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Title: The fundamental theorem of finite games.

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The paper provides a comprehensive examination of the fundamental theorem of finite games, a cornerstone of combinatorial game theory often referred to as Zermelo's theorem. The theorem asserts that in any finite two-player game of perfect information, one player must possess a winning strategy, or both players must possess drawing strategies. The author investigates the theorem by presenting five distinct proof methods (win by not losing, induction on the size of the game tree, back-propagation, De Morgan's law and game values recursion). Initially, the focus is on games without draws to establish the foundational logic before generalizing the result to the full version. To address the general version of the theorem, the author employs a mechanism of auxiliary games. By defining secondary games where draws are counted as wins for a specific player by default, the author demonstrates that the absence of a winning strategy for one player necessarily implies a drawing strategy for the other (or vice versa), thereby confirming the fundamental theorem for all finite games.

A significant portion of the work is dedicated to extending these finite methods to infinite games of perfect information, specifically regarding determinacy for clopen (where the winning and drawing conditions are clopen sets in the topology of move sequences), and open games (where the winning conditions are open sets). About clopen determinacy, the author notices that three of the five methods for finite games (win by not losing, induction and back-propagation), generalize to infinite games with the finite-play property (albeit infinite, the game features no infinite play and the game tree is well-founded). For open games, the author observes that while the win-by-not-losing method only works partially, game-value recursion remains robust, establishing determinacy through transfinite ordinal values.

It is noted that in the context of infinite games, these proofs rely on the axiom of choice for the amalgamation of strategies from child nodes.