EIGENVECTORS OF TENSORS AND ALGORITHMS FOR WARING DECOMPOSITION

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ABSTRACT. A Waring decomposition of a (homogeneous) polynomial f is a minimal sum of powers of linear forms expressing f. Under certain conditions, such a decomposition is unique. We discuss some algorithms to compute the Waring decomposition, which are linked to the equations of certain secant varieties and to eigenvectors of tensors. In particular we explicitly decompose a cubic polynomial in three variables as the sum of five cubes (Sylvester Pentahedral Theorem).

1. INTRODUCTION

In this article we shall be concerned with the following general problem: given a polynomial, what is its minimal decomposition as a sum of powers of linear forms?

Let V be a complex vector space of dimension $n + 1$ and let $S^d V$ denote the space of dth-order symmetric tensors. A choice of basis $\{x_0, \ldots, x_n\}$ for V induces a natural choice of basis for $S^d V$ and allows one to express $f \in S^d V$ as a polynomial in the variables x_i. Because of our interest in tensor decomposition, we often do not make a distinction between a symmetric tensor and a polynomial, and often use the terms interchangeably.

Let $f \in S^d V$. In this paper, our focus is on the symmetric tensor decomposition

$$(1) \qquad f = \sum_{i=1}^{r} c_i (v_i)^d,$$

where $v_i \in V$ have degree 1 and $c_i \in \mathbb{C}$. For historical reasons, we will refer to this as a *Waring decomposition* or simply a *decomposition* of f. The minimum number of summands occurring in such a decomposition is called the *(symmetric) rank* of f.

Remark 1.1. Note that when working over $\mathbb{C}$, the constants c_i in (1) may be assumed to be equal to 1, however this ambiguity will become important in our algorithms as initially we will only find the classes of the linear forms $[v_i] \in \mathbb{P}V$ (the projective space of lines in V) and we will have to solve an easy linear

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system to find appropriate c_i to resolve this indeterminacy in order to complete our decomposition algorithms.

Because of the vast amount of work done in this area by several authors in diverse areas of science, the Waring decomposition is also known by many other names such as canonical decomposition (CANDECOMP/PARAFAC (CP)), rank-1 decomposition, sum of powers decomposition, and so on.

We start with an example that illustrates the main results of this paper, which are Algorithm 4 and Thm. 4.7. A classical result, attributed to Hilbert, Richmond and Palatini, states that a general [1] form $f \in S^5\mathbb{C}^3$ can be decomposed in unique way as the sum of seven fifth powers. For reference, see [RS00] or [CS11b] and the references therein.

Set for the moment $V = \mathbb{C}^3$. In the formula (1) with $r = 7, d = 5$, we want to find an efficient algorithm to find the $v_i \in \mathbb{C}^3$ and $c_i \in \mathbb{C}$ which give the decomposition of f. To f we associate a linear map

$$P_f \colon \mathrm{Hom}(S^2V, V) \to \mathrm{Hom}(V, S^2V),$$

where $\mathrm{Hom}(S^2V, V)$ is the space of linear maps from S^2V to V. Note that the target space is the dual of the source one. If $f = v^5$ then the definition of P_{v^5} is the following

$$(2) \qquad P_{v^5}(M)(w) = \big(M(v^2) \wedge v \wedge w\big)(v^2),$$

where $M \in \mathrm{Hom}(S^2V, V)$ and $w \in V$. In general, if f is any element of S^5V, the definition of P_f is extended by linearity (which can always be done because S^dV has a basis consisting of powers of linear forms).

To see the map P_f explicitly as a matrix, consider the monomial basis $\{x_ix_j\}_{0 \le i \le j \le 2}$ of S^2V. It turns out that with an appropriate ordering of the basis, P_f is represented by the following 18×18 block matrix, where the nine depicted blocks have size 6×6

$$(3) \qquad \begin{bmatrix} 0 & C_{f_2} & -C_{f_1} \\ -C_{f_2} & 0 & C_{f_0} \\ C_{f_1} & -C_{f_0} & 0 \end{bmatrix},$$

and C_{f_i} is the matrix whose entries labeled by $((j,k),(p,q))$ (for $0 \le j \le k \le 2$, $0 \le p \le q \le 2$) are given by the fifth derivative f_{ikjpq}.

If f has rank 1, we may choose coordinates such that $f = x_0^5$. In this case rank $C_{f_0} = \delta_{0i}$ and the matrix $P_{x_0^5}$ has rank 2. By the linearity of P_f in the argument f and the sub-additivity of matrix rank, if f has rank r, P_f must have rank $\le 2r$. In fact, we will see that in addition to giving this bound on rank, P_f will also be the key to an efficient way to decompose f.

Now we have the following result (see Lemma 3.3): $P_{v^5}(M) = 0$ if and only if there exists λ such that $M(v^2) = \lambda v$. Such v is called an *eigenvector of the tensor M* (or simply *eigenvector* of M, when the context is clear), where the homomorphism $M \colon S^2V \to V$ is viewed as a tensor in $S^2V^* \otimes V$.

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[1]see Remark 2.1 for a discussion of the term general

Eigenvectors of tensors were introduced and studied in [Lim05, Qi05]. Recently, Cartwright and Sturmfels [CS11a] have found a formula computing the number of eigenvectors of a tensor $M \in \mathrm{Hom}(S^m\mathbb{C}^n, \mathbb{C}^n)$. (In Section 6 we review their formula and propose a geometric interpretation of this formula that leads to an alternative proof which generalizes to other cases.) The Cartwright-Sturmfels formula says that there are exactly *seven* eigenvectors $\{v_1, \ldots, v_7\}$ of a general tensor in $\mathrm{Hom}(S^2\mathbb{C}^3, \mathbb{C}^3)$. Later we will see that general elements of the kernel $\ker P_f$ share the same eigenvectors. *These seven eigenvectors appear exactly in the decomposition (1).* This is the basic novelty of this paper with respect to [LO11], where the same example was considered. Our methods also provide a solution to the pentahedral example (see Section 3.2), which was left open in [LO11].

In our case, choose a basis $\{y_0, y_1, y_2\}$ of V and its dual basis $\{x_0, x_1, x_2\}$ of V^*. Any $M \in S^2V^* \otimes V$ can be written as $\sum_{i=0}^{2} y_i q_i(x)$ where q_i is a quadratic polynomial. Recall that v is an eigenvector of $M \in S^2V^* \otimes V$ if $M(v^2) \wedge v = 0$, so the coordinates of the seven eigenvectors can be found by the vanishing of the 2×2 minors of the matrix

$$(4) \qquad \begin{bmatrix} x_0 & x_1 & x_2 \\ q_0 & q_1 & q_2 \end{bmatrix}.$$

Now we summarize the steps to compute the decomposition of general plane quintics, $f \in S^5\mathbb{C}^3$:

Algorithm 1.
   Input: $f \in S^5\mathbb{C}^3$.
   (1) Construct the matrix $P_f \colon \mathrm{Hom}(S^2\mathbb{C}^3, \mathbb{C}^3) \to \mathrm{Hom}(\mathbb{C}^3, S^2\mathbb{C}^3)$ as in (3).
   (2) Compute $\ker P_f$. Choose a general $M \in \ker P_f$ and write $M = \sum_{i=0}^{2} y_i q_i(x)$ as above.
   (3) Find eigenvectors $\{v_1, \ldots, v_7\} \in \mathbb{C}^3$ of M via the zero-set of the 2×2 minors of (4).
   (4) Solve the linear system $f = \sum_{i=1}^{7} c_i v_i^5$ in the unknowns $c_i \in \mathbb{C}$.
   Output: The unique Waring decomposition of f.

In this article, our approach to Waring decomposition uses algebraic geometry, starting with the classical Sylvester algorithm and the notion of eigenvectors of tensors. With the aid of recent progress [LO11], on equations of secant varieties using vector bundle techniques, we are able to go further. In fact, this paper can be seen as a constructive version for the symmetric case, of the techniques developed in [LO11]. The main result of this paper is a new algorithm for efficient Waring decomposition of symmetric tensors, stated in general in Algorithm 4. This algorithm is a consequence of the geometric facts contained inProp. 4.1 and Thm. 4.7.

Of course our algorithm will not always succeed to produce the Waring decomposition of symmetric tensors unless the rank is sufficiently small and the tensor is general. On the other hand, we can give precise bounds for the maximum rank

of a symmetric tensor which can be decomposed via our algorithm. In Thm. 2.4, we give an improvement to the Iarrobino-Kanev bound on the applicability of the catalecticant method for Waring decomposition. Going further, we state in Theorems 3.5 and 5.4 sufficient conditions for the success of our algorithm in the cases of symmetric tensors on 3 or more variables, respectively. We give a further discussion of what happens in the case that our algorithm fails in Remarks 4.6, 4.8. In particular there is at least one case where our algorithm fails to decompose a tensor, but, as a side effect, brings to light a new way to express a rational quartic curve through 7 (given) general points, see Remark 5.5.

In addition to finding a new algorithm for Waring decomposition, and bounds for its success, we find a new proof of a result of Cartwright-Sturmfels (Proposition 6.1) on the number of generalized eigenvectors that uses Chern classes and generalizes to other types of generalized eigenvectors (seeProp. 6.2), that we call simply eigenvectors of tensors.

The use of tensors is widespread throughout science and appears in areas such as Algebraic Statistics, Chemistry, Computer Science, Electrical Engineering, Neuroscience, Physics and Psychometrics. A common theme at the 2010 conference on Tensor Decomposition and Applications in Monopoli, Italy was the need for efficient and reliable algorithms to perform tensor decomposition. Indeed, we are certainly not the first to consider this problem. For a sample of related recent progress on tensor decomposition, see [BCMT10, CM96, BB11, BGI11, BB10]. Our aim is to use algebraic geometry as a basis for algorithms that can be used (either in place of or in combination with the previous algorithms) to improve efficiency and robustness. We make some comparison with our methods and those of [BCMT10] in Remark 3.10.

Kolda and Mayo [KM10] have recently studied an efficient way of computing eigenvectors of tensors, analogous to the usual iterative procedure to compute usual eigenvectors of matrices. Further work has been done by Ballard, Kolda and Plantenga [BKP11] to implement this method and they have achieved significant speed-ups utilizing a GPU when eigenvectors of many small tensors are to be computed. Since our tensor decomposition methods use eigenvectors of tensors, this indicates that these methods could be combined with our algorithms to improve efficiency.

Another aspect of using tensor eigenvectors in our algorithms is that it may be reasonable to try (in the tensor setting) to mimic a method to approximate a matrix by one of lower rank via eliminating the eigenspaces corresponding to small eigenvalues. We hope that this article can serve as a starting point for such a study. For more issues regarding low-rank approximation of tensors, and the well-posedness of this problem, see [dSL08].

We have structured this article for two diverse audiences; algebraic geometers and researchers from a variety of applied fields studying tensors. With algebraic geometers in mind, our goal is to show how well-known techniques in algebraic geometry can be used to solve problems in applied areas. Most of our constructions

are better explained with the geometric language of *vector bundles*. However, we did our best to first state the main results by using explicit matrices, without the language of vector bundles. For this reason, we defer the main proofs until Section 4, in the sense that almost all our results are particular cases of a general result (Thm. 4.7), which is stated with the language of vector bundles. Keeping in mind researchers studying applied tensor problems, in Section 2 we describe the history and state of the art of algebraic geometry concerning tensor decomposition from a practical point of view. Because in their original versions the statements may not have been so accessible for applications, we have restated results from algebraic geometry, hopefully in a more transparent language, concerning generic rank (see Thm. 2.2) and uniqueness of tensor decomposition (see Thm. 2.3). In Section 3, we go on to illustrate our new techniques that use Koszul matrices to compute Waring decomposition. As mentioned above, in Section 4 we give the generalization of our techniques. Finally, in Section 7 we describe our Macaulay2 implementation of our algorithms, and we hope that this will serve as a starting point for further implementations of these methods.

2. Classical methods for tensor decomposition: Sylvester's catalecticant method

2.1. **General results on the symmetric rank.**

Remark 2.1. We use the term "general element" in the sense of algebraic geometry to indicate that the element is chosen to avoid a (Zariski) closed set. So when we say that the general element in a variety X has a property, this means that X contains a dense subset X^0 such that every element in X^0 satisfies that property. We call a property "generic" if it holds for general elements. We consider the "general" assumption a mild assumption because in practice, most tensors we encounter in nature will actually be general. It is no loss to replace "generic" with "almost always" and "general" with "randomly chosen," or "up to certain non-degeneracy conditions."

Regarding ranks of tensors, the following capstone theorem answers the question completely for general symmetric tensors.

Theorem 2.2 ([AH95]). *Let V be a complex vector space of dimension $n + 1$. The general $f \in S^d V$ has rank*

$$\left\lceil \frac{\binom{n+d}{d}}{n + 1} \right\rceil,$$

which is called the generic rank, with the only exceptions

- *$d = 2$, where the generic rank is $n + 1$.*
- *$2 \leq n \leq 4$, $d = 4$, where the generic rank is $\binom{n+2}{2}$.*
- *$(n, d) = (4, 3)$, where the generic rank is 8.*

Note that we use the functions $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ respectively to indicate round-up and round-down. The elements of rank one in $S^d V$ are just the polynomials that are d-th powers of a linear polynomial. They form an irreducible algebraic variety, which is the cone over a projective variety which is called the d-Veronese variety of $\mathbb{P}^n$ and we denote $v_d(\mathbb{P}^n)$. For all values of k less than the generic rank, the (Zariski) closure of the elements of rank $\leq k$ is a irreducible variety, which is the cone over $\sigma_k(v_d(\mathbb{P}^n))$, the latter is called the k-th secant variety of $v_d(\mathbb{P}^n)$. A consequence of the Alexander-Hirschowitz theorem is that if $k \leq \left\lfloor \frac{\binom{n+d}{d}}{n+1} \right\rfloor$, then $\dim \sigma_k(v_d(\mathbb{P}^n)) = k(n+1) - 1$ (the expected dimension) with the only exceptions

- $d = 2$, $2 \leq k \leq n$
- $2 \leq n \leq 4$, $d = 4$, $k = \binom{n+2}{2} - 1$
- $(n, d) = (4, 3)$, $k = 7$.

The cases listed above are called the *defective cases*.

After the rank of a tensor is known, the next natural question is whether there is a unique decomposition (ignoring trivialities). The following represents the state of the art regarding this question.

Theorem 2.3 ([CC02], [Mel06], [Bal05])**.** *For all values of r smaller than the generic rank, the general element of rank r in $S^d V$ has a unique (up to scaling) decomposition $f = \sum_{i=1}^{r} c_i(v_i)^d$ with the only exceptions*

(1) the defective cases, where there are infinitely many decompositions
(2) rank 9 in $S^6 \mathbb{C}^3$, where there are exactly two decompositions
(3) rank 8 in $S^4 \mathbb{C}^4$, where there are exactly two decompositions.

The cases listed as (2) and (3) in the above theorem are called the *weakly defective cases*. For the generic rank, it is known that when $n+1$ does not divide $\binom{n+d}{d}$ then there are infinitely many decompositions. On the other hand, when $n+1$ divides $\binom{n+d}{d}$, apart from the defective cases, then there are finitely many decompositions. In the latter case, it is expected that the decomposition of a general element in $S^d V$ is rarely unique. In this situation, the only cases where uniqueness is known to hold are the following:

- $S^d \mathbb{C}^2$, for odd d, which was addressed by Sylvester in 1851.
- $S^5 \mathbb{C}^3$, which is the case addressed in the introduction.
- $S^3 \mathbb{C}^4$, the uniqueness result for the general rank, which is 5, is known as the *Sylvester Pentahedral Theorem*.

It is expected that these three are the only cases where uniqueness of decompositions hold for the general element. Partial results confirming this expectation are proved in [Mel09].

A consequence of our construction is that we can treat all three of these cases in a unified manner. The fact that our construction only finds these three cases gives further evidence that these may be the only exceptional cases for uniqueness.

Despite this beautiful theoretical picture, it is hard to compute the rank of a given symmetric tensor and to find explicitly its tensor decomposition. The

brute force attempt to solve (1) in the unknowns defining each v_i with a computer algebra system is time and memory consuming and often fails, even in small dimension. In fact, it is known that most tensor problems are extremely hard and often unsolvable [HL09]. So while our goal is to make improvements in efficiency and reliability of tensor decomposition algorithms, we know that the generic problem for large tensors will remain difficult.

2.2. **The catalecticant method.** The catalecticant method was developed in the XIX century, by Sylvester and others, to compute the rank of a symmetric tensor in $S^d\mathbb{C}^{n+1}$ and to compute its Waring decomposition. The method is completely successful in the case of binary forms, *i.e.* $n = 1$, but gains only partial success for $n \geq 2$. It is important to understand it deeply, because most of successive methods proposed, including our method developed in this paper, can be considered as a generalization of the catalecticant method.

Let $\{x_i\}_{1 \leq i \leq n+1}$ be a basis of a vector space V. Given $f \in S^dV$ one can define maps for each $m < d$

$$
(5) \qquad
\begin{aligned}
C_f^m : \qquad S^mV^* &\longrightarrow S^{d-m}V \\
x_{i_1} \cdots x_{i_m} &\mapsto \frac{\partial^m f}{\partial x_{i_1} \cdots \partial x_{i_m}}.
\end{aligned}
$$

Seen as a matrix, C_f^m is known as a catalecticant or (semi-)Hankel matrix. From (the more recent) point of view of tensors, C_f^m can be thought of as a *symmetric flattening* of a symmetric tensor, which is a symmetric version of a flattening of a tensor. For this and other types of flattenings see [LO11].

Note that if $f = l^d$ then rank $C_f^m = 1$, hence if f has rank r then rank $C_f^m \leq r$. This gives lower bounds on the rank of f. But in fact, in this case, when C_f^m is non-trivial we will be able to further exploit this construction to aid in decomposing f.

In their book [IK99], Iarrobino and Kanev described the key ingredients to the classical catalecticant method. In particular they describe an algorithm to find a Waring decomposition. We have implemented this approach, see Section 7.1. Here is a summary of the algorithm.

Algorithm 2 (Catalecticant Algorithm). [IK99, 5.4]
    Input: $f \in S^dV$, where $\dim V = n + 1$.

(1) Construct, via (5), the most square possible catalecticant $C_f^m = C_f$ with $m = \lceil \frac{d}{2} \rceil$.
(2) Compute ker C_f. Note that $\mathrm{rank}(f) \geq \mathrm{rank}(C_f)$.
(3) Find the zero-set Z' of the polynomials in ker C_f.
    (a) If Z' is not given by finitely many reduced points, stop; this method fails.
    (b) Else continue with $Z' = \{[v_1], \ldots, [v_s]\}$.
(4) Solve the linear system defined by $f = \sum_{i=1}^{s} c_i v_i^d$ in the unknowns c_i.

    Output: The unique Waring decomposition of f.

Algorithm 2 is a special case of Algorithm 4 below, where we take $E = \mathcal{O}(m)$ and $L = \mathcal{O}(d)$. The following theorem gives a sufficient condition that guarantee its success. It can be seen as a slight improvement (at least for $n \geq 3$) of the bound described in [IK99], see Theorems 4.10A and 4.10B.

Theorem 2.4. *Suppose $f = \sum_{i=1}^{r} v_i^d$ is a general form of rank r in $S^d V$, let $z_i = [v_i] \in \mathbb{P}(V)$ be the corresponding points and let $Z = \{z_1, \ldots, z_r\}$. Set $m = \lceil \frac{d}{2} \rceil$.*

(1) If d is even and $r \leq \binom{n+m}{n} - n - 1$ or if d is odd and $r \leq \binom{n+m-1}{n}$, then

(6) $$\ker C_f = I_{Z,m},$$

> *where $I_{Z,m} \subset S^m V^*$ denotes the subspace of polynomials of degree m vanishing on $Z \subset V$. Moreover Algorithm 2 produces the unique Waring decomposition of f.*
> *(2) Finally if d is even, and $r = \binom{n+m}{n} - n$, then it is possible that $Z \subsetneq Z'$, where Z' is obtained by Algorithm 2. But still when $n = 2$, the algorithm will produce the unique minimal Waring decomposition. Further when $n \geq 3$, the algorithm will succeed after repeating step (4) finitely many times using subsets $Z'' \subset Z'$ of size $\mathrm{rank}(C_f)$.*

As mentioned in the introduction, many of our statements in this section are consequences of more general results that are proved later in Section 4. While the reader may better understand the following arguments after the methods in Section 4 are presented, we believe that it is worthwhile to anticipate their use now so that the simple pattern in this case may be seen.

Proof. First notice that by Thm. 2.3, we know that in this case we have a unique decomposition.

Here we use the more general theory applied to this specific case. Now we prove (1). ByProp. 4.1, with $E = \mathcal{O}(m)$ and $L = \mathcal{O}(d)$, we have the inclusion $I_{Z,m} \subseteq \ker C_f$, and by the same Proposition, the equality holds if we can show that the map $H^0(E^* \otimes L) \longrightarrow H^0(E \otimes L_{|Z})$ is surjective. For our particular choice of E and L, we are considering the map $H^0(\mathcal{O}(-m+d)) \longrightarrow H^0(\mathcal{O}(-m+d)_{|Z})$. Let $m' = d - m = \lfloor \frac{d}{2} \rfloor$. So this amounts to showing that the natural evaluation map

$$S^{m'} V^* \longrightarrow \mathbb{C}^r$$
$$f \longmapsto (f(v_1), \ldots, f(v_r))$$

is surjective. But this is equivalent to the condition that $\mathrm{codim}(I_{Z,m'}) = r$. The preceding condition is satisfied because (by a basic dimension counting argument) r general points give independent conditions on hypersurfaces of degree m' when r is bounded by $\binom{n+m'}{m'}$. It follows that $\mathrm{rank}\, C_f = r$ and we can apply Prop. 4.3 and conclude that $I_{Z,m'} = \ker C_f$.

Next we will use the following theorem.

Theorem 2.5. [CC02, Theorem 2.6], *Assume that* $r \leq \binom{n+m}{m} - n - 1$. *Let X be an irreducible projective variety. For $r \geq 3$, if every $(r-2)$-plane spanned by general points $x_1, \ldots, x_{r-1}$ also meets X in an r-th point x_r different from $x_1, \ldots, x_{r-1}$, then X is contained in a linear subspace L, with $\mathrm{codim}_L(X) \leq r - 2$.*

Let $Z' = \mathrm{baseloc}(\ker C_f)$. We claim that $Z' = Z$, and the equality $I_{Z,m} = \ker C_f$ implies that the points Z' can be used to give the decomposition of f. Indeed, we just showed that $Z' = \mathrm{baseloc}(I_{Z,m})$, and the latter is equal to Z by applying Theorem 2.6 in [CC02], stated above.

In the odd case, note that we will have $C_f \colon S^m V \longrightarrow S^{m-1}V$, and we always have $\binom{n+m-1}{n} \leq \binom{n+m}{n} - n - 1$.

For the proof of (2), now Z' will be given by m^n points, complete intersection of n hypersurfaces of degree m. Note that for $n \geq 2$ we have $\binom{n+m}{m} - n \leq m^n$. For $n = 2$, the m^2 points impose independent conditions on the hypersurfaces (curves) of degree $2m$. Indeed, from the sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(m)^2 \longrightarrow \mathcal{I}_{Z'}(2m) \longrightarrow 0,$$

one shows that $h^1(\mathcal{I}_{Z'}(2m)) = 0$. Denote $Z' = \{v_1, \ldots, v_{m^2}\}$. The last vanishing implies that the powers $(v_i)^{2m}$ for $i = 1, \ldots, m^2$ are linearly independent. Then the linear system in step (4) in the algorithm has a unique solution , which moreover only uses $\mathrm{rank}(C_f)$ of the points.

For $n \geq 3$, since the v_i^d are no longer independent, the linear system in step (4) of the algorithm will no longer have finitely many solutions using all the forms. However we can overcome this problem by repeating step (4) with each subset of Z' of size $\mathrm{rank}(C_f)$ until we find the decomposition.

For the rest of the theorem, apply Algorithm 2 and refer to Thm. 4.7. $\square$

2.3. **Limits of the catalecticant method.** As in the proof of Thm. 2.4, let $m' = \lfloor \frac{d}{2} \rfloor$. Another way to see that the catalecticant method can work for polynomials of rank r only if $r < \binom{n+m'}{m'}$, is the following. The maximum rank of any catalecticant matrix $C_f^m \colon S^m V^* \longrightarrow S^{m'}V$ is $\binom{n+m'}{m'}$. If f has this rank or greater, we will either (in the even case) have no kernel to work with, or (in the odd case) fail to satisfy the equality $I_{Z,m'} = \ker C_f^m$.

Usually the general rank $\frac{\binom{n+d}{d}}{n+1}$ is larger than $\binom{n+m'}{m'}$, the first example being $v_3(\mathbb{P}^2)$, where the catalecticant matrices have size 3×6 or 6×3. The equation of $\sigma_3(v_3(\mathbb{P}^2))$, which is called the Aronhold invariant, cannot be found as a minor of any catalecticant matrix. It can be obtained as a Pfaffian of a Koszul flattening (see [Ott09]). In practice, one should start by applying the catalecticant method, and if it fails to produce the Waring decomposition, this implies that the rank is larger than the bounds listed in Thm. 2.4. Next we introduce new algorithms that will succeed to decompose tensors in a larger range of ranks.

3. New methods for tensor decomposition: Koszul flattening and eigenvectors of tensors

Note that the following definition also appeared in [Ott09] in a few special cases and was further developed in [LO11], both with a focus of finding equations of secant varieties. We also note that this construction was used in the case of partially symmetric tensors to find the ideal-theoretic defining equations of the k-th secant variety (with $k \leq 5$) of $Seg(\mathbb{P}^2 \times \mathbb{P}^n)$ embedded by $\mathcal{O}(1,2)$ in [CEO11]. Here our presentation is focused on using this construction to find decompositions of tensors via eigenvectors.

The general setting of this section is the following. Let $f \in S^d V$ and fix $0 \leq a \leq n$, $1 \leq m \leq d-1$. We construct a linear map

$$(7) \qquad P_f \colon \operatorname{Hom}(S^m V, \textstyle\bigwedge^a V) \to \operatorname{Hom}(\textstyle\bigwedge^{n-a} V, S^{d-m-1} V).$$

If $f = v^d$ then the definition of P_{v^d} is the following

$$(8) \qquad P_{v^d}(M)(w) = (M(v^m) \wedge v \wedge w) (v^{d-m-1}),$$

where $M \in \operatorname{Hom}(S^m V, \textstyle\bigwedge^a V)$, $w \in \textstyle\bigwedge^{n-a} V$ and we fixed an isomorphism $\textstyle\bigwedge^{n+1} V \simeq \mathbb{C}$. If f any element of $S^d V$, the definition of P_f is extended by linearity. This extension is guaranteed by the fact that on general decomposable tensors $f = v_1 \otimes \ldots \otimes v_d$ the map P_f has the following expression

$$P_{v_1 \otimes \ldots \otimes v_d}(M)(w) = \sum_\sigma \left(M(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(m)}) \wedge v_{\sigma(m+1)} \wedge w \right) (v_{\sigma(m+2)} \otimes \ldots \otimes v_{\sigma(d)}),$$

which is visibly linear, where the summation is performed for σ in the symmetric group of permutations on d elements.

Although this definition might seem artificial at first glance, we now explain how it can be used. We wait until Section 4.2 for a more formal treatment of P_f via a presentation of a vector bundle. The linear map P_f can be explicitly computed by using Koszul matrices, which motivates the name *Koszul flattening* that we give to P_f and is intended to mirror the term symmetric flattening which it generalizes.

In order to explicitly write down the matrix representing P_f, we need to recall the properties of the *Koszul complex*. It is the minimal resolution of the field $\mathbb{C}$ as an $R = \mathbb{C}[x_0, \ldots, x_n]$-module. Here we give some examples, but interested readers unfamiliar with the Koszul complex may wish to consult [Eis05]. For $n = 2$ the Koszul complex is

$$0 \longrightarrow R(-3) \xrightarrow{k_3} R(-2)^3 \xrightarrow{k_2} R(-1)^3 \xrightarrow{k_1} R \longrightarrow \mathbb{C} \longrightarrow 0,$$

where

$$k_1 = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}$$

$$k_2 = \begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & x_0 & 0 \end{pmatrix}.$$

For $n = 3$ it is

$$0 \longrightarrow R(-4) \xrightarrow{k_4} R(-3)^4 \xrightarrow{k_3} R(-2)^6 \xrightarrow{k_2} R(-1)^4 \xrightarrow{k_1} R \longrightarrow \mathbb{C} \longrightarrow 0,$$

where

$$k_1 = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix}$$

$$k_2 = \begin{pmatrix} -x_1 & -x_2 & 0 & -x_3 & 0 & 0 \\ x_0 & 0 & -x_2 & 0 & -x_3 & 0 \\ 0 & x_0 & x_1 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}.$$

In general we have

$$k_i \colon R(-i)^{\binom{n+1}{i}} \longrightarrow R(-i+1)^{\binom{n+1}{i-1}}.$$

The matrix k_i is a Koszul matrix. It corresponds to the presentation of the vector bundle $\bigwedge^{n+1-i} Q(-i)$ and on the point $\langle v \rangle \in \mathbb{P}V$ it corresponds to the wedge product $\bigwedge^{n+1-i} V \xrightarrow{\wedge v} \bigwedge^{n+2-i} V$. We note that the Koszul complexes can be easily computed by any standard computational algebra system such as Macaulay2 [GS10].

Next we recall a result from [LO11] that gives an explicit version of this construction.

Lemma 3.1. *Let $f \in S^d V$. The matrix $P_f \colon \mathrm{Hom}(S^m V, \bigwedge^a V) \to \mathrm{Hom}(\bigwedge^{n-a} V, S^{d-m-1} V)$ can be computed using the matrix k_{n+1-a} of the Koszul complex, of size $\binom{n+1}{a} \times \binom{n+1}{a+1}$, where at the place of the indeterminate x_i we substitute the catalecticant matrix $C_{f_i}^m$ of size $\binom{n+d-m-1}{n} \times \binom{n+m}{n}$, where $f_i = \frac{\partial f}{\partial x_i}$. The matrix P_f obtained has size*

$$\left[\binom{n+m}{n} \binom{n+1}{a} \right] \times \left[\binom{n+d-m-1}{n} \binom{n+1}{a+1} \right].$$

Proof. See [LO11, Section 8.3]. $\square$

Now we propose the following

Definition 3.2. *Given $M \in \mathrm{Hom}(S^m V, \bigwedge^a V)$, a vector $v \in V$ is called an* eigenvector *of the tensor M if*

$$M(v^m) \wedge v = 0.$$

For $m = a = 1$ this is a usual eigenvector, and for $a = 1$, any m this agrees with the notion of [Lim05, Qi05].

Now we have the following important lemma (whose proof is straightforward) that we would like to emphasize.

Lemma 3.3. *Let $M \in \mathrm{Hom}(S^m V, \bigwedge^a V)$.*

(1) *A vector $v \in V$ is an eigenvector of M if and only if $M \in \ker(P_{v^d})$.*
(2) *Let $f = \sum v_i^d$. If each v_i is an eigenvector of M, then $M \in \ker P_f$.*

Cartwright and Sturmfels have recently found a formula computing the number of eigenvectors of a general tensor $M \in \mathrm{Hom}(S^m V, V)$, which is the case $a = 1$ in our construction. The following theorem generalizes the Cartwright-Sturmfels formula to any a.

Theorem 3.4. *For a general $M \in \mathrm{Hom}(S^m V, \bigwedge^a V)$, the number of $[v] \in \mathbb{P}V$ such that $M(v^m) \wedge v = 0$ is given by*

- $\begin{cases} m \text{ for } a = 0, 2 \text{ and } n = 1, \\ \infty \text{ when } a = 0, n+1 \text{ and } n > 1. \end{cases}$
- $\frac{m^{n+1}-1}{m-1}$ *for $a = 1$*
- 0 *for $2 \le a \le n-2$*
- $\frac{(m+1)^{n+1}+(-1)^n}{m+2}$ *for $a = n-1$.*

Proof. The non-trivial cases follow from Propositions 6.1 and 6.2. □

For $a = 0$ the condition becomes $M(v_i^m) = 0$ and P_f reduces to the catalecticant. The cases $a = 1$ and $a = n-1$ are a bit special. Note that Hilbert's quintic example from the introduction fits the case $a = 1$ (and agrees with the case $a = n-1$ since $n = 2$), while we will see that Sylvester's pentahedral example fits in the case $a = n-1$.

As we mentioned in the introduction, in the case $a = 1$, the iterative methods of [KM10] may be used to find eigenvectors of tensors in $\mathrm{Hom}(S^m V, V)$, however for general a, more work needs to be done in order to efficiently find eigenvectors of tensors in $\mathrm{Hom}(S^m V, \bigwedge^a V)$. The idea, like in the matrix case, is to iterate the map

$$v_{k+1} = \frac{M(v_k^m)}{\|M(v_k^m)\|},$$

and to successively approximate the eigenvectors, starting from the dominant one and repeating until all eigenvectors are found.

Next we describe a general algorithm to decompose polynomials, which corresponds to the case $a = 1$, any m. This algorithm is especially effective in the case $n = 2$.

Algorithm 3 (Koszul Flattening Algorithm 1).
  Input: $f \in S^d V$, where $\dim V = n + 1$.
  (1) Construct, via (7) and (8), $P_f \colon \mathrm{Hom}(S^m V, V) \to \mathrm{Hom}(\bigwedge^{n-1} V, S^{d-m-1} V)$.
  (2) Compute $\ker P_f$ and note that $\mathrm{rank}(f) \ge \frac{\mathrm{rank}(P_f)}{n}$.
  (3) Find Z', the common (projective) eigenvectors of a basis of the kernel of P_f.
      (a) If Z' is not given by finitely many reduced points, stop; this method fails.
      (b) Else continue with $Z' = \{[v_1], \ldots, [v_s]\}$.
  (4) Solve the linear system on the constants c_i defined by setting $f = \sum_{i=1}^{s} c_i v_i^d$.
  Output: The unique Waring decomposition of f.

Now we state sufficient conditions in the case $n = 2$ and d is odd for Algorithm 3 to succeed. (In the case d is even we don't state conditions because our experiments show that the Catalecticant Algorithm covers all cases that the Koszul algorithm can cover).

Theorem 3.5. *Suppose $n = 2$ and set $d = 2m + 1$. Let $f = \sum_{i=1}^{r} v_i^d$ be a general form of rank r in $S^d V$, let $z_i = [v_i] \in \mathbb{P}(V)$ be the corresponding points and let $Z = \{z_1, \ldots, z_r\}$. Let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$.*

(1) If $2r \leq m^2 + 3m + 4$ then $Z' = Z$. Moreover Algorithm 3 produces the unique Waring decomposition of f.
(2) If $2r \leq m^2 + 4m + 2$, then it is possible that $Z \subsetneq Z'$. Even in this case, Algorithm 3 will produce the unique minimal Waring decomposition.

We postpone the proof of Thm. 3.5 until we have the tools from the next section.

Before going on, we note that the map P_f factors, and therefore it will have smaller rank than what may be first expected. In order to accurately explain this, we will need to use representation theory and the language of partitions and Young diagrams according to [FH91].

The map P_f always has a non-trivial kernel, which comes from an analogy to the matrix case and the fact that every vector is an eigenvector of a scalar multiple of the identity. First we notice that $\mathrm{Hom}(S^m V, \bigwedge^a V)$ splits as the direct sum of two $SL(V)$-modules, via the Pieri rule, (see [FH91, equation (6.9), p.79])

$$(9) \qquad \mathrm{Hom}(S^m V, \textstyle\bigwedge^a V) = \Gamma^{m^n} \otimes \textstyle\bigwedge^a V = \Gamma^{(m+1)^a, m^{n-a}} \oplus \Gamma^{(m)^{a-1}, (m-1)^{n-a+1}},$$

where in general for any partition π, Γ^π is the $SL(V)$-representation associated to π. In partitions, we use the notation m^i to denote m repeated i times. Note that Γ^π inherently depends on V, but we suppress this from the notation for simplicity.

Lemma 3.6. P_f *restricted to* $\Gamma^{(m)^{a-1}, (m-1)^{n-a+1}}$ *is zero.*

Proof. By the linearity of P_f in f, it suffices to prove the lemma for $f = v^d$. The essential fact that we use is that the representation $\Gamma^{m^{a-1}, (m-1)^{n-a+1}}$ is isomorphic to a subspace of $\mathrm{Hom}(S^{m-1} V, \bigwedge^{a-1} V)$, which indeed splits as

$$\mathrm{Hom}(S^{m-1} V, \textstyle\bigwedge^{a-1} V) = \Gamma^{m^{a-1}, (m-1)^{n-a+1}} \oplus \Gamma^{(m-1)^{a-2}, (m-2)^{n-a+2}}.$$

So consider $M \in \Gamma^{m^{a-1}, (m-1)^{n-a+1}} \subset \mathrm{Hom}(S^m V, \bigwedge^a V)$. There is a natural equivariant map

$$\begin{array}{ccc} \mathrm{Hom}(S^{m-1} V, \textstyle\bigwedge^{a-1} V) & \to & \mathrm{Hom}(S^m V, \textstyle\bigwedge^a V) \\ N & \mapsto & [v^m \mapsto N(v^{m-1}) \wedge v] \end{array}$$

which is nonzero, so by Schur lemma it identifies the summands $\Gamma^{m^{a-1}, (m-1)^{n-a+1}}$ in both sides. We write $\tilde{M}$ as the copy of M in $\mathrm{Hom}(S^{m-1} V, \bigwedge^{a-1} V)$ according to this identification.

Then $M(v^m) = \tilde{M}(v^{m-1}) \wedge v$, so for this M,

$$P_{v^d}(M)(w) = (M(v^m) \wedge v \wedge w)\,(v^{d-m-1}) = \left(\tilde{M}(v^{m-1}) \wedge v \wedge v \wedge w\right)(v^{d-m-1}) = 0.$$

$\square$

3.1. **The quotient bundle and eigenvectors of tensors.** We show in this section that the eigenvectors of a tensor can be interpreted as zero loci of sections of twists of the quotient bundle. Similar methods were recently used in [OS10] to study matrices with eigenvectors in a given subspace. For generalities about vector bundles we refer to [OSS80]. $\mathbb{P}V$ is the projective space of lines in V, therefore $H^0(\mathbb{P}V, \mathcal{O}(1)) = V^*$. The quotient bundle of $\mathbb{P}V$ which we will denote by Q, appears in the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}V}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}V} \otimes V \longrightarrow Q \longrightarrow 0,$$

taking wedge powers and tensoring by $\mathcal{O}(m)$ we get the sequence

$$\textstyle\bigwedge^{a-1}V \otimes \mathcal{O}(m-1) \longrightarrow \bigwedge^a V \otimes \mathcal{O}(m) \longrightarrow \bigwedge^a Q(m) \longrightarrow 0.$$

Taking the global sections we get

(10) $$\mathrm{Hom}(S^{m-1}V, \textstyle\bigwedge^{a-1}V) \longrightarrow \mathrm{Hom}(S^m V, \bigwedge^a V) \xrightarrow{\phi} H^0\left(\bigwedge^a Q(m)\right) \longrightarrow 0.$$

For any tensor $M \in \mathrm{Hom}(S^m V, \bigwedge^a V)$ we denote by s_M the section of $\bigwedge^a Q(m)$ corresponding to $\phi(M)$ in sequence (10). We want to show that the eigenvectors of the tensor M correspond to the zero locus of s_M. To make this construction precise, we recall the following straightforward lemma.

Lemma 3.7.

(1) *The fiber of $\bigwedge^a Q(m)$ at $x = \langle v \rangle$ is isomorphic to $\mathrm{Hom}(\langle v^m \rangle, \bigwedge^a V/\langle v \wedge \bigwedge^{a-1}V \rangle)$.*

(2) *The section s_M vanishes in $\langle v \rangle$ if and only if v is an eigenvector of the tensor M.*

Proof. The section $s_M \in H^0(\bigwedge^a Q(m))$ corresponds on the fiber of v to the composition $\langle v^m \rangle \xrightarrow{i} S^m V \xrightarrow{M} \bigwedge^a V \xrightarrow{\pi} \bigwedge^a V/\langle v \wedge \bigwedge^{a-1}V \rangle$ where i is the inclusion and π is the quotient map. Now s_M vanishes on $\langle v \rangle$ if and only if $\pi(M(v^m)) = 0$ if and only if $M(v^m) \wedge v = 0$. $\square$

Remark 3.8. Note that in the decomposition of formula (9), $\mathrm{Hom}(S^m V, V) = \Gamma^{m^n} \otimes V = \Gamma^{m+1,m^{n-1}} \oplus S^{m-1}V^*$, we have from Bott's theorem that $H^0(Q(m)) = \Gamma^{m+1,m^{n-1}}$.

Now we have the proper tools to anticipate the proof of Thm. 3.5, even if we need some results of Sections 4 and 5.

Proof of Thm. 3.5. First notice that by Thm. 2.3, we know that in this case we have a unique decomposition.

Now we prove (1) by applying Prop. 4.1. To match with the notation of Prop. 4.1, take $E = Q(m)$, a twist of the quotient bundle on $\mathbb{P}^2$, and $L = \mathcal{O}(d)$

with $d = 2m + 1$. (See also Example 4.2 for more details on the matrix version of this construction.) By Prop. 4.1, we have the inclusion $H^0(Q \otimes \mathcal{I}_Z(m)) \subseteq \ker A_f$. Note that $E^* \otimes L = Q^*(-m + 2m + 1) = Q^*(m + 1) = Q(m) = E$. The equality holds if the map $H^0(Q(m)) \longrightarrow H^0(Q_Z(m))$, is surjective. This is true by Thm. 5.1(i).

It follows that $\operatorname{rank} A_f = \operatorname{rank} P_f = 2r$ and we can apply Thm. 4.7. Moreover, in this case, $Z' = Z$. Indeed, we just showed that Z' is the base locus of $H^0(Q \otimes \mathcal{I}_Z(m))$, and the latter is equal to Z because of (ii) of Thm. 5.1.

For the proof of (2), now Z' is contained in the zero-locus Z'' of a section of $Q(m)$. It is enough to show that Z'' imposes independent conditions on the hypersurfaces of degree $2m + 1$. Indeed, from the sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow Q^*(m + 1) \longrightarrow \mathcal{I}_{Z''}(2m + 1) \longrightarrow 0,$$

and the vanishing $h^1(Q^*(m + 1)) = 0$, [OSS80] Ch.1 § 1, reading $Q^*(m + 1) = \Omega^1(m + 2)$, one shows that $h^1(\mathcal{I}_{Z''}(2m + 1)) = 0$. Then the linear system in step (4) in Algorithm 3 has a unique solution (because the powers v_i^d are linearly independent), which moreover only uses $\operatorname{rank}(P_f)/2$ of the points. $\qquad\square$

3.2. **Pentahedral example.** Here we treat the Sylvester pentahedral example, in an analogous way to the Hilbert quintic case treated in the introduction. Let $f \in S^3 \mathbb{C}^4$. The classical approach to the Pentahedral Theorem is to use the 10 points p_i which are the singular locus of the Hessian of f. These ten points are the vertices of the pentahedral formed by the five planes v_i. We do not know how to express the five planes rationally from f with this approach. Enriques and Chisini, in the third book of their textbook [EC18], attribute to Gordan the computation of the fifth degree covariant of f given by the five planes, in terms of symbolic calculus, but we found difficult to explicitly compute the Gordan covariant.

Our approach covers both the proof of the theorem and the possibility to find explicitly the five planes.

Theorem 3.9. Sylvester Pentahedral Theorem *For any general $f \in S^3 \mathbb{C}^4$, there exist unique $v_i \in \mathbb{C}^4$ (up to scalar) and $c_i \in \mathbb{C}$ for $i = 1, \ldots, 5$ such that*

$$f = \sum_{i=1}^{5} c_i v_i^3.$$

The algorithm to find the v_i is described in the proof.

Proof. In the pentahedral case we have $f \in S^3 \mathbb{C}^4$. Set $a = 2, m = 1$ in the Koszul flattening construction. This corresponds to constructing $P_f \colon \operatorname{Hom}(\mathbb{C}^4, \bigwedge^2 \mathbb{C}^4) \to \operatorname{Hom}(\mathbb{C}^4, \mathbb{C}^4)$. This is a 16×24 matrix coming from the 4×6 Koszul matrix

$$k_2 = \begin{pmatrix} -x_1 & -x_2 & 0 & -x_3 & 0 & 0 \\ x_0 & 0 & -x_2 & 0 & -x_3 & 0 \\ 0 & x_0 & x_1 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix},$$

and substituting the 4×4 catalecticant matrix $C^1_{f_i}$ at each occurrence of x_i. Using Macaulay2 [GS10] it can be checked that the kernel of P_f has dimension 9 and it is spanned by 9 vectors in $\mathbb{C}^{24} \simeq \mathrm{Hom}(\mathbb{C}^4, \bigwedge^2 \mathbb{C}^4)$ written as $\sum w_{ij} x^i \wedge x^j$ that can be grouped as $\{w_{01}, w_{02}, w_{12}, w_{03}, w_{13}, w_{23}\}$ with $w_{ij} \in \mathbb{C}^{4*}$ linear forms.

The general element M of the kernel can be computed, again with the help of Macaulay2, by a random linear combination of the nine elements of the basis and it has exactly *five* eigenvectors v_i (up to scale), in agreement with Prop. 6.2, thus proving the existence and uniqueness statement of the theorem. Explicitly the five eigenvectors (which are dual to the planes of the pentahedral) can be computed using the 4×4 minors of

$$
\begin{pmatrix}
x_2 & x_3 & 0 & 0 & w_{01} \\
-x_1 & 0 & x_3 & 0 & w_{02} \\
x_0 & 0 & 0 & x_3 & w_{12} \\
0 & -x_1 & -x_2 & 0 & w_{03} \\
0 & x_0 & 0 & -x_2 & w_{13} \\
0 & 0 & x_0 & x_1 & w_{23}
\end{pmatrix},
$$

where the first 4 columns express k_3.

The zero locus of these minors is indeed formed by the five points corresponding to v_i.   □

Remark 3.10. The recent results in the nice paper [BCMT10] makes use of a different generalization of the classical methods for tensor decomposition. Their methods make use of a (semi-)Hankel operator constructed from all possible catalecticant matrices, and compute the linear forms in the decomposition of a tensor utilizing a method of zero-dimensional root finding involving simultaneous eigenvectors of companion matrices. In theory, this method eventually covers all cases to decompose a general symmetric tensor, however there is an essential difference with our methods. That is, their method relies on numerical techniques in order to construct the Hankel operator. As a consequence, for example in the pentahedral case, their technique gives the ideal of the 5 points numerically, while in our algorithm, we have found the ideal of the 5 points symbolically, and numerical techniques are used only to compute the individual points.

4. New methods for tensor decomposition: bundle method

4.1. **The bundle construction.** Let L be the line bundle on X which gives the embedding $X \subset \mathbb{P}(H^0(X, L)^*) = \mathbb{P}W$. In particular $L = \mathcal{O}(d)$ on $\mathbb{P}^n = \mathbb{P}V$ gives the embedding of the Veronese variety $X = v_d(\mathbb{P}^n) = v_d(\mathbb{P}V)$, where in this case $S^d V = W$, defined in the introduction.

Let E be a vector bundle on $X \subset \mathbb{P}W$. In [LO11] a linear map A_f was constructed, depending linearly on $f \in W$, which comes from the natural contraction map

$$
(11) \qquad\qquad H^0(E) \otimes H^0(E^* \otimes L) \longrightarrow H^0(L).
$$

From (11) we get a linear map

$$H^0(E) \otimes H^0(L)^* \longrightarrow H^0(E^* \otimes L)^*,$$

and this can be seen as a linear map

(12) $$A_f \colon H^0(E) \longrightarrow H^0(E^* \otimes L)^*$$

depending linearly on $f \in H^0(L)^*$.

Our starting point is the following result from [LO11].

Proposition 4.1. [LO11, Proposition 5.4.1] *Let* $f = \sum_{i=1}^r v_i \in W$ *with* $z_i = [v_i] \in X \subset \mathbb{P}W$ *and put* $Z = \{z_1, \ldots, z_r\}$.

$$H^0(\mathcal{I}_Z \otimes E) \subseteq \ker A_f$$

$$H^0(\mathcal{I}_Z \otimes E^* \otimes L) \subseteq (\operatorname{im} A_f)^\perp .$$

The first inclusion is an equality if $H^0(E^* \otimes L) \longrightarrow H^0(E^* \otimes L_{|Z})$ *is surjective. The second inclusion is an equality if* $H^0(E) \longrightarrow H^0(E_{|Z})$ *is surjective.*

Recall that the *base locus* of a space of sections of a bundle is the common zero locus of all the sections of the space. Again, let E be a vector bundle on $X \subset \mathbb{P}(H^0(X, L)^*)$. Let $f = \sum_{[v_i] \in Z} v_i \in H^0(X, L)^*$. Assume that $H^0(E^* \otimes L) \longrightarrow H^0(E^* \otimes L_{|Z})$ is surjective. Then the kernel of $A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$ is equal $H^0(\mathcal{I}_Z \otimes E)$. So, if the base locus of $H^0(\mathcal{I}_Z \otimes E)$ is Z itself, then the decomposition of f can be computed from the base locus of $\ker A_f$.

Some advantages of our algorithm are now apparent. First, $\ker A_f$ can be computed by an explicit matrix construction that we give below and methods from linear algebra can be used to compute this kernel. Recall that there is always a brute force method where one guesses a rank r, chooses r linear forms $p_i = p_i(x_0, \ldots, x_n)$ each depending on $n+1$ parameters, and tries to solve the system of polynomials given by comparing coefficients on the expression $f = \sum_{i=1}^r p_i^d$.

So if one compares our method to the brute force method, $\ker A_f$ consists of polynomials of lower degree than the original polynomial f, and in general a system of polynomials of lower degree should be easier to solve than one consisting of polynomials of higher degree. We tested our algorithm using r randomly chosen linear forms l_i, and tried to decompose (the expanded form of) $f = \sum_{i=1}^r l_i^d$. The brute force method quickly fails (we run out of memory and time) even for the pentahedral example, where our algorithm succeeds in less than one second.

4.2. **A side remark on presentations.** A *presentation* of a bundle is what allows us to make the transition between vector bundles and matrices. For a bit more on presentations we invite the reader to consult [Eis05, Chapter 6C]. A presentation of a bundle E on $\mathbb{P}V$ is constructed as follows. We have the (finite) minimal resolution of E, which is

(13) $$\ldots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow E \longrightarrow 0,$$

where each L_i is a direct sum of line bundles, and has the property that the induced map $H^0(L_1) \longrightarrow H^0(E)$ is surjective. This follows because the resolution is constructed as the sheafification of the corresponding resolution of graded modules.

Moreover we have the (finite) minimal resolution of E^* which is

$$(14) \qquad \ldots \longrightarrow L^*_{-1} \longrightarrow L^*_0 \longrightarrow E^* \longrightarrow 0,$$

where each L_i is again a direct sum of line bundles, which has the property that, even tensoring with a line bundle L, the induced map $H^0(L^*_0 \otimes L) \longrightarrow H^0(E^* \otimes L)$ is surjective. Dualizing, we get a double resolution

$$\ldots \longrightarrow L_2 \longrightarrow L_1 \overset{p}{\longrightarrow} L_0 \longrightarrow L_{-1} \longrightarrow \ldots,$$

where $\operatorname{im}(p) = E$. The map $L_1 \overset{p}{\longrightarrow} L_0$ gives the *presentation* of E. We get the composition

$$(15) \qquad P_f = \beta \circ A_f \circ \alpha \colon H^0(L_1) \overset{\alpha}{\longrightarrow} H^0(E) \overset{A_f}{\longrightarrow} H^0(E^* \otimes L)^* \overset{\beta}{\longrightarrow} H^0(L^*_0 \otimes L)^*,$$

where α is surjective (because of (13), which is the sheafification of the minimal free resolution of the module $\oplus_m H^0(E(m)))$ and β is injective (because of (14)). It can be shown that the matrix of P_f can be constructed just by the presentation p with a block structure obtained as follows. If p has a linear entry depending on x_i, substitute the catalecticant matrix $C_{\frac{\partial f}{\partial x_i}}$ for each x_i (where the size of the matrix is to be determined by L_1 and L_0), and if p has non-linear entries, replace each monomial in the x_i by the catalecticant matrix of the associated derivative of f, for more details see [LO11, Section 8.3].

Hence $\operatorname{rank} A_f = \operatorname{rank} P_f$ and, even more important $\ker A_f$ and $\ker P_f$ have the same base locus. The advantage is that P_f can be explicitly computed from a matrix with entries homogeneous polynomials, and $\ker P_f$ is spanned by computable polynomials.

Example 4.2. For a basic example, suppose $d = 2m + 1$, and let $E = Q(m)$ be a twist of the quotient bundle on $\mathbb{P}^2 = \mathbb{P}V$, as in the case of Thm. 3.5

This gives a presentation

$$L_1 = \mathcal{O}(m) \otimes V \overset{p}{\longrightarrow} \mathcal{O}(m+1) \otimes V^* = L_0,$$

which is part of the Koszul complex

$$0 \longrightarrow \mathcal{O}(m-1) \longrightarrow \mathcal{O}(m) \otimes V \overset{p}{\longrightarrow} \mathcal{O}(m+1) \otimes V^* \longrightarrow \mathcal{O}(m+2) \longrightarrow 0$$

After an appropriate choice of basis, p may be represented by the matrix

$$\begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix},$$

which is one of the Koszul matrices we have already seen. Then $H^0(L_1) = S^m V^* \otimes V$.

Now let $L = \mathcal{O}(2m + 1)$. For any $f \in S^{2m+1}V^* = W$, A_f is the morphism from $H^0(Q(m))$ to its dual $H^0(Q^*(m + 1))^*$ and

$$P_f \colon H^0(L_1) = \mathrm{Hom}(S^m V, V) \longrightarrow \mathrm{Hom}(V, S^m V) = H^0(L_0^* \otimes L)^*$$

is represented by the matrix

$$\begin{pmatrix} 0 & C_{f_2}^m & -C_{f_1}^m \\ -C_{f_2}^m & 0 & C_{f_0}^m \\ C_{f_1}^m & -C_{f_0}^m & 0 \end{pmatrix},$$

where $C_{f_i}^m \colon S^m V \to S^m V^*$ are catalecticant matrices of the partial derivatives $f_i = \frac{\partial f}{\partial x_i}$. From this presentation, we can see also that P_f is skew-symmetric. Note that when $f = x_0^{2m+1}$ is the power of a linear form then the above matrix has rank 2, which indeed is the rank of $E = Q(m)$. We further remark that then the principal Pfaffians of this matrix give equations for secant varieties of $v_{2m+1}(\mathbb{P}^2)$, see [LO11].

4.3. **Vector bundles, statement and proofs of the main results.** For this section we assume the following general setup. Let X be an algebraic variety and let L be the line bundle on X which gives the embedding $X \subset \mathbb{P}(H^0(X, L)^*) = \mathbb{P}W$. For $f \in W$, let $f = \sum_{i=1}^k v_i$ be a minimal decomposition, let $z_i = [v_i] \in \mathbb{P}(W)$ be the corresponding points and let $Z = \{z_1, \dots, z_r\}$ For a vector bundle E over X, construct the map $A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$ as above.

Proposition 4.3. *Assume that* $\mathrm{rank}\, A_f = k \cdot \mathrm{rank}\, E$. *Then we have*

$$H^0(\mathcal{I}_Z \otimes E) = \ker A_f$$
$$H^0(\mathcal{I}_Z \otimes E^* \otimes L) = (\mathrm{im}\, A_f)^\perp .$$

Proof. Prop. 4.1 says that we always have the inclusion $H^0(\mathcal{I}_Z \otimes E) \subseteq \ker A_f$. But in fact we get equality because of the following dimension argument.

$$\mathrm{codim}(H^0(I_Z \otimes E)) \le k \cdot \mathrm{rank}\, E = \mathrm{rank}(A_f) = \mathrm{codim}(\ker A_f).$$

The same argument applies to the second equality. $\qquad\qquad\square$

We can give a general criterion

Theorem 4.4. *Assume that* $\mathrm{rank}\, A_f = k \cdot \mathrm{rank}\, E$ *and*

$$H^0(\mathcal{I}_Z \otimes E) \otimes H^0(\mathcal{I}_Z \otimes E^* \otimes L) \to H^0(\mathcal{I}_Z^2 \otimes L)$$

is surjective.
Assume that X is not k-weakly defective, then the common base locus of $\ker A_f$ and of $(\mathrm{im}\, A_f)^\perp$ is given by Z itself, hence Z can be reconstructed by f.

Note that the notion of k-weakly defective has been introduced in [CC02, Definition 1.2]. See also Thm. 2.3 and the paragraph thereafter.

Proof. We can use the equalities of Prop. 4.3. Assume that the common base locus of $\ker A_f$ and of $(\operatorname{im} A_f)^{\perp}$ contains $Z \cup \{z'\}$, then every element of $H^0(\mathcal{I}_{Z^2} \otimes L)$ vanishes doubly on z', that is the general hyperplane section (in the system $H^0(L)$) of X which is singular at Z is singular also at z', and this contradicts [CC02, Theorem 1.4]. □

Example 4.5. The example of plane sextics $v_6(\mathbb{P}^2)$, where Z is given by 8 points is instructive. In this case, set $E = \mathcal{O}(3)$ and $L = \mathcal{O}(6)$, it is a classical fact that the map $H^0(\mathcal{I}_Z \otimes E) \otimes H^0(\mathcal{I}_Z \otimes E^* \otimes L) \to H^0(\mathcal{I}_Z^2 \otimes L)$ is NOT surjective. Indeed, it is known from [LO11] that the catalecticant minors are not enough to give all the equations of $\sigma_8(v_6(\mathbb{P}^2))$, a Koszul flattening is needed in addition. In this case, the kernel of A_f is spanned by a pencil of plane cubics and has base locus in 9 points, one more than the original eight. This is linked to the known classical fact that the pencil of plane cubics through 8 points has an additional ninth base point. Still, in this case, by (2) of Thm. 2.4, Algorithm 2 succeeds to find the tensor decomposition.

The theorem applies in the presentation setting as well. In the case $E = \bigwedge^a Q(\delta)$, there is the presentation $p: L_1 = \bigwedge^a V \otimes \mathcal{O}(\delta) \to \bigwedge^{n-a} V^* \otimes \mathcal{O}(\delta+1) = L_0$ such that $\operatorname{im} p = E$. Then $H^0(L_1) = \operatorname{Hom}(S^\delta V, \bigwedge^a V)$ and an element of $H^0(L_1)$ goes to a section of E vanishing in v if it corresponds to $M \in \operatorname{Hom}(S^\delta V, \bigwedge^a V)$ such that $M(v^\delta) \in v \wedge (\bigwedge^{a-1} V)$, i.e. if v is an eigenvector of M. In this case, the base locus of the kernel can be studied directly in $H^0(L_1)$. The criterion applies to specific tensors.

We now come to the main practical result of this paper.

Algorithm 4 (General algorithm to find tensor decomposition).
Input: $f \in S^d V$ where $\dim V = n + 1$, E is a convenient vector bundle on $\mathbb{P}(V)$, to be chosen.
  (1) Construct the map A_f as defined in (12), where $L = \mathcal{O}(d)$.
  (2) Compute $\ker A_f$. If $\ker A_f$ is trivial, stop, this method fails.
      (a) note that $\operatorname{rank}(f) \geq \frac{\operatorname{rank}(A_f)}{\operatorname{rank}(E)}$.
  (3) Find the base locus Z' of $\ker A_f$ by explicitly computing $\ker P_f$ as in (15).
      (a) If Z' does not consist of a finite set of reduced points, stop; this method fails.
      (b) Otherwise continue with $Z' = \{[v_1], \ldots, [v_s]\}$.
  (4) Solve the linear system defined by $f = \sum_{i=1}^{s} c_i v_i^d$ in the unknowns c_i.
Output: If there is a unique solution to (4), this is the unique Waring decomposition of f. Else we possibly find many minimal Waring decompositions of f.

Note that Thm. 4.4 applies to step (3) in that it says that the baselocus computed actually consists of the linear forms used in the construction of f.

Remark 4.6. In practice one finds that the algorithm will fail if either $\ker P_f$ is trivial, in which case we must conclude that the rank of the input is too large

to be decomposed by this method (see also Rmk. 4.8 below), or the base locus of $\ker P_f$ contains infinitely many points, in which case we cannot determine the decomposition. In the latter case we conclude that there is a positive dimensional variety on which our input tensor lies. In some pathological cases it may happen that the base locus of $\ker P_f$ is non reduced, also in this case we cannot determine the decomposition.

The following theorem has already been applied in many specific examples, (starting from the case $\mathrm{rk}E = 1$ which corresponds to the catalecticant case), that show the versatility and power of the result.

Theorem 4.7. *Let $f \in S^d V$, and set $Z' = \mathrm{baseloc}(\ker A_f)$, and let $length(Z') = s$. Assume that $\mathrm{rank}(f) = \frac{\mathrm{rank}(A_f)}{\mathrm{rank}(E)}$. Then for any minimal decomposition $f = \sum_{i=1}^r c_i v_i$ with $z_i = [v_i] \in X$, $c_i \in \mathbb{C}$ and $Z = \{z_1, \ldots, z_r\}$, we have $Z \subset Z'$.*
If $length(Z') < \infty$, then Algorithm 4 produces all minimal Waring decompositions of f, in particular if the solution is unique, we find the unique Waring decomposition of f.

Proof. Indeed let $f = \sum_{i=1}^r \mu_i v_i$ be a decomposition with minimal rank and set $Z = \{z_1, \ldots, z_r\}$, with $z_i = [v_i]$. By Prop. 4.3 we have $H^0(\mathcal{I}_Z \otimes E) = \ker A_f$. It follows that Z' is also the base locus of $H^0(I_Z \otimes E)$, hence $Z' \supseteq Z$. If Z' is finite, then we can try the linear system in step (4), and with finitely many attempts we find a decomposition. Uniqueness implies that the linear system given by $f = \sum_{i=1}^r c_i v_i$ has a unique solution, thus producing the unique minimal Waring decomposition. $\square$

To further clarify, we restate the previous algorithm in a bit more detail in the case $E = \bigwedge^a Q(m)$, and in the presentation setting:

Algorithm 5 (Koszul Flattening Algorithm (General)).
  Input: $f \in S^d V$, where V has basis $\{x_0, \ldots, x_n\}$.
  (1) Compute $\delta_- = \lfloor \frac{d-1}{2} \rfloor$ and $\delta_+ = \lceil \frac{d-1}{2} \rceil$, and choose $a = \lceil \frac{n}{2} \rceil$, the Koszul flattening to use.
  (2) Construct the Koszul matrices k_p for $p = n + 1 - a, n + 2 - a$
  (3) Construct the catalecticants $C_{f_i}: S^{\delta_+} V \to S^{\delta_-} V$, of $f_i = \frac{\partial f}{\partial x_i}$ for each i.
  (4) Construct the matrix $P_f: \mathrm{Hom}(S^{\delta_+} V, \bigwedge^a V) \to \mathrm{Hom}(\bigwedge^{n-a} V, S^{\delta_-} V)$ by substituting C_{f_i} for x_i in the matrix k_{n+1-a}.
  (5) Compute a basis $\{M_1, \ldots, M_t\}$ of $\ker P_f$, and associate vectors of polynomials $\vec{w}_i$ to each M_i. If $\ker P_f$ is trivial, stop; this method fails.
  (6) Compute the eigenvectors $\{v_1, \ldots, v_s\}$ of a general element in $\ker P_f$ as follows:
    (a) For each $\vec{w}_i$ compute the $\binom{n+1}{a-1} \times \binom{n+1}{a-1}$ minors of the block matrix $\left(\, k_{n+2-a} \mid \vec{w} \, \right)$ and store these minors in an ideal J.
    (b) Find the set $\{v_1, \ldots, v_s\}$ of common eigenvectors of all M in $\ker P_f$ by computing the zero-set of J. If s is infinite, stop, the method fails. Otherwise continue.

(7) Solve the linear system $f = \sum_{i=1}^{s} c_i v_i^d$ in the unknowns c_i.

Output: The unique Waring decomposition of f.

Remark 4.8. Notice that in the middle of our algorithm (step (5)), the algorithm will fail if a certain matrix we construct has a trivial kernel. This is an indication that the rank of the tensor is higher than the ranks which we can successfully decompose. We give precise bounds on the rank of the tensor (depending on the number of variables) for the success of this algorithm in Section 5. So the failure of the algorithm provides a lower bound for the rank of the input tensor.

4.4. **Quintic and Pentahedral examples revisited.** Suppose $d = 2m+1$. In the quintic example we set $m = 2$ and $E = Q(2)$ on $\mathbb{P}^2$. The general section of $Q(2)$ vanishes on 7 points.

In the pentahedral example we set $m = 1$ and $E = Q^*(2)$ on $\mathbb{P}^3$. The general section of $Q^*(2)$ vanishes on 5 points.

An interesting remark is that asking that $h^0(Q(m)) \geq nc_n(Q(m))$ (necessary condition to get that the zero locus of a section of $Q(m)$ is given by $c_n(Q(m))^2$ arbitrary points, by counting parameters), one checks that the only solution for $m \geq 2$ to the diophantine equation resulting from Prop. 6.1 is given by $n = m = 2$ (the quintic example).

In the same way, asking that $h^0(Q^*(m)) \geq nc_n(Q^*(m))$, the only solution to the diophantine equation resulting from Prop. 6.2 for $m \geq 2$ is given by $n = 3, m = 2$ (pentahedral example). In particular, our methods find all of the already known cases where the decomposition of a general symmetric tensor is unique, but do not provide any more uniqueness than was already known classically. This gives evidence that there may not be any other cases where uniqueness holds for the general tensor.

5. Rank bounds for feasibility of our algorithms

In this section we give bounds, depending on the number of variables and the rank of the tensor, for when our algorithm will succeed to produce the Waring decomposition of a given symmetric tensor.

Below in Thm. 5.1 and Thm. 5.3 we study bundles (over $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively) twisted by an arbitrary integer m. More generally we consider $\mathbb{P}^n$ and apply these results to prove Thm. 3.5 and Thm. 5.4 in the specific case that $d = 2m + 1$.

Theorem 5.1. *Let Q be the quotient bundle on $\mathbb{P}^2$ and let Z be a finite collection of s general points in $\mathbb{P}^2$.*

  (*i*) *If $s \leq \frac{1}{2}(m + 3)(m + 1)$ and $m \geq 0$, then $h^0(Q \otimes \mathcal{I}_Z(m)) = (m + 3)(m + 1) - 2s$;*

      this is equivalent to the fact that $H^0(Q(m)) \longrightarrow H^0(Q_Z(m))$ is surjective.

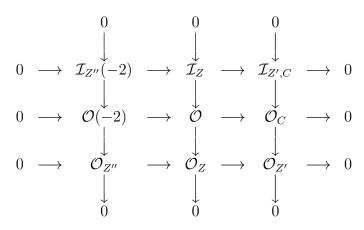[2]We recall a few facts about the Chern class c_n in Section 6.

(ii) If $s \leq \frac{1}{2}(m^2 + 3m + 4)$ $(m \geq 2)$ (or $s \leq 3$ for $m = 1$), then the base locus of $H^0(Q \otimes \mathcal{I}_Z(m))$ is given by Z itself.

Proof. First note that, since Z is a general collection of points, it is reduced. By semi-continuity, both statements can be proved for a special collection Z. We prove the theorem by induction from $m - 2$ to m, so we have to distinguish the even and odd cases. The starting cases $m = 0$ and $m = 1$ can be easily checked directly. If $m = 1$ then (i) says: if $s \leq 4$, then $h^0(Q \otimes \mathcal{I}_Z(1)) = 8 - 2s$; Indeed an element $A \in H^0(Q(1))$ corresponds to a traceless endomorphism of V and by Lemma 3.7 it vanishes exactly at the eigenvectors of A. The space $E(v_1, \ldots v_s)$ of traceless endomorphisms which have s general vectors $v_1, \ldots v_s$ as eigenvectors has dimension $8 - 2s$ for $s \leq 4$, which proves the initial case $m = 1$ of (i). Moreover the common eigenvectors of the endomorphisms in $E(v_1, \ldots v_s)$ for $s \leq 3$ are $v_1, \ldots v_s$ themselves, which proves the initial case $m = 1$ of (ii). The initial case $m = 0$ of (i) is easier, while the initial case $m = 2$ of (ii) corresponds to the plane quintic example developed in the introduction. In this case Z is given by seven points, $H^0(Q \otimes \mathcal{I}_Z(2))$ is one dimensional and it is spanned by the unique section of $Q(2)$ vanishing on Z, here it plays an essential role that $c_2(Q(2)) = 7$. This example may be checked also in Macaulay2.

First we prove (i). Let $C \simeq \mathbb{P}^1$ be a smooth conic. We recall that we have the splitting $Q(m)_{|C} = \mathcal{O}_{\mathbb{P}^1}(2m + 1) \oplus \mathcal{O}_{\mathbb{P}^1}(2m + 1)$. Indeed, from all the possible splittings $\mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$ with $\alpha + \beta = 4m + 2$ the most balanced one is the only compatible with the vanishing $H^0(Q(-1)_{|C}) = 0$ which follows from the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow Q(-1) \longrightarrow Q(-1)_{|C} \longrightarrow 0.$$

We choose to specialize $\tilde{s} = \min(s, 2m + 2)$ points as general points on C, and we call $Z' \subset \mathbb{P}^1$ the subcollection obtained. Let Z'' be given by the remaining points. Note that $\frac{1}{2}(m^2 + 4m + 3) - \frac{1}{2}(m - 2)^2 + 4(m - 2) + 3 = 2m + 2$, so that the length of Z'' satisfies the assumption of the inductive step. There is a natural exact diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{Z''}(-2) & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{I}_{Z',C} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-2) & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{Z''} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_{Z'} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Tensoring the first row by $Q(m)$ we get the exact sequence

(16) $0 \longrightarrow Q(m-2) \otimes \mathcal{I}_{Z''} \longrightarrow Q(m) \otimes \mathcal{I}_Z \longrightarrow \mathcal{I}_{Z'|\mathbb{P}^1}(2m+1) \oplus \mathcal{I}_{Z'|\mathbb{P}^1}(2m+1) \longrightarrow 0,$

and the associated cohomology sequence

$$0 \longrightarrow H^0(Q(m-2) \otimes \mathcal{I}_{Z''}) \longrightarrow H^0(Q(m) \otimes \mathcal{I}_Z) \longrightarrow \mathbb{C}^{4m+4-2\tilde{s}}.$$

Hence

$$h^0(Q(m) \otimes \mathcal{I}_Z) \le h^0(Q(m-2) \otimes \mathcal{I}_{Z''}) + 4m + 4 - 2\tilde{s},$$

and by the inductive assumption we get the inequality

$$h^0(Q \otimes \mathcal{I}_Z(m)) \le m^2 + 4m + 3 - 2s.$$

By considering the sequence

$$0 \longrightarrow Q \otimes \mathcal{I}_Z(m) \longrightarrow Q(m) \longrightarrow \mathbb{C}^2 \otimes \mathcal{O}_Z \longrightarrow 0,$$

we get the opposite inequality, then the result.

The proof of (ii) follows the same inductive step and we specialize the points in a similar manner. We choose to specialize now only $\min(s, 2m+1)$ points as general points on a smooth conic $C \simeq \mathbb{P}^1$. An anonymous referee pointed out that the case $m = 3$, $s = 11$ cannot be proved by induction because the case $m = 1$ our statement holds only for $s \le 3$. Indeed the case $m = 3$, $s = 11$ has to be proved separately (by direct computation). Our computation in Macaulay2 that accomplishes this can be checked and repeated using the file "General Kappa method.m2" (see Section 7) and setting the parameters $s = 11$, $d = 7$, $n = 3$.

For any $z' \notin Z$ we have to prove that there is a section $\sigma \in H^0(Q \otimes \mathcal{I}_Z(m))$ such that $\sigma(z') \ne 0$. Note that by the inductive assumption there is $\sigma' \in H^0(Q \otimes \mathcal{I}_{Z''}(m-2))$ such that $\sigma'(z') \ne 0$. Let f be the equation of C. If $z' \notin C$ then $\sigma = \sigma' f$ works. The proof of (i) (specifically the sequence (16) specialized to fewer points) shows that

$$H^0(Q \otimes \mathcal{I}_Z(m)) \longrightarrow H^0(Q \otimes \mathcal{I}_{Z',C}(m))$$

is surjective. If $z' \in C$ there is a section $\sigma'' \in H^0(Q \otimes \mathcal{I}_{Z',C}(m))$ which does not vanish in z' (here we need that length $Z' \le 2m+1$) and there is a section in $H^0(Q \otimes \mathcal{I}_Z(m))$ which restricts to σ'' and then does not vanish in z', as we wanted. $\square$

Remark 5.2. Note that when m is even $(m \ge 2)$ and $2s = m^2 + 4m + 2$ then $H^0(Q \otimes \mathcal{I}_Z(m))$ is generated by one section, and the number of points in the base locus is given by $c_2(Q(m)) = 1 + m + m^2 \ge \frac{1}{2}(m^2 + 4m + 2)$ and the equality holds only for $m = 2$. The case $m = 2$ is indeed special, and it is the case of $S^5\mathbb{C}^3$ addressed in the introduction. The corresponding tensor decomposition of $S^5\mathbb{C}^3$ (considered here in the introduction) was noted in [LO11].

The following theorem is a generalization of Thm. 5.1 from $\mathbb{P}^2$ to $\mathbb{P}^3$. In principle it is possible to obtain similar theoretical bounds for $\bigwedge^a Q$ on $\mathbb{P}^n$, but it does not seem easy to find sharp bounds like those in Thm. 5.1 on $\mathbb{P}^2$. See the

Rmk. 5.5 which shows that this topic can be tricky. We will continue to use Q for the quotient bundle on any $\mathbb{P}^3$ (and later we will use the same Q for the quotient bundle on $\mathbb{P}^n$), and when we restrict this quotient bundle to a smaller space we will indicate that as $Q_{|\mathbb{P}^2}$ for example – we hope the usage will be clear in context. Recall, from the Euler sequence, that $h^0(\mathbb{P}^3, Q(m)) = \frac{1}{2}(m+4)(m+2)(m+1)$ and $h^0(\mathbb{P}^3, Q^*(m+1)) = \frac{1}{2}(m+4)(m+3)(m+1)$ for $m \geq 0$.

In the proof of Thm. 3.5 there was extra symmetry (that the bundles E and $E^* \otimes L$ happened to be isomorphic in the case $E = Q(m)$ and $L = \mathcal{O}(2m+1)$). Without this symmetry, we will need two analogous pairs of statements (stated next) in order to apply Prop. 4.1 for the proof of Thm. 5.4.

Theorem 5.3. *Let Q be the quotient bundle on $\mathbb{P}^3$ and let Z be a finite collection of s general points in $\mathbb{P}^3$.*

(i) *If $s \leq \left\lfloor \frac{1}{3}(h^0(Q(m)) - m + 2) \right\rfloor = \left\lfloor \frac{1}{3}\left(\frac{1}{2}(m+4)(m+2)(m+1) - m + 2\right) \right\rfloor$ and $m \geq 0$, then $h^0(Q \otimes \mathcal{I}_Z(m)) = \frac{1}{2}(m+4)(m+2)(m+1) - 3s$. This is equivalent to the fact that $H^0(Q(m)) \longrightarrow H^0(Q_Z(m))$ is surjective.*

(ii) *If $s \leq \frac{1}{3}\left(\frac{1}{2}(m+4)(m+2)(m+1) - \frac{m^2}{2} - \frac{3m}{2} + 5\right)$ $(m \geq 2)$ (or $s \leq 4$ for $m = 1$), then the base locus of $H^0(Q \otimes \mathcal{I}_Z(m))$ is given by Z itself.*

(iii) *If $s \leq \left\lfloor \frac{1}{3}(h^0(Q^*(m+1)) - m + 1) \right\rfloor = \left\lfloor \frac{1}{3}\left(\frac{1}{2}(m+4)(m+3)(m+1) - m - 2\right) \right\rfloor$ and $m \geq 0$, then $h^0(Q^* \otimes \mathcal{I}_Z(m+1)) = \frac{1}{2}(m+4)(m+3)(m+1) - 3s$. This is equivalent to the fact that $H^0(Q^*(m+1)) \longrightarrow H^0(Q_Z^*(m+1))$ is surjective.*

(iv) *If $s \leq \frac{1}{3}\left(\frac{1}{2}(m+4)(m+3)(m+1) - \frac{m^2}{2} - \frac{m}{2} - 8\right)$ $(m \geq 2)$ (or $s \leq 5$ for $m = 1$), then the base locus of $H^0(Q^* \otimes \mathcal{I}_Z(m+1))$ is given by Z itself.*

Proof. By semi-continuity, each statement can be proved for a special collection Z. We prove the theorem by induction from $m-1$ to m. The starting cases $m = 1$ can be easily checked directly. Note that $Q(m)_{|\mathbb{P}^2} = Q_{\mathbb{P}^2}(m) \oplus \mathcal{O}_{\mathbb{P}^2}(m)$. We choose to specialize $\tilde{s} = \min\left(s, \left\lfloor \frac{1}{3}(h^0(Q(m)_{|\mathbb{P}^2})\right\rfloor\right)$ points as general points on a hyperplane $\mathbb{P}^2$ and we call $Z' \subset \mathbb{P}^2$ the subcollection obtained. Let Z'' be given by the remaining points. Let $g(m)$ and $f(m)$ respectively denote the numbers

$$\text{(17)} \qquad g(m) = \left\lfloor \frac{1}{3}(h^0(Q(m)_{|\mathbb{P}^2}) \right\rfloor$$

and

$$\text{(18)} \qquad f(m) = \left\lfloor \frac{1}{3}\left(\frac{1}{2}(m+4)(m+2)(m+1) - m + 2\right) \right\rfloor.$$

It is straightforward to check that $f(m) - f(m-1) = g(m)$, so that the length of Z'' satisfies the assumption of the inductive step.

Then we have the exact sequence

$$0 \longrightarrow Q(m-1) \otimes \mathcal{I}_{Z''} \longrightarrow Q(m) \otimes \mathcal{I}_Z \longrightarrow \mathcal{I}_{Z'|\mathbb{P}^2} \otimes Q_{|\mathbb{P}^2}(m) \longrightarrow 0,$$

and the associated cohomology sequence

$$0 \longrightarrow H^0(Q(m-1) \otimes \mathcal{I}_{Z''}) \longrightarrow H^0(Q(m) \otimes \mathcal{I}_Z) \longrightarrow \mathbb{C}^{h^0(Q(m)_{|\mathbb{P}^2})-3s}.$$

Hence

$$h^0(Q(m) \otimes \mathcal{I}_Z) \leq h^0(Q(m-1) \otimes \mathcal{I}_{Z''}) + h^0(Q(m)_{|\mathbb{P}^2}) - 3s,$$

and by the inductive assumption we get the inequality

$$h^0(Q \otimes \mathcal{I}_Z(m)) \leq (m+4)(m+2)(m+1) - 3s.$$

By considering the sequence

$$0 \longrightarrow Q \otimes \mathcal{I}_Z(m) \longrightarrow Q(m) \longrightarrow \mathbb{C}^3 \otimes \mathcal{O}_Z \longrightarrow 0,$$

we get the opposite inequality, then the result.

The proof of (ii) follows the same inductive step and we specialize the points in a similar manner. We choose to specialize now only $\min(s, \lfloor \frac{1}{3} \left((m^2 + 3m + 4) + \binom{m+2}{2} \right) \rfloor)$ points as general points on $\mathbb{P}^2$. Now let $g'(m)$ and $f'(m)$ respectively denote the numbers

$$(19) \qquad g'(m) = \left\lfloor \frac{1}{3} \left((m^2 + 3m + 4) + \binom{m+2}{2} \right) \right\rfloor$$

and

$$(20) \qquad f'(m) = \frac{1}{3} \left(\frac{1}{2}(m+4)(m+2)(m+1) - \frac{m^2}{2} - \frac{3m}{2} + 5 \right).$$

Also in this case it is possible to check that $f'(m) - f'(m-1) = g'(m)$.

For any $z' \notin Z$ we have to prove that there is a section $s \in H^0(Q \otimes \mathcal{I}_Z(m))$ such that $s(z') \neq 0$. Note that there is $s' \in H^0(Q \otimes \mathcal{I}_{Z''}(m-1))$ such that $s'(z') \neq 0$. Let h be the equation of $\mathbb{P}^2$. If $z' \notin C$ then $s = s'h$ works. The proof of (i) shows that

$$H^0(Q \otimes \mathcal{I}_Z(m)) \longrightarrow H^0(Q \otimes \mathcal{I}_{Z', \mathbb{P}^2}(m))$$

is surjective. If $z' \in \mathbb{P}^2$ there is a section $s'' \in H^0(Q \otimes \mathcal{I}_{Z', \mathbb{P}^2}(m))$ which does not vanish in z' (here we need (ii) of Thm. 5.1) and there is a section in $H^0(Q \otimes \mathcal{I}_Z(m))$ which restricts to s'' and then does not vanish in z', as we wanted. The proof of (iii) is very similar to the proof of (i) because $Q^*(m+1)$ restricts on every plane to $Q_{\mathbb{P}^2}(m) \oplus \mathcal{O}_{\mathbb{P}^2}(m+1)$. We set

$$(21) \qquad g(m) = \left\lfloor \frac{1}{3}(h^0(Q^*(m+1)_{|\mathbb{P}^2}) \right\rfloor$$

and

$$(22) \qquad f(m) = \left\lfloor \frac{1}{3} \left(\frac{1}{2}(m+4)(m+3)(m+1) - m - 2 \right) \right\rfloor.$$

We check that $f(m) - f(m-1) = g(m)$ and the initial case holds: $f(0) = 1$. Then we proceed exactly like in (i).

The proof of (iv) is very similar to the proof of (ii). So let

$$(23) \qquad g'(m) = \left\lfloor \frac{1}{3} \left((m^2 + 3m + 4) + \binom{m+3}{2} \right) \right\rfloor$$

and let $f'(m) = \lfloor \frac{1}{3} \left(\frac{1}{2}(m+4)(m+3)(m+1) - \frac{m^2}{2} - \frac{m}{2} - 8 \right) \rfloor$. Then one checks that $f'(m) - f'(m-1) = g'(m)$ and the initial case holds: $f'(2) = 11$.

$\square$

The following theorem generalizes Thm. 3.5 to the case $n \geq 3$. Again, when d is even, use Algorithm 2. In the following theorem we set $E = \wedge^{n-1}Q(m) = Q^*(m+1)$. Note that the pentahedral example of 4.4 corresponds to $n = 3$ and $m = 1$. So we have the maps $A_f \colon H^0(Q^*(m+1)) \to H^0(Q(m))^*$ and $P_f \colon \mathrm{Hom}(S^m V, \wedge^{n-1}V) \to \mathrm{Hom}(V, S^m V)$.

Theorem 5.4. *Suppose $n \geq 3$ and set $d = 2m + 1$. Let $f = \sum_{i=1}^{r} v_i^d$ be a form in $S^d V$ of rank r such that f is general among the forms in $S^d V$ of rank r, let $z_i = [v_i] \in \mathbb{P}(V)$ be the corresponding points and let $Z = \{z_1, \ldots, z_r\}$.*

    (1) *Let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$. If n is even and $r \leq \binom{m+n}{n}$, then Z' agrees with Z. Moreover Algorithm 5 produces the unique Waring decomposition of f.*

    (2) *If n is odd and $r \leq \binom{m+n}{n}$ let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$ and $(\mathrm{im}\, P_f)^\perp$. Then $Z' = Z$ and Algorithm 5, with this modification, produces the unique Waring decomposition of f.*

    (3) *If $n = 3$ and $r \leq \frac{1}{3} \left(\frac{1}{2}(m+4)(m+3)(m+1) - \frac{m^2}{2} - \frac{m}{2} - 8 \right)$ let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$. Then Algorithm 5 (with $a = 2$) produces the unique Waring decomposition of f.*

Proof. First notice that by Thm. 2.3, we know that in these cases we have a unique decomposition.

Now we prove (1). By [LO11, §7] and [LO11, Theorem 1.2.3] we have that the natural map

$$(24) \qquad H^0(\textstyle\bigwedge^a Q \otimes \mathcal{I}_Z(m)) \otimes H^0(\textstyle\bigwedge^{n-a} Q \otimes \mathcal{I}_Z(m)) \longrightarrow H^0(\mathcal{I}_{Z^2}(2m+1))$$

is surjective. Moreover, the dimension of the image of the map in (24) is equal to the rank of the normal bundle of the variety cut out by the minors of size $\binom{n}{a}r+1$ of P_f (where f is considered here as a polynomial with variable entries). By the Alexander-Hirschowitz Theorem 2.2 the space $H^0(\mathcal{I}_{Z^2}(2m+1))$ has the expected codimension $r(n+1)$ in the space $H^0(\mathcal{O}(2m+1))$, hence the rank of the normal bundle is the expected one $\binom{n+2m+1}{n} - r(n+1)$ and it follows that the dimension of the scheme cut out by the minors is the expected one $r(n+1) - 1$ at $[f]$. In particular this scheme has a reduced irreducible component containing $[f]$ which is the r-secant variety to the d-Veronese embedding of $\mathbb{P}^n$. It follows that the scheme cut out by minors is smooth of the expected dimension at f, hence we have the equality $\mathrm{rk}(P_f) = \binom{n}{a}r = \mathrm{rk}(\bigwedge^a Q) \cdot \mathrm{rk}(f)$, otherwise, if $\mathrm{rk}(P_f)$ is smaller,

then the variety cut out by minors should have been singular at $[f]$. Note that in [LO11, Theorem 1.2.3] it was fixed the value $a = \lfloor n/2 \rfloor$, but we may apply it as well in the case $a = \lceil n/2 \rceil$ because we get just the transpose map. If n is even, then the symmetry of P_f guarantees that $\ker P_f = (\operatorname{im} P_f)^\perp$. By Thm. 4.4, with $E = \bigwedge^a Q$ and $L = \mathcal{O}(2m+1)$, we get that $Z = Z'$ (the assumptions of Thm. 4.4 are satisfied by Thm. 2.3). The proof of (2) is analogous.

The proof of (3) follows the same lines of the proof of Thm. 3.5, but we use Thm. 5.3 at the place of Thm. 5.1.   □

Remark 5.5. An interesting case is $n = 4$, $d = 3$, $r = 7$, a defective case addressed in [Ott09]. Set $V = \mathbb{C}^5$, pick a general $f = \sum_{i=1}^7 v_i^3$ with $Z = \{[v_1], \ldots, [v_7]\}$ and construct $P_f \colon \operatorname{Hom}(V, \bigwedge^2 V) \to \operatorname{Hom}(\bigwedge^2 V, V)$. The locus Z' of common eigenvectors of $\ker P_f$ is the unique rational quartic curve in $\mathbb{P}V$ passing through Z. Note that a general element in $Hom(V, \bigwedge^2 V)$ has no eigenvectors, in agreement with Thm. 3.4. In more geometric terms, this means that all sections of $\bigwedge^2 Q(1)$ vanishing on Z also vanish on Z'. This follows easily by the construction and by Terracini Lemma. So, as a byproduct of Algorithm 5, we have found an algorithm to write down the unique rational quartic curve through 7 general points in $\mathbb{P}^4$. According to Ranestad-Schreyer (that we quote) the uniqueness is by Castelenuovo, (see the proof of Prop. 5.2 in their paper).

6. USING CHERN CLASSES TO COUNT THE NUMBER OF EIGENVECTORS OF A GENERAL TENSOR

To a vector bundle E on an algebraic variety X are associated its Chern classes $c_i(E) \in \mathcal{A}^i(X)$, where $\mathcal{A}(X)$ is the Chow ring of X. The reader unfamiliar with Chern classes may wish to consult [Har77] or [OSS80]. For vector bundles on $\mathbb{P}V$, we have $\mathcal{A}^i(\mathbb{P}V) = \mathbb{Z}$ and the Chern classes can be considered as integers.

The basic principle that we will use is the following. If a vector bundle E of rank r on a variety X has a section vanishing on Z, and the codimension of Z is equal to r, then the class of $[Z] \in \mathcal{A}^r(X)$ is computed by $[Z] = c_r(E)$.

The following proposition is a particular case of a more general result, proved recently by Cartwright and Sturmfels with toric techniques [CS11a], who proved a conjecture stated in [NQWW07], whereProp. 6.1 was proved in the case m odd.

Proposition 6.1. *(Cartwright-Sturmfels) The number of eigenvectors (counted with multiplicity) of a general* $M \in \operatorname{Hom}(S^m V, V)$ *is equal to*

$$\frac{m^{n+1} - 1}{m - 1}.$$

Proof. By Lemma 3.7, the number of eigenvectors of a general $M \in \operatorname{Hom}(S^m V, V)$ is equal to $c_n(Q(m))$. In order to compute this number we use the formula (see [OSS80] section 1.2)

$$c_n(E \otimes L) = \sum_{i=0}^n \binom{r-i}{n-i} c_i(E) c_1(L)^{n-i}$$

for a vector bundle E and a line bundle L.

In our case we have

$$c_n(Q(m)) = \sum_{i=0}^{n} c_i(Q)c_1(\mathcal{O}(m))^{n-i} = \sum_{i=0}^{n} m^{n-i} = \frac{1 - m^{n+1}}{1 - m},$$

where we have used the well known fact that $c_i(Q) = 1$, which follows immediately from the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes V \longrightarrow Q \longrightarrow 0. \qquad \square$$

Proposition 6.2. *The number of eigenvectors (counted with multiplicity) of a general $M \in \mathrm{Hom}(S^m V, \bigwedge^{n-1} V)$ is equal to*

$$\frac{(m+1)^{n+1} + (-1)^n}{m+2}.$$

Proof. By Lemma 3.7, the number of eigenvectors of a general $M \in \mathrm{Hom}(S^m V, \bigwedge^{n-1} V)$ is equal to $c_n(\bigwedge^{n-1} Q(m)) = c_n(Q^*(m+1))$, which can be computed in a similar way by noting that $c_i(Q^*) = (-1)^i$. To follow the previous proof, we set $m' = m + 1$.

$$c_n(Q^*(m')) = \sum_{i=0}^{n} c_i(Q^*)c_1(\mathcal{O}(m'))^{n-i} = \sum_{i=0}^{n} (-1)^i (m')^{n-i}$$

$$= (-1)^n \sum_{i=0}^{n} (-m')^{n-i} = (-1)^n \frac{1 - (-m')^{n+1}}{1 + m'} = \frac{(m')^{n+1} + (-1)^n}{m' + 1}. \qquad \square$$

7. Macaulay 2 Implementation

The tensor decomposition algorithms in this article could be implemented in a variety of computational algebra or computational linear algebra packages. We chose to implement our algorithms in Macaulay 2 because we found many of the procedures we would need were already implemented. Our algorithms may be easily adapted to other languages, depending on the desired features. Our code may be found online with the ancillary materials accompanying the arXiv version of our paper or by contacting either author. In this section we have tried to use the verbatim text style to indicate Macaulay2 input.

7.1. **The catalecticant algorithm implementation.** Our first example is an implementation of the catalecticant algorithm (Algorithm 2). For this example we work with degree d polynomials on $n + 1$ variables. This is the file "cat_method.m2". Our experiments show that working over a prime characteristic base field, or also the rational numbers, for relatively small d and n, the catalecticant method quickly computes the decomposition within the range of Thm. 2.4, and sometimes when the degree is low, even succeeds for slightly higher ranks than predicted by the bound. For this implementation the user can change the values of degree `d` and projective dimension `n` as well as the rank `s`

of the test polynomial at the beginning of the file. The ring `R` can be either over the rationals or over a prime characteristic field.

A polynomial `ff` over ground field `KK = ZZ/p` or `QQ` is constructed as the sum of `s` d^{th} powers by the following.

```
R  = KK[x_0..x_n]
ff=sum(s,i->(random(1,R))^d)
```

Next, we construct a map that computes the "most square" catalecticant matrix associated to a given input polynomial. We can do this conveniently by using ceiling and floor commands to define the degrees, the basis command to define vectors of appropriate sizes for what would be the labels of the rows and columns of the matrix and the `diff` command to construct the matrix.

```
af = floor(d/2)
ac = ceiling(d/2)
xaf = basis(af,R)
xac = basis(ac,R)
catalecticant=f->diff(transpose xac,diff(xaf,f))
```

The next step is to find the base locus of the kernel of the catalecticant matrix. First we compute the generators of the kernel, then we convert these integer vectors to an ideal of polynomials using the basis of the base space, and finally we decompose the radical of the ideal by first computing the saturation (this sometimes results in a speed-up, and is justified because the saturation has the same scheme-theoretic structure, and this is all that concerns us with this application).

```
K =gens kernel catalecticant ff
I = ideal(xaf*K)
L = decompose saturate I
```

The list `L` contains the apolars of the linear forms which will (up to scale) be used to write the decomposition.

Next we construct a ring `S` by appending constants `c_i` whose number is the number of points in the base locus of the kernel.

```
S  = KK[x_0..x_n,c_0..c_(length L -1)]
```

We construct the polynomial $\mathtt{fc} = \sum_{v_i \in \mathrm{L}} c_i v_i^*$, where the v_i^* are the apolar forms to the v_i. We accomplish the swap between a form its apolar form within the summation as follows.

```
bS = sub(basis(1,R),S)
fc = sum(length L,i->
c_i*((bS*(mingens kernel diff(bS, transpose mingens sub(L_i,S)))))_(0,0))^d );
```

Next we solve the linear system on the `c_i` obtained by setting equal to zero all d^{th} derivatives of the expression `fc-ff`.

```
Ic= ideal(sub(diff(basis(d,R),ff),S) - diff(sub(basis(d,R),S),fc));
Vc = decompose saturate Ic
```

Finally we substitute the found values for the `c_i` into the polynomial `fc` check to see if the decomposition succeeded.

```
FF =sub(substitute(fc,S/Vc_0),R)
FF-ff
```

Our tests succeeded to find decompositions for the following `n`, `d`, and `s` for example.

```
-- n=2: (d=3, s=1),  (d=4, s<=4),  (d=5, s<=4),  (d=6, s<=8)
-- n=3: (d=3, s=1),  (d=4, s<=7),  (d=5, s<=7),  (d=6, s<=16)
-- n=4: (d=3, s=1),  (d=4, s<=10), (d=5, s<=10), (d=6, s<=16).
```

7.2. **The Koszul flattening algorithm implementation.** Our second example is the implementation of the Koszul flattening algorithm (Algorithm 5). This is contained in the file "General Kappa Method.m2".

As before we tested our algorithm by taking a sum of a fixed number s of powers of random linear forms, expanding the resulting polynomial, and then testing to see if our algorithm gave the correct decomposition. Here we will describe the aspects of this algorithm that differ from the catalecticant algorithm.

As before, we construct a map that computes the "most square" catalecticant matrix associated to a given input polynomial of degree `d-1`. The degree drops because we will eventually feed this map the first partial derivatives of our input polynomial `ff`. Then we construct the Koszul complex.

```
M = ideal(basis(1,R))
RM = resolution M
```

Using the `diff` command again, we take a matrix from the Koszul complex and construct a block matrix, replacing each entry in the Koszul matrix with the catalecticant of the partial derivative of our test function with respect to the entry in the Koszul matrix. This matrix is our Koszul flattening, where `ka` indicates which map in the Koszul complex we are using. In this case `ka=n+1-ceiling(n/2)`. The matrix `K` corresponds to the map called P_f in this paper.

```
K = diff(transpose RM.dd_ka, catalecticant ff)
```

The base locus of the kernel of the Koszul Flattening `K` is a set of (generalized) eigenvectors. Later we will construct this base locus, for now we compute the generators of the kernel of the Koszul Flattening.

```
KM = generators kernel K
```

The kernel of the Koszul flattening should be a vector space of polynomials, but at present it is expressed as integer vectors. The Koszul flattening `K` is an `a`×`a` blocked matrix (in the case n even, with small variations in the odd case, when the matrix is no longer square) of `m`×`m` blocks where `a = binomial(n+1,ka)` and `m=binomial(n+d-ac-1,n)`. The kernel of `K` respects this block structure. So we convert each blocked integer vector in the kernel into a smaller vector where each of the `a` blocks becomes a polynomial written in the basis of monomials

previously defined at `xaf`. Macaulay 2 can do these computations simultaneously on matrices and not just individual vectors.

```
for i from 0 to a-1 do { G_i =submatrix(KM,{i*m..(i+1)*m-1},);}
pG =xaf* G_0; for i from 1 to a-1 do { pG = pG||(xaf*G_i);}
```

The outcome `pG` is a matrix, each row of which is a vector of polynomials in the kernel of `K`.

The kernel of `K` is a subspace of $\mathrm{Hom}(S^{\mathtt{ka}}V, V)$, and the zero-set of the `a` $\times$ `a` minors of the matrix representing a map in $\mathrm{Hom}(S^{\mathtt{ka}}V, V)$ are generalized eigenvectors. Therefore, for each basis vector of the kernel of `K`, we construct an ideal of `a` $\times$ `a` minors. Since we are interested in the common generalized eigenvectors to all basis vectors of the kernel of `K`, we construct an ideal `J` which is generated by all of the `a'` $\times$ `a'` minors we constructed as follows: `a'= binomial(n,ka)+1`.

```
J= ideal(0*x_0);
for i from 0 to r-1 do
            J= J + minors(a',RM.dd_(ka+1)|transpose(submatrix(pG,,{i})));
```

Next we want to compute the zero-set of the the ideal `J` above. In order to save time, we first compute the saturation of the ideal since an ideal and its saturation have the same zero-set (scheme).

```
L = decompose saturate J
```

The list `L` consists of linear forms which are generalized eigenvectors in the kernel of `K`. The solutions in `L` are the polar forms to those that we want. The rest of the implementation is identical to that of the catalecticant algorithm implementation.

This example code succeeds for the following initial parameters:

```
--n=2, (d=3, s<=3), (d=4, s<=3), (d=5, s<=7), (d=6, s<=7),
--n=3, (d=3, s<=5}), (d=4, s<=5), (d=5, s<=11),(d=6, s<=11),
--n=4, (d=3, s<=6), (d=4, s<=6), (d=5, s<=14).
```

We were able to test that over prime characteristic, all of these cases are sharp except for the case `n=4, d=5`, as in this case the algorithm slowed considerably as `s` grew. Note for example that when `n=3`, `d=5`, and `s=8` there are rationality problems.

Remark 7.1 (Remark on numerical methods for inexact solutions and practical issues). Many of the bounds in the examples we have presented could be improved if we allowed for irrational or complex solutions to our systems of polynomials. In addition, we could succeed in treating the generic case if we were to use numerical eigenvector methods on the Koszul flattening and to proceed by declaring the kernel of `K` to consist of those eigenvectors of `K` which are associated to small eigenvalues. We believe that the algorithms we have presented are well suited to such adaptations, however we leave this to future study.

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