elimination theorem (with context-sharing cut) requires a restricted version of contraction admissibility, that is, when cut gets formulated as a context-sharing rule contraction has to be proved admissible only for atomic formulas. The conjecture, first suggested by Roy Dyckhoff for the intuitionistic system **G3ip**, is that it is possible to prove cut elimination without any contraction at all, since contraction is a special case of context-sharing cut in which is cut formula is principal of an initial sequent. It must be interesting to prove this conjecture for the labelled system **G3K**.

As we already mentioned in the fourth chapter, a possible development of the system **G3PAL** for the logic of public announcements could be to add rules for dealing with the common knowledge operator. The problem is due to the iterative interpretation of common knowledge like an infinite conjunction or, equivalently, to the presence of an accessibility relation defined as the (reflexive and) transitive closure of each R_a. The same question arises for other logics like LTL (Linear Time Logic) and the results of finitization given in Boretti and Negri (2010) should lead the further research in this direction.

The style of analysis of the last chapter on the knowability paradox can be variously applied: it is there shown that the importance of the techniques of proof theory, as normalization and cut elimination, go beyond their immediate applications to logical calculi like, for instance, syntactic proof of consistency, disjunction property for intuitionsitic system, interpolation theorem. Indeed, when a normal derivation of the paradox is found, every inferential step is explicit and it is clear where the paradoxical conclusion comes from. This encourages us to consider other semantic paradoxes in the light of the method presented. From a more philosophical perspective, that work takes a stand on the revisionary approach to paradoxes: any revisionary approach to paradoxes should come after (or, at least, reckon with) the structural proof analysis of its derivation. The requirement of normal derivability is unavoidable in the presence of non-logical axioms, because such axioms could make the application of other logical rules problematic. For this purpose, natural deduction with general elimination rules and sequent systems with non-logical inference rules will play a central role in the diagnosis of the logical paradoxes.

References

Arrazola, X. and M. Ponte, eds (2010) *ILCLI 2010. Proceedings of the Second International Workshop on Logic and Philosophy of Knowledge, Communication and Action*, University of the Basque Country Press.

Artemov, S. and T. Protopopescu (2011) Knowability from a logical point of view, ms.

Avron, A. (1996) The method of hypersequents in the proof theory of propositional non-classical logic, in W. Hodges, M. Hyland, C. Steinhorn and J. Truss (eds), *Logic: From Foundations to Applications*, Oxford University Press, pp. 1–32.

Balbiani, P., H. van Ditmarsch, A. Herzig and T. de Lima (2010) Tableaux for public announcement logic, *Journal of Logic and Computation*, vol. 20, pp. 55–76.

Beall, J.C. (2000) Fitch's proof, verificationism, and the knower paradox, *Australasian Journal of Philosophy*, vol. 78, pp. 241–247.

Beall, J.C. (2009) Knowability and possible epistemic oddities. In Salerno (ed), pp. 105–125.

van Benthem, J. (2009) Actions that make us know. In Salerno (ed), pp. 129–146.

Blackburn, P., M. de Rijke and Y. Venema (2001) *Modal Logic*, Cambridge University Press.

Boretti B. and S. Negri (2010) On the finitization of Priorean linear time, in M. D'Agostino and C. Sinigaglia (eds), *New Essays in Logic and Philosophy of Science*, College Publications, London, pp. XXX–XXX.

Božić, M. and K. Došen (1984) Models for normal intuitionistic modal logics. *Studia Logica*, vol. 43, pp. 217–245.

Brogaard, B. and J. Salerno (2009) Fitch's paradox of knowability, in E. N. Zalta (ed) *The Stanford Encyclopedia of Philosophy (Fall 2009 Edition)*,
URL = <http://plato.stanford.edu/archives/sum2011/entries/fitch-paradox/>.

Brünnler, K. (2009) Deep Sequent Systems for Modal Logic, *Archive for Mathematical Logic*, vol. 48, pp. 551–77.

Bucheli, S., R. Kuznets, B. Renne, J. Sack, and T. Studer, (2010) *Justified Belief Change*, in X. Arrazola and M. Ponte (eds), pp. 135–155.

Burgess, J. (2009) Can truth out? In Salerno (ed), pp. 147–162.

Castellini, C. (2005) Automated Reasoning in Quantified Modal and Temporal Logic, Ph.D. thesis, University of Edinburgh.

Castellini, C. and A. Smaill (2002) A systematic presentation of quantified modal logics, *Logic Journal of the IGPL*, vol. 10, pp. 571–599.

Catach, L. (1991) TABLEAUX: A General theorem prover for modal logic, *Journal of Automated Reasoning*, vol. 7, pp. 489–510.

Cerrato, C. (1996) Modal sequents, in Wansing (ed), pp. 141–66.

van Ditmarsch, H., W. van der Hoek, and B. Kooi (2007) *Dynamic Epistemic Logic*, Springer, Berlin.

Došen, K. (1985) Sequent-systems for modal logic, *Journal of Symbolic Logic*, vol. 50, pp. 149–159.

Dummett, M. (2001) Victor's error, *Analysis*, vol. 61, pp. 1–2.

Dummett, M. (2009) Fitch's paradox of knowability, in Salerno (ed.), pp. 51–52.

Edington, D. (1985) The paradox of knowability, *Mind*, vol.94, pp. 557–568.

Fagin, R., J. Y. Halpern, Y. Moses and M. Y. Vardi, (1995) *Reasoning about Knowledge*, MIT Press, Cambridge (USA).

Fitch, F. (1963) A logical analysis of some value concepts, *Journal of Symbolic Logic*, vol. 28, pp. 135–42.

Fitch, F. (1966) Tree proofs in modal logic, *The Journal of Symbolic Logic*, vol. 31, pp. 152.

Fitting, M. (1983) *Proof Methods for Modal and Intuitionistic Logics*, Kluwer, Dordrecht.

Gabbay, D. M, (1996), *Labelled Deductive Systems: Volume 1. Foundations*, Oxford University Press.

Gentzen, G. (1969) Untersuchungen über das logische Schlissen, in M. Szabo (ed) *The collected Papers of Gerhard Gentzen*, North-Holland, Amsterdam, pp. 68–131.

Gerbrandy, J. and W. Groenveled (1997) Reasoning about information change, *Journal of Logic, Language and Information*, vol. 6, pp.147–169.

Giaretta, P. (2009), The paradox of knowability from a Russellian perspective, *Prolegomena*, vol. 8, pp. 141–158.

Girard, J. (1987) *Proof Theory and Logical Complexity*, Bibliopolis, Napoli.

Goré, R. (1998) Tableau methods for modal and temporal logics, in M. D'Agostino *et al.* (eds) *Handbook of Tableau Methods*, Kluwer.

Hakli, R. and S. Negri (2008) Proof theory for distributed knowledge, in M. Fisher, F. Sadri and M. Thielscher (eds) *Computational Logic in Multi-Agent Systems*, Springer, Berlin, pp. 100–16.

Hakli, R. and S. Negri (2011), Does the Deduction Theorem Fail for Modal Logic?, in S. Lindström, E. Palmgren and D. Westerståhl (eds), *The Philosophy of Logical Consequence and Inference*, Synthese, Berlin, in press.

Hakli, R. and S. Negri (2011a), Reasoning about collectively accepted group beliefs, *Journal of Philosophical Logic*, vol 40, pp. 531–555.

Indrzejczak, A. (1998) Cut-free Double Sequent Calculus for S5, *Logic Journal of the IGPL*, vol 6, pp. 505–16.

Ketonen, O. (1944) *Untersuchungen zum Prädikatenkalkül*, Annales Acad. Sci. Fenn. Ser. A.I. 23, Helsinki.

Kleene, S. (1952) *Introduction to Metamathematics*, North Holland, Amsterdam.

Kripke S. (1963) Semantical analysis of modal logic I. Normal modal propositional calculi, *Zeitschrift f. math. Logik und Grund. d. Math.*, vol. 9, pp. 67–96.

Kripke, S. (1965) Semantical Analysis of Intuitionistic Logic I, in *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, vol. 40, pp. 92–130.

Kushida, H. and M. Okada (2003) A proof-theoretic study of the correspondence of classical logic and modal logic, *The Journal of Symbolic Logic*, vol. 68, pp. 1403–1414.

Hintikka J. (1962) *Knowledge and Belief. And introduction to the logic of the two notions*, Cornell University Press, Ithaca & London.

Hughes G. and M. Cresswell (1996) *A New Introduction to Modal Logic*, Routledge, London.

Maffezioli P. and S. Negri (2010) A Gentzen-style Analysis of Public Announcement Logic, in X. Arrazola and M. Ponte (2010), pp. 293–313.

Maffezioli P. and S. Negri (2011), A proof theoretical perspective on Public Announcement Logic, in *SILFS 2010. Proceedings of the International Conference of the Italian Society for Logic and Philosophy of Science*, to appear.

Maffezioli, P., A. Naibo and S. Negri (2011) The Church-Fitch knowability paradox in the light of structural proof theory, *Synthese*, in press.

Martin-Löf, P. (1998) Truth and knowability: On the principles C and K of Michael Dummett. In G. Dales and G. Oliveri (eds) *Truth in Mathematics*, pp. 105–114, Oxford University Press.

Martini S. and A. Masini (1996) A Computational Interpretation of Modal Proofs, in H. Wansing (ed), *Proof Theory of Modal Logic*, Kluwer, Dordrecht, pp. 213–41.

Massacci, F. (2000) Single step tableaux for modal logics: methodology, computations, algorithms, *Journal of Automated Reasoning*, vol. 24, pp. 319–64.

Meyer J.C. and W. van der Hoek (2004), *Epistemic Logic for AI and Computer Science*, Cambridge University Press.

Mints, G. (1997) Indexed systems of sequents and cut-elimination, *Journal of Philosophical Logic*, vol. 26, pp. 67–696.

Negri, S. (2003) Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem, *Archive for Mathematical Logic*, vol. 42, pp. 389–401.

Negri S. (2005) Proof analysis in modal logic, *Journal of Philosophical Logic*, vol. 34, pp. 507–44.

Negri S. (2009) Kripke completeness revisited, in G. Primiero and S. Rahman (eds), *Acts of Knowledge – History, Philosophy and Logic*, pp. 247–282, College Publications, London

Negri S. (2011), Proof theory for modal logic, *Philosophy Compass*, vol. 6/8, pp. 523–38.

Negri S. and J. von Plato (1998), Cut elimination in presence of axioms, *Bullitin of Symbolic Logic*, vol. 4, pp. 418–35.

Negri, S. and J. von Plato (2001) *Structural Proof Theory*, Cambridge University Press.

Negri, S. and J. von Plato (2001a), Sequent calculus in natural deduction style, *Journal of Symbolic Logic*, **66**:1803–16.

Negri, S. and J. von Plato (2011), *Proof Analysis: A Contribution to Hilbert's Last Problem*, Cambridge University Press.

Nerode, A. (1991) Some lectures on modal logic, in F. L. Bauer (ed), *Logic, Algebra, and Computation*, Springer, Berlin.

Ohnishi, M. and K. Matsumoto (1957), Gentzen Method in Modal Calculi, *Osaka Mathematical Journal*, vol. 9, pp. 113–30.

Ono, H. (1998) Proof-Theoretic Methods in Nonclassical Logic. An Introduction, in M. Takahashi, M. Okada and M. Dezani-Ciancaglini (eds), *Theories of Types and Proofs, Mathematical Society of Japan Memoirs*, Mathematical Society of Japan, Tokyo, vol. 2, pp. 207–54.

von Plato, J. (2001) Natural deduction with general elimination rules, *Archive for Mathematical Logic*, vol. 40, pp. 541–67.

von Plato, J. (2001a) A proof of Gentzen's *Hauptsatz* without multicut, *Archive for*

Mathematical Logic, vol. 40, pp. 9–18.

von Plato, J. (2003) Translation from natural deduction to sequent calculus, *Mathematical Logic Quarterly*, vol. 49, pp. 435–43.

von Plato, J. (2009) Gentzen's logic, in D. Gabbay and J. Woods (eds) *Handbook of the History of Logic*, pp. 667–721, Elsevier. Amsterdam.

Plaza J. (1989) Logics of public communications, in M.L. Emrich *et al.* (eds), *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems: Poster Session Program*, pp. 201–216. Reprinted in *Synthese*, vol. 158, pp. 165–179, 2007.

Poggiolesi, F. (2008) A cut-free simple sequent calculus for modal logic S5, *Review of Symbolic Logic*, vol. 1, pp. 3–15.

Poggiolesi, F. (2009) The Tree-hypersequent Method for Modal Propositional Logic, in D. Makinson, J. Malinowski and H. Wansing (eds), *Trends in Logic: Towards Mathematical Philsophy*, Springer, Berlin, pp. 31–51.

Poggiolesi, F. (2010) *Gentzen Calculi for Modal Propositional Logic*, Springer, Berlin.

Prawitz, D. (1965) *Natural Deduction. A proof-theoretical study*, Almqvist & Wiksell, Stockholm.

Priest, G. (2009), Beyond the limits of knowledge. In Salerno (ed.), pp. 93–104.

Proietti, C. (2011) Intuitionistic epistemic logic, Kripke models and Fitch's paradox, *Journal of Philosophical Logic*, in press.

Proietti, C. and G. Sandu (2010) Fitch's paradox and *ceteris paribus* modalities. *Synthese*, vol. 173, pp. 75–87.

Rabinowicz, W. and K. Segerberg (1994) Actual truth, possible knowledge. *Topoi*, vol. 13, pp. 101–115.

Read, S. (2008) Harmony and Modality, in C. Dégremont, L. Kieff and H. Rückert (eds), *Dialogues, Logics and Other Strong Things: Essays in Honour of Shahid Rahman*, College Publications, London, pp. 285–303.

Restall, G. (2009) Not every truth can be known (at least, not all at once). In Salerno (ed.), pp. 339–354.

Salerno, J., ed (2009) *New Essays on the Knowability Paradox*. Oxford University Press.

Schwichtenberg, H. (1977) Proof theory: some application of cut-elimination, in J. Barwise (ed), *Handbook of Mathematical logic*, North Holland, Amsterdam, pp. 868–95.

Schroeder-Heister, P. (1984) A natural extension of natural deduction, *The Journal of Symbolic Logic*, vol. 49, pp. 1284–1300.

Schroeder-Heister, P. (2010) Generalized elimination inferences, higher-level rules, and the implications-as-rules interpretation of the sequent calculus, in E. H. Haeusler, L. C. Pereira and V. de Paiva (eds), *Advances in Natural Deduction*.

Simpson, A. (1994) *The Proof Theory and Semantics of Intuitionistic Modal Logic*, Ph.D. thesis, University of Edinburgh.

Stewart, C. and P. Stouppa (2005) A systematic proof theory for several modal logics, in R. Schmidt, I. Pratt-Hartmann, M. Reynolds and H. Wansing (eds.) *Advances in Modal Logic 5*, King's College Publication, London, pp. 309–33.

Takeuti, G. (1987) *Proof Theory*, North Holland, Amsterdam.

Tennant, N. (1997) *The Taming of the True*. Oxford University Press.

Tennant, N. (2009) Revamping the restriction strategy. In Salerno (ed.), pp. 223–238.

Troelstra, A.S. and H. Schwichtenberg (2000) *Basic Proof Theory*, Cambridge University Press.

Viganò, L. (2000) *Labelled Non-Classical Logics*, Kluwer, Dordrecht.

H. Wansing ed. (1996) *Proof Theory of Modal Logic*, Kluwer, Dordrecht.

Wansing, H. (2002) Sequent Systems for Modal Logic, in D. Gabbay and F. Guenther (eds), *Handbook of Philosophical Logic*, Kluwer, Dordrecht, pp.61–145.

Wansing, H. (2002a) Diamonds are philosopher's best friends. *Journal of Philosophical Logic*, vol. 31, pp. 591–612.

Williamson, T. (1982) Intuitionism disproved? *Analysis*, vol. 42, pp. 203–207.

von Wright, G. H. (1951) *An Essay in Modal Logic*, North-Holland, Amsterdam.