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Research Article

Fourth-Order Differential Equation with Deviating Argument

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We consider the fourth-order differential equation with middle-term and deviating argument $x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f(x(\varphi(t))) = 0$, in case when the corresponding second-order equation $h'' + q(t)h = 0$ is oscillatory. Necessary and sufficient conditions for the existence of bounded and unbounded asymptotically linear solutions are given. The roles of the deviating argument and the nonlinearity are explained, too.

1. Introduction

The aim of this paper is to investigate the fourth-order nonlinear differential equation with middle-term and deviating argument

$$x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f\left(x\left(\varphi(t)\right)\right) = 0. \tag{1.1}$$

The following assumptions will be made.

(i) q is a continuously differentiable bounded away from zero function, that is, $q(t) \geq q_0 > 0$ for large t such that

$$\int_0^\infty \left|q'(t)\right| dt < \infty. \tag{1.2}$$

(ii) r, φ are continuous functions for $t \geq 0$, r is not identically zero for large t, $\varphi(t) \geq 0$, and $\varphi(0) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$.

(iii) f is a continuous function such that $f(u)u > 0$ for $u \neq 0$.

Observe that (i) implies that there exists a positive constant Q such that $q(t) \leq Q$ and the linear second-order equation

$$h''(t) + q(t)h(t) = 0 \tag{1.3}$$

is oscillatory. Moreover, solutions of (1.3) are bounded together with their derivatives, see for example, [1, Theorem 2].

By a solution of (1.1) we mean a function x defined on $[T_x, \infty)$, $T_x \geq 0$, which is differentiable up to the fourth order and satisfies (1.1) on $[T_x, \infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq T_x$.

A solution x of (1.1) is said to be *asymptotically linear (AL-solution)* if either

$$\lim_{t\to\infty} x(t) = c_x \neq 0, \qquad \lim_{t\to\infty} x'(t) = 0, \tag{1.4}$$

or

$$\lim_{t\to\infty} |x(t)| = \infty, \qquad \lim_{t\to\infty} x'(t) = d_x \neq 0, \tag{1.5}$$

for some constants c_x, d_x.

Fourth-order nonlinear differential equations naturally appear in models concerning physical, biological, and chemical phenomena, such as, for instance, problems of elasticity, deformation of structures, or soil settlement, see, for example, [2, 3].

When (1.3) is nonoscillatory and h is its eventually positive solution, it is known that (1.1) can be written as the two-term equation

$$\left(h^2(t)\left(\frac{x''(t)}{h(t)}\right)'\right)' + h(t)r(t)f(x(t)) = 0. \tag{1.6}$$

In this case, the question of oscillation and asymptotics of such class of equations has been investigated with sufficient thoroughness, see, for example, the papers [3–10] or the monographs [11, 12] and references therein.

Nevertheless, as far we known, there are only few results concerning (1.1) when (1.3) is oscillatory. For instance, the equation without deviating argument

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)f(x(t)) = 0 \tag{1.7}$$

has been investigated by Kiguradze in [13] in case $q(t) \equiv 1$ and by the authors in [14, 15] when q satisfies (i). In particular, in [14] the oscillation of (1.1) in the case $n = 3$ is studied. In [15], the existence of positive bounded and unbounded solutions as well as of oscillatory solutions for (1.7) has been considered and the case $n = 4$ has been analyzed in detail. Other results can be found in [16] and references therein, in which the existence and uniqueness of almost periodic solutions for equations of type (1.1) with almost periodic coefficients q, r are studied.

Motivated by [14, 15], here we study the existence of AL-solutions for (1.1). The approach is completely different from the one used in [15], in which an iteration process, jointly with a comparison with the linear equation $y^{(4)} + q(t)y^{(2)} = 0$, is employed. Our tools are based on a topological method, certain integral inequalities, and some auxiliary functions. In particular, for proving the continuity in the Fréchet space $C[t_0, \infty)$ of the fixed point operators here considered, we use a similar argument to that in the Vitali convergence theorem.

Our results extend to the case with deviating argument analogues ones stated in [15] for (1.7) when $n = 4$. We obtain sharper conditions for the existence of unbounded AL-solutions of (1.1), and, in addition, we show that under additional assumptions on q, r, these conditions become also necessary for the existence of AL-solutions, in both the bounded and unbounded cases. In the final part, we consider the particular case

$$f(u) = |u|^{\lambda} \operatorname{sgn} u \qquad (\lambda > 0) \tag{1.8}$$

and we study the possible coexistence of bounded and unbounded AL-solutions. The role of deviating argument and the one of the growth of the nonlinearity are also discussed and illustrated by some examples.

2. Unbounded Solutions

Here we study the existence of unbounded AL-solutions of (1.1). Our first main result is the following.

Theorem 2.1. *For any c, $0 < c < \infty$, there exists an unbounded solution x of* (1.1) *such that*

$$\lim_{t\to\infty} x'(t) = c, \qquad \lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 2, 3, \tag{2.1}$$

provided

$$\int_0^{\infty} |r(t)| F(\varphi(t))\, dt < \infty, \tag{2.2}$$

where for $u > 0$

$$F(u) = \max\left\{ f(v) : |v - u| \le \frac{1}{2}u \right\}. \tag{2.3}$$

Proof. Without loss of generality, we prove the existence of solutions of (1.1) satisfying (2.1) for $c = 1$.

Let u and v be two linearly independent solutions of (1.3) with Wronskian $d = 1$. Denote

$$w(s,t) = u(s)v(t) - u(t)v(s), \qquad z(s,t) = \frac{\partial}{\partial t} w(s,t). \tag{2.4}$$

As claimed by the assumptions on q, all solutions of (1.3) and their derivatives are bounded. Thus, put

$$M = \sup\{|w(s,t)| + |z(s,t)| : s \ge 0,\ t \ge 0\}, \qquad L = \frac{2(2M+1)}{q_0}. \tag{2.5}$$

Let $\bar{t} \ge t_0$ be such that $\varphi(t) \ge t_0$ for $t \ge \bar{t}$. Define

$$\overline{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \ge \bar{t}, \\ \varphi(\bar{t}) & \text{if } t_0 \le t \le \bar{t}, \end{cases} \tag{2.6}$$

and choose $t_0 \ge 0$ large so that

$$\int_{t_0}^{\infty} |r(s)| F(\overline{\varphi}(s))\, ds \le \frac{1}{2L}, \qquad \frac{1}{q_0}\int_{t_0}^{\infty} |q'(t)|\, dt \le \frac{1}{2}. \tag{2.7}$$

Denote by $C[t_0,\infty)$ the Fréchet space of all continuous functions on $[t_0,\infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$, and consider the set $\Omega \subset C[t_0,\infty)$ given by

$$\Omega = \left\{ x \in C[t_0,\infty) : \frac{t}{2} \le x(t) \le \frac{3t}{2} \right\}. \tag{2.8}$$

Let $T > t_0$ and define on $[t_0,T]$ the function

$$g(t) = \gamma''(t) + q(t)\gamma(t), \tag{2.9}$$

where

$$\gamma(t) = -\int_t^T \int_\tau^\infty r(s) f(x(\overline{\varphi}(s))) w(s,\tau)\, ds\, d\tau \tag{2.10}$$

and $x \in \Omega$. Then,

$$\gamma'(t) = \int_t^\infty r(s) f(x(\overline{\varphi}(s))) w(s,t)\, ds, \tag{2.11}$$

$$\begin{gathered} \gamma''(t) = \int_t^\infty r(s) f(x(\overline{\varphi}(s))) z(s,t)\, ds, \\ \gamma'''(t) = -r(t) f(x(\overline{\varphi}(t))) - q(t)\gamma'(t). \end{gathered} \tag{2.12}$$

Moreover, $g(T) = \gamma''(T)$, and it holds for $t \in [t_0,T]$ that

$$g'(t) = \gamma'''(t) + q(t)\gamma'(t) + q'(t)\gamma(t) = -r(t) f(x(\overline{\varphi}(t))) + q'(t)\gamma(t). \tag{2.13}$$

Integrating, we obtain

$$g(t) = g(T) - \int_t^T g'(s)ds = \gamma''(T) + \int_t^T r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)ds - \int_t^T q'(s)\gamma(s)ds. \tag{2.14}$$

From here and (1.3), we get

$$\gamma(t) = \frac{1}{q(t)}\left(\gamma''(T) - \gamma''(t) + \int_t^T r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)ds - \int_t^T q'(s)\gamma(s)ds\right). \tag{2.15}$$

Thus,

$$\left|\gamma(t)\right| \leq \frac{1+2M}{q_0}\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds + \frac{1}{q_0}\max_{t\leq s\leq T}\left|\gamma(s)\right|\int_{t_0}^\infty \left|q'(s)\right|ds, \tag{2.16}$$

and so

$$\left(1 - \frac{1}{q_0}\int_{t_0}^\infty \left|q'(s)\right|ds\right)\max_{t\leq s\leq T}\left|\gamma(s)\right| \leq \frac{1+2M}{q_0}\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds, \tag{2.17}$$

or, in view of (2.7),

$$\left|\gamma(t)\right| \leq L\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds. \tag{2.18}$$

Thus, from (2.10), as $T \to \infty$, we get

$$\left|\int_t^\infty \int_\tau^\infty r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)w(s,\tau)ds\,d\tau\right| \leq L\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds. \tag{2.19}$$

Hence, the operator $\mathcal{T} : \Omega \to \Omega$ given by

$$\mathcal{T}(x)(t) = t - \int_{t_0}^t \int_\sigma^\infty \int_\tau^\infty r(s)f\left(x\left(\overline{\varphi}(s)\right)\right)w(s,\tau)ds\,d\tau\,d\sigma \tag{2.20}$$

is well defined for any $x \in \Omega$. Moreover, in view of (2.19), we have

$$\left|\mathcal{T}'(x)(t) - 1\right| \leq L\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds. \tag{2.21}$$

From here, in virtue of (2.7) we get

$$|\mathcal{T}(x)(t) - t| \leq Lt\int_{t_0}^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds \leq \frac{1}{2}t. \tag{2.22}$$

Hence, $\mathcal{T}(\Omega) \subset \Omega$. From (2.5) and (2.11), we have

$$\left|\mathcal{T}''(x)(t)\right| = \left|\gamma'(t)\right| \leq M \int_t^{\infty} |r(s)| F(\overline{\varphi}(s)) ds, \tag{2.23}$$

and so $\lim_{t\to\infty} \mathcal{T}''(x)(t) = 0$. Similarly,

$$\left|\mathcal{T}'''(x)(t)\right| = \left|\gamma''(t)\right| \leq M \int_t^{\infty} |r(s)| F(\overline{\varphi}(s)) ds, \tag{2.24}$$

and thus, $\lim_{t\to\infty} \mathcal{T}'''(x)(t) = 0$, too. In addition,

$$\mathcal{T}^{(4)}(x)(t) = \gamma'''(t) = -q(t)\mathcal{T}''(x)(t) - r(t) f(x(\overline{\varphi}(t))). \tag{2.25}$$

Hence, any fixed point of $\mathcal{T}$ is a solution of (1.1) for large t.

Let us show that $\mathcal{T}(\Omega)$ is relatively compact, that is, $\mathcal{T}(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval of $[t_0, \infty)$. Because $\mathcal{T}(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, in view of (2.7), $\mathcal{T}'(u)(t)$ is bounded for any $u \in \Omega$, which yields the equicontinuity of the elements in $\mathcal{T}(\Omega)$.

Now we prove the continuity of $\mathcal{T}$ in Ω. Let $\{x_n\}$, $n \in \mathbb{N}$, be a sequence in Ω, which uniformly converges to $\overline{x} \in \Omega$ on every compact interval of $[t_0, \infty)$. Fixing $T > t_0$, in virtue of (2.23), the dominated convergence Lebesgue theorem gives

$$\lim_{n\to\infty} \int_\sigma^T \int_\tau^{\infty} r(s)(f(x_n(\overline{\varphi}(s))))w(s,\tau) ds\, d\tau = \int_\sigma^T \int_\tau^{\infty} r(s)(f(x(\overline{\varphi}(s))))w(s,\tau) ds\, d\tau. \tag{2.26}$$

Moreover,

$$\begin{aligned} &\left|\int_\sigma^{\infty} \int_\tau^{\infty} r(s)(f(x_n(\overline{\varphi}(s))) - f(\overline{x}(\overline{\varphi}(s))))w(s,\tau) ds\, d\tau\right| \\ &\quad \leq \left|\int_\sigma^{T} \int_\tau^{\infty} r(s)(f(x_n(\overline{\varphi}(s))) - f(\overline{x}(\overline{\varphi}(s))))w(s,\tau) ds\, d\tau\right| \\ &\quad\quad + \int_T^{\infty} \int_\tau^{\infty} |r(s)|(f(x_n(\overline{\varphi}(s))) + f(\overline{x}(\overline{\varphi}(s))))|w(s,\tau)| ds\, d\tau. \end{aligned} \tag{2.27}$$

In view of (2.19), we have

$$\int_T^{\infty} \int_\tau^{\infty} |r(s)|(f(x_n(\overline{\varphi}(s))) + f(\overline{x}(\overline{\varphi}(s))))|w(s,\tau)| ds\, d\tau \leq 2M \int_T^{\infty} |r(s)| F(\overline{\varphi}(s)) ds. \tag{2.28}$$

Thus, choosing T sufficiently large, we get from (2.27)

$$\lim_{n\to\infty} \int_\sigma^{\infty} \int_\tau^{\infty} r(s) f(x_n(\overline{\varphi}(s)))w(s,\tau) ds\, d\tau = \int_\sigma^{\infty} \int_\tau^{\infty} r(s) f(\overline{x}(\overline{\varphi}(s)))w(s,\tau) ds\, d\tau, \tag{2.29}$$

and so the continuity of $\mathcal{T}$ in Ω follows. By the Tychonoff fixed point theorem, the operator $\mathcal{T}$ has a fixed point x, which is an unbounded solution of (1.1) satisfying (2.1). □

Remark 2.2. With minor modifications, Theorem 2.1 gives also the existence of eventually negative unbounded AL-solutions. The details are omitted.

Remark 2.3. When $\varphi(t) \equiv t$, Theorem 2.1 is related with Theorem 1 in [15], from which the existence of unbounded AL-solutions of (1.1) can be obtained under stronger assumptions. A comparison between Theorem 1 in [15] and Theorem 2.1 is given in Section 4.

Our next result gives a necessary condition for the existence of unbounded solutions x of (1.1) satisfying for large t and some α and β

$$0 < \alpha \le x'(t) \le \beta. \tag{2.30}$$

Theorem 2.4. *Assume either $r(t) \ge 0$ or $r(t) \le 0$.*

Equation (1.1) *does not have eventually positive solutions x satisfying* (2.30) *for large t and some α and β provided*

$$\int_0^\infty |r(t)|\overline{F}(\varphi(t))dt = \infty, \tag{2.31}$$

where for $u > 0$

$$\overline{F}(u) = \min\left\{f(v) : \frac{\alpha}{2}u \le v \le 2\beta u\right\}. \tag{2.32}$$

Proof. Assume $r(t) \ge 0$, and let x be an eventually positive solution of (1.1) satisfying (2.30). Then, there exists τ such that

$$\frac{\alpha}{2}t \le x(t) \le 2t\beta \quad \text{for } t \ge \tau. \tag{2.33}$$

Consequently, in view of (2.31), we have

$$\lim_{t\to\infty}\int_\tau^t r(s)f(x(\varphi(s)))ds = \infty. \tag{2.34}$$

Thus, integrating (1.1), we get

$$\lim_{t\to\infty}\left(x'''(t) + \int_\tau^t q(s)x''(s)ds\right) = -\infty. \tag{2.35}$$

Furthermore,

$$\begin{aligned}\left|\int_{\tau}^{t} q(s)x''(s)ds\right| &= \left|q(t)x'(t) - q(\tau)x'(\tau) - \int_{\tau}^{t} q'(s)x'(s)ds\right| \\ &\leq 2\beta Q + \beta\int_{\tau}^{\infty} \left|q'(s)\right| ds < \infty,\end{aligned} \tag{2.36}$$

where $Q = \sup_{s\geq 0} q(s)$. Hence $\lim_{t\to\infty} x'''(t) = -\infty$, which gives a contradiction with the boundedness of x'. Finally, if $r(t) \leq 0$, the argument is similar and the details are left to the reader. □

3. Bounded Solutions

In this section we study the existence of bounded AL-solutions of (1.1). The following holds.

Theorem 3.1. *If*

$$\int_{0}^{\infty} |r(t)| t\, dt < \infty, \tag{3.1}$$

then, for any $c \in \mathbb{R} \setminus \{0\}$, there exists a solution x of (1.1) *satisfying*

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 1, 2. \tag{3.2}$$

Proof. Without loss of generality, we prove the existence of solutions of (1.1) satisfying (3.2) for $c = 1$.

We proceed by a similar way to that in the proof of Theorem 2.1, and we sketch the proof.

Let M be the constant given in (2.5), and let

$$K = \max\left\{ f(u) : \frac{1}{2} \leq u \leq \frac{3}{2} \right\}, \qquad L_1 = \frac{2K(2M+1)}{q_0}. \tag{3.3}$$

Choose $t_0 \geq 0$ large so that

$$\int_{t_0}^{\infty} t|r(t)|dt \leq \frac{1}{2L_1}, \qquad \frac{1}{q_0}\int_{t_0}^{\infty} \left|q'(s)\right| ds \leq \frac{1}{2}, \tag{3.4}$$

and define $\overline{\varphi}$ as in (2.6). Denote by $C[t_0, \infty)$ the Fréchet space of all continuous functions on $[t_0, \infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$, and consider the set $\Omega \subset C[t_0, \infty)$ given by

$$\Omega = \left\{ x \in C[t_0, \infty) : \frac{1}{2} \leq x(t) \leq \frac{3}{2} \right\}. \tag{3.5}$$

Let $T > t_0$, and, for any $x \in \Omega$, consider again the function γ given in (2.10). Reasoning as in the proof of Theorem 2.1, with minor changes, we obtain

$$\left|\int_t^\infty \int_\tau^\infty r(s) f(x(\overline{\varphi}(s))) w(s,\tau) ds\, d\tau\right| \le L_1 \int_t^\infty |r(s)| ds. \tag{3.6}$$

Hence, in virtue of (3.1), the operator $\mathcal{H} : \Omega \to \Omega$ given by

$$\mathcal{H}(x)(t) = 1 + \int_t^\infty \int_\sigma^\infty \int_\tau^\infty r(s) f(x(\overline{\varphi}(s))) w(s,\tau) ds\, d\tau\, d\sigma \tag{3.7}$$

is well defined and $\lim_{t\to\infty} \mathcal{H}(x)(t) = 1$. In view of (3.6), we get

$$\left|\mathcal{H}'(x)(t)\right| \le L_1 \int_t^\infty |r(s)| ds. \tag{3.8}$$

A similar estimation holds for $|\mathcal{H}''(x)|$. Thus, $\lim_{t\to\infty} \mathcal{H}^{(i)}(x)(t) = 0,\ i = 1,2$. In view of (3.4), from (3.8), we obtain

$$|\mathcal{H}(x)(t) - 1| \le L_1 \int_t^\infty s|r(s)| ds \le \frac{1}{2}, \tag{3.9}$$

that is, $\mathcal{H}(\Omega) \subset \Omega$. Moreover, a standard calculation gives

$$\mathcal{H}^{(4)}(x)(t) = -q(t)\mathcal{H}^{(2)}(x)(t) - r(t) f(x(\overline{\varphi}(s))), \tag{3.10}$$

and so any fixed point of $\mathcal{H}$ is, for large t, a solution of (1.1). Proceeding by a similar way to that in the proof of Theorem 2.1, we obtain that $\mathcal{H}(\Omega)$ is relatively compact.

Now we prove the continuity of $\mathcal{H}$ in Ω. Let $\{x_n\}$, $n \in \mathbb{N}$, be a sequence in Ω, which uniformly converges to $\overline{x} \in \Omega$ on every compact interval of $[t_0, \infty)$. Since

$$\left|\int_\tau^\infty r(s)(f(x_n(\overline{\varphi}(s)))) w(s,\tau) ds\right| \le KM \int_\tau^\infty |r(s)| ds, \tag{3.11}$$

in virtue of (3.1), the dominated convergence Lebesgue theorem gives

$$\lim_{n\to\infty} \int_\sigma^\infty \int_\tau^\infty r(s)(f(x_n(\overline{\varphi}(s)))) w(s,\tau) ds\, d\tau = \int_\sigma^\infty \int_\tau^\infty r(s)(f(x(\overline{\varphi}(s)))) w(s,\tau) ds\, d\tau. \tag{3.12}$$

Moreover, fixing $T > t_0$, we have

$$\begin{aligned}
&\left|\int_t^\infty \int_\sigma^\infty \int_\tau^\infty r(s)\left(f\left(x_n(\overline{\varphi}(s))\right) - f\left(\overline{x}(\overline{\varphi}(s))\right)\right)w(s,\tau)ds\,d\tau\,d\sigma\right| \\
&\quad\le \left|\int_t^T \int_\sigma^\infty \int_\tau^\infty r(s)\left(f\left(x_n(\overline{\varphi}(s))\right) - f\left(\overline{x}(\overline{\varphi}(s))\right)\right)w(s,\tau)ds\,d\tau\,d\sigma\right| \\
&\qquad + \int_T^\infty \int_\sigma^\infty \int_\tau^\infty |r(s)|\left(f\left(x_n(\overline{\varphi}(s))\right) + f\left(\overline{x}(\overline{\varphi}(s))\right)\right)|w(s,\tau)|ds\,d\tau\,d\sigma.
\end{aligned} \tag{3.13}$$

In view of (3.9), we have

$$\int_T^\infty \int_\sigma^\infty \int_\tau^\infty |r(s)|\left(f\left(x_n(\overline{\varphi}(s))\right) + f\left(\overline{x}(\overline{\varphi}(s))\right)\right)|w(s,\tau)|ds\,d\tau\,d\sigma \le 2L_1 \int_T^\infty s|r(s)|ds, \tag{3.14}$$

and so, choosing T sufficiently large, from (3.13) we obtain the continuity of $\mathscr{H}$ in Ω. Hence, by the Tychonoff fixed point theorem, the operator $\mathscr{H}$ has a fixed point x, which is a bounded solution of (1.1) satisfying (3.2). □

Remark 3.2. When $n = 4$, Theorem 3.1 extends to equations with deviating argument of a similar result stated in [15] for (1.7). Observe that our approach used here is completely different from that in [15].

The next result shows that, under additional assumptions, condition (3.1) can be also necessary for the existence of bounded AL-solutions of (1.1).

Theorem 3.3. *Assume either*

$$r(t) \ge 0, \qquad q''(t) \ge 0 \quad \text{for large } t \tag{3.15}$$

or

$$r(t) \le 0, \qquad q''(t) \le 0 \quad \text{for large } t. \tag{3.16}$$

If

$$\int_0^\infty |r(t)|t\,dt = \infty, \tag{3.17}$$

then (1.1) *does not have solutions x satisfying*

$$0 < \alpha \le x(t) \le \beta, \tag{3.18}$$

for large t and some α and β. Consequently, every bounded solution x of (1.1) *satisfies*

$$\liminf_{t\to\infty}|x(t)| = 0. \tag{3.19}$$

The following lemmas are needed for proving Theorem 3.3.

Lemma 3.4. *Assume $q''(t) \geq 0$ for $t \geq T \geq 0$, and let x be a solution of* (1.1) *satisfying* (3.18) *for $t \geq T$. Then, there exist two constants M_1, M_2 such that for $t \geq T$*

$$-\int_T^t sq(s)x''(s)ds < tq'(t)x(t) - tq(t)x'(t) + M_1, \tag{3.20}$$

$$\int_T^t (s-T)q'(s)x'(s)ds < M_2. \tag{3.21}$$

If $q''(t) \leq 0$ for $t \geq T \geq 0$, inequalities (3.20), (3.21) *hold in the opposite order.*

Proof. Suppose $q''(t) \geq 0$ on $[T, \infty)$. We have

$$\int_T^t sq(s)x''(s)ds = tq(t)x'(t) - Tq(T)x'(T) - \int_T^t q(s)x'(s)ds - \int_T^t sq'(s)x'(s)ds. \tag{3.22}$$

Since

$$\begin{gathered}\int_T^t sq'(s)x'(s)ds = tq'(t)x(t) - Tq'(T)x(T) - \int_T^t q'(s)x(s)ds - \int_T^t sq''(s)x(s)ds, \\ \int_T^t q(s)x'(s)ds = q(t)x(t) - q(T)x(T) - \int_T^t q'(s)x(s)ds,\end{gathered} \tag{3.23}$$

from (3.22), we get

$$\begin{aligned}-\int_T^t sq(s)x''(s)ds = {}& tq'(t)x(t) - tq(t)x'(t) + q(t)x(t) \\ & - 2\int_T^t q'(s)x(s)ds - \int_T^t sq''(s)x(s)ds + K_1,\end{aligned} \tag{3.24}$$

where K_1 is a suitable constant. Since q, x are bounded, $q''(t) \geq 0$, in view of (1.1), inequality (3.20) follows.

Moreover, q' is nondecreasing for $t \geq T$. Because q is a positive bounded function, then $q'(t) \leq 0$ on $[T, \infty)$. Thus, inequality (3.21) follows integrating by parts and using (1.1). Finally, if $q''(t) \leq 0$ on $[T, \infty)$, the argument is similar. □

Lemma 3.5. *Let x be a solution of* (1.1) *satisfying* (3.18) *for large t. If*

$$\int_0^\infty |r(t)|dt < \infty, \tag{3.25}$$

then x'' is bounded. If, in addition, $r(t) \geq 0$, $q''(t) \geq 0$ for $t \geq T \geq 0$ and (3.17) *holds, then for large t*

$$x'''(t) + q(t)x'(t) < q'(t)x(t). \tag{3.26}$$

If $r(t) \leq 0$, $q''(t) \leq 0$ for $t \geq T \geq 0$, inequality (3.26) *holds in the opposite order.*

Proof. Since $\lim_{t\to\infty}\varphi(t) = \infty$, there exists τ such that for $t \geq \tau$

$$0 < \alpha \leq x(\varphi(t)) \leq \beta. \tag{3.27}$$

Without loss of generality, let $\tau = T$. Thus, $\inf_{t\geq T} f(x(\varphi(t))) > 0$.

Let u and v be two linearly independent solutions of (1.3) with Wronskian $d = 1$. By assumptions on q, all solutions of (1.3) and their derivatives are bounded. Thus, by the variation constant formula, there exist constants c_1 and c_2 such that

$$x''(t) = c_1u(t) + c_2v(t) - \int_T^t (u(s)v(t) - u(t)v(s))r(s)f(x(\varphi(s)))ds, \tag{3.28}$$

and, in view of (3.25), x'' is bounded.

Let us prove (3.26), and suppose $r(t) \geq 0$, $q''(t) \geq 0$ on $[T,\infty)$. Multiplying (1.1) by t and integrating from T to t, we get

$$tx'''(t) - x''(t) + \int_T^t sq(s)x''(s)ds = Tx'''(T) - x''(T) - \int_T^t sr(s)f(x(\varphi(s)))ds, \tag{3.29}$$

or, in view of Lemma 3.4,

$$tx'''(t) \leq x''(t) + tq'(t)x(t) - tq(t)x'(t) - \int_T^t sr(s)f(x(\varphi(s)))ds + K_2, \tag{3.30}$$

where K_2 is a suitable constant. Since x'' is bounded and

$$\int_T^t sr(s)f(x(\varphi(s)))ds \geq \inf_{t\geq T} f(x(\varphi(t))) \int_T^t sr(s)ds, \tag{3.31}$$

from (3.17) and (3.30), we have

$$\lim_{t\to\infty} t(x'''(t) - q'(t)x(t) + q(t)x'(t)) = -\infty, \tag{3.32}$$

which gives the assertion. The case $r(t) \leq 0$, $q''(t) \leq 0$ on $[T,\infty)$ can be treated in a similar way. □

Proof of Theorem 3.3. Suppose $r(t) \geq 0$, $q''(t) \geq 0$ for $t \geq T \geq 0$. Without loss of generality, assume also that (3.27) holds for $t \geq T$. Define

$$v(t) = x''(t) + q(t)x(t), \tag{3.33}$$

$$z(t) = x'''(t) + q(t)x'(t) - \int_T^t q'(s)x'(s)ds. \tag{3.34}$$

Then, $z'(t) = -r(t)f(x(\varphi(t))) \leq 0$ and

$$z(t) = z(T) - \int_T^t r(s)f\big(x(\varphi(s))\big)ds. \tag{3.35}$$

Since $q'(t) \leq 0$ for $t \geq T$, we have

$$v'(t) \leq z(t) + \int_T^t q'(s)x'(s)ds = z(T) - \int_T^t r(s)f\big(x(\varphi(s))\big)ds + \int_T^t q'(s)x'(s)ds. \tag{3.36}$$

Case I. Assume

$$\int_0^\infty r(t)dt = \infty. \tag{3.37}$$

Since for $t \geq T$ we have $q''(t) \geq 0$ and, as claimed, $q'(t) \leq 0$, we get

$$\int_T^t q'(s)x'(s)ds = q'(t)x(t) - q'(T)x(T) - \int_T^t q''(s)x(s)ds \leq -q'(T)x(T). \tag{3.38}$$

Thus, from (3.36), we obtain $\lim_{t\to\infty} v'(t) = -\infty$, that is, v is unbounded. Hence, in view of (3.33), we obtain a contradiction with the boundedness of x.

Case II. Now assume (3.17) and (3.25). In view of Lemma 3.5, without loss of generality, we can suppose that (3.26) holds for $t \geq T$. Then,

$$z(T) = x'''(T) + q(T)x'(T) < q'(T)x(T). \tag{3.39}$$

Hence, $z(T) < 0$. Integrating (3.36), we get

$$v(t) \leq v(T) + z(T)(t-T) - \int_T^t (s-T)r(s)f\big(x(\varphi(s))\big)ds + \int_T^t (s-T)q'(s)x'(s)ds, \tag{3.40}$$

and, in view of Lemma 3.4, we have

$$v(t) \leq v(T) + z(T)(t-T) + M_2. \tag{3.41}$$

Thus, $\lim_{t\to\infty} v(t) = -\infty$, that is, as before, a contradiction. Finally, the case $r(t) \le 0$, $q''(t) \le 0$ for large t follows in a similar way. □

4. Applications

Here we present some applications of our results to a particular case of (1.1), namely, the equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)\left|x(\varphi(t))\right|^{\lambda} \operatorname{sgn} x(\varphi(t)) = 0 \quad (\lambda > 0), \tag{4.1}$$

jointly with some suggestions for future research.

4.1. Coexistence of Both Types of AL-Solutions

Applying Theorems 2.1–3.3 to this equation, we obtain the following.

Corollary 4.1. (a) *Let $r(t) \neq 0$ for large t. Equation* (4.1) *has unbounded AL-solutions if and only if*

$$\int_0^{\infty} |r(t)| \varphi^{\lambda}(t) dt < \infty. \tag{4.2}$$

(b) *Assume either* (3.15) *or* (3.16). *Equation* (4.1) *has bounded AL-solutions if and only if* (3.1) *holds.*

Corollary 4.1 shows also that the deviating argument can produce a different situation concerning the unboundedness of solutions with respect to the corresponding equation without delay, as the following example illustrates.

Example 4.2. In view of Corollary 4.1(a), the equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^2}\left|x\left(\sqrt{t}\right)\right|^{3/2} \operatorname{sgn} x\left(\sqrt{t}\right) = 0, \tag{4.3}$$

where q satisfies (i), has unbounded AL-solutions, while the corresponding ordinary equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^2}|x(t)|^{3/2} \operatorname{sgn} x(t) = 0, \tag{4.4}$$

in view of Theorem 2.4, does not have unbounded AL-solutions. Moreover, if in addition $q''(t) > 0$ for large t, then from Corollary 4.1(b) (4.3) does not have bounded AL-solutions.

The following example shows that the opposite situation to the one described in Example 4.2 can occur.

Example 4.3. Consider the equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^3}x\left(t^2\right) = 0, \tag{4.5}$$

where q satisfies (i). From Theorem 3.1, (4.5) has bounded AL-solutions and the same occurs for the corresponding ordinary equation. Nevertheless, in view of Corollary 4.1(a), (4.5) has no unbounded AL-solutions.

Examples 4.2 and 4.3 illustrate also that the coexistence of both AL-solutions for (4.1) can fail. Sufficient conditions for the coexistence of these solutions immediately follow from Corollary 4.1.

Corollary 4.4. *Let $r(t) \neq 0$ for large t.*

(a) *Assume for large t*

$$\varphi(t) \geq t^{1/\lambda}. \tag{4.6}$$

If (4.1) *has unbounded AL-solutions, then* (4.1) *also has AS bounded solutions.*

(b) *Assume for large t*

$$\varphi(t) \leq t^{1/\lambda}, \qquad \operatorname{sgn} r(t) = \operatorname{sgn} q''(t). \tag{4.7}$$

If (4.1) *has bounded AL-solutions, then* (4.1) *also has unbounded AL-solutions.*

For the equation without deviating argument

$$x^{(4)}(t) + q(t)x''(t) + r(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0 \quad (\lambda > 0), \tag{4.8}$$

from Corollary 4.4 *we get the following.*

Corollary 4.5. *Let $r(t) \neq 0$ for large t.*

(a) *Assume $\lambda \geq 1$. If* (4.8) *has unbounded AL-solutions, then* (4.8) *has also bounded AL-solutions.*

(b) *Assume $0 < \lambda \leq 1$ and $\operatorname{sgn} r(t) = \operatorname{sgn} q''(t)$ for large t. If* (4.8) *has bounded AL-solutions, then* (4.8) *has also unbounded AL-solutions.*

4.2. Comparison with Some Results in [15]

As claimed, the existence of unbounded AL-solutions for (4.8) follows also from Theorem 1 in [15]. For $n = 4$ this result reads as follows.

Theorem A. *If*

$$\int_0^{\infty} |r(t)| t^{\lambda+1} dt < \infty, \tag{4.9}$$

then there exists a solution x of (4.8) *such that*

$$x^{(i)}(t) = t^{(i)} + \varepsilon_i(t), \quad i = 0, \ldots, 3, \tag{4.10}$$

where ε_i are functions of bounded variation for large t and $\lim_{t\to\infty}\varepsilon_i(t) = 0$.

Therefore, when $\varphi(t) \equiv t$, Theorem 2.1 ensures the existence of unbounded AL-solutions of (4.8) under a weaker condition than (4.9), namely,

$$\int_0^\infty |r(t)|t^\lambda dt < \infty. \tag{4.11}$$

On the other hand, Theorem A gives an asymptotic formula for such solutions.

4.3. An Open Problem

Equation (1.1) can admit also other types of nonoscillatory solutions, as the following examples show.

Example 4.6. Consider the equation

$$x^{(4)}(t) + x^{(2)}(t) - \frac{2t^2 + 4t + 26}{(t+1)^{7/2}}|x(t)|^{3/2}\operatorname{sgn} x(t) = 0. \tag{4.12}$$

In virtue of Corollary 4.1(b), (4.12) has no bounded AL-solutions. Nevertheless, this equation admits nonoscillatory bounded solutions because $x(t) = (1+t)^{-1}$ is a solution of (4.12).

Example 4.7. Consider the equation

$$x^{(4)}(t) + x^{(2)}(t) + \frac{t^2 + 4t + 10}{(t+2)^4(\log(t+2))^3}x^3(t) = 0. \tag{4.13}$$

Thus, (3.1) holds, while $\int_0^\infty t^3 r(t)dt = \infty$. Hence, in virtue of Corollary 4.1, (4.13) has bounded AL-solutions, but no unbounded AL-solutions. Nevertheless, this equation admits nonoscillatory unbounded solutions because $x(t) = \log(t+2)$ is a solution of (4.13).

The existence of nonoscillatory solutions x satisfying either $\lim_{t\to\infty}x(t) = 0$ or $\lim|x(t)| = \infty$, $\lim_{t\to\infty}x'(t) = 0$ will be a subject of our next research.

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