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Physics Phd thesis

Thermodynamics and the relativistic spin tensor

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Contents

Notation

In this work we adopt natural units, where the Planck constant, the speed of light in the vacuum and the Boltzmann constant are dimensionless an equal to one, $\hbar = c = K_B = 1$.

For ease of writing we identify tensors with their components in particular frames, so we will have "the stress-energy-momentum tensor $T^{\mu\nu}$". We use greek indices for four-vectors' components, while latin indices i, j, k, l, m, n correspond to the spatial part, so we have the four current j^μ, with $\mu \in \{0, 1, 2, 3\}$, including the charge density, whilst j^i is only the spatial current.

We also adopt the Einstein summation convention to sum over indices that appears at least once in an upper and once in lower position, and we contract vectors with a dot:

$$v \cdot w = v_\mu w^\mu = v^\mu w^\nu g_{\mu\nu} = \sum_{\mu,\nu=0}^{3} v^\mu w^\nu g_{\mu\nu}.$$

Bolded letters are classical (three-)vectors or pseudo-vectors like the (three-)velocity $\mathbf{v}$ or the spatial coordinate $\mathbf{x}$, and contractions of the spatial part of a tensor with the Levi-Civita symbol. So, being $J^{\mu\nu}$ the angular momentum, $\mathbf{J}|^i = \epsilon_{ijk} J^{jk}$ is the angular momentum pseudo-vector.

We use the usual notation ∂_μ for partial derivatives, for the spatial gradient we use $\nabla_\mathbf{x}$, so the the four-divergence of a four-vector is $\partial \cdot u$ and the three-divergence of a three-vector is $\nabla_\mathbf{x} \cdot \mathbf{v}$.

We write versors with little hats, so $\hat{\omega}$ is the direction of $\boldsymbol{\omega}$. To distinguish between quantum operators and their classical counterparts we write operators with a large hat, except the Dirac field operator which is denoted with a capital Ψ, thus $\widehat{J}^{\mu\nu}$

is the quantum angular momentum, the generator of proper orthochronous Lorentz transformations.

We take diag$(1, -1, -1, -1)$ as the Minkowski metric signature, so time-like vectors have positive inner product $u_\mu u^\mu > 0$.

Fore the levi-Civita symbol in four dimension $\epsilon^{\mu\nu\rho\sigma}$ we use the convention:

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{for even permutations of}(0,1,2,3) \\ -1 & \text{for odd permutations ofi } (0,1,2,3) \\ 0 & \text{otherwise,} \end{cases}$$

so $\epsilon^{0ijk} = \epsilon_{ijk}$ or, lowering indices using the metric tensor $g_{\mu\nu}$, $\epsilon_{0ijk} = -\epsilon_{ijk}$.

Introduction

Relativistic hydrodynamics is an important tool in many aspects of modern physics, from laboratory physics to astrophysics and cosmology. During the last decades we have seen a renewed interest in the topic, stemming from both its phenomenological applications and theoretic results. For instance at the very high energies and densities reached in heavy ion collisions we can produce quark gluon plasma, a deconfined state of matter where quark and gluons behave like an almost perfect fluid. On the other hand the Maldacena conjecture of AdS/CFT duality provides universal limits for transport coefficients, like the widely known relation between shear viscosity and entropy density $\eta/s \geq 1/4\pi$.

Relativistic hydrodynamics refers both to special and general relativity. A fluid is relativistic if it has large enough energy and speed that length contraction, time dilatation and relativistic covariance in general can not be neglected. It is a term even used for gravitating fluid systems, not even having necessarily large energies or velocity. Two important examples are the expansion of the universe and oscillations of neutron stars. In this work we will only consider a flat space-time, limiting the discussion to Minkowski space. However our findings have many theoretical implications on gravitation and cosmology.

Hydrodynamics arises as an effective theory valid in the long-wavelength, low-frequency limit where the equations of motion can be expressed as conservation laws for the total four-momentum $\partial_\mu T^{\mu\nu} = 0$, as well as conserved charge(s) $\partial_\mu j^\mu = 0$ if present. To our best knowledge the right description of microscopic interactions is quantum mechanics. Being relativistic hydrodynamics a classical theory with only classical degrees of freedom, it is usually understood that the stress-energy-momentum tensor, like any other macroscopic object, is the average value of the corresponding quantum operator $T^{\mu\nu} = \langle \widehat{T}^{\mu\nu} \rangle$.

The quantum stress-energy-momentum tensor is not uniquely defined in quantum field theories, unlike its spatial integrals, the hamiltonian $\widehat{H}$ and linear momentum $\widehat{\mathbf{P}}$ operators. An interesting question is thus if different quantum $\widehat{T}^{\mu\nu}$ operators will give the same macroscopic, average, results. Or otherwise, what observables change if we take a different fundamental tensor.

The main purpose of this work is to answer these questions. We will se that the problem is strictly related to the presence, or absence, of a fundamental spin tensor $\widehat{\mathcal{S}}^{\lambda,\mu\nu}$. The spin tensor is an operator which contributes to the total angular momentum along with $\widehat{T}^{\mu\nu}$ in the operator sense. Once we take the average value for the macroscopic system, it amounts to the internal angular momentum of fluid cells.

We found that previously thought equivalent pairs of stress-energy-momentum and spin tensors are actually inequivalent. The inequivalence does not show for the familiar grand grand-canonical ensemble, but it is enough to have a rigidly rotating system at global equilibrium to find different linear and angular momentum density. In addition we found that total entropy and transport coefficients are different, out of equilibrium, if we take different couples of quantum tensors.

An important consideration is that we provided in principle a way to distinguish between couples of tensors without considering gravity. A measurement proving that the usual stress-energy-momentum tensor is not the right one would have major consequences in gravitation and cosmology. The presence of a spin tensor and, more importantly, a non vanishing antisymmetric part of the stress-energy-momentum tensor, for example, would point toward theories with non vanishing torsion, namely affine theories.

Chapter 1

Relativistic fluids with spin

This chapter is a brief introduction to relativistic fluids with internal angular momentum. We will start in the next section from the general notions of relativistic hydrodynamics, and the link between quantum field theories and relativistic fluids. Later we will show a canonical procedure to build the stress-energy-momentum operator $\widehat{T}^{\mu\nu}$ knowing the action of a system. From the very same procedure the fundamental spin tensor $\widehat{\mathcal{S}}^{\lambda,\mu\nu}$ is introduced. We will see that there is class of transformations, called pseudo-gauge transformations, which allows to change the fundamental operators $\{\widehat{T}, \widehat{\mathcal{S}}\}$. One particular transformation eliminates the spin tensor. Since pairs of tensors linked by a pseudo-gauge transformation are commonly believed to be physically equivalent, we can ask if it is necessary to have a fundamental $\widehat{\mathcal{S}}^{\lambda\mu\nu}$ operator. The aim of this work is to study the effects of these transformations on observable quantities, and we will show that different pairs are actually inequivalent.

1.1 Quantum field theories and relativistic fluids

Relativistic hydrodynamics is a model to describe the behavior of a continuous, macroscopic, system with classical -*i.e.* non quantum- degrees of freedom[1]. The equation of motion is the local conservation of four-momentum:

[1]Causality forbid rigid bodies in special relativity, as they require superluminal propagation of information to maintain the same relative distances during acceleration. Every relativistic system can be seen as a fluid or, in general, as a viscoelastic system.

$$\partial_\mu T^{\mu\nu}(x) = 0.$$

This means that the stress-energy-momentum tensor $T^{\mu\nu}$ corresponds to four-momentum flux. The total four-momentum P^μ then reads:

$$P^\mu(t) = \int_V \mathrm{d}^3\mathbf{x}\, T^{0\mu}(x),$$

and it is constant, provided that the four-momentum flux at the spatial boundary vanishes, as it is usually required:

$$\int_{\partial V} \mathrm{d}S\, n_i\, T^{i\mu}(x) = 0,$$

where $\hat{n}$ is the direction of the infinitesimal surface element.The stress-energy-momentum tensor thus includes the energy density T^{00}, that is the energy of a microscopic cell; the linear momentum density T^{0i}, and the four-momentum flux $T^{i\mu}$. The ideal fluid [1] reads:

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p\eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ is the Minkowski metric, u^μ is the four-velocity of the elementary cell, ε is the proper energy density, namely the energy density as measured in the comoving frame, and p is the pressure. As the presence of the pressure suggests, the stress-energy-momentum tensor contains the relativistic generalization of the stress tensor (pressure, and dissipative currents for the non ideal fluid), whence "stress". We will return in detail to the topic of the general form[2] of $T^{\mu\nu}$ in chapter 4, showing that the local four-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ for perfect fluids corresponds, in the non-relativistic limit, to the continuity equation and the Euler equation.

Quantum field theory embodies both special relativity and quantum Mechanics axioms. It is the most common tool to deal with microscopic interactions at relativistic energies. Relativistic hydrodynamics on the other hand is a classical theory with classical degrees of freedom. We know that classical observables amounts to the average value of corresponding quantum operators:

[2]including the non-ideal case.

$$\mathcal{O}_{\mathrm{cl.}} = \mathrm{tr}\left(\widehat{\rho} : \widehat{\mathcal{O}} :\right),$$

where $\widehat{\rho}$ is the density operator, which describe the state, either pure or mixed, of the system. So the average value is taken both with respect to the quantum and to the statistical fluctuations. The normal ordering $: \cdots :$ is introduced to avoid zero point infinities and, in general, it may be necessary to use renormalized operators.

The most straightforward way to study relativistic fluids taking into account microscopic quantum interactions is therefore to take as the stress-energy-momentum tensor the mean value of the corresponding operator in quantum field theory:

$$T^{\mu\nu}(x) = \mathrm{tr}\left(\widehat{\rho} : \widehat{T}^{\mu\nu}(x) :\right).$$

where the operator $\widehat{T}^{\mu\nu}(x)$, being the quantum counterparts of the stress-energy-momentum tensor, must fulfill the equations:

$$\partial_\mu \widehat{T}^{\mu\nu} = 0 \qquad\qquad \widehat{P}^\mu = \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{0\mu}(x),$$

so that the mean value fulfills the classical counterpart. In the next section we will see a canonical way to build a quantum tensors fulfilling these equations, but, as we will see in the following one, there is not only one candidate.

1.2 Canonical derivation of quantum tensors

Differently from classical physics in quantum field theory there is not a straightforward definition of total energy, momentum and angular momentum from the fundamental objects, field operators. The situation is even worse if we want a definition of local quantities like four-momentum and angular momentum densities.

In Newtonian mechanics we can write the action of a system (the time integral of the Lagrangian, or the space-time integral of the Lagrangian density for continuos systems) so that the principle of least action gives the equation of motion for each degree of freedom. Noether's theorem states that the invariance of the action with respect to a group of global transformations corresponds to a set of conserved currents. The corresponding conserved charge is the generator of the starting transformation.

So if we have an invariant action:

$$\mathcal{A} = \int_{\Omega} \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x), x) \mathrm{d}^4 x$$

where the infinitesimal transformation reads:

$$\begin{cases} x^\mu \to \xi^\mu = x^\mu + \delta x^\mu = x^\mu + \varepsilon X^\mu \\ \phi^A(x) \to \alpha^A(\xi) = \phi^A(x) + \delta\phi^A(x) = \phi^A(x) + \varepsilon \Phi^A, \end{cases}$$

Noether's theorem guarantees that the four current j^μ has a vanishing four-divergence $\partial \cdot j = 0$:

$$j^\mu = \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^A\right)} (X \cdot \partial)\phi^A - \mathcal{L} X^\mu \right] - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^A\right)} \Phi^A,$$

If we consider the the invariance under space time translation, having the total four-momentum as the generator, we have:

$$\begin{cases} \delta x^\mu = \varepsilon e^\mu \\ \delta \phi^A(x) = 0, \end{cases}$$

where e^μ is the direction of the space-time translation. The conserved current is therefore:

$$j^\mu = \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^A\right)} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L} \right] e_\nu,$$

where $\eta_{\mu\nu}$ is naturally the Minkowski metric. From the last formula we get the canonical stress-energy momentum tensor $T^{\mu\nu}_{\text{c.}}$:

$$T^{\mu\nu}_{\text{c.}} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^A\right)} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L}.$$

The procedure remains valid even in the quantum case [4], following the same steps we have therefore the canonical operator:

$$\widehat{T}^{\mu\nu}_{\text{c.}} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \widehat{\phi}^A\right)} \partial^\nu \widehat{\phi}^A - \eta^{\mu\nu} \mathcal{L},$$

where the fields $\widehat{\phi}^A$ are now operators, and the integral of the time component correspond to the generator of quantum space-time translations $\widehat{P}^\mu$.

It is important now to extend a little further the discussion. For ease of reading we will not write an upper hat on the fields, understanding that the same arguments hold for the quantum system.

Let us consider at this point rotations and boosts, which have the total angular momentum as the generator. The infinitesimal transformation are:

$$\begin{cases} \delta x^\mu = -\frac{i}{2}\varepsilon e^{\rho\sigma}(J_{\rho\sigma})^{\mu\nu}x_\nu = \frac{1}{2}\varepsilon e^{\rho\sigma}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\nu_\rho\delta^\mu_\sigma)^x_\nu = \varepsilon e^{\mu\nu}x_\nu \\ \delta\phi^A(x) = -\frac{i}{2}\varepsilon e^{\mu\nu}(\Sigma_{\mu\nu})^A_{A'}\phi^{A'}, \end{cases}$$

with $e^{\mu\nu}$ constant, unitary and skew symmetric. $(J_{\rho\sigma})^{\mu\nu}$ and $(\Sigma_{\mu\nu})^A_{A'}$ are the representation of the algebra of the proper orthochronous Lorenz group. $(J_{\rho\sigma})^{\mu\nu} = i(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\nu_\rho\delta^\mu_\sigma)$ is the vector representation, used for positions x, whilst $(\Sigma_{\mu\nu})^A_{A'}$ depends on the spin of the fields.

The conserved current is:

$$j^\lambda = \left[\frac{i}{2}\frac{\partial\mathcal{L}}{\partial\left(\partial_\lambda\phi^A\right)}(\Sigma^{\mu\nu})^A_{A'}\phi^{A'} - x^\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\lambda\phi^A\right)}\partial^\nu\phi^A - \eta^{\lambda\nu}\mathcal{L}\right)\right]e_{\mu\nu}.$$

Using the definition of the canonical stress-energy-momentum tensor, the latter reads:

$$j^\lambda = -\frac{1}{2}\left[-i\frac{\partial\mathcal{L}}{\partial\left(\partial_\lambda\phi^A\right)}(\Sigma^{\mu\nu})^A_{A'}\phi^{A'} + x^\mu T^{\lambda\nu}_{\text{c.}} - x^\nu T^{\lambda\mu}_{\text{c.}}\right]e_{\mu\nu},$$

so we have the canonical angular momentum four-current:

$$\mathcal{J}^{\lambda,\mu\nu} = -i\frac{\partial\mathcal{L}}{\partial\left(\partial_\lambda\phi^A\right)}(\Sigma^{\mu\nu})^A_{A'}\phi^{A'} + x^\mu T^{\lambda\nu}_{\text{c.}} - x^\nu T^{\lambda\mu}_{\text{c.}}.$$

the part which does not explicitly depend on the stress-energy momentum tensor is called the spin tensor:

$$\mathcal{S}^{\lambda,\mu\nu}_{\text{c.}} = -i\frac{\partial\mathcal{L}}{\partial\left(\partial_\lambda\phi^A\right)}(\Sigma^{\mu\nu})^A_{A'}\phi^{A'}.$$

Excluding possible quantum anomalies, from the action of a quantum field theory we can build at least a couple of canonical tensors $\{\widehat{T}_{\text{c.}}, \widehat{\mathcal{S}}_{\text{c.}}\}$ that fulfills the conservation equations:

$$\begin{cases} \partial_\mu \widehat{T}^{\mu\nu} = 0 \\ \partial_\lambda \left(\widehat{\mathcal{S}}^{\lambda,\mu\nu} - x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) = 0 \end{cases} \Rightarrow \begin{cases} \partial_\mu \widehat{T}^{\mu\nu} = 0 \\ \partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} = \widehat{T}^{\nu\mu} - \widehat{T}^{\mu\nu}, \end{cases} \tag{1.1}$$

and gives the Poincaré algebra generators, total four-momentum and angular momentum operators, when spatially integrated:

$$\int_V \mathrm{d}^3\mathbf{x}\, \widehat{T}^{0\mu} = \widehat{P}^\mu \qquad \int_V \mathrm{d}^3\mathbf{x}\, \widehat{\mathcal{J}}^{0\mu\nu} = \int_V \mathrm{d}^3\mathbf{x} \left(\widehat{\mathcal{S}}^{0,\mu\nu} + x^\mu \widehat{T}^{0\nu} - x^\nu \widehat{T}^{0\mu} \right) = \widehat{J}^{\mu\nu}. \tag{1.2}$$

1.2.1 Meaning of the spin tensor

We have just seen that Noether's theorem, when we use it to find the conserved current linked to Lorentz boost and rotation, provides a new operator, the spin tensor, a part of the angular momentum current. Total angular momentum and angular momentum conservation are, in the operator sense:

$$\widehat{J}^{\mu\nu} = \int_V \mathrm{d}^3\mathbf{x} \left(\widehat{\mathcal{S}}^{0,\mu\nu} + x^\mu \widehat{T}^{0\nu} - x^\nu \widehat{T}^{0\mu} \right) \qquad \partial_\lambda \left(\widehat{\mathcal{S}}^{\lambda,\mu\nu} - x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right) = 0.$$

Therefore for the average value, we have as the total angular momentum:

$$J^{\mu\nu} = \int_V \mathrm{d}^3\mathbf{x} \left(\mathcal{S}^{0,\mu\nu} + x^\mu T^{0\nu} - x^\nu T^{0\mu} \right),$$

where we recognize the orbital angular momentum in the last two terms. Being $T^{0\mu}$ the linear momentum density, $x^\mu T^{0\nu} - x^\nu T^{0\mu}$ is the orbital angular momentum of a microscopic fluid cell[3]. Accordingly the spin tensor correspond to the internal angular momentum of the fluid cell, *i.e.* the angular momentum about the position of the cell as the origin, hence the name spin tensor.

From a kinetic point of view we can build a spin tensor using the average polar-

[3] The cross product of two vectors is not defined in four dimension. The covariant extension of angular momentum is thus $x^\mu p^\nu - x^\nu p^\mu$. If we consider the spatial components and contract with the three-dimensional Levi-Civita symbol we recover the familiar angular momentum pseudo-vector $l^i = 1/2 \epsilon_{ijk}(x^j p^k - x^k p^j)$.

ization of particles, as we have the stress-energy-momentum tensor from the particle distribution function [1]. When there is a non vanishing polarization of particles, it is natural to think that the spin tensor represents the density of spin per unit volume but, as we will see in more detail on the discussion at the end of chapter 2, particle polarization contribution to total angular momentum can be included in a fluid having only orbital angular momentum. As we shall see in the next section, the canonical pair given by the Noether's theorem is not the only one fulfilling the local conservation equation and giving the Poincaré algebra generators, and it is possible to build a pair $\{\widehat{T}, \widehat{\mathcal{S}}\}$ with a vanishing spin tensor $\widehat{\mathcal{S}} \equiv 0$.

It is important to stress how the presence, or absence, of a fundamental spin tensor should be seen from observable quantities, like, for example, energy density, momentum density and *total* angular momentum density. The presence of a non-vanishing internal angular momentum density in a macroscopic system is not conclusive. For instance let us start with a spin-less fluid, a system which does not need a fundamental spin tensor to be described. We can still measure a non-vanishing internal angular momentum density because of physical reasons. Any measuring device will perform an angular momentum measure over a small but finite volume, it will not perform a direct measure of density. Thus we can not distinguish between a fundamental spin tensor $\mathcal{S}^{0,\mu\nu} = \langle \widehat{\mathcal{S}}^{0,\mu\nu} \rangle$ and small vortices encompassed in the volume of integration.

1.3 Pseudo-gauge transformations

Even if Noether's theorem gives a canonical couples $\widehat{T}^{\mu\nu}_{\rm c.}$ and $\widehat{\mathcal{S}}^{\lambda,\mu\nu}_{\rm c.}$, in quantum field theory the stress-energy-momentum tensor and the spin tensor are not uniquely defined. The canonical couple of tensors is not the only one fulfilling the conservation equations (1.1) and giving the generators of Poincaré algebra trough the spatial integral of time components (1.2). Once a particular couple $\{\widehat{T}, \widehat{\mathcal{S}}\}$ of this tensors is found, *e.g.* the canonical couple from Noether's theorem, it is possible to generate new couple $\{\widehat{T}', \widehat{\mathcal{S}}'\}$ using a pseudo-gauge transformation:

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha \widehat{\phi}^{\alpha\mu,\mu} \qquad \widehat{\mathcal{S}}'^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu} + \partial_\alpha \widehat{Z}^{\alpha\lambda,\mu\nu}, \qquad (1.3)$$

where:

$$\widehat{\phi}^{\alpha\mu,\nu} = \widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu},$$

and $\widehat{\Phi}^{\lambda,\mu\nu}$, usually called superpotential, is an arbitrary rank three tensor antisymmetric in the last two indices. The last operator $\widehat{Z}^{\alpha\lambda,\mu\nu}$ is a rank four tensor, antisymmetric both in the $\{\alpha\beta\}$ and the $\{\mu\nu\}$ indices. This is an auxiliary supepotential, it is seldom considered, but is still part of the most general pseudo-gauge transformation.

It is straightforward to check that the new couple still fulfills the continuity equations (1.1). If we take the four-divergence $\partial_\mu \widehat{T}'^{\mu\nu}$ we have the, vanishing, four-divergence of the starting stress tensor $\partial_\mu \widehat{T}^{\mu\nu}$, and the divergence:

$$\frac{1}{2}\partial_\mu\partial_\alpha\left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu}\right),$$

but even the last term vanishes because $\partial_\mu\partial_\alpha$ is symmetric in the $\alpha \leftrightarrow \mu$ exchange whereas $\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu}$ is antisymmetric by construction. In a similar manner the term we add to the total angular momentum is:

$$-\widehat{\Phi}^{\lambda,\mu\nu} + \frac{1}{2}x^\mu\partial_\alpha\widehat{\phi}^{\alpha\lambda,\nu} - \frac{1}{2}x^\nu\widehat{\phi}^{\alpha\lambda,\mu} + \partial_\alpha\widehat{Z}^{\alpha\lambda,\mu\nu} =$$

$$= \frac{1}{2}\partial_\alpha\left[x^\mu\widehat{\phi}^{\alpha\lambda,\nu} - x^\nu\widehat{\phi}^{\alpha\lambda,\mu} + 2\widehat{Z}^{\alpha\lambda,\mu\nu}\right].$$

It is the divergence on the α index of of an antisymmetric term in $\alpha \leftrightarrow \lambda$ exchange, so it vanishes if we take the divergence on the λ index. The spatial integrals $\int_V \widehat{T}'^{0\mu}\mathrm{d}^3\mathbf{x}$ and $\int_V \widehat{\mathcal{J}}'^{0,\mu\nu}\mathrm{d}^3\mathbf{x}$ of the new tensors are invariant, thus yielding the same generators, if the following integrals of the superpotentials $\widehat{\Phi}$ and $\widehat{Z}$ vanish:

$$\int_V \partial_\alpha\left(\widehat{\Phi}^{\alpha,0\mu} - \widehat{\Phi}^{0,\alpha\mu} - \widehat{\Phi}^{\mu,\alpha 0}\right)\mathrm{d}^3\mathbf{x} = 0$$

$$\int_V \partial_\alpha\left[x^\mu\left(\widehat{\Phi}^{\alpha,0\nu} - \widehat{\Phi}^{0,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha 0}\right) - x^\nu\left(\widehat{\Phi}^{\alpha,0\mu} - \widehat{\Phi}^{0,\alpha\mu} - \widehat{\Phi}^{\mu,\alpha 0}\right) + 2\widehat{Z}^{\alpha 0,\mu\nu}\right]\mathrm{d}^3\mathbf{x} = 0,$$

because of symmetry (the temporal derivative is always vanishing since all the operators are antisymmetric in the α and 0 indices) and because of the divergence theorem in three dimension we can write the integrals as boundary conditions:

$$\int_{\partial V} \mathrm{d}S\, \hat{n}_i \left(\widehat{\Phi}^{i,0\mu} - \widehat{\Phi}^{0,i\mu} - \widehat{\Phi}^{\mu,i0} \right) = 0 \tag{1.4}$$

$$\int_{\partial V} \mathrm{d}S\, \hat{n}_i \left[x^\mu \left(\widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{\nu,i0} \right) - x^\nu \left(\widehat{\Phi}^{i,0\mu} - \widehat{\Phi}^{0,i\mu} - \widehat{\Phi}^{\mu,i0} \right) + 2\widehat{Z}^{\alpha 0,\mu\nu} \right] = 0.$$

These conditions are usually ensured taking the superpotential as a combination of the fields of the theory and enforcing boundary conditions for them; for instance, the familiar periodic boundary conditions for a box, or requiring some expression involving the field or its normal derivatives to vanish at the boundary. Usually the same conditions on the fields ensures both the vanishing of the boundary integrals for the suerpotentials and for the original tensors:

$$\int_{\partial V} \mathrm{d}S\, \hat{n}_i \widehat{T}^{i\mu} = 0 \qquad \int_{\partial V} \mathrm{d}S\, \hat{n}_i \widehat{J}^{i,\mu\nu} = 0. \tag{1.5}$$

It is convenient to remind that the last condition ensure the constancy of the total four momentum and angular momentum. The simplest case is to take fields (and their derivatives up to the relevant order[4]) that vanish at the boundary of the region occupied by the system, this is a covariant assumption and guarantees that in every inertial frame the tensors are constant and both $\widehat{P}^\mu$ and $\widehat{J}^{\mu\nu}$ transform as four tensors[5]. This is not always possible, for instance the MIT bag model which we will use in the next chapter. Many times the boundary conditions ensures the vanishing of the flux only on a particular inertial frame or class of inertial frames but non all of them. The generators are thus constant only on a class of frames and they do not change like four-tensors under boost and rotations. We will assume in such cases to perform calculations in the fixed frame for mathematical convenience.

In conclusion, a pseudo-gauge transformation is always possible, provided that

[4]We understand here that superpotentials and operators depend only on the fields and field derivatives, up to a certain order. So they are all vanishing provided fields and derivatives are.

[5]They transform like tensors under boost and rotations, through the Jacobian matrix. This does not mean they behave like tensor for general transformations, *e.g.* the total angular momentum operator changes its components under translations whilst an actual tensor only change the the application point, not the components.

suitable boundary conditions are ensured on the superpotential. In this case the couple $\{\widehat{T},\widehat{\mathcal{S}}\}$ and $\{\widehat{T}',\widehat{\mathcal{S}}'\}$ are regarded as equivalent in quantum field theory because they fulfill the same equations, give the same Poincaré algebra generators (*i.e.* total energy, momentum and angular momentum, in the operator sense) including the Hamiltonian and thus the time evolution of the system.

An important transformation of this class is the Belinfante symmetrization, where the superpotential is $\widehat{\Phi}^{\lambda,\mu\nu}$ is the spin tensor itself, and $\partial_\alpha \widehat{Z}^{\alpha\lambda,\mu\nu}$ is vanishing. After the transformation the resulting spin tensor $\widehat{\mathcal{S}}_{\mathrm{B.}}$ is vanishing:

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu}_{\mathrm{B.}} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\mathcal{S}}^{\lambda,\mu\nu} \equiv 0,$$

and the new stress-energy-momentum tensor $\widehat{T}^{\mu\nu}_{\mathrm{B.}}$ is symmetric in the $\mu \leftrightarrow \nu$ exchange because of four-momentum and angular momentum conservation:

$$\partial_\lambda \widehat{\mathcal{J}}^{\lambda,\mu\nu}_{\mathrm{B.}} = \partial_\lambda \left(x^\mu \widehat{T}^{\lambda\nu}_{\mathrm{B.}} - x^\nu \widehat{T}^{\lambda\mu}_{\mathrm{B.}} \right) = \widehat{T}^{\mu\nu}_{\mathrm{B.}} - \widehat{T}^{\nu\mu}_{\mathrm{B.}},$$

thus the name symmetrization.

This transformation has a particular significance in gravitation [2]. Einstein Field equation is a classical -that is non-quantum- equation that links geometry and matter. Geometry through the Einstein tensor, and matter through the -non quantum- stress-energy-momentum tensor.

Being quantum mechanics our best description of microscopic interaction it is usually assumed that the classical tensor is the average value of the corresponding quantum operator. The classical tensor, the average value of the operator, must be equal to the Einstein tensor. General relativity in its simplest formulation uses the Levi-Civita connection as the generalization of ordinary partial derivatives to curvilinear space. Levi-Civita connection only depends on the metric tensor, and it is both torsion free and compatible with the metric. These properties entails that the Einstein tensor is symmetric by construction, and so $T^{\mu\nu}$ has to be symmetric too.

Using the Belinfante procedure the resulting tensor, other than being symmetric, correspond to the Hilbert sress-energy-momentum tensor[6]:

[6]This tensor, unlike the canonical tensor, does not stem from space-time translation invariance and it is not trivial that gives the Poincaré algebra generators once integrated.

$$-\frac{2}{\sqrt{-g}}\frac{\partial(\mathcal{L}_{matter}\sqrt{-g})}{\partial g_{\mu\nu}},$$

where g is the determinant of the metric $g^{\mu\nu}$ and $\mathcal{L}_{matter}$ is the matter term in the Einstein-Hilbert action:

$$S_{\mathrm{E.H.}} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2k}R - \mathcal{L}_{matter}\right).$$

The Euler-Lagrange equation for the Einstein-Hilbert action is the Einstein field equation with the Hilbert stress-energy-momentum tensor.

In our work space-time curvature and gravitational coupling have been disregarded. For each tensors created with the transformation (1.3), it should be shown that an extension of general relativity exists having it as a source, this is possible for example with the canonical tensors, namely Einstein-Cartan theory [3], and Belinfante tensors (Einstein-Hilbert), but could not be always possible.

Chapter 2

Thermodynamical inequivalence: equilibrium

As seen in the previous chapter, within quantum field theory on a flat space-time, it is possible to generate apparently equivalent couples of stress-energy-momentum and spin tensors. In particular we can have different stress-tensors which are *e.g.* symmetric or non-symmetric. Indeed, gravitational coupling provides an unambiguous way of defining the stress-energy tensor; in General Relativity, it is symmetric by construction and the spin tensor vanishes. However, in a likely extension known as Einstein-Cartan theory (not excluded by present observations) the spin tensor is non-vanishing and the stress-energy tensor is non-symmetric. A non gravitational way to distinguish between different tensors would have major consequences in hydrodynamics, gravity and cosmology.

During this chapter we will see that, despite giving the same total four momentum and angular momentum, couples of microscopic tensors linked by a pseudo-gauge transformation (1.3) have different momentum and angular momentum densities[1] in general. Thermodynamics can be used therefore to prove if a particular couple of microscopic tensors is wrong, without the need to resort to gravitation.

In classical physics we have a stronger requirement with respect to a quantum theory: we would like the energy, momentum and angular momentum content of any arbitrary macroscopic spatial region to be well defined concepts; otherwise stated, we

[1] As reported in our work [5].

would like to have objective values for the energy, momentum and angular momentum densities. If these quantities are to be the components of the stress-energy-momentum and spin tensors, such a requirement strongly limits the freedom to change these tensors. It is important to stress how these requirements are for the classical tensors, the average values of the microscopic quantum underlying tensors.

2.1 The equivalence condition

The classical counterpart of a pseudo-gauge transformation can be easily calculated by applying the expectation value to both sides of the quantum transformation. This obviously leads to:

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\alpha \left(\Phi^{\alpha,\mu\nu} - \Phi^{\mu,\alpha\nu} - \Phi^{\nu,\alpha\mu}\right) \tag{2.1}$$

$$\mathcal{S}'^{\lambda,\mu\nu} = \mathcal{S}^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\alpha Z^{\alpha\lambda,\mu\nu},$$

so the new angular momentum $\mathcal{J}'$ is:

$$\begin{aligned} \mathcal{J}'^{\lambda,\mu\nu} &= \mathcal{J}^{\lambda,\mu\nu} + \\ &+\frac{1}{2}\partial_\alpha \left[x^\mu \left(\Phi^{\alpha,\lambda\nu} - \Phi^{\lambda,\alpha\nu} - \Phi^{\nu,\alpha\lambda}\right) - x^\nu \left(\Phi^{\alpha,\lambda\mu} - \Phi^{\lambda,\alpha\mu} - \Phi^{\mu,\alpha\lambda}\right) + 2Z^{\alpha\lambda,\mu\nu}\right]. \end{aligned} \tag{2.2}$$

For a macroscopic system, we would like the mean values of the stress-energy momentum $T^{\mu\nu}$ and angular momentum tensors $\mathcal{J}^{\lambda,\mu\nu}$ to be invariant under a pseudo-gauge transformation, and not just their integrals P^μ and $J^{\mu\nu}$. This is because, as previously mentioned, energy, momentum and total angular momentum densities classically must take on objective values, independent of the particular quantum tensors. Actually only four momentum and the spatial part of angular momentum are observable quantity, there is no clear physical way to measure the boost generator. A minimal requirement would be the invariance of the aforementioned densities, that is:

$$T'^{0\mu} = T^{0\mu} \qquad \mathcal{J}'^{0,ij} = \mathcal{J}^{0,ij},$$

however, this is a frame-dependent requirement; a Lorenzt-boosted frame would measure a different energy-momentum density if only the first row of the stress-energy tensor was invariant under pseudo-gauge transformations in one particular frame:

$$(T')^{0\mu}_{\mathsf{R}} = \Lambda^0_\alpha \Lambda^\nu_\beta T'^{\alpha\beta} = \Lambda^0_0 \Lambda^\nu_\beta T^{0\beta} + \Lambda^0_i \Lambda^\nu_\beta T'^{i\beta} \neq \Lambda^0_0 \Lambda^\nu_\beta T^{0\beta} + \Lambda^0_i \Lambda^\nu_\beta T^{i\beta} = (T)^{0\mu}_{\mathsf{R}},$$

where $(\cdots)^{0\mu}_{\mathsf{R}}$ means that the 0μ components are taken in the boosted reference frame $)_{\mathsf{R}}$, and Λ is the Jacobian matrix of the boost. In a similar manner the spatial components of the angular momentum density are not invariant in every frame of reference.

We are thus to enforce a stricter requirement, namely:

$$T'^{\mu\nu} = T^{\mu\nu}, \tag{2.3}$$

whilst for the rank three angular momentum tensor, we can make a looser request:

$$\mathcal{J}'^{\lambda,\mu\nu} = \mathcal{J}^{\lambda,\mu\nu} + \eta^{\lambda\mu} K^\nu - \eta^{\lambda\nu} K^\mu, \tag{2.4}$$

where K is a vector field. Being the Minkowski metric $\eta^{\mu\nu}$ diagonal in every inertial frame, if we limit ourselves to spatial indices $\mu, \nu = 1, 2, 3$, the above equation is enough to ensure that the angular momentum densities, with $\lambda = 0$, remain the same.

The procedure used to get the primed tensors impose an additional constrain on the auxiliary vector field K^μ. Being T' given by the average of the quantum tensor (2.1), equation (2.3) leads to:

$$\frac{1}{2}\partial_\alpha \left(\Phi^{\alpha,\mu\nu} - \Phi^{\mu,\alpha\nu} - \Phi^{\nu,\alpha\mu} \right) = 0, \tag{2.5}$$

while comparing equation (2.4) with equation (2.2), if we take the last formula into account, we obtain as a condition for the angular momentum:

$$
\begin{aligned}
&\eta^{\lambda\mu}K^{\nu} - \eta^{\lambda\nu}K^{\mu} = \qquad (2.6)\\
&= \frac{1}{2}\partial_\alpha\left[x^\mu\left(\Phi^{\alpha,\lambda\nu} - \Phi^{\lambda,\alpha\nu} - \Phi^{\nu,\alpha\lambda}\right) - x^\nu\left(\Phi^{\alpha,\lambda\mu} - \Phi^{\lambda,\alpha\mu} - \Phi^{\mu,\alpha\lambda}\right)\right] + \partial_\alpha Z^{\alpha\lambda,\mu\nu} =\\
&= \frac{1}{2}\left(\Phi^{\mu,\lambda\nu} - \Phi^{\lambda,\mu\nu} - \Phi^{\nu,\mu\lambda} - \Phi^{\nu,\lambda\mu} + \Phi^{\lambda,\nu\mu} + \Phi^{\mu,\nu\lambda}\right) + \partial_\alpha Z^{\alpha\lambda,\mu\nu} =\\
&= -\Phi^{\lambda,\mu\nu} + \partial_\alpha Z^{\alpha\lambda,\mu\nu}.
\end{aligned}
$$

Plugging this last result back into equation (2.5) one obtains:

$$\partial^\nu K^\mu - \eta^{\mu\nu}\left(\partial\cdot K\right) + \partial_\alpha\partial_\beta\left(Z^{\alpha\mu,\beta\nu} + Z^{\alpha\nu,\beta\mu}\right) = 0. \qquad (2.7)$$

If we take the divergence of the last equation, contracting with ∂_μ we get the trivial equation $0 = 0$, but contracting with ∂_ν we have:

$$\Box K^\mu - \partial^\nu\left(\partial\cdot K\right) = 0, \qquad (2.8)$$

where naturally $\Box = \partial_\mu\partial^\mu$ is the D'Alembert operator. Whilst if we instead take the trace of (2.7), we have:

$$3\left(\partial\cdot K\right) + 2\partial_\alpha\partial_\beta Z^{\alpha\mu,\beta}{}_{\mu} = 0. \qquad (2.9)$$

Equations (2.5), (2.6), (2.7), (2.8) and (2.9) define a set of non trivial conditions to be met for the average values of the superpotentials. However in order to prove that different tensors give different macroscopic results we can consider a subset of pseudo-gauge transformations, namely the ones with a vanishing[2] $\widehat{Z}^{\alpha\lambda,\mu\nu}$. This subset include the Belinfante transformation and, as we will see in this chapter, it is enough to prove the inequivalence. Being the terms on the auxiliary superpotential $\widehat{Z}$ vanishing, we have a simpler condition on the mean value of $\widehat{\Phi}$, as the (2.5) reads:

$$-\Phi^{\lambda,\mu\nu} = \eta^{\lambda\mu}K^\nu - \eta^{\lambda\nu}K^\mu,$$

[2]Actually only $\partial_\alpha Z^{\alpha\lambda,\mu\nu}$ have to be vanishing, but this depends on the state of the system, being Z the mean value of an operator. Taking $\partial_\alpha\widehat{Z}^{\alpha\mu\nu} = 0$, or directly a vanishing $\widehat{Z}$, ensures that the average value is vanishing too, even if it is not the most general case.

whislst (2.9) became:

$$(\partial \cdot K) = 0,$$

and thus (2.7) simply reads:

$$\partial^\nu K^\mu = 0. \tag{2.10}$$

Therefore the equations (2.5) and (2.6) imply that the vector field K is a constant field. Therefore, as long as we consider only transformations with a vanishing $\widehat{Z}$ term, in order to verify if the equivalence conditions (2.3) and (2.4) are fulfilled, it is sufficient to compute the mean value Φ and check if it is of the form:

$$\Phi^{\lambda,\mu\nu} = -\left(\eta^{\lambda\mu} K^\nu - \eta^{\lambda\nu} K^\mu\right), \tag{2.11}$$

with a constant K^μ, otherwise the equivalence condition is not fulfilled and inequivalence of the microscopic couples is proven.

It should be emphasized how the conditions on the superpotential, from the simpler condition for the restricted class (2.11) to the generic conditions from (2.5) and (2.9), do not need to apply to the quantum tensors $\widehat{\Phi}$ and $\widehat{Z}$, which only have to meet the boundary conditions (1.4) we have seen on the previous chapter. In fact, it may happen that the mean value Φ fufills equation (2.11), or more generally that the superpotentials fulfill (2.5) and (2.6), even though their quantum correspondents $\widehat{\Phi}$ and $\widehat{Z}$ do not. This can be possible because of specific features of the density operator $\widehat{\rho}$. In this case, the couples $\{\widehat{T}, \widehat{\mathcal{S}}\}$ and $\{\widehat{T}', \widehat{\mathcal{S}}'\}$ are to be considered equivalent only with regard to a particular density operator, that is, for a specific quantum state. We will see in the next two sections that the equivalence between couples of tensors crucially depends on the symmetry properties of the physical state $\widehat{\rho}$ (either mixed or pure). Particularly, we shall see that if $\widehat{\rho}$ is the usual thermodynamical equilibrium operator, proportional to $\exp\left(-\widehat{H}/T + \mu\widehat{Q}/T\right)$, any quantum tensors $\widehat{\Phi}$ and $\widehat{Z}$ will result in a mean value fulfilling equations (2.5) and (2.6). This means that all possible quantum microscopic stress-energy and spin tensors will yield the same physics in terms of macroscopically observable quantities.

2.2 Grand-canonical ensemble

In non-relativistic statistical mechanics the grand-canonical ensemble is a system at equilibrium with determined average energy and particle number, in contrast with a constant, non fluctuating, particle number (canonical and micro-canonical ensemble) and energy (micro-canonical). Particle number is not conserved at relativistic energies, because of pair production and annihilation, so the chemical potential refers to a conserved charge $\widehat{Q}$, like electric charge or baryonic number, instead of particle number. The thermodynamical equilibrium distribution (in the thermodynamical limit $V \to \infty$) of a quantum system is then:

$$\widehat{\rho} = \frac{1}{Z} \mathrm{e}^{-\widehat{H}/T + \mu \widehat{Q}/T}, \tag{2.12}$$

where $\widehat{H}$ is the Hamiltonian, and Z is the grand-canonical partition function:

$$Z = \mathrm{tr}\left(\mathrm{e}^{-\widehat{H}/T + \mu \widehat{Q}/T} \right).$$

The state of the system is remarkably symmetric. It is space-time translationally invariant, since both $\widehat{Q}$ and $\widehat{H}$ commute with translation operators $\widehat{\mathsf{T}}(a) = \exp[ia \cdot \widehat{P}]$. This entails that the mean value of any space-time dependent operator $\widehat{A}(x)$, including stress-energy and spin tensor, are independent of the space-time position:

$$\begin{aligned} \mathrm{tr}\left(\widehat{\rho} : \widehat{A}(x+a) : \right) &= \mathrm{tr}\left(\widehat{\rho} : \widehat{\mathsf{T}}(a) \widehat{A}(x) \widehat{\mathsf{T}}(a)^{-1} : \right) = \mathrm{tr}\left(\widehat{\rho}\, \widehat{\mathsf{T}}(a) : \widehat{A}(x) : \widehat{\mathsf{T}}(a)^{-1} \right) = \\ &= \mathrm{tr}\left(\widehat{\mathsf{T}}(a)^{-1} \widehat{\rho}\, \widehat{\mathsf{T}}(a) : \widehat{A}(x) : \right) = \mathrm{tr}\left(\widehat{\rho} : \widehat{A}(x) : \right), \end{aligned} \tag{2.13}$$

where the ciclicity of the trace and the transparency of the normal ordering with respect to translations have been used [3]. As a consequence, the average value of any

[3] Here a comment is in order. The transparency of the normal ordering with respect to a conjugation transformation, that is $: \mathsf{A} F(\Psi) \mathsf{A}^{-1} := \mathsf{A} : F(\Psi) : \mathsf{A}^{-1}$ where A is a translation or a Lorentz transformation and F a function of the fields and its derivatives, is guaranteed for free fields provided that the vacuum $|0\rangle$ is an eigenstate of the same transformation, which is always the case. For interacting fields, we will assume that the definition of normal ordering (for this problem, see e.g. ref. [6]) is such that transparency for conjugation holds; anyhow, for the examined case in Sect. 3 we will just need transparency for a free field.

space-time derivative vanishes, and so will do the divergences on the right hand side of equation (2.1). Therefore the classical, macroscopic stress-energy-momentum tensor, the the expectation value of the quantum operator, will be the same regardless of the particular microscopic quantum tensor used. For instance the canonical and the Belinfante symmetrized tensors will result in the same average, macroscopic tensor, despite having in general different algebraic properties in $\mu \leftrightarrow \nu$ exchange. In addition the mean value $\partial_\alpha Z^{\alpha\lambda,\mu\nu}$ is vanishing, therefore all couple of tensor give the same macroscopic in the gand-canonical ensembe if Φ fulfills (2.11), as we will see later.

Before that we can use the symmetry properties of the state of the system to obtain the components of $T^{\mu\nu}$. The density operator (2.12) manifestly enjoys rotational symmetry, for $\widehat{H}$ and $\widehat{Q}$ commute with rotation operators $\widehat{\mathsf{R}}_{\hat{\theta}}(\theta) = \exp\left(i\boldsymbol{\theta}\cdot\widehat{\mathbf{J}}\right)$[4]. This implies that most components of average tensors vanish. To show that, it is sufficient to choose suitable rotation operators and repeat the same reasoning as in equation (2.13). For instance, choosing the $\mathsf{R}_2(\pi)$ operator, i.e. the rotation of 180 degrees around the 2 (or y) axis, changing the sign of 1 (or x) and 3 (or z) components and leaving 2 and 0 unchanged, in the $x = (t, \mathbf{0})$ one has:

$$\begin{aligned}
T^{12}(x) &= \mathrm{tr}\left(\widehat{\rho}\,:\widehat{T}^{12}(x):\right) = \mathrm{tr}\left(\widehat{\mathsf{R}}_2(\pi)\widehat{\rho}\,\widehat{\mathsf{R}}_2(\pi)^{-1}:\widehat{T}^{12}(x):\right) = \\
&= \mathrm{tr}\left(\widehat{\rho}\,\widehat{\mathsf{R}}_2(\pi)^{-1}:\widehat{T}^{12}(x):\widehat{\mathsf{R}}_2(\pi)\right) = \mathrm{tr}\left(\widehat{\rho}\,\mathsf{R}_2(\pi)^1_\mu \mathsf{R}_2(\pi)^2_\nu :\widehat{T}^{\mu\nu}(\mathsf{R}_2(\pi)^{-1}(x)):\right) = \\
&= -\mathrm{tr}\left(\widehat{\rho}\,:\widehat{T}^{12}(\mathsf{R}_2(\pi)^{-1}(x)):\right) = -\mathrm{tr}\left(\widehat{\rho}\,:\widehat{T}^{12}(x):\right) = -T^{12}(x),
\end{aligned} \tag{2.14}$$

where, in the last equality, we have taken advantage of the point independence of all average values shown in eq. (2.13); thus, $T^{12}((t,\mathbf{0})) = 0$ and, in view of the translational invariance $T^{12}(x) = 0\ \forall x$. Similarly, by choosing other rotation operators, it can be shown that all off-diagonal elements of a rank two tensor vanish. The only non-vanishing components are the diagonal ones, which, again owing to the rotational symmetry (if we choose $\mathsf{R}_i(\pi/2)$ and repeat the above reasoning it follows immediately), are equal:

[4]Here $\widehat{\mathbf{J}}^i = \epsilon_{ijk}\widehat{J}^{jk}$, with ε_{ijk} the Levi-Civita symbol in three dimensions, and $\widehat{J}$ is the angular momentum operator.

$$T^{11}(x) = T^{22}(x) = T^{33}(x),$$

The component $T^{00}(x)$ can also be non-vanishing and its value is not related to the other diagonal ones. Altogether, the average stress-energy-momentum tensor can only have the diagonal (symmetric) form in the rest frame:

$$T^{\mu\nu} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} = (\varepsilon + p)\hat{t}^{\mu}\hat{t}^{\nu} - pg^{\mu\nu},$$

where $\hat{t}$ is the unit time vector with components $(1, \mathbf{0})$ and ε and p have the physical meaning of proper energy density and pressure. It should be stressed that, for a system at full thermodynamical equilibrium described by $\widehat{\rho}$ in equation (2.12) they would be the same regardless of the particular form of the quantum stress-energy tensor; *e.g.* for the free Dirac field the canonical tensor is not symmetric but the average value in the grand-canonical ensemble is nonetheless diagonal, and so symmetric, like the average value of any other possible microscopic tensor.

As far as the superpotential is concerned, it is easy to calculate, by using suitable rotations as in the previous case, that the only non-vanishing components are:

$$\Phi^{1,01}(x) = \Phi^{2,02}(x) = \Phi^{3,03}(x) = -\Phi^{1,10}(x) = -\Phi^{2,20}(x) = -\Phi^{3,30}(x).$$

Hence, being the average of any tensor point independent, one scalar function B independent of x, is sufficient to determine difference in angular momentumthat stems from the pseudo-gauge transformation, for a system at full thermodynamical equilibrium:

$$\Phi^{\lambda,\mu\nu} = B(\eta^{\lambda\nu}\hat{t}^{\mu} - \eta^{\lambda\mu}\hat{t}^{\nu})$$

This tensor has exactly the form for a "good" superpotential because using eq.s (2.1) and (2.2) the new primed tensors automatically fulfill the conditions (2.3) and (2.4). In conclusion, *any* possible pseudo-gauge transformation will yield the same energy, momentum and angular momentum density for all inertial frames and so, all quantum stress-energy-momentum and spin tensors are equivalent as far as the density operator (2.12) is concerned.

2.3 Finite angular momentum

In the previous section there was equivalence between any possible couple of microscopic tensors $\{\widehat{\mathcal{S}}, \widehat{T}\}$ and $\{\widehat{\mathcal{S}}', \widehat{T}'\}$ because of the symmetry properties of the density matrix (2.12) of the grand canonical ensemble. It is reasonable therefore to expect a remarkably different situation for a less symmetric state, *e.g.* a thermodynamical system having a macroscopic non-vanishing total angular momentum. In this case, in its rest frame (defined as the one in which the total momentum vanishes) the density operator reads [7, 9] :

$$\widehat{\rho} = \frac{1}{Z_\omega} \mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T + \mu\widehat{Q}/T} \tag{2.15}$$

where $\boldsymbol{\omega}$ has the physical meaning of a constant, fixed angular velocity around which the system rigidly rotates. The factor Z_ω is the *rotational* grand-canonical partition function:

$$Z_\omega = \mathrm{tr}\left(\mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T + \mu\widehat{Q}/T}\right) \tag{2.16}$$

The density operator (2.15) is much less symmetric than that in (2.12), it has a privileged direction $\hat{\omega}$, and this has remarkable and interesting consequences on the allowed transformations of stress-energy and spin tensor. The surviving symmetries in (2.15) are time-translations $\mathsf{T}(t)$ and translations along the $\boldsymbol{\omega}$ axis ($\hat{\omega}$ which we choose to label z axis, without any loss of generality) $\mathsf{T}(z)$, rotations around the $\boldsymbol{\omega}$ axis $\mathsf{R}_{\hat{\omega}}(\varphi)$ and reflection $\Pi_{\hat{\omega}}$ with respect to planes orthogonal to $\boldsymbol{\omega}$ (assuming for simplicity a parity-invariant hamiltonian $\widehat{H}$).

The density operator (2.15) can be obtained as a limiting macroscopic case of a quantum statistical system with finite volume and fixed angular momentum in its rest frame in an exact quantum sense, *i.e.* belonging to a specific representation of the rotation group SO(3) (or its universal covering group SU(2)). This point of view was presented and the calculations were carried out in ref. [10]. However, for a truly macroscopic system, it can be more easily obtained by extending to the relativistic case an argument used by Landau [7], or by maximizing the entropy with the constrain of fixed angular momentum [11]. It should be pointed out that $\widehat{H}, \widehat{Q}$ and the angular momentum operator along the $\boldsymbol{\omega}$ direction commute with each other, so that the exponential in (2.15) also factorizes.

The density operator (2.15) implies that, in its rest frame, the system is rigidly rotating with a velocity field $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$. The classical, non-relativistic derivation by Landau [7] shows this in a very simple fashion by assuming that the system is made of macroscopic cells. To show the same thing within a quantum formalism, implies a little more effort, which is nevertheless quite enlightening.

If we consider a vector field $\widehat{V}(x)$ and calculate its mean value at a point $x + a$ by using space-time translation operators, we have:

$$
\begin{aligned}
&\mathrm{tr}\left(\widehat{\rho} : \widehat{V}^\nu(x+a) :\right) = \mathrm{tr}\left(\widehat{\rho}\,\widehat{\mathsf{T}}(a) : \widehat{V}^\nu(x) : \widehat{\mathsf{T}}(a)^{-1}]\right) = \mathrm{tr}\left(\widehat{\mathsf{T}}(a)^{-1}\widehat{\rho}\,\widehat{\mathsf{T}}(a) : \widehat{V}^\nu(x) :\right) \\
&= \frac{1}{Z_\omega}\mathrm{tr}\left(\widehat{\mathsf{T}}(a)^{-1}\, \mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T + \mu\widehat{Q}/T}\, \widehat{\mathsf{T}}(a) : \widehat{V}^\nu(x) :\right) = \\
&= \frac{1}{Z_\omega}\mathrm{tr}\left(\mathrm{e}^{-\widehat{\mathsf{T}}(a)^{-1}\widehat{H}\widehat{\mathsf{T}}(a)/T + \boldsymbol{\omega}\cdot\widehat{\mathsf{T}}(a)^{-1}\widehat{\mathbf{J}}\widehat{\mathsf{T}}(a)/T + \mu\widehat{\mathsf{T}}(a)^{-1}\widehat{Q}\widehat{\mathsf{T}}(a)/T} : \widehat{V}^\nu(x) :\right) \\
&= \frac{1}{Z_\omega}\mathrm{tr}\left(\mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega}\cdot\widehat{\mathsf{T}}(a)^{-1}\widehat{\mathbf{J}}\widehat{\mathsf{T}}(a)/T + \mu\widehat{Q}/T} : \widehat{V}^\nu(x) :\right),
\end{aligned}
\tag{2.17}
$$

where known commutation relations $[\widehat{Q}, \widehat{P}^\mu] = 0$ and $[\widehat{H}, \widehat{P}^\mu] = 0$ have been used in the last step. Now the angular momentum is not translation invariant, from the theory of Poincaré group is known that:

$$
\widehat{\mathsf{T}}(a)^{-1}\widehat{\mathbf{J}}\widehat{\mathsf{T}}(a) = \widehat{\mathbf{J}} + \mathbf{a} \times \widehat{\mathbf{P}},
$$

so(2.17) reads:

$$
\begin{aligned}
\mathrm{tr}\left(\widehat{\rho} : \widehat{V}^\nu(x+a) :\right) &= \frac{1}{Z_\omega}\mathrm{tr}\left(\mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega}\cdot(\widehat{\mathbf{J}} + \mathbf{a}\times\widehat{\mathbf{P}})/T + \mu\widehat{Q}/T} : \widehat{V}^\nu(x) :\right) = \\
&= \frac{1}{Z_\omega}\mathrm{tr}\left(\mathrm{e}^{-\widehat{H}/T + (\boldsymbol{\omega}\times\mathbf{a})\cdot\widehat{\mathbf{P}}/T + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T + \mu\widehat{Q}/T} : \widehat{V}^\nu(x) :\right).
\end{aligned}
\tag{2.18}
$$

We can now use the four-temperature, a four vector defined as:

$$
\beta = \frac{1}{T}(1, \boldsymbol{\omega} \times \mathbf{a}),
$$

or equivalently:

$$\beta = \frac{1}{T_0} u = \frac{1}{T_0}(\gamma, \gamma \mathbf{v}), \tag{2.19}$$

where $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{a}$, $\gamma = 1/\sqrt{1-v^2}$ and $T_0 = \gamma T$. The vector $\mathbf{v}$ is manifestly the tangential velocity field of a rigid rotation, while T_0 is the inverse modulus of β, i.e. the comoving temperature which differs by the constant uniform T by a γ factor [12, 13]. The mean value of $V^\nu(x+a)$ in eq. (2.18) becomes:

$$\mathrm{tr}\left(\widehat{\rho} : \widehat{V}^\nu(x+a) :\right) = \frac{1}{Z_\omega} \mathrm{tr}\left(\mathrm{e}^{-\beta(a)\cdot \widehat{P} + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T + \mu \widehat{Q}/T} : \widehat{V}^\nu(x) :\right) \tag{2.20}$$

As long as β is timelike, that means provided that $v < 1$, it is possible to find a Lorentz transformation Λ such that:

$$T_0 \beta_\mu = u_\mu = \Lambda_{0\mu} = g_{0\lambda} \Lambda^\lambda_{\ \mu}. \tag{2.21}$$

The most convenient choice is the pure Lorentz boost along the $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{a}$ direction. Being $\hat{v}$, the direction of the velocity $\boldsymbol{\omega} \times \mathbf{a}$, ortogonal to $\hat{\omega}$, the boost leaves the product $\widehat{\mathbf{J}} \cdot \boldsymbol{\omega}$ invariant:

$$\Lambda = \exp[-i \operatorname{arccosh}(\gamma) \hat{\mathbf{v}} \cdot \mathsf{K}],$$

where K_i $(i = 1, 2, 3)$ are the generators of pure Lorentz boosts. Thereby, the trace on the right hand side of the eq. (2.20) can be rewritten:

$$\begin{aligned}
&\mathrm{tr}\left(\mathrm{e}^{-\Lambda_{0\mu}\widehat{P}^\mu/T_0 + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T + \mu\widehat{Q}/T} : \widehat{V}^\nu(x) :\right) = \mathrm{tr}\left(\mathrm{e}^{-\widehat{\Lambda}^{-1}\widehat{P}_0\widehat{\Lambda}/T_0 + \boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T + \mu\widehat{Q}/T} : \widehat{V}(x) :\right) = \\
&= \mathrm{tr}\left(\mathrm{e}^{-\widehat{\Lambda}^{-1}(\widehat{P}^0/T_0 + \gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0 + \gamma\mu\widehat{Q}/T_0)\widehat{\Lambda}} : \widehat{V}^\nu(x) :\right) = \\
&= \mathrm{tr}\left(\widehat{\Lambda}^{-1}\mathrm{e}^{-\widehat{P}^0/T_0 + \gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0 + \gamma\mu\widehat{Q}/T_0}\widehat{\Lambda} : \widehat{V}^\nu(x) :\right) = \\
&= \mathrm{tr}\left(\mathrm{e}^{-\widehat{P}^0/T_0 + \gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0 + \gamma\mu\widehat{Q}/T_0}\widehat{\Lambda} : \widehat{V}(x) : \widehat{\Lambda}^{-1}\right) = \\
&= (\Lambda^{-1})^\nu_\mu \mathrm{tr}\left(\mathrm{e}^{-\widehat{P}^0/T_0 + \gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0 + \gamma\mu\widehat{Q}/T_0} : \widehat{V}^\mu(\Lambda(x)) :\right)
\end{aligned} \tag{2.22}$$

Finally, from (2.20) and (2.22) we get:

$$\mathrm{tr}\left(\widehat{\rho} : \widehat{V}^\nu(x+a) :\right) = \frac{1}{Z_\omega} (\Lambda^{-1})^\nu_\mu \mathrm{tr}\left(\mathrm{e}^{-\widehat{P}^0/T_0 + \gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0 + \gamma\mu\widehat{Q}/T_0} : \widehat{V}^\mu(\Lambda(x)) :\right), \tag{2.23}$$

which tells us how to calculate the mean value of a vector field at any space-time point given its value in some other specific point. The most interesting feature of eq. (2.23) is that the density operator on the right hand side the same as $\widehat{\rho}$ on the left hand side with the replacement:

$$T \to T_0 = \gamma(a)T \qquad \boldsymbol{\omega} \to \gamma(a)\boldsymbol{\omega} \qquad \mu \to \gamma(a)\mu \tag{2.24}$$

If we choose $x = (0, \mathbf{0})$, the origin of Minkowski coordinates, and a purely spatial $a = (0, \mathbf{a})$ the eq. (2.23) implies:

$$\mathrm{tr}\left(\widehat{\rho} : \widehat{V}^\nu(0, \mathbf{a}) :\right) = \frac{1}{Z_\omega}(\Lambda^{-1})^\nu_\mu \mathrm{tr}\left(\mathrm{e}^{-\widehat{P}^0/T_0 + \gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0 + \gamma\mu\widehat{Q}/T_0} : \widehat{V}^\mu(0, \mathbf{0}) :\right). \tag{2.25}$$

That is the average of the vector field at any space-time point (it should be kept in mind that $\widehat{\rho}$ is invariant by time translation and so any mean value is stationary) is completely determined by the mean value at the origin of the coordinates, with the same density operator, modulo the replacement of thermodynamical parameters in (2.24). This particular value is strongly constrained by the symmetries of $\widehat{\rho}$. We already identified the $\boldsymbol{\omega}$ direction as that of the z (or 3) axis (see fig. 2.1) and consider the reflection Π_z with respect to $z = 0$ plane and the rotation $\mathsf{R}_3(\pi)$ of an angle π around the z axis; by repeating the same reasoning as for eq. (2.14) for $V^\nu(0)$ we can easily conclude that the time component $V^0(0)$ is the only one having a non-vanishing mean value. Note, though, that the mean value on the right-hand side of (2.25) depends on the distance r from the axis because the density operator is modified by the replacement of the uniform temperature T with a radius-dependent $T_0 = \gamma T$. Therefore, according to eq. (2.25) and using (2.21), the mean value of the vector field can be written:

$$\begin{aligned} V_\nu(x) &= \mathrm{tr}\left(\widehat{\rho} : \widehat{V}_\nu(x) :\right) = \\ &= \frac{1}{Z_\omega}(\Lambda^{-1})_{\nu 0} \mathrm{tr}\left(\mathrm{e}^{-\widehat{P}^0/T_0(r) + \gamma(r)\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0(r) + \gamma(r)\mu\widehat{Q}/T_0(r)} : \widehat{V}^0(0) :\right) \equiv \Lambda_{0\nu} V(r) = \\ &= V(r) u_\nu. \end{aligned} \tag{2.26}$$

This means it has to be collinear with the four-velocity field $u = (\gamma, \gamma\mathbf{v})$ in eq. (2.19) and, therefore, its field lines are circles centered on the z axis and orthogonal to it.

Similarly, we can obtain the general form of tensor fields of various rank and specific symmetry properties as a function of the basic four-velocity field.

However, the previous derivation relies on the fact that the system is infinitely extended in space. Indeed, at a distance from the axis such that $|\boldsymbol{\omega} \times \mathbf{x}| > 1$ the velocity surpass the speed of light in the vacuum and the system has a singularity. We cannot, therefore, take the strict thermodynamical limit $V \to \infty$ for a system with macroscopic angular momentum. Instead, we have to enforce a spatial cut-off at some distance and figure out how this reflects on the most general forms of vector and tensor fields.

Enforcing a bounded region V for a thermodynamical system implies the replacement of all traces over the full set of states with a trace over a complete set of states $|h_V\rangle$ of the fields for this region V, that we indicate with a subscript V:

$$\mathrm{tr} \to \mathrm{tr}_V = \sum_{h_V} \langle h_V | \ldots | h_V \rangle.$$

The density operator $\widehat{\rho}$ is still the same as in (2.15), with the difference that the partition function is now obtained by tracing over the localized state. It may be convenient to introduce the projection operator:

$$\mathsf{P}_V = \sum_{h_V} |h_V\rangle\langle h_V|$$

which allows us to maintain the trace over the full set of states, provided that we replace the density matrix $\widehat{\rho}$ with $\mathsf{P}_V \widehat{\rho}$; indeed for a generic operator $\widehat{A}$ we have:

$$\mathrm{tr}_V \left(\widehat{\rho} \widehat{A} \right) = \mathrm{tr} \left(\mathsf{P}_V \widehat{\rho} \widehat{A} \right),$$

which amounts to state that the effective density operator is now $\widehat{\rho}_V$:

$$\widehat{\rho}_V = \frac{1}{Z_\omega} \mathsf{P}_V \mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega} \cdot \widehat{\mathbf{J}}/T + \mu \widehat{Q}/T} \tag{2.27}$$

where now, as the *rotational* partition function, we have:

$$Z_\omega = \mathrm{tr} \left(\mathsf{P}_V \mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega} \cdot \widehat{\mathbf{J}}/T + \mu \widehat{Q}/T)} \right) = \mathrm{tr}_V \left(\mathrm{e}^{-\widehat{H}/T + \boldsymbol{\omega} \cdot \widehat{\mathbf{J}}/T + \widehat{Q}/T} \right).$$

In order to maintain the same symmetry of the density operator in (2.15), P_V has to commute with $\widehat{J}_z$, $\widehat{H}$, $\widehat{P}_z$, the Lorentz boost along z $\widehat{\mathsf{K}}_z$ and the reflection operator with respect to any plane parallel to $z = 0$, $\widehat{\Pi}_z$ (see fig. 2.1). These requirements are met if the region V is a static longitudinally indefinite cylinder with finite radius R and axis $\hat{\omega}$, and we will henceforth take this assumption.

There are two important consequences of having a finite radius R. As first, because of the presence of the projector P_V, the previous derivation which led us to express the vector field according to the simple formula (2.26) cannot be carried over to the case of finite (though macroscopic) radius. The reason is that P_V does not commute with the Lorentz boost along $\mathbf{v}$ or, otherwise stated, a Lorentz boost along a direction other than z will not transform the set of states $|h_V\rangle$ into themselves, as needed for completeness. So, one the one of the crucial steps in eq. (2.22) no longer holds and specifically:

$$\begin{aligned}\mathrm{tr}\left(\mathsf{P}_V\widehat{\Lambda}^{-1}\mathrm{e}^{-\widehat{P}^0/T_0+\gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0+\gamma\mu\widehat{Q}/T_0}\widehat{\Lambda} :\widehat{V}^\nu(x):\right) \neq \\ \neq \mathrm{tr}\left(\mathsf{P}_V\mathrm{e}^{-\widehat{P}^0/T_0+\gamma\boldsymbol{\omega}\cdot\widehat{\mathbf{J}}/T_0+\gamma\mu\widehat{Q}/T_0}\widehat{\Lambda} :\widehat{V}^\nu(x): \widehat{\Lambda}^{-1}\right).\end{aligned}$$

As a consequence, general vector and tensor fields will be more complicated than in the unphysical infinite radius case and get additional components. The most general expressions of mean value of fields in the cases of interest for the stress-energy and spin tensor will be systematically determined in the next section. The second consequence is that boundary conditions for the quantum fields must be specified at a finite radius value R, but we will see that those conditions alone cannot ensure the validity of of the equivalence conditions, which are local conditions.

2.4 Tensor fields in an axisymmetric system

In this section we will write down the most general forms of vector and tensor fields in an axisymmetric system, i.e. a system with the same symmetry features of the thermodynamical rotating system at equilibrium studied in the previous section. The goal of this section is to establish the conditions, if any, to be fulfilled by the super-

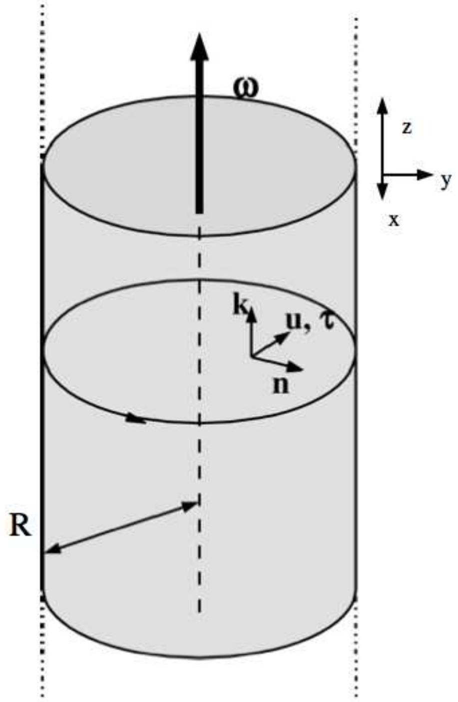

Figure 2.1: Rotating cylinder with finite radius R at temperature T. Also shown the inertial frame axes and the spatial parts of the vectors of tetrad (2.28).

potential to generate a good transformation of the stress-energy-momentum and spin tensors.

2.4.1 Vector field

The decomposition of a vector field will serve as a paradigm for more complicated cases. The idea is to take a suitable tetrad of space-time dependent orthonormal four-vectors and decompose the vector field onto this basis. The tetrad we choose is dictated by the cylindrical symmetry:

$$u = (\gamma, \gamma \mathbf{v}) \qquad \tau = (\gamma v, \gamma \hat{\mathbf{v}}) \qquad n = (0, \hat{\mathbf{r}}) \qquad k = (0, \hat{\mathbf{k}}), \tag{2.28}$$

where $\hat{\mathbf{r}}$ is the radial versor in cylindrical coordinates, while $\hat{\mathbf{k}}$ is the versor of the z axis, that is the axis of the cylinder (see fig. 2.1).

Due to symmetry for reflections with respect to $z = const$ planes, the most general vector field V has vanishing component on k, and therefore:

$$V = A(r)u + B(r)\tau + C(r)n, \tag{2.29}$$

where A, B, C are scalar functions which can only depend on the radial coordinate r, owing to the cylindrical symmetry. Note the presence of two additional components with respect to the infinitely extended cylinder case in eq. (2.26). For symmetry reasons the only surviving component of the field at the axis is the time component, so $B(0) = C(0) = 0$.

If the field is divergence-free, then $C(r) \equiv 0$.

2.4.2 Rank 2 antisymmetric tensor field

Any antisymmetric tensor field of rank 2 can be decomposed first as:

$$A^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} X_\rho u_\sigma + Y^\mu u^\nu - Y^\nu u^\mu,$$

where:

$$X^\rho = -\frac{1}{2}\epsilon^{\rho\alpha\beta\gamma} A_{\alpha\beta} u_\gamma \qquad Y^\rho = A^{\rho\alpha} u_\alpha,$$

and, thus, X and Y are two space-like vector fields such that $X \cdot u = Y \cdot u = 0$. Because of the reflection symmetry with respect to $z = \mathit{const}$ planes, one has $A_{xz} = A_{yz} = A_{0z} = 0$ and this in turn entails that, being $u_z = 0$, the only non-vanishing component of the pseudo-vector X is along k. Conversely, Y is a polar vector and it has components along τ and n which must vanish in $r = 0$. Altogether:

$$A^{\mu\nu} = D(r)\epsilon^{\mu\nu\rho\sigma} k_\rho u_\sigma + E(r)(\tau^\mu u^\nu - \tau^\nu u^\mu) + F(r)(n^\mu u^\nu - n^\nu u^\mu), \tag{2.30}$$

with $E(0) = F(0) = 0$. Since:

$$\epsilon^{\mu\nu\rho\sigma} k_\rho u_\sigma = n^\mu \tau^\nu - n^\nu \tau^\mu,$$

(which can be easily checked), the expression (2.30) can be rewritten as:

$$A^{\mu\nu} = D(r)(n^\mu \tau^\nu - n^\nu \tau^\mu) + E(r)(\tau^\mu u^\nu - \tau^\nu u^\mu) + F(r)(n^\mu u^\nu - n^\nu u^\mu). \tag{2.31}$$

2.4.3 Rank 2 symmetric tensor field

For the symmetric tensor $S^{\mu\nu}$ we will employ an iteration method in order to write down the most general decomposition. First, we project the tensor onto the u field:

$$S^{\mu\nu} = G(r)u^\mu u^\nu + q^\mu u^\nu + q^\nu u^\mu + \Theta^{\mu\nu},$$

where $q \cdot u = 0$ and $\Theta^{\mu\nu} u_\nu = 0$. Then, we decompose the space-like polar vector field q according to (2.29):

$$q = H(r)\tau + I(r)n,$$

with $H(0) = I(0) = 0$, and we project the tensor Θ in turn onto the vector field τ:

$$S^{\mu\nu} = G(r)u^\mu u^\nu + H(r)(\tau^\mu u^\nu + \tau^\nu u^\mu) + I(r)(n^\mu u^\nu + n^\nu u^\mu) + J(r)\tau^\mu \tau^\nu + h^\mu \tau^\nu + h^\nu \tau^\nu + \Xi^{\mu\nu},$$

being $h \cdot u = h \cdot \tau = 0$ (whence $h = K(r)n$ with $K(0) = 0$) and $\Xi^{\mu\nu}\tau_\nu = \Xi^{\mu\nu}u_\nu = 0$. This procedure can be iterated projecting Ξ onto n and the thus-obtained new symmetric tensor onto k. Thereby, we get:

$$\begin{aligned} S^{\mu\nu} &= G(r)u^\mu u^\nu + H(r)(\tau^\mu u^\nu + \tau^\nu u^\mu) + I(r)(n^\mu u^\nu + n^\nu u^\mu) \\ &+ J(r)\tau^\mu \tau^\nu + K(r)(n^\mu \tau^\nu + n^\nu \tau^\mu) + L(r)n^\mu n^\nu + M(r)k^\mu k^\nu. \end{aligned} \tag{2.32}$$

However, since $u^\mu u^\nu - \tau^\mu \tau^\nu - n^\mu n^\nu - k^\mu k^\nu = g^{\mu\nu}$ the last term can be replaced with a linear combination of all other diagonal terms plus a term in $g^{\mu\nu}$ and the most general symmetric tensor can be rewritten, after a suitable redefinition of the scalar coefficients, as:

$$
\begin{aligned}
S^{\mu\nu} &= G(r)u^\mu u^\nu + H(r)(\tau^\mu u^\nu + \tau^\nu u^\mu) + I(r)(n^\mu u^\nu + n^\nu u^\mu) \\
&+J(r)\tau^\mu\tau^\nu + K(r)(n^\mu\tau^\nu + n^\nu\tau^\mu) + L(r)n^\mu n^\nu - M(r)g^{\mu\nu}
\end{aligned}
$$

where $H(0) = I(0) = K(0) = 0$.

2.4.4 Rank 3 spin-like tensor field

The decomposition of a rank 3 tensor is carried out in an iterative way, similarly to what we have just done for the rank 2 symmetric tensor. First, we project the tensor onto the vector u and, taking the antisymmetry of $\mu\nu$ indices into account, one obtains:

$$
\Phi^{\lambda,\mu\nu} = u^\lambda(f^\mu u^\nu - f^\nu u^\mu) + u^\lambda\Gamma^{\mu\nu} + \Sigma^{\lambda\mu}u^\nu - \Sigma^{\lambda\nu}u^\mu + \Upsilon^{\lambda,\mu\nu} \tag{2.33}
$$

where all vector and tensor fields have vanishing contractions with u for any index. Particularly, using the general expressions (2.29) and (2.31), the vector field f and the antisymmetric tensor Γ read:

$$
f = E(r)\tau^\mu + F(r)n^\mu \qquad\qquad \Gamma^{\mu\nu} = D(r)(n^\mu\tau^\nu - n^\nu\tau^\mu) \tag{2.34}
$$

with $E(0) = F(0) = 0$. The tensor Σ can be decomposed as the sum of a symmetric and an antisymmetric part; having vanishing contractions with u, according to eqs. (2.31) and (2.32), it can be written as:

$$
\Sigma^{\lambda\mu} = N(r)(n^\lambda\tau^\mu - n^\mu\tau^\lambda) + P(r)\tau^\lambda\tau^\mu + Q(r)(n^\lambda\tau^\mu + n^\mu\tau^\lambda) + R(r)n^\lambda n^\mu + S(r)k^\lambda k^\mu \tag{2.35}
$$

wiht $Q(0) = 0$. The tensor Υ is projected in turn onto n and the above procedure is iterated. Then, similarly to eq. (2.33):

$$
\Upsilon^{\lambda,\mu\nu} = \chi^{\mu\nu}n^\lambda + (h^\mu n^\nu - h^\nu n^\mu)n^\lambda + \Theta^{\lambda\mu}n^\nu - \Theta^{\lambda\nu}n^\mu + \Lambda^{\lambda,\mu\nu} \tag{2.36}
$$

where all tensors have vanishing contractions with u and n. The antisymmetric tensor χ must be orthogonal to u and n and, therefore, according to eq. (2.31), vanishes. On the other hand, the vector field h can only have non-vanishing component on τ and so $h = T(r)\tau$ with $T(0) = 0$. Finally, the tensor Θ must be orthogonal to n, besides u, hence, using eqs. (2.31) and(2.32), can only be of the form:

$$\Theta^{\lambda\mu} = U(r)\tau^\lambda\tau^\mu + V(r)k^\lambda k^\mu \tag{2.37}$$

Likewise, the tensor Λ can be decomposed onto τ and, because of vanishing contractions with u and n, it can be written as:

$$\Lambda^{\lambda,\mu\nu} = W(r)k^\lambda(k^\mu\tau^\nu - k^\nu\tau^\mu) \tag{2.38}$$

Putting together eqs. (2.33), (2.34), (2.35), (2.36), (4.11) and (2.38), the general decomposition of a rank 3 tensor with antisymmetric $\mu\nu$ indices is obtained:

$$\begin{aligned}\Phi^{\lambda,\mu\nu} &= D(r)(n^\mu\tau^\nu - n^\nu\tau^\mu)u^\lambda + E(r)(\tau^\mu u^\nu - \tau^\nu u^\mu)u^\lambda + F(r)(n^\mu u^\nu - n^\nu u^\mu)u^\lambda + \\ &+N(r)(n^\lambda\tau^\mu - n^\mu\tau^\lambda)u^\nu - N(r)(n^\lambda\tau^\nu - n^\nu\tau^\lambda)u^\mu + P(r)\tau^\lambda(\tau^\mu u^\nu - \tau^\nu u^\mu) + \\ &+Q(r)(n^\lambda\tau^\mu + n^\mu\tau^\lambda)u^\nu - Q(r)(n^\lambda\tau^\nu + n^\nu\tau^\lambda)u^\mu + R(r)n^\lambda(n^\mu u^\nu - n^\nu u^\mu) + \\ &+S(r)k^\lambda(k^\mu u^\nu - k^\nu u^\mu) + T(r)(\tau^\mu n^\nu - \tau^\nu n^\mu)n^\lambda + U(r)\tau^\lambda(\tau^\mu n^\nu - \tau^\nu n^\mu) + \\ &+V(r)k^\lambda(k^\mu n^\nu - k^\nu n^\mu) + W(r)k^\lambda(k^\mu\tau^\nu - k^\nu\tau^\mu)\end{aligned} \tag{2.39}$$

with $E(0) = F(0) = Q(0) = T(0) = 0$.

We are now in a position to find out the conditions to be fulfilled by the superpotential Φ to be a good transformation of the stress-energy and spin tensors in a thermodynamically equilibrated system with angular momentum, as derived in Sect. 2.1.

Let us start from eq. (2.6), which is the most constraining. Since:

$$u^\lambda u^\mu - \tau^\lambda\tau^\mu - n^\lambda n^\mu - k^\lambda k^\mu = g^{\lambda\mu},$$

we can write a rank 3 tensor (2.39) in the form of eq. (2.11) as long as:

$$
\begin{aligned}
V(r) &= U(r) = F(r) \\
P(r) &= R(r) = S(r) \\
E(r) &= W(r) = -T(r) \\
D(r) &= N(r) = Q(r) = 0
\end{aligned} \tag{2.40}
$$

which are definitely non-trivial conditions. If these are fulfilled, then the superpotential (2.39) reduces to:

$$
\Phi^{\lambda,\mu\nu} = (F(r)n^\mu + E(r)\tau^\mu + P(r)u^\mu)g^{\lambda\nu} - (F(r)n^\nu + E(r)\tau^\nu + P(r)u^\nu)g^{\lambda\mu} \equiv K^\mu g^{\lambda\nu} - K^\nu g^{\lambda\mu},
$$

Now, the field $K^\mu = F(r)n^\mu + E(r)\tau^\mu + P(r)u^\mu$ ought to be a constant one, according to eq. (2.10). Since its divergence vanishes, then $F(r) = 0$ and, by using the definitions (2.28), we readily obtain the conditions:

$$
F(r) = 0 \qquad P(r)/\gamma = \text{const} \qquad E(r) = -P(r)\omega r \tag{2.41}
$$

In conclusion, only if a quantum superpotential is such that its mean value, calculated with the density operator (2.27), fulfills conditions (2.40) and (2.41), are the corresponding transformations (2.1) and (2.2) possible. Otherwise, the original and transformed stress-energy and spin tensors are inequivalent because they imply different values of mean energy, momentum or angular momentum densities. Since the most general form of the mean value of the superpotential, i.e. eq. (2.39) is highly non-trivial, the inequivalence will occur far more often than equivalence. To demonstrate this, in the next chapter we will consider a specific instance involving the most familiar quantum field endowed with a spin tensor.

Chapter 3

An example: the free Dirac field

To prove the inequivalence we had to find a significant case where the equivalence condition is not fulfilled, therefore we studied the simplest quantum field theory endowed with a spin tensor, namely the free Dirac field, and we compared the canonical tensors with the widely used Belinfante symmetrized couple.

From the Lagrangian density of the free Dirac field:

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\Psi - m\overline{\Psi}\Psi \tag{3.1}$$

we obtain, by means of the Noether theorem, the canonical stress-energy and spin tensors [14]:

$$\begin{aligned} \widehat{T}^{\mu\nu} &= \frac{i}{2}\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\Psi \\ \widehat{\mathcal{S}}^{\lambda,\mu\nu} &= \frac{1}{2}\overline{\Psi}\{\gamma^{\lambda},\Sigma^{\mu\nu}\}\Psi = \frac{i}{8}\overline{\Psi}\{\gamma^{\lambda},[\gamma^{\mu},\gamma^{\nu}]\}\Psi \end{aligned} \tag{3.2}$$

where:

$$\Sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma_k/2 & 0 \\ 0 & \sigma_k/2 \end{pmatrix}$$

and σ_k are Pauli matrices. Dirac motion equations ensure the angular momentum conservation and thus the spin tensor obeys:

$$\partial_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu} = \widehat{T}^{\nu\mu} - \widehat{T}^{\mu\nu} = \frac{i}{2}\overline{\Psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi - \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi$$

The couple of quantum tensors in (3.2) can be changed through the psuedo-gauge transformation. Accordingly we remind that taking $\widehat{\Phi} = \widehat{\mathcal{S}}$, namely the superpotential as the original spin tensor itself, a symmetrized stress-energy tensor and a vanishing spin tensor are obtained:

$$\begin{aligned} \widehat{T}'^{\mu\nu} &= \frac{i}{4}\left[\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi + \overline{\Psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi\right] \\ \widehat{\mathcal{S}}'^{\lambda,\mu\nu} &= 0 \end{aligned} \tag{3.3}$$

This transformation is well known as Belinfante's symmetrization procedure.

One may wonder whether these tensors, fulfilling continuity equations, still exist in a bounded region breaking the global translational and Lorentz symmetry, such as our cylinder with finite radius; or if, because of the boundary, they get additional terms with respect to the usual form. The problem of Dirac field with boundary has been tackled and solved by the authors of the MIT bag model [15]. First of all, it should be pointed out that the continuity equations (1.1) certainly apply to tensors (3.2) and (3.3) on-shell, i.e. for fields obeying the free Dirac equation within the cylinder. Furthermore, it is possible to find suitable boundary conditions, discussed in the next subsection, such that the fluxes (1.5) vanish, as needed, without introducing an ad-hoc discontinuity in the Dirac field. Thereby, the stress-energy and spin tensors retain the same form as in the usual no-boundary case and the integrals over the bounded region of the time components have the same physical meaning of conserved generators.

The spin tensor in eq. (3.2) has a remarkable feature which makes it easier to check the equivalence of the two couples in eq. (3.2) and (3.3): because of the special properties of gamma matrices, the spin tensor is also antisymmetric in the first two indices:

$$\widehat{\mathcal{S}}^{\lambda,\mu\nu} = -\widehat{\mathcal{S}}^{\mu,\lambda\nu} \tag{3.4}$$

and thus the mean value of this tensor is greatly simplified. The antisymmetry in the indices (λ,μ) dictates that all coefficients of symmetric $\lambda\mu$ terms of the general form of this kind of tensor found in eq. (2.39) vanish:

$$E(r) = F(r) = P(r) = Q(r) = R(r) = S(r) = T(r) = U(r) = V(r) = W(r) = 0$$

and that $D(r) = N(r)$, so that $\mathcal{S}$ is simply given by:

$$\mathcal{S}^{\lambda,\mu\nu} = D(r)[(n^\mu \tau^\nu - n^\nu \tau^\mu)u^\lambda + (n^\lambda \tau^\mu - n^\mu \tau^\lambda)u^\nu - (n^\lambda \tau^\nu - n^\nu \tau^\lambda)u^\mu] \tag{3.5}$$

and it is described by just one unknown radial function $D(r)$. Therefore, according to the conditions (2.40), the Belinfante tensors (3.3) are equivalent to the canonical ones (3.2) only if $D(r) = 0$, i.e. only if the spin tensor has a vanishing mean value.

For $\lambda = 0$, eq. (3.5) reads:

$$\mathcal{S}^{0,\mu\nu} = D(r)[(n^\mu \tau^\nu - n^\nu \tau^\mu)u^0 - \tau^0(n^\mu u^\nu - n^\nu u^\mu)]$$

and, because of the antisymmetry, the only non-vanishing components are those with both μ and ν equal to 1,2,3, indices that we denote with i, j. We can then write, using (2.28):

$$\begin{aligned} \mathcal{S}^{0,ij} &= D(r)[(n^i \tau^j - n^j \tau^i)u^0 - \tau^0(n^i u^j - n^j u^i)] = \\ &= D(r)[\gamma^2(n^i \hat{v}^j - n^j \hat{v}^i) - \gamma^2 v^2 (n^i \hat{v}^j - n^j \hat{v}^i) = \\ &= D(r)(n^i \hat{v}^j - n^j \hat{v}^i) = D(r)\epsilon_{ijk}\hat{k}^k \end{aligned} \tag{3.6}$$

Therefore, as expected, the time part of the spin tensor, contributing to the angular momentum density, is equivalent to a pseudo-vector field $\mathbf{D}(r)$ directed along z axis.

According to eq. (2.1), the variation of energy-momentum density reads:

$$\frac{1}{2}\partial_\alpha \left(\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}\right) = \frac{1}{2}\partial_\alpha \left(\mathcal{S}^{\alpha,0\nu} - \mathcal{S}^{0,\alpha\nu} - \mathcal{S}^{\nu,\alpha 0}\right) = -\frac{1}{2}\partial_\alpha \mathcal{S}^{0,\alpha\nu} \tag{3.7}$$

which implies at once that the energy density is unchanged because $^{0,\alpha 0} = 0$ in view of (3.4), whereas the momentum density varies by a derivative. Using (3.6):

$$\begin{aligned} T^{0i}_{\text{Belinfante}} &= T^{0i}_{\text{canonical}} - \frac{1}{2}\partial_\alpha \mathcal{S}^{0,\alpha i} = T^{0i}_{\text{canonical}} - \frac{1}{2}\partial_\alpha \epsilon_{\alpha i k} D(r)\hat{k}^k \\ &= T^{0i}_{\text{canonical}} + \frac{1}{2}(\text{rot}\mathbf{D})^i = T^{0i}_{\text{canonical}} - \frac{1}{2}\frac{\mathrm{d}D(r)}{\mathrm{d}r}\hat{v}^i \end{aligned} \tag{3.8}$$

Note that this last equation implies that the mean value of the canonical stress-energy-momentum tensor of the Dirac field has a non-trivial antisymmetric part if $D'(r) \neq 0$ as, according to local angular momentum conservation:

$$\partial_\alpha \mathcal{S}^{0,\alpha i} = -\partial_\alpha \mathcal{S}^{\alpha,0i} = T^{0i} - T^{i0}$$

Now we can write the angular momentum density variation using eq. (2.2) with $\Phi = \mathcal{S}$:

$$\begin{aligned} \mathcal{J}^{0,\mu\nu}_{\text{Belinfante}} &= \mathcal{J}^{0,\mu\nu}_{\text{canonical}} + \frac{1}{2}\partial_\alpha \left[x^\mu \left(\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}\right) - x^\nu \left(\Phi^{\alpha,0\mu} - \Phi^{0,\alpha\mu} - \Phi^{\mu,\alpha 0}\right)\right] \\ &= \frac{1}{2}\left[x^\mu \partial_\alpha \left(\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}\right) + \Phi^{\mu,0\nu} - \Phi^{0,\mu\nu} - \Phi^{\nu,\mu 0} - (\mu \leftrightarrow \nu)\right] \end{aligned}$$

The sum of all terms linear in the superpotential returns a $-\Phi^{0,\mu\nu}$ while for the derivative terms we can use eq. (3.7):

$$\begin{aligned} \mathcal{J}^{0,\mu\nu}_{\text{Belinfante}} &= \mathcal{J}^{0,\mu\nu}_{\text{canonical}} + \frac{1}{2}\left[x^\mu \partial_\alpha \left(\Phi^{\alpha,0\nu} - \Phi^{0,\alpha\nu} - \Phi^{\nu,\alpha 0}\right) - (\mu \leftrightarrow \nu)\right] - \Phi^{0,\mu\nu} \\ &= \mathcal{J}^{0,\mu\nu}_{\text{canonical}} - \frac{1}{2}\left[x^\mu \partial_\alpha \mathcal{S}^{0,\alpha\nu} - x^\nu \partial_\alpha \mathcal{S}^{0,\alpha\mu}\right] - \mathcal{S}^{0,\mu\nu} \end{aligned} \tag{3.9}$$

Therefore, by plugging the expression of the mean value of the spin tensor in eq. (3.6), the angular momentum pseudo-vector corresponding to the angular momentum density in (3.9) can be finally written:

$$\boldsymbol{\mathcal{J}}_{\text{Belinfante}} = \boldsymbol{\mathcal{J}}_{\text{canonical}} - \frac{1}{2}\left(\mathbf{x} \times \frac{\mathrm{d}D(r)}{\mathrm{d}r}\hat{\mathbf{v}}\right) - \mathbf{D}(r) = \boldsymbol{\mathcal{J}}_{\text{canonical}} - \left(\frac{1}{2}r\frac{\mathrm{d}D(r)}{\mathrm{d}r} + D(r)\right)\hat{\mathbf{k}} \tag{3.10}$$

In order for the canonical and Belinfante tensors to be equivalent, as has been mentioned and as it is apparent from eqs. (3.8) and (3.10) the function $D(r)$ ought to vanish everywhere. If $D'(r) \neq 0$, the two stress-energy tensors give two different momentum densities and are thus inequivalent; if, on top of that, $D'(r) \neq -2D(r)/r$ then the angular momentum densities are inequivalent as well. In the rest of this section we will prove that this is exactly the case, *i.e.* neither of these conditions is fulfilled. In order to show that this is not a problem arising from peculiar values of the field at the boundary, we will conservatively enforce boundary conditions such that the *total* energy, momentum and angular momentum operators obtained by integrating the fields within the cylinder are invariant under pseudogauge transformation. Note that for this to be true, in the case under consideration, it is necessary that the function $D(r)$ vanishes at the boundary, *i.e.* $D(R) = 0$, because the difference between total angular momenta is:

$$\int_V \mathrm{d}^3\mathbf{x}\ \left(\boldsymbol{\mathcal{J}}_{\text{Belinfante}} - \boldsymbol{\mathcal{J}}_{\text{canonical}}\right) = -\int_V \mathrm{d}z\,\mathrm{d}\varphi\,\mathrm{d}r\ r\ \left(\frac{1}{2}r\frac{\mathrm{d}D(r)}{\mathrm{d}r} + D(r)\right)\hat{\mathbf{k}} =$$

$$= -2\pi\int_{-\infty}^{+\infty}\mathrm{d}z\int_0^R \mathrm{d}r\ \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{2}D(r)\right)\hat{\mathbf{k}}$$

Thereby, we will demonstrate that, although the stress-energy and spin tensors in (3.2) and (3.3) lead to the same quantum generators, their respective mean densities are inconsistent. The problem we are facing is then to solve the Dirac equation within a cylinder with finite radius and second-quantize the field.

3.1 The Dirac field in a cylinder

The problem of the Dirac field within a cylinder with finite radius has been tackled by several authors in the context of the MIT bag model [16]. One of the major issues is the choice of appropriate boundary conditions, not an easy task because the Dirac equation is a first-order partial differential equation. The authors of the bag model [15] have shown that the following condition[1]:

[1]Actually, in the paper [15], the boundary condition chosen is $i\not{n}\Psi(R) = \Psi(R)$, but the change of sign is indeed immaterial.

$$i\not{n}\Psi(R) = in^\mu\gamma_\mu\Psi(R) = -\Psi(R), \tag{3.11}$$

ensures the vanishing of the fluxes (1.5) through the border and allows non-trivial solutions of the Dirac equation within the cylinder (see fig. 2.1) which, however, extend to the whole space without any discontinuity in the field. The above equation, in the non-relativistic limit, entails the vanishing of the "large" components of the Dirac field at the boundary, that is one is left with the Schrödinger equation with Dirichlet boundary conditions. We can readily verify the vanishing of fluxes implied by eq. (3.11) by first noting that:

$$\begin{aligned} i\overline{\Psi}(R)\not{n} = \overline{\Psi}(R) \quad &\Rightarrow \quad \overline{\Psi}(R)\Psi(R) = (i\overline{\Psi}(R)\not{n})(-i\not{n}\Psi(R)) = \\ &\qquad = -\overline{\Psi}(R)\Psi(R) \\ &\Rightarrow \quad \overline{\Psi}(R)\Psi(R) = 0, \end{aligned} \tag{3.12}$$

being $\not{n}\not{n} = n^2 = -1$. The equations above imply that the current flux also vanishes at the boundary because:

$$\widehat{j}^\mu(R)n_\mu = \overline{\Psi}(R)\not{n}\Psi(R) = i\overline{\Psi}(R)\Psi(R) = 0.$$

Since $\overline{\Psi}\Psi(R) = 0$, the outer surface of the cylinder must be such that $\partial^\mu\overline{\Psi}\Psi|_R = \widehat{\Xi}\, n^\mu$ or:

$$n^\mu\partial_\mu(\overline{\Psi}\Psi)(R) = \frac{\partial}{\partial r}\overline{\Psi}\Psi\Big|_{r=R} = -\widehat{\Xi}(R). \tag{3.13}$$

Since $\overline{\Psi}\Psi = 0$ for $r = R$ for any φ, t, z, then the operator Ξ can only depend on the radial coordinate, hence on R. The flux of energy-momentum of the canonical tensor at the boundary is vanishing because, using eq. (3.2) and eqs. (3.11),(3.13):

$$\begin{aligned} \int_{\partial V} \mathrm{d}S\; \widehat{T}^{\mu\nu} n_\mu \quad &= \quad \frac{i}{2}\int_{\partial V} \mathrm{d}S\; \overline{\Psi}\not{n}\partial^\nu\Psi - \partial^\nu\overline{\Psi}\not{n}\Psi = \frac{1}{2}\int_{\partial V}\mathrm{d}S\; \overline{\Psi}\partial^\nu\Psi + \partial^\nu\overline{\Psi}\Psi = \\ &= \quad \frac{1}{2}\int_{\partial V}\mathrm{d}S\; \partial^\nu(\overline{\Psi}\Psi) = -\frac{\widehat{\Xi}(R)}{2}\int_{\partial V}\mathrm{d}S\; n^\nu = 0. \end{aligned}$$

Likewise, for the orbital part of the angular momentum flux:

$$\begin{aligned}\int_{\partial V} \mathrm{d}S\; x^\mu \widehat{T}^{\lambda\nu} n_\lambda - (\mu \leftrightarrow \nu) &= \frac{1}{2}\int_{\partial V} \mathrm{d}S\; x^\mu \partial^\nu (\overline{\Psi}\Psi) - (\mu \leftrightarrow \nu) = \\ &= \frac{\widehat{\Xi}(R)}{2}\int_{\partial V} \mathrm{d}S\; (x^\mu n^\nu - x^\nu n^\mu) = 0.\end{aligned}$$

where the last integral vanishes because of the geometrical symmetry $z \to -z$. Finally, the flux of the spin tensor also vanishes at the boundary because, using (3.2) and (3.11):

$$n_\lambda \widehat{\mathcal{S}}^{\lambda,\mu\nu}(R) = \frac{1}{2}\left(\overline{\Psi}\not{n}\Sigma^{\mu\nu}\Psi + \overline{\Psi}\Sigma^{\mu\nu}\not{n}\Psi\right) = -\frac{i}{2}\left(\overline{\Psi}\Sigma^{\mu\nu}\Psi - \overline{\Psi}\Sigma^{\mu\nu}\Psi\right) = 0. \tag{3.14}$$

Therefore, the eq. (1.5) applies and the integrals:

$$\widehat{P}^\nu = \int_V \mathrm{d}^3\mathbf{x}\; \widehat{T}^{0\nu} \qquad\qquad \widehat{J}^{\mu\nu} = \int_V \mathrm{d}^3\mathbf{x}\; \widehat{\mathcal{J}}^{0,\mu\nu}, \tag{3.15}$$

are conserved. Since, we also have, from the Lagrangian, the usual anticommutation relations at equal times:

$$\{\Psi_a(t,\mathbf{x}), \Psi^\dagger_b(t,\mathbf{x}')\} = \delta_{ab}\delta^3(\mathbf{x}-\mathbf{x}') \quad \{\Psi_a(t,\mathbf{x}), \Psi_b(t,\mathbf{x}')\} = \{\Psi^\dagger_a(t,\mathbf{x}), \Psi^\dagger_b(t,\mathbf{x}')\} = 0,$$

it is easy to check that the conserved hamiltonian $i/2 \int \mathrm{d}^3\mathbf{x}\;\; \Psi^\dagger \overleftrightarrow{\partial}_t \Psi$ is indeed, as expected, the generator of time translations, i.e.:

$$[\widehat{H}, \Psi] = -i\frac{\partial}{\partial t}\Psi \quad [\widehat{H}, \Psi^\dagger] = -i\frac{\partial}{\partial t}\Psi^\dagger,$$

and, therefore, putting together the above equation with eq. (3.15) and (3.2), the conclusion is that:

$$[\widehat{H}, \widehat{J}_i] = 0,$$

for the case under examination.

The complete solution of the free Dirac equation for a massive particle in a longitudinally unlimited cylinder with finite transverse radius, with boundary conditions of the kind (3.11) has been obtained by Bezerra de Mello *et al* in ref. [17] and we summarize it here. In a longitudinally unlimited cylinder, but with finite transverse radius R, the field is expanded in terms of eigenfunctions of the longitudinal momentum, third component of angular momentum, transverse momentum and an additional "spin" quantum number [17]. The relevant quantum numbers $\mathbf{n} = (p_z, M, \zeta_{(M,\xi,l)}, \xi)$ take on continuous (p_z) and discrete values ($M, \zeta_{(M,\xi,l)}, \xi$). The third component of the angular momentum M takes on all semi-integer values $\pm 1/2, \pm 3/2, \ldots$; the "spin" quantum number ξ can be ± 1 and the transverse momentum quantum number:

$$\zeta_{(M,\xi,l)} = p_{Tl} R, \tag{3.16}$$

takes on discrete values which are zeroes, sorted in ascending order with the label $l = 1, 2, \ldots$ and depending on M and ξ, of the equation:

$$J_{\left|M-\frac{1}{2}\right|}(p_T R) \; + \; \mathrm{sgn}(M)\, b_\xi^{(+)}\, J_{\left|M+\frac{1}{2}\right|}(p_T R) = 0, \tag{3.17}$$

where J are Bessel functions and:

$$b_\xi^{(\pm)} = \frac{\pm m + \xi m_T}{p_T}. \tag{3.18}$$

m being the mass and:

$$m_T = \sqrt{p_T^2 + m^2},$$

the transverse mass [2]; we note in passing that $b_\xi^{(+)} = 1/b_\xi^{(-)}$. The Dirac field itself can be written as an expansion:

$$\Psi(x) = \sum_{\mathbf{n}} U_{\mathbf{n}}(x) a_{\mathbf{n}} + V_{\mathbf{n}}(x) b_{\mathbf{n}}^\dagger, \tag{3.19}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are destruction operators of, in order, particles and antiparticles having quantum numbers $\mathbf{n}$, while:

[2] In the rest of this section the symbol p_{Tl} stands for a discrete variable taking on (M, ξ, l)-dependent values given by the eq. (3.16) or, later on, by eq. (3.38).

$$\sum_{\mathbf{n}} \equiv \sum_M \sum_{\xi=\pm 1} \sum_{\zeta_{(M,\xi,l)}} \int_{-\infty}^{+\infty} \mathrm{d}p_z = \sum_M \sum_{\xi=-1,1} \sum_{l=1}^{\infty} \int_{-\infty}^{+\infty} \mathrm{d}p_z.$$

The eigenspinors $U_{\mathbf{n}}$ and $V_{\mathbf{n}}$ read, in the Dirac representation of the γ matrices and in cylindrical coordinates (t, r, φ, z):

$$U_{\mathbf{n}}(x) = C_{\mathbf{n}} \begin{pmatrix} J_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) \\ i\,\mathrm{sgn}(M)\kappa_\xi b_\xi^{(+)} J_{\left|M+\frac{1}{2}\right|}(p_{Tl}r)\mathrm{e}^{i\varphi} \\ \kappa_\xi J_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) \\ -i\,\mathrm{sgn}(M) b_\xi^{(+)} J_{\left|M+\frac{1}{2}\right|}(p_{Tl}r)\mathrm{e}^{i\varphi} \end{pmatrix} \frac{\mathrm{e}^{i[(M-1/2)\varphi+p_z z-\varepsilon t]}}{\sqrt{2\pi}}$$

$$V_{\mathbf{n}}(x) = \frac{C_{\mathbf{n}}}{b_\xi^{(-)}} \begin{pmatrix} J_{\left|M+\frac{1}{2}\right|}(p_{Tl}r) \\ i\,\mathrm{sgn}(M)\kappa_\xi b_\xi^{(-)} J_{\left|M-\frac{1}{2}\right|}(p_{Tl}r)\mathrm{e}^{i\varphi} \\ \kappa_\xi J_{\left|M+\frac{1}{2}\right|}(p_{Tl}r) \\ -i\,\mathrm{sgn}(M) b_\xi^{(-)} J_{\left|M-\frac{1}{2}\right|}(p_{Tl}r)\mathrm{e}^{i\varphi} \end{pmatrix} \frac{\mathrm{e}^{-i[(M+1/2)\varphi+p_z z-\varepsilon t]}}{\sqrt{2\pi}}, \quad (3.20)$$

with:

$$\kappa_\xi = \frac{\varepsilon + \xi\sqrt{\varepsilon^2 - p_z^2}}{p_z},$$

and $\varepsilon = \sqrt{p_z^2 + p_{Tl}^2 + m^2}$ being the energy. The eigenspinors (3.20) are normalized so as to:

$$\int_V \mathrm{d}^3\mathbf{x}\ \Psi^\dagger \Psi = \sum_{\mathbf{n}} a_{\mathbf{n}}^\dagger a_{\mathbf{n}} + b_{\mathbf{n}} b_{\mathbf{n}}^\dagger,$$

that is with:

$$\int_V \mathrm{d}^3\mathbf{x}\ U_{\mathbf{n}}^\dagger(x) U_{\mathbf{n}'}(x) = \int_V \mathrm{d}^3\mathbf{x}\ V_{\mathbf{n}}^\dagger(x) V_{\mathbf{n}'}(x) = \delta_{\mathbf{n}\mathbf{n}'}; \quad \int_V \mathrm{d}^3\mathbf{x}\ U_{\mathbf{n}}^\dagger(x) V_{\mathbf{n}'}(x) = 0, \quad (3.21)$$

being $\delta_{\mathbf{n}\mathbf{n}'} = \delta_{MM'}\delta_{\xi\xi'}\delta_{ll'}\delta(p_z - p_z')$ and the anticommutation relations of creation and destruction operators:

$$\{a_{\mathbf{n}}, a^{\dagger}_{\mathbf{n}'}\} = \{b_{\mathbf{n}}, b^{\dagger}_{\mathbf{n}'}\} = \delta_{\mathbf{n}\mathbf{n}'} \qquad \{a_{\mathbf{n}}, b_{\mathbf{n}'}\} = \{a^{\dagger}_{\mathbf{n}}, b_{\mathbf{n}'}\} = 0. \tag{3.22}$$

The normalization coefficient in (3.20) obtained from the condition (3.21) reads [17]:

$$(C_{\mathbf{n}})^{-2} = 2\pi R^2 J^2_{|M-\frac{1}{2}|}(p_{Tl}R)\frac{\kappa_\xi^2+1}{p_{Tl}^2R^2}\big(2R^2m_{Tl}^2 + 2\xi MRm_{Tl} + mR\big). \tag{3.23}$$

3.2 Proving the inequivalence

For what we have seen so far, from a purely quantum field theoretical point of view, the Belinfante tensors (3.3) for the Dirac field in the cylinder could be regarded as equivalent to the canonical ones in eq. (3.2) because they give, once integrated, the same generators (3.15). This happens because the condition (1.4) is met for $\widehat{\Phi} = \widehat{\mathcal{S}}$ (what follows from eq. (3.14)) and this implies, taking eq. (3.4) into account, that all the integrands of (1.4) vanish at the boundary. Yet, these two set of tensors are thermodynamically inequivalent because, as it will be shown hereafter, it turns out that, using eq. (3.6):

$$\mathcal{S}^{0,ij} = \frac{1}{2}\mathrm{tr}\left(\widehat{\rho} :\overline{\Psi}\{\gamma^0, \Sigma^{ij}\}\Psi:\right) = D(r)\epsilon_{ijk}k^k \neq 0 \Rightarrow D(r) \neq 0, \tag{3.24}$$

at some $r \neq R$ (we have used the eq. (3.6)), with $\widehat{\rho}$ we mean $\widehat{\rho}_V$ written in eq. (2.27), omitting the $\cdots_V$ for ease of reading. This will be enough to conclude that either the energy-momentum or the angular momentum densities or both have different values for different sets of quantum tensors, as previously discussed. Note that the boundary condition (3.14) together with the general expression of the mean value of the spin tensor (3.5) implies that $D(R) = 0$, i.e. its vanishing at the boundary.

We can rewrite the inequality (3.24) by taking advantage of the commutation relation:

$$\left[\gamma^\lambda, \Sigma^{\mu\nu}\right] = i\eta^{\lambda\mu}\gamma^\nu - i\eta^{\lambda\nu}\gamma^\mu,$$

implying:

$$\mathcal{S}^{0,ij} = \mathrm{tr}\left(\widehat{\rho} : \Psi^\dagger \Sigma^{ij} \Psi :\right) - i\eta^{0i} \mathrm{tr}\left(\widehat{\rho} : \overline{\Psi} \gamma^j \Psi :\right) + i\eta^{0j} \mathrm{tr}\left(\widehat{\rho} : \overline{\Psi} \gamma^i \Psi :\right) = \mathrm{tr}\left(\widehat{\rho} : \Psi^\dagger \Sigma^{ij} \Psi :\right) \neq 0,$$

or, equivalently:

$$D(r) = \frac{1}{2} \epsilon_{ij3} \mathcal{S}^{0,ij} = \frac{1}{2} \mathrm{tr}\left(\widehat{\rho} : \Psi^\dagger \epsilon_{ij3} \Sigma^{ij} \Psi :\right) \equiv \mathrm{tr}\left(\widehat{\rho} : \Psi^\dagger \Sigma_3 \Psi :\right) \neq 0, \tag{3.25}$$

where the indices i, j can only take on the value 1 or 2. In the above equation and henceforth, we can take the Heisenberg field operators at some fixed time $t = 0$ because of the stationarity of density operator $\widehat{\rho}$. Hence we just need to show that:

$$\mathrm{tr}\left(\widehat{\rho} : \Psi^\dagger(0, \mathbf{x}) \Sigma_z \Psi(0, \mathbf{x}) :\right) \neq 0 \tag{3.26}$$

with:

$$\Sigma_z = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.27}$$

for some point $\mathbf{x}$ within the cylinder and our goal is achieved.

To calculate the mean value of the spin density in eq. (3.26), we start by observing that (see Appendix A for the proof):

$$\begin{aligned} &\mathrm{tr}\left(\widehat{\rho}\, a^\dagger_{\mathbf{n}} a_{\mathbf{n}'}\right) = \frac{\delta_{\mathbf{n}\mathbf{n}'}}{\mathrm{e}^{(\varepsilon - M\omega + \mu)/T} + 1} \quad \mathrm{tr}\left(\widehat{\rho}\, b^\dagger_{\mathbf{n}} b_{\mathbf{n}'}\right) = \frac{\delta_{\mathbf{n}\mathbf{n}'}}{\mathrm{e}^{(\varepsilon - M\omega - \mu)/T} + 1} \\ &\mathrm{tr}\left(\widehat{\rho}\, a^\dagger_{\mathbf{n}} b_{\mathbf{n}'}\right) = \mathrm{tr}\left(\widehat{\rho}\, a_{\mathbf{n}} b_{\mathbf{n}'}\right) = 0 \end{aligned} \tag{3.28}$$

which allows us to work it out by plugging in there the field expansion (3.19):

$$\begin{aligned}
&\mathrm{tr}\left(\widehat{\rho}\,{:}\Psi^\dagger(0,\mathbf{x})\Sigma_z\Psi(0,\mathbf{x}){:}\right) = \qquad (3.29)\\
&= \sum_{\mathbf{n}} \frac{1}{\mathrm{e}^{(\varepsilon - M\omega+\mu)/T}+1}[U^\dagger_{\mathbf{n}}(x)\Sigma_z U_{\mathbf{n}}(x)] - \frac{1}{\mathrm{e}^{(\varepsilon - M\omega-\mu)/T}+1}[V^\dagger_{\mathbf{n}}(x)\Sigma_z V_{\mathbf{n}}(x)].
\end{aligned}$$

where we have taken into account that the normal ordering of fermions is such that ${:}b_{\mathbf{n}} b^\dagger_{\mathbf{n}'}{:} = -b^\dagger_{\mathbf{n}'} b_{\mathbf{n}}$. By using eq. (3.20) and (3.27):

$$\begin{aligned}
U^\dagger_{\mathbf{n}}(x)\Sigma_z U_{\mathbf{n}}(x) &= \frac{C^2_{\mathbf{n}}}{4\pi}\Big[J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) - \kappa_\xi^2 b_\xi^{(+)^2} J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl}r) + \\
&\qquad + \kappa_\xi^2 J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) - b_\xi^{(+)^2} J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl}r)\Big] = \\
&= \frac{C^2_{\mathbf{n}}}{4\pi}\Big[J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) - b_\xi^{(+)^2} J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl}r)\Big]\left(1+\kappa_\xi^2\right)\\
V^\dagger_{\mathbf{n}}(x)\Sigma_z V_{\mathbf{n}}(x) &= \frac{C^2_{\mathbf{n}}}{4\pi} b_\xi^{(-)^2}\Big[J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl}r) - b_\xi^{(-)^2} J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}r)\Big]\left(1+\kappa_\xi^2\right) = \\
&= \frac{C^2_{\mathbf{n}}}{4\pi}\Big[-J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) + b_\xi^{(+)^2} J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl}r)\Big]\left(1+\kappa_\xi^2\right) = -U^\dagger_{\mathbf{n}}(x)\Sigma_z U_{\mathbf{n}}(x).
\end{aligned}$$

hence, by using eqs. (3.23) and (3.25), we can rewrite eq. (3.29) as:

$$\begin{aligned}
&\mathrm{tr}\left(\widehat{\rho}\,{:}\Psi^\dagger(0,\mathbf{x})\Sigma_z\Psi(0,\mathbf{x}){:}\right) = D(r) = \\
&= \sum_M \sum_{\xi=\pm1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_z \left[\frac{1}{\mathrm{e}^{(\varepsilon - M\omega+\mu)/T}+1} + \frac{1}{\mathrm{e}^{(\varepsilon - M\omega-\mu)/T}+1}\right] \times \\
&\times \frac{p^2_{Tl}\Big[J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) - b_\xi^{(+)^2} J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl}r)\Big]}{8\pi^2 R J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}R)(2Rm^2_{Tl} + 2\xi M m_{Tl} + m)}. \qquad (3.30)
\end{aligned}$$

The mean value of the spin tensor is therefore given by the sum of a particle and an antiparticle term which are equal only for $\mu = 0$. As expected, it vanishes for $r = R$ in view of the eq. (3.17), yet our goal is to show that it is non-vanishing at some point $\mathbf{x}$ not belonging to the boundary. It is worth pointing out that, if this is the case, the spin tensor has a *macroscopic* value because, as it is apparent from (2.27), it is proportional to the number density (in phase space) of quanta $1/\exp[(\varepsilon - M\omega \pm \mu)/T + 1]$. It is most convenient to consider a point belonging to the rotation axis, i.e. with radial coordinate $r = 0$ because Bessel functions of all orders but 0 vanish therein.

We will show that $D(0) = 0$ for $\omega = 0$ first and that it is an increasing function of ω/T thereafter. In general, the whole function $D(r)$ must be vanishing for $\omega = 0$ for symmetry reasons. In fact, if $\omega = 0$, the density operator (2.27) has an additional symmetry, that is the rotation of an angle π around any axis orthogonal to the cylinder axis, say $\mathsf{R}_2(\pi)$. This transformation corresponds to flip over the cylinder, which leaves the system invariant provided that $\omega = 0$, and has the consequence that any pseudo-vector field directed along the axis must vanish. Therefore:

$$\mathrm{tr}\left(\widehat{\rho} : \Psi^\dagger(0,\mathbf{x})\Sigma_z\Psi(0,\mathbf{x}) :)\Big|_{\omega=0}\right) = D(r)|_{\omega=0} = 0. \tag{3.31}$$

We will, however, explicitly work out $D(0)|_{\omega=0}$, because it will be needed to calculate the derivative as a function of ω/T and also in order to check that the expression (3.30) fulfills the eq. (3.31) in $r = 0$. We first not that the function $D(r)$ in eq. (2.27) is the sum of a particle $D(r)^+$ and an antiparticle $D(r)^-$ term. We focus on the particle term $D(r)^+$, the calculation for $D(r)^-$ will be a trivial extension. Let us define (see eq. (3.16)):

$$p_{\pm,\xi} = \frac{\zeta_{(\pm\frac{1}{2},\xi,l)}}{R},$$

and recalling that $J_0(0) = 1$, from eq. (2.27):

$$D(0)^+ = \frac{1}{8\pi^2 R}\sum_{\xi=\pm1}\sum_{l=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}p_z \tag{3.32}$$
$$\left\{\frac{p_{+,\xi}^2}{\left(\mathrm{e}^{\left(\varepsilon-\frac{1}{2}\omega+\mu\right)/T}+1\right)J_0(p_{+,\xi}R)^2\left[2R\left(p_{+,\xi}^2+m^2\right)+2\xi\sqrt{p_{+,\xi}^2+m^2}+m\right]}\right.$$
$$\left.-\frac{p_{-,\xi}^2 b_\xi^{(+)2}}{\left(\mathrm{e}^{\left(\varepsilon+\frac{1}{2}\omega+\mu\right)/T}+1\right)J_1(p_{-,\xi}R)^2\left[2R\left(p_{-,\xi}^2+m^2\right)-2\xi\sqrt{p_{-,\xi}^2+m^2}+m\right]}\right\}.$$

We can rearrange the above sum by noting that the equation (3.17), depending on indices (M,ξ) is the *same* for $(-M,-\xi)$. In fact:

$$J_{|-M-\frac{1}{2}|}(\zeta) \;+\; \mathrm{sgn}(-M)\, b^{(+)}_{-\xi}\, J_{|-M+\frac{1}{2}|}(\zeta) = J_{|M+\frac{1}{2}|}(\zeta) - \mathrm{sgn}(M) b^{(+)}_{-\xi} J_{|M-\frac{1}{2}|}(\zeta),$$

However, because of (3.18), $-b^{(+)}_{-\xi} = b^{(-)}_{\xi} = 1/b^{(+)}_{\xi}$, and so multiplying the right hand side of above equation by $\mathrm{sgn}(M) b^{(+)}_{\xi}$ one gets the left hand side of eq. (3.17). Hence, the zeroes of eq. (3.17) and the one with "reflected" indices $(-M,-\xi)$ are the same:

$$\zeta_{(-M,-\xi,l)} = \zeta_{(M,\xi,l)}, \tag{3.33}$$

for any $l = 1, 2, \ldots$. Now we can redefine the indices in the second term of the sum in eq. (3.32) by turning ξ into $-\xi$, which changes nothing as $\xi = -1, +1$ and write:

$$\begin{aligned} D(0)^+ &= \frac{1}{8\pi^2 R} \sum_{\xi=\pm1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_z \times \\ &\times \left\{ \frac{p^2_{+,\xi}}{\left(\mathrm{e}^{\left(\varepsilon-\frac{1}{2}\omega+\mu\right)/T}+1\right) J_0(p_{+,\xi}R)^2 \left[2R\left(p^2_{+,\xi}+m^2\right) + 2\xi\sqrt{p^2_{+,\xi}+m^2}+m\right]} + \right. \\ &\left. - \frac{p^2_{-,-\xi} {b^{(+)}_{-\xi}}^2}{\left(\mathrm{e}^{\left(\varepsilon+\frac{1}{2}\omega+\mu\right)/T}+1\right) J_1(p_{-,-\xi}R)^2 \left[2R\left(p^2_{-,-\xi}+m^2\right) + 2\xi\sqrt{p^2_{-,-\xi}+m^2}+m\right]} \right\}. \end{aligned}$$

We can replace $p_{-,-\xi}$ with $p_{+,\xi}$ because of (3.33) and therefore:

$$\begin{aligned} D(0)^+ &= \frac{1}{8\pi^2 R} \sum_{\xi=\pm1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_z \times \\ &\times \left\{ \frac{p^2_{+,\xi}}{\left(\mathrm{e}^{\left(\varepsilon-\frac{1}{2}\omega\right)/T}+1\right) J_0(p_{+,\xi}R)^2 \left[2R\left(p^2_{+,\xi}+m^2\right) + 2\xi\sqrt{p^2_{+,\xi}+m^2}+m\right]} + \right. \\ &\left. - \frac{p^2_{+,\xi} {b^{(+)}_{-\xi}}^2}{\left(\mathrm{e}^{\left(\varepsilon+\frac{1}{2}\omega+\mu\right)/T}+1\right) J_1(p_{+,\xi}R)^2 \left[2R\left(p^2_{+,\xi}+m^2\right) + 2\xi\sqrt{p^2_{+,\xi}+m^2}+m\right]} \right\}. \end{aligned} \tag{3.34}$$

We are now going to prove that this latter expression is non-vanishing when $\omega \neq 0$. First, we note that it does vanish when $\omega = 0$. In this case eq. (3.34) yields:

$$D(0)^+\Big|_{\omega=0} = \frac{1}{8\pi^2 R}\sum_{l,\xi}\int_{-\infty}^{\infty} \mathrm{d}p_z \times$$
$$\times \frac{p_{+,\xi}^2\left[J_1(p_{+,\xi}R)^2 - {b_{-\xi}^{(+)}}^2 J_0(p_{+,\xi}R)^2\right]}{(\mathrm{e}^{(\varepsilon+\mu)/T+}+1)J_1(p_{+,\xi}R)^2 J_0(p_{+,\xi}R)^2\left[2R\big(p_{+,\xi}^2+m^2\big)+2\xi\sqrt{p_{+,\xi}^2+m^2}+m\right]}.$$

By using again (3.33) to replace $p_{+,\xi}$ with $p_{-,-\xi}$ it is easy to show that the numerator of the integrand vanishes as:

$$\begin{aligned} J_1(p_{+,\xi}R)^2 - {b_{-\xi}^{(+)}}^2 J_0(p_{+,\xi}R)^2 &= J_1(p_{-,-\xi}R)^2 - {b_{-\xi}^{(+)}}^2 J_0(p_{-,-\xi}R)^2 = \\ &= J_1(\zeta_{(-1/2,-\xi,l)})^2 - {b_{-\xi}^{(+)}}^2 J_0(\zeta_{(-1/2,-\xi,l)})^2 = 0, \end{aligned}$$

in view of the eq. (3.17). Therefore, the spin tensor density in $r = 0$ vanishes for a non-rotating system, as expected. To show that it is no longer zero for $\omega \neq 0$ we just need to show that the derivative with respect to ω/T in $\omega = 0$ is not zero. One has:

$$\frac{\partial}{\partial(\omega/T)}D(0)^+\Big|_{\omega=0} = \frac{1}{16\pi^2 R}\sum_{l,\xi}\int_{-\infty}^{\infty} \mathrm{d}p_z$$
$$\frac{\mathrm{e}^{(\varepsilon+\mu)/T}p_{+,\xi}^2\left[J_1(p_{+,\xi}R)^2 + {b_{-\xi}^{(+)}}^2 J_0(p_{+,\xi}R)^2\right]}{(\mathrm{e}^{(\varepsilon+\mu)/T}+1)^2 J_1(p_{+,\xi}R)^2 J_0(p_{+,\xi}R)^2\left[2R\big(p_{+,\xi}^2+m^2\big)+2\xi\sqrt{p_{+,\xi}^2+m^2}+m\right]}.$$

All terms are mainfestly positive except $\left[2R(p_{Tl}^2+m^2)+2\xi\sqrt{p_{Tl}^2+m^2}+m\right]$ in the denominator when $\xi = -1$. However, in this case:

$$\begin{aligned} R\big(p_{Tl}^2+m^2\big) - \sqrt{p_{Tl}^2+m^2} &= \sqrt{p_{Tl}^2+m^2}\left(R\sqrt{p_{Tl}^2+m^2}-1\right) > \\ &> \sqrt{p_{Tl}^2+m^2}\,(Rm-1), \end{aligned}$$

which is positive for a radius greater than the Compton wavelength of the particle, that is positive for any actually macroscopic value of the radius R. The very same argument applies to the antiparticle term $D(0)^-$ of the $D(r)$ function in eq. (3.30) with the immaterial replacement $\mu \to -\mu$, hence:

$$D(0)|_{\omega=0} = 0 \qquad\qquad \frac{\partial}{\partial(\omega/T)} D(0)\Big|_{\omega=0} > 0$$

It thence follows that the inequality (3.26) must be true for small, yet finite, values of ω/T and therefore:

$$D(r) \neq 0 \qquad \text{for } \ \omega/T > 0$$

in a finite region around $r = 0$. Moreover, it is easy to show, by using eq. (3.30), that the derivative of the function $D(r)$ vanishes in $r = 0$ for it is proportional to terms, with $N \geq 0$:

$$2J_N(0)J_N'(0) = J_N(0)\left(J_{N-1}(0) - J_{N+1}(0)\right)$$

which all vanish because of the known properties of Bessel functions. Hence, the mean angular momentum density in $r = 0$ differs between canonical and Belinfante tensors, *i.e.* rewriting the eq. (3.10) for $r = 0$:

$$\boldsymbol{\mathcal{J}}_{\text{Belinfante}}(0) = \boldsymbol{\mathcal{J}}_{\text{canonical}}(0) - D(0)\hat{\mathbf{k}}$$

where $D(0)$ is finite for finite ω and positive. Thus, the Belinfante angular momentum density is lower than the canonical one by some finite and macroscopic amount.

It's important to stress how, had we used the definition of average values without normal ordering, this conclusion would be unaffected. Indeed, the spin tensor in (3.2) is a bilinear in the fields and therefore the difference between the two definitions is a vacuum expectation value of the spin tensor:

$$\operatorname{tr}\left(\widehat{\rho} : \widehat{\mathcal{S}} :\right) = \operatorname{tr}\left(\widehat{\rho}\, \widehat{\mathcal{S}}\right) - \langle 0|\widehat{\mathcal{S}}|0\rangle. \tag{3.35}$$

Vectorial irreducible parts of the spin tensor, such as $\widehat{\mathcal{S}}^{0ij}$ have a vanishing vacuum expectation value if the vacuum is invariant under general rotations. The vacuum of the free Dirac field in the cylinder, dened by $a_{\mathbf{n}}|0\rangle = |0\rangle$, is indeed rotationally invariant. If degenerate vacua existed, the commutation of the Hamiltonian with angular momentum operators that was shown before (see Eq. (60)) would ensure that they belong to some irreducible representation of the $SU(2)$ group. However, for the free Dirac field in the cylinder, the angular momentum operator along the z axis turns out to be:

$$\widehat{J}_z = \sum_{\mathbf{n}} M \left(a^\dagger_{\mathbf{n}} a_{\mathbf{n}} + b^\dagger_{\mathbf{n}} b_{\mathbf{n}} \right) ,$$

so that $\widehat{J}_z|0\rangle = 0$ on all possible degenerate vacua. This means that the only possible multiplet is one-dimensional and, thereby, the vacuum is non degenerate and the second term in the Eq. (3.35) vanishes.

3.2.1 The non-relativistic limit

It would be very interesting to calculate the function $D(r)$ numerically to "see" the difference between the Belinfante and the canonical tensors and to make sure that this difference is not a rapidly oscillating function on a microscopic scale, which would render the macroscopic observation of the difference impossible. This is, though, very hard in the fully relativistic case but relatively easy in the non-relativistic limit $m/T \gg 1$, because in this case the eq. (3.17) yielding the quantized transverse momenta reduces to the vanishing of one single Bessel function. This happens because in the non-relativistic limit:

$$b^{(+)}_\xi = \frac{\xi m_T + m}{p_T} = \begin{cases} \frac{m_T+m}{p_T} \simeq \frac{2m+p_T^2/2m}{p_T} \simeq \frac{2m}{p_T} \gg 1 & \text{for } \xi = 1 \\ \frac{m-m_T}{p_T} \simeq -\frac{p_T^2/2m}{p_T} = -\left(\frac{p_T}{2m} \ll 1\right), & \text{for } \xi = -1 \end{cases} \tag{3.36}$$

so that the eq. (3.17) in fact reduces to:

$$\begin{cases} J_{|M+\frac{1}{2}|}(p_T R) = -\mathrm{sgn}(M)\frac{p_T}{2m} J_{|M-\frac{1}{2}|}(p_T R) \simeq 0 & \text{for } \xi = 1 \\ J_{|M-\frac{1}{2}|}(p_T R) = \mathrm{sgn}(M)\frac{p_T}{2m} J_{|M+\frac{1}{2}|}(p_T R) \simeq 0. & \text{for } \xi = -1 \end{cases} \tag{3.37}$$

Altogether, we can solve the equation $J_L(p_T R) = 0$ for all integers $L = M + \xi/2$ and take the quantized transverse momenta:

$$p_{Tl} = \frac{\zeta_{L,l}}{R}, \tag{3.38}$$

where $\zeta_{L,l}$ $l = 1, 2, \ldots$ are now the familiar zeroes of the Bessel function of integer order L.

We can now write the particle and antiparticle terms in the eq. (3.30):

$$D(r)^{\pm} = \sum_M \sum_{\xi=\pm1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_z \frac{1}{\mathrm{e}^{(\varepsilon - M\omega \pm \mu)/T} + 1} \frac{p_{Tl}^2 \left[J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl} r) \; - \; b_\xi^{(+)^2} J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl} r) \right]}{8\pi^2 R J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl} R)(2Rm_{Tl}^2 + 2\xi M m_{Tl} + m)},$$

with, as has been mentioned, $D(r) = D(r)^+ + D(r)^-$. In the non-relativistic limit one has:

$$2Rm_T^2 + 2\xi M m_T + m \simeq 2Rm^2 + 2\xi M m + m \simeq 2Rm^2, \tag{3.39}$$

where the last approximation is due to the obvious assumption $Rm \gg 1$ and that the term $|\xi M m|$ can be comparable to Rm^2 only if $|M|$ is very large. However, terms with large $|M|$ are either suppressed by the exponential $\exp[\omega M/T]$ or by the Bessel functions, which effectively implements the semiclassical equality $M \approx Rp_T$; since non-relativistically $Rm^2 \gg Rp_T m \approx |M|m$, the approximation (3.39) is justified. We then calculate the terms with $\xi = 1$ and $\xi = -1$ in the sum in eq. (3.37) separately. For $\xi = 1$ one sets $M + 1/2 = L$ and writes the integrand of eq. (3.30), including approximation (3.39) and taking into account (3.36):

$$\begin{aligned} &\frac{1}{\mathrm{e}^{(\varepsilon - L\omega + \omega/2 \pm \mu)/T} + 1} \frac{p_T^2}{16\pi^2 R^2 m^2} \frac{J^2_{|L-1|}(p_{Tl} r) \; - \; \frac{4m^2}{p_{Tl}^2} J^2_{|L|}(p_{Tl} r)}{J^2_{|L-1|}(p_{Tl} R)} \simeq \\ &\simeq -\frac{1}{\mathrm{e}^{(\varepsilon - L\omega + \omega/2 \pm \mu)/T} + 1} \frac{1}{4\pi^2 R^2} \frac{J^2_{|L|}(p_{Tl} r)}{J^2_{|L-1|}(p_{Tl} R)}, \end{aligned} \tag{3.40}$$

where p_{Tl} is a solution of the first equation in (3.37). Similarly, for $\xi = -1$ one sets

$M - 1/2 = L$ and obtains, by using the second of the equations (3.37):

$$
\begin{aligned}
&\frac{1}{\mathrm{e}^{(\varepsilon - L\omega + \omega/2 \pm \mu)/T} + 1} \frac{p_T^2}{16\pi^2 R^2 m^2} \frac{J_{|L|}^2(p_{Tl} r) \; - \; \frac{p_{Tl}^2}{4m^2} J_{|L+1|}^2(p_{Tl} r)}{J_{|L|}^2(p_{Tl} R)} \simeq \\
&\simeq \frac{1}{\mathrm{e}^{(\varepsilon - L\omega - \omega/2 \pm \mu)/T} + 1} \frac{p_T^2}{16\pi^2 R^2 m^2} \frac{J_{|L|}^2(p_{Tl} r)}{\frac{p_{Tl}^2}{4m^2} J_{|L+1|}^2(p_{Tl} R)} = \\
&= \frac{1}{\mathrm{e}^{(\varepsilon - L\omega - \omega/2 \pm \mu)/T} + 1} \frac{1}{4\pi^2 R^2} \frac{J_{|L|}^2(p_{Tl} r)}{J_{|L+1|}^2(p_{Tl} R)}. \qquad (3.41)
\end{aligned}
$$

Now, by using approximations (3.39),(3.40) and (3.41) we can write the non-relativistic limit of $D(r)^\pm$ as:

$$
\begin{aligned}
D(r)^\pm \;\; &= \;\; \frac{1}{4\pi^2 R^2} \sum_{L=-\infty}^{\infty} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_z \times \qquad (3.42) \\
&\times \frac{1}{\mathrm{e}^{(\varepsilon - L\omega - \omega/2 \pm \mu)/T} + 1} \frac{J_{|L|}^2(p_{Tl} r)}{J_{|L+1|}^2(p_{Tl} R)} - \frac{1}{\mathrm{e}^{(\varepsilon - L\omega + \omega/2 \pm \mu)/T} + 1} \frac{J_{|L|}^2(p_{Tl} r)}{J_{|L-1|}^2(p_{Tl} R)},
\end{aligned}
$$

where the first term is to be associated to particles with spin projection $+1/2$ along the z axis and the second term to those with projection $-1/2$. Finally, the integral over p_z in eq. (3.42) can be worked out by first introducing the non-relativistic approximation $\varepsilon = m + p_T^2/2m + p_z^2/2m$ and then expanding the Fermi distribution. The final result is:

$$
\begin{aligned}
D(r)^\pm \;\; &= \;\; \frac{1}{4\pi^2 R^2} \sum_{L=-\infty}^{\infty} \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt{\frac{2\pi m K T}{n}} \mathrm{e}^{-n(mc^2 \pm \mu + p_T^2/2m - L\hbar\omega)/KT} \\
&\times \;\; \left\{ \mathrm{e}^{n\hbar\omega/2KT} \frac{J_{|L|}^2(p_{Tl} r/\hbar)}{J_{|L+1|}^2(p_{Tl} R/\hbar)} - \mathrm{e}^{-n\hbar\omega/2KT} \frac{J_{|L|}^2(p_{Tl} r/\hbar)}{J_{|L-1|}^2(p_{Tl} R/\hbar)} \right\},
\end{aligned}
$$

where we have purposely restored, for reasons to become clear shortly, the natural constants.

It is very interesting to observe that the functions $D(r)^\pm$, hence $D(r)$, are non-vanishing in the exact non-relativistic limit $c \to \infty$. Indeed, it can be seen from eq. (3.43) that no factor $\hbar$ or c or powers thereof appear as proportionality constants in front of it, because the $D(r)^{\pm}$ dimension is already - in natural units - that of an angular momentum; the only c^2 needed is in the exponent, which is compensated by a shift of the chemical potential, and the only $\hbar$'s needed are those multiplying ω and in the argument of Bessel functions. Since $\hbar$ multiplies ω everywhere and $D(r)$ vanishes for $\omega = 0$, we also see that the difference between canonical and Belinfante densities is essentially a quantum effect, as it vanishes in the limit $\hbar \to 0$; this is expected as the spin tensor exists only for quantum fields.

For very small values of $\hbar\omega/KT$ these two functions are proportional to $\hbar\omega/KT$ itself since $D(r)|_{\omega=0} = 0$, as shown in eq. (3.31). Retaining only the $n = 1$ term of the series, corresponding to the Boltzmann limit of Fermi-Dirac statistics, and expanding the exponentials $\exp(\pm n\hbar\omega/2KT)$ at first order, one obtains the noteworthy equality:

$$D(r)^{\pm} = \hbar \mathrm{tr}\left(\widehat{\rho}\left(:\Psi^{\dagger}\Sigma_z\Psi:\right)^{\pm}\right) \simeq \frac{1}{2}\frac{\hbar\omega}{KT}\hbar \mathrm{tr}\left(\widehat{\rho}\left(:\Psi^{\dagger}\Psi:\right)^{\pm}\right) = \hbar\frac{1}{2}\frac{\hbar\omega}{KT}\left(\frac{\mathrm{d}n}{\mathrm{d}^3\mathbf{x}}\right)^{\pm}, \quad (3.43)$$

where the superscript $\pm$ implies that one retains either the particle or theantiparticle term in the expansion of the free field and $(\mathrm{d}n/\mathrm{d}^3\mathbf{x})^{\pm}$ is, apparently, the particle or antiparticle density. The eq. (3.43) can be shown by retracing all the steps of the calculations carried out for the spin tensor just replacing Σ_z with the identity matrix.

The function $D(r)$ can be computed with available numerical routines finding a sufficient number of zeroes of Bessel functions, according to eq. (3.37). For the numerical computation to be accurate enough one has to make the series in L, l and n quickly convergent at any r. For the series in L, two requirements should be met: first (in natural units) $\omega/T \ll 1$ in order to keep the exponential $\exp[L\omega/T]$ relatively small and, secondly, the radius R should be such that $R\sqrt{mT}$ is not too large; this condition stems from the fact that, as the Bessel functions effectively implement the semiclassical approximation $|L| \simeq p_T R$ and $p_T \approx \sqrt{mT}$, the effective maximal value of L is of the order of $R\sqrt{mT}$. For the series in l, one has to set $m/T \gg 1$, so that large p_T's are strongly suppressed; this is also the non-relativistic limit condition. For the series in n, one has to choose μ so as to keep far from the degenerate Fermi gas case. The function $D(r)$ as a function of r is shown in fig. 3.1 for $\mu = 0$, $R = 300$,

$T = 0.01$, $m = 1$ and two different values of ω, 10^{-4} and $2 \cdot 10^{-4}$; the function $(r/2)D'(r) - D(r)$, which is the difference between angular momentum densities for the canonical and Belinfante tensors, is shown in fig. 3.2.

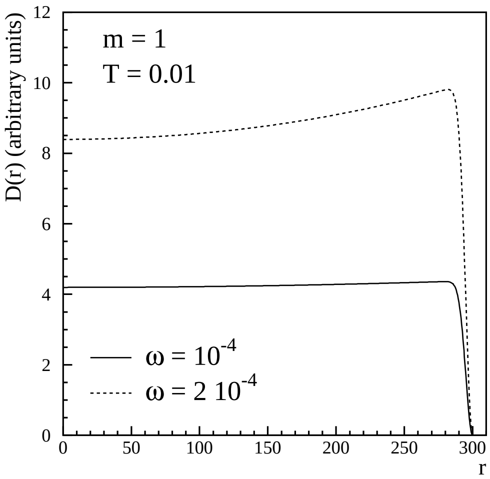

Figure 3.1: Function $D(r)$, corresponding to the mean value of the canonical spin tensor for the free Dirac field, in a rotating cylinder at thermodynamical equilibrium as a function of radius r, in the non-relativistic limit.

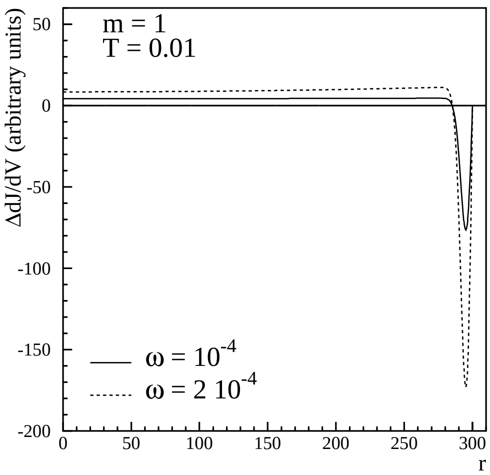

Figure 3.2: Difference between the mean value of the canonical angular momentum density and the Belinfante angular momentum density for the free Dirac field, in a rotating cylinder at thermodynamical equilibrium as a function of radius r, in the non-relativistic limit.

The plots in figs. 3.1,3.2 show that the angular momentum density is larger in the canonical than in the Belinfante case almost everywhere, except for a narrow space near the boundary, whose thickness is plausibly determined by the microscopic scales of the problem (thermal wavelength or Compton wavelenght). Thereby, the observable macroscopic value of the differences between angular momentum densities, for a rotating system of free fermions, is the slowly varying positive one in the bulk. While the boundary conditions are needed to ensure the invariance of the total angular momentum, the rapid drop to zero within a microscopic distance from the cylinder surface tells us that the chosen boundary conditions at a macroscopic scale of observation correspond to a discontinuity or a surface effect. Any *macroscopic* coaxial sub-cylinder of the full cylinder with a radius $r < R$ will therefore have different total angular momenta whether one chooses the canonical or the Belinfante tensors in eqs. (3.2) and (3.3) respectively. Such an ambiguity is physically unacceptable and can be solved only by admitting that these tensors are in fact inequivalent.

3.3 Discussion

We have shown that, in general, couples of stress-energy-momentum and spin tensors related by a pseudo-gauge transformation and allegedly equivalent in quantum field theory, are in fact thermodynamically inequivalent. The inequivalence doesn't show up for systems - familiar in thermal field theory - globally and locally at rest in an inertial frame. The symmetry of the density matrix (2.12) ensures that any couple linked by a pseudo-gauge transformation has the same average values. In this case symmetry constraints are so strong that is not even necessary that the transformation is a valid one; provided that the Hamiltonian $\widehat{H}$ remains the same, even when the boundary condition (1.4) are not fulfilled, and thus changing some of the quantum Poincaré group generators, symmetry ensures that the average four momentum and angular momentum remain the same.

On the other hand it is enough to add a rigid rotation to have a drastically different behavior. The - looser - symmetry of the system is not enough anymore to ensure the equivalence for the average values. We have worked out exhaustively an instance of such inequivalence involving the free Dirac field and shown that, surprisingly, the canonical and Belinfante tensors imply the same average energy density but different mean densities of momentum and angular momentum. Particularly, the latter is almost everywhere larger in the canonical than in the Belinfante case for a small, yet macroscopic, amount even in the non-relativistic limit.

It is a very important issue what is the right couple of tensors; for instance, if it was found that the quantum spin tensor is not the trivial Belinfante one (*i.e.* vanishing) this would have major consequences in hydrodynamics and gravity, even more if its associated stress-energy tensor had a non-symmetric part, because this could imply a torsion of the space-time (for a recent discussion see e.g. ref. [18]). From a theoretical viewpoint we cannot, for the present, determine a thermodynamically "best" couple of stress-energy and spin tensor, but from an experimental viewpoint, in principle we could decide if a specific stress-energy or a spin tensor is *wrong* by measuring with sufficient accuracy the angular momentum density of a rotating system at full thermodynamical equilibrium kept at fixed temperature T and angular velocity ω. This measurement would, for instance, be able to reject the canonical or Belinfante tensor without even the need of resorting to relativistic systems as their difference has a non-vanishing non-relativistic limit, as has been discussed at the end of last section. In practice, at a glance, this measurement would not seem an easy one. According to eq. (3.43), in the non-relativistic limit the difference between these two tensors is of the order of $\hbar\omega/KT$ times $\hbar$ times the particle density, that is particles have a polarization of the order of $\hbar\omega/KT$. This ratio is extremely small for ordinary macroscopic systems; assuming a large angular velocity ω, say 100 Hz, at room temperature $T = 300\,^{\circ}\mathrm{K}$ it turns out to be of the order of 10^{-12}. Notwithstanding, this is precisely the polarization responsible for the observed magneto-mechanical phenomena, the Barnett [19] (magnetization induced by a rotation) and Einstein-De Haas (rotation induced by magnetization) effects. It is therefore possible that with some suitable experiment of this sort one can discriminate between spin tensors, the effect could be enhanced lowering the temperature so much to increase the ratio $\hbar\omega/KT$, *e.g.* with cold atom techniques.

Since the difference between couples of tensors corresponds, at least for the case taken into account, to an average polarization of particles, a legitimate question would be if the presence particle polarization itself requires a fundamental spin tensor. Indeed the canonical spin tensor directly depends on the representation of $SU(2)$ the fields are in. More over that the canonical angular momentum density is mostly, except near the border, the Belinfante angular momentum density plus the average spin tensor, the polarization of particles.

The issue is whether a phenomenological macroscopic spin density, like the magnetization in the Barnett and Einstein-De Haas can be reproduced by the quantum Belinfante stress-energy-momentum tensor or, more generally, by a purely orbital angular momentum, thus eliminating the need of a fundamental spin tensor.

Considering only the free Dirac field for simplicity's sake, it is easier and enlightening to work in the non-relativistic limit. If we take the canonical (3.2) and Belinfante (3.3) couples of tensors, and perform the non-relativistic limit (following [20]) we have:

$$\widehat{T}^{0i}_{\text{canonical}} \xrightarrow{\text{n.r.}} -\frac{i}{2}\mathbf{\Phi}^\dagger \overleftrightarrow{\nabla} \mathbf{\Phi}\Big|^i$$

$$\widehat{T}^{0i}_{\text{Belinfante}} \xrightarrow{\text{n.r.}} \widehat{T}^{0i}_{\text{canonical}} + \frac{1}{2}\nabla \times \mathbf{\Phi}^\dagger \mathbf{\Sigma} \mathbf{\Phi}\Big|^i$$

$$\widehat{\mathcal{S}}^{0,ij}_{\text{canonical}} \xrightarrow{\text{n.r.}} \varepsilon_{ijk} \mathbf{\Phi}^\dagger \Sigma^k \mathbf{\Phi},$$

where Φ is the two-component field (the “large” upper components of the Dirac spinor field in the $c \to \infty$ limit, in the spinorial representation of the gamma matrices), and $\mathbf{\Sigma}$ are the Pauli matrices divided by 2. From the above equation it ensues that the total angular momentum density reads:

$$\begin{aligned} \mathbf{j}_{\text{canonical}} &= \mathbf{x} \times \left(-\frac{i}{2}\mathbf{\Phi}^\dagger \overleftrightarrow{\nabla} \mathbf{\Phi} \right) + \mathbf{\Phi}^\dagger \mathbf{\Sigma} \mathbf{\Phi} \\ \mathbf{j}_{\text{Belinfante}} &= \mathbf{x} \times \left(-\frac{i}{2}\mathbf{\Phi}^\dagger \overleftrightarrow{\nabla} \mathbf{\Phi} \right) + \mathbf{x} \times \left[\frac{1}{2}\nabla \times \mathbf{\Phi}^\dagger \mathbf{\Sigma} \mathbf{\Phi} \right]. \end{aligned}$$

The difference between the Belinfante and canonical angular momentum density can be easily re-expressed as a total derivative, as expected:

$$\mathbf{j}_{\text{Belinfante}} = \mathbf{j}_{\text{canonical}} + \frac{1}{2}\nabla \left(\mathbf{x} \cdot \mathbf{\Phi}^\dagger \mathbf{\Sigma} \mathbf{\Phi} \right) - \frac{1}{2} \sum_j \partial_j \left(x^j \mathbf{\Phi}^\dagger \mathbf{\Sigma} \mathbf{\Phi} \right).$$

Despite the lack of a microscopic spin tensor, even in the Belinfante case there is a contribution of the polarization to the total angular momentum. We wonder at this point why the very fact that the expectation value of $\Phi^\dagger \mathbf{\Sigma} \Phi$ is non-zero (as it is indeed in the Barnett experiment) should rule out the Belinfante tensors or other possible couples with a vanishing spin tensor. It is not even enough to show that $\frac{1}{2}\nabla \left(\mathbf{x} \cdot \Phi^\dagger \mathbf{\Sigma} \Phi \right) - \frac{1}{2} \sum_j \partial_j \left(x^j \Phi^\dagger \mathbf{\Sigma} \Phi \right)$ has a non-vanishing expectation value. Showing that singles terms are non-vanishing does not lead to conclude that a fundamental spin tensor exists unless we show (and this is the crucial point) how much they contribute to total angular momentum density.

Chapter 4

Decomposition of the stress tensor

Thus far, in this work, we have been focusing on momentum densities and equilibrium properties of relativistic fluids. The information of the stress-energy-momentum tensor however is not limited to the four momentum density, as the name implies. Pressure and dissipative forces like shear stress play an important role in hydrodynamics and thermodynamics. The fundamental quantum tensor themselves are related to the entropy production out of equilibrium. These are the main topic of our work [27] and will be discussed in more detail in chapter 6, while in this chapter we will briefly present the decomposition of the macroscopic stress tensor in terms of four velocity proper energy density, pressure and dissipative terms.

We will start from the simplest case, the perfect fluid, and we will limit the discussion to an isotropic fluid for mathematical convenience. It will be understood that anisotropies will require a more general treatment.

4.1 The perfect fluid

The simplest relativistic fluid is the relativistic generalization of the perfect fluid. In non relativistic mechanics a perfect fluid is a continuous system with vanishing shear stress at equilibrium[1] and out of equilibrium, it doesn't have heat flux and all the energy transport occurs through the speed of fluid cells and mechanical transport.

The non relativistic fluid is described by five degrees of freedom. Three for the

[1] Up to this point this is the generic definition of a fluid.

fluid velocity $\mathbf{v}$, one for the density ρ and one for the pressure p. Being a perfect fluid there is no shear stress and, since we assume isotropy as already said, the only source of surface forces is the pressure, a single degree of freedom.

For the relativistic perfect fluid mass density is replaced by proper energy density ε, namely the energy density in the rest frame of the fluid cell[2]. In the non relativistic limit the mass term mc^2 become the dominant part of particles' energy over the kinetic energy term $p^2/2m + \cdots$, therefore $\varepsilon/c^2 \to \rho$ when $c \to \infty$. Three-velocity $\mathbf{v}$ is replaced by four-velocity u^μ. This is a four component vector, but there are still only three degrees of freedom, since the vector field has to be time-like $u \cdot u = 0$. The relativistic perfect fluid is thus described by five degrees of freedom, proper energy density ε, four-velocity u^μ and pressure p.

Having only the four velocity u^μ and the Minkowski metric $\eta^{\mu\nu}$ to build a rank two tensor[3], $T^{\mu\nu}$ will be symmetric. Indeed the only rank two tensor we have are $u^\mu u^\nu$ and $\eta^{\mu\nu}$. If we want to write two linearly independent tensors, we have $u^\mu u^\nu$ and $\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$, namely the projectors along the four velocity and orthogonal. The stress-energy-momentum tensor is therefore:

$$T^{\mu\nu} = \mathsf{A} u^\mu u^\nu + \mathsf{B} \Delta^{\mu\nu}.$$

Now T^{00} is the energy density, using the definition of proper energy density in the local rest frame (where $u(x) = (1, \mathbf{0})$) we have $\varepsilon(x) = T^{00}(x)$, but in the local rest frame Δ^{00} is vanishing, so $\mathsf{A} \equiv \varepsilon$.

Concerning the last coefficient B, we can show that it is the pressure using the local four momentum conservation equation:

$$0 = \partial_\mu T^{\mu\nu} = (D\varepsilon)u^\nu + \varepsilon\, \theta\, u^\nu + \varepsilon A^\nu + \nabla^\nu \mathsf{B} - \mathsf{B}\theta u^\nu - \mathsf{B} A^\nu,$$

where $\theta = \partial \cdot u$ is the divergence of four-velocity, $D = u^\mu \partial_\mu$ is the convective derivative, $\nabla^\nu = \Delta^{\mu\nu}\partial_\mu$ is the spatial derivative[4] and $A^\mu = Du^\mu$ is the four-acceleration.

Contracting with the projector Δ^μ_ν the last equation reads:

[2] Opposed to the laboratory frame.

[3] As we will see later, gradients of u^μ correspond to dissipative terms.

[4] In the comoving frame, they correspond to the temporal derivative and the spatial gradient since u^i and ∇^0 vanish.

$$(\varepsilon - \mathsf{B})\, A^\mu = -\nabla^\mu \mathsf{B}. \tag{4.1}$$

Now we need to see the non-relativistic limit $c \to \infty$ and write explicitly the speed of light in the vacuum c. In the non relativistic limit $u = (\gamma, \gamma \mathbf{v}/c) \to (1, \mathbf{0})$, where $\gamma = (1 - v^2/c^2)$ is the Lorentz factor. Therefore:

$$\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \nabla^\mu \to (0, -\nabla_{\mathbf{x}}),$$

here we wrote $\nabla_{\mathbf{x}}$ to distinguish the actual spatial gradient from relativistic spatial gradient ∇_ν which, in general, is a combination of space and time derivatives. The right hand side of (4.1) correspond in the non-relativistic limit to $\nabla_{\mathbf{x}}\mathsf{B}$. On the other hand the four acceleration is:

$$A^\mu = Du^\mu = \begin{pmatrix} D\gamma \\ (D\gamma)\mathbf{v}/c + \gamma D\mathbf{v}/c \end{pmatrix},$$

and accordingly[5]:

$$D\gamma = D\left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] = -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \frac{2\mathbf{v}\cdot D\mathbf{v}}{c^2} = -\frac{1}{c^2}\gamma^3 \mathbf{v}\cdot D\mathbf{v}$$

$$D\mathbf{v} = u^\mu \partial_\mu \mathbf{v} = \left(\gamma\partial_0 + \gamma\frac{\mathbf{v}}{c}\cdot\nabla_{\mathbf{x}}\right)\mathbf{v} = \frac{1}{c}\gamma\left(\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{x}}\right)\mathbf{v} = \frac{1}{c}\gamma\mathbf{a},$$

where $\mathbf{a}$ is the three-acceleration[6] of a fluid cell. The four-acceleration then reads:

$$A^\mu = \frac{1}{c^2}\begin{pmatrix} -\gamma^4 \mathbf{v}\cdot\mathbf{a}/c \\ -\gamma^4(\mathbf{v}\cdot\mathbf{a})\mathbf{v}/c^2 + \gamma^2\mathbf{a} \end{pmatrix}.$$

Equation (4.1) can be rewritten:

[5] It is worth to remind that $\partial_0 = \frac{1}{c}\partial_t$ if we write explicitly the speed of light.
[6] It is formally identical to the definition of non-relativistic acceleration of a fluid cell.

$$\frac{\varepsilon - \mathsf{B}}{c^2} \begin{pmatrix} -\gamma^4 \mathbf{v} \cdot \mathbf{a}/c \\ -\gamma^4 (\mathbf{v} \cdot \mathbf{a})\mathbf{v}/c^2 + \gamma^2 \mathbf{a} \end{pmatrix} = -\nabla^\nu \mathsf{B},$$

in the non-relativistic limit the time component becomes the trivial equation $0 = 0$, but, being $\lim_{c\to\infty} \varepsilon/c^2 = \rho$, the spatial parts become:

$$\rho\, \mathbf{a} - \lim_{c\to\infty} \frac{\mathsf{B}}{c^2}\, \mathbf{a} = \nabla_{\mathbf{x}} \mathsf{B},$$

taking $\mathsf{B} = -p$ we get the Euler equation, namely the equation of motion of a non-relativistic perfect fluid. Accordingly the stress-energy-momentum tensor of a relativistic perfect fluid is:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu}.$$

We have already seen that three of the components of the local four momentum conservation $\partial_\mu T^{\mu\nu} = 0$ (namely the projection orthogonal to the four-velocity) correspond to the relativistic Euler equation. The non-relativistic limit of the remaining (the projection along the four-velocity) is the continuity equation. If we write it explicitly:

$$0 = u_\nu \partial_\mu T^{\mu\nu} = D\varepsilon + (\varepsilon + p)\,\theta = \partial_\mu (\varepsilon u^\mu) + p\theta.$$

It is convenient here to divide both members by c and write the explicit dependence on the speed of light:

$$\partial_t \left(\gamma \frac{\varepsilon}{c^2}\right) + \nabla_{\mathbf{x}} \left(\gamma \frac{\varepsilon}{c^2} \mathbf{v}\right) + \frac{1}{c^2} p \left[\, \partial_t \gamma + \nabla_{\mathbf{x}} \cdot (\gamma \mathbf{v}) \right] = 0,$$

the first two terms in the non relativistic limit become $\partial_t \rho$ and $\nabla_{\mathbf{x}} \cdot (\rho \mathbf{v})$, while the last one vanishes because $p/c^2 \to 0$, even if the term in brackets ($\nabla_{\mathbf{x}} \cdot \mathbf{v}$ for $c \to \infty$) does not necessarily vanish. The non-relativistic limit is therefore the continuity equation:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0.$$

4.2 Symmetric stress-energy tensor

The next simplest case is that of a general spin-less fluid, namely a non ideal fluid. It is not only defined by proper energy density, pressure and four velocity. It is important at this point to discuss the physical interpretation of four-velocity before going on.

In non-relativistic hydrodynamics the velocity field $\mathbf{v}$ is taken as the average velocity of (classical) particles in the small but finite region of the fluid cell. In relativistic quantum mechanics, which we want to use to get the macroscopic quantities as mean values of corresponding operators, however we can not use the same definition. In quantum mechanics there is not a clear definition of velocity, particles are delocalized and, especially at relativistic energies, there is not even a fixed number of particles[7].

There is not a unique definition of four velocity in relativistic hydrodynamics. We can start from the previous example of a perfect fluid to introduce one. For the perfect fluid the energy flux $T^{i0}(x)$ seen from an observer locally at rest is vanishing, being the stress-energy-momentum tensor diagonal:

$$T^{\mu\nu}|_{\mathsf{R}} = (\varepsilon u^\mu u^\nu - p\Delta^{\mu\nu})\,|_{\mathsf{R}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

where $|_{\mathsf{R}}$ means that we are taking the components on the local rest frame. Therefore we can define the four-velocity of a relativistic fluid as the time-like direction that fulfills:

$$\Delta^\mu_{\ \alpha}\, T^{\alpha\nu} u_\nu = 0 \qquad \Rightarrow \qquad T^{i0}|_{\mathsf{R}} = 0. \tag{4.2}$$

This amounts to the Landau prescription [21] and is usually referred to as the Landau frame. Another widely used definition is that of Eckart [22], where the four-velocity is taken as the direction of a conserved time-like four current j^μ:

$$u^\mu = \frac{j^\mu}{\sqrt{j \cdot j}},$$

[7]Because of pair creation-annihilation and because the state of the system is not constrained to be an eigenstate of particle number operator.

the four current reads then $j^\mu = nu^\mu$ where $n = \sqrt{j \cdot j}$ is the proper charge density[8]. In the Eckart frame (4.2) is not fulfilled an there is an energy flux even in the rest frame (the heat flux), on the other hand the conserved current j^μ in the Landau frame is not always parallel to four-velocity and the right decomposition of the four current is $j^\mu = nu^\mu + \nu^\mu$, where $n = j \cdot u$ is still the proper charge density (having a different four velocity we changed the comoving frame and it is different from the Eckart frame proper charge density) while ν^μ is the diffusion current, non vanishing in the Landau comoving frame. They are the most common choices, nevertheless it is possible to take other definitions of four-velocity and have both a heat flux and a diffusion current.

In the remainder of this work, in order to be as general as possible, we will not make assumptions on the choice of four velocity definition. Therefore we will have a non vanishing heat flux and diffusion current.

As in the previous case we can use the definition of proper energy density to see that $\varepsilon = u_\mu u_\nu T^{\mu\nu}$, and we can decompose the remaining parts of the stress-energy-momentum tensor projecting along the four-velocity direction and perpendicularly. The proper energy density is the projection of both indices along the four-velocity direction, we already saw the heat flux, usually called q:

$$q^\mu = u_\rho T^{\rho\nu} \Delta^\mu_{\ \nu} = \Delta^\mu_{\ \rho} T^{\rho\nu} u_\nu,$$

the last equality is due to the symmetry of $T^{\mu\nu}$. The perfect fluid has a vanishing heat flux. This means that every definition of four-velocity is equivalent for the perfect fluid, and every four current is parallel to the energy flux.

The remaining part, usually referred to as the stress tensor is:

$$\Pi^{\mu\nu} = \Delta^\mu_{\ \rho} \Delta^\nu_{\ \sigma} T^{\rho\sigma},$$

this is usually divided in the trace and traceless part[9]:

$$-\frac{1}{3}\Pi^{\mu\nu}\eta_{\mu\nu} = -\frac{1}{3}\Pi^{\mu\nu}\Delta_{\mu\nu} = p - \pi_0$$

[8] The charge density in a particular frame is j^0, the proper charge density is the density in the comoving frame and is a Lorentz scalar.

[9] In order to have al the parts of $T^{\mu\nu}$ divided in tensors on the same representation of the rotation group.

$$\pi^{\mu\nu} = \Pi^{\mu\nu} - \frac{1}{3}\Pi^{\rho\sigma}\eta_{\rho\sigma}\Delta^{\mu\nu},$$

the perfect fluid has both $\pi^{\mu\nu}$ and π_0 vanishing, which we will see to correspond to a vanishing shear and bulk viscosity. Pressure p can not be isolated here from algebraic consideration, and it needs thermodynamics to be separated from the dissipative part π_0.

The most general form of a symmetric stress-energy-momentum tensor is thus:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \pi_0\Delta^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \pi^{\mu\nu}. \tag{4.3}$$

We can easily recognize the perfect fluid part, and using thermodynamics we can clarify the role of the other terms. Let us start from the four-momentum conservation equation:

$$0 = \partial_\mu T^{\mu\nu} = D\epsilon\, u^\nu + \epsilon\,\theta\, u^\nu + \epsilon A^\nu - \nabla^\nu p + p\,\theta\, u^\nu + pA^\nu + \nabla^\nu \pi_0 - \pi_0\,\theta\, u^\nu - \pi_0 A^\nu +$$

$$+(\partial\cdot q)u^\nu + q\cdot\partial u^\nu + \theta q^\nu + Dq^\nu + \partial_\mu \pi^{\mu\nu},$$

contracting all the terms with the four-velocity, the last equation reads:

$$D\epsilon + (\epsilon + p)\theta + \partial\cdot q + u\cdot Dq + u_\nu\partial_\mu\pi^{\mu\nu} - \pi_0\theta = 0. \tag{4.4}$$

It is worth now to rewrite some terms in a more convenient form:

$$\partial\cdot q = \partial\cdot\left(T_0\frac{q}{T_0}\right) = T\partial\cdot\left(\frac{q}{T_0}\right) + \frac{1}{T_0}q\cdot\partial T_0 = T_0\partial\cdot\left(\frac{q}{T_0}\right) + \frac{1}{T_0}q\cdot\nabla T_0$$

$$u\cdot Dq = D\left(u\cdot q\right) - q\cdot Du = -q\cdot A$$

$$u_\nu\partial_\mu\pi^{\mu\nu} = \partial_\mu\left(\pi^{\mu\nu}u_\nu\right) - \pi^{\mu\nu}\partial_\mu u_\nu = -\frac{1}{2}\pi^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle},$$

where the scalar T_0 is the local temperature $T_0 = 1/\sqrt{\beta\cdot\beta}$, *i.e.* the temperature measured in a comoving frame; and $\nabla_{\langle\mu}u_{\nu\rangle}$ denotes:

$$\nabla_{\langle\mu}u_{\nu\rangle} = \left(\Delta^{\rho}_{\ \mu}\Delta^{\sigma}_{\ \nu} + \Delta^{\rho}_{\ \nu}\Delta^{\sigma}_{\ \mu} - 2/3\Delta_{\mu\nu}\Delta^{\rho\sigma}\right)\partial_\rho u_\sigma,$$

which has the same algebraic properties of $\pi^{\mu\nu}$, symmetric in he $\mu \leftrightarrow \nu$ exchange, orthogonal to the four-velocity in both indices and traceless.

Equation (4.4) now reads:

$$D\epsilon + (\epsilon + p)\theta + T_0\partial \cdot \left(\frac{q}{T_0}\right) + \frac{1}{T_0}q \cdot (\nabla T_0 - T_0 A) - \frac{1}{2}\pi^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle} - \pi_0\theta = 0. \quad (4.5)$$

Now we can use the known thermodynamic relations between proper entropy density, temperature, proper energy density and pressure to rewrite the last equation as the divergence of the entropy flux, which must be non negative. The relations between proper energy density, pressure temperature and proper entropy density, for a system at equilibrium are the relativistic generalizations of the usual thermodynamic relations for volumetric densities:

$$T_0 s = \epsilon + p \qquad T_0 \mathrm{d}s = \mathrm{d}\epsilon. \quad (4.6)$$

These relations only hold for thermodynamic equilibrium, but an usual approximation is to assume that the same relations even hold for a system out of equilibrium. This amounts to neglect other contributions to entropy, being them small as long as the system is close enough to equilibrium, since they have to vanish at equilibrium. Using the last relations out of equilibrium, the first and the second term of (4.5) read:

$$\begin{cases} (\epsilon + p)\theta = T_0 s\,\theta \\ \\ D\epsilon = T_0 D s \end{cases} \quad \Rightarrow \quad D\epsilon + (\epsilon + p)\theta = T_0\partial \cdot (su), \quad (4.7)$$

therefore we have:

$$T_0\partial_\mu \left(su^\mu + \frac{1}{T_0}q^\mu\right) = \pi_0\theta + \frac{1}{2}\pi^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle} - \frac{1}{T_0}q_\mu\left(\nabla^\mu T_0 - T_0 A^\mu\right). \quad (4.8)$$

We can recognize in the left hand side the temperature times the divergence of the entropy flux[10]. The right hand side must be therefore non-negative and vanish at thermodynamical equilibrium. The latter can be easily verified considering that the equilibrium four temperature fulfills a Killing equation [23]:

$$\Big(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu\Big)_{\text{eq.}} = 0. \tag{4.9}$$

Indeed if we write the derivatives $\nabla_\mu u_\nu$ explicitly in their β_μ dependence:

$$\nabla_\mu u_\nu = \nabla_\mu \left(\frac{\beta_\nu}{\sqrt{\beta\cdot\beta}}\right) = T_0 \Delta^\rho_{\ \mu}\delta^\sigma_\nu \partial_\rho\beta_\sigma - \frac{1}{2}T_0^3 \Delta^\rho_{\ \mu}\beta_\nu 2\beta^\sigma\partial_\rho\beta_\sigma = T_0\Delta^\rho_{\ \mu}\Delta^\sigma_{\ \nu}\partial_\rho\beta_\sigma, \tag{4.10}$$

we can write all the multiplying factors of π_0, $\pi^{\mu\nu}$ and q_μ in the right hand side of (4.8) as derivatives of the four temperature:

$$\theta = \nabla_\mu u^\mu = T_0 \Delta^\rho_{\ \mu}\Delta^{\sigma\mu}\partial_\rho\beta_\sigma = T_0\Delta^{\rho\sigma}\partial_\rho\beta_\sigma \equiv \frac{1}{2}T_0\Delta^{\rho\sigma}\left(\partial_\rho\beta_\sigma + \partial_\sigma\beta_\rho\right), \tag{4.11}$$

while:

$$\nabla_{\langle\mu}u_{\nu\rangle} = T_0\left(\Delta^\rho_{\ \mu}\Delta^\sigma_{\ \nu} - \frac{1}{3}\Delta_{\mu\nu}\Delta^{\rho\sigma}\right)\left(\partial_\rho\beta_\sigma + \partial_\sigma\beta_\rho\right), \tag{4.12}$$

and finally:

$$\begin{aligned}
&\nabla^\mu T_0 - T_0 A^\mu = \left(g^{\mu\rho} - \frac{\beta^\mu\beta^\rho}{\beta\cdot\beta}\right)\partial_\rho\left[(\beta\cdot\beta)^{-\frac{1}{2}}\right] - (\beta\cdot\beta)^{-\frac{1}{2}}(\beta\cdot\beta)^{-\frac{1}{2}}\beta^\rho\partial_\rho\left[(\beta\cdot\beta)^{-\frac{1}{2}}\beta^\mu\right] = \\
&= -\frac{1}{2}(\beta\cdot\beta)^{-\frac{3}{2}}\left(g^{\mu\rho} - \frac{\beta^\mu\beta^\rho}{\beta\cdot\beta}\right)2\beta^\sigma\partial_\rho\beta_\sigma - (\beta\cdot\beta)^{-\frac{3}{2}}\left(g^{\mu\sigma}\partial_\rho\beta_\sigma - \frac{\beta^\mu\beta^\sigma}{\beta\cdot\beta}\right)\beta^\rho\partial_\rho\beta_\sigma = \\
&= -T^3\left[\left(g^{\mu\rho} - \frac{\beta^\mu\beta^\rho}{\beta\cdot\beta}\right)\beta^\sigma + \left(g^{\mu\sigma}\partial_\rho\beta_\sigma - \frac{\beta^\mu\beta^\sigma}{\beta\cdot\beta}\right)\beta^\rho\right]\partial_\rho\beta_\sigma = \\
&= -T^3\Delta^{\mu\rho}\beta^\sigma\left(\partial_\rho\beta_\sigma + \partial_\sigma\beta_\rho\right).
\end{aligned} \tag{4.13}$$

Plugging (4.9) into (4.11), (4.12) and (4.13) we find that the right hand side of (4.8) is vanishing at equilibrium.

[10]This amounts to the entropy transport, su^μ entropy density time four velocity, plus the heat flux. We remind here that q^μ is the energy flux in the rest frame.

To ensure that (4.8) has a non-negative right hand side out of equilibrium we take the same approach of non relativistic hydrodynamics and insert transport coefficients:

$$\pi_0 = \zeta\theta = \zeta\left[\partial_t\gamma + \nabla_{\mathbf{x}}\cdot(\gamma\mathbf{v})\right],$$

where ζ is the relativistic generalization of bulk viscosity,

$$\pi_{\mu\nu} = \eta\nabla_{\langle\mu}u_{\nu\rangle},$$

while η is the relativistic shear viscosity,

$$q^\mu = \kappa\left(\nabla^\mu T_0 - T_0 A^\mu\right),$$

and κ is the relativistic thermal conductivity. We can write then (4.3) using transport coefficients:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \zeta\,\theta\,\Delta^{\mu\nu} + \kappa\left[\left(\nabla^\mu T_0 - T_0 A^\mu\right)u^\nu + \mu\leftrightarrow\nu\right] + \eta\nabla_{\langle\mu}u_{\nu\rangle}. \quad (4.14)$$

It can be easily checked that in the non relativistic limit the heat flux become[11] $(0, -\kappa\nabla_{\mathbf{x}}T)$ as we would expect from the Fourier law, the four-acceleration contribution vanishes because A^μ is of higher order in $1/c$. In a similar way the bulk viscosity term become $\zeta\nabla_{\mathbf{x}}\cdot\mathbf{v}\delta_{ij}$ and the last one become $\eta\left(\partial_i v_j + \partial_j v_i - 2/3\nabla_{\mathbf{x}}\cdot\mathbf{v}\,\delta_{ij}\right)/2$. With the pressure term they form the non relativistic stress tensor. This is the relativistic Navier-Stokes theory, the relativistic covariant extension of the ordinary Navier-Stokes equations.

Before continuing with the next section and the generalization to a non symmetric stress-energy-momentum tensor, it is worth to note that the approximation used in (4.7), namely to use the equilibrium relations between entropy pressure and temperature out of equilibrium, lead to causality violation and instabilities [24], it is possible to obtain a causal theory if we add other terms to non-equilibrium entropy [25] thereby obtaining new transport coefficients, usually relaxation times of dissipative currents. The relativistic Navier-Stokes theory however is still used as a low-frequency approx-

[11] In non-relativistic thermodynamics both the local temperature $T_0 = 1/\sqrt{\beta\cdot\beta}$ and the laboratory temperature $T = 1/\beta^0$ correspond to the ordinary temperature of the fluid cell.

imation of causal dissipative theories. We are only interested in the lowest approximation, therefore in the rest of this work we will not take into account causality and stability problems, and we will limit the discussion to the relativistic Navier-Stokes dissipative equations.

4.3 The general case

In the most general case, the fluid system have a non vanishing spin tensor and, due to local angular momentum conservation, the divergence of the spin tensor will be the non symmetric part of the stress-energy-momentum tensor:

$$\partial_\lambda \mathcal{S}^{\lambda,\mu\nu} = -2T_{\rm a}^{\mu\nu} \qquad T^{\mu\nu} = T_{\rm s}^{\mu\nu} + T_{\rm a}^{\mu\nu} \qquad \begin{cases} T_{\rm s}^{\mu\nu} = \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right) \\ T_{\rm a}^{\mu\nu} = \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right). \end{cases}$$

In this section we want to see if we can extend the relativistic Navier-Stokes dissipative theory to a non-symmetric stress tensor. We can decompose the symmetric part $T_{\rm s}^{\mu\nu}$ as in (4.3):

$$T_{\rm s}^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \pi_0\Delta^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \pi^{\mu\nu},$$

and, using the same approach, we can decompose in a similar manner the antisymmetric part $T_{\rm a}^{\mu\nu}$:

$$T_{\rm a}^{\mu\nu} = R^{\mu\nu} + r^\mu u^\nu - u^\mu r^\nu,$$

where:

$$R^{\mu\nu} = \Delta^\mu_{\ \rho}\, T_{\rm a}^{\rho\sigma}\, \Delta^\nu_{\ \sigma} \qquad\qquad r^\mu = \Delta^\mu_{\ \rho}\, T_{\rm a}^{\rho\nu}\, u_\nu = -u_\nu\, T_{\rm a}^{\nu\rho}\, \Delta^\mu_{\ \rho}.$$

Therefore the stress-energy-momentum tensor reads:

$$\begin{aligned} T^{\mu\nu} = T_{\rm s}^{\mu\nu} + T_{\rm a}^{\mu\nu} &= \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \pi_0\Delta^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \pi^{\mu\nu} + \\ &\quad + R^{\mu\nu} + r^\mu u^\nu - u^\mu r^\nu. \end{aligned} \tag{4.15}$$

We could follow the same procedure used for the symmetric tensor, taking the local four-momentum conservation equation, contracting with the four-velocity and, assuming the equilibrium thermodynamical relations hold even out of equilibrium, finally extract the divergence of entropy current. It is important however to stress that the relations (4.6), even at equilibrium, only hold for a spin-less fluid, we can expect a contribution to entropy from the spin tensor (for instance in [26] they have an additional term for both relations). Furthermore we can expect another term in the entropy current. The energy flux in the local rest frame is not anymore q^μ being there another non vanishing term r^μ, so we expect:

$$s^\mu = su^\mu + \frac{1}{T_0}\left(q^\mu + r^\mu\right) + \textsf{other possible terms.}$$

In order to extend the relativistic Navier-Stokes theory to a system with a non symmetric part of $T^{\mu\nu}$ we will use a new approach. Let us start with the general entropy curent (see B.1 in Appendix B):

$$s^\mu = \phi^\mu + T^{\mu\nu}\beta_\nu - \frac{1}{2}\mathcal{S}^{\mu,\alpha\beta}\omega_{\alpha\beta}. \tag{4.16}$$

The divergence of entropy current, which must be non-negative, reads:

$$\begin{aligned} \partial\cdot s &= \partial\cdot\phi + \left(\partial_\mu T^{\mu\nu}\right)\beta_\nu + T^{\mu\nu}\partial_\mu\beta_\nu - \frac{1}{2}\left[\left(\partial_\lambda\mathcal{S}^{\lambda,\mu\nu}\right)\omega_{\alpha\beta} + \mathcal{S}^{\lambda,\mu\nu}\left(\partial_\lambda\omega_{\mu\nu}\right)\right] = \\ &= \partial\cdot\phi + T^{\mu\nu}_{\mathrm{s}}\partial_{\{\mu}\beta_{\nu\}} + T^{\mu\nu}_{\mathrm{a}}\partial_{[\mu}\beta_{\nu]} - \frac{1}{2}\mathcal{S}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu} + T^{\mu\nu}_{\mathrm{a}}\omega_{\mu\nu} = \\ &= \partial\cdot\phi + T^{\mu\nu}_{\mathrm{s}}\partial_{\{\mu}\beta_{\nu\}} + T^{\mu\nu}_{\mathrm{a}}\left(\partial_{[\mu}\beta_{\nu]} + \omega_{\mu\nu}\right) - \frac{1}{2}\mathcal{S}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}, \end{aligned} \tag{4.17}$$

where local four-momentum and angular momentum conservation have been used, and $\partial_{\{\mu}\beta_{\nu\}}$ and $\partial_{[\mu}\beta_{\nu]}$ stand for the symmetrization and antisymmetrization of the μ,ν indices. Hereafter we will show how to recover the relativistic Navier-Stokes theory from this equation if there is no internal angular momentum, then the same procedure will be applied to the general case in order to extend the theory.

Recovering the relativistic Navier-Stokes theory

When we have a vanishing spin tensor the last equation become:

$$\partial \cdot s = \partial \cdot \phi + T_{\rm s}^{\mu\nu} \partial_{\{\mu} \beta_{\nu\}}, \tag{4.18}$$

but the last term is, due to the decomposition (4.3):

$$T_{\rm s}^{\mu\nu} \partial_{\{\mu} \beta_{\nu\}} = \varepsilon u^\mu u^\nu \partial_\mu \beta_\nu + \Delta^{\mu\nu} \partial_\mu \beta_\nu \left(\pi_0 - p \right) + \pi^{\mu\nu} \partial_\mu \beta_\nu + q^\mu u^\nu \left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu \right),$$

reminding that $\beta = 1/T_0 u$ and the algebraic properties of q and $\pi_{\mu\nu}$, we have:

$$\varepsilon u^\mu u^\nu \partial_\mu \beta_\nu = \left[\frac{1}{T_0} \varepsilon u^\mu u^\nu \partial_\mu u_\nu \equiv 0 \right] + \varepsilon D \left(\frac{1}{T_0} \right) = \varepsilon D \left(\frac{1}{T_0} \right)$$

$$\pi^{\mu\nu} \partial_\mu \beta_\nu = \frac{1}{2} \pi^{\mu\nu} \left(\Delta^{\rho}_{\ \mu} \Delta^{\sigma}_{\ \nu} - \frac{1}{3} \Delta_{\mu\nu} \Delta^{\rho\sigma} \right) \left(\partial_\rho \beta_\sigma + \partial_\sigma \beta_\rho \right)$$

$$q^\mu u^\nu \left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu \right) = q_\mu \Delta^{\mu}_{\ \rho} u_\sigma \left(\partial_\rho \beta_\sigma + \partial_\sigma \beta_\rho \right),$$

and using (4.11), (4.12) and (4.13) we have:

$$\Delta^{\mu\nu} \partial_\mu \beta_\nu \left(\pi_0 - p \right) = \frac{1}{T_0} \pi_0 \, \theta - \frac{1}{T_0} p \, \theta$$

$$\pi^{\mu\nu} \partial_\mu \beta_\nu = \frac{1}{2T_0} \pi^{\mu\nu} \nabla_{<\mu} u_{\nu>}$$

$$q^\mu u^\nu \left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu \right) = -\frac{1}{T_0^2} q_\mu \left[\nabla^\mu T_0 - T_0 A^\mu \right].$$

Plugging all in (4.18) and multiplying T_0 we have:

$$T_0 \partial \cdot s = T_0 \partial \cdot \phi + T_0 \varepsilon D \left(\frac{1}{T_0} \right) - p \, \theta + \pi_0 \, \theta + \frac{1}{2} \pi^{\mu\nu} \nabla_{<\mu} u_{\nu>} - \frac{1}{T_0} q_\mu \left[\nabla^\mu T_0 - T_0 A^\mu \right].$$

Provided that the first three terms on the right hand side vanish, this is the entropy production for the relativistic Navier-Stokes theory. Let us see now the proper entropy density:

$$s = u_\mu s^\mu \quad \Rightarrow \quad T_0 s = T_0 \phi \cdot u + \varepsilon,$$

this is the first of the (4.6), as long as[12]:

$$T_0 \phi \cdot u = p,$$

and it actually can be used as a thermodynamical definition of pressure (which we remind that it can not be separated from π_0 only using algebraic properties). Thus:

$$\phi^\mu = p\beta^\mu + \tilde{\phi}^\mu,$$

therefore we have:

$$\partial \cdot \phi = \partial \cdot \tilde{\phi} + \frac{1}{T_0} p\,\theta + D\left(\frac{p}{T_0}\right).$$

and accordingly:

$$T_0 \partial \cdot \phi + T_0 \varepsilon D\left(\frac{1}{T_0}\right) - p\,\theta = T_0 \partial \cdot \tilde{\phi} + T_0 D\left(\frac{p}{T_0}\right) + T_0 \varepsilon D\left(\frac{1}{T_0}\right) =$$

$$= T_0 \partial \cdot \tilde{\phi} + T_0 D\left(\frac{\varepsilon + p}{T_0}\right) - D\varepsilon = T_0 \partial \cdot \tilde{\phi} + T_0 D s - D\varepsilon.$$

The entropy production is then:

$$T_0 \partial \cdot s = T_0 \partial \cdot \tilde{\phi} + T_0 D s - D\varepsilon + \pi_0\,\theta + \frac{1}{2}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T_0} q_\mu \left[\nabla^\mu T_0 - T_0 A^\mu\right].$$

Entropy production must vanish at equilibrium. Since we have already seen that the last three terms vanish at equilibrium, it means that:

$$\left[T_0 \partial \cdot \tilde{\phi} + T_0 D s - D\varepsilon\right]_{\text{eq.}} = 0,$$

to recover the entropy production of the relativistic Navier-Stokes theory we assume

[12] This is true for homogeneous equilibrium [23]

that this expression is vanishing even out of equilibrium. This is consistent with the theory and we fully recover the relativistic Navier-Stokes theory if we assume that both $\partial \cdot \tilde{\phi}$ and $T_0 Ds - D\varepsilon$ are individually vanishing.

Internal angular momentum

Using the general decomposition (4.15), when we have a non vanishing spin tensor, the entropy current reads:

$$s^\mu = \phi^\mu - \frac{1}{2}\mathcal{S}^{\mu,\alpha\beta}\omega_{\alpha\beta} + \frac{1}{T_0}\left[\, \varepsilon u^\mu + q^\mu + r^\mu \right].$$

We can already find the proper energy density contracting with the four-velocity:

$$s = u_\mu s^\mu = \phi \cdot u + \frac{1}{T_0}\varepsilon - \frac{1}{2}\sigma^{\mu\nu}\omega_{\mu\nu},$$

where $\sigma^{\mu\nu} = u_\lambda \mathcal{S}^{\lambda,\mu\nu}$ is the proper internal angular momentum density. If we multiply all members of the last equation with the local temperature T_0 we get:

$$T_0 s = \varepsilon + T_0 \phi \cdot u - \frac{1}{2}\sigma^{\mu\nu}\Omega_{\mu\nu},$$

with $\Omega_{\mu\nu} = T_0\omega_{\mu\nu}$. We use now the same thermodynamical definition for pressure used in the last paragraph, then:

$$T_0 s = \varepsilon + p - \frac{1}{2}\sigma^{\mu\nu}\Omega_{\mu\nu}.$$

This provides the generalization of the first relation in (4.6), and, as expected, it includes a spin contribution. We can rearrange the first terms of (4.17) like in the last paragraph but we have to remind that the proper entropy density is different:

$$T_0\partial \cdot s = T_0\partial \cdot \phi + T_0\varepsilon D\left(\frac{1}{T_0}\right) - p\,\theta + \pi_0\,\theta + \frac{1}{2}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>} - \frac{1}{T_0}q_\mu\left[\, \nabla^\mu T_0 - T_0 A^\mu\right] +$$

$$-\frac{1}{2}T_0\mathcal{S}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu} + T_0 T^{\mu\nu}_{\rm a}\left(\, \partial_{[\mu}\beta_{\nu]} + \omega_{\mu\nu}\right),$$

whilst the last term includes:

$$T_{\rm a}^{\mu\nu}\left(\partial_{[\mu}\beta_{\nu]}+\omega_{\mu\nu}\right)=R^{\mu\nu}\Delta^{\rho}_{\mu}\Delta^{\sigma}_{\nu}\left(\partial_{[\rho}\beta_{\sigma]}+\omega_{\rho\sigma}\right)+r^{\mu}\Delta^{\rho}_{\mu}u^{\sigma}\left[\left(\partial_{\rho}\beta_{\sigma}-\partial_{\sigma}\beta_{\rho}\right)+2\omega_{\rho\sigma}\right],$$

Since the four-temperature fulfills (4.9), these terms vanish at global equilibrium. We have already seen that the terms in π_0, $\pi^{\mu\nu}$ and q^μ on the entropy production equation vanish at equilibrium too. On the other hand, the other terms on the right hand side read, following the steps in the last paragraph:

$$\begin{aligned}
&T_0\partial\cdot\phi+T_0\varepsilon D\left(\frac{1}{T_0}\right)-p\,\theta-\frac{1}{2}T_0\mathcal{S}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}=\\
&=T_0\partial\cdot\tilde{\phi}+T_0D\left(\frac{\varepsilon+p}{T_0}\right)-D\varepsilon-\frac{1}{2}T_0\sigma^{\mu\nu}D\omega_{\mu\nu}-\frac{1}{2}T_0\tilde{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}=\\
&=T_0\partial\cdot\tilde{\phi}-\frac{1}{2}T_0\tilde{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}+T_0D\left(\frac{\varepsilon+p-\frac{1}{2}\sigma^{\mu\nu}\Omega_{\mu\nu}}{T_0}\right)-D\varepsilon+\frac{1}{2}\Omega_{\mu\nu}D\sigma^{\mu\nu}=\\
&=T_0\partial\cdot\tilde{\phi}-\frac{1}{2}T_0\tilde{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}+T_0Ds-D\varepsilon+\frac{1}{2}\Omega_{\mu\nu}D\sigma^{\mu\nu},
\end{aligned}$$

where $\tilde{\mathcal{S}}^{\lambda,\mu\nu}=\Delta^{\lambda}_{\alpha}\mathcal{S}^{\lambda,\mu\nu}=\mathcal{S}^{\lambda,\mu\nu}-u^\lambda\sigma^{\mu\nu}$. At global equilibrium this term has to vanish. Since $\omega_{\mu\nu}$ is constant at equilibrium [23], we need:

$$\left[T_0\partial\cdot\tilde{\phi}+T_0Ds-D\varepsilon+\frac{1}{2}\Omega_{\mu\nu}D\sigma^{\mu\nu}\right]_{\rm eq.}=0.$$

We assume now that the equilibrium relation between thermodynamical quantities even holds for a system out of equilibrium, as we did for the spin-less fluid, in order to have the generalization of Navier-Stokes theory. The entropy production thus reads:

$$\begin{aligned}
T_0\,\partial\cdot s\;=\;&\pi_0\,\theta+\frac{1}{2}\pi^{\mu\nu}\nabla_{<\mu}u_{\nu>}-\frac{1}{T_0}q_\mu\left[\nabla^\mu T_0-T_0A^\mu\right]+\\
&+T_0R^{\mu\nu}\Delta^{\rho}_{\mu}\Delta^{\sigma}_{\nu}\left(\partial_{[\rho}\beta_{\sigma]}+\omega_{\rho\sigma}\right)+T_0r^{\mu}\Delta^{\rho}_{\mu}u^{\sigma}\left[\left(\partial_\rho\beta_\sigma-\partial_\sigma\beta_\rho\right)+2\omega_{\rho\sigma}\right]+\\
&-\frac{1}{2}T_0\tilde{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}.
\end{aligned}$$

It is important to stress how adding internal angular momentum preserve the $T_{\rm s}^{\mu\nu}$

contribution to entropy production, therefore we still have shear viscosity η bulk viscosity ζ and thermal conductivity κ. However, as we will see later in detail, the presence of a fundamental spin tensor in general change the *values* of these coefficients.

The other terms can provide a new generation of transport coefficients. Being more interested however on the effects of pseudo-gauge transformations on the known transport coefficients here we will limit the discussion to the simplest case, namely of a system with a vanishing $\tilde{\mathcal{S}}$. The antisymmetric part $T^{\mu\nu}_{\rm a}$ amounts, up to a constant, to the divergence of $\mathcal{S}^{\lambda,\mu\nu}$, which includes $\tilde{\mathcal{S}}^{\lambda,\mu\nu}$. Therefore $R^{\mu\nu}$ and r^μ are not independent of $\tilde{\mathcal{S}}^{\lambda,\mu\nu}$ in principle. If $\tilde{\mathcal{S}}^{\lambda,\mu\nu}$ is vanishing the contribution of $R^{\mu\nu}$ and r^μ to entropy are:

$$T_0 R^{\mu\nu}\Delta^\rho_{\ \mu}\Delta^\sigma_{\ \nu}\left[\omega_{\rho\sigma}-\left(-\frac{1}{2}\right)(\partial_\rho\beta_\sigma-\partial_\sigma\beta_\rho)\right]$$

$$2T_0^2 r_\mu\Delta^{\mu\rho}\beta^\sigma\left[\left(-\frac{1}{2}\right)(\partial_\rho\beta_\sigma-\partial_\sigma\beta_\rho)-\omega_{\rho\sigma}\right].$$

As we have seen the equilibrium relation between $\omega_{\mu\nu}$ and β_μ (4.9) ensures that the equilibrium contribution of the two terms vanish. If $\omega_{\mu\nu}=-\partial_{[\mu}\beta_{\nu]}$ even out of equilibrium r^μ and $R^{\mu\nu}$ do not contribute to entropy production, otherwise the last two terms have to be non negative out of equilibrium, thus we have two new transport coefficients, λ and τ:

$$R^{\mu\nu}=\lambda\left\{T_0\,\Delta^\rho_{\ \mu}\Delta^\sigma_{\ \nu}\left[\omega_{\rho\sigma}-\left(-\frac{1}{2}\right)(\partial_\rho\beta_\sigma-\partial_\sigma\beta_\rho)\right]\right\}$$

$$r^\mu=\tau\left\{2T_0^3\Delta^{\mu\rho}\beta^\sigma\left[\left(-\frac{1}{2}\right)(\partial_\rho\beta_\sigma-\partial_\sigma\beta_\rho)-\omega_{\rho\sigma}\right]\right\}.$$

In order to be consistent with thermodynamics, $\omega_{\mu\nu}$ still have to be equivalent to $-\partial_{[\mu}\beta_{\nu]}$ at global equilibrium. It is not difficult to find examples of antisymmetric tensors which reduce to the antisymmetric part of four-temperature gradient. For instance we can take $\omega_{\mu\nu}=-\partial_{[\mu}\beta_{\nu]}$ plus higher order derivatives of four temperature[13].

[13] As already said four temperature gradient is constant at equilibrium [23].

Chapter 5

Relativistic response theory

5.1 Introduction

In the last chapter we have seen the decomposition of the stress-energy-momentum tensor and how transport coefficients, like viscosity and thermal conductivity, arise if we take into account thermodynamic relations. During this chapter we will show how we can get transport coefficients from the microscopic theory studying the linear response, namely the response of the hydrodynamic system to a deviation from global equilibrium, up to the linear order in the perturbation, allegedly small.

Here we consider for mathematical simplicity the example of shear viscosity for a fluid without internal angular momentum. However it is understood that the same arguments hold for the other transport coefficients. Furthermore when we consider a non symmetric stress-energy-momentum tensor, the arguments are still valid using the symmetric part $T_{\rm s}^{\mu\nu}$ instead of the full tensor $T^{\mu\nu}$. Other transport coefficients which may stem from the antisymmetric part $T_{\rm a}^{\mu\nu}$ will not be considered during this work, and may be an interesting subject for further investigations.

Starting from the general stress-energy-momentum tensor (4.14) we write explicitly the temperature and four-velocity dependence[1]:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \zeta\left(\partial_\rho u^\rho\right)\Delta^{\mu\nu} + \kappa\left[\left(\nabla^\mu T - Tu^\rho\partial_\rho u^\mu\right)u^\nu + \mu\leftrightarrow\nu\right] +$$

[1] We understand however the u^μ dependence of the projector $\Delta^{\mu\nu}$ to ease the notation.

$$+\eta\left[\nabla^{\mu}u^{\nu}+\nabla^{\nu}u^{\mu}-\frac{2}{3}\left(\partial_{\rho}u^{\rho}\right)\Delta^{\mu\nu}\right].$$

A very familiar global equilibrium configuration is the homogeneous equilibrium, in which the system is static and the macroscopic degrees of freedom do not depend on the position. Looking from the rest frame the four velocity is $u=(1,\mathbf{0})$ and the temperature T is constant. This corresponds to the canonical equilibrium in statistical mechanics[2], as we have already seen on chapter 2.2 it entails that the only surviving terms are the first two of the right hand side:

$$T^{\mu\nu}_{\text{h.e.}}=\varepsilon\delta^{\mu}_{0}\delta^{\nu}_{0}+p\delta^{\mu}_{i}\delta^{\nu}_{j}\delta_{ij}=\begin{pmatrix}\varepsilon & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & p\end{pmatrix},$$

where $|_{\text{h.e.}}$ stands for homogeneous equilibrium. We consider now a small deviation from global equilibrium in the form of a small perturbation of the y component of four velocity[3] δu^{y}. Furthermore we take a perturbation with a non vanishing gradient only on the x direction:

$$\delta u^{y}(x)\Rightarrow\partial_{x}\delta u^{y}\neq 0.$$

This entails that the four-divergence $\partial_{\rho}u^{\rho}$ is vanishing, being the derivative on the same direction of the vector index. Even the four-acceleration $A^{\mu}=u^{\rho}\partial_{\rho}u^{\mu}$ is vanishing, since the only non vanishing element of the gradient, namely $\partial_{x}u^{y}\equiv\partial_{x}\delta u^{y}$, is multiplied by $u^{x}=0$. Therefore the remaining parts of the stress-energy-momentum tensor out of equilibrium only are:

$$\varepsilon u^{\mu}u^{\nu}-p\Delta^{\mu\nu}+\eta\left(\nabla^{\mu}u^{\nu}+\nabla^{\nu}u^{\mu}\right),$$

but being:

$$\nabla^{\mu}u^{\nu}=\partial^{\mu}u^{\nu}-u^{\mu}u^{\rho}\partial_{\rho}u^{\nu},$$

[2]Or grand canonical, if we consider conserved charges at equilibrium with a thermal bath.
[3]Here we assume a Cartesian reference, meanwhile the direction of the axes has no effect.

and having seen that $u^\rho \partial_\rho u^\nu = 0$, it reads:

$$\varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \eta \left(\partial^\mu u^\nu + \partial^\nu u^\mu\right) = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} - \eta \partial_x \delta u^y \left(\delta^\mu_x \delta^\nu_y + \delta^\nu_x \delta^\mu_y\right).$$

Being the homogeneous equilibrium tensor diagonal, the deviation from equilibrium of the xy component of the stress-energy-momentum tensor is just:

$$\delta T^{xy} = T^{xy} - T^{xy}|_{\text{h.e.}} = -\eta\, \partial_x \delta u^y = \eta\, \partial_x \delta u_y,$$

the multiplication of the shear viscosity η, times the known gradient of the perturbation of the four-velocity $\partial_x \delta u_y$. This is already linear in the perturbation δu, but in general it may be necessary to take only the lowest order. For instance if we had u^0 in the last formula we would have:

$$u^0 = \sqrt{1 + \mathbf{u}\cdot\mathbf{u}} = \sqrt{1 + (\delta u_y)^2} \simeq 1 + \frac{1}{2}(\delta u_y)^2,$$

therefore it remains unitary up to the linear order in the perturbations. In a similar way it is possible to check that we can identify four-velocity perturbation with four-temperature perturbations, still up to the linear order. The previous four-velocity perturbation corresponds to a four-temperature perturbation $\delta\beta_y$ which only depends on the x coordinate, since $\beta_\mu = u_\mu / T_0$ and the local temperature T_0 is:

$$T_0 = \frac{1}{\sqrt{\beta\cdot\beta}} = \frac{1}{\sqrt{(\beta_0)^2 - (\delta\beta_y)^2}} = \frac{T_{\text{h.e.}}}{\sqrt{1 - (\delta\beta_y/\beta_0)^2}} \simeq \frac{T_{\text{h.e.}}}{1 - (T_{\text{h.e.}}\delta\beta_y)^2/2} \simeq$$

$$\simeq T_{\text{h.e.}}\left[1 + \frac{1}{2}T^2_{\text{h.e.}}(\delta\beta_y)^2\right] = T_{\text{h.e.}} + \frac{1}{2}T^3_{\text{h.e.}}(\delta\beta_y)^2.$$

It differs only by a quadratic term in the perturbation. Using the same arguments we can identify temperature perturbations with $\delta\beta_0$.

We can thus study the response of the stress-energy-momentum tensor, up to the linear order in the four-temperature perturbation, to find transport coefficients. We want to use the mean values of quantum operators as the macroscopic tensors, and so we need both the equilibrium expectation values and the non-equilibrium ones. It is not difficult, at least at the conceptual level, to get the equilibrium expectation values of operators. What we need, to find transport coefficients from the microscopic underlying theory, is a method to compute average values for a system out of equilibrium. We need way to describe microscopically a non equilibrium state which is close to global equilibrium and differing for a small but known perturbation of the four temperature.

5.2 The Zubarev method

A suitable formalism to calculate transport coefficients for relativistic quantum fields, studying the linear response instead of going through kinetic theory, was developed by Zubarev [30, 31]. The method extends to the relativistic domain a formalism already introduced by Kubo [32] for non relativistic statistical mechanics. In this approach it is introduced the non-equilibrium density operator that we need for the non-equilibrium average values, which reads [33][4]:

$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{\Upsilon}] = \frac{1}{Z}\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\ \widehat{T}^{0\nu}(x)\beta_\nu(x)\right], \tag{5.1}$$

where we understand that the position x, and later the derivatives ∂_μ, refer to the variables of integration, and not t'; the Z factor is analogous to a partition function, a normalization factor to have $\mathrm{tr}\,(\widehat{\rho}) = 1$:

$$Z = \mathrm{tr}\left(\mathrm{e}^{-\widehat{\Upsilon}}\right) = \mathrm{tr}\left(\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\ \widehat{T}^{0\nu}(x)\beta_\nu(x)\right]\right).$$

The operators in the exponential of Eq. (5.1) are in the Heisenberg representation. It should be stressed that in the formula (5.1) covariance is broken from the very beginning by the choice of a specific inertial frame with its time. However, it can be shown that the operator $\widehat{\rho}$ is in fact time-independent, hence independent of t', which

[4]Here ε clearly stands for a small parameter and is not related to the proper energy density. We assume that the difference is big enough to avoid misunderstandings.

can be arbitrarily chosen. Indeed if we take the time derivative of $\widehat{\Upsilon}$ we have:

$$\partial_{t'}\widehat{\Upsilon} = \lim_{\varepsilon\to 0}\varepsilon\left[\int \mathrm{d}^3\mathbf{x}\,\widehat{T}^{0\nu}(t',\mathbf{x})\beta_\nu(t',\mathbf{x}) - \varepsilon\int_{-\infty}^{t'}\mathrm{d}t\;\mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\,\widehat{T}^{0\nu}(x)\beta_\nu(x)\right] = 0,$$

thus, $\widehat{\rho}$ is a good density operator in the Heisenberg representation. When we consider the familiar homogeneous equilibrium four-temperature $\beta_\mu = \delta^0_\mu 1/T$, with constant temperature T, the exponent $\widehat{\Upsilon}$ reads:

$$\widehat{\Upsilon} = \lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\;\mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\,\widehat{T}^{0\nu}(x)\beta_\nu(x) = \frac{1}{T}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\;\mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\,\widehat{T}^{00}(x) =$$

$$= \frac{1}{T}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\;\mathrm{e}^{\varepsilon(t-t')}\widehat{H}.$$

Being the hamiltonian $\widehat{H}$ time independent[5], the limit is easily computed:

$$\widehat{\Upsilon} = \frac{1}{T}\widehat{H},$$

so the density matrix (5.1) reads:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\widehat{H}/T\right].$$

This is the canonical equilibrium distribution, as we expected. The non-equilibrium operator (5.1) is thus consistent with the equilibrium distribution, and in general it is a state that explicitly depends on the four temperature β_μ. We can use it for states arbitrarily close to global equilibrium and, at the end, find transport coefficients using the linear response.

[5]This is not always true, but for simplicity we assume the first equation on (1.5) is always met, ensuring the time independence of the Hamiltonian operator $\widehat{H}$.

5.2.1 Linear response using Zubarev method

We have seen that (5.1) is a good density matrix to compute the components of $T^{\mu\nu}$ out of equilibrium, but typically it is not an easy task to directly compute the non-equilibrium components, and some approximations are usually adopted. As first we note that the ε dependent part in (5.1) can be written as a time derivative $\varepsilon \mathrm{e}^{\varepsilon(t-t')} = \partial_0 \left[\mathrm{e}^{\varepsilon(t-t')}\right]$. Therefore exponent $\widehat{\Upsilon}$ reads:

$$\widehat{\Upsilon} = \lim_{\varepsilon\to 0} \int_{-\infty}^{t'} \mathrm{d}t\, \partial_0 \left[\mathrm{e}^{\varepsilon(t-t')}\right] \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{0\nu}(x)\beta_\nu(x) = \lim_{\varepsilon\to 0} \int_{-\infty}^{t'} \mathrm{d}t\, \partial_\mu \left[\mathrm{e}^{\varepsilon(t-t')}\right] \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{\mu\nu}(x)\beta_\nu(x),$$

reminding the continuity equation $\partial_\mu \widehat{T}^{\mu\nu} = 0$, after integrating by parts we have:

$$\widehat{\Upsilon} = \lim_{\varepsilon\to 0} \left\{ \int_{-\infty}^{t'} \mathrm{d}t\, \partial_\mu \left[\mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{\mu\nu}(x)\beta_\nu(x)\right] - \int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{\mu\nu}(x)\partial_\mu\beta_\nu(x) \right\} =$$

$$\begin{aligned} &= \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{0\nu}(t',\mathbf{x})\, \beta_\nu(t',\mathbf{x}) + \lim_{\varepsilon\to 0} \int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}S\; n_i\, \widehat{T}^{i\nu}(x)\, \beta_\nu(x) \\ &\quad - \lim_{\varepsilon\to 0} \int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{\mu\nu}(x)\, \partial_\mu\beta_\nu(x). \end{aligned}$$

The first term is the so-called *local thermodynamical equilibrium* one, which is defined by the same formula of the global equilibrium with an x-dependent four-temperature, whereas the term dependent on their derivatives is interpreted as a perturbation. We have already seen that at equilibrium the density matrix correspond to the usual canonical equilibrium distribution, that means the surface term in the last equation vanishes for the equilibrium four temperature. In order to neglect the boundary term it is usually taken a non-equilibrium four-temperature that matches the equilibrium four temperature at the boundary, or, calling $\delta\beta_\mu$ the difference between the four temperature and the homogeneous equilibrium four-temperature $\delta\beta_\mu|_{\partial V} \equiv 0$. In this way the last equation reads:

$$\widehat{\Upsilon} = \int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{0\nu}(t',\mathbf{x})\,\beta_\nu(t',\mathbf{x}) - \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{\mu\nu}(x)\,\partial_\mu\delta\beta_\nu(x).$$

Dividing the exponent of the density matrix in two operators $\widehat{A}$ and $\widehat{B}$:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\widehat{\Upsilon}\right] = \frac{1}{\mathrm{tr}\left(\exp\left[\widehat{A}+\widehat{B}\right]\right)}\exp\left[\widehat{A}+\widehat{B}\right]$$

with:

$$\widehat{A} = -\int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{0\nu}(t',\mathbf{x})\,\beta_\nu(t',\mathbf{x}) \qquad \widehat{B} = \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\, \widehat{T}^{\mu\nu}(x)\,\partial_\mu\delta\beta_\nu(x),$$

we can take the lowest order in $\widehat{B}$, being the perturbation $\delta\beta_\mu$ small by hypothesis, hence:

$$Z = \mathrm{tr}\left(\mathrm{e}^{\widehat{A}+\widehat{B}}\right) \simeq \mathrm{tr}\left(\mathrm{e}^{\widehat{A}}\left[1+\widehat{B}\right]\right) = Z_{\mathrm{LE}}\left(1+\langle\widehat{B}\rangle_{\mathrm{LE}}\right) \Rightarrow \frac{1}{Z} \simeq \frac{1}{Z_{\mathrm{LE}}}\left(1-\langle\widehat{B}\rangle_{\mathrm{LE}}\right)$$

and, using the Kubo formula:

$$\mathrm{e}^{\widehat{A}+\widehat{B}} = \left[1+\int_0^1 \mathrm{d}z\ \mathrm{e}^{z\left(\widehat{A}+\widehat{B}\right)}\widehat{B}\mathrm{e}^{-z\widehat{A}}\right]\mathrm{e}^{\widehat{A}} \simeq \left[1+\int_0^1 \mathrm{d}z\ \mathrm{e}^{z\widehat{A}}\widehat{B}\,\mathrm{e}^{-z\widehat{A}}\right]\mathrm{e}^{\widehat{A}},$$

where the subscript LE stands for Local Equilibrium and implies the *local equilibrium* partition function for $Z_{\mathrm{LE}} = \mathrm{tr}\left(\mathrm{e}^{\widehat{A}}\right)$, and the calculation of average values with the local equilibrium density operator, that is $\widehat{\rho}_{\mathrm{LE}} = \mathrm{e}^{\widehat{A}}/Z_{\mathrm{LE}}$, for $\langle\cdots\rangle_{\mathrm{LE}}$. Thereby, putting all together and retaining only first-order terms in $\widehat{B}$ we get:

$$\widehat{\rho} \simeq \left(1-\langle\widehat{B}\rangle_{\mathrm{LE}}\right)\widehat{\rho}_{\mathrm{LE}} + \int_0^1 \mathrm{d}z\ \mathrm{e}^{z\widehat{A}}\widehat{B}\mathrm{e}^{-z\widehat{A}}\widehat{\rho}_{\mathrm{LE}}.$$

Therefore we can approximate the average value of a generic operator $\widehat{\mathcal{O}}$ with:

$$\langle\widehat{\mathcal{O}}\rangle \simeq \left(1-\langle\widehat{B}\rangle_{\rm LE}\right)\langle\widehat{\mathcal{O}}\rangle_{\rm LE}+\langle\widehat{\mathcal{O}}\int_0^1 {\rm d}z\ {\rm e}^{z\widehat{A}}\widehat{B}{\rm e}^{-z\widehat{A}}\rangle_{\rm LE}.$$

Being interested in linear response and transport coefficients, we take[6] $\widehat{\mathcal{O}} = \widehat{T}^{\mu\nu}(y)$. Even after these approximations calculating the local equilibrium expectation values usually proves to be a daunting task. However we can use our starting assumption of a small deviation from equilibrium $\delta\beta$, and replace the local equilibrium averages with the global equilibrium. More precisely we replace the operator $\widehat{A} = -\int {\rm d}^3\mathbf{x}\widehat{T}^{0\nu}\beta_\nu$ with the equilibrium operator $-\widehat{H}/T$, then the local equilibrium density matrix $\widehat{\rho}_{\rm L.E.}$ becomes homogeneous equilibrium distribution $\widehat{\rho}_0 = \exp\left[-\widehat{H}/T\right]/Z$, and the mean value of the stress-energy-momentum tensor reads:

$$T^{\mu\nu} = \left(1-\langle\widehat{B}\rangle_0\right)T^{\mu\nu}|_0+\langle\widehat{T}^{\mu\nu}\int_0^1 {\rm d}z\ {\rm e}^{-z\widehat{H}/T}\,\widehat{B}\,{\rm e}^{z\widehat{H}/T}\rangle_0\,.$$

Using these approximations we can already write the linear response $\delta T^{\mu\nu} = T^{\mu\nu} - \widehat{T}^{\mu\nu}_0$:

$$\delta T^{\mu\nu} = \langle\widehat{T}^{\mu\nu}\int_0^1 {\rm d}z\ {\rm e}^{-z\widehat{H}/T}\,\widehat{B}\,{\rm e}^{z\widehat{H}/T}\rangle_0 - \ \langle\widehat{B}\rangle_0\, T^{\mu\nu}|_0\ , \tag{5.2}$$

we do not even need to explicitly compute the equilibrium values $T^{\mu\nu}|_0$ since the first term in the right hand side has a part that simplifies with the second term. Indeed the operator $\widehat{B}$ is an integral of $\widehat{T}^{\rho\sigma}\partial_\rho\delta\beta_\sigma$, so the first term on the right hand side of the last equation consists on the combination of integrals of:

$$\langle\widehat{T}^{\mu\nu}(y)\int_0^1 {\rm d}z\ {\rm e}^{-z\widehat{H}/T}\ \widehat{T}^{\rho\sigma}(x)\,{\rm e}^{z\widehat{H}/T}\rangle_0 = \frac{1}{\overline{\beta}}\int_0^1 {\rm d}u\langle\widehat{T}^{\mu\nu}(y){\rm e}^{i(iu)\widehat{H}}\widehat{T}^{\rho\sigma}(t,\mathbf{x}){\rm e}^{-i(iu)\widehat{H}}\rangle_0 =$$

$$= \frac{1}{\overline{\beta}}\int_0^1 {\rm d}u\langle\widehat{T}^{\mu\nu}(y)\widehat{\mathsf{T}}^{-1}(iu,\mathbf{0})\widehat{T}^{\rho\sigma}(t,\mathbf{x})\widehat{\mathsf{T}}(iu,\mathbf{0})\rangle_0 = \frac{1}{\overline{\beta}}\int_0^1 {\rm d}u\langle\widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(t+iu,\mathbf{x})\rangle_0,$$

where $\overline{\beta} = 1/T$ is the inverse of the (constant) equilibrium temperature T, and we performed a change of variables $z\overline{\beta} = u$, and a time translation of a complex time

[6] We are using y here because we already used x as an integration variable in $\widehat{B}$.

interval iu. The last term can be rewritten in a more convenient form:

$$\frac{1}{\overline{\beta}}\int_0^1 \mathrm{d}u \langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(t+iu,\mathbf{x})\rangle_0 = \frac{1}{\overline{\beta}}\int_0^1 \mathrm{d}u \int_{-\infty}^{t} \mathrm{d}\tau\, \partial_\tau \langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau+iu,\mathbf{x})\rangle_0 + \qquad (5.3)$$
$$+ \lim_{\tau\to-\infty}\frac{1}{\overline{\beta}}\int_0^1 \mathrm{d}u \langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau+iu,\mathbf{x})\rangle_0,$$

using the fact that correlation at infinite time distances vanishes:

$$\lim_{\Delta t\to\infty}\langle \widehat{\mathcal{O}}_1(t)\widehat{\mathcal{O}}_2(t+\Delta t)\rangle = \lim_{\Delta t\to\infty}\langle \widehat{\mathcal{O}}_1(t)\rangle\langle \widehat{\mathcal{O}}_2(t+\Delta t)\rangle,$$

and time translation invariance of the equilibrium distribution, the last term of (5.3), once integrated following the definition of $\widehat{B}$, exactly simplifies with the last term on the right hand side of (5.2). Furthermore the first term on the right hand side of (5.3) can be written as a commutator of $\widehat{T}^{\mu\nu}$ components. Indeeed:

$$\frac{1}{\overline{\beta}}\int_0^1 \mathrm{d}u \int_{-\infty}^{t} \mathrm{d}\tau\, \partial_\tau \langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau+iu,\mathbf{x})\rangle_0 = \frac{1}{i\overline{\beta}}\int_{-\infty}^{t} \mathrm{d}\tau \int_0^1 \mathrm{d}u\, \partial_u \langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau+iu,\mathbf{x})\rangle_0 =$$

$$= \frac{1}{i\overline{\beta}}\int_{-\infty}^{t} \mathrm{d}\tau \left[\langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau+i\overline{\beta},\mathbf{x})\rangle_0 - \langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau,\mathbf{x})\rangle_0\right],$$

but, using the cyclic property of trace and the definition of time translation operator $\widehat{\mathsf{T}}(t_0) = \exp\left[-it_0\widehat{H}\right]$ and equilibrium distribution $\widehat{\rho}_0 = \exp\left[-\overline{\beta}\widehat{H}\right]/Z_0$, we have:

$$\begin{aligned}
\langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau+i\overline{\beta},\mathbf{x})\rangle_0 &= \mathrm{tr}\left(\widehat{\rho}_0\widehat{T}^{\mu\nu}(y)\widehat{\mathsf{T}}^{-1}(i\overline{\beta})\widehat{T}^{\rho\sigma}(\tau,\mathbf{x})\widehat{\mathsf{T}}(i\overline{\beta})\right) = \\
&= \frac{1}{Z_0}\mathrm{tr}\left(\mathrm{e}^{-\overline{\beta}\widehat{H}}\widehat{T}^{\mu\nu}(y)\mathrm{e}^{-\overline{\beta}\widehat{H}}\widehat{T}^{\rho\sigma}(\tau,\mathbf{x})\mathrm{e}^{\overline{\beta}\widehat{H}}\right) = \\
&= \frac{1}{Z_0}\mathrm{tr}\left(\mathrm{e}^{-\overline{\beta}\widehat{H}}\widehat{T}^{\rho\sigma}(\tau,\mathbf{x})\widehat{T}^{\mu\nu}(y)\right) = \\
&= \langle \widehat{T}^{\rho\sigma}(\tau,x)\widehat{T}^{\mu\nu}(y)\rangle_0,
\end{aligned}$$

hence:

$$\frac{1}{\overline{\beta}}\int_0^1 \mathrm{d}u \int_{-\infty}^{t} \mathrm{d}\tau\, \partial_\tau \langle \widehat{T}^{\mu\nu}(y)\widehat{T}^{\rho\sigma}(\tau+iu,\mathbf{x})\rangle_0 = \frac{1}{i\overline{\beta}}\int_{-\infty}^{t} \mathrm{d}\tau \langle \Big[\widehat{T}^{\rho\sigma}(\tau,\mathbf{x}),\widehat{T}^{\mu\nu}(y)\Big]\rangle_0 \,.$$

The linear response for a small deviation $\delta\beta_\mu$ from equilibrium, after some mathematical steps, is thus:

$$\delta T^{\mu\nu}(y) = \lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}}\int_{-\infty}^{t'} \mathrm{d}t\; \mathrm{e}^{\varepsilon(t-t')}\int_{-\infty}^{t}\mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \langle \Big[\widehat{T}^{\rho\sigma}(\tau,\mathbf{x}),\widehat{T}^{\mu\nu}(y)\Big]\rangle_0 \partial_\rho \delta\beta_\sigma(x). \qquad (5.4)$$

These arguments can e extended to more general cases. For instance, if we take into account conserved charges, we can start from the density matrix:

$$\widehat{\rho} = \frac{1}{Z}\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\; \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\; \left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{j}^0\xi(x)\right)\right],$$

where $\xi = \mu/T$ is a scalar function whose physical meaning is that of a point-dependent ratio between comoving chemical potential μ and comoving temperature T; the partition function here is:

$$Z = \mathrm{tr}\left(\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\; \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\; \left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{j}^0\xi(x)\right)\right]\right).$$

The global equilibrium here is $\beta_{\mathrm{eq.}} = (\overline{\beta},\mathbf{0})$ and $\xi = \overline{\beta}\mu$, both constant. If we consider a four temperature perturbation like in the previous case we still get (5.4), but, due to the richer structure, we can also treat chemical potential perturbations.

Chapter 6

Non-equilibrium inequivalence

During this chapter we will show that different pairs of stress-energy-momentum and spin tensors $\{\widehat{T}, \widehat{\mathcal{S}}\}$ imply different average values of physical quantities in thermodynamical non-equilibrium situations. Most notably, transport coefficients and the total entropy production rate, both obtained using the non-equilibrium statistical operator, are affected by the choice of tensors of the relativistic quantum field theory under consideration[1].

In order to use the Zubarev density matrix, introduced in the previous chapter, to calculate the average values $\widehat{T}^{\mu\nu}$ out of equilibrium, we will start with an extension of the method in the general case of a theory with a spin tensor and, possibly, a non symmetric stress-energy-momentum tensor. Then we will show the effects of pseudo-gauge transformations on the non-equilibrium density matrix, and on total entropy. At last we will show that shear viscosity, and in general transport coefficients, depend on the choice of microscopic tensors.

6.1 Nonequilibrium density operator

To extend the Zubarev formalism to the case of a generic couple of microscopic operators $\{\widehat{T}, \widehat{\mathcal{S}}\}$, we start from the formula previous formula:

[1]This is essentially our work in [27].

$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{\Upsilon}] = \frac{1}{Z}\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int\mathrm{d}^3\mathbf{x}\ \left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{j}^0\xi(x)\right)\right]. \quad (6.1)$$

In the formula (6.1) the possible contribution of a spin tensor is simply disregarded; therefore, the formula is correct only if the stress-energy tensor is the symmetrized Belinfante one, whose associated spin tensor is vanishing. To find the appropriate extension of the formula (6.1) with a spin tensor we will follow the same steps seen in the last chapter to find what lack in order to have the right operator form for the equilibrium configuration of four-temperature β_μ. Using the identity:

$$\mathrm{e}^{\varepsilon(t-t')}\left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{j}^0\xi(x)\right) = \left(\frac{\partial}{\partial x^\mu}\frac{\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\right)\left(\widehat{T}^{\mu\nu}\beta_\nu(x) - \widehat{j}^\mu\xi(x)\right),$$

integrating by parts and taking into account the continuity equations $\partial_\mu\widehat{T}^{\mu\nu} = \partial_\mu\widehat{j}^\mu = 0$, the operator $\widehat{\Upsilon}$ in Eq. (6.1) can be rewritten as:

$$\begin{aligned}\widehat{\Upsilon} &= \int\mathrm{d}^3\mathbf{x}\left(\widehat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \widehat{j}^0\xi(t',\mathbf{x})\right) + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int\mathrm{d}S\, n_i\left(\widehat{T}^{i\nu}\beta_\nu(x) - \widehat{j}^i\xi(x)\right)\\ &- \lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int\mathrm{d}^3\mathbf{x}\ \left(\widehat{T}^{\mu\nu}\partial_\mu\beta_\nu(x) - \widehat{j}^\mu\partial_\mu\xi(x)\right). \qquad (6.2)\end{aligned}$$

At equilibrium, the right hand side should reduce to the known form, which, at least for the most familiar form of thermodynamical equilibrium where $\beta^{\mathrm{eq}} = (\overline{\beta}, \mathbf{0}) = (1/T, \mathbf{0})$ with constant global[2] temperature T, and $\xi^{\mathrm{eq}} = \mu/T = const$ is readily recognized in the first term setting $\beta = \beta^{\mathrm{eq}}$ and $\xi = \xi^{\mathrm{eq}}$:

[2]In this case the local temperature $1/\sqrt{\beta\cdot\beta}$ coincides with the global temperature.

$$\begin{aligned}\widehat{\Upsilon}^{\rm eq} &= \int {\rm d}^3{\bf x}\,\left(\widehat{T}^{0\nu}\beta_\nu^{\rm eq} - \widehat{j}^0\xi^{\rm eq}\right) + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\;{\rm e}^{\varepsilon(t-t')}\int {\rm d}S\,n_i\left(\widehat{T}^{i\nu}\beta_\nu^{\rm eq} - \widehat{j}^i\xi^{\rm eq}\right) \\ &\quad - \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\;{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\bf x}\,\left(\widehat{T}^{\mu\nu}\partial_\mu\beta_\nu^{\rm eq} - \widehat{j}^\mu\partial_\mu\xi^{\rm eq}\right) = \\ &= \widehat{H}/T - \mu\widehat{Q}/T + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\;{\rm e}^{\varepsilon(t-t')}\int {\rm d}S\,n_i\left(\widehat{T}^{i\nu}\beta_\nu^{\rm eq} - \widehat{j}^i\xi^{\rm eq}\right) + \\ &\quad - \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\;{\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\bf x}\,\left(\widehat{T}^{\mu\nu}\partial_\mu\beta_\nu^{\rm eq} - \widehat{j}^\mu\partial_\mu\xi^{\rm eq}\right). \end{aligned} \tag{6.3}$$

Hence, the two rightmost terms of (6.3) must vanish at equilibrium. Indeed, the surface term is supposed to vanish through a suitable choice of the field boundary conditions while the third term vanishes in view of the constancy of $\beta^{\rm eq}$ and $\xi^{\rm eq}$. However, this is not the case for the most general form of equilibrium; in the most general form (see discussion in ref. [23]), while the scalar $\xi^{\rm eq}$ stays constant, the four-vector β fulfills a Killing equation, whose solution is [1]:

$$\beta_\nu^{\rm eq}(x) = b_\nu^{\rm eq} + \omega_{\nu\mu}^{\rm eq}x^\mu, \tag{6.4}$$

with the four-vector $b^{\rm eq}$ and the antisymmetric tensor $\omega^{\rm eq}$ constants. Therefore:

$$\partial_\mu\beta_\nu^{\rm eq} = -\omega_{\mu\nu}^{\rm eq},$$

which in general is non-vanishing, so that the third term on the right hand side of Eq. (6.3) survives. For instance, for the thermodynamical equilibrium with rotation [23], the tensor ω turns out to be:

$$\omega_{\lambda\nu}^{\rm eq} = \omega/T\left(\delta_\lambda^1\delta_\nu^2 - \delta_\lambda^2\delta_\nu^1\right), \tag{6.5}$$

ω being the angular velocity and T the temperature measured by the inertial frame. even for rotating equilibrium the global temperature $1/\beta^0_{\rm eq.}$ the same as the local temperature$1/\sqrt{\beta^2_{\rm eq.}}$.

In order to find the appropriate generalization of the operator $\widehat{\Upsilon}$, let us plug the formula (6.4) of general thermodynamical equilibrium into the (6.3):

$$\widehat{\Upsilon}^{\rm eq} = \int {\rm d}^3{\rm x} \left(\widehat{T}^{0\nu}\beta_\nu^{\rm eq} - \widehat{j}^0\xi^{\rm eq}\right)\Big) + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')} \int {\rm d}S\, n_i\Big(\widehat{T}^{i\nu}(b_\nu^{\rm eq} + \omega_{\nu\mu}^{\rm eq}x^\mu) - \widehat{j}^i\xi^{\rm eq}\Big) + \\ + \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')} \int {\rm d}^3{\rm x}\ \widehat{T}^{\mu\nu}\omega_{\mu\nu}^{\rm eq} \tag{6.6}$$

where the $\partial_\mu\xi^{\rm eq} = 0$ has been taken into account. For a symmetric stress-energy-momentum tensor $\widehat{T}$, the last term vanishes, but if a spin tensor is present $\widehat{T}$ may have an antisymmetric part. Particularly, from the angular momentum continuity equation:

$$\widehat{T}^{\mu\nu}\omega_{\mu\nu}^{\rm eq} = \frac{1}{2}(\widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu})\omega_{\mu\nu}^{\rm eq} = -\frac{1}{2}\partial_\lambda\widehat{\mathcal{S}}^{\lambda,\mu\nu}\omega_{\mu\nu}^{\rm eq}, \tag{6.7}$$

so that the last term on the right hand side of Eq. (6.6) can be rewritten as:

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')} \int {\rm d}^3{\rm x}\ \widehat{T}^{\mu\nu}\omega_{\mu\nu}^{\rm eq} = -\frac{1}{2}\omega_{\mu\nu}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')} \int {\rm d}^3{\rm x}\ \partial_\lambda\widehat{\mathcal{S}}^{\lambda,\mu\nu} \tag{6.8}$$
$$= -\frac{1}{2}\omega_{\mu\nu}^{\rm eq}\lim_{\varepsilon\to 0}\int {\rm d}^3{\rm x}\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')}\frac{\partial}{\partial t}\widehat{\mathcal{S}}^{0,\mu\nu} - \frac{1}{2}\omega_{\mu\nu}^{\rm eq}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')} \int {\rm d}S\, n_i\widehat{\mathcal{S}}^{i,\mu\nu}.$$

The first term on the right hand side can be integrated by parts, yielding:

$$-\frac{1}{2}\omega_{\mu\nu}^{\rm eq}\lim_{\varepsilon\to 0}\int {\rm d}^3{\rm x}\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')}\frac{\partial}{\partial t}\widehat{\mathcal{S}}^{0,\mu\nu} = \tag{6.9}$$
$$= -\frac{1}{2}\omega_{\mu\nu}^{\rm eq}\int {\rm d}^3{\rm x}\,\widehat{\mathcal{S}}^{0,\mu\nu}(t',{\rm x}) + \frac{1}{2}\omega_{\mu\nu}^{\rm eq}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} {\rm d}t\ {\rm e}^{\varepsilon(t-t')} \int {\rm d}^3{\rm x}\ \widehat{\mathcal{S}}^{0,\mu\nu}(x).$$

Plugging the Eq. (6.9) into (6.8) and this in turn into (6.6) we obtain:

$$
\begin{aligned}
\widehat{\Upsilon}^{\rm eq} \;=\; & \int {\rm d}^3{\rm x} \, \left(\widehat{T}^{0\nu} \beta_\nu^{\rm eq} - \widehat{j}^0 \xi^{\rm eq} - \frac{1}{2} \omega_{\mu\nu}^{\rm eq} \widehat{\mathcal{S}}^{0,\mu\nu} \right) + \\
& + \lim_{\varepsilon \to 0} \int_{-\infty}^{t'} {\rm d}t \; {\rm e}^{\varepsilon(t-t')} \left[b_\nu^{\rm eq} \int {\rm d}S \, n_i \widehat{T}^{i\nu} - \xi^{\rm eq} \int {\rm d}S \, n_i \widehat{j}^i \right. \\
& \left. - \frac{1}{2} \omega_{\mu\nu}^{\rm eq} \int {\rm d}S \, n_i (x^\mu \widehat{T}^{i\nu} - x^\nu \widehat{T}^{\mu i} + \widehat{\mathcal{S}}^{i,\mu\nu}) \right] + \\
& - \frac{1}{2} \omega_{\mu\nu}^{\rm eq} \lim_{\varepsilon \to 0} \varepsilon \int_{-\infty}^{t'} {\rm d}t \; {\rm e}^{\varepsilon(t-t')} \int {\rm d}^3{\rm x} \, \widehat{\mathcal{S}}^{0,\mu\nu}(x), \qquad (6.10)
\end{aligned}
$$

where the surface term involving $\widehat{T}$ in Eq. (6.6) has been rearranged taking advantage of the antisymmetry of the ω tensor. The surface terms in the above equations now are manifestly the total momentum flux, the charge flux and the *total* angular momentum flux through the boundary. All of these terms are supposed to vanish at thermodynamical equilibrium through suitable conditions enforced on the field operators at the boundary, so that the (6.10) reduces to:

$$
\begin{aligned}
\widehat{\Upsilon}^{\rm eq} \;=\; & \int {\rm d}^3{\rm x} \, \left(\widehat{T}^{0\nu} \beta_\nu^{\rm eq} - \widehat{j}^0 \xi^{\rm eq} - \frac{1}{2} \omega_{\mu\nu}^{\rm eq} \widehat{\mathcal{S}}^{0,\mu\nu} \right) + \\
& - \frac{1}{2} \omega_{\mu\nu}^{\rm eq} \lim_{\varepsilon \to 0} \varepsilon \int_{-\infty}^{t'} {\rm d}t \; {\rm e}^{\varepsilon(t-t')} \int {\rm d}^3{\rm x} \, \widehat{\mathcal{S}}^{0,\mu\nu}(x). \qquad (6.11)
\end{aligned}
$$

The first term on the right hand side just gives rise to the desired form of the equilibrium operator. For instance, for a rotating system with ω as in Eq. (6.5) one has [23]:

$$
\int {\rm d}^3{\rm x} \, \left(\widehat{T}^{0\nu} \beta_\nu^{\rm eq} - \widehat{j}^0 \xi^{\rm eq} - \frac{1}{2} \omega_{\mu\nu}^{\rm eq} \widehat{\mathcal{S}}^{0,\mu\nu} \right) = \widehat{H}/T - \mu \widehat{Q}/T - \omega \widehat{J}/T,
$$

$\widehat{J}$ being the total angular momentum, which is the known form [11]. Nevertheless, the second term in Eq. (6.11) does not vanish and, thus, must be subtracted away with a suitable modification of the definition of the $\widehat{\Upsilon}$ operator. The form of the unwanted term demands the following modification of (6.1):

$$\begin{aligned}\widehat{\rho} &= \frac{1}{Z}\exp[-\widehat{\Upsilon}] = \qquad (6.12)\\ &= \frac{1}{Z}\exp\left[-\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int\mathrm{d}^3\mathbf{x}\ \left(\widehat{T}^{0\nu}\beta_\nu(x)-\widehat{j}^0\xi(x)-\frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x)\right)\right],\end{aligned}$$

where $\omega_{\mu\nu}(x)$ is an antisymmetric tensor field which must reduce to the constant $\omega_{\mu\nu}^{\mathrm{eq}}$ tensor at equilibrium. It is easy to check, by tracing the previous calculations, that the equilibrium form of $\widehat{\Upsilon}$ reduces to the desired form:

$$\widehat{\Upsilon}^{\mathrm{eq}} = \int\mathrm{d}^3\mathbf{x}\ \left(\widehat{T}^{0\nu}\beta^{\mathrm{eq}}-\widehat{j}^0\xi^{\mathrm{eq}}-\frac{1}{2}\omega_{\mu\nu}^{\mathrm{eq}}\widehat{\mathcal{S}}^{0,\mu\nu}\right),$$

as the spin tensor term in Eq. (6.11) cancels out. Therefore, the operator (6.12) is the only possible extension of the non-equilibrium density operator with a spin tensor.

The new operator $\widehat{\Upsilon}$ can be worked out the same way as we have done when obtaining Eq. (6.2) from Eq. (6.1):

$$\begin{aligned}\widehat{\Upsilon} &= \int\mathrm{d}^3\mathbf{x}\left[\widehat{T}^{0\nu}\beta_\nu(t',\mathbf{x})-\widehat{j}^0\xi(t',\mathbf{x})-\frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x})\right]+ \qquad (6.13)\\ &+ \lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int\mathrm{d}S\,n_i\left[\widehat{T}^{i\nu}\beta_\nu(x)-\widehat{j}^i\xi(x)-\frac{1}{2}\widehat{\mathcal{S}}^{i,\mu\nu}\omega_{\mu\nu}(x)\right]+\\ &- \frac{1}{2}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\int\mathrm{d}^3\mathbf{x}\left\{\widehat{T}_{\mathrm{s}}^{\mu\nu}\left[\partial_\mu\beta_\nu(x)+\partial_\mu\beta_\nu(x)\right]+\right.\\ &\left.+\widehat{T}_{\mathrm{a}}^{\mu\nu}\left[\partial_\mu\beta_\nu(x)-\partial_\mu\beta_\nu(x)+2\omega_{\mu\nu}(x)\right]-\widehat{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\omega_{\mu\nu}(x)-2\widehat{j}^\mu\partial_\mu\xi(x)\right\},\end{aligned}$$

where:

$$\widehat{T}_{\mathrm{s}}^{\mu\nu} = \frac{1}{2}(\widehat{T}^{\mu\nu}+\widehat{T}^{\nu\mu}) \qquad\qquad \widehat{T}_{\mathrm{a}}^{\mu\nu} = \frac{1}{2}(\widehat{T}^{\mu\nu}-\widehat{T}^{\nu\mu}),$$

and the continuity equation for angular momentum has been used. The first term on the right hand side is the new local thermodynamical term whilst the third term can be further expanded to derive the relativistic Kubo formula of transport coefficients (see Appendix C).

6.2 Density operator and pseudo-gauge transformations

A natural requirement for the density operator (6.12) would be its independence of the particular couple of stress-energy and spin tensor, because one would like the mean value of any observable $\widehat{\mathcal{O}}$:

$$\mathcal{O} \equiv \mathrm{tr}(\widehat{\rho}\,\widehat{\mathcal{O}}),$$

to be an objective one. In ref. [28] we showed that even at thermodynamical equilibrium with rotation this is not the case for the components of the stress-energy and spin tensor themselves because they change through the pseudo-gauge transformation (1.3). However, at equilibrium, $\widehat{\rho}$ itslef is a function of just integral quantities (total energy, angular momentum, charge) which are invariant under a transformation (1.3) provided that boundary fluxes (1.4) vanish as it is usually assumed, so a specific observable $\widehat{\mathcal{O}}$, including the components of a *specific* stress-energy tensor, does not change under (1.3). However, it is not obvious that this feature persists in a non-equilibrium case, in fact we are going to show that, in general, this is not the case.

Let us consider the operator $\widehat{\Upsilon}$ in (6.12) and how it gets changed under a pseudo-gauge transformation (1.3). It is trivial to see that a transformation involving only the $\widehat{Z}$ superpotential will typically change the density matrix changing only the spin tensor $\Delta\widehat{\mathcal{S}}^{\lambda,\mu\nu} = \partial_\alpha \widehat{Z}^{\alpha\lambda,\mu\nu}$:

$$\widehat{\Upsilon}' = \widehat{\Upsilon} \lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x} \left[\partial_i \widehat{Z}^{i0,\mu\nu}(x)\right] \omega_{\mu\nu}(x),$$

where we used the antisymmetry of $\widehat{Z}$. We know that the spatial integral of $\partial_i \widehat{Z}^{i0,\mu\nu}$ is vanishing as long as the boundary conditions (1.4), needed to preserve the generators of Poincaré algebra, are fulfilled. In the general case for an arbitrary $\omega_{\mu\nu}$ the non-equilibrium density matrix will change.

A more interesting case is that of transformation with a vanishing $\widehat{Z}$ superpotential and non-vanishing $\widehat{\Phi}$, namely the subset of pseudo-gauge transformation which includes the Belinfante symmetrization procedure. Since the superpotential changes both the stress-energy-momentum tensor $\widehat{T}$ and the spin tensor $\widehat{\mathcal{S}}$, after a transformation we have:

$$\widehat{\Upsilon}' = \widehat{\Upsilon} + \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}^3\mathbf{x}\, \left(\partial_\lambda \widehat{\varphi}^{\lambda 0,\nu}\beta_\nu(x) + \widehat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}(x)\right), \tag{6.14}$$

where, we remind:

$$\widehat{\varphi}^{\lambda\mu,\nu} = \widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu}, \tag{6.15}$$

is antisymmetric in the first two indices. We can rewrite Eq. (6.14) as:

$$\begin{aligned}\widehat{\Upsilon}' - \widehat{\Upsilon} &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t\int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{\varepsilon(t-t')}\left[\partial_\lambda(\widehat{\varphi}^{\lambda 0,\nu}\beta_\nu(x)) - \widehat{\varphi}^{\lambda 0,\nu}\partial_\lambda\beta_\nu + \widehat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}(x)\right] \qquad (6.16)\\ &= \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')}\left[\int \mathrm{d}S\, n_i\, \widehat{\varphi}^{i0,\nu}\beta_\nu(x) - \int \mathrm{d}^3\mathbf{x}\, \left(\widehat{\varphi}^{\lambda 0,\nu}\partial_\lambda\beta_\nu - \widehat{\Phi}^{0,\mu\nu}\omega_{\mu\nu}(x)\right)\right],\end{aligned}$$

where we performed an integration by parts. Let us now write the general fields β and ω as the sum of the equilibrium values and a perturbation, that is:

$$\beta(x) = \beta^{\mathrm{eq}}(x) + \delta\beta(x) \qquad\qquad \omega(x) = \omega^{\mathrm{eq}} + \delta\omega(x), \tag{6.17}$$

and work out first the equilibrium part of the right hand side of Eq. (6.16). As $\partial_\lambda\beta^{\mathrm{eq}}_\nu = -\omega^{\mathrm{eq}}_{\lambda\nu}$ one has:

$$\begin{aligned}(\widehat{\Upsilon}' - \widehat{\Upsilon})|_{\mathrm{eq}} =& \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')}\left[\int \mathrm{d}S\, n_i\widehat{\varphi}^{i0,\nu}\beta^{\mathrm{eq}}_\nu + \int \mathrm{d}^3\mathbf{x}\left(\widehat{\varphi}^{\lambda 0,\nu}\omega^{\mathrm{eq}}_{\lambda\nu} + \widehat{\Phi}^{0,\mu\nu}\omega^{\mathrm{eq}}_{\mu\nu}(x)\right)\right] =\\ =& \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}S\, n_i\, \widehat{\varphi}^{i0,\nu}\beta^{\mathrm{eq}}_\nu +\\ & + \int \mathrm{d}^3\mathbf{x}\, \left(\widehat{\Phi}^{\lambda,0\nu}\omega^{\mathrm{eq}}_{\lambda\nu} - \widehat{\Phi}^{0,\lambda\nu}\omega^{\mathrm{eq}}_{\lambda\nu} - \widehat{\Phi}^{\nu,\lambda 0}\omega^{\mathrm{eq}}_{\lambda\nu} + \widehat{\Phi}^{0,\mu\nu}\omega^{\mathrm{eq}}_{\mu\nu}\right) =\\ =& \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'} \mathrm{d}t\, \mathrm{e}^{\varepsilon(t-t')}\int \mathrm{d}S\, n_i\widehat{\varphi}^{i0,\nu}\beta^{\mathrm{eq}}_\nu,\end{aligned} \tag{6.18}$$

where we have used the Eq. (6.15) and the antisymmetry of indices of the superpotential $\widehat{\Phi}$. By using the Eq. (6.4), the last expression can be rewritten as:

$$\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\left[b_\nu^{\mathrm{eq}}\int\mathrm{d}S\,n_i\widehat{\varphi}^{i0,\nu}+\frac{1}{2}\omega_{\nu\mu}^{\mathrm{eq}}\int\mathrm{d}S\,n_i(x^\mu\widehat{\varphi}^{i0,\nu}-x^\nu\widehat{\varphi}^{i0,\mu})\right].$$

The two surface integrals above are the additional four-momentum and the additional *total* angular momentum in the operator sense, after having made a pseudo-gauge transformation with a vanishing $\widehat{Z}$ superpotential. If the boundary conditions ensure that the momentum and total angular momentum fluxes vanish (in order to have conserved energy and momentum operators) for any couple $(\widehat{T},\widehat{\mathcal{S}})$ of tensors, then the two fluxes in the above equations must vanish. Therefore, we can conclude that:

$$\widehat{\Upsilon}'|_{\mathrm{eq}}=\widehat{\Upsilon}|_{\mathrm{eq}}.$$

Now, let us focus on the non-equilibrium perturbation of the $\widehat{\Upsilon}$ operator.

$$\begin{aligned}(\widehat{\Upsilon}'-\widehat{\Upsilon})|_{\mathrm{non-eq}}&=\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\left[\int\mathrm{d}S\,n_i\,\widehat{\varphi}^{i0,\nu}\delta\beta_\nu+\right.\\&\left.-\int\mathrm{d}^3\mathbf{x}\ \widehat{\varphi}^{\lambda0,\nu}\partial_\lambda\delta\beta_\nu-\widehat{\Phi}^{0,\mu\nu}\delta\omega_{\mu\nu}\right]=\\&=\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\left[\int\mathrm{d}S\,n_i\,\widehat{\varphi}^{i0,\nu}\delta\beta_\nu+\right.\\&\left.-\int\mathrm{d}^3\mathbf{x}\ (\widehat{\Phi}^{\lambda,0\nu}-\widehat{\Phi}^{0,\lambda\nu}-\widehat{\Phi}^{\nu,\lambda0})\partial_\lambda\delta\beta_\nu-\widehat{\Phi}^{0,\mu\nu}\delta\omega_{\mu\nu}\right]=\\&=\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{t'}\mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\left[\int\mathrm{d}S\,n_i\,\widehat{\varphi}^{i0,\nu}\delta\beta_\nu+\right.\\&\left.-\int\mathrm{d}^3\mathbf{x}\ \widehat{\Phi}^{\lambda,0\nu}(\partial_\lambda\delta\beta_\nu+\partial_\nu\delta\beta_\lambda)-\widehat{\Phi}^{0,\lambda\nu}\left(\frac{1}{2}(\partial_\lambda\delta\beta_\nu-\partial_\nu\delta\beta_\lambda)+\delta\omega_{\lambda\nu}\right)\right],\end{aligned}\tag{6.19}$$

where the dependence of $\delta\beta$ and $\delta\omega$ on x is now understood. It can be seen that it is impossible to make this difference vanishing in general. One can get rid of the surface term by choosing a perturbation which vanishes at the boundary and the last term by locking the perturbation of the tensor ω to that of the inverse temperature four-vector:

$$\delta\omega_{\lambda\nu}(x) = -\frac{1}{2}(\partial_\lambda \delta\beta_\nu(x) - \partial_\nu \delta\beta_\lambda(x)), \tag{6.20}$$

but it is impossible to cancel out the term:

$$\delta\widehat{\Upsilon} \equiv -\frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x} \; \widehat{\Phi}^{\lambda,0\nu}(\partial_\lambda \delta\beta_\nu(x) + \partial_\nu \delta\beta_\lambda(x)), \tag{6.21}$$

unless in special cases, e.g. when the tensor $\widehat{\Phi}$ is also antisymmetric in the first two indices. Using the same arguments it is possible to prove that the density matrix will change in the most general case of peudogauge transformations (1.3).

We have thus come to the conclusion that the non-equilibrium density operator does depend, in general, on the particular choice of stress-energy and spin tensor of the quantum field theory under consideration. Therefeore, the mean value of any observable in a non-equilibrium situation shall depend on that choice. It is worth stressing that this is a much deeper dependence on the particular stress-energy and spin tensor than what we showed in chapter (2) for thermodynamical equilibrium with rotation. Therein, mean values of the angular momentum densities and momentum densities were found to be dependent on the pseudo-gauge transformation because the relevant quantum operators could be varied, but not because the density operator $\widehat{\rho}$ was dependent thereupon. In fact, at non-equilibrium, even $\widehat{\rho}$ varies under a transformation (1.3). Note that, in principle, even the mean values of the total energy and momentum could be dependent on the quantum stress-energy tensor choice although boundary conditions ensure, as we have assumed, that the total energy and momentum *operators* are invariant under a pseudogauge transformation. This happens, again, because the density operator is not invariant under (1.3). In formula:

$$\mathrm{tr}(\widehat{\rho}'\widehat{P}'^{\mu}) = \mathrm{tr}(\widehat{\rho}'\widehat{P}^{\mu}) \neq \mathrm{tr}(\widehat{\rho}\widehat{P}^{\mu}).$$

It must be pointed out that the variation of the Zubarev non-equilibrium density operator (6.21) depends on the gradients of the four-temperature field and it can be thus a small one if we are close to thermodynamical equilibrium. In the next Section we will show in more details how the mean values of observables change under a small change of the nonequilibrium density operator, or, in other words, when the system is close to thermodynamical equilibrium.

6.3 Variation of mean values and linear response

We will first study the general dependence of the mean value of an observable $\widehat{\mathcal{O}}$ on the spin tensor by denoting by $\delta\widehat{\Upsilon}$ the supposedly small variation, under a transformation of the operator $\widehat{\Upsilon}$. Henceforth, we will consider only the subset of pseudo-gauge transformations with a vanishing $\widehat{Z}$ term, since using the most general transformation would result in complicating the mathematics only, without changing the results.

The variation $\delta\widehat{\Upsilon}$ can be either the one in Eq. (6.21) or the more general (only bulk terms) in Eq. (6.19). In formula:

$$\mathrm{tr}(\widehat{\rho}'\widehat{\mathcal{O}}) = \frac{1}{Z'}\mathrm{tr}(\exp[-\widehat{\Upsilon}']\widehat{\mathcal{O}}) = \frac{1}{Z'}\mathrm{tr}(\exp[-\widehat{\Upsilon} - \delta\widehat{\Upsilon}]\widehat{\mathcal{O}}), \tag{6.22}$$

being $Z' = \mathrm{tr}(\exp[-\widehat{\Upsilon} - \delta\widehat{\Upsilon}])$. We can expand in $\delta\widehat{\Upsilon}$ at the first order (Zassenhaus formula):

$$\begin{aligned} & Z' \simeq Z - \mathrm{tr}\left(\exp[-\widehat{\Upsilon}]\delta\widehat{\Upsilon}\right) \qquad (6.23)\\ & \mathrm{tr}\left(\exp[-\widehat{\Upsilon} - \delta\widehat{\Upsilon}]\widehat{\mathcal{O}}\right) \simeq \mathrm{tr}\left(\exp[-\widehat{\Upsilon}](I - \delta\widehat{\Upsilon} + \frac{1}{2}[\widehat{\Upsilon}, \delta\widehat{\Upsilon}] - \frac{1}{6}[\widehat{\Upsilon}, [\widehat{\Upsilon}, \delta\widehat{\Upsilon}]] + \ldots)\widehat{\mathcal{O}}\right), \end{aligned}$$

hence, denoting by $\langle . \rangle = \mathrm{tr}(\widehat{\rho} .)$, at the first order in $\delta\widehat{\Upsilon}$:

$$\mathrm{tr}(\widehat{\rho}'\widehat{\mathcal{O}}) \equiv \langle\widehat{\mathcal{O}}\rangle' \simeq \langle\widehat{\mathcal{O}}\rangle(1+\langle\delta\widehat{\Upsilon}\rangle) - \langle\widehat{\mathcal{O}}\delta\widehat{\Upsilon}\rangle + \frac{1}{2}\langle[\widehat{\Upsilon}, \delta\widehat{\Upsilon}]\widehat{\mathcal{O}}\rangle - \frac{1}{6}\langle[\widehat{\Upsilon}, [\widehat{\Upsilon}, \delta\widehat{\Upsilon}]]\widehat{\mathcal{O}}\rangle + \ldots \tag{6.24}$$

which makes manifest the dependence of the mean value on the choice of the super-potential $\widehat{\Phi}$.

As has been mentioned, close to thermodynamical equilibrium, the operator $\delta\widehat{\Upsilon}$ is "small" and one can write an expansion of the mean value of the observable $\widehat{\mathcal{O}}$ in the gradients of the four-temperature field, according to relativistic linear response theory [33]. This method, just based on Zubarev's nonequilibrium density operator method, allows to calculate the variation between the actual mean value of an operator and its value at local thermodynamical equilibrium for small deviations from it. In fact, it can be seen from Eq. (6.21) that the operator $\delta\widehat{\Upsilon}$, from the linear response theory viewpoint, is an additional perturbation in the derivative of the four-temperature field and therefore the difference between actual mean values at first order turns out be (see Appendix C for reference):

$$\Delta\langle\widehat{\mathcal{O}}\rangle \simeq -\lim_{\varepsilon\to 0}\frac{T}{2i}\int_{-\infty}^{t'}\mathrm{d}t\,\mathrm{e}^{\varepsilon(t-t')}\int\mathrm{d}^3\mathbf{x}\,\langle[\widehat{\Phi}^{\lambda,0\nu}(x),\widehat{\mathcal{O}}]\rangle_0(\partial_\lambda\delta\beta_\nu(x)+\partial_\nu\delta\beta_\lambda(x)), \quad (6.25)$$

where $\langle\ldots\rangle_0$ stands for the expectation value calculated with the equilibrium density operator, that is:

$$\widehat{\rho}_0 = \frac{1}{Z_0}\exp[-\widehat{H}/T+\mu\widehat{Q}/T]. \quad (6.26)$$

Since $\mathrm{tr}(\widehat{\rho}_0[\widehat{\Phi}^{\lambda,0\nu},\widehat{\mathcal{O}}]) = \mathrm{tr}(\widehat{\Phi}^{\lambda,0\nu}[\widehat{\mathcal{O}},\widehat{\rho}_0])$ the right hand side of (6.25) vanishes for all quantities commutating with the equilibrium density operator, notably total energy, momentum and angular momentum. Nevertheless, in principle, even the mean values of the conserved quantities are affected by the choice of a specific quantum stress-energy tensor, though at the second order in the perturbation $\delta\beta$.

We now set out to study the effect of a pseudo-gauge transformation on the total entropy. In non-equilibrium conditions, entropy is usually defined as [11]:

$$S = -\mathrm{tr}(\widehat{\rho}_{\mathrm{LE}}\log\widehat{\rho}_{\mathrm{LE}}), \quad (6.27)$$

where $\widehat{\rho}_{\mathrm{LE}}$ is the local thermodynamical equilibrium operator, namely:

$$\widehat{\rho}_{\mathrm{LE}}(t) = \frac{\exp[-\int\mathrm{d}^3\mathbf{x}\,\left(\widehat{T}^{0\nu}\beta_\nu(x)-\widehat{\jmath}^0\xi(x)-\frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x)\right)]}{\mathrm{tr}(\exp[-\int\mathrm{d}^3\mathbf{x}\,\left(\widehat{T}^{0\nu}\beta_\nu(x)-\widehat{\jmath}^0\xi(x)-\frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x)\right)])}, \quad (6.28)$$

which - as emphasized in the above equation - is explicitely dependent on time, unlike the Zubarev stationary nonequilibrium density operator (6.12); of course the time dependence is crucial to make entropy increasing in nonequilibrium situation. In order to study the effect of pseudo-gauge transformations on the entropy it is convenient to define:

$$\widehat{\Upsilon}_{\rm LE} = \int {\rm d}^3{\bf x} \, \left(\widehat{T}^{0\nu}\beta_\nu(x) - \widehat{\jmath}^0\xi(x) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(x) \right), \tag{6.29}$$

for which it can be shown that, with calculations similar to those in the previous section, the variation induced by the transformation is:

$$\delta\widehat{\Upsilon}_{\rm LE} = \frac{1}{2}\Bigg\{ \int {\rm d}S \, n_i \, \widehat{\varphi}^{i0,\nu}\delta\beta_\nu + \tag{6.30}$$
$$- \int {\rm d}^3{\bf x} \left[\widehat{\Phi}^{\lambda,0\nu}(\partial_\lambda\delta\beta_\nu + \partial_\nu\delta\beta_\lambda) - \widehat{\Phi}^{0,\lambda\nu}\left(\frac{1}{2}(\partial_\lambda\delta\beta_\nu - \partial_\nu\delta\beta_\lambda) + \delta\omega_{\lambda\nu}\right)\right]\Bigg\}.$$

As has been mentioned, it is possible to get rid of the first and last term through a suitable choice of the perturbations, but not all of them.

Since $\delta\widehat{\Upsilon}_{\rm LE}$ is a small term compared to $\widehat{\Upsilon}_{\rm LE}$ we can determine the variation of the entropy (6.27). First, we observe that, expanding the trace in $\delta\widehat{\Upsilon}_{\rm LE}$ at first order (see also Eq. (6.23):

$$Z'_{\rm LE} \equiv {\rm tr}\left(\exp[-\widehat{\Upsilon}_{\rm LE} - \delta\widehat{\Upsilon}_{\rm LE}]\right) \simeq {\rm tr}\left(\exp[-\widehat{\Upsilon}_{\rm LE}](I - \delta\widehat{\Upsilon}_{\rm LE})\right) =$$

$$= Z_{\rm LE}(1 - \langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}),$$

where $\langle\rangle_{\widehat{\Upsilon}}$ stands for the averaging with the original $\widehat{\Upsilon}_{\rm LE}$ local equilibrium operator. Hence, the new entropy reads:

$$\begin{aligned} S' &= \frac{1}{Z'_{\rm LE}}{\rm tr}\left(\exp[-\widehat{\Upsilon}_{\rm LE} - \delta\widehat{\Upsilon}_{\rm LE}](\widehat{\Upsilon}_{\rm LE} + \delta\widehat{\Upsilon}_{\rm LE})\right) + \log Z'_{\rm LE} \simeq \\ &\simeq \frac{1}{Z_{\rm LE}}(1 + \langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}})\,{\rm tr}\left(\exp[-\widehat{\Upsilon}_{\rm LE} - \delta\widehat{\Upsilon}_{\rm LE}](\widehat{\Upsilon}_{\rm LE} + \delta\widehat{\Upsilon}_{\rm LE})\right) + \\ &\quad + \log Z_{\rm LE} + \log(1 - \langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}). \end{aligned} \tag{6.31}$$

We can now further expand the exponentials as we have done in Eq. (6.23). First:

$$
\begin{aligned}
\operatorname{tr}\left(\exp[-\widehat{\Upsilon}_{\rm LE}-\delta\widehat{\Upsilon}_{\rm LE}]\widehat{\Upsilon}_{\rm LE}\right) &\simeq \operatorname{tr}\Bigg(\exp[-\widehat{\Upsilon}_{\rm LE}](I-\delta\widehat{\Upsilon}_{\rm LE}+\frac{1}{2}[\widehat{\Upsilon}_{\rm LE},\delta\widehat{\Upsilon}_{\rm LE}]+ \\
&\quad -\frac{1}{6}[\widehat{\Upsilon}_{\rm LE},[\widehat{\Upsilon}_{\rm LE},\delta\widehat{\Upsilon}_{\rm LE}]]+\ldots)\widehat{\Upsilon}_{\rm LE}\Bigg) = \\
&= \operatorname{tr}(\exp[-\widehat{\Upsilon}_{\rm LE}]\widehat{\Upsilon}_{\rm LE})-\operatorname{tr}(\exp[-\widehat{\Upsilon}_{\rm LE}]\delta\widehat{\Upsilon}_{\rm LE}\,\widehat{\Upsilon}_{\rm LE}) = \\
&= Z_{\rm LE}\langle\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}-Z_{\rm LE}\langle\delta\widehat{\Upsilon}_{\rm LE}\,\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}, \qquad (6.32)
\end{aligned}
$$

where, in the second equality, we have taken advantage of commutativity and ciclicity of the trace. Then:

$$
\begin{aligned}
\operatorname{tr}\left(\exp[-\widehat{\Upsilon}_{\rm LE}-\delta\widehat{\Upsilon}_{\rm LE}]\delta\widehat{\Upsilon}_{\rm LE}\right) &\simeq \operatorname{tr}\Bigg(\exp[-\widehat{\Upsilon}_{\rm LE}](I-\delta\widehat{\Upsilon}_{\rm LE}+\frac{1}{2}[\widehat{\Upsilon}_{\rm LE},\delta\widehat{\Upsilon}_{\rm LE}]+ \\
&\quad -\frac{1}{6}[\widehat{\Upsilon}_{\rm LE},[\widehat{\Upsilon}_{\rm LE},\delta\widehat{\Upsilon}_{\rm LE}]]+\ldots)\delta\widehat{\Upsilon}_{\rm LE}\Bigg) \\
&\simeq \operatorname{tr}(\exp[-\widehat{\Upsilon}_{\rm LE}]\delta\widehat{\Upsilon}_{\rm LE}) = Z_{\rm LE}\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}, \qquad (6.33)
\end{aligned}
$$

keeping only first order terms. Thus, Eq. (6.31) can be rewritten as:

$$
\begin{aligned}
S' &\simeq \frac{1}{Z_{\rm LE}}(1+\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}})\operatorname{tr}\left(\exp[-\widehat{\Upsilon}_{\rm LE}-\delta\widehat{\Upsilon}_{\rm LE}](\widehat{\Upsilon}_{\rm LE}+\delta\widehat{\Upsilon}_{\rm LE})\right)+ \\
&\quad +\log Z_{\rm LE}+\log(1-\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}) \simeq \\
&\simeq \frac{1}{Z_{\rm LE}}(1+\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}})\left(Z_{\rm LE}\langle\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}-Z_{\rm LE}\langle\delta\widehat{\Upsilon}_{\rm LE}\,\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}+Z_{\rm LE}\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}\right)+ \\
&\quad +\log Z_{\rm LE}+\log(1-\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}) \simeq \\
&= (1+\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}})\left(\langle\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}-\langle\delta\widehat{\Upsilon}_{\rm LE}\,\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}+\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}\right)+ \\
&\quad +\log Z_{\rm LE}+\log(1-\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}). \qquad (6.34)
\end{aligned}
$$

Retaining only the first order terms in $\delta\widehat{\Upsilon}_{\rm LE}$, expanding the logarithm for $\langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\rm LE} \ll 1$ and inserting the original expression of entropy:

$$S' \simeq S - \langle\delta\widehat{\Upsilon}_{\rm LE}\,\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}} + \langle\delta\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}\langle\widehat{\Upsilon}_{\rm LE}\rangle_{\widehat{\Upsilon}}. \tag{6.35}$$

Therefore, the variation of the total entropy is, to the lowest order, proportional to the correlation between $\widehat{\Upsilon}$ and $\delta\widehat{\Upsilon}$, which is generally non-vanishing. It is now a sensible physical question to ask where does this variation arise. Indeed, this variation stems from the variation of transport coefficients, which affect the entropy production rate, hence its total amount. We have seen in the last chapter the link between the stress-energy-momentum tensor and transport coefficients and how to calculate them studying the linear response. We can thus see the difference in transport coefficients using the modified Zubarev density matrix (6.12).

6.4 Transport coefficients: shear viscosity as an example

In this section we will prove that a transformation on the microscopic tensors results in a difference in the predicted values of transport coefficients calculated with the relativistic Kubo formula, which is obtained by working out the mean value of the stress-energy tensor itself with the linear response theory and the nonequilibrium density operator in Eq. (6.1). For this purpose, the derivation in ref. [33] must be extended to the most general expression of the nonequilibrium density operator including a spin tensor, that is, Eq. (6.12); it can be found in Appendix C.

The equation (6.25), yielding the difference of mean values of a general observable under pseudogauge transformations, cannot be straightforwardly used to calculate the mean value of the stress-energy tensor setting $\widehat{O} = \widehat{T}^{\mu\nu}(y)$ because $\widehat{T}^{\mu\nu}(y)$ gets transformed itself. It is therefore more convenient to work out the general expression of the Kubo formula and study how it is modified by transformations thereafter.

We will take shear viscosity as example, the transformation of other transport coefficients being derivable with the same reasoning. As we have shown, the symmetric part of the, macroscopic, stress-energy-momeum tensor has the same structure for systems with or without internal angular momentum, including the shear stress part and shear viscosity.

For the symmetric part of the stress-energy tensor $T_{\rm s}^{\mu\nu} \equiv 1/2(T^{\mu\nu} + T^{\nu\mu})$, using the general formula of relativistic linear response theory (Eq. C.14 of Appendix C), the difference $\delta T_{\rm s}^{\mu\nu}(y)$ between actual mean value and the equilibrium value reads, at the lowest order in gradients:

$$\begin{aligned}
\delta T_{\rm s}^{\mu\nu}(y) \;=\;& \lim_{\varepsilon\to 0} \frac{T}{i}\int_{-\infty}^{t'} {\rm d}t\, \frac{1-{\rm e}^{\varepsilon(t-t')}}{\varepsilon} \int {\rm d}^3{\bf x}\, \langle\Big[\widehat{T}^{\rho\sigma}(x), \widehat{T}_{\rm s}^{\mu\nu}(y)\Big]\rangle_0 \partial_\rho \delta\beta_\sigma(x) \\
&-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{T}{i}\int_{-\infty}^{t'} {\rm d}t\; {\rm e}^{\varepsilon(t-t')}\int {\rm d}^3{\bf x}\,\langle\Big[\widehat{\mathcal{S}}^{0,\rho\sigma}(x),\widehat{T}_{\rm s}^{\mu\nu}(y)\Big]\rangle_0 \delta\omega_{\rho\sigma}(x) \\
&-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{T}{i}\int_{-\infty}^{t'} {\rm d}t\; {\rm e}^{\varepsilon(t-t')}\int_{-\infty}^{t} {\rm d}\tau \int {\rm d}^3{\bf x}\,\langle\Big[\widehat{\mathcal{S}}^{0,\rho\sigma}(\tau,{\bf x}),\widehat{T}_{\rm s}^{\mu\nu}(y)\Big]\rangle_0 \frac{\partial}{\partial t}\delta\omega_{\rho\sigma}(x).
\end{aligned} \tag{6.36}$$

In order to obtain transport coefficients, a suitable perturbation must be chosen which can be eventually taken out from the integral. Physically, it corresponds to the enforcement of a particular hydrodynamical motion and observe the response in the stress-energy tensor to infer the dissipative coefficient. The perturbation $\delta\beta$ is taken to be a stationary one and non-vanishing only within a finite region V, at whose boundary it goes to zero in a continuous and derivable fashion. The perturbation $\delta\omega$ is also taken to be stationary and it can be chosen either to vanish or like in Eq. (6.20); in both cases, one gets to the same final result.

Let us then set $\delta\omega = 0$ and expand the perturbation $\delta\beta = (0, 0, \delta\beta\cdot\beta, 0)$ dependent on x^1 in a Fourier series (it vanishes at some large, yet finite boundary). Since we want the higher order gradients of the perturbation to be negligibly small (the so-called hydrodynamic limit), the Fourier components with short wavelength must be correspondingly suppressed. The component with the longest wavelength will then be much larger than any other and, therefore, $\delta\beta\cdot\beta({\bf x})$ can be approximately written, at least far from the boundary, as $A\sin(\pi x^1/L)$ where L is the size of the region V in the x^1 direction and A is a constant. The derivative of this perturbation reads:

$$\partial_1\delta\beta_2({\bf x}) = \frac{\pi}{L}A\cos(\pi x^1/L) = \partial_1\delta\beta_2({\bf 0})\cos(\pi x^1/L) \equiv \partial_1\delta\beta_2({\bf 0})\cos(kx^1),$$

where $k \equiv \pi/L$. Therefore, by defining ${\bf k} = (k, 0, 0)$ and plugging the last equation in

Eq. (6.36):

$$\begin{aligned}\delta T_{\rm s}^{\mu\nu}(y) &= \lim_{\varepsilon\to 0}\frac{T}{i}\partial_1\delta\beta_2(\mathbf{0})\int_{-\infty}^{t'}{\rm d}t\;\frac{1-{\rm e}^{\varepsilon(t-t')}}{\varepsilon}\int_V {\rm d}^3\mathbf{x}\;\cos\mathbf{k}\cdot\mathbf{x}\langle\Big[\widehat{T}^{12}(x),\widehat{T}_{\rm s}^{\mu\nu}(y)\Big]\rangle_0\\ &= \lim_{\varepsilon\to 0} T\partial_1\delta\beta_2(\mathbf{0})\;{\rm Im}\int_{-\infty}^{t'}{\rm d}t\;\frac{1-{\rm e}^{\varepsilon(t-t')}}{\varepsilon}\int_V {\rm d}^3\mathbf{x}\;{\rm e}^{i\mathbf{k}\cdot\mathbf{x}}\langle\Big[\widehat{T}^{12}(x),\widehat{T}_{\rm s}^{\mu\nu}(y)\Big]\rangle_0,\end{aligned} \tag{6.37}$$

taking into account that the commutator is purely imaginary. To extract shear viscosity we have to evaluate the stress-energy tensor in $\mathbf{y}=0$ to make it proportional to the derivative of the four-temperature field in the same point and we have to take the limit $L\to\infty$ which implies $V\to\infty$ and $\mathbf{k}\to 0$ at the same time:

$$\begin{aligned}&\delta T_{\rm s}^{\mu\nu}(t_y,\mathbf{0}) = \\ &= \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}\;T\,\partial_1\delta\beta_2(\mathbf{0})\;{\rm Im}\int_{-\infty}^{t'}{\rm d}t\;\frac{1-{\rm e}^{\varepsilon(t-t')}}{\varepsilon}\int {\rm d}^3\mathbf{x}\;{\rm e}^{i\mathbf{k}\cdot\mathbf{x}}\langle\Big[\widehat{T}^{12}(x),\widehat{T}_{\rm s}^{\mu\nu}(t_y,\mathbf{0})\Big]\rangle_0,\end{aligned} \tag{6.38}$$

where it has been assumed that the integration domain goes to its thermodynamic limit independently of the integrand. Because of the time-translation symmetry of the equilibrium density operator $\widehat{\rho}_0$, the mean value in the integral only depends on the time difference $t-t_y$. Thus, choosing the arbitrary time $t'=t_y$ and redefining the integration variables, the Eq. (6.38) can be rewritten as:

$$\begin{aligned}&\delta T_{\rm s}^{\mu\nu}(t_y,\mathbf{0}) = \\ &= \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}\;T\,\partial_1\delta\beta_2(\mathbf{0})\;{\rm Im}\int_{-\infty}^{0}{\rm d}t\;\frac{1-{\rm e}^{\varepsilon t}}{\varepsilon}\int {\rm d}^3\mathbf{x}\;{\rm e}^{i\mathbf{k}\cdot\mathbf{x}}\langle\Big[\widehat{T}^{12}(x),\widehat{T}_{\rm s}^{\mu\nu}(0)\Big]\rangle_0,\end{aligned} \tag{6.39}$$

which shows that the mean value $\delta T_{\rm s}^{\mu\nu}(t_y,\mathbf{0})$ is indeed independent of t_y, which is expected as $\delta\beta$ is stationary.

We can now take advantage of the well known Curie symmetry "principle" which states that tensors belonging to some irreducible representation of the rotation group will only respond to perturbations belonging to the same representation and with the same components [3]. In our case the Curie principle implies that only the same component of the symmetric part of the stress-energy tensor, i.e. $\widehat{T}_S^{12}$, will give a non-vanishing value:

$$\begin{aligned} \delta T_{\rm s}^{12}(t_y, \mathbf{0}) &= \qquad (6.40)\\ &= \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0} T\,\partial_1\delta\beta_2(\mathbf{0})\,{\rm Im}\int_{-\infty}^{0}{\rm d}t\,\frac{1-{\rm e}^{\varepsilon t}}{\varepsilon}\int {\rm d}^3\mathbf{x}\;{\rm e}^{i\mathbf{k}\cdot\mathbf{x}}\langle\Big[\widehat{T}_{\rm s}^{12}(x),\widehat{T}_{\rm s}^{12}(0)\Big]\rangle_0. \end{aligned}$$

From the above expression, a Kubo formula for shear viscosity can be extracted setting $\delta\beta = (1/T)\delta u$:

$$\eta = \lim_{\varepsilon\to 0}\lim_{\mathbf{k}\to 0}\,{\rm Im}\int_{-\infty}^{0}{\rm d}t\,\frac{1-{\rm e}^{\varepsilon t}}{\varepsilon}\int {\rm d}^3\mathbf{x}\;{\rm e}^{i\mathbf{k}\cdot\mathbf{x}}\langle\Big[\widehat{T}_{\rm s}^{12}(x),\widehat{T}_{\rm s}^{12}(0)\Big]\rangle_0, \qquad (6.41)$$

which, after a little algebra, can be shown to be the same expression obtained in ref. [33]. Because of the rotational invariance of the equilibrium density operator, shear viscosity is independent of the particular couple $(1,2)$ of chosen indices. It is worth pointing out that if we started from Eq. (C.15) instead of Eq. (C.14), choosing $\delta\omega = 0$ or like in Eq. (6.20), we would have come to the same formula for shear viscosity; in the latter case, the third contributing term in Eq. (C.15) would have been of higher order in derivatives of $\delta\beta$, hence negligible.

Now, the question we want to answer is whether the equation (6.41) is invariant by a pseudo-gauge transformation, which turns the symmetric part of thestress-energy tensor into:

$$\widehat{T}_{\rm s}^{\prime\mu\nu} = \widehat{T}_{\rm s}^{\mu\nu} - \frac{1}{2}\partial_\lambda(\widehat{\Phi}^{\mu,\lambda\nu} + \widehat{\Phi}^{\nu,\lambda\mu}) = \widehat{T}_{\rm s}^{\mu\nu} - \partial_\lambda\widehat{\Xi}^{\lambda\mu\nu}, \qquad (6.42)$$

where:

$$\frac{1}{2}(\widehat{\Phi}^{\mu,\lambda\nu} + \widehat{\Phi}^{\nu,\lambda\mu}) \equiv \widehat{\Xi}^{\lambda\mu\nu}, \qquad (6.43)$$

$\widehat{\Xi}$ being symmetric in the last two indicess. We will study the effect of the transformation on the mean value of the stress-energy tensor in the point $y = 0$ starting from

[3] This is true provided that the right hand side of Eq. (6.39) is a continuous function of $\mathbf{k}$ for $\mathbf{k} = 0$ or that its limit for $\mathbf{k}\to 0$ exists, i.e. it is independent of the direction of $\mathbf{k}$

the formula Eq. (C.15) instead of Eq. (C.14) with $\delta\omega = 0$ or like in Eq. (6.20), which allows us to retain only the first contributing term to $\delta T_S^{12}(0)$. The perturbation $\delta\beta$ is taken to be stationary and t' is set to be equal to $t_y = 0$. Eventually,the appropriate limits will be calculated to get the new shear viscosity. Thus:

$$\delta T_{\rm s}^{\prime 12}(0) = \delta T_{\rm s}^{12}(0) + \lim_{\varepsilon\to 0}\int_{-\infty}^{0} {\rm d}\tau\, \frac{1-{\rm e}^{\varepsilon\tau}}{\varepsilon} \int {\rm d}^3{\bf x} \langle \left[\partial_\alpha \widehat{\Xi}^{\alpha 12}(\tau,{\bf x}), \partial_\beta \widehat{\Xi}^{\beta 12}(0,{\bf 0})\right]\rangle_0 \times$$
$$\times (\partial_1 \delta\beta_2({\bf x}) + \partial_2\delta\beta_1({\bf x})) + \tag{6.44}$$
$$-\lim_{\varepsilon\to 0}\int_{-\infty}^{0} {\rm d}\tau \frac{1-{\rm e}^{\varepsilon\tau}}{\varepsilon} \int {\rm d}^3{\bf x}\Big(\langle \left[\partial_\alpha \widehat{\Xi}^{\alpha 12}(\tau,{\bf x}), \widehat{T}_{\rm s}^{12}(0,{\bf 0})\right]\rangle_0 + \langle \left[\widehat{T}_{\rm s}^{12}(\tau,{\bf x}), \partial_\alpha \widehat{\Xi}^{\alpha 12}(0,{\bf 0})\right]\rangle_0 \Big) \times$$
$$\times (\partial_1 \delta\beta_2({\bf x}) + \partial_2\delta\beta_1({\bf x})).$$

We can simplify the above formula by noting that the mean value of two operators at equilibrium can oly depend on the difference of the coordinates, so:

$$\langle \left[\widehat{\mathcal{O}}_1(y), \partial_\mu \widehat{\mathcal{O}}_2(x)\right]\rangle_0 = \frac{\partial}{\partial x^\mu}\langle \left[\widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2\right]\rangle_0(y-x) = -\frac{\partial}{\partial y^\mu}\langle \left[\widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2\right]\rangle_0(y-x),$$

hence, the Eq. (6.44) can be rewritten as:

$$\delta T_{\rm s}^{\prime 12}(0) = \delta T_{\rm s}^{12}(0) + \tag{6.45}$$
$$-\lim_{\varepsilon\to 0}\int_{-\infty}^{0} {\rm d}\tau\, \frac{1-{\rm e}^{\varepsilon\tau}}{\varepsilon} \int {\rm d}^3{\bf x} \left\{ \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \langle \left[\widehat{\Xi}^{\alpha 12}(\tau,{\bf x}), \widehat{\Xi}^{\beta 12}(0,{\bf 0})\right]\rangle_0 \,(\partial_1 \delta\beta_2({\bf x}) + \partial_2\delta\beta_1({\bf x})) + \right.$$
$$\left. + \frac{\partial}{\partial x^\alpha} \Big(\langle \left[\widehat{\Xi}^{\alpha 12}(\tau,{\bf x}), \widehat{T}_{\rm s}^{12}(0,{\bf 0})\right]\rangle_0 - \langle \left[\widehat{T}_{\rm s}^{12}(\tau,{\bf x}), \widehat{\Xi}^{\alpha 12}(0,{\bf 0})\right]\rangle_0 \Big) \,(\partial_1 \delta\beta_2({\bf x}) + \partial_2\delta\beta_1({\bf x})) \right\}.$$

We are now going to inspect the two terms on the right-hand side of the above equation. If the hamiltonian is time-reversal invariant, it can be shown (see Appendix D):

$$\langle \left[\widehat{T}_{\rm s}^{ij}(\tau,{\bf x}), \widehat{\Xi}^{\alpha ij}(0,{\bf 0})\right]\rangle_0 = (-1)^{n_0} \langle \left[\widehat{\Xi}^{\alpha ij}(0,{\bf 0}), \widehat{T}_{\rm s}^{ij}(-\tau,{\bf x})\right]\rangle_0 = (-1)^{n_0} \langle \left[\widehat{\Xi}^{\alpha ij}(\tau,-{\bf x}), \widehat{T}_{\rm s}^{ij}(0,{\bf 0})\right]\rangle_0,$$

where n_0 is the total number of time indices among those in the above expression. Similarly, if the hamiltonian is parity invariant, then:

$$\langle \left[\widehat{\Xi}^{\alpha ij}(\tau, -\mathbf{x}), \widehat{T}_{\mathrm{s}}^{ij}(0, \mathbf{0}) \right] \rangle_0 = (-1)^{n_s} \langle \left[\widehat{\Xi}^{\alpha ij}(\tau, \mathbf{x}), \widehat{T}_{\mathrm{s}}^{ij}(0, \mathbf{0}) \right] \rangle_0,$$

where n_s is the total number of space indices. Using the last two equations, the (6.45) becomes:

$$\begin{aligned} \delta T_{\mathrm{s}}^{'12}(0) &= \delta T_{\mathrm{s}}^{12}(0) + \\ &- \lim_{\varepsilon \to 0} \int_{-\infty}^{0} \mathrm{d}\tau \, \frac{1 - \mathrm{e}^{\varepsilon \tau}}{\varepsilon} \int_V \mathrm{d}^3\mathbf{x} \left\{ \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \langle \left[\widehat{\Xi}^{\alpha 12}(\tau, \mathbf{x}), \widehat{\Xi}^{\beta 12}(0, \mathbf{0}) \right] \rangle_0 (\partial_1 \delta\beta_2(\mathbf{x}) + \partial_2 \delta\beta_1(\mathbf{x})) + \right. \\ &\left. +2 \, \frac{\partial}{\partial x^\alpha} \langle \left[\widehat{\Xi}^{\alpha 12}(\tau, \mathbf{x}), \widehat{T}_{\mathrm{s}}^{12}(0, \mathbf{0}) \right] \rangle_0 (\partial_1 \delta\beta_2(\mathbf{x}) + \partial_2 \delta\beta_1(\mathbf{x})) \right\} . \end{aligned} \tag{6.46}$$

The two terms on the right hand side of (6.45) can be worked out separately. Using invariance by time-reversal and parity, one has:

$$\begin{aligned} &\langle \left[\widehat{\Xi}^{\alpha ij}(\tau, \mathbf{x}), \widehat{\Xi}^{\beta ij}(0, \mathbf{0}) \right] \rangle_0 = (-1)^{n_0} \langle \left[\widehat{\Xi}^{\beta ij}(0, \mathbf{0}), \widehat{\Xi}^{\alpha ij}(-\tau, \mathbf{x}) \right] \rangle_0 = \\ &= (-1)^{n_0} \langle \left[\widehat{\Xi}^{\beta ij}(\tau, -\mathbf{x}), \widehat{\Xi}^{\alpha ij}(0, \mathbf{0}) \right] \rangle_0 = (-1)^{n_0 + n_s} \langle \left[\widehat{\Xi}^{\beta ij}(\tau, \mathbf{x}), \widehat{\Xi}^{\alpha ij}(0, \mathbf{0}) \right] \rangle_0 = \\ &= \langle \left[\widehat{\Xi}^{\beta ij}(\tau, \mathbf{x}), \widehat{\Xi}^{\alpha ij}(0, \mathbf{0}) \right] \rangle_0, \end{aligned} \tag{6.47}$$

being $n_0 + n_s = 6$. Hence, the first term on the right hand side of (6.45) can be decomposed as:

$$\begin{aligned} &- \lim_{\varepsilon \to 0} \int_{-\infty}^{0} \mathrm{d}\tau \frac{1 - \mathrm{e}^{\varepsilon \tau}}{\varepsilon} \int_V \mathrm{d}^3\mathbf{x} \, \left(\partial_\tau^2 \langle \left[\widehat{\Xi}^{0ij}(\tau, \mathbf{x}), \widehat{\Xi}^{0ij}(0, \mathbf{0}) \right] \rangle_0 + 2\partial_\tau \frac{\partial}{\partial x^k} \langle \left[\widehat{\Xi}^{kij}(\tau, \mathbf{x}), \widehat{\Xi}^{0ij}(0, \mathbf{0}) \right] \rangle_0 \right. \\ &\left. + \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^l} \langle \left[\widehat{\Xi}^{kij}(\tau, \mathbf{x}), \widehat{\Xi}^{lij}(0, \mathbf{0}) \right] \rangle_0 \right) (\partial_i \delta\beta_j(\mathbf{x}) + \partial_j \delta\beta_i(\mathbf{x})), \end{aligned} \tag{6.48}$$

and, similarly, the second term as:

$$-2\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}\tau\frac{1-\mathrm{e}^{\varepsilon\tau}}{\varepsilon}\int_V \mathrm{d}^3\mathbf{x}\ \Big(\partial_\tau\langle\Big[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{T}_{\mathrm{s}}^{12}(0,\mathbf{0})\Big]\rangle_0+ \\ +\frac{\partial}{\partial x^k}\langle\Big[\widehat{\Xi}^{k12}(\tau,\mathbf{x}),\widehat{T}_{\mathrm{s}}^{12}(0,\mathbf{0})\Big]\rangle_0(\partial_1\delta\beta_2(\mathbf{x})+\partial_2\delta\beta_1(\mathbf{x}))\Big). \tag{6.49}$$

All terms in Eqs. (6.48) and (6.49) with a space derivative do not yield any contribution to first-order transport coefficients. This can be shown by, firstly, integrating by parts and generating two terms, one of which is a total derivative and the second involves the second derivative of the perturbation $\delta\beta$. The total derivative term can be transformed into a surface integral on the boundary of V which vanishes because therein the perturbation $\delta\beta$ is supposed to vanish along with all of its derivatives (see previous discussion). The second term, involving higher derivatives, does not give contribution to transport coefficients at first order in the derivative expansion. Therefore, the Eq. (6.46) turns into:

$$\delta T_{\mathrm{s}}^{'12}(0)=\delta T_{\mathrm{s}}^{12}(0)+ \\ -\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}\tau\ \frac{1-\mathrm{e}^{\varepsilon\tau}}{\varepsilon}\int_V \mathrm{d}^3\mathbf{x}\ \partial_\tau^2\langle\Big[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{\Xi}^{012}(0,\mathbf{0})\Big]\rangle_0(\partial_1\delta\beta_2(\mathbf{x})+\partial_2\delta\beta_1(\mathbf{x})) \\ -2\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}\tau\ \frac{1-\mathrm{e}^{\varepsilon\tau}}{\varepsilon}\int_V \mathrm{d}^3\mathbf{x}\ \partial_\tau\langle\Big[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{T}_{\mathrm{s}}^{12}(0,\mathbf{0})\Big]\rangle_0(\partial_1\delta\beta_2(\mathbf{x})+\partial_2\delta\beta_1(\mathbf{x}))+ \\ +\mathcal{O}(\partial^2\delta\beta), \tag{6.50}$$

which can be further integrated by parts in the time τ, yielding:

$$\delta T_{\mathrm{s}}^{'12}(0)=\delta T_{\mathrm{s}}^{12}(0)+ \\ -\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}\tau\ (\delta(\tau)-\varepsilon\mathrm{e}^{\varepsilon\tau})\int_V \mathrm{d}^3\mathbf{x}\ \langle\Big[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{\Xi}^{012}(0,\mathbf{0})\Big]\rangle_0(\partial_1\delta\beta_2(\mathbf{x})+\partial_2\delta\beta_1(\mathbf{x})) \\ -2\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\mathrm{d}\tau\ \mathrm{e}^{\varepsilon\tau}\int_V \mathrm{d}^3\mathbf{x}\ \langle\Big[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{T}_{\mathrm{s}}^{12}(0,\mathbf{0})\Big]\rangle_0(\partial_1\delta\beta_2(\mathbf{x})+\partial_2\delta\beta_1(\mathbf{x}))+ \\ +\mathcal{O}(\partial^2\delta\beta), \tag{6.51}$$

provided that, for general space-time dependent operators $\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_2$

$$\lim_{\tau\to-\infty}\int_V \mathrm{d}^3\mathbf{x}\ \mathrm{e}^{n\varepsilon\tau}\partial_\tau\langle\left[\widehat{\mathcal{O}}_1(\tau,\mathbf{x}),\widehat{\mathcal{O}}_2(0,\mathbf{0})\right]\rangle_0 = 0$$

$$\lim_{\tau\to-\infty}\int_V \mathrm{d}^3\mathbf{x}\ \mathrm{e}^{n\varepsilon\tau}\langle\left[\widehat{\mathcal{O}}_1(\tau,\mathbf{x}),\widehat{\mathcal{O}}_2(0,\mathbf{0})\right]\rangle_0 = 0,$$

with $n = 0, 1$, which is reasonable because thermodynamical correlations are expected to vanish exponentially as a function of time for fixed points in space [4].

From Eq. (6.51) the variation of the shear viscosity can be inferred with the very same reasoning that led us to formula (6.41), that is:

$$\begin{aligned}\Delta\eta = \eta' - \eta = \\ = -\lim_{\varepsilon\to 0}\lim_{k\to 0}\ \mathrm{Im}\int_{-\infty}^{0}\mathrm{d}\tau\ (\delta(\tau)-\varepsilon\mathrm{e}^{\varepsilon\tau})\int\mathrm{d}^3\mathbf{x}\ \mathrm{e}^{ikx^1}\langle\left[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{\Xi}^{012}(0,\mathbf{0})\right]\rangle_0 + \\ -2\lim_{\varepsilon\to 0}\lim_{k\to 0}\ \mathrm{Im}\int_{-\infty}^{0}\mathrm{d}\tau\ \mathrm{e}^{\varepsilon\tau}\int\mathrm{d}^3\mathbf{x}\ \mathrm{e}^{ikx^1}\langle\left[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{T}^{12}_{\mathrm{s}}(0,\mathbf{0})\right]\rangle_0.\end{aligned} \tag{6.52}$$

If the first integral is regular, then the $\varepsilon \to 0$ limit kills one term and the (6.52) reduces to:

$$\begin{aligned}\Delta\eta = \eta' - \eta = \\ = -\lim_{k\to 0}\int_V \mathrm{d}^3\mathbf{x}\ \cos kx^1\langle\left[\widehat{\Xi}^{012}(0,\mathbf{x}),\widehat{\Xi}^{012}(0,\mathbf{0})\right]\rangle_0 \\ -2\lim_{\varepsilon\to 0}\lim_{k\to 0}\ \mathrm{Im}\int_{-\infty}^{0}\mathrm{d}\tau\ \mathrm{e}^{\varepsilon\tau}\int\mathrm{d}^3\mathbf{x}\ \mathrm{e}^{ikx^1}\langle\left[\widehat{\Xi}^{012}(\tau,\mathbf{x}),\widehat{T}^{12}_{\mathrm{s}}(0,\mathbf{0})\right]\rangle_0.\end{aligned} \tag{6.53}$$

In general, this difference is non-vanishing. Therefore, the existence of a spin tensor in the underlying quantum field theory affects the transport coefficients.

[4] There might be singularities on the light cone, however for fixed $\mathbf{x}$ and $\mathbf{0}$ and integration over a finite region V, in the limit $\tau \to -\infty$ light cone is not involved

6.5 Discussion

It is important to stress that the dependence of the transport coefficients on the particular set of stress-energy and spin tensor of the theory is physically meaningful. This means that the variation of some coefficient is not compensated by a corresponding variation of another coefficient so as to eventually leave measurable quantities unchanged. This has been proved in Sect. 6.3 where we showed that total entropy itself undergoes a variation under a transformation of the stress-energy and spin tensor (see Eq. (6.35).

A second question is whether at least one specific physical system exists for which the transformation (1.3) actually leads to different values for e.g. transport coefficients or other quantities in non-equilibrium situations. We do not carry out full calculations, however we provide some general considerations about a specific instance, spinor electrodynamics.

Starting from the symmetrized gauge-invariant Belinfante tensor, having $\widehat{\mathcal{S}} = 0$:

$$\widehat{T}^{\mu\nu} = \frac{i}{4}\left(\overline{\Psi}\gamma^{\mu}\overleftrightarrow{\nabla}^{\mu}\Psi + \overline{\Psi}\gamma^{\nu}\overleftrightarrow{\nabla}^{\mu}\Psi\right) + \widehat{F}^{\mu}{}_{\lambda}\widehat{F}^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}\widehat{F}^{2} \tag{6.54}$$

where $\nabla_{\mu} = \partial_{\mu} - ieA_{\mu}$ is the gauge covariant derivative, one can generate other stress-energy tensors with suitable gauge-invariant rank three tensors and then setting $\widehat{\Phi} = -\widehat{\mathcal{S}}'$ where $\widehat{\mathcal{S}}'$ is the new spin tensor, having thus a vanishing $\widehat{Z}$ term. One of the best known is the *canonical* Dirac spin tensor:

$$\widehat{\Phi} = -\frac{i}{8}\overline{\Psi}\{\gamma^{\lambda}, [\gamma^{\mu}, \gamma^{\nu}]\}\Psi$$

($\{\ \}$ stands for anticommutator) which is gauge-invariant and transforms the Belinfante tensor (6.54) back to the canonical one obtained from the spinor electrodynamics lagrangian (see also [28] for a detailed discussion). However, this is totally antysimmetric in the three indices λ, μ, ν and thus the variation of $\widehat{\Upsilon}$ operator (see Eq. (6.21) as well as transport coefficients, which depend on the symmetrized $\widehat{\Xi}$ tensor (6.43) vanish.

Nevertheless, other gauge-invariant $\widehat{\Phi}$-like tensors can be found. For instance, one could add a superpotential like:

$$\widehat{\Phi} = \frac{\hbar^2}{m_e^2 c^4} \overline{\Psi} \gamma^\lambda \Psi \widehat{F}^{\mu\nu}$$

(we have purposely restored natural constants) which is gauge invariant and which gives rise to a non-vanishing variation of the symmetrized tensor $\widehat{\Xi}$ (6.43), thus a variation of thermodynamics. It is also interesting to note that the "improved" stress-energy tensor by Callan, Coleman and Jackiw [34] with renormalizable matrix elements at all orders of perturbation theory, is obtained from the Belinfante's symmetrized one in Eq. (6.54) with a transformation of the kind (1.3) setting (for the Dirac field and vanishing constants [34]):

$$\widehat{Z}^{\alpha\lambda,\mu\nu} = -\frac{1}{6} \left(g^{\alpha\mu} g^{\lambda\nu} - g^{\alpha\nu} g^{\lambda\mu} \right) \overline{\Psi} \Psi$$

and requiring $\widehat{\mathcal{S}}' = \widehat{\mathcal{S}} = 0$ so that $\widehat{\Phi}^{\lambda,\mu\nu} = \partial_\alpha \widehat{Z}^{\alpha\lambda,\mu\nu}$, hence:

$$\begin{aligned}
\widehat{\Phi}^{\lambda,\mu\nu} &= -\frac{1}{6} \left(g^{\lambda\nu} \partial^\mu - g^{\lambda\mu} \partial^\nu \right) \overline{\Psi} \Psi \\
\widehat{\Xi}^{\lambda\mu\nu} &= \frac{1}{2} (\widehat{\Phi}^{\mu,\lambda\nu} + \widehat{\Phi}^{\nu,\lambda\mu}) = -\frac{1}{6} \left[g^{\mu\nu} \partial^\lambda - \frac{1}{2} (g^{\lambda\nu} \partial^\mu + g^{\lambda\mu} \partial^\nu) \right] \overline{\Psi} \Psi \\
\widehat{T}'^{\mu\nu} &= \widehat{T}^{\mu\nu} - \partial_\lambda \widehat{\Xi}^{\lambda\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{6} (g^{\mu\nu} \Box - \partial^\mu \partial^\nu) \overline{\Psi} \Psi
\end{aligned}$$

which is just the improved stress-energy tensor [34].

It is likely (to be verified though) that the aforementioned modified stress-energy tensors imply a different thermodynamics with respect to the original Belinfante symmetrized tensor. More in general, once the implied different values of e.g. transport coefficients are theoretically calculated and known, it is possible, at least in principle, to pin them down with a suitably designed thermodynamics experiment and thus confirm or disprove a particular stress-energy tensor.

In order to have an order of magnitude estimate, we could consider as a superpotential for (6.54) the operator[5]:

$$\widehat{\Phi}^{\lambda,\mu\nu} = \frac{1}{8m} \overline{\Psi} \left(\left[\gamma^\mu, \gamma^\lambda \right] \overleftrightarrow{\nabla}^\nu - \left[\gamma^\nu, \gamma^\lambda \right] \overleftrightarrow{\nabla}^\mu \right) \Psi,$$

[5] This amounts to the gauge-invariant version of the one used in ref. [1] to obtain a conserved spin current

therefore the symmetric part that provide a different shear viscosity after the transformation reads:

$$\widehat{\Xi}^{\lambda,\mu\nu} = \frac{1}{16m}\overline{\Psi}\left(\left[\gamma^{\lambda},\gamma^{\mu}\right]\overleftrightarrow{\nabla}^{\nu} + \left[\gamma^{\lambda},\gamma^{\nu}\right]\overleftrightarrow{\nabla}^{\mu}\right)\Psi.$$

Noting that the structure of the above tensor is very similar to the Belinfante stress-energy tensor itself (6.54), it is not difficult to find a rough estimate of the variation of shear viscosity induced by the transformation. If we look at (6.53) we note that $\widehat{\Xi}^{012}$ mainly differs from $\widehat{T}^{12}$ in (6.54) by the factor $1/m$. The physical dimension of the superpotential is that of a stress-energy-momentum tensor multiplied by a time, and therefore this term must be of the order of $\eta\hbar/mc^2\tau$ where τ is the microscopic correlation time scale of the original stress-energy-momentum tensor or the collisional time scale in the kinetic language and η the shear viscosity obtained from the original tensor. Thus, the expected relative variation of shear viscosity in this case is of the order:

$$\frac{\Delta\eta}{\eta} \approx \mathcal{O}\left(\frac{\hbar}{mc^2\tau}\right),$$

which is (as it could have been expected) a quantum relativistic correction governed by the ratio $(\lambda_{\mathrm{c}}/c)/\tau$ being λ_{c} the Compton wavelength. For the electron, the ratio $(\lambda_{\mathrm{c}} \approx 10^{-21}$ sec, which is a very small time scale compared to the usual kinetic time scales, yet it could be detectable for particular systems with very low shear viscosity.

Chapter 7

Conclusions

In this work we have studied the effect of changing the pairs of fundamental stress-energy-momentum and spin tensor $\{\widehat{T}, \widehat{\mathcal{S}}\}$ with another one linked by a pseudo-gauge transformation $\{\widehat{T}', \widehat{\mathcal{S}}'\}$. Despite the common belief that such pairs are physically equivalent, or at least that only gravitational effects can distinguish between them, we found that they are in fact inequivalent, both for system at thermodynamical equilibrium and transport coefficients.

We have seen that his inequivalence is strongly linked with the symmetries of the state of the system, which we use to take mean values and to find the macroscopic, classical, tensors $\{T, \mathcal{S}\}$. Even though a system at homogeneous equilibrium have the same mean values for every pair of fundamental tensors, being it canonical or grand canonical global equilibrium. The looser symmetry of a rigidly rotating system allows to see the inequivalence.

Besides in chapter 6 we have seen that a pseudo-gauge transformation change the value of transport coefficients, even for a system arbitrarily close to homogeneous equilibrium, which does not show inequivalence in mean densities. Farther, this difference can not be compensated by a variation in other coefficients in order to leave observable quantities unchanged, as the entropy itself change (out of equilibrium) if we change the couple of fundamental tensors.

For the time being we lack a theoretical way to identify the right pair of tensors, but could prove if a couple is wrong with high precision measurements, for instance, of angular momentum density and transport coefficients.

In both cases we have seen that the difference is a small quantum effect but is not, in principle, out of the range of possible experimental investigations. The greatest difficulties in developing possible tests lie in the calculations of the effects in relevant cases. The explicit calculation in chapter 3 was for a very ideal case. In cold gas experiments it is possible to reach very low temperatures, and thus a ratio $\hbar\omega/KT$ of the order of a few precent, but mutual interactions of particles and interactions with external fields (trapping the atoms in a spatial region) have to be taken into account. Regarding transport coefficients, quark gluon plasma has a very low viscosity, making it an interesting candidate for a shear viscosity tests, but it may prove necessary to take into account the high external magnetic fields produced in peripheral heavy ion collisions. These strong magnetic fields, providing a privileged direction, break the isotropy assumption thus requiring a more general framework.

Appendix A

Creation and destruction operators, average

We follow the argument used in [1]. The aim is to calculate:

$$\mathrm{tr}\left(\widehat{\rho}\, a^\dagger_{\mathbf{n}} a_{\mathbf{n}'}\right),$$

with $\widehat{\rho}$ given by eq. (2.27). For this purpose we define, with $\beta = 1/T$:

$$a^\dagger_{\mathbf{n}}(\beta) = \mathrm{e}^{-\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})} a^\dagger_{\mathbf{n}} \mathrm{e}^{\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})} \tag{A.1}$$

and similarly for $a_{\mathbf{n}}$, $b_{\mathbf{n}}$ and $b^\dagger_{\mathbf{n}}$. Now, from the above equation one can readily check that:

$$\frac{\partial a^\dagger_{\mathbf{n}}(\beta)}{\partial \beta} = [a^\dagger_{\mathbf{n}}(\beta), \widehat{H} - \omega\widehat{J} - \mu\widehat{Q}], \tag{A.2}$$

and, since:

$$[\widehat{H}, a^\dagger_{\mathbf{n}}] = \varepsilon a^\dagger_{\mathbf{n}} \qquad [\widehat{J}, a^\dagger_{\mathbf{n}}] = M a^\dagger_{\mathbf{n}} \qquad [\widehat{Q}, a^\dagger_{\mathbf{n}}] = q a^\dagger_{\mathbf{n}}$$

one readily obtains that eq. (A.2) is equivalent to:

$$\frac{\partial a^\dagger_{\mathbf{n}}(\beta)}{\partial \beta} = (-\varepsilon + M\omega + \mu q) a^\dagger_{\mathbf{n}}(\beta)$$

which is solved by, being $a^\dagger_{\mathbf{n}}(0) = a^\dagger_{\mathbf{n}}$:

$$a^{\dagger}_{\mathbf{n}}(\beta) = a^{\dagger}_{\mathbf{n}} \mathrm{e}^{-\beta(\varepsilon - M\omega - \mu q)} \tag{A.3}$$

We can now write:

$$\begin{aligned}
\mathrm{tr}\left(\widehat{\rho}\, a^{\dagger}_{\mathbf{n}} a_{\mathbf{n}'}\right) &= \mathrm{tr}\left(\widehat{\rho}\, a^{\dagger}_{\mathbf{n}} \mathrm{e}^{\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})} \mathrm{e}^{-\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})} a_{\mathbf{n}'}\right) = \\
&= \mathrm{tr}\left(\mathrm{e}^{-\beta(\widehat{H}+\omega\widehat{J}-\mu\widehat{Q})} a_{\mathbf{n}'}\, \widehat{\rho}\, a^{\dagger}_{\mathbf{n}} \mathrm{e}^{\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})}\right) = \\
&= \frac{1}{Z_\omega} \mathrm{tr}\left(\mathrm{e}^{-\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})} a_{\mathbf{n}'}\, \mathsf{P}_V\, \mathrm{e}^{-\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})} a^{\dagger}_{\mathbf{n}} \mathrm{e}^{\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})}\right) = \\
&= \frac{1}{Z_\omega} \mathrm{tr}\left(\mathsf{P}_V\, \mathrm{e}^{-\beta(\widehat{H}-\omega\widehat{J}-\mu\widehat{Q})} a_{\mathbf{n}'}\, a^{\dagger}_{\mathbf{n}}(\beta)\right) = \mathrm{tr}\left(\widehat{\rho}\, a_{\mathbf{n}'} a^{\dagger}_{\mathbf{n}}(\beta)\right),
\end{aligned}$$

where we have used the ciclicity of the trace, the definition of $\widehat{\rho}$ in eq. (2.27), the commutation relations with the volume projector P_V, and the eq (A.1). It should be pointed out that the ciclicity of the trace can be used safely because a complete set of states for the cylinder with finite radius can be constructed with eigenvectors of the operators $\widehat{H}$,$\widehat{J}_z$ and $\widehat{Q}$. By using eq. (A.3) and the anticommutation relation (3.22), the above equation can also be written as:

$$\begin{aligned}
\mathrm{tr}\left(\widehat{\rho}\, a^{\dagger}_{\mathbf{n}} a_{\mathbf{n}'}\right) &= \mathrm{tr}\left(\widehat{\rho}\, a_{\mathbf{n}'} a^{\dagger}_{\mathbf{n}}(\beta)\right) = \mathrm{tr}\left(\widehat{\rho}\, a_{\mathbf{n}'} a^{\dagger}_{\mathbf{n}}\right) \mathrm{e}^{-\beta(\varepsilon - M\omega - \mu q)} = \\
&= \left(-\mathrm{tr}\left(\widehat{\rho}\, a^{\dagger}_{\mathbf{n}} a_{\mathbf{n}'}\right) + \delta_{\mathbf{n}\mathbf{n}'}\right) \mathrm{e}^{-\beta(\varepsilon - M\omega - \mu q)},
\end{aligned}$$

whence:

$$\mathrm{tr}\left(\widehat{\rho}\, a^{\dagger}_{\mathbf{n}} a_{\mathbf{n}'}\right) = \frac{\delta_{\mathbf{n}\mathbf{n}'}}{\mathrm{e}^{\beta(\varepsilon - M\omega - \mu q)} + 1}.$$

The above method can be used for the calculation of other bilinear combinations of creation and destruction operators, leading to the equalities reported in eq. (3.28).

Appendix B

General form of the entropy current

Here we will present the formula for entropy current we use to introduce transport coefficients for a generic stress-energy tensor. We will follow mainly [23], while we disregard charge currents for mathematical ease.

The global equilibrium density matrix reads, if we consider rotations:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\frac{\widehat{H}}{T} + \frac{\boldsymbol{\omega}\cdot\widehat{J}}{T}\right] \qquad\qquad Z = \mathrm{tr}\left(\exp\left[-\frac{\widehat{H}}{T} + \frac{\boldsymbol{\omega}\cdot\widehat{J}}{T}\right]\right).$$

Without loss of generality, we can call $\hat{\omega}$ the z (or 3) axis and write the exponent using the stress-energy-momentum and spin tensors, understanding the position dependence of operators:

$$-\frac{\widehat{H}}{T}+\frac{\boldsymbol{\omega}\cdot\widehat{J}}{T} = -\frac{1}{T}\int \mathrm{d}^3\mathbf{x}\widehat{T}^{00}+\frac{1}{T}\int \mathrm{d}^3\mathbf{x}\left[\frac{1}{2}\omega\left(\delta^1_\mu\delta^2_\nu - \delta^1_\nu\delta^2_\mu\right)\left(x^\mu\widehat{T}^{0\nu} - x^\nu\widehat{T}^{0\mu} + \widehat{\mathcal{S}}^{0,\mu\nu}\right)\right] =$$

$$= -\int \mathrm{d}^3\mathbf{x}\left(\widehat{T}^{0\nu}\beta_\nu(x) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\alpha\beta}\omega_{\alpha\beta}(x)\right),$$

where $\beta = 1/T(1, \boldsymbol{\omega}\times\mathbf{x})$ is the four temperature of the rigidly rotating system, and $\omega_{\mu\nu} = \omega/T\left(\delta^1_\mu\delta^2_\nu - \delta^1_\nu\delta^2_\mu\right)$. Calling $\mathrm{d}\Sigma^\mu$ the space-like hyper-surface element of the three dimensional space (embedded in Minkowski space-time), the last equation reads:

$$-\frac{\widehat{H}}{T} + \frac{\boldsymbol{\omega} \cdot \widehat{J}}{T} = -\int \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \frac{1}{2} \omega_{\alpha\beta} \widehat{\mathcal{S}}^{\mu,\alpha\beta} \right).$$

Total entropy S thus reads:

$$S = -\mathrm{tr}\,(\widehat{\rho} \ln \widehat{\rho}) = \ln Z + \int \mathrm{d}\Sigma_\mu \left[T^{\mu\nu}(x) \beta_\nu(x) - \frac{1}{2} \mathcal{S}^{\mu,\alpha\beta}(x) \omega_{\alpha\beta}(x) \right].$$

For an entropy current to exist in relativistic thermodynamics the logarithm of the partition function must be written as an integral over the same hypersurface of a vector eld, hereby defined as the thermodynamic potential current:

$$\ln Z = \int \mathrm{d}\Sigma_\mu \, \phi^\mu,$$

so that the entropy current reads:

$$s^\mu(x) = \phi^\mu(x) + T^{\mu\nu}(x) \beta_\nu(x) - \frac{1}{2} \mathcal{S}^{\mu,\alpha\beta}(x) \omega_{\alpha\beta}(x). \tag{B.1}$$

The last formula holds for rotating equilibrium, it naturally holds for vanishing angular velocity $\boldsymbol{\omega}$ and, as reported in [23], can be extended to a more general class of thermodynamical equilibrium. The equilibrium relation between $\omega_{\mu\nu}$ and four-temperature is still $\omega_{\mu\nu} = -1/2 \partial_{[\mu} \beta_{\nu]}$. The interpretation of the thermodynamic potential current ϕ^μ is not yet clear in the most general case but corresponds to pressure times four temeprature $\phi^\mu = p\beta^\mu$ for homogeneous (non-rotating) equilibrium.

In relativistic Navier-Stokes theory it is assumed that equilibrium thermodynamic relations even hold if the system is out of equilibrium. So we will use (B.1) as a general form of entropy current, though the relation between $\omega_{\mu\nu}$ and four-temperature out of equilibrium is unclear.

Appendix C

Linear response theory with spin tensor

We extend the relativistic linear response theory in the Zubarev's approach to the case of a non-vanishing spin tensor. The (stationary) nonequilibrium density operator is the one in Eq. (6.12), with $\widehat{\Upsilon}$ expanded as in Eq. (6.13). As has been shown in Sect. 6.1, at equilibrium only the first term of the $\widehat{\Upsilon}$ operator survives in Eq. (6.13); therefore, one can rewrite that equation using the perturbations $\delta\beta$, $\delta\xi$ and $\delta\omega$ which are defined as the difference between the actual value and their value at thermodynamical equilibrium:

$$
\begin{aligned}
\widehat{\Upsilon} \;&= \int \mathrm{d}^3\mathbf{x} \, \left[\widehat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \widehat{j}^0\xi(t',\mathbf{x}) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x}) \right] \\
&+ \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}S \, n_i \left[\widehat{T}^{i\nu}\delta\beta_\nu(x) - \widehat{j}^i\delta\xi(x) - \frac{1}{2}\widehat{\mathcal{S}}^{i,\mu\nu}\delta\omega_{\mu\nu}(x) \right] \\
&- \frac{1}{2}\lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t \int \mathrm{d}^3\mathbf{x} \; \mathrm{e}^{\varepsilon(t-t')} \Big\{ \widehat{T}_{\mathrm{s}}^{\mu\nu} \left[\partial_\mu\delta\beta_\nu(x) + \partial_\mu\delta\beta_\nu(x) \right] + \\
&\qquad + \widehat{T}_{\mathrm{a}}^{\mu\nu} \left[\partial_\mu\delta\beta_\nu(x) - \partial_\mu\delta\beta_\nu(x) + 2\delta\omega_{\mu\nu}(x) \right] + \\
&\qquad - \widehat{\mathcal{S}}^{\lambda,\mu\nu}\partial_\lambda\delta\omega_{\mu\nu}(x) - 2\widehat{j}^\mu\partial_\mu\delta\xi(x) \Big\} ,
\end{aligned} \tag{C.1}
$$

where it is henceforth understood that $x = (t, \mathbf{x})$.
In fact, we will use a rearrangement of the right-hand-side expression which is more

convenient if one wants to work with an unspecified, yet small, $\delta\omega$. Therefore, the above equation is rewritten as:

$$\begin{aligned}\widehat{\Upsilon} &= \int \mathrm{d}^3\mathbf{x} \left(\widehat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \widehat{j}^0\xi(t',\mathbf{x}) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x})\right) \\ &- \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\frac{\partial}{\partial t}\int \mathrm{d}^3\mathbf{x} \left(\widehat{T}^{0\nu}\delta\beta_\nu(x) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x) - \widehat{j}^0\delta\xi(x)\right), \qquad \text{(C.2)}\end{aligned}$$

what it can be easily obtained from Eq. (6.12) integrating by parts in time.

For the sake of simplicity we calculate the linear response with $\xi_{\rm eq} = \delta\xi = 0$, but it can be shown that our final expressions hold for $\xi_{\rm eq} \neq 0$ (in other words with a non-vanishing chemical potential $\mu \neq 0$). Let us now define:

$$\widehat{A} = -\int \mathrm{d}^3\mathbf{x} \left(\widehat{T}^{0\nu}\beta_\nu(t',\mathbf{x}) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\omega_{\mu\nu}(t',\mathbf{x})\right),$$

and:

$$\widehat{B} = \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t\ \mathrm{e}^{\varepsilon(t-t')}\frac{\partial}{\partial t}\int \mathrm{d}^3\mathbf{x} \left(\widehat{T}^{0\nu}\delta\beta_\nu(x) - \frac{1}{2}\widehat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x)\right),$$

so that:

$$\widehat{\rho} = \frac{1}{Z}\exp[-\widehat{\Upsilon}] = \frac{1}{Z}\exp[\widehat{A}+\widehat{B}], \qquad \text{(C.3)}$$

with $Z = \mathrm{tr}\exp[\widehat{A}+\widehat{B}]$.

The operator $\widehat{B}$ is the small term in which $\widehat{\rho}$ will be expanded, according to the linear response theory. It can can be rewritten in a way which will be useful later on. Since:

$$\begin{aligned}\int \mathrm{d}^3\mathbf{x}\,\frac{\partial}{\partial t}\widehat{T}^{0\nu}(x)\,\delta\beta_\nu(x) &= \int \mathrm{d}^3\mathbf{x}\,\partial_\mu\left(\widehat{T}^{\mu\nu}(x)\,\delta\beta_\nu(x)\right) - \int \mathrm{d}^3\mathbf{x}\,\partial_i\widehat{T}^{i\nu}(x)\,\delta\beta_\nu(x) = \\ &= \int \mathrm{d}^3\mathbf{x}\,\widehat{T}^{\mu\nu}(x)\partial_\mu\delta\beta_\nu(x) - \int_{\partial V}\mathrm{d}S\ \hat{n}_i\widehat{T}^{i\nu}(x)\,\delta\beta_\nu(x),\end{aligned}$$

then:

$$\widehat{B} = \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x} \, \left(\widehat{T}^{\mu\nu}\partial_\mu\delta\beta_\nu(x) - \frac{1}{2}\frac{\partial}{\partial t}\left(\widehat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x)\right)\right) - \int_{\partial V} \mathrm{d}S \; \hat{n}_i \widehat{T}^{i\nu}(x)\,\delta\beta_\nu(x).$$

The perturbation $\delta\beta$ must be chosen such that $\delta\beta|_{\partial V} = 0$ so that only the bulk term survives in the above equation:

$$\widehat{B} = \lim_{\varepsilon\to 0}\int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x} \, \left(\widehat{T}^{\mu\nu}\partial_\mu\delta\beta_\nu(x) - \frac{1}{2}\frac{\partial}{\partial t}\left(\widehat{\mathcal{S}}^{0,\mu\nu}\delta\omega_{\mu\nu}(x)\right)\right). \tag{C.4}$$

At the lowest order in $\widehat{B}$:

$$Z = \mathrm{tr}\left(\mathrm{e}^{\widehat{A}+\widehat{B}}\right) \simeq \mathrm{tr}\left(\mathrm{e}^{\widehat{A}}\left[1+\widehat{B}\right]\right) = Z_{\mathrm{LE}}\left(1+\langle\widehat{B}\rangle_{\mathrm{LE}}\right) \Rightarrow \frac{1}{Z} \simeq \frac{1}{Z_{\mathrm{LE}}}\left(1-\langle\widehat{B}\rangle_{\mathrm{LE}}\right), \tag{C.5}$$

and:

$$\mathrm{e}^{\widehat{A}+\widehat{B}} = \left[1 + \int_0^1 \mathrm{d}z \; \mathrm{e}^{z\left(\widehat{A}+\widehat{B}\right)}\widehat{B}\mathrm{e}^{-z\widehat{A}}\right]\mathrm{e}^{\widehat{A}} \simeq \left[1 + \int_0^1 \mathrm{d}z \; \mathrm{e}^{z\widehat{A}}\widehat{B}\,\mathrm{e}^{-z\widehat{A}}\right]\mathrm{e}^{\widehat{A}}, \tag{C.6}$$

where the subscript LE stands for Local Equilibrium and implies the calculation of mean values with the local equilibrium density operator, that is $\widehat{\rho}_{\mathrm{LE}} = \mathrm{e}^{\widehat{A}}/\mathrm{tr}(\mathrm{e}^{\widehat{A}})$. Thereby, putting together (C.5) and (C.6) and retaining only first-order terms in $\widehat{B}$:

$$\widehat{\rho} \simeq \left(1 - \langle\widehat{B}\rangle_{\mathrm{LE}}\right)\widehat{\rho}_{\mathrm{LE}} + \int_0^1 \mathrm{d}z \; \mathrm{e}^{z\widehat{A}}\widehat{B}\mathrm{e}^{-z\widehat{A}}\widehat{\rho}_{\mathrm{LE}},$$

hence the mean value of an operator $\widehat{\mathcal{O}}(y)$ becomes:

$$\langle\widehat{\mathcal{O}}(y)\rangle \simeq \left(1 - \langle\widehat{B}\rangle_{\mathrm{LE}}\right)\langle\widehat{\mathcal{O}}(y)\rangle_{\mathrm{LE}} + \langle\widehat{\mathcal{O}}(y)\int_0^1 \mathrm{d}z \; \mathrm{e}^{z\widehat{A}}\widehat{B}\mathrm{e}^{-z\widehat{A}}\rangle_{\mathrm{LE}}. \tag{C.7}$$

Let us focus on the last term, which, by virtue of (C.4), contains expressions of this sort:

$$\langle \widehat{\mathcal{O}}(x)\widehat{X}'(z,t,\mathbf{x})\rangle_{\rm LE} \equiv \langle \widehat{\mathcal{O}}(x){\rm e}^{z\widehat{A}}\widehat{X}(t,\mathbf{x}){\rm e}^{-z\widehat{A}}\rangle_{\rm LE},$$

where $\widehat{X}$ is a general notation for components of either $\widehat{T}$ or $\widehat{\mathcal{S}}$ or $\partial_0\widehat{\mathcal{S}}$. From the identity:

$$\langle \widehat{\mathcal{O}}(y)\widehat{X}'(z,t,\mathbf{x})\rangle_{\rm LE} = \int_{-\infty}^{t} {\rm d}\tau\ \langle \widehat{\mathcal{O}}(y)\partial_\tau\widehat{X}'(z,\tau,\mathbf{x})\rangle_{\rm LE} + \lim_{\tau\to-\infty}\langle \widehat{\mathcal{O}}(y)\widehat{X}'(z,\tau,\mathbf{x})\rangle_{\rm LE},$$

and the observation that correlations vanish for very distant times, one obtains:

$$\langle \widehat{\mathcal{O}}(y)\widehat{X}'(z,t,\mathbf{x})\rangle_{\rm LE} = \int_{-\infty}^{t} {\rm d}\tau\langle \widehat{\mathcal{O}}(y)\partial_\tau\widehat{X}'(z,\tau,\mathbf{x})\rangle_{\rm LE} + \lim_{\tau\to-\infty}\langle \widehat{\mathcal{O}}(y)\rangle_{\rm LE}\langle \widehat{X}(\tau,\mathbf{x})\rangle_{\rm LE}, \quad \text{(C.8)}$$

where we have also taken advantage of the commutation between $\exp[\widehat{A}]$ and $\exp[\pm z\widehat{A}]$.

We now approximate (see [33]) the local equilibrium density operator with the nearest equilibrium operator $\widehat{\rho}_0$ in Eq. (6.26), which also implies that:

$$\widehat{A} \simeq -\widehat{H}/T,$$

where $\widehat{H}$ is the hamiltonian operator (which ought to exists given the chosen boundary conditions). The straightforward consequence of this approximation is that the second term on the right hand side in Eq. (C.8) can be written as:

$$\langle \widehat{X}(-\infty,\mathbf{x})\rangle_{\rm LE} \simeq \langle \widehat{X}(-\infty,\mathbf{x})\rangle_0 = \langle \widehat{X}(t,\mathbf{x})\rangle_0$$

because the mean value is stationary under the equilibrium distribution. Therefore, the Eq. (C.8) can be approximated as:

$$\langle \widehat{\mathcal{O}}(y)\widehat{X}'(z,t,\mathbf{x})\rangle_{\rm LE} \simeq \int_{-\infty}^{t} {\rm d}\tau\ \langle \widehat{\mathcal{O}}(y)\partial_\tau\widehat{X}'(z,\tau,\mathbf{x})\rangle_0 + \langle \widehat{\mathcal{O}}(y)\rangle_0\langle \widehat{X}(t,\mathbf{x})\rangle_0, \quad \text{(C.9)}$$

and the (C.7) as:

$$\langle\widehat{\mathcal{O}}(y)\rangle \simeq \left(1-\langle\widehat{B}\rangle_0\right)\langle\widehat{\mathcal{O}}(y)\rangle_0 + \int_0^1 \mathrm{d}z\ \langle\widehat{\mathcal{O}}(y)\mathrm{e}^{-z\widehat{H}/T}\widehat{B}\mathrm{e}^{z\widehat{H}/T}\rangle_0. \tag{C.10}$$

Once integrated, the second term in (C.9) gives rise to a term which cancels out exactly the $\langle\widehat{B}\rangle_0\langle\widehat{\mathcal{O}}(y)\rangle_0$ in the equation above, which then becomes:

$$\langle\widehat{\mathcal{O}}(y)\rangle \simeq \langle\widehat{\mathcal{O}}(y)\rangle_0 + \langle\widehat{\mathcal{O}}(y)\widehat{B}''\rangle_0, \tag{C.11}$$

where $\widehat{B}''$ is an operator built flollowing (C.4), but substituting in any instance the terms $\widehat{X}'(z,t,\mathbf{x})$ with the integrals $\int_{-\infty}^t \mathrm{d}\tau\ \partial_\tau\widehat{X}'(z,\tau,\mathbf{x})$.

Let us now take care of the first term in Eq. (C.9) and integrate it in z:

$$\int_0^1 \mathrm{d}z \int_{-\infty}^t \mathrm{d}\tau\ \langle\widehat{\mathcal{O}}(y)\partial_\tau\widehat{X}'(z,\tau,\mathbf{x})\rangle_0 = \frac{1}{\overline{\beta}}\int_0^{\overline{\beta}} \mathrm{d}u \int_{-\infty}^t \mathrm{d}\tau\ \langle\widehat{\mathcal{O}}(y)\partial_\tau\mathrm{e}^{-u\widehat{H}}\widehat{X}(\tau,\mathbf{x})\mathrm{e}^{u\widehat{H}}\rangle_0,$$

where $\overline{\beta}=1/T$ and $\overline{\beta}z=u$. As $\widehat{H}$ is the generator of time translations:

$$\begin{aligned}
&\frac{1}{\overline{\beta}}\int_0^{\overline{\beta}} \mathrm{d}u \int_{-\infty}^t \mathrm{d}\tau\ \langle\widehat{\mathcal{O}}(y)\partial_\tau\mathrm{e}^{-u\widehat{H}}\widehat{X}(\tau,\mathbf{x})\mathrm{e}^{u\widehat{H}}\rangle_0 = \frac{1}{\overline{\beta}}\int_0^{\overline{\beta}} \mathrm{d}u \int_{-\infty}^t \mathrm{d}\tau\ \langle\widehat{\mathcal{O}}(y)\partial_\tau\widehat{X}(\tau+iu,\mathbf{x})\rangle_0 = \\
&= \frac{1}{i\overline{\beta}}\int_0^{\overline{\beta}} \mathrm{d}u \int_{-\infty}^t \mathrm{d}\tau\ \langle\widehat{\mathcal{O}}(y)\frac{\partial}{\partial u}\widehat{X}(\tau+iu,\mathbf{x})\rangle_0 = \frac{1}{i\overline{\beta}}\int_0^{\overline{\beta}} \mathrm{d}u \int_{-\infty}^t \mathrm{d}\tau\ \frac{\partial}{\partial u}\left(\langle\widehat{\mathcal{O}}(y)\widehat{X}(\tau+iu,\mathbf{x})\rangle_0\right) = \\
&= \frac{1}{i\overline{\beta}}\int_{-\infty}^t \mathrm{d}\tau \int_0^{\overline{\beta}} \mathrm{d}u\ \frac{\partial}{\partial u}\left(\langle\widehat{\mathcal{O}}(y)\widehat{X}(\tau+iu,\mathbf{x})\rangle_0\right) = \frac{1}{i\overline{\beta}}\int_{-\infty}^t \left(\langle\widehat{\mathcal{O}}(y)\widehat{X}(\tau+i\overline{\beta},\mathbf{x})\rangle_0 - \langle\widehat{\mathcal{O}}(y)\widehat{X}(\tau,\mathbf{x})\rangle_0\right).
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\langle\widehat{\mathcal{O}}(y)\widehat{X}(\tau+i\overline{\beta},\mathbf{x})\rangle_0 = \mathrm{tr}(\widehat{\rho}_0\widehat{\mathcal{O}}(y)\mathrm{e}^{-\overline{\beta}\widehat{H}}\widehat{X}(\tau,\mathbf{x})\mathrm{e}^{+\overline{\beta}\widehat{H}}) = \frac{1}{Z_0}\mathrm{tr}(\mathrm{e}^{-\overline{\beta}\widehat{H}}\widehat{\mathcal{O}}(y)\mathrm{e}^{-\overline{\beta}\widehat{H}}\widehat{X}(\tau,\mathbf{x})\mathrm{e}^{\overline{\beta}\widehat{H}}) = \\
&= \frac{1}{Z_0}\mathrm{tr}(\widehat{\mathcal{O}}(y)\mathrm{e}^{-\overline{\beta}\widehat{H}}\widehat{X}(\tau,\mathbf{x})) = \mathrm{tr}(\widehat{X}(\tau,\mathbf{x}))\widehat{\rho}_0\widehat{\mathcal{O}}(y)) = \langle\widehat{X}(\tau,\mathbf{x})\widehat{\mathcal{O}}(y)\rangle_0.
\end{aligned}$$

Hence, putting the last three equations together, we have:

$$\int_0^1 \mathrm{d}z \int_{-\infty}^t \mathrm{d}\tau \; \langle \widehat{\mathcal{O}}(y)\partial_\tau \widehat{X}'(z,\tau,\mathbf{x})\rangle_0 = \frac{1}{i\overline{\beta}} \int_{-\infty}^t \mathrm{d}\tau \; \langle [\widehat{X}(\tau,\mathbf{x}), \widehat{\mathcal{O}}(y)]\rangle_0. \tag{C.12}$$

Substituting now $\widehat{X}$ with its specific operators, Eq. (C.11) can be expanded as:

$$\begin{aligned}
\delta\langle\widehat{\mathcal{O}}(y)\rangle = \langle\widehat{\mathcal{O}}(y)\rangle - \langle\widehat{\mathcal{O}}(y)\rangle_0 \simeq \lim_{\varepsilon\to 0} \frac{1}{i\overline{\beta}} \int_{-\infty}^{t'}\!\mathrm{d}t \, \mathrm{e}^{\varepsilon(t-t')}\!\int_{-\infty}^{t}\!\mathrm{d}\tau \int\!\mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{T}^{\mu\nu}(\tau,\mathbf{x}), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \partial_\mu \delta\beta_\nu(x) \, + \\
-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}} \int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \frac{\partial}{\partial t} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{\mathcal{S}}^{0,\mu\nu}(\tau,\mathbf{x}), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \delta\omega_{\mu\nu}(x) = \\
= \lim_{\varepsilon\to 0} \frac{1}{i\overline{\beta}} \int_{-\infty}^{t'}\!\mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')}\!\int_{-\infty}^{t}\!\mathrm{d}\tau \int\!\mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{T}^{\mu\nu}(\tau,\mathbf{x}), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \partial_\mu \delta\beta_\nu(x) \, + \\
-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}} \int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int \mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{\mathcal{S}}^{0,\mu\nu}(t,\mathbf{x}), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \delta\omega_{\mu\nu}(x) \, + \\
-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}} \int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{\mathcal{S}}^{0,\mu\nu}(\tau,\mathbf{x}), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \frac{\partial}{\partial t}\delta\omega_{\mu\nu}(x).
\end{aligned} \tag{C.13}$$

The first term on the right hand side of the above equation can be integrated by parts using:

$$\begin{aligned}
\int_{-\infty}^{t'} \mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int_{-\infty}^{t} \mathrm{d}\tau \; f(\tau) = \int_{-\infty}^{t'} \mathrm{d}t \; \frac{\partial}{\partial t}\left(\frac{\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\right) \int_{-\infty}^{t} \mathrm{d}\tau \; f(\tau) = \\
= \frac{1}{\varepsilon}\int_{-\infty}^{t'} \mathrm{d}\tau \; f(\tau) - \int_{-\infty}^{t'} \mathrm{d}t \; \frac{\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon} f(t) = \int_{-\infty}^{t'} \mathrm{d}t \frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon} f(t),
\end{aligned}$$

so that the Eq. (C.13) can be finally written:

$$\begin{aligned}
\delta\langle\widehat{\mathcal{O}}(y)\rangle = \lim_{\varepsilon\to 0} \frac{1}{i\overline{\beta}} \int_{-\infty}^{t'}\!\mathrm{d}t \; \frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon} \int\!\mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{T}^{\mu\nu}(x), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \partial_\mu \delta\beta_\nu(x) \, + \\
-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}} \int_{-\infty}^{t'}\!\mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int\!\mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{\mathcal{S}}^{0,\mu\nu}(x), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \delta\omega_{\mu\nu}(x) \, + \\
-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}} \int_{-\infty}^{t'}\!\mathrm{d}t \; \mathrm{e}^{\varepsilon(t-t')} \int_{-\infty}^{t}\!\mathrm{d}\tau \int\!\mathrm{d}^3\mathbf{x} \; \langle \Big[\widehat{\mathcal{S}}^{0,\mu\nu}(\tau,\mathbf{x}), \widehat{\mathcal{O}}(y)\Big]\rangle_0 \frac{\partial}{\partial t}\delta\omega_{\mu\nu}(x).
\end{aligned} \tag{C.14}$$

There is another useful expression of the $\delta\langle\widehat{\mathcal{O}}(y)\rangle$ which can be obtained starting from the expression (6.13) of $\widehat{\Upsilon}$, where the continuity equation for angular momentum was implemented from the beginning. Repeating the same reasoning as above, it can be shown that one gets to:

$$\begin{aligned}
\delta\langle\widehat{\mathcal{O}}(y)\rangle = &\lim_{\varepsilon\to 0}\ \frac{1}{2i\overline{\beta}}\int_{-\infty}^{t'}\!\mathrm{d}t\,\frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\int\!\mathrm{d}^3\mathbf{x}\langle\Big[\widehat{T}_S^{\mu\nu}(x),\widehat{\mathcal{O}}(y)\Big]\rangle_0(\partial_\mu\delta\beta_\nu(x)+\partial_\nu\delta\beta_\mu(x))\,+\\
&+\lim_{\varepsilon\to 0}\frac{1}{2i\overline{\beta}}\int_{-\infty}^{t'}\!\mathrm{d}t\,\frac{1-\mathrm{e}^{\varepsilon(t-t')}}{\varepsilon}\int\!\mathrm{d}^3\mathbf{x}\langle\Big[\widehat{T}_A^{\mu\nu}(x),\widehat{\mathcal{O}}(y)\Big]\rangle_0(\partial_\mu\delta\beta_\nu(x)-\partial_\nu\delta\beta_\mu(x)+2\delta\omega_{\mu\nu}(x))\,+\\
&-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}}\int_{-\infty}^{t'}\!\mathrm{d}t\,\mathrm{e}^{\varepsilon(t-t')}\int_{-\infty}^{t}\!\mathrm{d}\tau\int\!\mathrm{d}^3\mathbf{x}\langle\Big[\widehat{\mathcal{S}}^{\lambda,\mu\nu}(\tau,\mathbf{x}),\widehat{\mathcal{O}}(y)\Big]\rangle_0\partial_\lambda\delta\omega_{\mu\nu}(x)\,+\\
&-\frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{i\overline{\beta}}\int_{-\infty}^{t'}\!\mathrm{d}t\,\mathrm{e}^{\varepsilon(t-t')}\int_{-\infty}^{t}\!\mathrm{d}\tau\int\!\mathrm{d}^3\mathbf{x}\,\langle\Big[\widehat{\mathcal{S}}^{0,\mu\nu}(\tau,\mathbf{x}),\widehat{\mathcal{O}}(y)\Big]\rangle_0\frac{\partial}{\partial t}\delta\omega_{\mu\nu}(x)
\end{aligned}\tag{C.15}$$

As we have pointed out, these expressions hold when $\widehat{\rho}_0$ has a non-vanishing chemical potential.

Appendix D

Discrete symmetries

During the calculations for the linear response we found the average values of terms of the form:

$$\langle \left[\widehat{\mathcal{O}}_1^{\mu_1 \cdots \mu_n}(\tau, \mathbf{x}), \widehat{\mathcal{O}}_2^{\nu_1 \cdots \nu_n}(0, \mathbf{0}) \right] \rangle_0$$

where $\widehat{\mathcal{O}}_2$ and $\widehat{\mathcal{O}}_2$ are tensors of rank respectively m and n and hermitian operators, e.g. the commutator $\langle \left[\widehat{T}_S^{12}(\tau, \mathbf{x}), \widehat{\Xi}^{\alpha 12}(0, \mathbf{0}) \right] \rangle_0$ in Eq. (6.45)*(46)*.

As long as we take as the equilibrium distribution the non rotating density matrix $\rho_0 = \exp[-\widehat{H}/T]/Z$, the system enjoys translation, rotation, time reversal and parity invariance, provided the the hamiltonian $\widehat{H}$ is itself parity an time reversal invariant. The symmetry under this class of transformations allows to write the mean value in more convenient forms. For any linear unitary transformation $\widehat{\mathsf{T}}$ which doesn't change the state of the system he have:

$$\langle \widehat{\mathcal{O}} \rangle_0 = \mathrm{tr}\left(\widehat{\rho}_0 \, \widehat{\mathcal{O}} \right) = \mathrm{tr}\left(\widehat{\mathsf{T}}^{-1} \widehat{\rho}_0 \widehat{\mathsf{T}} \, \widehat{\mathcal{O}} \right) = \mathrm{tr}\left(\widehat{\rho}_0 \, \widehat{\mathsf{T}} \widehat{\mathcal{O}} \widehat{\mathsf{T}}^{-1} \right) = \langle \widehat{\mathsf{T}} \widehat{\mathcal{O}} \widehat{\mathsf{T}}^{-1} \rangle_0$$

the average value of the transformed operator. Henceforth translation invariance imply in our case that we can translate both operators of the same distance without changing the mean value:

$$\langle \left[\widehat{\mathcal{O}}_1^{\mu_1 \cdots \mu_n}(\tau, \mathbf{x}), \widehat{\mathcal{O}}_2^{\nu_1 \cdots \nu_n}(0, \mathbf{0}) \right] \rangle_0 = \langle \left[\widehat{\mathcal{O}}_1^{\mu_1 \cdots \mu_n}(\tau + a_0, \mathbf{x} + \mathbf{a}), \widehat{\mathcal{O}}_2^{\nu_1 \cdots \nu_n}(a_0, \mathbf{a}) \right] \rangle_0$$

and so, for $(a_0, \mathbf{a}) = (-\tau, -\mathbf{x})$ it reads:

$$\langle\left[\widehat{\mathcal{O}}_1^{\mu_1\cdots\mu_n}(\tau,\mathbf{x}), \widehat{\mathcal{O}}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})\right]\rangle_0 = \langle\left[\widehat{\mathcal{O}}_1^{\mu_1\cdots\mu_n}(0,\mathbf{0}), \widehat{\mathcal{O}}_2^{\nu_1\cdots\nu_n}(-\tau,-\mathbf{x})\right]\rangle_0$$

On a similar manner parity invariance of the state means that performing a parity transformation on the argument doesn't change the mean values, hence:

$$\langle\left[\widehat{\mathcal{O}}_1^{\mu_1\cdots\mu_n}(\tau,\mathbf{x}), \widehat{\mathcal{O}}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})\right]\rangle_0 = (-1)^{n_s}\langle\left[\widehat{\mathcal{O}}_1^{\mu_1\cdots\mu_n}(\tau,-\mathbf{x}), \widehat{\mathcal{O}}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})\right]\rangle_0$$

The overall sign in the above formula depends on the number of total spatial indices n_s among $\mu_1, \cdots \nu_m, \nu_1, \cdots \nu_n$ since parity, besides changing the sign of the spatial variables $\mathbf{x}$, change the sing of tensors with an odd number of spatial indices.

Time reversal works in a similar way, but an important difference is that time reversal Θ is antiunitary and antilinear. An operator $\widehat{A}$ transforms thus using:

$$\widehat{A} \to \Theta \widehat{A}^\dagger \Theta^{-1}$$

instead of the more familiar rule for linear transformations. It follows that operators in a product invert their relative order, hence for commutators:

$$\left[\widehat{A}, \widehat{B}\right] \to \Theta\left[\widehat{A}, \widehat{B}\right]^\dagger \Theta^{-1} = \Theta\left[\widehat{B}^\dagger, \widehat{A}^\dagger\right]\Theta^{-1}$$

Hermitian operators are self adjoint $\widehat{\mathsf{H}}^\dagger = \widehat{\mathsf{H}}$, but the hermitian conjugation still change the order of hermitian operators in a commutator. Time reversal invariance for the density matrix ensures that the mean value at the beginning reads:

$$\langle\left[\widehat{\mathcal{O}}_1^{\mu_1\cdots\mu_n}(\tau,\mathbf{x}), \widehat{\mathcal{O}}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0})\right]\rangle_0 = (-1)^{n_0}\langle\left[\widehat{\mathcal{O}}_2^{\nu_1\cdots\nu_n}(0,\mathbf{0}), \widehat{\mathcal{O}}_1^{\mu_1\cdots\mu_n}(-\tau,\mathbf{x})\right]\rangle_0$$

where n_0 is the number of temporal indices.

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