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2013 Inverse Problems 29 125012

(http://iopscience.iop.org/0266-5611/29/12/125012)

IOP PUBLISHING INVERSE PROBLEMS

Inverse Problems **29** (2013) 125012 (17pp) doi:10.1088/0266-5611/29/12/125012

Doubling inequalities for anisotropic plate equations and applications to size estimates of inclusions

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Received 23 July 2013, in final form 16 October 2013
Published 21 November 2013
Online at stacks.iop.org/IP/29/125012

Abstract

We prove the upper and lower estimates of the area of an unknown elastic inclusion in a thin plate by one boundary measurement. The plate is made of non-homogeneous linearly elastic material belonging to a general class of anisotropy and the domain of the inclusion is a measurable subset of the plate. The size estimates are expressed in terms of the work exerted by a couple field applied at the boundary and of the induced transversal displacement and its normal derivative taken at the boundary of the plate. The main new mathematical tool is a doubling inequality for solutions to fourth-order elliptic equations whose principal part $P(x, D)$ is the product of two second-order elliptic operators $P_1(x, D)$, $P_2(x, D)$ such that $P_1(0, D) = P_2(0, D)$. The proof of the doubling inequality is based on the Carleman method, a sharp three-spheres inequality and a bootstrapping argument.

1. Introduction

This paper deals with the inverse problem of detecting, inside a thin elastic plate, an unknown inclusion of a different elastic material, in terms of measurements taken at the boundary of the plate. Concerning the basic issue of uniqueness, let us recall that, even for the analogous inverse problem in conductivity, which involves a second-order elliptic equation instead of the fourth-order elliptic equation governing the static equilibrium of a plate, a general result under a finite number of boundary measurements has not yet been proved. In the electrostatic context, Isakov [Is] proved that the inclusion is uniquely determined when all possible measurements are at disposal, but, unfortunately, the inverse problem shows a weak rate of convergence (logarithmic) (see [A-DiC] and [DiC-Ron]), which represents a strong obstruction for reconstruction techniques. For these reasons, it is of interest for applications to find constructive stable estimates of some relevant geometrical parameters of the unknown inclusion, such as its measure.

Following a line of research on 'size estimates' initiated in [Ka-S-S, Al-Ro, Al-Ro-Se] in the electrostatic context and developed in [Ik] and [Al-Mo-Ro02] in linear elasticity (see also the review paper [Al-Mo-Ro03]), in this paper we derive constructive upper and lower bounds for the area of the inclusion in terms of the difference between the work exerted in deforming the defective plate and a reference plate (i.e., a plate without inclusion) by applying the same couple field at the boundary.

Size estimates of this type for general inclusions, that is, measurable subsets of the plate, have been proved in [Mo-Ro-Ve09] when the material of the reference plate is isotropic. Analogous results for the case of inclusions in shells have been recently obtained in [DiC-Li-Ve-Wa, DiC-Li-Wa]. The assumption of an isotropic material for the reference plate is rather restrictive, since in an increasing number of practical applications the use of materials with various degrees of anisotropy is required to achieve better structural performance. In [Mo-Ro-Ve13], the isotropy condition on the reference plate was removed and size estimates were obtained for a large class of anisotropic materials satisfying an algebraic condition including our condition (2.17). We refer to [Mo-Ro-Ve11, remark 3.3] for concrete examples of anisotropic materials satisfying condition (2.17) which are significant for engineering applications. Finally, for the sake of completeness, we mention an interesting approach to size estimates recently developed in [Ka-Ki-Mi, Ka-Mi] and in [Mi-Ng] where the translation method and the splitting method were introduced, respectively.

The bounds in [Mo-Ro-Ve13] were derived under the so-called *a priori fatness condition* on the unknown inclusion E, namely that, for a given $h_1 > 0$,

$$\text{area}\left(\{x \in E \,|\, \text{dist}\{x, \partial E\} > h_1\}\right) \geqslant \tfrac{1}{2}\text{area}(E).$$

In this paper (see theorem 2.2, section 2) we remove this geometrical assumption and we prove the size estimates for an inclusion which is a measurable subset of the plate, under condition (2.17) on the background material.

Under various perspectives, a single unifying theme has been used in order to deal with the class of inverse boundary value problems posed by the 'size estimates' approach, namely quantitative estimates of unique continuation. In this paper we study the so-called doubling inequality property of solutions. A connection between this property and the strong unique continuation principle for elliptic partial differential equations was originally investigated in [Ga-Li86, Ga-Li87]. Subsequently, the doubling inequality property has been widely used in inverse boundary value problems to obtain volume bounds of unknown cavities and inclusions.

More generally, we consider a class of fourth-order differential equations, which includes the plate equation under condition (2.17). Precisely, let u be a solution of the differential equation

$$P(x, D)u = Qu \qquad \text{in } B_1 := B_1(0), \tag{1.1}$$

where for $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x) = \{y \in \mathbb{R}^n \,:\, |x - y| < r\}$ and

$$P(x, D)u = (P_1(x, D)P_2(x, D))u, \tag{1.2}$$

with

$$P_k(x, D)u = g_k^{ij}(x)D_{ij}^2 u, \qquad k = 1, 2, \tag{1.3}$$

with g_k^{ij} being a tensor such that

$$g_1^{ij}(0) = g_2^{ij}(0), \tag{1.4}$$

$$\lambda|\xi|^2 \leqslant g_k^{ij}(x)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^2,\ \forall\, x \in \overline{B}_1 \tag{1.5}$$

and

$$\sum_{|\alpha|\leqslant 2}\sum_{i,j=1}^{2} \|D^\alpha g_k^{ij}\|_{L^\infty(B_1)} \leqslant C_0, \tag{1.6}$$

for a given positive constant $C_0 > 0$ and $\lambda \in (0, 1)$. Here Q is a third-order differential operator such that there exists a positive constant M such that

$$|Qu| \leqslant M \sum_{|\alpha|\leqslant 3} |D^\alpha u|, \qquad \forall\, u \in H^4(B_1). \tag{1.7}$$

We want to study the doubling property of solutions u of (1.1) which says that for any compact subset G of B_1 and any concentric balls $B_r, B_{2r} \subset G$, the following inequality holds

$$\int_{B_{2r}} u^2 \leqslant K \int_{B_r} u^2, \tag{1.8}$$

where K depends on G, the ellipticity, the regularity bounds and also, necessarily, on the solution u.

Under the above assumptions, we prove the following theorem.

Theorem 1.1. *Let $P(x, D)$ be defined as (1.2) satisfying (1.3)–(1.6). Let $u \in H^4(B_1)$ be such that*

$$|P(x, D)u| \leqslant M \sum_{|\alpha|\leqslant 3} |D^\alpha u|, \qquad \text{in } \overline{B}_1.$$

Then there exists a constant $R \in (0, 1)$, only depending on λ, M_1 and C_0, and there exist constants $\theta \in (0, 1/2)$ and $K > 0$, only depending on λ, M, C_0 and

$$\mathcal{F}_{\text{loc}} = \frac{\|u\|_{L^2(B_R)}}{\|u\|_{L^2(B_{\lambda^2 R^2})}}, \tag{1.9}$$

such that

$$\int_{B_{2r}} |u|^2 \leqslant K \int_{B_r} |u|^2, \quad \forall\, r,\ 0 < r < \theta. \tag{1.10}$$

In order to apply the doubling inequality (1.10) to our inverse problem, it is crucial to estimate the constant K in terms of the available boundary data instead of the interior values of the solution u, which may not be known.

The paper is organized as follows. In section 2 we derive doubling inequalities for a class of anisotropic plate equations and we apply them to the size estimate problem. In section 3 we provide a detailed proof of theorem 1.1.

2. Doubling inequalities and size estimates of inclusions in plates

Let us state some notation and definitions.

For $P = (x_1(P), x_2(P))$, a point in $\mathbb{R}^2$, we denote by $R_{a,b}(P)$ the rectangle of center P and sides parallel to the coordinate axes of length $2a$ and $2b$, namely $R_{a,b}(P) = \{x = (x_1, x_2) : |x_1 - x_1(P)| < a,\ |x_2 - x_2(P)| < b\}$. We set also $R_{a,b}(0) = R_{a,b}$.

Definition 2.1 ($C^{k,\alpha}$ regularity)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^2$*. Given* k, α *with* $k \in \mathbb{N}$, $0 < \alpha \leqslant 1$, *we say that a portion* S *of* $\partial\Omega$ *is of class* $C^{k,\alpha}$ *with constants* $\rho_0, M_0 > 0$ *if for any* $P \in S$ *there exists a rigid transformation of coordinates under which we have* $P = 0$ *and*

$$\Omega \cap R_{\frac{\rho_0}{M_0},\rho_0} = \left\{x = (x_1, x_2) \in R_{\frac{\rho_0}{M_0},\rho_0} : x_2 > \psi(x_1)\right\},$$

where ψ *is a* $C^{k,\alpha}$ *function on* $\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)$ *satisfying*

$$\psi(0) = 0, \qquad \psi^{(k)}(0) = 0, \ \text{when } k \geqslant 1,$$

$$\|\psi\|_{C^{k,\alpha}(-\rho_0/M_0,\rho_0/M_0)} \leqslant M_0\rho_0.$$

When $k = 0$, $\alpha = 1$ *we say that* S *is of the Lipschitz class with constants* ρ_0, M_0.

Hereafter, we shall consider a bounded domain Ω satisfying

$$|\Omega| \leqslant M_1\rho_0^2, \tag{2.11}$$

where $|\Omega|$ denotes the area of Ω.

Working in the framework of the Kirchhoff–Love theory, the transversal displacement u of the middle plane Ω of the plate $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ satisfies the following fourth-order equation

$$\operatorname{div}(\operatorname{div}(\mathbb{P}D^2u)) = 0, \quad \text{in } \Omega, \tag{2.12}$$

with

$$\mathbb{P} = \frac{h^3}{12}\mathbb{C}, \tag{2.13}$$

where h is the uniform thickness of the plate and $\mathbb{C} = \{C_{ijkl}\}$, $\mathbb{C} \in L^\infty(\Omega)$, is the elastic tensor of the material. On $\mathbb{C}$ we shall assume

$$C_{ijkl} = C_{klij} = C_{klji}, \qquad i, j, k, l = 1, 2, \quad \text{a.e. in } \Omega, \tag{2.14}$$

$$\gamma|A|^2 \leqslant \mathbb{C}A \cdot A \leqslant \gamma^{-1}|A|^2, \quad \text{a.e. in } \Omega, \tag{2.15}$$

$$\sum_{i,j,k,l=1}^{2} \sum_{|\alpha|\leqslant 2} \rho_0^{|\alpha|}\|D^\alpha C_{ijkl}\|_{L^\infty(\Omega)} \leqslant M_2, \tag{2.16}$$

where $\gamma \in (0, 1)$ and $M_2 > 0$ are given constants. Note that condition (2.14) implies that instead of 16 coefficients we actually deal with 6 coefficients. Denoting by $a_0 = C_{1111}$, $a_1 = 4C_{1112}$, $a_2 = 2C_{1122} + 4C_{1212}$, $a_3 = 4C_{2212}$, $a_4 = C_{2222}$ and by $S(x)$ the following 7×7 matrix

$$S(x) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \end{pmatrix},$$

we define

$$\mathcal{D}(x) = \frac{1}{a_0}|\det S(x)|$$

and we assume that

$$\mathcal{D}(x) = 0, \qquad \forall x \in \mathbb{R}^2. \tag{2.17}$$

Under the above conditions on the plate tensor $\mathbb{P}$, we can state the following doubling inequality.

Proposition 2.1 (Doubling inequality for the plate equation)**.** *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of the Lipschitz class with constants ρ_0, M_0, and let $u \in H^4_{\mathrm{loc}}(\Omega)$ be a non trivial solution of the equation (2.12). Then, there exists a constant $\theta \in (0, 1)$, depending on γ and M_2 only, such that for every $\overline{r} > 0$ and $x_0 \in \Omega_{\overline{r}\rho_0}$ we have*

$$\int_{B_{2r}(x_0)} u^2 \leqslant K \int_{B_r(x_0)} u^2, \qquad \forall\, r,\ 0 < r \leqslant \frac{\theta}{2}\overline{r}\rho_0, \tag{2.18}$$

where K only depends on γ, M_0, M_1, M_2, $\overline{r}$ and $\|u\|_{H^{1/2}(\Omega)}/\|u\|_{L^2(\Omega)}$.

Proof. Without loss of generality, we consider the case $\rho_0 = 1$. In [Mo-Ro-Ve11, section 6] it was shown that, under condition (2.17), there exists a matrix $\{g^{ij}((x)\}_{i,j=1}^2$ such that

$$\lambda|\xi|^2 \leqslant g^{ij}(x)\xi_i\xi_j \leqslant \lambda^{-1}|\xi|^2, \qquad \forall\, x \in \Omega, \forall\, \xi \in \mathbb{R}^2,$$

$$\sum_{i,j=1}^{2} \sum_{|\alpha|\leqslant 2} \|D^\alpha g^{ij}\|_{L^\infty(\Omega)} \leqslant C,$$

where λ, $0 < \lambda \leqslant 1$, C depend on γ and M_2, and

$$\mathrm{div}(\mathrm{div}(\mathbb{P}D^2\cdot)) = P(x, D\cdot) + Q(\cdot),$$

with $P(x, D) = (g^{ij}(x)D^2_{ij})(g^{lk}(x)D^2_{lk})$, and Q is a third-order operator such that

$$|Q(v)| \leqslant cM_2 \sum_{2\leqslant|\alpha|\leqslant 3} |D^\alpha v|, \qquad \forall v \in H^4(\Omega),$$

where c is an absolute constant. Now, since u is solution to (2.12), u satisfies the doubling inequality (1.10) of theorem 1.1. In order to prove the proposition, it suffices to estimate the local frequency (1.9).

To this aim, note that under our assumptions the solution u satisfies the following *the Lipschitz propagation of smallness property*

$$\int_{B_\rho(x)} u^2 \geqslant C_\rho \int_\Omega u^2, \qquad \forall\, \rho > 0 \text{ and } \forall\, x \in \Omega_{s\rho}, \tag{2.19}$$

where $s > 1$ only depends on γ, M_2 and $C_\rho > 0$ only depends on ρ_0, M_0, M_1, γ, M_2, $\frac{\|u\|_{H^{1/2}(\Omega)}}{\|u\|_{L^2(\Omega)}}$ and ρ. The proof of (2.19) is essentially based on a three-spheres inequality for the solution u, which has been derived in [Mo-Ro-Ve11, section 6]; for details see the arguments in [Al-Mo-Ro-Ve, theorem 3.2]. □

Next, we consider the problem of the detection of an unknown inclusion E in Ω. Let E be a measurable, possible disconnected subset of Ω satisfying

$$dist(E, \partial\Omega) \geqslant d_0\rho_0, \tag{2.20}$$

for some positive constant d_0. Let us assume that the plate tensor $\widetilde{\mathbb{P}} = \frac{h^3}{12}\widetilde{\mathbb{C}}$ in the inclusion belongs to $L^\infty(\Omega)$, where $\widetilde{\mathbb{C}}$ satisfies symmetry conditions analogous to (2.14). Moreover, we assume the following jump conditions on the elastic tensors $\mathbb{P}$ and $\widetilde{\mathbb{P}}$: either there exist $\eta > 0$ and $\delta > 1$ such that

$$\eta\mathbb{P} \leqslant \widetilde{\mathbb{P}} - \mathbb{P} \leqslant (\delta - 1)\mathbb{P}, \quad \text{a.e. in } \Omega \text{ (hard inclusion)}, \tag{2.21}$$

or there exist $\eta > 0$ and $0 < \delta < 1$ such that

$$-(1 - \delta)\mathbb{P} \leqslant \widetilde{\mathbb{P}} - \mathbb{P} \leqslant -\eta\mathbb{P}, \quad \text{a.e. in } \Omega \text{ (soft inclusion)}. \tag{2.22}$$

Let us apply a couple field $\widehat{M} = \widehat{M}_2 e_1 + \widehat{M}_1 e_2$ on the boundary of Ω such that:

$$\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2), \tag{2.23}$$

$$supp(\widehat{M}) \subset \Gamma, \tag{2.24}$$

where Γ is an open subarc of $\partial\Omega$ such that its length $|\Gamma|$ satisfies

$$|\Gamma| \leqslant (1-\delta_0)|\partial\Omega|, \tag{2.25}$$

for some positive constant δ_0. Moreover, the couple field $\widehat{M}$ is assumed to satisfy the obvious *compatibility conditions*

$$\int_{\partial\Omega} \widehat{M}_\alpha = 0, \quad \alpha = 1, 2. \tag{2.26}$$

When the inclusion E is absent, the transversal displacement $w_0 \in H^2(\Omega)$, normalized by $\int_\Omega w_0 = 0$ and $\int_\Omega \nabla w_0 = 0$, satisfies the Neumann boundary value problem

$$\begin{cases} \mathrm{div}(\mathrm{div}(\mathbb{P}D^2 w_0)) = 0, & \text{in } \Omega, \\ (\mathbb{P}D^2 w_0)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \mathrm{div}(\mathbb{P}D^2 w_0)\cdot n + ((\mathbb{P}D^2 w_0)n\cdot\tau)_{,s} = (\widehat{M}_\tau)_{,s} & \text{on } \partial\Omega. \end{cases} \tag{2.27}$$

In the above equations, $\widehat{M}_\tau = \widehat{M}\cdot n$ is the twisting moment and $\widehat{M}_n = \widehat{M}\cdot\tau$ is the bending moment, where τ and n are respectively the tangent and the normal vector to the boundary $\partial\Omega$, with $n \times \tau = e_3$. Moreover, $(\cdot)_{,s}$ stands for the derivative with respect to the arc length s.

The equilibrium problem for the defective plate is governed by the boundary value problem

$$\begin{cases} \mathrm{div}(\mathrm{div}((\chi_{\Omega\setminus E}\mathbb{P} + \chi_E\widetilde{\mathbb{P}})\nabla^2 w)) = 0, & \text{in } \Omega, \\ (\mathbb{P}D^2 w)n\cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \mathrm{div}(\mathbb{P}D^2 w)\cdot n + ((\mathbb{P}D^2 w)n\cdot\tau)_{,s} = (\widehat{M}_\tau)_{,s} & \text{on } \partial\Omega, \end{cases} \tag{2.28}$$

where w is normalized by $\int_\Omega w = 0$ and $\int_\Omega \nabla w = 0$.

Our size estimates of $|E|$ are given in terms of works W, W_0 exerted by the boundary couple field $\widehat{M}$ when E is present or absent, respectively, namely

$$W = -\int_{\partial\Omega} (\widehat{M}_{\tau,s} w + \widehat{M}_n w_{,n}), \tag{2.29}$$

$$W_0 = -\int_{\partial\Omega} (\widehat{M}_{\tau,s} w_0 + \widehat{M}_n w_{0,n}). \tag{2.30}$$

Theorem 2.2 (Size estimates of $|E|$)**.** *Under the above hypotheses, and assuming in addition that $\partial\Omega$ is of class $C^{4,1}$ with constants ρ_0, M_0, if (2.21) holds then*

$$\frac{1}{\delta-1} C_1^+ \rho_0^2 \frac{W_0 - W}{W_0} \leqslant |E| \leqslant \left(\frac{\delta}{\eta}\right)^{\frac{1}{p}} C_2^+ \rho_0^2 \left(\frac{W_0 - W}{W_0}\right)^{\frac{1}{p}}. \tag{2.31}$$

Conversely, if (2.22) holds then

$$\frac{\delta}{1-\delta} C_1^- \rho_0^2 \frac{W - W_0}{W_0} \leqslant |E| \leqslant \left(\frac{1}{\eta}\right)^{\frac{1}{p}} C_2^- \rho_0^2 \left(\frac{W - W_0}{W_0}\right)^{\frac{1}{p}}, \tag{2.32}$$

where C_1^+, C_1^- depend only on M_0, M_1, M_2, d_0, γ, whereas C_2^+, C_2^-, $p > 1$ only depend on the same quantities and also on δ_0 and on

$$\frac{\|\widehat{M}\|_{L^2(\partial\Omega,\mathbb{R}^2)}}{\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}. \tag{2.33}$$

Sketch of the proof. As a first step, the variational formulation for the reference and defective plate problems, both for hard and soft inclusions, leads to the following double inequality

$$C_l \int_E |D^2 w_0|^2 \leqslant |W - W_0| \leqslant C_u \int_E |D^2 w_0|^2, \tag{2.34}$$

where C_l, C_u are positive constants only depending on η, δ, h and γ. The above inequalities mean that the strain energy of the reference plate, stored in the set E, is comparable with the work gap $|W - W_0|$.

The lower bounds in (2.31) and (2.32) follow from the right-hand side (RHS) of (2.34) and from the regularity estimates for the solution to the unperturbed plate problem (2.27).

The derivation of the upper bounds for $|E|$ requires a lower estimate of $\int_E |D^2 w_0|^2$ in (2.34). It is exactly at this point that the doubling inequality (2.18) plays a crucial role. In fact, arguing similarly to [Mo-Ro-Ve09, section 4], one can derive from (2.18) that there exists a constant $\widetilde{\theta} \in (0, 1)$, depending on γ and M_2 only, such that for every $\overline{r} > 0$ and $x_0 \in \Omega_{\overline{r}\rho_0}$ we have

$$\int_{B_{2r}(x_0)} |D^2 w_0|^2 \leqslant K \int_{B_r(x_0)} |D^2 w_0|^2, \qquad \forall\, r,\ 0 < r \leqslant \frac{\widetilde{\theta}}{2}\overline{r}\rho_0, \tag{2.35}$$

where K only depends on M_0, M_1, δ_0, M_2, γ, $\overline{r}$ and $\|\widehat{M}\|_{L^2(\partial\Omega)}/\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}$.

By the general theory of Garofalo and Lin [Ga-Li86, Ga-Li87], inequality (2.35) ensures that $|D^2 w_0|^2$ is an A_p-weight for some $p > 1$ only depending on M_0, M_1, δ_0, M_2, γ and $\|\widehat{M}\|_{L^2(\partial\Omega)}/\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}$, that is $|D^2 w_0|^{-\frac{2}{p-1}}$ is locally integrable. Such an integrability property enables, through Hölder's inequality, to get the upper bounds in (2.31) and (2.32), see for details [Mo-Ro-Ve09]. □

3. Proof of theorem 1.1

In order to prove theorem 1.1, we will first prove a doubling type inequality (proposition 3.1) where (1.4) will be replaced by the slightly stronger assumption

$$g_1^{ij}(0) = g_2^{ij}(0) = \delta^{ij}. \tag{3.36}$$

Afterwards, we derive theorem 1.1 by a suitable change of variables.

Proposition 3.1. *Let $P(x, D)$ be as in (1.2) satisfying (1.3), (1.5), (1.6) and (3.36). Let $u \in H^4(B_1)$ be such that*

$$|P(x, D)u| \leqslant M \sum_{|\alpha|\leqslant 3} |D^\alpha u| \qquad \text{in } \overline{B}_1, \tag{3.37}$$

where M is a given positive constant. There exists a constant $R_1 \in (0, 1)$, only depending on λ, M, C_0, and there exist constants $\theta_1 \in (0, 1/2)$ and $K_1 > 0$, only depending on λ, M, C_0 and

$$\mathcal{F}^{(1)}_{\mathrm{loc}} = \frac{\|u\|_{L^2(B_{R_1^2})}}{\|u\|_{L^2(B_{R_1^4})}}, \tag{3.38}$$

such that

$$\int_{B_{2r}} |u|^2 \leqslant K_1 \int_{B_r} |u|^2 \qquad \forall\, r, 0 < r < \theta_1. \tag{3.39}$$

We shall prove proposition 3.1 by a bootstrapping argument based on the three-spheres inequalities developed in [Li-Nak-Wa] and [Li-Nag-Wa]. It may be possible to derive a doubling inequality like (3.39) using a Carleman estimate with more sophisticated weight functions, which have been obtained in [Co-Ko]. However, we would like to emphasize again that our ultimate goal is to apply doubling inequalities to the inverse problems described above. Therefore, it is crucial to know precisely how the constant K_1 of (3.39) depends on u. The quantity $\mathcal{F}^{(1)}_{\mathrm{loc}}$ of (3.38) is important in the investigation of inverse problems. In the bootstrapping argument, we use a simple Carleman estimate (see (3.42)) and the derivation of doubling inequalities is rather elementary.

We begin by stating two results that will be used later. The first ones are two Caccioppoli type inequalities which are simple consequences of [Ho, theorem 17.1.13]. Let $u \in H^4(B_1)$ such that

$$|P(x,D)u| \leqslant L \sum_{|\alpha|\leqslant 3} |D^\alpha u|, \qquad \text{in } B_1,$$

where $P(x,\partial)$ is defined in (1.2)–(1.4), then

$$\sum_{|\alpha|\leqslant 4} \int_{a_1 r<|x|<a_2 r} ||x|^\alpha D^\alpha u|^2 \leqslant C' \int_{a_3 r<|x|<a_4 r} |u|^2, \tag{3.40}$$

with $0 < a_3 < a_1 < a_2 < a_4 < 1$, $r < 1$, where $C' > 1$ depends on L, λ, C_0, $a_1 - a_3$ and $a_4 - a_2$ only. Let us stress here that the smaller the differences $a_1 - a_3$ and $a_4 - a_2$, the larger is the constant C';

$$\sum_{|\alpha|\leqslant 3} \int_{B_{r/2}} |D^\alpha u|^2 \leqslant C'' \left(\frac{2}{r}\right)^{2|\alpha|} \int_{B_r} u^2, \tag{3.41}$$

where $C'' > 1$ depends on L, λ and C_0 only.

Moreover we recall the following Carlemann type estimate derived in [LeB] and [Li-Nag-Wa]. For any $v \in C_0^\infty(B_1 \setminus \{0\})$ and $m = j + 1/2 \geqslant j_* + 1/2 =: m_*$, $j \in \mathbb{N}$, there exists a constant $C_1 > 1$ such that

$$\sum_{|\alpha|\leqslant 4} m^{4-2|\alpha|} \int |x|^{-2m+2|\alpha|-n} |D^\alpha v|^2 \leqslant C_1 \int |x|^{-2m+8-n} |\Delta^2 v|^2. \tag{3.42}$$

Let R_0 and $\bar{R}$, $R_0, \bar{R} \in (0,1)$, be numbers that will be chosen later and assume $0 < \bar{R} \leqslant R_0$. Setting $r_4 = \frac{R_0(R_0+1)}{4}$ (which implies $r_4 < R_0/2$) and picking an arbitrary δ such that

$$0 < \delta \leqslant \tfrac{1}{4} R_0^2 \bar{R}, \tag{3.43}$$

we define a function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned}
&0 \leqslant \chi \leqslant 1, && \chi = 1 \text{ on } B_{r_4\bar{R}} \setminus B_{\delta/2},\\
&\chi = 0, && \text{for } |x| \leqslant \frac{\delta}{3} \text{ and } |x| \geqslant 2r_4\bar{R},\\
&|D^\alpha \chi(x)| \leqslant \frac{C_3}{\delta^{|\alpha|}}, && \text{for } \frac{\delta}{3} \leqslant |x| \leqslant \frac{\delta}{2},\\
&|D^\alpha \chi(x)| \leqslant \frac{C_3}{(r_4\bar{R})^{|\alpha|}}, && \text{for } r_4\bar{R} \leqslant |x| \leqslant 2r_4\bar{R},
\end{aligned}$$

where C_3 is an absolute constant. After a possible regularization, we insert in (3.42) the function $v = u\chi$, where $u \in H^4(B_1)$ satisfies (1.1). We have

$$\sum_{|\alpha|\leqslant 4} m^{4-2|\alpha|} \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+2|\alpha|-n} |D^\alpha u|^2 \leqslant C_1 \int |x|^{-2m+8-n} |\Delta^2(\chi u)|^2, \tag{3.44}$$

for every $m \geqslant m_*$. Now splitting the integral on the RHS

$$\int |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 = \int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 + \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2 u|^2 + \int_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2$$

and using the following chain of inequalities

$$\begin{aligned}
|\Delta^2 u| &\leqslant |P(0,D)u - P(x,D)u| + |P(x,D)u| \\
&\leqslant cC_0^2|x| \sum_{|\alpha|=4} |D^\alpha u| + (cC_0^2 + M) \sum_{|\alpha|\leqslant 3} |D^\alpha u| \\
&\leqslant cC_0^2 r_4 \sum_{|\alpha|=4} |D^\alpha u| + (cC_0^2 + M) \sum_{|\alpha|\leqslant 3} |D^\alpha u|,
\end{aligned}$$

which follows, for $|x| \leqslant r_4\bar{R}$, by the Lipschitz continuity of coefficients and by (1.7), we obtain

$$\begin{aligned}
\int |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 &\leqslant \int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 \\
&+ C_4\Bigg[(r_4\tilde{R})^2 \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|=4} |D^\alpha u|^2 \\
&+ \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|\leqslant 3} |D^\alpha u|^2\Bigg] \\
&+ \int_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2,
\end{aligned}$$

where $C_4 = c(C_0^2 + M^2)$. Now by properties of χ we have

$$\int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 \leqslant cC_3^2 \sum_{|\alpha|\leqslant 4} \delta^{2(|\alpha|-4)} \int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}} |x|^{-2m+8-n}|D^\alpha u|^2 := I^{(\delta)}$$

and

$$\begin{aligned}
&\int_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|\Delta^2(\chi u)|^2 \\
&\qquad \leqslant cC_3 \sum_{|\alpha|\leqslant 4} (r_4\bar{R})^{2(|\alpha|-4)} \int_{r_4\bar{R}<|x|<2r_4\bar{R}} |x|^{-2m+8-n}|D^\alpha u|^2 =: I^{(\bar{R})}.
\end{aligned}$$

Inserting everything in (3.44) we get

$$\begin{aligned}
&\sum_{|\alpha|\leqslant 4} m^{4-2|\alpha|} \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+2|\alpha|-n}|D^\alpha u|^2 \\
&\qquad \leqslant I^{(\delta)} + I^{(\bar{R})} + C_4(r_4\bar{R})^2 \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|=4} |D^\alpha u|^2 \\
&\qquad + C_4 \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m+8-n} \sum_{|\alpha|\leqslant 3} |D^\alpha u|^2, \qquad \forall\, m \geqslant m_*,
\end{aligned} \tag{3.45}$$

that we write in the form

$$S_1 \leqslant I^{(\delta)} + I^{(R)}, \qquad \forall\, m \geqslant m_*, \tag{3.46}$$

where by S_1 we denote the left-hand side (LHS) integral of (3.45) minus the two remaining integrals of RHS. Our aim now is to estimate the five orders of derivatives of S_1, starting from the fourth-order one (i.e. $\alpha = 4$), which decays to zero faster than the others, up to the zero-order integral ($\alpha = 0$), which like the first-order term ($\alpha = 1$), can be estimated easily taking $\bar{R}$ small enough and m large enough.

Let $\mu \geqslant 1$ (to be chosen later) and set $\bar{R} = \frac{1}{\mu m^2}$, we have

$$\sum_{|\alpha|=4}\left(\frac{1}{m^4} - C_4(r_4\bar{R})^2\right)\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}|x|^{-2m+8-n}|D^\alpha u|^2$$
$$= \sum_{|\alpha|=4}\frac{1}{m^4}\left(1 - \frac{C_4 r_4^2}{\mu^2}\right)\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}|x|^{-2m+8-n}|D^\alpha u|^2.$$

In order to have this quantity greater or equal to zero it suffices to take

$$\mu \geqslant \sqrt{C_4} r_4. \tag{3.47}$$

Let us consider now the third-order term. By (3.42) and (3.47), we have by the choice of $\bar{R}$,

$$\sum_{|\alpha|=3}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}\left(\frac{1}{m^2} - C_4|x|^2\right)|x|^{-2m+6-n}|D^\alpha u|^2$$
$$\geqslant \sum_{|\alpha|=3}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}\left(\frac{1}{m^2} - C_4(r_4\bar{R})^2\right)|x|^{-2m+6-n}|D^\alpha u|^2 \geqslant 0.$$

To estimate the second-order term we can proceed as follows:

$$\sum_{|\alpha|=2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}(1 - C_4|x|^4)|x|^{-2m+4-n}|D^\alpha u|^2$$
$$\geqslant \sum_{|\alpha|=2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}\left(1 - \frac{C_4}{m^8}\right)|x|^{-2m+4-n}|D^\alpha u|^2.$$

To have this term positive, we take $m \geqslant C_4^{1/8}$. The first-order term can be estimated similarly, whereas for the zero-order term we have

$$\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}(m^4 - C_4|x|^8)|x|^{-2m-n}|u|^2 \geqslant \frac{m^4}{2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}|x|^{-2m-n}|u|^2,$$

as long as $m \geqslant (2C_4)^{1/4}$. Summarizing, picking $m_1 := \max\{(2C_4)^{n/2} + 1, m_*\}$, if $\mu \geqslant \sqrt{C_4} r_4$,

$$S_1 \geqslant \frac{m^4}{2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}|x|^{-2m-n}|u|^2,$$

which, recalling (3.46), leads to

$$\frac{m^4}{2}\int_{\frac{\delta}{2}<|x|<r_4\bar{R}}|x|^{-2m-n}|u|^2 \leqslant I^{(\delta)} + I^{(\bar{R})}, \qquad \forall\, m \geqslant m_1, \tag{3.48}$$

with

$$\bar{R} = \frac{1}{\mu m^2} \qquad \text{and} \qquad \mu = \sqrt{C_4}. \tag{3.49}$$

We now use (3.40) to get rid of terms of zero derivatives in $I^{(\delta)}$ and $I^{(\bar{R})}$ appearing in the inequality (3.48). Let us consider $I^{(\delta)}$ first; we have

$$I^{(\delta)} = cC_3^2\sum_{|\alpha|\leqslant 4}\delta^{2(|\alpha|-4)}\int_{\frac{\delta}{3}<|x|<\frac{\delta}{2}}|x|^{-2m+8-n-2|\alpha|}||x|^{|\alpha|}|D^\alpha u||^2$$
$$\leqslant c' C_3^2 C_2\left(\frac{\delta}{3}\right)^{-2m-n}\int_{\frac{\delta}{6}<|x|<\delta}|u|^2. \tag{3.50}$$

Similarly, choosing in (3.40) $a_1 = 1$, $a_2 = 2$, $a_3 = 1/2$ and $a_4 = \frac{4}{R_0+1}$ with the further restriction that $R_0 \leqslant 1/2$, entailing $a_4 - a_2 \geqslant 2/3$,

$$I^{(\bar{R})} \leqslant c'' C_2 C_3^2 (r_4\bar{R})^{-2m-n} \int_{\frac{r_4\bar{R}}{2}<|x|<\bar{R}R_0} |u|^2. \tag{3.51}$$

Inserting (3.50) and (3.51) into (3.48) we get

$$m^4 \int_{\frac{\delta}{2}<|x|<r_4\bar{R}} |x|^{-2m-n}|u|^2 \leqslant C_5\left(\frac{\delta}{3}\right)^{-2m-n} \int_{|x|<\delta} |u|^2 + C_5(r_4\bar{R})^{-2m-n}\int_{|x|<R_0\bar{R}} |u|^2, \tag{3.52}$$

for every $m \geqslant m_1$, $\bar{R}$ and μ satisfying (3.49) and $R_0 \leqslant 1/2$, where $C_5 = \max\{C_2, c'C_3^2, c''C_2, C_3^2\}$. Now, observing that for $R_0 \leqslant 1/3$ we get $R_0^2 \leqslant \frac{R_0(R_0+1)}{4} = r_4$, we have that the LHS of (3.52) can be trivially bounded from below by an integral over the set $\{\frac{\delta}{2} < |x| < R_0^2\bar{R}\}$, which yields to the following inequality:

$$m^4 \int_{\frac{\delta}{2}<|x|<2\delta} |x|^{-2m-n}|u|^2 + m^4\int_{2\delta<|x|<R_0\bar{R}} R_0^2|x|^{-2m-n}|u|^2 \leqslant C_5\left(\frac{\delta}{3}\right)^{-2m-n} \int_{|x|<\delta} |u|^2 + C_5(r_4\bar{R})^{-2m-n}\int_{|x|<R_0\bar{R}} |u|^2,$$

for every $m \geqslant m_1$. Let us consider now the LHS of the above inequality. Observing that $|x| \leqslant 2\delta$ in the first integral and $|x| \leqslant R_0^2\bar{R}$ in the second one and adding to both sides the term $m^4(2\delta)^{-2m-n}\int_{|x|<\delta/2}|u|^2$, we have

$$m^4(2\delta)^{-2m-n}\int_{|x|<2\delta}|u|^2 + m^4(R_0^2\bar{R})^{-2m-n}\int_{2\delta<|x|<R_0^2\bar{R}}|u|^2 \leqslant \left[\left(C_5\frac{\delta}{3}\right)^{-2m-n} + m^4(2\delta)^{-2m-n}\right]\int_{|x|<\delta}|u|^2 + C_5(r_4\bar{R})^{-2m-n}\int_{|x|<R_0\bar{R}}|u|^2, \tag{3.53}$$

for every $m \geqslant m_1$, with $R_0 \leqslant 1/3$, and R, μ as in (3.49). Now by (3.43) we have $(2\delta)^{-2m-n} \geqslant \left(\frac{R_0^2\bar{R}}{2}\right)^{-2m-n}$, which allows us to estimate from below the LHS of (3.53) by

$$\text{LHS} \geqslant m^4\frac{1}{2}(2\delta)^{-2m-n}\int_{|x|<2\delta}|u|^2 + m^4(R_0^2\bar{R})^{-2m-n}\int_{|x|<R_0^2\bar{R}}|u|^2,$$

whereas to estimate from above the second integral of the RHS let us note that

$$C_5 m^{-4}\left(\frac{R_0^2}{r_4}\right)^{2m+n} \leqslant (4R_0)^{2m} \leqslant \mathrm{e}^{-2m},$$

with $R_0 \leqslant \frac{1}{4\mathrm{e}}$ and $m \geqslant m_2 := \max\{m_1, [\sqrt[4]{C_5}] + 1\}$. Now inserting all previous inequalities in (3.53) we get

$$\frac{1}{2}m^4(2\delta)^{-2m-n}\int_{|x|<2\delta}|u|^2 + m^4(R_0^2\bar{R})^{-2m-n}\int_{|x|<R_0^2\bar{R}}|u|^2 \leqslant \left(C_5\left(\frac{\delta}{3}\right)^{-2m-n} + m^4(2\delta^{-2m-n})\right)\int_{|x|<\delta}|u|^2 + m^4(R_0^2\bar{R})^{-2m-n}\mathrm{e}^{-2m}\int_{|x|<R_0\bar{R}}|u|^2. \tag{3.54}$$

We use now (5.36) of [Mo-Ro-Ve11, theorem 5.3]. Let us point out that, according to their notation, by (1.4) we have $\nu_* = \nu^* = \mu_* = \mu^* = 1$, which means that for every $\beta > 0$, there exists $s_1 \in (0, 1)$ and $C \geqslant 1$, depending on λ, C_0, M and β such that for every $\rho_1 \in (0, s_1)$ and r, ρ such that $r < \rho < \rho_1\lambda^2/2$ we have

$$\sum_{|\alpha|\leqslant 3} \rho^{2|\alpha|} \int_{B_\rho} |D^\alpha u|^2 \leqslant C \max\{1, \rho^{-(5\beta-2)}\} e^{C\left((\lambda^{-1}\rho)^{-\beta} - \left(\frac{\rho_1\lambda}{2}\right)^{-\beta}\right)}$$
$$\times \left(r^{5\beta-2} \sum_{|\alpha|\leqslant 3} r^{2|\alpha|} \int_{B_r} |D^\alpha u|^2 \right)^{\theta_0} \left(\rho_1^{5\beta-2} \sum_{|\alpha|\leqslant 3} \rho_1^{2|\alpha|} \int_{B_{\rho_1}} |D^\alpha u|^2 \right)^{1-\theta_0}, \tag{3.55}$$

where

$$\theta_0 = \frac{(\lambda^{-1}\rho)^{-\beta} - \left(\frac{\lambda\rho_1}{2}\right)^{-\beta}}{\left(\frac{\lambda r}{2}\right)^{-\beta} - \left(\frac{\lambda\rho_1}{2}\right)^{-\beta}}.$$

Now, assuming $r < \rho < \frac{\rho_1\lambda^2}{4}$, using the Caccioppoli inequality (3.41) and taking $\beta \leqslant 2/5$, we can write (3.55) as follows. There exists a constant C_6 depending on λ, C_0, M and β only such that if $r_1 < r_2 < \frac{r_3}{C_6}$, with $r_3 \leqslant \frac{1}{C_6}$, then

$$\int_{B_{r_2}} u^2 \leqslant e^{C_6 r_2^{-\beta}} \left(\int_{B_{r_1}} u^2 \right)^{\theta} \left(\int_{B_{r_3}} u^2 \right)^{1-\theta} \tag{3.56}$$

where

$$\theta = \frac{(\lambda^{-1}r_2)^{-\beta} - \left(\frac{\lambda r_3}{4}\right)^{-\beta}}{\left(\frac{\lambda r_1}{4}\right)^{-\beta} - \left(\frac{\lambda r_3}{4}\right)^{-\beta}}. \tag{3.57}$$

Proposition 3.2. *Let $R_j = \frac{1}{\mu(j+1/2)^2}$, $j \in \mathbb{N}$. We have*

$$R_{j+1} < R_j < 2R_{j+1}, \qquad \text{for } j \geqslant 3. \tag{3.58}$$

Furthermore, for every $R_{j+1} < R < R_j$, $j \geqslant 3$, we have

$$e^{-\frac{2}{\sqrt{\mu R_j}}} \int_{|x|<R_0R_j} u^2 \leqslant e^{-\frac{\sqrt{2}}{\sqrt{\mu R}}} \int_{|x|<R} u^2. \tag{3.59}$$

Proof. Inequality (3.58) is easy to check. Let us prove (3.59). We first observe that, since $R_0 \leqslant \frac{1}{4e}$, we have

$$R_0R_j \leqslant \frac{1}{4e}R_j < \frac{1}{2e}R_{j+1},$$

thus if $R_{j+1} < R < R_j$, we get the thesis. □

We can write (3.54) with $\bar{R} = R_j$ and, recalling that $m = \frac{1}{\sqrt{\mu R_j}}$, by replacing the term e^{-2m} with $e^{-\frac{2}{\sqrt{\mu R_j}}}$. By (3.59) for every $R \in (R_{j+1}, R_j]$, with $j \geqslant j_3$, where $j_3 = \max\{j_2, 3\}$, we have

$$\frac{1}{2}m^4(2\delta)^{-2m-n} \int_{|x|<2\delta} u^2 + m^4\left(R_0^2R_j\right)^{-2m-n} \int_{|x|<R_0^2R} u^2$$
$$\leqslant \left(C_5 \left(\frac{\delta}{3}\right)^{-2m-n} + m^4(2\delta)^{-2m-n} \right) \int_{|x|<\delta} u^2$$
$$+m^4\left(R_0^2R_j\right)^{-2m-n} e^{-\frac{\sqrt{2}}{\sqrt{\mu R}}} \int_{|x|<R} u^2, \tag{3.60}$$

for every $m \geqslant m_3 := j_3 + 1/2$. Now by (3.60), if there exist $s \in \mathbb{N}$ and $\hat{j} \geqslant j_3$ such that

$$R_{\hat{j}+1} \leqslant R_0^{2s} \leqslant R_{\hat{j}}, \tag{3.61}$$

then, setting $\hat{m} = \hat{j} + \frac{1}{2}$, we have

$$\begin{aligned}\frac{1}{2}\hat{m}^4(2\delta)^{-2\hat{m}-n}\int_{|x|<2\delta} u^2 + \hat{m}^4\left(R_0^2 R_{\hat{j}}\right)^{-2\hat{m}-n}\int_{|x|<R_0^{2s+2}} u^2 \\ \leqslant \left(C_5\left(\frac{\delta}{3}\right)^{-2\hat{m}-n} + \hat{m}^4(2\delta)^{-2\hat{m}-n}\right)\int_{|x|<\delta} u^2 \\ +\hat{m}^4\left(R_0^2 R_{\hat{j}}\right)^{-2\hat{m}-n} \mathrm{e}^{-\frac{\sqrt{2}}{\sqrt{\mu R_0^{2s}}}}\int_{|x|<R_0^{2s}} u^2.\end{aligned} \tag{3.62}$$

Proposition 3.3. *Let* $\bar{R}_0 = \frac{\lambda^8}{2^{18}C_6^5(2\mu)^{5/2}}$ *and*

$$s_0 = 1 + \lfloor \max\{\log[(\lambda/2)^{4/5}\sqrt{2\mu}\log(\mathrm{e}N)], 2\}\rfloor,$$

where $\lfloor\cdot\rfloor$ *stands for the integer part, then*

$$\mathrm{e}^{-\frac{\sqrt{2}}{(\mu R_0^{2k})^{1/2}}}\left(\frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2}\right) \leqslant 1, \tag{3.63}$$

for every $k \geqslant s_0$ *and* $R_0 \leqslant \bar{R}_0$.

Proof. We begin by using the three-sphere inequality (3.56) with $r_1 = R_0^{2k+2}$, $r_2 = R_0^{2k}$, $r_3 = R_0^{2k-2}$, $k \in \mathbb{N}$, $k > 1$. Thus we require $R_0 \leqslant \min\left\{\frac{1}{4\mathrm{e}}, \frac{1}{\sqrt{C_6}}\right\}$. Our goal is to estimate the exponent θ. By (3.57) we have

$$\frac{1-\theta}{\theta} = \frac{(R_0^4)^{-\beta} - (4\lambda^{-2}R_0^2)^{-\beta}}{(4\lambda^{-2}R_0^2)^{-\beta} - 1}.$$

Setting

$$\alpha_{k+1} = \frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2},$$

by (3.56) we have

$$\alpha_{k+1} \leqslant \left(\mathrm{e}^{C_6(R_0^{2k})^{-\beta}}\right)^{\frac{1}{\theta}}\alpha_k^{\frac{1-\theta}{\theta}}. \tag{3.64}$$

Now if $R_0 \leqslant \left[\frac{1}{4\lambda^{-2}}\left(\frac{1}{2}\right)^{1/\beta}\right]^{1/2}$, we have

$$\frac{1-\theta}{\theta} \leqslant 2\left(\frac{\lambda^2 R_0^2}{4}\right)^{-\beta} =: \omega.$$

Setting $E_k = \mathrm{e}^{C_6(R_0^{2k})^{-\beta}}$, (3.64) can be written as

$$\alpha_{k+1} \leqslant E_k^{\omega}\alpha_k^{\omega}, \qquad k \geqslant 2,$$

that, iterating, leads to

$$\alpha_{k+1} \leqslant G_k \alpha_2^{\omega^{k-1}}, \quad k \geqslant 2, \tag{3.65}$$

where

$$G_k := \exp\left(C_6 2^{k-1}k\left(\frac{R_0\lambda}{2}\right)^{-2\beta(k+1)}\right)\alpha_2^{\omega^{k-1}}, \quad k \geqslant 2.$$

Defining

$$N := N(R_0^4) := \frac{\int_{|x|<R_0^2} u^2}{\int_{|x|<R_0^4} u^2},$$

inequality (3.65) can be written as

$$\frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2} \leqslant G_k N^{\omega^{k-1}}, \quad k \geqslant 2.$$

Now by the previous inequality we have

$$\mathrm{e}^{-\frac{\sqrt{2}}{\sqrt{\mu R_0^{2k}}}} \left(\frac{\int_{|x|<R_0^{2k}} u^2}{\int_{|x|<R_0^{2(k+1)}} u^2} \right) \leqslant G_k \mathrm{e}^{-\frac{1}{\sqrt{2\mu R_0^{2k}}}} N^{\omega^{k-1}} \mathrm{e}^{-\frac{1}{\sqrt{2\mu R_0^{2k}}}}. \tag{3.66}$$

Now taking $\beta = \frac{2}{5} < \frac{1}{2}$ and requiring $4^{4/5}\lambda R_0 \leqslant 1$ and $4^{4/5}\lambda R_0 \leqslant \frac{1}{2}(2\mu)^{-1/2}C_6^{-1}$, it is simple to check that

$$G_k \mathrm{e}^{-(2\mu R_0^{2k})^{-1/2}} \leqslant 1, \qquad \forall k \geqslant 2.$$

Let us consider now the other part of the RHS side of (3.66). Recalling that, by our choice of $\tilde{R}_0$

$$\frac{2R_0^{1/2}}{\left(\frac{\lambda}{2}\right)^{4/5}} \leqslant \frac{1}{\mathrm{e}},$$

taking

$$k \geqslant \max\left\{ \log\left[\left(\frac{\lambda}{2}\right)^{4/5} \sqrt{2\mu}\log(\mathrm{e}N) \right], 2 \right\},$$

we easily get

$$\mathrm{e}^{-\frac{1}{(2\mu R_0^{2k})^{1/2}}} N^{\omega^{k-1}} \leqslant 1,$$

which gives us the thesis. □

To complete the proof of the theorem, we have to check (3.61), that is we have to determine $\hat{j} \geqslant j_3$ and $s \geqslant s_0$, where s_0 has been defined in proposition 3.3, such that

$$R_{\hat{j}+1} < \bar{R}_0^{2s} \leqslant R_{\hat{j}}. \tag{3.67}$$

Now, let

$$s_1 = 2 + \left\lfloor \max\left\{ s_0, \frac{\log\sqrt{\mu}(j_3 + 1/2)}{|\log \bar{R}_0|} \right\} \right\rfloor,$$

we have

$$\bar{R}_0^{2s_1} \leqslant \frac{1}{\mu(j_3 + 1/2)^2}.$$

Let

$$J = \left\{ j \in \mathbb{N}, j \geqslant j_3 : \bar{R}_0^{2s_1} \leqslant \frac{1}{\mu(j + 1/2)^2} \right\},$$

clearly $J \neq \emptyset$. Setting $j_4 = \max J$, (3.67) holds for $s = s_1$ and $\hat{j} = j_4$. We can now conclude the proof of theorem 3.1. Namely, defining $\hat{m}_4 = j_4 + 1/2$ and $k = s_1$, by (3.61), (3.62) and (3.63) with $R_0 = \bar{R}_0$ we obtain

$$\frac{1}{2}\hat{m}_4^4(2\delta)^{-2\hat{m}_4 - n} \int_{|x|<2\delta} u^2 \leqslant \left[C_5 \left(\frac{\delta}{3}\right)^{-2\hat{m}_4 - n} + \hat{m}_4^4(2\delta)^{-2\hat{m}_4 - n} \right] \int_{|x|<\delta} u^2,$$

for every δ such that $0 < \delta \leqslant \frac{1}{4}\bar{R}_0^2 R_{j_4}$. Now (3.39) follows dividing the previous inequality by $\frac{1}{2}\hat{m}_4^4(2\delta)^{-2\hat{m}_4-n}$, with

$$R_1 = \bar{R}_0, \quad K_1 = \frac{6^n(2C_5+1)6^{2\hat{m}_4}}{\hat{m}_4^4} \quad \text{and} \quad \theta_1 = \frac{1}{4}\bar{R}_0^2 R_{j_4}.$$

We can now prove theorem 1.1.

Proof of theorem 1.1. Let $J = \sqrt{g_1^{-1}(0)}$, where

$$g_1^{-1}(0) = \left\{g_1^{ij}(0)\right\}_{i,j=1}^n = \left\{g_2^{ij}(0)\right\}_{i,j=1}^n = g_2^{-1}(0)$$

and

$$\psi : \mathbb{R}^n \to \mathbb{R}^n, \qquad \psi(x) = Jx.$$

Setting $\tilde{g}_k^{-1}$ such that $\tilde{g}_k^{-1}(\psi(x)) = Jg_k^{-1}(x)J^T$, $k = 1, 2$, we have $\tilde{g}_1^{-1}(0) = \tilde{g}_2^{-1}(0) = Id$. Let $u \in H^4(B_1)$ be a solution to (3.37). We define

$$\begin{aligned}
&U(y) := u(\psi^{-1}(y)),\\
&\tilde{P}(y, D) := \tilde{P}_2(y, D)\tilde{P}_1(y, D),\\
&\tilde{P}_k(y, D) := \textstyle\sum_{i,j=1}^n \tilde{g}_k^{ij}(y)D_{ij}^2, \qquad k = 1, 2
\end{aligned}$$

and for any $r > 0$,

$$\mathcal{E}_r := \left\{x \in \mathbb{R}^n : g_1^{-1}(0)x \cdot x < r^2\right\}.$$

We have that $\mathcal{E}_r = \psi^{-1}(B_r)$ (see [Al-Ro-Ro-Ve, page 16]),

$$B_{\sqrt{\lambda}r} \subset \mathcal{E}_r \subset B_{r/\sqrt{\lambda}} \qquad \text{for every } r > 0 \tag{3.68}$$

and (see [Mo-Ro-Ve11, page 1523])

$$|\tilde{P}(y, D)U(y)| \leqslant cM \sum_{|\alpha|\leqslant 3} |D^\alpha U|, \qquad \text{in } \mathcal{E}_1 \supset B_{\sqrt{\lambda}},$$

where c depends on λ only. By theorem 3.1, performing the variable change $y \to \sqrt{\lambda}y$, we have that there exist constants $\tilde{R}_1$ and $\tilde{\theta}_1$, $\tilde{R}_1 \in (0, 1)$, $\tilde{\theta}_1 \in (0, 1/2)$, such that

$$\int_{B_{2^l r}} |U(y)|^2 \mathrm{d}y \leqslant \tilde{K}_1^l \int_{B_r} |U(y)|^2 \mathrm{d}y, \qquad \forall\, r,\ 0 < r < \tilde{\theta}_1/2^l, \tag{3.69}$$

where l will be chosen later on, $\tilde{R}_1$ depends on λ, M and C_0 only and $\tilde{\theta}_1$ and $\tilde{K}_1$ depend on λ, M, C_0 and

$$\tilde{\mathcal{F}}_{\mathrm{loc}}^{(1)} = \frac{\|U\|_{L^2(B_{\tilde{R}_1})}}{\|U\|_{L^2(B_{\tilde{R}_1^2})}} \tag{3.70}$$

only. By (3.68) and (3.69) we have

$$\begin{aligned}
\int_{B_{2^l r}} |u|^2 \mathrm{d}x &= \int_{B_{\frac{2^l r}{\sqrt{\lambda}}}} |u(\psi^{-1}(y))|^2 \left|\det \frac{\partial \psi^{-1}}{\partial y}(y)\right| \mathrm{d}y \\
&\leqslant \lambda^{-1} \int_{B_{2^l r/\sqrt{\lambda}}} |U(y)|^2 \mathrm{d}y \leqslant \lambda^{-1}\tilde{K}_1^l \int_{B_{r/\sqrt{\lambda}}} |U(y)|^2 \mathrm{d}y \\
&= \lambda^{-1}\tilde{K}_1^l \int_{\mathcal{E}_{r/\sqrt{\lambda}}} |U(\psi(x))|^2 \left|\det \frac{\partial \psi}{\partial x}(x)\right| \mathrm{d}x \leqslant \lambda^{-2}\tilde{K}_1^l \int_{B_{r/\lambda}} |u(x)|^2 \mathrm{d}x.
\end{aligned} \tag{3.71}$$

Let $l = [\log_2 \lambda^{-1}] + 2$, which implies $2^l r \geqslant 2r\lambda^{-1}$ and by (3.71)

$$\int_{B_{2r/\lambda}} |u|^2 \mathrm{d}x \leqslant \lambda^{-2} (\tilde{K}_1)^{[\log_2 \lambda^{-1}]+2} \int_{B_{r/\lambda}} |u|^2 \mathrm{d}x,$$

that can be written as

$$\int_{B_{2s}} |u|^2 \mathrm{d}x \leqslant \lambda^{-2} (\tilde{K}_1)^{[\log_2 \lambda^{-1}]+2} \int_{B_s} |u|^2 \mathrm{d}x, \qquad \text{for } 0 < s < \frac{\tilde{\theta}_1}{\sqrt{\lambda} 2^{[\log_2 \lambda^{-1}]+2}}.$$

Now by the inequalities

$$\int_{B_{\tilde{R}_1}} |U(y)|^2 \mathrm{d}y \leqslant \lambda^{-1} \int_{B_{\tilde{R}_1/\sqrt{\lambda}}} |u(x)|^2 \mathrm{d}x,$$

$$\int_{B_{\tilde{R}_1^2}} |U(y)|^2 \mathrm{d}y \geqslant \lambda \int_{B_{(\tilde{R}_1 \sqrt{\lambda})^2}} |u(x)|^2 \mathrm{d}x,$$

we get the thesis with $R = \tilde{R}_1/\sqrt{\lambda}$. □

Acknowledgments

AM is partially supported by the Carlos III University of Madrid-Banco de Santander Chairs of Excellence Programme for the 2013-2014 academic year. ER is supported by Università degli Studi di Trieste FRA 2012 'Problemi Inversi'. JW is supported in part by the National Science Council of Taiwan.

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